Teaching High School Students Parametric Functions Through Covariation

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By

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INTRODUCTION

The mathematical topic of parametric functions is not one that would typically be found in a high school curriculum. However, we felt that this topic is one that not only could be taught in a high school curriculum, but it can be an appropriate and natural “next phase” if the emphasis in the course is covariational reasoning. Moreover, the understandings that students build in learning about functions defined parametrically can be leveraged to introduce other topics normally not included in the secondary curriculum, such as functions three-dimensional parametric functions and functions of two variables.

Covariational reasoning refers to the ability to form an image of two varying quantities and coordinating their changes in relation to each other (Oerhtman, Carlson & Thompson, 2008). The curriculum of a typical high school mathematics course unfortunately puts very little emphasis on the idea of covariation. It is an idea that can possibly be found (if you’re looking hard enough) in the proportional reasoning unit and loosely in the linear function unit. However, even within those contexts the emphasis is not on covariation as defined above, but instead on performing very specific calculations, manipulations, and disjoint representations. Students who think about functions only in terms of symbolic manipulations and procedural techniques are unable to comprehend a more general mapping of a set of input values to a set of output values (Carlson, 1998; Monk & Nemirovsky, 1994; Thompson 1994 as cited in Oerhtman et al., 2008). This idea of general mapping is exactly what we DO want our students to be able to do while reasoning covariationally and right now, students are not receiving that support. Teachers are frustrated to have students who year after year do not understand the idea of functional relationships and who have limited understanding of the multiple representations of function; but it is not surprising when you examine what the curriculum emphasizes. Students get lost in the notation, the manipulation and the "rules" of functions because they are presented as just that: notations, manipulations and rules. Students do not understand why the notation is a powerful and meaningful representation. Students are not given the opportunity to explore different representations of function definitions and how manipulating them can either maintain or change their meaning. Students are not held accountable for understanding why various rules regarding functions exist, and how each rule was established or developed.

The ideas of function and variation also rely on students having a rich conception of variable. Research on high school and beginning college mathematics students suggests that students often possess "very limited superficial concept of variable" (Trigueros & Jacobs, 2008 p. 4). Trigueros and Jacobs (2008) describe a study conducted by Ursini and Trigueros (1997) that brought these superficial understandings out in the open. Students in this study demonstrated limitations in their understanding of variable by demonstrating comfort in solving for x in $x + 3 = 7$, as well as investigating in a tabular nature the relationship $y = 2x$. However, the students fell short when asked to discuss the role of a variable in a complex equation or function definition. The findings of this research are consistent with our observations of students' understandings of functions and variables. Trigueros and Ursini's (1997) study points to the obvious—students learn what we teach and do not learn what we neither teach them nor expect of them.

Research on obstacles students meet because of weak conceptions of variable, variation, covariation, and function led to our decision to not only make covariation and function the heart of our mathematics course, but to see just how much high school students could do when trained to think covariationally. Specifically, we decided to see in what way students who began with a foundation in covariational reasoning would respond to instruction on parametric functions that leveraged covariational reasoning.

The concept of parametric function is nothing more than covariation at its core. In order to understand a function defined parametrically, like $(x,y) = (f(t), g(t))$, one must be imagining scanning through values of one function and keeping track of values of another function (Oerhtman, Carlson, & Thompson, 2008). When $f(t) = t$ you have standard covariation between a function and its argument. Depending on the complexity of functions $f(t)$ and $g(t)$ this idea can indeed become challenging, but with covariation as the primary way of thinking, even a complex parametric function can be reasoned through.

With covariation as our foundation, we planned to have a group of high school students reason about and define functions parametrically in a computer graphing program, as well as create graphs that were results of reasoning parametrically. However, before we could begin a unit on parametric functions it was imperative that we lay the groundwork of ideas that would be necessary for students to reason covariationally.

THEORETICAL FRAMEWORK

Covariation will end up being the all-encompassing goal that will allow our students to reason about parametric functions. Reasoning covariationally is beneficial on a number of levels: (1) It supports and aids the student's ability to reason about functions with a process view. This is, to view a function as a general, dynamic process that defines a mapping between input and output values (Oerhtman et al., 2008). (2) It fosters a student's ability to understand graphical, tabular, formulaic, and inverse aspects of a given function. Covariational Reasoning is a tool that can be applied to any mathematical model or situation (unlike all the disjoint formulaic approaches the text would teach).

From past experience we know that a vast majority of students come to us with what is referred to as an *action view of a function* (Dubinsky & Harel, 1992). An action view of function refers to a static conception of function in that the subject will tend to think about evaluating it one calculation at a time (Oerhtman et al., 2008). A student with an action conception, for example would be comfortable with plugging in specific values into a function definition and performing calculations. They would not however, be able to explain the overall behavior of the function. An action view of function has many limitations: (1) Students may be unable to disregard specific computations and therefore are unable to imagine running rapidly through all input-output pairs. Instead the student is too focused on the need to do each computation one at a time, which in theory is impossible! (2) Students will often think of a graph as a *picture* instead of the general mapping of all input-output pairs (Oerhtman et al., 2008 pg. 32). (3) The ideas of inverse, composition, piecewise, domain and range are all ideas that are full of holes and misconceptions if the student has an action understanding of function.

Students who have a weak understanding of function are on a track that will only worsen as their math careers progress. Many students will have a hard time distinguishing between a functional relationship definition and an equation (Carlson, 1998). The difference between a covarying relationship and a situation of two expressions being equal to each other is not a topic that students are often required to think about in the current math curriculum. Problems are usually presented as pre-set-up functions or equations with an immediate set of directions as to what to do with them. If students are not asked to consider the implications of these representations, their differences, their meanings etc, it comes at no surprise that this is an area of confusion as they progress through mathematics courses. Students also struggle with the transformation of functions. In one study 43% of A students at the college algebra level could not correctly find *f(x + a)* for a given function (Carlson, 1998). Again, when the focus of a math curriculum is not to require students to understand the ideas of domain and input/output of a function, but instead is a set of rules regarding shifts, picture translations etc this finding is no surprise. Many current assessments even enforce these misconceptions. Students are asked questions like:

> *What will happen to the graph of* $y = x^2$ *if the definition is changed to* $y = (x-2)^2 - 3$*?*

The "correct" answer as the text answer key would put it, is: shifted 2 units right and 3 units down. There are no questions regarding the inputs and outputs of the function. There is no scenario that goes with it that asks about the implications of variable definition between the two definitions. The goal of this type of question is solely to asses a students ability to memorize the set of rules that define the shifts, reflections, stretches etc that are a result of changing a function's definition.

A more powerful way of thinking about the original function as well as the altered definition would be in the context of function behavior and variable definition. For instance, if the first function were named *f(x),* then the second function's would be given by *f(x – 2) – 3*. The conversations that can now take place would first be centered around the behavior of *f(x)*. The fact that *f(x)* has a linear rate of change and has concavity based on the rate of the rate of change, and how the direction of the curve will change from decreasing to increasing (or visa versa) whenever the rate of change has it's zero. Once all of these phenomenon are fully understood one can begin to investigate the effect of changing *f(x)* to be *f(x – 2)*.

Each and every input value on *f(x)* are now linked with input value *(x – 2)*. It's like changing your point of view. The function *f(x)* has it's turning point at an input *of x = 0*. This new function *f(x)* will have it's turning point at an input of *(x – 2) = 0* or *x = 2*. The overall behavior of the graph will still be the same, due to the fact that they are both quadratic in nature, the difference is simply that they're input and output values are based off of different set of input variable definitions. If variable *x* were defined as hours since noon, then *(x-2)* would be the same phenomenon only from the perspective of hours since 2pm. Having a solid understanding of the implications of variable definition and the parent function's behavior will be able to answer any "shift, stretch & flip" question you could come into contact with! Again, unfortunately these are not the ideas taught, implied or even mentioned in a traditional text setting.

As a result of all of the current limitations, one of the things that would need to play a dominant role in the design and implementation of this course would be to help students move from an action view of function to a process view of function as defined earlier. Cultivating a process view will allow the students to move beyond a felt need to compute every input to get resulting outputs. Instead they will be able to imagine a function rule being applied over a continuum of input values and begin to imagine the overall behavior of a function.

This process view of imagining all values at once and how they coordinate is precisely what covariational reasoning makes possible. Students must be thinking about how one variable changes (often the dependent variable) while imagining changes in the other (often the independent variable) (Oerhtman et al., 2008, pg. 35). The types of problems and activities that we choose to nurture this will play a crucial role in this goal being met. Standard textbook questions will not be helpful. In the sections within our text aimed towards understanding function, the types of questions you'll find are:

Evaluate f(5), f(-2) and f(10) for $f(x) = x^2 - 1$

Create a table of values and sketch a graph for $y = 4x + 6$

Determine if the following graphs represent a functional relationship (Where a selection of graphs will be given)

Determine the domain and range for the relation {(-3, 2), (-5, 1), (8, -3), (-3, 0)} then state whether or not the relation is a function

All of these questions support the action view of functions and therefore contradict the goals that we have with our students. Specifically, these questions make the notion of function rule seem like a very technical process of plugging in values, the properties of function are an application of definitions, and the graphs of functions are a result of a very limited set of solutions that are somehow magically expanded to represent all values on a graph.

In order to meet our goals of covariationally reasoning about functions, we will need to provide activities where discussing a function rule as a general mapping is *natural* in a given context—situations where simply knowing a few solution points does not do the function rule justice. Models that can be reasoned about one small input at a time, while all along keeping track of successive changes in output so that a general trend can be reasoned through and the "shape" of a graph is not a shape at all, but a trend of changes. One technique that we will make the most of in order to cultivate this way of thinking is called the *finger tool.*

The finger tool is a way for students to imagine two quantities varying simultaneously and to have their imaginings come to fruition by keeping track of each instance, as ordered pairs, that represent those two varying quantities. Envisioning a single quantity's variation having momentary states is the first step to achieving this goal. (Thompson, 2002, p. 207). Students might first be asked to keep track of an independent variable, such as time, by sweeping their pointer finger to create an imaginary segment whose length would represent the elapsed time. The next phase is to repeat the process to keep track of the dependent variable by sweeping their other pointer finger vertically. After solidifying the two quantities independently, the students are asked to sweep both fingers simultaneously, keeping the independent sweeps horizontal and the vertical sweeps vertical. Finally the simultaneous record is repeated only this time dependent finger is aligned over the independent finger in order to record the two quantities simultaneously (Saldanha & Thompson, 1998; Thompson, 2002).

With this way of thinking, domain and range will need not to be definitions, but to be necessary things to consider when having discussions of independent and dependent variable. Each function definition and analysis will need to provide opportunities for students to talk about what the varying quantities are, *how* those quantities vary and how that analysis is apparent in both a graphical and tabular sense.

Once students have formed this image of co-varying quantities, they can then begin to discuss rate of change in very sophisticated ways. They can investigate ideas of average rate of change, the changes in the rates of change and the instantaneous rates of change. Analysis of rate of change will allow students to imagine very dynamic relationship between the varying quantities. This general yet powerful approach will be imperative in order for students to reason about a function parametrically.

METHOD

In order to accomplish the goal of reasoning about functions defined parametrically, we knew we would need to take our students current level of action view and transition them into process view. In order to accomplish this we took our students through a series of activities that focused on modeling situations with unevaluated expressions, analyzing changing quantities, modeling situations that involve changing quantities, and on to parametric functions. As an initial bridge to cross over into process view we began with building a foundation of writing unevaluated expressions to model complex situations.

Building a foundation of writing unevaluated expressions was done so that students developed a strategy as to how to approach modeling. We presented them with situations like:

Light travels at a speed of approximately 186,282 mi/sec.

1. The earth is approximately 25,000 miles around at the equator.
    a. How many seconds would it take to travel around the earth at the speed of light?
    b. How is this question like finding your height in miles?

Students were expected to write out an expression that would calculate the number of seconds it takes light to travel around the equator, without actually calculating this value. This was done in order to allow the students to begin seeing how to represent this situation symbolically. Having our beginning focus be on modeling, also allowed us to strengthen the student's concept of variable and foster their ability to overcome a merely computational way of thinking about symbols (Trigueros & Jacobs, 2008). One thing we did that is consistent with the research conducted by Trigueros and Jacobs as well as our goal, was to have students explain what each aspect of a symbolic expression means. This is not only to have them define their variables meaningfully, but to talk about what various products, sums, quotients etc mean along the way. For example, the students were given the following question:

Suppose CJ could walk at a constant speed of 3.28 ft/sec.

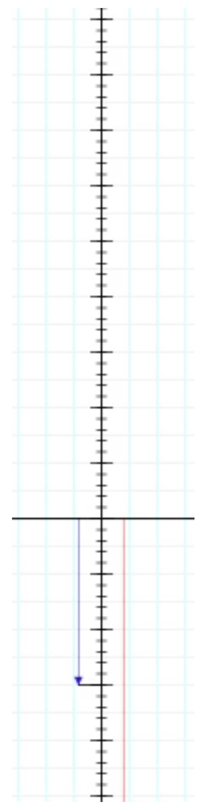
Is it reasonable to say that someone walks at 3.28 ft/sec? Why?

Explain what these expressions represent in regard to CJ walking at 3.28 ft/sec.

$$S_1 = 3.28$$
$$D_1 = \frac{(12S_1)2.54}{100 \cdot 1000}$$
$$\frac{1000D_1}{60 \cdot 60}$$

As part of their explanation they were expected to state not only that D_1 represents CJ's speed in meters per second, but also that $12S_1$ represents CJ's speed in inches per second, that $(12S_1)2.54$ represents CJ's speed in centimeters per second, that $\frac{(12S_1)2.54}{100}$ represents CJ's speed in meters per second, and so on. Having students explain all of these things, not only keeping track of what the variable means but what the units are on various outputs along the way.

After students had opportunities to write and explain all parts of expressions written both numerically and symbolically, it was time to move on to varying quantities. In order to make connections between writing expressions and writing functions, where something is now varying, students were asked to view animations and complete tasks like the following:



> The animation expresses a relationship between two magnitudes (amounts of quantities). They are in the folder *Function Relationships* on your desktop.
>
> You are to do two things:
>
> 1) Devise a general strategy for determining what the relationship is in any animation.
>
> 2) Represent the relationship using any letters (variables) you wish.

The animation mentioned in the above questions was a movie of two arrows that moved in a way that their magnitudes were related by a linear function. That is, changes in the magnitude of the red arrow (*r*) were always some number of times as large as changes in the magnitude of the blue arrow (*b*). Thus, the magnitudes were related by the function *r* = *mb* + c, where *c* is the magnitude of the red arrow when the magnitude of the blue arrow is zero.

These tasks were designed to allow the students to use the strategy that had been developed previously when writing expressions and apply it to varying quantities. This bridged the gap between simply writing expressions and analyzing changes in quantities. Therefore the students were now ready to model situations that involve changing magnitudes. For example:

> A Paris newspaper wishes to help Americans with converting from mi/gal to liters/100km by publishing a function definition for the conversion. Pretend you are an employee who is given the task of producing this function. Let *u* stand for a car's fuel efficiency measured in mi/gal and let *f* stand for the car's fuel efficiency measured in liters/100km. Write a function definition to express *f* in terms of *u*. Explain your definition.
>
> Function definition:
>
> Explanation:

Students were expected to write a function definition and then go through the process of explaining what each part of their definition represents with the units. Tasks like these allowed students to model situations that involved variation. Since the mathematics of variation involves imagining a quantity whose magnitude varies, the very process of imagining a magnitude having different values at different moments in time is at the core of understanding parametric functions (Thompson, 2008, personal communication). Once the students began to understand varying magnitude we completed all of our pre-unit goals of fostering students ability to use and understand the idea of variable, represent situations, and think covariationally. We then introduced the ideas of parametric functions in order to allow the students to build on these ideas.

Parametric functions were introduced using a program called Graphing Calculator (GC). We were able to use this program as a way to introduce parametric functions since each student has a laptop to work with. We have discovered that students tend to work more productively when they have someone to bounce ideas off of, so students were assigned to work with a partner and each pair had a laptop. The student's interests were peaked by suggesting that they were going to learn how to create some of the interesting graphs that had been used and presented earlier in the course. Students were led through a series of activities beginning with how to plot a point using the program. They were asked question like:

> Plot three points on the graph of the function $y = x^2 - 6x + 7$. Give their coordinates below. How do you know your points are on the graph of $y = x^2 - 6x + 7$?
>
> $$\begin{bmatrix} \_ \\ \_ \end{bmatrix} \begin{bmatrix} \_ \\ \_ \end{bmatrix} \begin{bmatrix} \_ \\ \_ \end{bmatrix}$$
>
> Explanation:

One solution to this question is: $\begin{bmatrix} 3 \\ -2 \end{bmatrix} \begin{bmatrix} 4.3 \\ -0.31 \end{bmatrix} \begin{bmatrix} 9 \\ 34 \end{bmatrix}$, which students made by calculating each value of y for each respective value of x. A second solution to this question is to define a function f as $f(x) = x^2 - 6x + 7$ and then list the points $\begin{bmatrix} 3 \\ f(3) \end{bmatrix} \begin{bmatrix} 4.3 \\ f(4.3) \end{bmatrix} \begin{bmatrix} 9 \\ f(9) \end{bmatrix}$. This type of question was designed not only to teach students how to use GC but also introduced them to the idea of using functional notation $f(x)$ in order to easily re-use a computational rule. We then showed students how to animate a point and we asked them to perform different tasks with the animated point. For example:

> **Animating a point in Graphing Calculator**
>
> Now it's time to animate some points of your own. I've given you some tasks to

get you started, but don't limit yourself to them. If something interests you, try it out! If you have an idea, go with it. Test things out and see what Graphing Calculator can do.

1. Using Graphing Calculator, animate a point that moves on a horizontal line.

2. Animate a point that moves on a diagonal line.

3. How could we make two points move simultaneously on the same diagonal line?

4. Move a point on a path that is not a line. Be creative. Describe how you created your Graphing Calculator program; include a description of any changes that you made that wouldn't show up on a print out.

This is where the student's ability to reason covariationally was tested. In order to create a point that can move in GC one must define points using *n.* GC uses *n* as a parameter whose value changes sequentially from the lower end of its domain to the upper end in a stipulated number of steps. GC then updates the value of *n* in all of its uses in that document. If one defines a point in GC as $\begin{bmatrix} n \\ 3n \end{bmatrix}$, then it will plot the point $(n,3n)$ each time it updates the value of *n*, thus giving the appearance that the point is animated. Thus to imagine the behavior of the point, one must imagine how both functions would behave (the function that controls the movement vertically and the function that controls the movement horizontally) when being evaluated for successive values of *n*. Now that the students were able to plot and animate points it was now necessary for us to introduce the idea of defining a function using *t.* We told students that GC uses the variable *t* to represent all possible values of *n*. Thus, re-defining $\begin{bmatrix} n \\ 3n \end{bmatrix}$ as $\begin{bmatrix} t \\ 3t \end{bmatrix}$ would cause GC to plot all points having these coordinates instead of animating (a subset of) these points one at a time. Students were then given tasks like:

> Play your Point Animation (task 4 from above) again. If you were to use *t* to show every possible location of the animated point in that animation, what would the result look like? Draw a sketch.

> Use *t* to show every possible location of the animated point in your Point Animation task. How closely does your sketch match Graphing Calculator?

These tasks were designed to test student's ability to covary, since the variable *t* is used to show the path that the animated point travels. The students now had all of the skills required to define functions parametrically, they were given a project to show just how much they really understood.

The project was introduced as a favor to another teacher. The teacher was asking my class to design a program that her students could use in order to visualize the following scenario:

> *The city of Northport maintains a very active shipping business. Due to the high volume of traffic in the harbor, all ships in Northport water must register their location and direction with the Northport shipping authority. Northport defines these locations and speed using a grid system of horizontal and vertical distances from the center of the harbor. Currently, Northport shipping authority is tracking with concern two small boats: the USS Spirit of the Blue, and the USS Greensboro.*

> *The USS Spirit of the Blue has registered a horizontal distance from center of 5.4km, and a vertical distance from center of 3.55km. The Greensboro's registered speed is a horizontal change of -1km per minute, and a vertical change of -0.8km per minute.*

> *The USS Greensboro has registered a horizontal distance from center of -0.5km, and a vertical distance from center of 3km. The Greensboro's registered speed is a horizontal change of 2 km per minute, and a vertical change of -0.5km per minute.*

> *Upon running a simulation of these two paths, the Northport shipping authority has become concerned that the two boats might crash. The boats are traveling in heavy fog and are unable to see each other, so Northport needs to*

advise the boats if a course change is needed.

Determine whether or not the two boats will crash.

Students were asked to create an animation that other teachers could use to help students understand that although the ships' paths always cross, the ships might not actually crash. The animation could also help the teacher hold conceptual conversations about how one might determine ahead of time, by just knowing the ships' initial locations and rates of change along horizontal and vertical axes, whether the ships will crash.

As part of the project the students were to create a user manual for how their program worked as well as an appendix. Due to the complexity of the project we allowed the students three days to complete it. Students' responses to this project will be discussed in the Findings section.

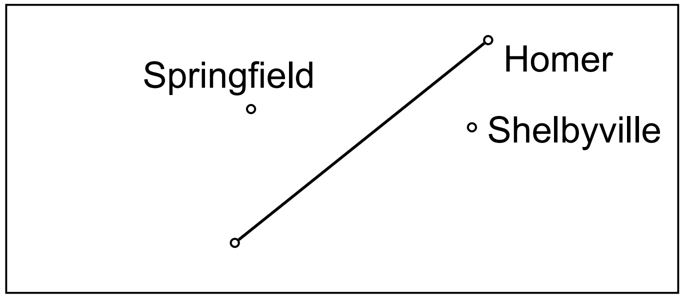
Since all situations in this unit prior to this project had been linear we decided that we should expand the students ability to reason covariationally and introduced them to non-linear situations. The non-linear situations were introduced to the students through the classic City A - City B problem with a small tweak: Intro the Simpsons. The students were given a road that runs parallel to two cities; they were asked to keep track of a Homer's distance from the cities as he traveled along the road. The students were asked to analyze situations like:

For each arrangement:

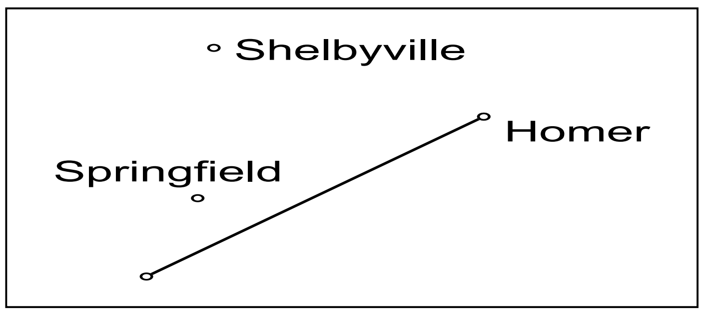
- Imagine the behavior of *Distance from Shelbyville* relative to Homer's position as you move the car from one end of the road to the other.
- Imagine the behavior of *Distance from Springfield* relative to Homer's position as you move the car from one end of the road to the other.
- Imagine the behavior of *Correspondence Point* as you move Homer from one end of the road to the other.

- Sketch your prediction of the graph that *Correspondence Point* will make with this arrangement.
- Test your prediction

A.

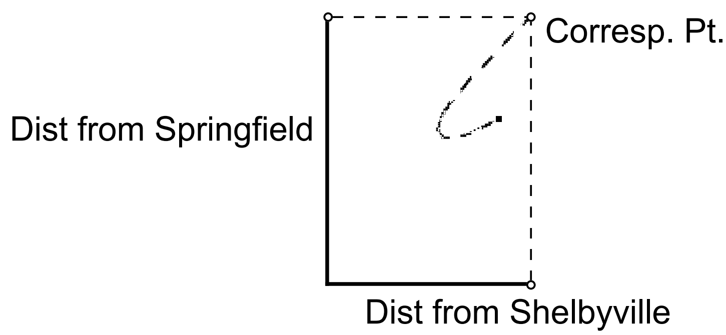


B.



These situations where designed to allow the student to practice their covariational reasoning and to re-emphasize thinking in terms of parametric functions. Let *h* represent Homer's distance along the road from his start, and *Sp*(*h*) and *Sh*(*h*) represent Homer's distances from Springfield and Shelbyville, respectively. Then the parametric function (*Sh*(*h*),*Sp*(*h*)) represents the two distances simultaneously for each value of *h*, and also represents a point in the Cartesian coordinate system.

The students were re-introduced to the finger tool as a method to help them reason through situations involving covarying quantities. They were asked to use their right index finger to track Homer's distance from Shelbyville along the bottom of their desk. Allowing the students to focus their attention on only one thing that is varying. After the students could keep track of the distance from Shelbyville no matter how quickly or in what direction Homer drove, we then asked them to use their left index finger to track Homer's distance from Springfield along the left side of their desk. Again allowing the students to focus their attention on only one thing that is varying. Finally the students were ready to demonstrate their ability to track the two quantities simultaneously as they covaried. They were to use both left and right index fingers to track both Homer's distance from Shelbyville and Homer's distance from Springfield. However instead of keeping their left index finger along the right side of their desk they needed to keep it perpendicular to with their right index finger. In essence the students were graphing the parametric function on their desktops. They were sketching representations like:



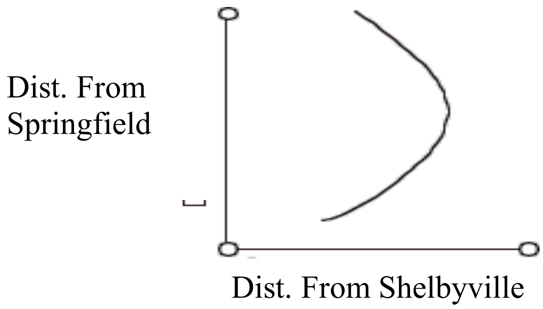
Unlike the ships in the fog unit where students were able to think about each ships time and distance varying independent of the other ship, this activity forces them to think about how the changing distances vary together.

We then gave student parametric graphs of various arrangements of the road, Shelbyville, and Springfield, asking them to draw the diagrams of where the cities and road could be so that they would generate a particular graph. They were presented with questions like:

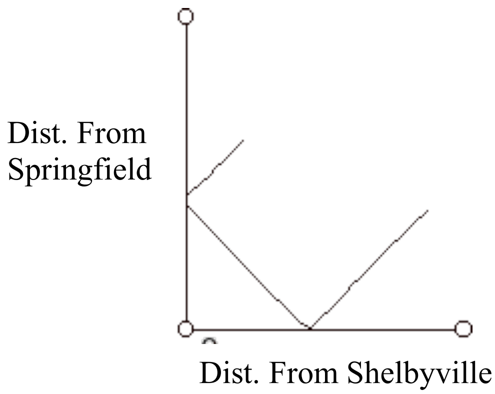
For each graph, do the following. Do this without the computer.

- Describe key features of the graphs and what this tells you about the possible locations for the cities relative to the road. (For example: The graph intersects the horizontal axis, so I know that at some point the road passes through Shelbyville.) Describe as many things as you can think of, including observations about how the distance from city appears to change as Homer gets closer or further from the other city.
- Draw a picture of the road and the location of the two cities that you believe will create the given graph of the correspondence point.

A.



B.



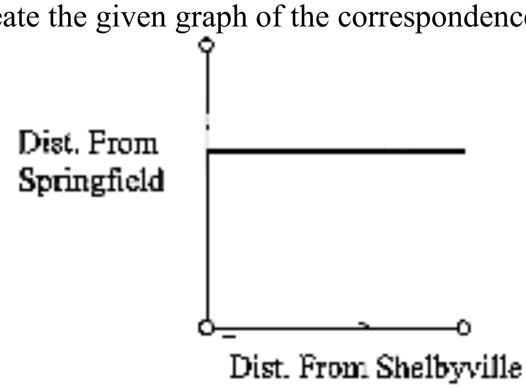
Students were expected to analyze the changing distances in order to create the original setup of the cities. All that was left was to interview the students to see what they know about variation. All materials used in this unit are given in Appendix A.

Each student was interviewed once and the interviews lasted approximately 50 minutes. The interviews followed a detailed protocol (Appendix B) that was written by Arizona State University professor Dr. Patrick Thompson, Arizona State University PhD student Carlos Castillo-Garsow, and Arizona State University PhD student Sharon Lima. The interviews began by asking the students how they felt the Ships in the Fog project went. Then the interviewer proceeded to ask students a list of task based questions designed to learn what the students understood about function notation, defining functions, defining functions parametrically, and how restricting the domain of a function affects the range of a function. The interview protocol can be found in the appendices.

FINDINGS

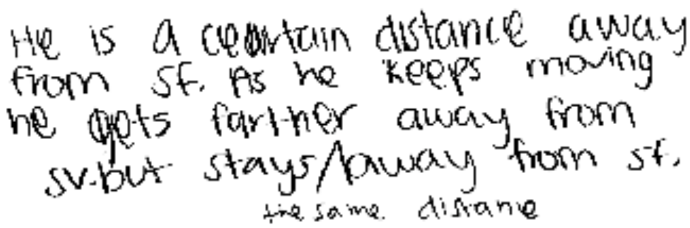
One major finding was that students had in fact learned to reason covariationally during the parametric function unit. This was evident within the context of a homework assignment designed to assess just that. There were two days where covariational reasoning was the primary concept being addressed. One the first day 77% of the students demonstrated either strong understandings or understandings with moderate flaw and on the second day 80% of the students demonstrated either strong or moderately flawed understanding. The type of questions that the homework posed was as follows:

> 1) Describe key features of the graphs and what this tells you about the possible locations for the cities relative to the road. (For example: The graph intersects the horizontal axis, so I know that at some point the road passes through Shelbyville.) Describe as many things as you can think of, including observations about how the distance from city appears to change as Homer gets closer or further from the other city.
>
> 2) Draw a picture of the road and the location of the two cities that you believe will create the given graph of the correspondence point.



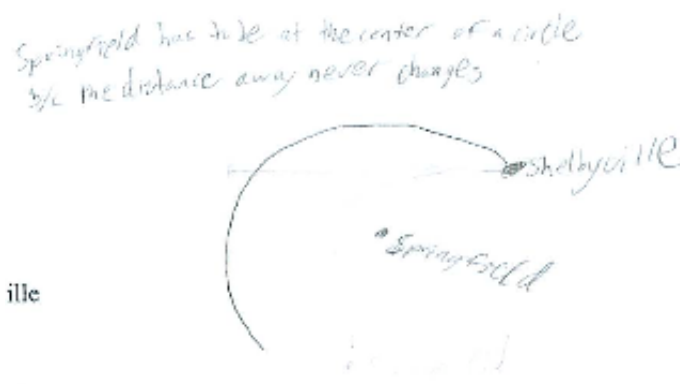
In order to answer this question, students need to thinking about the distance magnitudes involved, how they change with respect to each individual city, and also how they change together. Once they make sense of the covariation that is taking place, they also then need to imagine the orientation of the cities as well as the road that is being traveled on so that this graph would be an accurate portrayal of the variation. Here are some sample student responses that support covariational reasoning:

Student Response 1



This response shows that the student understands that the given implies that one of distances is increasing, while the other distance is staying constant.

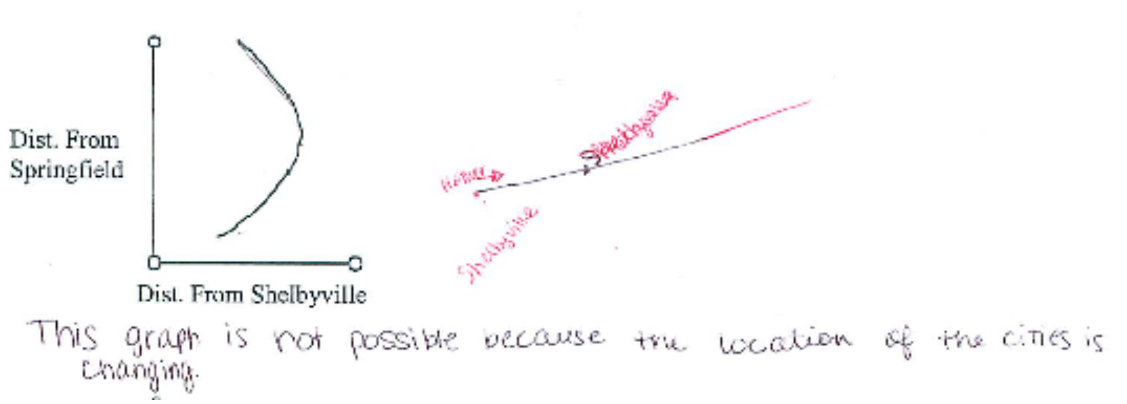
Student Response 2



This student response confirms covariational reasoning for the same scenario in that they have set up the road so that if you travel along the road you will get farther away from Shelbyville while always remaining the same distance from Springfield.

Another graph scenario that was given under the same set of directions is shown below, along with a student's response to that scenario:

Student Response 3



This response confirms covariational reasoning in that the student realizes that the given graph implies that for one particular distance away from Shelbyville, there are two potential distances away from Springfield. This is not possible (assuming the cities stay fixed) and the student realizes that!

Another important understanding the students gained was the concept of variable. Throughout the Ships in the Fog project the students dealt with variables and their definitions. Students were asked to write an instruction manual so that another student/class could use their program. One group gave this response when defining the axes:

Group 1

The X axis is the horizontal distance from the center of the harbor which everyone has to agree on and which direction to (east or west).
The Y axis is the vertical distance from the center of the harbor which everyone has to agree on the direction of (north and south).

While this response has some minor grammatical errors it does show that the students understand that the axes not only represent a distance but also a distance with respect to a location (and a direction from that location).

Another group gave this response regarding the Rate of Change of the Ships:

> Group 2
>
> If you want to change the slope or ROC of the USS Greensboro, the -0.5 represents its Vertical (North/South) rate of change in value (Y). Whereas the 2 represents the horizontal (East/West) change in value (X). If you change one of these values you will change the slope of the segment.

This response shows that not only does the group understand what is being tracked on the axes but is also mentally tracking the change in the variables and interpreting these changes and their effects on the function as well as the graph. At the conclusion of this project 16% of groups showed a thorough conceptual understanding of variable and the affects of small changing upon the variable and 84% of groups showed a somewhat flawed understanding of these same concepts.

A third important finding was that students did not fully grasp the concept that when a function is defined parametrically it has a rate of change that maintains the ratio of the change of y to the change of x. This became apparent when the students were asked the following question on a quiz and later during the interview process:

> Amanda entered commands into Graphing Calculator. I could see the definition of one of her functions (q), but not the definition of her function p. When I asked her for the definition of p, she said "I'll give you a hint. When you define p as I did, the GC commands below produce the same graph as produced by $y = 2x + 1$.

$p(s) =$
$q(w) = 0.5w + 1$
$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$
$t : -3 \ldots 3$

What was Amanda's definition of *p*?

This question is intended to get students to think about how the changes between *q* and *t* (the *y* coordinate) and the changes between *p* and *t* (the *x* coordinate) work together on the function's graph to result in the function's overall rate of change. Evidence through interviews lead us to see how many students were able to reason about the rates of the two given functions *q(t)* and *y(x),* but that they were not able to then conclude how they were linked or what the rate of *p(t)* would need to be in order to hold the relationship together. An ideal answer would be to imagine *t* varying by some small amount, then thinking that however much $q(t)$ changed, $p(t)$ had to change half as much. Also, when $q(t)$ is 1, $p(t)$ must be 0. Therefore $p(t) = 0.25t$. Only 13% of the students were able to demonstrate thorough understanding of this idea. 47% of the students were able to vocalize some understanding of the relationship, but with holes in their reasoning and another 40% of the students could not reason much at all about the ideas and relationships in this problem. Here are some sample student responses that supports this finding:

Interviewer: How fast does *y* change with respect to changes in *x*?

Student 1: Two times as fast.

Interviewer: How fast does *y* change with respect to changes in *t*, according to Amanda's definition?"

Student 1: Still two times as fast.

Interviewer: Earlier you mentioned that the *q* function has a role in the *y* coordinate… so how fast does that function change with respect to t?

Student 1: Oh! 0.5 times as much

Interviewer: What is the role of *q* function?

Student 1: Determines the *y* value of the graph

Interviewer: What's the role of the $y = 2x + 1$?

Student 1: The graph that you come up with from *q(t)* and *p(s)* is going to be the same as the $y = 2x+ 1$. … The q function will determine the *y* coordinate of $y = 2x + 1$ … but I don't know how that helps find the *x* coordinates function…

This student understood how rate of change works within each individual function, but she did not understand that the rate of $y = 2x + 1$ is also the link between the rates of the functions defined parametrically. Another student got the "right answer" intuitively, but only in the interview itself did he make explicit the relationships that would allow him to infer the rate of change of *y* with respect to *x* based on each coordinate's rate of change with respect to *t*.

Interviewer: How did you come up with the answer you found?

Student 2: First I made the graph of $y = 2x + 1$ so that I could figure out what I could put in [*q(w)* and *p(s)*] that would give the same answer. And well, I knew that *q(w)* was going to be the y value and it goes from – 3 to 3, so I plugged in -3 and got – 0.5, so I plotted/found that *y* value on my graph and then I used the line to follow it up and get x and just see what function would give me that *x* as an answer.

Interviewer: And you did that all in your head?

Student 2: Yeah, maybe… it was right!! ☺

Interviewer: Tell me more about the role of *q(w)*

Student 2: It tells you that if you plug in a number for w, then it will give you a location on the *y*-axis, and then you'd have to plug in the same number to the *x* function (*p(s)*) it will give you the location on the *x*-axis and you'd have a point.

Interviewer: How fast does *y* change with respect to changes in *x* according to her definition?

Student 2: By 0.5. You'd take a value for *t* and to know what's going to come next you'd multiply it by 0.5 to know how *y* will change

Interviewer: Ah that's with respect to changes in *t*, what about with respect to changes in *x*?

Student 2: Oh, twice as fast

Interviewer: And then you found some relationship for *x* with respect to *t* as well?

Student 2: Yeah I found that $p(s) = 0.25s$ so it changes a quarter as much as *t*.

Interviewer: Ok so you have all these relationships, how do they all those pieces fit together?

Student 2: Well, if *y* is ½ as fast as *t* and *x* is ¼ as fast as *t*, then the slope would be 2!

By conducting these interviews we were able to gain a more clear understanding of what the students understood and what was still an area of struggle. Taking these struggles into consideration along with all of the triumphs, we were extremely pleased with the student's achievement overall. The sophistication with which students were able to reason covariationally was impressive to say the least. Witnessing their covariational reasoning get leveraged into parametric variation as it did in the Simpsons activity was a big confirmation that students are capable of understanding more sophisticated topics than we often give them credit for. Our main hope now is that the ideas that were built into this high school course will prove to be beneficial to those students that pursue higher levels of mathematics at the college level.

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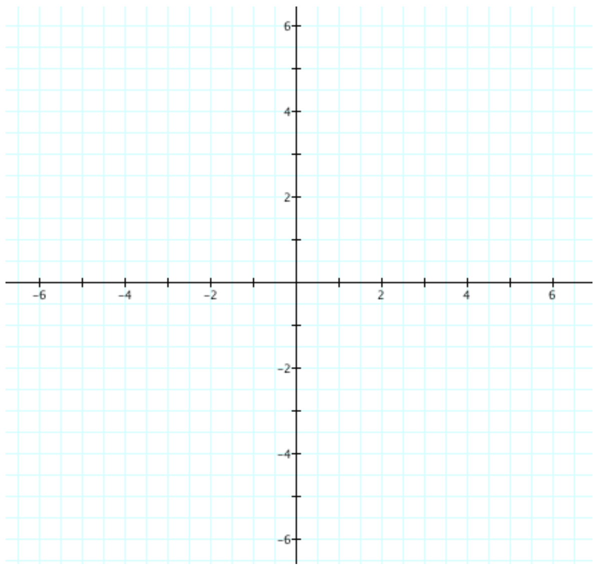
APPENDIX A: UNIT MATERIALS

Using Graphing Calculator to plot points

In order to create a vertical ordered pair, type ⌘+2, then enter the numbers

Below is a Graphing Calculator screen.

Where would the points (4,-1), (-3, 2.5), (-4, -1.5), and (-2.5, 4.7) appear on it?



Make Graphing Calculator plot the points (4,-1), (-3, 2.5), (-4, -1.5), and (-2.5, 4.7).
Did they appear where you expected?

Plot three points in GC that all lie on one line. Give their coordinates below. How do you know that they are all on one line?

$$\begin{bmatrix} \_\_\_ \\ \_\_\_ \end{bmatrix} \begin{bmatrix} \_\_\_ \\ \_\_\_ \end{bmatrix} \begin{bmatrix} \_\_\_ \\ \_\_\_ \end{bmatrix}$$

Explanation:

Plot three points on the graph of the function $y = x^2 - 6x + 7$. Give their coordinates below. How do you know your points are on the graph of $y = x^2 - 6x + 7$?

$$\begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix} \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix}$$

Explanation:

Plot three points on the graph of $y = x^3 - 2x^2 + 4x - 1$ where x has the values 1.15, 2.03, and 3.27. Hint: Thank about how you can have GC plot those three points without you having to calculate any values of y?

Animating a point in Graphing Calculator

Now it's time to animate some points of your own. I've given you some tasks to get you started, but don't limit yourself to them. If something interests you, try it out! If you have an idea, go with it. Test things out and see what Graphing Calculator can do.

5. Using Graphing Calculator, animate a point that moves on a horizontal line.

6. Animate a point that moves on a diagonal line.

7. How could we make two points move simultaneously on the same diagonal line?

8. Move a point on a path that is not a line. Be creative. Describe how you created your Graphing Calculator program here, include a description of any changes that you made that wouldn't show up on a print out.

9. Print out your work and keep it. We will be using it again later.

10. **The past, present and future of an animated point**

For these activities, you will need to refer to your previous worksheet, "Animating a point in Graphing Calculator."

1. Describe the difference between the use of n and the use of t in representing points within Graphing Calculator.

2. Use t to show every possible location of the animated point in your Point Animation task 2 (a point that moves on a diagonal line).

3. Play your Point Animation task 4 (your creative animation) again. If you were to use t to show every possible location of the animated point in that animation, what would the result look like? Draw a sketch.

4. Use t to show every possible location of the animated point in your Point Animation task 4. How closely does your sketch match Graphing Calculator?

Animating a point to order

Carlos often has to create specific animations for specific purposes. Here's a list Carlos made of some animations he needs to create for Mrs. Bishop. Give them a try yourself.

1. Make GC graph 3y + 2x = 7.

2. Animate a point moving on the graph of 3y + 2x = 7.

3. Animate a point along the graph of y = x(x-1)(x-2)(x-3).

4. Have one point chase another along the graph of y = x(x-1)(x-2)(x-3). Make this point a different color.

5. a. Use *t* to make GC show all the locations that the animated point in task 3 will have. Use a different color than the ones used in task 3 and task 4.

   b. How does GC use your *t* representation in #5a generate the same graph as when you enter y = x(x-1)(x-2)(x-3) into GC?

6. *Challenge Problem*: Recall your trigonometry from Geometry: Have one point chase another along a circle. (How can you represent a point's *x*- and *y*-coordinates when a point is on a circle?)

From: Ms. John
To: Math 5 Students
Re: An exciting project!

Mrs. Bishop, another MHS Algebra teacher, is preparing to teach a unit on linear functions. She found an interesting mathematics problem about two small boats traveling in heavy fog which asks the question, “Here are the starting locations, speeds, and directions of two boats sailing in a dense fog. Will they crash?”

She also found a video that helps students visualize the problem. It shows the paths the ships take, and the locations of the ships along their respective paths at each moment of time. Mrs. Bishop wishes to use this and similar problems in her class, but she also wants the video to reflect the changed situation so that students can test their predictions about whether the boats will crash or pass safely.

Now, I learned that the video is really just a Graphing Calculator program that someone captured to video. I told Mrs. Bishop, “Don’t worry! My students can make a GC program that you can use to generate any video you want!” So, now you must deliver! You must construct a graphing calculator program that will display the paths of the two boats and the boats’ locations along their paths over time for any starting locations, speeds, and directions.

I also had another idea! We could write directions for how to use the program you create so that ANY teacher can use it to teach these problems! That means that you need to write an “instruction manual” that tells the teacher how to use it to make a simulation for *any* “ships in fog” scenario.

Your instruction manual needs to contain two things: (1) Directions for how to use the GC file you create; (2) an appendix that explains what each part of your GC program does. The user manual should be written using MS Word. It doesn’t need to be long, but it does need to be written well and give clear instructions and clear explanations.

Insert text blocks inside your GC program wherever you need to remind yourself of what a part of the program is supposed to do or what an expression represents.

Ships in the Fog

The city of Northport maintains a very active shipping business. Due to the high volume of traffic in the harbor, all ships in Northport water must register their location and direction with the Northport shipping authority. Northport defines these locations and speed using a grid system of horizontal and vertical distances from the center of the harbor. Currently, the port is heavily fogged in, and Northport shipping authority is tracking with concern two small fishing boats: the USS Spirit of the Blue, and the USS Greensboro.

The USS Spirit of the Blue has registered a horizontal distance from center of 5.4km, and a vertical distance from center of 3.55km. The Spirit of the Blue's registered speed is a horizontal change of -1km per hour, and a vertical change of -0.8km per hour.

The USS Greensboro has registered a horizontal distance from center of -0.5km, and a vertical distance from center of 3km. The Greensboro's registered speed is a horizontal change of 2 km per hour, and a vertical change of -0.5km per hour.

Upon running a simulation of these two paths, the Northport shipping authority has become concerned that the two boats might crash. Since the boats are traveling in heavy fog they will unable to see each other until it is too late. So Northport needs to advise the boats if a course change is needed.

Determine whether or not the two boats will crash.

Equivalence Handout

1. In terms of what Graphing calculator does, what is different about the meaning of these three sets of GC commands?

First set of commands:

$f(x) = 5 - 3x^2$

$y = f(x), -2 \le x \le 2$

Second set of commands:

$y = 5 - 3x^2, -2 \le x \le 2$

Third set of commands:

$x(t) = t$

$y(t) = 5 - 3t^2$

$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

t: ¯2 …2

2. In terms of what Graphing Calculator does, why do these three sets of commands produce the same graph?

Animation Handout

I've entered the following commands into Graphing Calculator:

$$x(t) = 5t - 1$$

$$y(t) = {}^{-}3t + 3$$

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

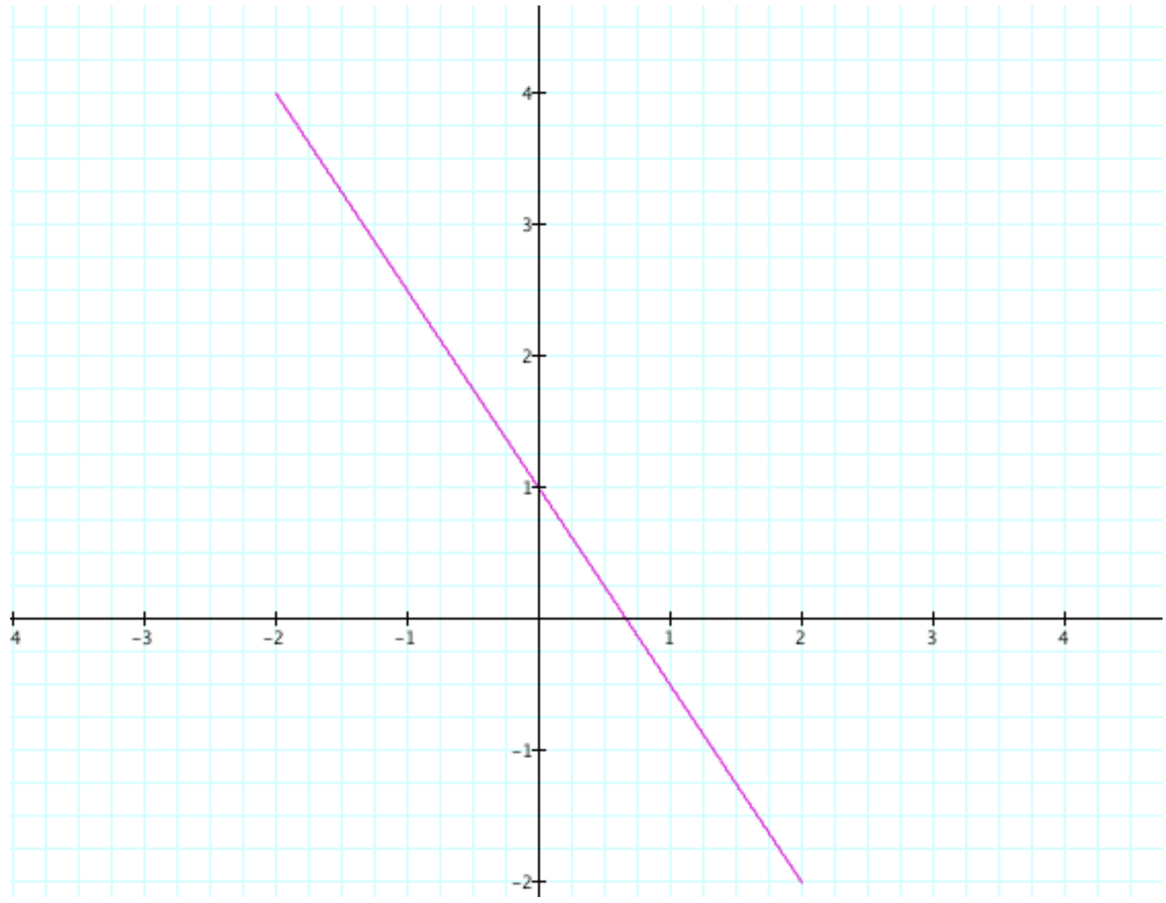
I've also set the slider values of *n*:

Lowest Value: -0.5
Highest Value: 1.2
Number of Steps: 201

Explain the behavior of $\begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$ when I press play on the *n* slider. Sketch a general graph to support your explanation.

Animation Handout (cont)

Write out a set of GC commands that will animate a point along this line:



Command Interpretation Handout

Imagine you've typed the following into Graphing Calculator already

$$f(x) = 5x$$

$$g(x) = 1 - \frac{2}{3}x$$

1. What point would $\begin{bmatrix} f(3) \\ g(3) \end{bmatrix}$ represent?

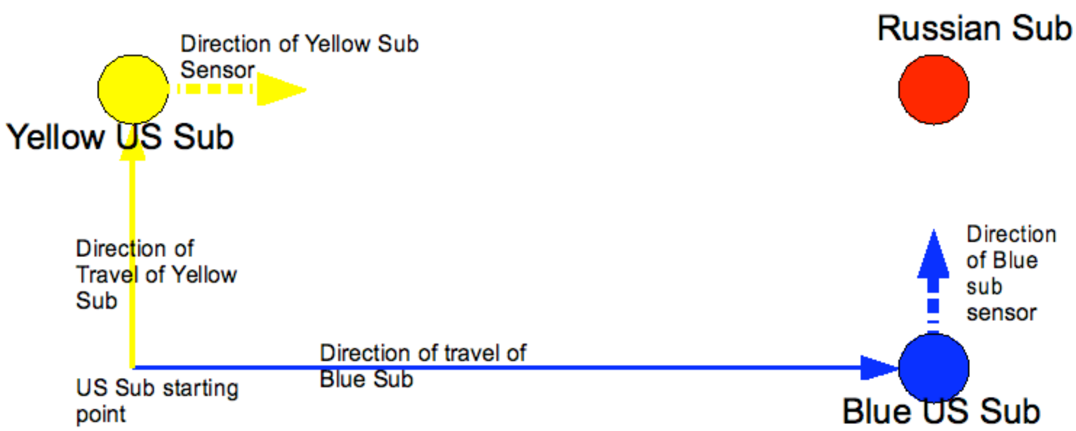
2. What does $\begin{bmatrix} f(n) \\ g(n) \end{bmatrix}$ represent, where n varies from 0 to 1?

3. What does $\begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$ represent, where t varies from -0.5 to 1?

Russian Submarine

Two American submarines are tracking a Russian submarine. They don't know how far away the Russian sub is, but each American sub has equipment that can detect the direction from the American sub to the Russian sub. The two American captains devise a plan to track the path of the Russian sub:

The American subs will move along a path perpendicular to the other's path. Each American sub will use its direction finding equipment to keep the Russian sub at a 90° angle from its path. Each American sub will keep track of its speed.



The Blue US sub reports moving at 3mph, and the Yellow US sub reports moving at 2mph. Write a set of Graphing Calculator commands that will show the location of the Russain Sub and its path relative to the paths of the American subs.

Quiz

Graphing Calculator Animations

Question 1

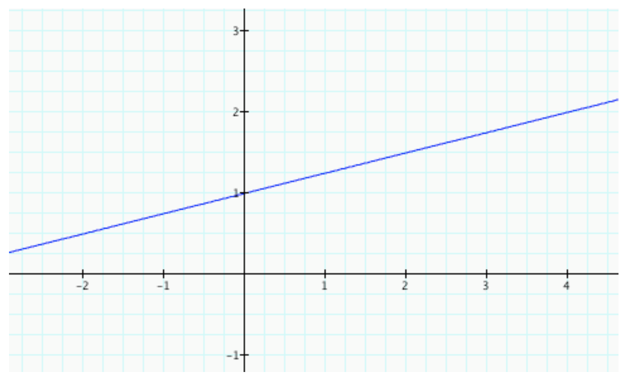
You've defined the functions *g* and *f* as:

$$g(m) = 2m + 1$$

$$f(s) = 0.5s + 1.25$$

a. What are the coordinates of the point represented by $\begin{bmatrix} f(^{-}0.25) \\ g(^{-}0.25) \end{bmatrix}$?

b. Write a Graphing Calculator command using the functions *f* and *g* that will produce the line shown here.



c. Write a Graphing Calculator command, using the functions *f* and *g*, to animate a point along the line shown above.

d. Write a second Graphing Calculator command that will animate a second point along the same line.

Question 2

a. Fill in these GC commands so that GC will animate a point. (The point can be animated any way you wish.)

$r(u) =$ ________________

$q(u) =$ ________________

$$\begin{bmatrix} \_\_\_ \\ \\ \_\_\_ \end{bmatrix}$$

b. Fill in this GC command so that GC will show all the locations over which your animated point in (a) will pass.

$$\begin{bmatrix} \_\_\_ \\ \\ \_\_\_ \end{bmatrix}$$

Question 3

Amanda entered commands into Graphing Calculator. I could see the definition of one of her functions (q), but not the definition of her function p. When I asked her for the definition of p, she said "I'll give you a hint. When you define p as I did, the GC commands below produce the same graph as produced by $y = 2x + 1$."

$$p(s) =$$
$$q(w) = 0.5w + 1$$
$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$$
$$t : -3\ldots3$$

What was Amanda's definition of p?

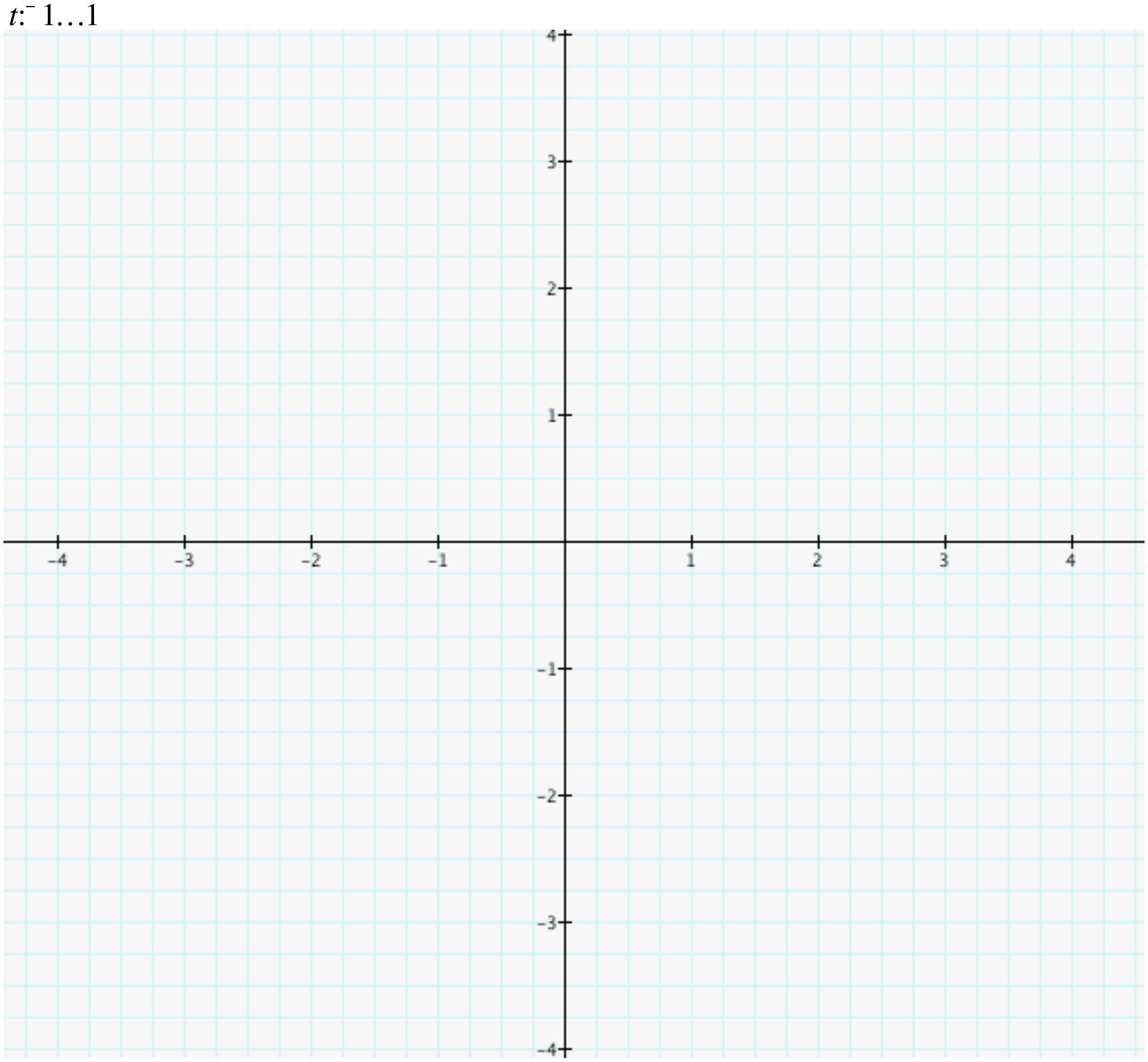
Question 4

Sketch the graph that will be produced by the following set of GC commands:

$a(b) = 2b + 1$

$\begin{bmatrix} a(t) \\ a(t) \end{bmatrix}$

$t: ^{-}1 \ldots 1$

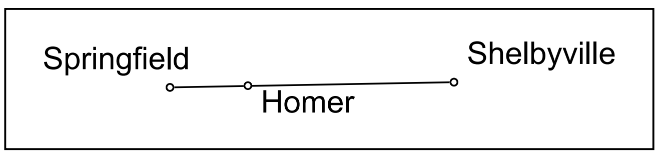


1. Here are five arrangements of the road, Shelbyville, and Springfield.

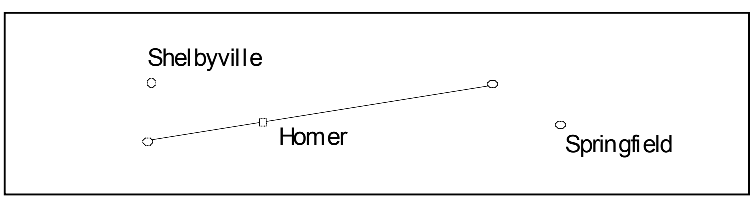
For each arrangement:

- Imagine the behavior of *Distance from Shelbyville* relative to Homer's position as you move the car from one end of the road to the other.
- Imagine the behavior of *Distance from Springfield* relative to Homer's position as you move the car from one end of the road to the other.
- Imagine the behavior of *Correspondence Point* as you move Homer from one end of the road to the other.
- Sketch your prediction of the graph that *Correspondence Point* will make with this arrangement.
- Test your prediction

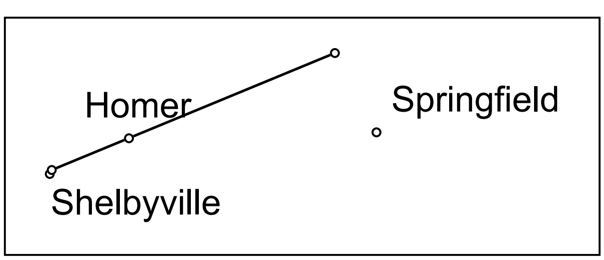
A.



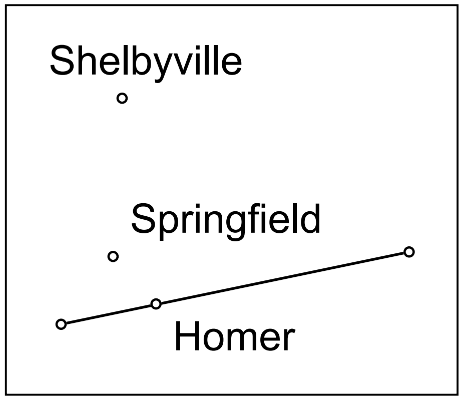
B.



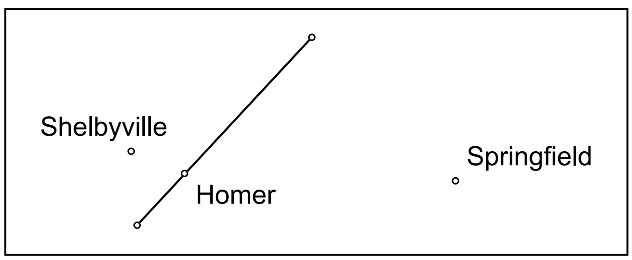
C.



D.



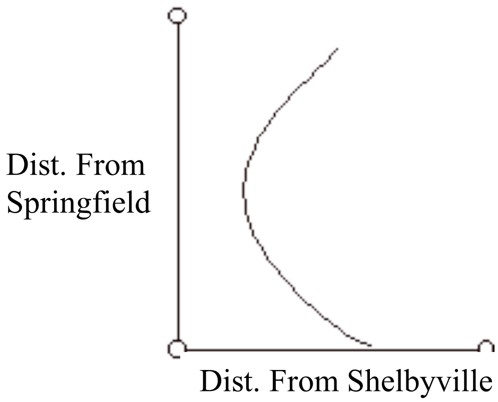
E.

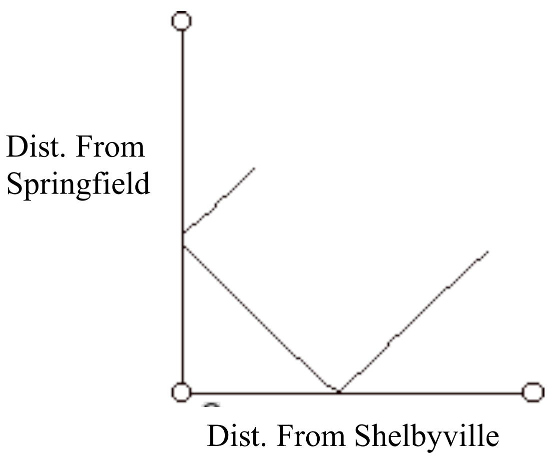


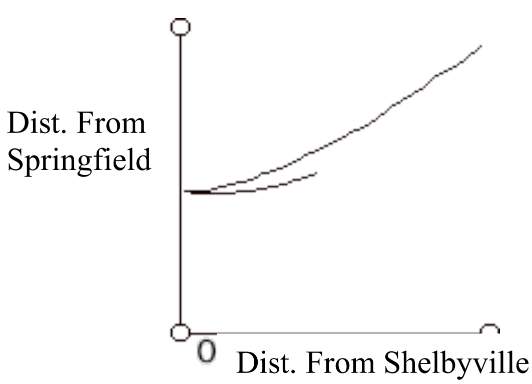
2. Create two arrangements that you will give to someone else. Try to make them as "tricky" as you can. Explain why you think each will be tricky.

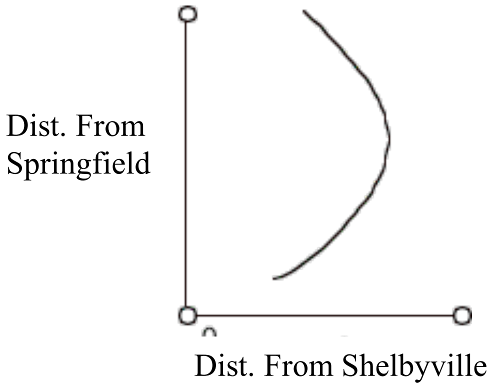
3. Give your arrangements to a neighboring student and have them try to predict the correspondence graphs.

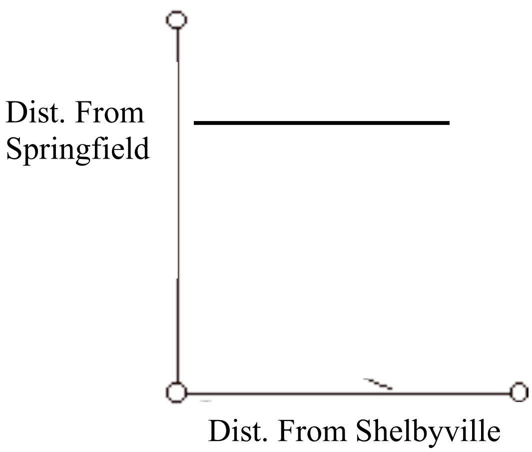
1. For each graph, do the following. Do this without the computer.
   - Describe key features of the graphs and what this tells you about the possible locations for the cities relative to the road. (For example: The graph intersects the horizontal axis, so I know that at some point the road passes through Shelbyville.) Describe as many things as you can think of, including observations about how the distance from city appears to change as Homer gets closer or further from the other city.
   - Draw a picture of the road and the location of the two cities that you believe will create the given graph of the correspondence point.











After making all your predictions:

Test your predictions on a computer. If you were way off on a graph, explore where you went wrong. Explain the error(s) in your thinking that led to your incorrect diagram.

2. Make up an "impossible" graph for this situation (i.e., a graph that could not be duplicated by drawing a road and the positions of the two cities). Explain why it is impossible. Please make your graph “plausible”, meaning that it is a graph that one might think is possible before giving it more thought. *For example, don’t draw a self-portrait and say that the correspondence point will never trace that shape because it is too complex*.

APPENDIX B: INTERVIEW PROTOCOL

- ❑ TURN OFF AIR CONDITIONER if in Room 22
- ❑ Greetings and small talk (Hi, Pat!)
- ❑ Ask about the Graphing Calculator unit
  - ❑ How do you feel about it?
  - ❑ Is this work different from what we were working on before?
- ❑ Give scratch paper to student
- ❑ Preliminary Questions
  - ❑ (Place paper with "g(s)=3t+5" on it. Say
    - ❑ A student in 0-hour math 5 defined the function g like this. Do you notice anything about her definition?
      - ❑ If "s on one side, t on the other": Why does that matter?
      - ❑ If not: Does it matter that she has s on one side and t on the other?
    - ❑ What is g(2)? g(4)?
  - ❑ (Show, on paper, two defined functions r and s, defined in w, and the pair [r(t), s(t)]. Ask
    - ❑ How does this set of GC commands work to draw a graph?
- ❑ Quiz 3
  - ❑ Question 4 (graph GC's output)
    - ❑ What is the question asking?
    - ❑ How did you choose which points to plot? (All students plotted points)
    - ❑ How did you decide where the graph began and ended?
    - ❑ (Choose an interesting point, endpoint, intercept, etc) Tell me how you found this point in particular.
    - ❑ Is this the way that GC would have figured the graph out?
  - ❑ Question 3
    - ❑ What is Amanda saying?
    - ❑ Does it matter that down here it says q(t) but up here it says q(w)?
    - ❑ What is the role of q(w)=0.5w+1 in drawing this graph?

      Make sure that something like "q(t) will control how the y co-ordinates of points on the graph will behave" is said explicitly before moving on.
    - ❑ What will be the role of p(s) in drawing the graph
    - ❑ What is the role of t?
    - ❑ With respect to what Amanda said:

- ❑ How fast does y change with respect to changes in x in y=2x+1?
- ❑ How fast does y change with respect to changes in t according to Amanda's definition?
- ❑ How fast must x change with respect to t so that y changes twice as fast as x with respect to changes in t?

- ❑ How does knowing all this help you figure out how Amanda defined p(s)?
- ❑ -------------------- (Perhaps the remaining questions are unnecessary now)
- ❑ Describe the relationship between changes in t and changes in q.
- ❑ Can you find a relationship between changes in q and changes in p?
- ❑ How does this help you to answer the problem?