

The Role of Quantitative Reasoning in Precalculus Students Learning  
Central Concepts of Trigonometry

by

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## ABSTRACT

Past research has revealed that both students and teachers have difficulty understanding and using the sine and cosine functions. They also hold weak understandings of ideas foundational for learning trigonometry (e.g., angle measure and the unit circle) and disconnected conceptions of the various contexts of trigonometry (e.g., unit circle and right triangle). This dissertation reports results of an investigation into the understandings and reasoning abilities involved in learning ideas of angle measure and the sine and cosine functions. The data was collected using a teaching experiment methodology. The instructional sequence was designed to support precalculus students in constructing understandings of angle measure and the radius as a unit for measuring an angle. Students were then supported in reasoning about how an angle measure and a distance vary in tandem. The instruction leveraged these reasoning abilities to introduce the sine and cosine functions in a unit circle context. Findings from the investigation revealed the importance of students' conceptualizing measurable (and varying) attributes of a situation (quantities) when conceptualizing angles and their measures. The idea of angle measure, and particularly the radius as a unit for measuring an angle, was also found to be foundational for learning and using the sine and cosine functions. When conceptualizing the sine and cosine functions, students needed to reason about how an angle measure and a varying distance change in tandem to model the periodic behavior between these two quantities. A process conception of function was also necessary for understanding and using the sine and cosine functions. This study's findings characterized the critical role that quantitative and covariational reasoning played in students developing the dynamic imagery needed to generate a sine or cosine



graph representing periodic motion. Finally, there was a wide variation in the students' willingness to engage in making meaning of the context of a problem. The findings revealed that if a student relies on imitating others' actions and carrying out non-quantitative procedures, and is not willing, curious, or confident enough to engage in meaning making, the student will likely have difficulty understanding new ideas.

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## TABLE OF CONTENTS

	Page
LIST OF TABLES .....	xiii
LIST OF FIGURES.....	xvi
 CHAPTER	
1 INTRODUCTION .....	1
Statement of the Problem .....	3
Research Questions .....	4
Outline of Study.....	5
2 THEORETICAL PERSPECTIVE AND LITERATURE REVIEW.....	8
Theoretical Perspective .....	8
Background for the Investigation.....	11
Function, Covariational Reasoning, and Quantitative Reasoning ..	13
Quantitative Reasoning in Problem Solving .....	26
Research Literature on Trigonometry .....	30
Students' Understandings of Trigonometry .....	30
Teachers' Understandings of Trigonometry.....	33
Summary of Chapter .....	35
3 METHODOLOGY .....	38
Subjects and Setting .....	38
Pre-Interviews.....	39
Teaching Experiment .....	40
Exploratory Teaching Interviews.....	45

	Data Collection and Analysis Overview .....	47
	Data Analysis Procedure .....	50
	Summary of Chapter .....	52
4	EXPLORATORY STUDY AND INSTRUCTIONAL DESIGN .....	53
	Exploratory Study .....	53
	Subjects and Setting .....	54
	Results .....	55
	Course assessment .....	55
	Pre-interviews .....	55
	Exploratory teaching interviews.....	56
	Post-interviews.....	59
	Summary of Findings.....	66
	Implications of the Exploatory Study .....	67
	Conceptual Analysis and Instructional Design .....	69
	Trigonometric Functions and Angle Measure .....	72
	Instructional Sequence .....	83
	Summary of Chapter .....	91
5	RESULTS OF ZAC .....	92
	Pre- and Post-Course Assessment.....	92
	Zac's Conception of Angle Measure Prior to Instruction.....	93
	Summary of Pre-interview .....	98
	Teaching Experiment Sessions One Through Four .....	99
	Teaching Experiment Session One .....	99

The cannon problem .....	100
The protractor problem .....	100
The protractor applet.....	105
Teaching Experiment Session Two .....	106
The protractor applet.....	106
The angle measurement problem .....	106
The circumference problem.....	107
Teaching Experiment Session Three .....	109
The fan problem.....	110
Teaching Experiment Session Four .....	114
The fan problem.....	114
The positions on a circle problem .....	115
The finding an arc length problem .....	117
Summary of the First Four Teaching Experiment Sessions.....	118
Exploratory Teaching Interview One.....	120
Angle Measure Interview Observations .....	120
The arc length problem.....	120
The radian measurements and pi problem .....	125
The arc problem.....	127
Summary of Angle Measure Observations.....	129
Trigonometric Function Interview Observations .....	130
The ferris wheel problem.....	130
The ski trail problem – version I .....	138

	The empire state building problem.....	144
	Summary of Exploratory Teaching Interview One .....	149
	Teaching Experiment Sessions Five and Six .....	152
	Teaching Experiment Session Five .....	152
	The finding an arc length problem .....	152
	The determining an output problem .....	153
	Teaching Experiment Session Six .....	156
	The airplane problem.....	156
	Summary of the Last Two Teaching Experiment Sessions .....	158
	Exploratory Teaching Interview Two .....	159
	The Adding Two Angle Measures Problem.....	159
	The Arc Problem.....	163
	The Enemy Approaches Problem .....	165
	The Trangent Function and Graphing Problem.....	169
	The Ski Trail Problem – Version II .....	165
	Summary of Exploratory Teaching Interview Two .....	178
	Summary and Discussion of Zac.....	179
6	RESULTS OF AMY.....	184
	Pre- and Post-Course Assessment.....	184
	Amy’s Conception of Angle Measure Prior to Instruction.....	185
	Amy’s Ways of Thinking During the Instructional Tasks .....	188
	The Role of Amy’s Problem Solving Behaviors .....	218
	Summary and Discussion of Amy .....	231

7	RESULTS OF JUDY .....	235
	Pre- and Post-Course Assessment.....	235
	Judy's Conception of Angle Measure Prior to Instruction .....	236
	Judy's Ways of Thinking During the Instructional Tasks .....	238
	The Role of Judy's Problem Solving Behaviors.....	264
	Summary and Discussion of Judy .....	268
8	CONCLUSIONS.....	272
	Quantitative Reasoning in Trigonometry.....	272
	Students' Conceptions of Angle Measure .....	273
	Students' Conceptions of Trigonometric Functions.....	279
	Trigonometric Function Framework .....	285
	Implications of the Students' Problem Solving Behaviors.....	290
	Suggestions for Curriculum and Instruction .....	294
	Limitations of the Study .....	297
	Directions for Future Research .....	299
	REFERENCES .....	302
	APPENDIX	
A	COVARIATION FRAMEWORK AND TRIGONOMETRY .....	306
B	EXPLORATORY STUDY INTERVIEW TASKS .....	310
C	INSTRUCTIONAL SEQUENCE.....	312
D	ZAC'S INTERVIEW TASKS .....	323
E	AMY'S INTERVIEW TASKS .....	328
F	JUDY'S INTERVIEW TASKS .....	332



G	HUMAN SUBJECTS APPROVAL LETTER .....	335
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## LIST OF TABLES

Table	Page
1. The Ferris Wheel Problem (Exploratory Study) .....	61
2. The Protractor Problem (Task 1) .....	83
3. The Circumference Problem.....	85
4. The Fan Problem (Sine).....	86
5. The Positions on a Circle Problem .....	88
6. The Finding an Arc Length Problem.....	89
7. The Determining an Output Problem .....	89
8. The Airplane Problem.....	90
9. Results of the PCA Pre- and Post-test (n = 16) .....	93
10. The Traversed Arc Problem .....	96
11. The Cannon Problem .....	100
12. The Protractor Problem (Task 1).....	101
13. The Angle Measurement Problem.....	107
14. The Circumference Problem.....	108
15. The Fan Problem (Sine).....	110
16. The Positions on a Circle Problem .....	115
17. The Finding an Arc Length Problem.....	117
18. The Arc Length Problem .....	121
19. The Arc Problem.....	127
20. The Ferris Wheel Problem.....	130
21. The Ski Trail Problem – Version I .....	139

22.	The Empire State Building Problem .....	144
23.	The Determining an Output Problem .....	154
24.	The Airplane Problem.....	156
25.	The Adding Two Angle Measures Problem.....	160
26.	The Enemy Approaches Problem.....	165
27.	The Tangent Function and Graphing Problem.....	169
28.	The Ski Trail Problem – Version II.....	172
29.	Results of the PCA Pre- and Post-test (n = 16) .....	185
30.	The Traversed Arc Problem .....	186
31.	The Protractor Problem.....	189
32.	The Circumference Problem.....	190
33.	The Arc Length Problem .....	191
34.	The Missing Measurement Problem.....	198
35.	Amy’s Ferris Wheel Problem.....	206
36.	Results of the PCA Pre- and Post-test (n = 16) .....	236
37.	The Protractor Problem.....	238
38.	The Circumference Problem.....	241
39.	The Arc Length Problem .....	243
40.	The Inches or Radians Problem.....	246
41.	The Radian Measurements and Pi Problem .....	248
42.	The Rotating Problem.....	251
43.	The Fan Problem (Sine).....	253
44.	The Ferris Wheel Problem.....	254

45.	The Enemy Approaches Problem.....	261
46.	The Trigonometric Function Framework.....	286

## LIST OF FIGURES

Figure	Page
1. Angle measure and units.....	76
2. An arc length image of angle measure and unit circle trigonometry .....	80
3. An arc length image of angle measure and triangle trigonometry .....	80
4. An arc length image of angle measure and the contexts of trigonometry .....	81
5. The fan applet.....	87
6. Zac's initial angle measure diagram.....	95
7. The protractor applet.....	105
8. The fan applet.....	110
9. The researcher's proposed graph.....	111
10. The students' graph.....	112
11. Zac's positions on a circle diagram .....	117
12. Zac's angle measure diagram .....	127
13. Zac's diagram of April's trip .....	131
14. Zac's Ferris wheel graph .....	133
15. Zac's initial diagram on The Empire State Building Problem.....	144
16. Zac's work on The Empire State Building Problem .....	147
17. Zac's refined diagram on The Empire State Building Problem.....	147
18. The right triangle applet.....	155
19. The students' diagram on The Airplane Problem .....	157
20. Zac's vertical segment .....	164
21. Zac's initial diagram on The Enemy Approaches Problem .....	166

22.	Zac's second diagram on The Enemy Approaches Problem .....	166
23.	Zac's third diagram on The Enemy Approaches Problem.....	167
24.	Zac's graph of the tangent function.....	172
25.	Amy's diagram of The Ferris Wheel Problem.....	210
26.	Amy's graph on The Ferris Wheel Problem .....	211
27.	Amy's diagram for describing the sine function.....	228
28.	The protractor applet.....	240
29.	Judy's diagram for linear measurements.....	246
30.	Judy's diagram for angle measures .....	250
31.	Judy's graph for The Ferris Wheel Problem .....	256
32.	Judy's diagram on The Ferris Wheel Problem.....	258
33.	Judy's refined graph on The Ferris Wheel Problem .....	259
34.	Judy's diagram on The Enemy Approaches Problem.....	262
35.	Judy connecting a right triangle to the unit circle .....	263

## Chapter 1

### Introduction

This dissertation describes an investigation into three students' understandings of angle measure and trigonometric functions<sup>1</sup>. Trigonometry and trigonometric functions have been an important part of the high school and undergraduate mathematics curriculum for the past century. In addition to numerous mathematical topics (e.g., Fourier series and integration techniques), various topics of science are reliant on trigonometric functions (e.g., projectile velocity and modeling wave behavior). Trigonometry and trigonometric functions also offer one of the earlier mathematical experiences that combine geometric, symbolic, and graphical reasoning about functions that cannot be calculated through algebraic computations. Though trigonometry has been a central part of mathematics and science curriculum for over a century, it is often the case that students and teachers have difficulty reasoning about topics dependent upon trigonometric functions (Brown, 2005; Fi, 2003; Thompson, Carlson, & Silverman, 2007; Weber, 2005). To further complicate this issue, few studies have investigated the reasoning abilities needed to understand and use trigonometric functions.

Students' difficulties relative to trigonometric functions may be related to the approach that curricular materials take when introducing the sine and cosine functions. In many mainstream textbooks used in the United States, trigonometric functions are explored in multiple contexts (e.g., triangle trigonometry and unit circle trigonometry). In

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<sup>1</sup> This investigation into student reasoning specifically focuses on the sine and cosine functions. Thus, this dissertation's use of the phrase *trigonometric functions* is in reference to the sine and cosine functions.

each of these contexts, textbooks present different purposes for using trigonometric functions (e.g., determining the side of a triangle or finding a coordinate). It is also the case that curricula typically treat these contexts as unrelated or only slightly related. For instance, right triangle trigonometry is often used as a lead into unit circle trigonometry, but curriculum does not appear to *leverage* common foundations to these contexts (e.g., angle measure) in order to develop coherence between each context. This treatment of trigonometry may hinder students from developing understandings that contain strong connections across the various trigonometry contexts. As Thompson (2008) has recently argued, curriculum should be designed such that it builds on meanings and foundations that are common across the various contexts in order to promote students in constructing coherent and flexible understandings of trigonometric functions. A major focus of this study stems from the consideration of building on common meanings and foundations between the various trigonometries.

This dissertation was designed to provide new research knowledge into students' learning of angle measure and trigonometric functions (areas where little research has been done) by investigating students solving instructional tasks in an undergraduate precalculus course. The instructional sequence for the study was designed to support the subjects (three students) in learning and using foundational ideas of trigonometry to reason about trigonometric functions. The activities also intended to promote students developing understandings of angle measure, the radius as a unit of measurement, and the unit circle. Due to the design of the instructional sequence, this investigation seeks to



offer insights into students' conceptions of trigonometric functions and their conceptions of topics that are foundational to learning trigonometry.

### **Statement of the Problem**

The NCTM *Standards* have called for connections between the various trigonometry contexts (NCTM, 1989, 2000), but trigonometry remains a highly difficult topic for students and teachers to grasp (Brown, 2005; Thompson, 2008; Thompson, et al., 2007; Weber, 2005). Furthermore, mathematics educators and curriculum developers have given limited attention to trigonometry in spite of the continuing difficulties encountered by students (Brown, 2005; Weber, 2005). The scholars that have given attention to trigonometry commonly identify that an increased focus needs to be given to developing understandings that are foundational to trigonometry and promoting coherence between the multiple contexts of trigonometry (Brown, 2005; Thompson, 2008; Thompson, et al., 2007; Weber, 2005).

The difficulties that students encounter in developing coherent trigonometric understandings are likely multifaceted. First, trigonometric functions require sophisticated reasoning relative to the function concept. Trigonometric functions are often one of a student's initial experiences with functions that cannot be computationally evaluated. Reasoning about trigonometric functions relies on reasoning about function in a manner that one can *anticipate* input values to a function being evaluated and output values being produced without performing numerical computations. The action of conceptualizing function as a process has been revealed to be a difficult task for students (M. Carlson, 1998; M. Carlson & Oehrtman, 2004; Harel & Dubinsky, 1992; Oehrtman,

Carlson, & Thompson, 2008; Sierpinska, 1992; Thompson, 1994b). Trigonometric functions offer the opportunity to develop and promote students reasoning in this way; yet, mainstream mathematics curricula do not appear to address or develop this foundational way of reasoning.

The sine and cosine functions may also be a difficult topic for students due to their limited conceptions of topics foundational to understanding these functions. As a result of these limited understandings, students do not have the tools necessary to construct meaningful and coherent understandings of the sine and cosine function. For instance, it is necessary that students construct conceptions of angle measure and the radius as a unit of measurement such that these conceptions support connected understandings of trigonometry (Moore, 2009). As Weber (2005) suggests, students also must be able to leverage the geometric objects of trigonometry (e.g., the unit circle and right triangles) to support their reasoning about the relationships formalized by the sine and cosine functions.

This investigation attempts to study these conjectures and research findings more deeply in order to contribute to the limited body of research literature on students developing understandings of trigonometry. The insights gained into how students' understandings develop are intended to identify the critical reasoning abilities necessary for learning ideas of trigonometry, while also informing the future design of trigonometry curriculum.

### **Research Questions**

The primary research question driving this study is:

- What understandings of trigonometric functions do students develop during a trigonometry instructional sequence that emphasizes quantitative and covariational reasoning?

Supporting research questions derived from the theoretical foundation and design of the study include the following:

- What roles do quantitative reasoning and covariational reasoning play in students developing understandings of trigonometric functions?
- What understandings of the topics foundational to trigonometric functions (e.g., angle measure and the radius as a unit of measure) do students develop during the trigonometry instructional sequence?
- How do understandings of these foundational trigonometry topics influence students' conceptions of trigonometric functions?

### **Outline of the Study**

The theoretical perspective of radical constructivism (Glaserfeld, 1995) forms the foundation for this study. The central premise of radical constructivism is that an individual's knowledge is fundamentally unknowable to any other individual. An individual constructs knowledge from experiences and reflecting on these experiences, where these experiences are entirely unique to the individual. An implication of this stance is that a researcher can only construct *models* of a student's understandings, where the researcher's goal is to construct, test, and refine models of a student's understandings. The model is considered viable when the student acts in ways consistent with the model of the student's understandings. However, there is no one-to-one correspondence between

the model of a student's understandings and the student's understandings. Thus, the researcher is always striving to test the model of a student's understandings in order to determine where this model breaks down and requires further modifications.

In order to accomplish the iterative process of constructing a model of students' understandings, this dissertation used a teaching experiment methodology (Steffe & Thompson, 2000) to gain insights into students' conceptions of trigonometric functions and angle measure. The subjects (Amy, Judy, and Zac) of this study were three students enrolled in an undergraduate precalculus course at a large public university in the southwestern United States. The precalculus classroom was part of a design research study where the initial classroom intervention (M. P. Carlson & Oehrtman, 2009) was informed by theories on the processes of covariational reasoning and quantitative reasoning, as well as select literature about mathematical discourse and problem solving (M. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; M. P. Carlson & Bloom, 2005; Clark, Moore, & Carlson, 2008; Smith III & Thompson, 2008). The principle investigator of the precalculus curriculum design project was the professor of the precalculus course.

The three students from the precalculus course volunteered to participate in a teaching experiment with the researcher (myself) acting as the students' instructor during the study. Six ninety-minute teaching experiment sessions were conducted with the three students. Additionally, multiple interviews were individually conducted with each student throughout the study. The interview sessions, which served as one-on-one teaching experiment sessions, offered a setting where the researcher tested models of each student's understandings that were constructed based on the students' actions during the

teaching experiment sessions. Upon completion of the study, the data was analyzed using the principles of qualitative data analysis outlined by Strauss and Corbin (1998). Also, the system of ideas driving the instructional design formed a foundation for the data analysis, where this analysis offered insights into the critical reasoning abilities needed to understand angle measure and the sine and cosine functions.

Chapter 2 provides the theoretical perspective for the study, as well as an overview of the research literature on trigonometry, covariational reasoning, and quantitative reasoning. Chapter 3 presents the methodology for data collection and data analysis used for this study. Chapter 4 presents a summary of the findings from an exploratory study into students' conceptions of angle measure. Chapter 4 also outlines the instructional sequence used in the study's teaching experiment, while describing the instructional activities in the context of the findings from the exploratory study and a conceptual analysis of trigonometry. Then, chapters 5-7 present the results and analysis of each student's actions during the study. Chapter 5 provides a detailed narrative of Zac's thinking over the course of the study. Chapters 6 and 7 summarize the progress of Amy and Judy, respectively. Finally, chapter 8 outlines this study's findings by describing and comparing the students' ways of thinking. Chapter 8 also presents the limitations of the study, and provides suggestions for curriculum and instruction. This is followed by offering future directions for building on and extending the research in this study.

## Chapter 2

### Theoretical Perspective And Literature Review

This chapter presents the theoretical perspective for the study. Glasersfeld's radical constructivism (1995) and Piaget's theory of genetic epistemology (Chapman, 1988; Piaget, 2001) informed the design, implementation, and analysis of data for this investigation. This chapter also discusses the relevant research literature on learning and understanding angle measure and trigonometric functions, which has commonly documented that students and teachers hold limited and fragmented understandings of these concepts.

#### **Theoretical Perspective**

The study is based on the radical constructivism premise that all learning begins and ends with the learner (Glasersfeld, 1995). This approach to learning views the classroom as a place for exploration that involves all participants asking questions, creating conjectures, making discoveries, and building understandings, where each student's knowledge is considered fundamentally unknowable to any other individual. An individual gains knowledge, or knowing, through experiences that are unique to the individual; in a class of thirty students, thirty different sets of experiences *will* occur. It is through an individual's reflection on their experiences that knowing is achieved, but this knowledge is not *of* anything and there is no one-to-one correspondence of knowledge and what it is about. Knowledge is what comes together through the processes of an individual altering his or her knowing (mental schema) in response to a cognitive perturbation or disequilibrium. The process of an individual reorganizing and

constructing cognitive structures and connections between these structures in response to a perturbation is referred to as *accommodation* (Glaserfeld, 1995).

It may appear that attributing learning completely to the learner creates a roadblock in teaching, as this stance can be interpreted as implying the role of the teacher is non-essential. Contrary to such an interpretation, this stance can be interpreted to imply that the act of teaching does not involve the direct transmission of thoughts. A teacher has influence on the classroom by acting as a catalyst for learning and it is the role of a teacher to create situations in which learning can happen through repetitive reasoning and reflecting on this reasoning. I also note that radical constructivism can be interpreted as dismissing a teacher lecturing in class. Again, I believe this is a misinterpretation. A teacher lecturing is often necessary in a classroom in order to provide various formalisms (e.g., the notation of  $\sin(\theta)$ ), but the learner is best prepared for lecturing when he or she has the mental structures available to process the information described by the teacher. The cognitive structures and foundations must be in place such that the students will be able to reflect on, make sense of, and construct meaning from the teacher's utterances and actions.

Reflection is a major aspect of learning and possibly the most important aspect of building knowledge (Piaget, 2001). Contrary to empiricists who deny the mind and its operations, and thus reduce all knowing to the reception of "sense data," reflection attributes learning to the ability of the mind to "stand still" and attempt to make sense of an experience. Ernst von Glasersfeld (1995) described reflection as:

[T]he mysterious capability that allows us to step out of the stream of direct experience, to re-present a chunk of it, and to look at it as though it were direct experience, while remaining aware of the fact that it is not. (p. 90)

In addition to the idea of reflection, Piaget, Glasersfeld, and others have identified the notion of abstraction, which is made possible through the comparison, separation, and connection of experiences. Ernst von Glasersfeld (1995) attributes John Locke with a description of abstraction:

This is called Abstraction, whereby ideas taken from particular beings become general representations of all the same kind; and their names general names, applicable to whatever exists conformable to such abstract ideas. (p. 91)

Through the mental activities of reflection and abstraction, the reorganization and construction of cognitive structures is achieved. Again, it is important to emphasize that these processes are completely dependent on the individual and are unseen by any other observer. Also, note that a student's experiences depend on the current model of knowing or schema of the student, where this current model defines the experience.

The necessity of reflection and abstraction implies that the purpose of instruction is to create opportunities for individuals to participate in (mental) actions and reflect on these actions such that mathematical structures and meanings can be constructed.

Instruction must create situations in which students face perturbations in their reasoning (what observers may call incorrect or undeveloped reasoning) and become aware of and resolve these conflictions. An individual becoming aware of and facing these conflictions requires reflecting on one's own thinking, a difficult and unnatural action.



The unnaturalness of reflecting on one's own thinking and actions is most seen in the natural occurrence of students incorrectly applying learned procedures. For instance, given the equation  $(x - 3)(x + 2) = 9$ , it is not uncommon that students solve  $x - 3 = 0$  and  $x + 2 = 0$  to arrive at an answer. In such a case, attention is not given to why the procedure is applied or what the application of the procedure does. Instead, it may be the case that students know that the procedure has worked before (e.g., problems solving the product of two linear expressions equal to zero), and that the problem appears to be the same (to the student). Thus, the procedure is applied without understanding the mechanism of the procedure. Students' actions such as this emphasize the necessity of a conceptual understanding that supports the application of procedural solutions. Also, it is not simply enough for a teacher to *show* procedures or *reveal* mathematical constructs. Although a teacher can show a procedure, this by no means implies the student will interpret the procedure as the teacher intends. The students must have experiences such that a need arises to develop their own understandings in a coherent manner consistent with the instructional purposes.

### **Background for the Investigation**

The research literature on student thinking in trigonometry is sparse, with Markel (1982) describing trigonometry as "forgotten and abused" due to the little attention given to trigonometry in mathematics education research and teaching. The lack of focus on students' conceptions of trigonometry may be due to the somewhat small portion of mathematics curriculum that trigonometry fills, though the amount of students taking courses that contain trigonometry has steadily increased over the past century (Brown,

2005). The research that is available focuses on both teachers' and students' understandings of trigonometry.

In addition to the limited research on student thinking in trigonometry, this investigation is guided by research on quantitative reasoning (Smith III & Thompson, 2008; Thompson, 1989, 1993) and covariational reasoning (M. Carlson, 1998; M. Carlson, et al., 2002; Oehrtman, et al., 2008). Also, research on the function concept (Harel & Dubinsky, 1992) influenced this study. Due to the limited research available in trigonometry, research on quantitative reasoning, covariational reasoning, and the function concept formed a significant foundation for this study.

In general, quantitative reasoning (Smith III & Thompson, 2008) refers to a student imagining a situation, conceptualizing measurable attributes (called quantities) within this imagined situation, and constructing relationships between these quantities. The mental structure that results from a student constructing quantitative relationships provides a foundation for the student to reflect upon and develop mathematical understandings. Quantitative reasoning also emphasizes that calculations and formulas emerge from and reflect relationships between quantities. This type of reasoning has been shown to be critical for understanding important topics of mathematics, such as the fundamental theorem of calculus (Thompson, 1994a) and rate of change (Thompson, 1994c).

Reasoning about the rate of change of two quantities is a mental activity that entails coordinating the values of two quantities that are changing in tandem. A mental activity such as this is referred to as covariational reasoning (M. Carlson, et al., 2002).

Such reasoning ranges from identifying a general correspondence between two quantities to reasoning about the rate of change of one quantity with respect to another quantity. Covariational reasoning has been suggested to be critical for success in calculus (M. Carlson, 1998; M. Carlson, et al., 2002; Oehrtman, et al., 2008) and research has revealed that students often have difficulty engaging in covariational reasoning and constructing the relevant quantities of a situation to covary (M. Carlson, et al., 2002; Moore, Carlson, & Oehrtman, 2009). The following section further explores relationships between the function concept, covariational reasoning, and quantitative reasoning in the context of learning angle measure and the sine and cosine functions.

### **Function, Covariational Reasoning, and Quantitative Reasoning**

Trigonometric functions are frequently one of a student's first mathematical experiences in which he or she is required to reason about a relationship between the (varying) values of two quantities that does not lend itself to being computed by hand. Thus, developing a conception of trigonometric functions as processes is essential for reasoning about these relationships. As Dubinsky and Harel (1992) describe,

*A process conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result. (p. 85)*

A student with a process conception of function does not see a function or an expression as a call to evaluate, but rather they see a function or expression as representing a

relationship or a mapping that is “self-evaluating.” In other words, a student with a process conception can anticipate the evaluation of input values and the resulting output values without actually performing the computations. This is precisely the reasoning needed to conceptually understand trigonometric functions, as these functions cannot be computationally evaluated in an efficient manner without the aid of a calculator or memorized values.

Contrary to a process conception of function is the case that a student is limited to reasoning that is reliant on *performing* specific actions such as calculations. If this is the dominating function conception a student holds, reasoning about trigonometric functions becomes a daunting task. A student can memorize a subset of the input-output values for trigonometric functions, but this may not promote an image of trigonometric functions as accepting an input and producing an output. According to Harel and Dubinsky (1992),

An *action*...conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one-step at a time (e.g., one evaluation of an expression). (p. 85)

A student with an action conception of function focuses on procedures and calculations when reasoning about functions. Rather than reasoning about the quantities of a situation, these students are quick to perform calculations without considering the contextual meanings of these calculations. A student with an action conception of function is unable to look past specific computations in order to view a function as accepting inputs and producing outputs, regardless of the algorithm or representation. An action conception

inhibits the student's ability to reason dynamically about the relationship between two quantities (M. Carlson, 1998; M. Carlson, et al., 2002; Oehrtman, et al., 2008), such as imagining a varying angle measure and a varying length.

A self-evaluating, or process view of function supports a student's ability to coordinate intervals of inputs and outputs and reason dynamically about an input-output relationship. The "cognitive activities [of an individual] involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" are referred to as covariational reasoning (M. Carlson, et al., 2002, p. 354). The mental actions involved in covarying quantities have been found to be important for understanding the function concept (M. Carlson, et al., 2002; M. Carlson & Oehrtman, 2004; Oehrtman, et al., 2008). Thompson (1994b) further described that the groundwork for reflecting on a set of possible inputs in relation to a set of corresponding outputs is laid when students are able to imagine an expression being evaluated continually as they "run rapidly" over a continuum.

In addition to the description of covariation provided by Thompson (1994b), Saldanha and Thompson (1998) argued that images of covariation are developmental. In other words, a student first coordinates two quantities' values (e.g., think of an angle measure and then a length, repeat). Then, as a student's image of the covariational relationship develops, her or his understanding of covariation begins to involve imagining continuous and simultaneously changing quantities (e.g., as an angle's measure changes, one has the *realization* that a length also changes simultaneously). Here, *continuous covariation* implies an image that includes an understanding that all intermediate values

of the quantities are obtained as the two quantities change in tandem. Furthermore, continuous covariation implies that for any interval of input considered, that interval can be divided into subintervals where the quantities covary continuously. The ability to imagine continuous changing quantities also parallels a process conception of function in that the student is able to *imagine* or *anticipate* simultaneous changes without having to determine the changes in one quantity and then the changes in the other quantity; the student does not actually think of all intermediate values, but he or she has the understanding that all of the intermediate values occur independent of calculating these values.

A study by Carlson et al. (2002) gained additional insights into the complexity of students' mental actions when engaging in covariational reasoning. Initially, the authors identified multiple behaviors in undergraduate students as the students attempted to interpret and represent dynamic function situations. In order to classify the different behaviors exhibited, a framework that consists of five *mental actions* and five *levels* of covariational reasoning was developed (Appendix A). The five mental actions are specific to behaviors exhibited. However, the mental actions alone were found to be insufficient in classifying the collection of behaviors a student exhibited. For instance, students engaged in higher mental actions but were unable to unpack these actions in lower mental actions. Thus, in order to describe a student's covariational reasoning ability relative to a situation or problem, the framework was extended to include five levels of covariational reasoning that parallel the five mental actions. A student is said to reason covariationally at a certain level when they are able to reason not only using the

mental action associated with that level, but also with all mental actions associated with lower levels (e.g., Level 3 implies MA1-MA3 abilities).

In an attempt to link the mental actions of the framework presented by Carlson et al. (2002) to trigonometric functions, consider the situation of tracking a point on the end of a fan blade rotating counterclockwise, with the point starting in the standard position (e.g., the 3 o'clock position). Specifically, consider the covariation of the distance traveled by the point and the point's vertical distance above the center of the fan. A student exhibiting behavior corresponding to Mental Action 1 (MA1) would focus on the *coordination of quantities*. Such a student may describe that as the distance traveled by the point changes, the vertical distance also changes. If the student's description were to also include a *coordination of direction of change* (e.g., as the distance traveled increases, the vertical distance from the center of the fan also increases), this would be indicative of MA2. MA3 refers to *coordinating amounts of change*. This would involve actions considering specific changes in the distance traveled and identifying the corresponding changes of vertical distance. Next, MA4 includes a focus on the *average rate of change* of one quantity with respect to the other quantity. This action could include a number of behaviors. For instance, a student may consider how the average rate of change of vertical distance with respect to distance traveled varies over successive intervals of distance traveled. Lastly, a behavior associated with MA5 would be describing the *instantaneous rate of change* over an interval of the domain. In the fan situation this would include a description that the rate at which the vertical distance is increasing with respect to the distance traveled is decreasing (in the first quadrant). Graphically, this

description may include a student sketching a smooth, concave down graph. Note that this mental action alone does not entail an understanding of rate, as a student who is describing rate may not be able to unpack this concept in terms of MA3 behaviors. Such instances of student reasoning were observed by Carlson et al. (2002). A summary of these mental actions and their verbal manifestations relative to trigonometric functions can be found in Appendix A.

The problems presented to the students under study by Carlson et al. (2002) were set in a context. Thus, the students were asked to reason covariationally about quantities they constructed from an imagined situation. Although it was not the focus of the investigation by Carlson et al., the students in the study constructed mental images of the problem situations, where the quantities composing these situations possibly impacted the students' abilities to reason covariationally. For instance, relative to this study, if a student conceived of the openness of an angle as a static quantity rather than a varying quantity, it is hopeless for the student to conceive of a dynamic situation consisting of a changing angle measure. The manner in which a student conceives of a situation and the quantities composing a situation is at the heart of quantitative reasoning.

Quantitative reasoning (Smith III & Thompson, 2008; Thompson, 1989) refers to a type of reasoning that is situation sensitive and places an emphasis on students constructing conceptual objects (quantities) that can be reasoned about. Quantitative reasoning emphasizes the mental actions of a student making sense of a situation, constructing an image of the quantities of a situation, and reasoning about relationships between these quantities. A mental structure that consists of quantities and relationships



between quantities creates a foundation that the student can reflect upon, and this reflection can result in mathematical reasoning and conceptual development.

Thompson (Thompson, 1989) argued that quantitative reasoning can lead to students constructing mathematical understandings, but warned that this type of reasoning is not necessarily natural or prevalent in mathematics curriculum and instruction. For instance, tools of algebra are often taught as an approach to complicated mathematical problems, where these tools are used to both simplify and solve the problem. Yet, this approach can result in the separation of the actual context of the problem (i.e., the relationships and intuitive properties of the situation) from the process of solving the problem. Although this may become a natural process for some, it is not an appropriate or natural approach for those who view these formalities of numerical and algebraic manipulations as “magical” and devoid of a situational reference. Furthermore, in a situation where algebraic manipulations are nontrivial or not possible, such as within the contexts of trigonometry and the use of trigonometric functions, reasoning about a problem’s context becomes necessary to develop deep and connected understandings.

To further explain quantitative reasoning’s role in reasoning about trigonometric functions and angle measure, a few working definitions of quantitative reasoning are given and explained using various topics of trigonometry. As Thompson (Thompson, 1989) explained, the definitions presented are intended to be constructs of a system composed of notions of quantitative and algebraic reasoning. It is important to note that this system of working definitions does not allege to represent the way people reason

quantitatively. Rather, it is one model that works in describing the cognitive processes and conceptual structures that enable quantitative reasoning<sup>2</sup>.

A *quantity* is defined as a conceived attribute of something, where the attribute is conceived such that it admits a measurement process (e.g., the something could be an image of a problem's context or a mathematical object, such as a graph) (Thompson, 1989). Relative to the focus of this study, an angle is an object that has a measurable attribute of openness. The place at which a quantity lies is subtle in this definition. Rather than existing in the experiential world independent of a student, a quantity is a conceptual entity. A quantity is constructed and this construction consists of a situation, an object of a situation, and an attribute of the object that admits an explicit or implicit measurement process, which includes the result of the measurement<sup>3</sup>. This definition of quantity implies that a student has *cognitively* identified an object or objectified a phenomenon that has attributes that can be measured, where this measure may or may not vary. This cognitive object can and will differ from individual to individual; a student may conceive of an angle's openness such that they do or do not imagine measuring along an arc subtended by the angle.

*Quantification* is the process by which a student assigns values to measurable attributes (Thompson, 1989). In other words, quantification is a process of either the

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<sup>2</sup> This study investigates the way students reason (or not reason) quantitatively within the context of trigonometry. The nature of this reasoning will differ from student to student, as well as topic to topic.

<sup>3</sup> An implicit or explicit measurement act implies that it is not actually making the measurement that results in a quantity. Rather, it is the *conceived ability* to make the measurement, whether or not it is carried out, that results in a quantity.

direct or indirect measurement of a quantity. It is in the process of quantifying an attribute that a quantity becomes conceptualized. In order for a student to comprehend a quantity, the individual must have a mental image of an object and attributes of this object that can be measured, an implicit or explicit act of measurement that produces the quantity, and a value, which is the result of that measurement. It is in this last characteristic that the process of quantification occurs. Quantification is the mapping of a result of a measurement process to the characteristic. In other words, quantification is the function relationship between the result of measuring and the attribute that is measured. The result of measurement, whether it is an implicit or explicit result, is *not* the attribute itself. Rather, it is a result of a measurement of the attribute. For instance, a measurement of two radians is not *the* arc length subtended by an angle; it is a result of measuring the arc length subtended by an angle in a number of radius lengths.

It is necessary to briefly describe what the terms *result* (of measurement) and *value* refer to when describing the role of mapping a result to a characteristic. If taken literally, the term result implies a number (e.g., 3.3 or  $2\pi$ ). However, result is used here in a more general sense. Result can be used to reference both determined and undetermined (or indeterminate) values. In addition to referring to an explicit value such as 3.3, the term result could refer to an imagined value. For instance, in the process of measuring a quantity, it is possible that one may not have the necessary physical tools to measure the quantity, but mentally they are able to imagine a measurement process resulting in 3.3 of something. This is especially the case with angle measure. Although a student may not

always have a protractor at hand, a student can imagine measuring the openness of an angle with the result representing a number of something (e.g., radians).

The term *value* refers to the result of a quantification process applied to the quantity. In other words, a value is the result of the function relationship created in the quantification process. For instance, if a student conceives of the *number* 2.2 as the result of measuring a subtended arc in a number of radius lengths, then the number 2.2 has become a value. However, this definition of value can cause confusion, as it creates the question, “What do we call a number or a variable that does not appear to be the result of a measurement process?” Thus, a number or a variable that is not the result of a measurement will be referenced as a *number*, as it has remained a number without a related measurement process. A *number* or *variable* that is conceived as a result of an implicit or explicit measurement process will be referenced as a value from this point forward.

An additional definition of use is that of a *quantitative operation*. A quantitative operation can be defined as the conception of two quantities being taken to produce a new quantity. An example of a quantitative operation is dividing an arc length by the length of a radius to create a multiplicative comparison that is the angle measure in radians. It is important to contrast a quantitative operation and an arithmetic, or numerical, operation. A numerical operation is the operation used to *calculate* a quantity’s value while a quantitative operation is the operation by which a quantity is *created*. For instance, an angle of measure 3.1 radians could double in openness, resulting in performing the arithmetic operation of  $3.1 \cdot 2$  and obtaining a measure of 6.2 radians.

However, the 6.2 radians can still be conceived as the result of a quantitative operation, the ratio of arc length to the length of a radius.

In direct relation to the conception of a quantitative operation is the conception of a *quantitative relationship*. A quantitative relationship is the image of three quantities, two of which determine the third by a quantitative operation. The difference between the conceptions of a quantitative operation and a quantitative relationship is the focus a student places on the result of operating. The conception of a quantitative operation focuses primarily on the *operation*. In other words, the attention is placed on a calculation. However, the conception of a quantitative relationship focuses on the result and its *relationship* to its operands. The attention is placed on a relationship, rather than a calculation. Consider the following distinction between a quantitative operation and a quantitative relationship using an arc length, the length of a radius, and the measure of an angle's openness in radians. In order to determine the measure of an angle's openness in radians, one determines the ratio of arc length to the length of a radius (e.g., a quantitative operation). However, a student's conception of the relationship between the three quantities may change. For instance, a student may conceive of the ratio as defining how many radius lengths lay along the arc length or they may conceive of the ratio as defining a magnitude that is so many times the magnitude of a radius. Or, the individual may conceive of the ratio as merely a calculation that is performed when attempting to determine a number of radians.

With these definitions in place, *quantitative reasoning* is defined as the analysis of a situation into a *quantitative structure*. A quantitative structure is the network of

quantities and quantitative relationships constructed, which forms a foundation for reasoning and reflection. *Quantity-based arithmetic* consists of quantitative reasoning, determination of appropriate operations (inferred according to relationships among quantities) to calculate quantities' values, and the propagation of calculations. Also, *quantity-based algebra* can now be described as the same as quantity-based arithmetic, except representations of situations are under-constrained in terms of quantities' values (i.e., there is not enough numerical information to propagate calculations), some value or values are represented symbolically, and formulas are propagated instead of values being propagated.

Overall, quantitative reasoning stresses the importance of students conceiving of situations and measurable attributes of a situation as a foundation for mathematical reasoning. Although each individual in a classroom develops unique understandings and images, this is not to say that students cannot develop understandings and images that are consistent with the instructional goals. Instruction must account for this initial development of situations and quantities that the students are to reason about, as research has revealed that students often have difficulty constructing situations consistent with the purpose of a problem (Moore, et al., 2009).

Allowing students the opportunity to construct a situation in which they can conceptualize quantities and their relationships also enables formulas and functions to emerge in a meaningful way. By first promoting the cognitive development of quantities and their relationships such that these images include how the quantities covary, formulas and representations of functions can emerge as a reflection and generalization of these

relationships (Moore, et al., 2009). This is opposed to the approach of developing a formula or function and *then* attempting to attach meaning and understandings to the formula. The approach that functions be a reflection of a student's understanding of a situation is necessary for trigonometric functions. The symbolic representations of trigonometry functions do not lend themselves to algebraic computations; a student must first be able to relate trigonometric functions to a context (e.g., the unit circle or a right triangle) if they are to understand trigonometric functions beyond a memorized set of values. Also, the student must conceptualize these contexts in ways that the formalisms of trigonometry can emerge from images of quantitative structures that include covarying quantities. In the case that a student does not construct a mental image consistent with the instructional goals, it is likely that the student will not reason correctly when attempting to relate the relevant quantities of the situation (Moore, et al., 2009).

Note that a process conception of function is also related to the type of reasoning that can occur when conceiving of a situation composed of quantities. To say more, a student's initial act of constructing measurable attributes of a situation can lead to an image of these quantities taking on indeterminate values. Also, if a student's image of a situation is dynamic in that he or she can imagine the two quantities varying in tandem previous to tracking or calculating numerical values, a student has an image from which he or she can formalize "a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity" (Harel & Dubinsky, 1992). With a properly developed image of a situation, the foundations are laid for a self-evaluating view of function that supports the

ability to coordinate entire intervals of inputs and outputs. However, for this type of reasoning to occur, it becomes critical that a student orients to a problem by constructing a mental image of the problem's context that promotes this reasoning (Moore, et al., 2009).

### **Quantitative Reasoning in Problem Solving**

Quantitative reasoning refers to a student identifying and conceptualizing quantities that compose a situation, which is an action central to contextual problem solving. When engaged in a novel contextual problem, the mental processes of creating a mental image occur. Objects of this mental image may be imagined and attributes of these objects can be identified and quantified. These mental actions (which are unique to each individual) of constructing an image of the problem's context may be part of the *orientation* phase of problem solving (M. P. Carlson & Bloom, 2005).

Carlson and Bloom's *Multidimensional Problem Solving Framework* (2005) emerged from a study that examined the mental process and knowledge influencing mathematicians' problem solving actions. Analysis of the data revealed four distinct phases of problem solving: *orienting*, *planning*, *executing*, and *checking*. The orientation phase is the phase in which a problem solver situates himself to a problem and constructs an initial mental image of the problem's context. During this phase the solver engages in sense making, organizing and constructing. Within the planning phase, conjectures about the solution's approach are created and tested. During the planning phase, the sub-cycle of *conjecture—imagine—verify* takes place. This sequence can be defined as follows: a) construct a conjecture, b) imagine how the solution will play out, and c) evaluate the



viability of the conjectured approach. This sub-cycle allows the problem solver a more efficient approach to the problem because it avoids formally executing each conjecture. The executing phase involves the problem solver making formal constructions and carrying out computations. The checking phase can be described as a process of verification. During this phase, the problem solver analyzes the reasonableness of his solution and computations. This results in a rejection (and a cycle back to orienting or planning) or an acceptance (and a move to a new solving cycle if the problem is not completed).

Before continuing, it is noted that for descriptive purposes the problem solving process is discussed in a somewhat linear progression. However, this is not necessarily how the problem solving process unfolds. As observed by Carlson and Bloom (2005), a problem solver may jump from phase to phase in any manner and is not restricted to, for example, only participating in orienting once and then moving on without participating in later orienting actions.

According to Carlson and Bloom (2005), the various engagement processes of orientation can be categorized as *sense making*, *organization*, and *constructing*. Resources, heuristics, and affect all have an influence on these behaviors. Resources are formal and informal knowledge, facts, and procedures used during the problem solving processes. Heuristics include actions such as the use of constructing a diagram or attempting a parallel problem. Affect is a description of beliefs and attitudes of the nature of mathematics, problem solving, testing, etc. that the problem solver holds. Examples of affect are enjoyment, confidence, frustration, and mathematical integrity. Mathematical

integrity refers to the nature of a problem solver's action of determining when a solution is correct, a problem is solved satisfactorily, or his or her understanding is sufficient.

Another aspect of orientation is sense making. Sense making is defined as the process in which a problem solver reads and interprets a problem<sup>4</sup>. During sense making, a problem solver identifies characteristics of an object and a situation to be modeled. These characteristics are possibly attributes of an object that will need to be quantified during future mental actions. Also, the problem solver may simply build a mental image of the physical situation being interpreted (e.g., a dog chasing a fox). During sense making, the problem solver also identifies what questions are to be answered and the problem solver develops the image of some goal, or goals, to accomplish.

While not a specific focus of this dissertation, goal(s) identification and self-reflection on these goals can have a significant impact on problem solving behaviors. The goals identified can shape the path a problem solver takes and contribute to determining the characteristics of an interpreted situation a problem solver identifies as being or not being important. Furthermore, as a problem is interpreted and solved, goals may be refined, new goals may be built, and existing goals may be abandoned. This is another example of the non-linear and possibly iterative nature of problem solving. Important to note is the influence of mathematical conceptions and beliefs about mathematics on goal identification. For instance, if a problem requires the creation of a symbolic representation of a function, a problem solver's conception of function can significantly

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<sup>4</sup> Note that the descriptions that follow are directed at problems that are context based. Also, a problem is defined as a non-routine task that requires the student to reason beyond applying procedures comfortably (an exercise).

influence the problem solver's interpretation of the problem's goal. In relation to beliefs, if a student views mathematics as only consisting of formulas and calculations and a problem asks for an algebraic function, it may be the case that quantifying a situation may not occur (except at an undeveloped level) as this may not be mathematics to the problem solver. Or, if a problem solver's view of mathematics is obtaining correct solutions, they may focus on the result of their solution rather than the actual solution process.

Another mental action that may occur during sense making is accessing existing concepts and experiences in an attempt to relate them to the interpreted situation. The word *experiences* is not used haphazardly here; rather than solely use the word concepts, which may only imply mathematical concepts to the reader, experiences is used to add attention to physical experiences and observances. For instance, in a question regarding building a box and creating volume, the solver may recall an image of a past experience of building and forming a box. Or, a student may recall an observable action or calculation made by him or herself, or another individual, during a similar problem or situation.

Carlson and Bloom (2005) emphasized that the orientation phase formed a critical component of an individual's problem solving behaviors; however, the authors did not provide a fine grained description of what is being imagined and constructed in the mind of the solver during this phase. The authors also noted that the strong abilities of the mathematicians under study made many of the mental actions in problem solving unobservable, particularly during the orientation phase. Thus, this dissertation attempts to

offer insights into the possible mental actions involved in the orientation phase and how these are connected to quantifying a situation and constructing understandings of trigonometric functions and angle measure. This study explores the role of quantitative reasoning during the problem solving processes of the students when solving the tasks composing the instructional unit on trigonometry.

### **Research Literature on Trigonometry**

Although limited, the research available on students' and teachers' conceptions of trigonometric functions revealed valuable insights that informed the design of this study. A common observance is that both students and teachers hold limited and fragmented understandings of trigonometric functions and topics fundamental to trigonometry.

### **Students' Understandings of Trigonometry**

The research literature on students' understandings of trigonometric functions is sparse, but there have been multiple studies that present findings pertinent to this investigation (Brown, 2005, 2006; Weber, 2005). These studies on student thinking have revealed students holding limited and narrow understandings. Students have also been labeled as having a fragile conception of angle measure (Brown, 2005).

In an attempt to gain insights into student thinking in trigonometry, a study conducted by Weber (2005) compared a lecture-based course versus experimental instruction in the context of two undergraduate trigonometry courses. The experimental instruction focused on investigating trigonometric functions by physically (or mentally) constructing situations and making (or estimating) measurements from these constructions. This experimental instruction rested on the stance that when students

reason about the values of trigonometric functions or expressions, a student must have a developed image of the geometric processes<sup>5</sup> used to obtain those values regardless of the context. The stance taken by Weber is consistent with the suggestions of quantitative reasoning; students must develop conceptions of the objects and quantities they are asked to reason about. In the case of Weber's study, the students who received the experimental instruction were found to develop deeper and more connected understandings of trigonometric functions.

The students composing the traditional, lecture-based group of Weber's study (2005) were often unable to discuss various properties of trigonometric functions or estimate their output values for various input values. The author identified that these students were unable to construct the geometric objects needed to reason about trigonometric functions. For instance, the students in the traditional group were unable to approximate  $\sin(\theta)$  for various values of  $\theta$ . Instead, the students claimed that they were not given enough information to accomplish this task and that they needed an appropriately labeled triangle. The author also revealed that the students in the lecture-based course frequently spoke of the sine function as a cue for finding an answer rather than as a function or process between the values of two quantities.

Weber (2005) also observed that when the students were asked why  $\sin(x)$  is a function, none of the students from the lecture-based class were able to provide a meaningful answer. This finding is consistent with the research literature that has revealed students having difficulty reasoning about function as a process (M. Carlson,

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<sup>5</sup> Angle measure did not appear to be an explicit focus of the experimental instruction.

1998; M. Carlson & Oehrtman, 2004; Harel & Dubinsky, 1992; Oehrtman, et al., 2008; Sierpinska, 1992; Thompson, 1994b). Relative to the experimental course, three of the four students described the sine function in terms of a process between an input and output quantity. Weber attributed the students' improved performance and reasoning to their use of the unit circle. That is, students who showed improved performance often revealed reasoning that was based in the context of the unit circle. However, he noted that not all approaches to trigonometric functions that use the unit circle will result in improved student understandings. He stressed the importance of students understanding the *process* of creating the unit circle in relation to the corresponding trigonometric functions. This important suggestion by Weber may explain why Kendal and Stacey (1997) found that students who were taught using a unit circle model learned less than those using a right triangle model, or Brown's (2005) finding of students having difficulty relating a point on the unit circle to the graph of the sine or cosine function.

In general, the reports on students' trigonometric understandings have revealed students encountering many difficulties when reasoning about trigonometric functions. Students are often observed having limited cognitive connections between the various contexts of trigonometry. Students also appeared to lack the foundational understandings necessary to build these connections. These foundational understandings include conceptions of angle measure, the radian as a unit of measurement, and the role of the unit circle in trigonometry. These findings suggest that more time needs to be devoted to developing the foundational understandings necessary for trigonometry and that investigations are needed to determine how to promote coherence between the various

trigonometric contexts. As Weber (2005) suggested, careful attention must be given to promote students constructing the unit circle and its relationship to trigonometric functions such that this context becomes a reasoning tool. This suggestion can be applied to any understanding considered foundational to trigonometry and trigonometric functions (e.g., angle measure and the radian).

### **Teachers' Understandings of Trigonometry**

Multiple studies have described teachers' understandings of trigonometry as narrow, limited, and entrenched (Akkoc, 2008; Fi, 2003, 2006; Thompson, et al., 2007; Topçu, Kertil, Akkoç, Kamil, & Osman, 2006). These investigations observed teachers lacking meaningful understandings of the radian as a unit of angle measure and found that teachers were much more comfortable with degree angle measures. For instance, Fi (2003, 2006) observed that secondary teachers used procedures that were not meaningful when converting between radian and degree angle measures. Also, the teachers were unable to describe a meaning of radian measure beyond these conversion procedures. Multiple studies (Akkoc, 2008; Fi, 2003, 2006; Tall & Vinner, 1981; Topçu, et al., 2006) have reported that teachers do not view  $\pi$  as a real number when discussed in a trigonometry context. Rather, these teachers were observed graphing  $\pi$  radians as equal to 180 (as a number, not degrees), where other teachers described  $\pi$  as the unit for radian measure (e.g., a radian is so many multiples of  $\pi$ ).

Akkoc (2008) also reported that the pre-service teachers with the most developed conception of the radius as a unit of measurement used the unit circle to relate various concepts of trigonometry, while teachers with less developed understandings relied on

using a right triangle (devoid of a circle) to explain concepts of trigonometry. The author suggested that the geometric underpinnings of trigonometry and the introduction of the cosine and sine functions in the context of right triangles might be the root of teachers' degree dominated images of angle measure. The degree is the typical unit of angle measure used in right triangle trigonometry, which becomes the image that dominates an individual's way of reasoning. In response to this finding, Akkoc suggested that instructional activities promote conceptions that enable understanding the radian as a unit of measurement.

In response to the limited and fragmented understandings of trigonometry often constructed by teachers, Thompson, Carlson, and Silverman (2007) engaged teachers in tasks designed to necessitate their re-conception of the mathematics they teach. The authors focused on using length (or magnitude) as a foundational concept. For instance, angle measure was developed in terms of an arc length's fraction of a circle's circumference in order to promote the teachers constructing a *process* for measuring an angle. The teachers' activity on the tasks implied that they held strong commitments to their current high school curriculum and the meanings they had attached to that curriculum. Specifically, the teachers were attached to introducing trigonometry using right triangles, rather than angle measure and the unit circle. The teachers also maintained a belief that trigonometry is mainly about solving for measurements of a triangle. These understandings, regardless of their incoherence, dominated what the teachers imagined themselves teaching even after the authors' brought the incoherence of these meanings to the teachers' attention. This emphasizes the importance of remaining attentive at all times



to the conceptions students (who may end up becoming teachers) build, especially at the outset of a mathematical topic.

A commonality between each of the above research reports is that teachers we found to be strongly committed to meanings that did not reflect coherent understandings of trigonometry. These reports revealed that a majority of the teachers lacked the foundational understandings necessary for trigonometry, which possibly contributed to the incoherence of their meanings. A majority of the teachers constructed a very limited conception of the radian as a unit of measure and the teachers' images of angle measure were frequently dominated by degree measures<sup>6</sup>. These images did not appear to promote the reasoning necessary for meaningful and connected understandings of trigonometric functions.

### **Summary of Chapter**

This chapter provided an overview of the theoretical perspective that forms the foundation for this study, which stems from a combination of radical constructivism and Piaget's theory of genetic epistemology. A central component of this theoretical perspective is that an individual's knowledge is unique and fundamentally unknowable to any other individual. In such a case, the goal of the researcher becomes building models of a student's knowledge, testing these models, and subsequently refining these models.

This chapter also addressed areas of research that offer insights into various reasoning abilities deemed critical to understanding trigonometric functions. Because the sine and cosine functions cannot be trivially calculated through numerical calculations, it

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<sup>6</sup> Other than the Thompson, Carlson, and Silverman (2007) report, no studies reported on teachers' images of degree measurement.

is important that students construct a process conception of the dynamic relationships formalized by trigonometric functions. The ability to covary two quantities has also been shown to be a critical and complex reasoning ability that students must apply when learning concepts of precalculus and calculus. This study attempts to leverage the research on covariation and function to promote students constructing meaningful and flexible understandings of trigonometric functions.

In order to engage in covariational reasoning, a student is expected to reason about dynamic relationships between two quantities. This implies that the student first constructs each quantity he or she is to reason about, which likely occurs during the orientation process of problem solving. Quantitative reasoning provides a model of a student conceptualizing quantities and relationships between these quantities. Also, quantitative reasoning stresses the value of engaging students in situations that offer them the *opportunity*<sup>7</sup> to construct quantitative relationships and reflect on these relationships in a manner that they develop understandings and ways of reasoning consistent with instructional goals. Also, a student that constructs quantitative structures composed of relationships between varying quantities is prepared to create formulas and mathematical representations (e.g., graphs) that reflect these relationships.

This chapter concluded by providing a synthesis of the research available on teachers' and students' understandings of trigonometry. The limited research available has identified both teachers and students as holding very fragmented and limited understandings of trigonometric functions. Additionally, students and teachers were

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<sup>7</sup> Merely offering contextual situations does not imply that students will engage in quantitative reasoning.

observed lacking foundational reasoning abilities necessary for understanding trigonometric functions, although few research studies have investigated the role of these foundational understandings. The next chapter provides the methodology for this dissertation, which intended to gain insights into the reasoning abilities and foundational understandings needed to construct meaningful conceptions of the sine and cosine functions, as well as angle measure.

## Chapter 3

### Methodology

The primary goal of this research was to investigate precalculus students' thinking and learning in the context of angle measure and the sine and cosine functions for the purpose of improving trigonometry instruction. If an intention of research is to improve mathematics instruction, it is necessary to gain insights into the conceptions students develop and the ways of reasoning they exhibit as they encounter instructional activities. These insights can then be used to improve instruction and future research ventures for the purpose of informing ongoing efforts to improve student learning.

This investigation into precalculus students' thinking and learning was carried out using a teaching experiment methodology (Steffe & Thompson, 2000) with three students that included a sequence of clinical interviews (Clement, 2000), teaching sessions, and exploratory teaching interviews (e.g., one-on-one teaching experiment sessions). This chapter describes the subjects and setting for this study. This is followed by an explanation of the methods for data collection, which includes a description of the theoretical principles driving each of the data collection methods. The chapter concludes with a description of the methods used to analyze the data.

### **Subjects and Setting**

The subjects<sup>8</sup> for this study were three precalculus students from a large public university in the southwest United States. The students were chosen on a volunteer<sup>9</sup> basis

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<sup>8</sup> The three subjects will be referenced as students from this point forward. Details of each student are provided in each chapter dedicated to the student.

and they were monetarily compensated for their time (\$20/hr.). The precalculus classroom from which the students were chosen was part of a design research study where the classroom intervention (M. P. Carlson & Oehrtman, 2009) was informed by theory on the processes of covariational reasoning and select literature about mathematical discourse and problem solving (M. Carlson, et al., 2002; M. P. Carlson & Bloom, 2005; Clark, et al., 2008). The head researcher of the precalculus redesign project was the professor of the class, which met for 75 minutes twice each week over a fifteen-week period. The classroom instruction consisted of lecturing, whole class discussion, and collaborative activity. Each module's design was based on a conceptual analysis of the cognitive activities conjectured to be necessary for developing understandings of the module's topic. Specific topics of focus for the course were rate of change, proportionality, functions, linear functions, exponential functions, logarithmic functions, rational functions, unit circle trigonometry, and right triangle trigonometry. Enrollment in the precalculus redesign course was voluntary and, upon enrollment, the students could not distinguish the redesign course from the other precalculus sections at the university.

### **Pre-Interviews**

The researcher conducted a 60-minute pre-interview with each student prior to the beginning of the teaching experiment sessions. The intention of these interviews was to gain information about each student's reasoning abilities and understandings of angle measure by having them solve various tasks involving angles and their measure. The pre-

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<sup>9</sup> The human subjects approval letter can be found in Appendix G.

interview followed the design of a clinical interview (Clement, 2000) and Goldin's (2000) principles of structured, task-based interviews.

### **Teaching Experiment**

The primary methodology for this study was a teaching experiment (Steffe & Thompson, 2000) conducted by the researcher. In principle, a teaching experiment not only tries to identify the beginning and ending points of student progress, but also the construction and reorganizations made by the student that enables this progress. The teaching experiment involves a sequence of teaching episodes that include: one or more students (this study has three), a teaching agent (the researcher), a witness of these episodes, and a method of recording what occurs during each episode.

The teaching experiment sessions for the study took place separate from the course in which the students were drawn. The students did not attend the precalculus course during the study, but instead met as a group with the researcher (myself) for six 90-minute sessions. Each teaching experiment session was videotaped and digitized and all student work from the teaching experiment sessions was collected.

A teaching experiment methodology was chosen in order to offer the researcher a first hand experience of the students' reasoning and development. According to Steffe and Thompson (2000), teaching experiments allow the researcher to experience constraints of the language and actions of students and of students' mistakes, especially those that persist regardless of the researcher's best efforts to both uncover and advance student thinking. The use of the word *experience* is not meant to imply that the researcher has direct access to the students' realities, as the students' mathematical realities are

entirely independent of those of the researcher. Rather, Steffe and Thompson use *experience* to refer to the researcher's experiences that allow insights into the *students' mathematics*, where a student's mathematics is fundamentally unknowable to the researcher. These insights or interpretations by the researcher of the students' mathematics allow the researcher to construct models of the students' mathematics, which Steffe and Thompson define as the *mathematics of students*. This distinction emphasizes the autonomy of individuals, where the most basic goal of a teaching experiment is to build, test, and refine tentative models of each student's mathematics, which are based on the researcher's interpretations of the students' actions and behaviors.

The teaching experiment methodology described by Steffe and Thompson (2000) enables a bridge between research and teaching. Rather than suppressing conceptual analysis in investigations into students' sense making, teaching experiments take the stance that students are in a constant mode of construction and the researcher's goal is to understand these constructions and how they are made. This contradicts classical experimental design in which students' mathematical achievement is often focused upon in a manner that does not consider the unique meanings constructed by the students involved in the experiment.

Due to students being in a constant mode of construction, a teaching experiment is not only designed to test hypotheses, but also *continually* generate them (Steffe & Thompson, 2000). Although one does not begin a teaching experiment without major hypotheses to test (e.g., a developed image of angle measure and the unit circle enables the construction of coherent understandings of trigonometric functions), these hypotheses

are forgotten during the teaching sessions of the experiment. This is done in order to continuously adapt to the constraints presented when interacting with students. The researcher attempts to generate hypotheses during interactions with the students with the goal of trying to promote the maximal development in each student participating in the experiment. Just as a researcher attempts to create perturbations in the students, a student's actions may lead to perturbations in the researcher's models of the student's mathematics that the researcher must consider and pursue.

The continual generation of hypotheses also allows the formulation of new situations of learning not considered in the initial design of the teaching experiment. As Steffe and Thompson (2000) articulate, this allows the researcher to push the boundaries of the students' mathematics to where students make what the researcher considers essential mistakes. This highlights the purpose of a teaching experiment: determining the reasoning a student can engage in (regardless of correct or incorrect) and the possible mental actions behind the student's reasoning. The continual generating and testing of hypotheses also stresses the importance and difficulty of the role of the researcher. The researcher will often engage in interactions with students where he or she will not know the direction the interactions are heading. Because of this, the researcher must engage in interactions such that he or she makes no intentional distinctions between his or her knowledge and a student's knowledge. The researcher's attempt to immerse himself in the interactions with the students stresses the importance of the role of a witness, as the witness's perspective may offer unique insights not considered by the researcher.



The continual hypothesis generating, testing, and reconstructing also highlights the recursive nature of Steffe and Thompson's teaching experiment methodology (2000). A researcher's inferences of a student's mental actions inform further interactions and interventions with the students, which may result in more crystallized models of each student's mathematics. But, it is important to note that although these models of students' mathematics may become more precise over time, the models are *never* to be interpreted as one-to-one representations of the students' thinking and knowing. Rather, the models are merely models that work in explaining possible cognitive processes behind the students' actions. These models remain viable as long as they explain the students' contributions. However, it is *always* the case that researchers should attempt to build models that supersede current ones. A model of a student's mathematics should never be thought of as perfect, as this would contradict the notion of the students being autonomous individuals.

Relative to this study's implementation of the teaching experiment methodology, the researcher met with a colleague previous to each teaching experiment session. The colleague also witnessed each teaching session. These meetings were used to discuss and document the instructional goals of the lesson and the design of the lesson in light of these instructional goals. These instructional goals and design decisions were informed by the hypotheses of each student's understandings that the researcher developed by reflecting on video data between sessions. This allowed the witness to develop a model of how the lesson was originally planned to unfold. The witness took notes on the behaviors exhibited by the students during each teaching experiment session. He also noted when

the instruction appeared to deviate from the planned lesson and what he initially believed caused this deviation.

The researcher also met with the witness to debrief after each teaching experiment session. This debriefing included discussing and documenting what were seemingly important moments of the teaching session and the students' behaviors that were noteworthy during the lesson. The researcher and witness also discussed and documented apparent deviations from the intended lesson and why the researcher made these deviations. This allowed an immediate documentation and later retrospective analysis of the reasoning for various instructor moves during the teaching sessions relative to the models of students' mathematics generated by the researcher. These discussions were critical in helping the researcher to articulate and validate the rationale for the researcher's decisions.

The behaviors of the students that were identified by the researcher and witness during the debriefing sessions were then analyzed in more detail between teaching experiment sessions by reviewing the video recording of the session. This recording was not transcribed between sessions due to time constraints, although instances that were identified as particularly revealing of a student's reasoning were (conceptually) analyzed in order to generate hypotheses of the student's mathematics. The implications of these hypotheses were considered relative to the future instructional goals, thus informing the design of the next instructional session and the exploratory teaching interviews. These sessions also facilitated the researcher's documentation of the reasoning that led to the design and selection of instructional tasks for the subsequent lessons. This documentation

and generation of instructional tasks enabled testing and modifying hypotheses of the students' mathematics while attempting to promote continued development in the students' understandings.

### **Exploratory Teaching Interviews**

In addition to the teaching experiment sessions conducted with the three students in a classroom setting, teaching experiment sessions were conducted with each individual student at critical points during the study. These interviews were also videotaped and digitized with all student work collected.

As discussed by Steffe and Thompson (2000), and illustrated in practice by Thompson (1994c), a teaching experiment offers a way to use teaching as a scientific method of investigation where models of students' mathematics are generated and tested. A sequence of one-on-one teaching experiment sessions was implemented in order to gain additional insights into the developing conceptions of the students and to test hypotheses (developed previous to the interviews and *on the fly* during the interviews) of each student's understandings. As the classroom sessions included all three students interacting with each other and the researcher, the researcher could only gain glimpses into each student's understandings. Thus, the individualized interviews were designed to gain deeper insights into each student's thinking. The study conducted by Thompson (1994c) illustrates the benefits of using a sequence of such interviews. In order to characterize the nature of these interviews and distinguish them from multi-student

teaching experiment sessions, I propose to call this type of interview created by Steffe and Thompson an *exploratory teaching interview*<sup>10</sup>.

As outlined by the teaching experiment methodology (Steffe & Thompson, 2000), the exploratory teaching interviews involved the posing of tasks and instruction based on each student's actions. This approach enabled the researcher to make decisions during the interview for the purpose of pushing the student to a point of disequilibrium by presenting situations to test the models of the student's mathematics. Included in this approach is presenting situations in which the student's current ways of thinking may result in the student encountering perturbations as a result of their current understandings. This approach allowed the researcher to determine possible limitations to the student's current ways of thinking that may not be revealed in other interview or classroom situations. It also provided an opportunity for the researcher to make instructional decisions aimed at promoting further development of the student. However, it is noted that these instructive actions of the researcher did not occur until the researcher deemed that the student was unable to proceed on the task or had completed the task using her or his current way of thinking.

Through this approach to the interviews, student development outside of the classroom sessions was unavoidable. However, the main intention of this study was not to solely judge the effectiveness of the instructional unit; that is, the overall intention of this study was not to conclude that the lesson worked or did not work. Rather, the main intention of this study was to gain insights into each student's reasoning abilities and

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<sup>10</sup> The use of the term *exploratory teaching interview* will also enable the reader to differentiate between the one-on-one interview settings and the group settings.

conceptions. Relative to this goal, these interviews played a critical role in this investigation by offering the researcher additional insights into the students' reasoning and ways to promote the students' development.

Lastly, due to the approach of considering each student's knowledge as entirely unique, the tasks of each exploratory teaching interview were unique to the interviewee. The tasks were chosen based on the models of that student's mathematics that were generated during the group sessions. This included designing tasks to gain insights into the student's current ways of reasoning, where her or his reasoning breaks down, and the student's ability to construct knowledge that allows her or him to overcome the obstacles encountered as a result of her or his current ways of reasoning. The researcher also documented his rationale for the design of each task relative to the conjectured models of the student's mathematics. This documentation allowed a retrospective analysis that included considering the tentative models of the students' mathematics that informed the instructional and interview design.

### **Data Collection and Analysis Overview**

The data collected for the study included:

- Videotaped teaching experiment classroom sessions (six 1.5-hour sessions), briefing, and debriefing sessions
- Videotaped pre-interviews (sixty minutes per student)
- Videotaped exploratory interviews (totaling approximately four hours per student)
- Student written work from the teaching experiment sessions and interviews
- Precalculus Concept Assessment (pre-course and post-course) and course grades

Each interview and teaching experiment session was videotaped. The teaching experiment sessions included two cameras, in addition to a computer feed capture. One camera was an overhead shot of the students' table that captured the gestures of each student. The second camera captured the behaviors of the researcher and all work produced on whiteboards. The computer feed captured all instructional moves that utilized dynamic applets.

All interviews were conducted in an interview room with two cameras. One camera captured a view from above the table that enabled the chronological documentation of each student's written work. The second camera was directed at the student in order to capture each student's gestures.

The quantitative aspect of this data utilized the performance of all students in the precalculus classrooms relative to the Precalculus Concept Assessment (PCA) and their course grades. The PCA (M. Carlson, Oehrtman, & Engelke, 2010) consisted of 25 multiple-choice questions focused on various reasoning abilities deemed critical for success in calculus (e.g. the function concept and rate of change). This instrument had been validated to reveal a correlation between success in calculus and student PCA scores. The quantitative data collected from all students provided information about the reasoning abilities and understandings of the students relative to the other participants in the course. This allowed the researcher to gauge the abilities of the study's students relative to those of the other students in the course.

The qualitative data was coded following an open (generative) and axial (convergent) coding approach (Strauss & Corbin, 1998) in order to both develop and

refine the models of students' mathematics. Specifically, the data was coded in an attempt to identify emerging student behaviors and patterns or connections between these behaviors. First, this grounded approach consisted of identifying episodes of a student's behaviors and actions that offered insights into the student's reasoning and thinking relative to the topics of instruction. These episodes were used to generate tentative models of the student's mathematics. Next, these tentative models were tested by searching and analyzing the data for evidence that either supported or contradicted the generated models. This analysis resulted in the refinement, extension, or reconstruction of the hypothesized models of each student's mathematics.

This approach of analyzing the data reflects both the generative and convergent purposes of this study. Very little has been documented regarding how students reason about the topics of trigonometry; hence, it was highly important that the researcher remained open to the continual refinement and reconstructing of the hypotheses developed. It was in this process of identifying contradictory or supportive evidence that hypotheses of a student's understandings were altered.

In order to generate hypotheses of a student's understandings, instances believed to reveal insights into a student's reasoning and understanding were analyzed in an attempt to determine the mental actions that contributed to the emerging behaviors. In other words, a conceptual analysis, as described by Thompson (2000), was performed using the data collected. The mathematical constructions and interactions that occurred between the students and researcher were examined in an attempt to model and understand the thinking of the student. However, these generated models were not

replications of the understandings of the student. That is, the generated models were only hypotheses that worked in explaining the behaviors and actions of the students. Because each individual's knowing is completely unknowable to any other individual, the generated models only offered explanations of how the students *may* have been thinking, opposed to claiming how they *were* thinking. It is through the analytical processes of generating, refining, extending, and reconstructing hypotheses that the researcher hoped to achieve validity in the model of each student's mathematics.

### **Data Analysis Procedure**

This section describes the approach to data analysis that involved the generation, refinement, extension, and reconstruction of models of the students' mathematics. The collected data was first organized in a chronological order of its production and an initial analysis consisted of viewing all video data in the order by which the data was produced. During this first viewing of these videos, notes were recorded that provided an overview of each student's behaviors and reasoning. This baseline viewing of the data resulted in a low-level preliminary analysis of the data.

After this preliminary analysis, all interview video data was transcribed and reanalyzed. At this stage the analysis was more fine-grained and included transcribing the student's and researcher's utterances, gestures, and characteristics of speech. Capturing each student's gestures was especially important when attempting to identify the mental images and quantities they constructed. For instance, a student's hand gestures often offered insights into the attributes of a situation they were describing.



The process of transcribing was used as a preliminary level of analysis. During the transcription process, the researcher developed an initial model of each student's understandings and reasoning when attempting to determine a student's utterances and what they meant by those utterances. When transcribing the videos, the initial notes from the first analysis of the data were elaborated and refined. As a result, a more detailed set of notes that were grounded in a careful transcription of the video data was created.

Next, with the textual product from the transcription available, the data was analyzed in a chronological order in an attempt to describe the possible reasoning and understandings that contributed to the student's actions and utterances. During this stage of the analysis, each student's actions were analyzed separately. Relative to the classroom sessions, instances previously deemed to reveal insights into a student's reasoning were reanalyzed in an attempt to identify the mental actions driving a student's products. Relative to the interview sessions, a line-by-line conceptual analysis was conducted in an attempt to identify the mental actions behind a student's behaviors. Furthermore, the perspectives of quantitative and covariational reasoning provided a lens for this analysis, as well as the system of ideas driving the instructional sequence.

With specific classroom instances and interview behaviors analyzed, connections were then sought between a student's actions. This was achieved by comparing and contrasting a student's actions over the course of the study for consistencies in her or his reasoning, conceptual development, and the possible implications of various ways of reasoning. As an example, the data was analyzed to determine the student's conception of the radius as a unit of measurement. This analysis first consisted of identifying and

analyzing specific instances that offered insight to this conception. Then, these models of the student's mathematics were compared and contrasted, while also identifying the role these conceptions played in the student constructing understandings of other topics. The result of this approach to the analysis of the data also generated insights about the role that the instructional sequence played in affecting a student's emerging understandings. Finally, the models of each student's mathematics were compared and contrasted. This analysis shed light on the critical reasoning abilities needed for constructing connected and coherent understandings of angle measure and trigonometric functions.

### **Summary of Chapter**

This chapter described the research methodology of this investigation into precalculus students' conceptions of the sine and cosine functions and topics foundational to trigonometry. A teaching experiment (Steffe & Thompson, 2000), which included a series of exploratory teaching interviews, was implemented in order to provide the researcher opportunities to construct and refine models of students' mathematics. A majority of the data analysis included selective conceptual analysis of the teaching experiment sessions and a line-by-line conceptual analysis of the students' actions during the interview sessions. This analysis enabled the researcher to document the progress of the students over the course of the instructional sequence.

The next chapter presents the findings from an exploratory study of students' conceptions of angle measure. Illustrations of how the exploratory study informed the teaching experiment design are highlighted and the instructional activities used during the teaching experiment are described in terms of the activities' intended outcomes.

## Chapter 4

### Exploratory Study And Instructional Design

This chapter provides an overview of an exploratory study into three students' conceptions of angle measure. The findings from this study are presented along with discussions of how these findings informed the instructional sequencing and tasks used in the study. This includes providing a detailed conceptual analysis of the ideas of angle measure, and the sine and cosine functions. The chapter concludes by presenting the instructional goals of the study and specific tasks that were designed to achieve these goals.

#### **Exploratory Study**

An exploratory study was conducted previous to this dissertation in order to gain insights into three precalculus students' understandings of angle measure. The following section briefly describes the methodology of this exploratory study and then provides a summary of the findings of the study.

The methodology used during the exploratory study was mostly consistent with the methodology of this dissertation, which was described in detail in chapter 3. Clinical interviews (2000) were conducted with each student before the teaching experiment sessions and immediately after the teaching experiment sessions. A teaching experiment (Steffe & Thompson, 2000) was conducted with the three students and consisted of three 65 minute sessions. The three sessions occurred within a time period of eight days. An exploratory teaching interview (one-on-one teaching experiment session) was conducted with each student between the first and second teaching experiment sessions. The focus

of these interviews was on angle measure (in degrees). During these interviews each student encountered the same interview tasks, while the researcher's questioning varied depending on each student's actions. All interview tasks used in the exploratory study can be found in Appendix B.

### **Subjects and Setting**

The exploratory study was conducted with three students from an undergraduate precalculus course at a large public university in the southwest United States in which the researcher (myself) was the instructor. The students were chosen on a volunteer basis and monetarily compensated. The precalculus classroom from which the students were drawn was part of a design research study where the classroom intervention (M. P. Carlson & Oehrtman, 2009) was informed by theory on the processes of covariational reasoning and select literature about mathematical discourse and problem solving (M. Carlson, et al., 2002; M. P. Carlson & Bloom, 2005; Clark, et al., 2008). The classroom instruction consisted of direct instruction, whole class discussion, and collaborative activity. Specific topics of focus were proportionality, functions, linear functions, exponential functions, logarithmic functions, rational functions, unit circle trigonometry, and right triangle trigonometry.

All three students were full-time students at the time of the study and all three students were males (Brad, Charles, and Travis). The first student, Travis, was in his mid-twenties and an architecture student. The second student, Brad, was in his late teens and a computer systems engineer major. The third student, Charles, was in his late teens and a psychology major.

## Results

This section presents an overview of the findings from the exploratory study. First, quantitative data is presented to situate the students within the precalculus classroom from which they were chosen. Then, analysis of the qualitative data is used to discuss each student's thinking exhibited during the study.

**Course assessment.** Brad received a 'C' for his final course grade, Travis received a 'C' for his final course grade, and Charles received an 'A' for his final course grade. Overall, two students from the course received a grade of 'D', five students received a 'C', three students received a 'B', and three students received an 'A'.

**Pre-interviews.** All three students revealed a loose coordination of arc length and angle measure upon entering the exploratory study. Both Travis and Brad measured an angle by constructing a circle and calculating the arc length measured in inches that corresponded to one degree. This way of reasoning (e.g., an arc length per one degree) offered obstacles when describing the measure of an angle relative to circles of different radius lengths. For instance, they had difficulty describing a meaning for angle measure when multiple circles were centered at the vertex of an angle.

The third student, Charles, attempted to recall a formula that related an arc length and an angle measure in order to determine the measure of an angle. He recalled a formula that was correct symbolically ( $s = r\theta$ ), but this formula did not emerge from his conception of a quantitative relationship. He was unable to justify the formula and eventually described an incorrect unit of measure for one of the variables (a degree measure for  $\theta$ ). Also, he did not attempt to explain the formula relative to a quantitative

relationship or operation. Charles then described that angle measures are found by using a formula or trigonometric functions. Charles provided the only mention of trigonometric functions by any of the students during the pre-interview, and he described trigonometric functions as providing an angle measure based on a coordinate on the unit circle. However, he was unable to expand on this statement.

***Exploratory teaching interviews.*** The exploratory teaching interviews occurred after the first teaching session, which introduced measuring angles in a number of degrees. During the interviews, both Travis and Brad first provided explanations of angle measure that referenced an arc length, a circumference, and an area of a circle. Their explanations vaguely referenced the quantities and how these quantities were related. Contrary to this, Charles immediately identified angle measure as determining the fraction of a circle's circumference subtended by the angle and that any circle could be used to determine this measure. He also described the unit circle (which had not been introduced to this point) as the easiest circle to use because the coordinates of the circle are related to the arc length. But, he was unsure how to use these coordinates, which was consistent with his description of coordinates on the unit circle during the pre-interview.

As the interviews progressed, each student refined his image of angle measure and its relationship to a subtended arc length and circumference of a circle. The students altered and refined their descriptions as they encountered situations where they needed to find various measurements by creating circles and using other given measurements. Furthermore, after the students completed the problems, the researcher often provided the students a calculation in open form and asked the students to interpret the expression

previous to calculating the value of the expression. This resulted in the students comparing how they could use the various calculations to solve the problem and how the provided expressions were related to the quantities of the situation.

For instance, the researcher asked the students to explain how another student might use the expression of  $\frac{22.3}{360} + \frac{3.1}{16}$  to solve a problem asking for the relative measurement (e.g., the percentage of a circle's circumference subtended by an angle) of an angle that measures 22.3 degrees plus 3.1 quips<sup>11</sup>. This resulted in the students describing each term of the expression relative to a quantitative relationship (e.g., 22.3/360 corresponds to the *percentage* that 22.3 degrees is of 360 degrees), while relating these values to a percentage of a circle's circumference. The students' conception of angle measure as subtending a *percentage* of a circle's circumference was spontaneous and not promoted previous to the interviews. This image of angle measure became a powerful aid for the students, as they were frequently observed reasoning about angle measure in this manner to complete the interview tasks. The students' percentage image appears to have promoted their reasoning about a fractional amount of a whole (e.g., 24% of a total 100%).

The students' actions during the exploratory study also revealed the importance of their ability to distinguish between linear measurements and angular measurements of arc length and circumference. As the students refined their images of angle measure such that these images included an explicit distinction of the two measurements, they were seen generalizing their statements about a subtended arc length's percentage of a circle's

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<sup>11</sup> Sixteen quips (a fictitious unit) rotate through the circumference of a circle.

circumference to include any unit of measurement and any circle used to measure the angle. For instance, Brad described that the percentage of a circle's circumference (e.g., 26%) subtended by an angle was not reliant on the unit of measure and that this percentage could be used to determine the angle measure in any unit. Also, the ability to distinguish and relate linear and angular measurements enabled all three students to describe how a varying radius would change the angular and linear speeds of an individual riding a Ferris wheel at 2.5 revolutions per minute. Consider Excerpt 1 for the Brad's description.

Excerpt 1

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1	KM:	So if we maintain the same angular speed, what happens as the radius
2		increases?
3	Brad:	You'll still have to travel, well, if you're looking at angular speed and
4		your radius is increasing then your radial speed is going to stay the
5		same, you'll still have to keep that to make it. But, as your looking at
6		it, your linear speed, in order to keep that exact same radial speed,
7		you'd have to increase your linear speed.
8	KM:	Ok, so you'll have to increase, and if we shrunk the radius?
9	Brad:	You'd have to decrease.
10	KM:	Decrease. Ok, but lets say we kept the same linear speed...as we
11		increase our radius what happens to our...
12	Brad:	You'd have to decrease your radial speed.
13	KM:	Decrease your radial speed. Ok, good, and if we decrease the radius?

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14 Brad: You'd have to increase your radial speed.

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It appears that Brad was able to coordinate the relationship of angular speed, or radial speed, to the linear speed of the individual. This reasoning enabled him to consider multiple lengths of a radius while distinguishing linear measurements from angular measurements.

***Post-interviews.*** The post-interviews were conducted after the last teaching experiment session. The second session focused on the use of a radian as a unit of angle measure. The third session explored circular motion (e.g., graphing the relationship of a traversed arc length and a vertical distance above the center of the circle) and the unit circle as the result of using the length of a radius as a unit of measurement.

All three students initially defined angle measure in terms of a subtended arc length's percentage of the corresponding circle's circumference. They further described that this percentage remained constant as the radius of the circle increased or decreased. This image of angle measure supported the students describing that the ratio of the linear measure of arc length and circumference would not change when using various circles to measure the angle.

As the students engaged in the activities that required reasoning about the radius as a unit of measure (of an angle or a vertical distance), they also spontaneously reasoned about radian measures as a percentage of the length of a radius. Initially, each student used a multiplicative relationship between arc length and circumference to determine an angle measure in radians, opposed to using a comparison between the arc length and the length of a radius. In response, the researcher posed the question of the meaning of a

number of radians (e.g., what does 0.628 radians mean?). Travis explained that the measure represented a percentage of “all the radians...6.28” and then added that the measure was also a percentage of one radius (e.g., 62.8%). As he continued, he described that the angle measure can be found by dividing the arc length by the length of a radius to find “how many radii were in the arc length.” Charles also explained a radian measure in terms of a percentage of a circle’s circumference. After prompting from the researcher, he described the measurement as the ratio of arc length to the length of a radius, where this ratio remained constant regardless of the size of the circle used. Brad described radian measures in terms of an arc length’s percentage of the entire circumference, while also describing that he could calculate a radian measure by dividing the linear measurement of arc length by the linear measurement of the radius to find how many radius lengths rotate along the arc.

Each student reasoned about a relationship between an arc length and a radius when discussing the radian as a unit of measure, but the students were most frequently observed reasoning about radian measurements as conveying a percentage of a circle’s circumference subtended by an angle of that measure. This may have been a result of emphasizing angle measure as a fraction of a circle’s circumference during the introduction to angle measure. In addition, it may have been an easier construction for students to relate arc length to circumference given that the circumference of a circle is an arc length; relating the length of a radius to an arc length requires conceiving of the magnitude of the radius as a unit for measuring a subtended arc.

As an example of the students' propensity to reason about a percentage of a circle's circumference, each student utilized ratios that represented a percentage of a circle's circumference (e.g.,  $\frac{s}{2\pi r} = \frac{\theta}{2\pi}$ ) in order to determine a formula relating the arc length subtended by an angle, the radius of the corresponding circle, and the angle measure in radians. Then, after simplifying their formulas and obtaining  $s = \theta r$  or  $\theta = \frac{s}{r}$ , each student discussed that their simplified formula stemmed from a radian measure conveying the number of radius lengths rotating along an arc length. Thus, it appears that the students constructed a relationship between an angle measure in radians and the length of a radius, but an image of angle measure as a percentage of a circle's circumference was more prominent in their reasoning.

Table 1
<i>The Ferris Wheel Problem (Exploratory Study)</i>
Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full rotation. April boards the Ferris wheel and begins a continuous ride on the Ferris wheel. If the platform to board the Ferris wheel is 8 feet off of the ground, sketch a graph that relates the total distance traveled by April and her vertical distance from the ground.

The students' solutions on The Ferris Wheel Problem (Table 1) also offered various insights into the students' abilities to leverage reasoning about a varying arc length. First, each student had no difficulty identifying the number of radians, degrees, and feet that corresponded to one minute of elapsed time. The students' ability to reason

about a varying arc length enabled them to construct a relationship between an elapsed time and a traversed arc length, while relating the traversed arc length to angle measure.

When prompted to graph the April's (the rider) vertical distance above the ground versus the distance traveled, each student did not develop an image of a vertical distance consistent with the researcher's intentions. Each individual correctly identified the minimum vertical distance from the ground as 8 feet. However, Charles explained that 72 feet was the maximum vertical distance (the diameter of the Ferris wheel). Brad identified 36 and 42 feet as maximum vertical distances (the radius of the Ferris wheel and an incorrect addition of the radius and 8 feet). Travis explained that 64 feet was the maximum vertical distance (the difference of the diameter of the Ferris wheel and 8 feet). As the students continued to work the problem, each identified the proper maximum value for the vertical distance. Their correction came as a result of identifying the initial vertical distance and the diameter of the Ferris wheel on a diagram of the situation. The inconsistencies in the students' images of the problem's context emphasize the importance of a researcher remaining attentive to a student's constructed mental image.

When graphing the relationship between the vertical distance from the ground and the total distance traveled, all three students initially drew incorrect graphs relative to concavity, but each graph was correct in terms of directional change. Each student justified his graph by reasoning that as the total distance increased, the vertical distance from the ground increased and then decreased (MA2). When asked to explain the curvature of the graph, each student then refined his explanation to include amounts of

change (MA3) *based on the graph*, opposed to a diagram of the situation. That is, they gave correct explanations relative to the relationship conveyed by their graphs.

As an example, after Charles drew a graph that was concave down for the first three-quarters of a revolution, he explained that his graph was concave down because the vertical distance was increasing and the change of vertical distance was decreasing as the total distance increased. Charles' explanation of the concavity of his produced graph was correct, but the graph itself was incorrect over the first quarter of a revolution (e.g., his graph was concave down opposed to concave up). Also, Charles did not refer to a diagram of the situation to explain the covariational relationship (and neither did the other two students without prompting). In response to Charles' actions, the researcher asked Charles to explain the shape of the graph using a diagram of the Ferris wheel (Excerpt 2).

#### Excerpt 2

- 
- |   |          |   |
|---|----------|---|
| 1 | Charles: | So, as the total distance is increasing ( <i>tracing the arc length</i> ), we   |
| 2 |          | notice that height is increasing but for every successive change of             |
| 3 |          | total distance ( <i>making marks at equal changes of arc length</i> ), lets say |
| 4 |          | right here it's eight, well if I drew a bigger one, I'd be able to show it      |
| 5 |          | more precise.   |
| 6 | KM:      | Here, go ahead and, uh, I've got some extra pieces of paper. Go ahead           |
| 7 |          | and if you want to draw it on there somewhere ( <i>handing him a sheet</i>      |
| 8 |          | <i>of paper</i> ) a little bigger.  |
| 9 | Charles: | ( <i>Drawing a larger circle and drawing a vertical-horizontal crosshair</i>    |
-

- 10                    *in the middle of the circle*) So, as, I guess we can assume this is  
11                    ninety degrees (*referring to the compass*) we can make an angle  
12                    (*attempts to use protractor on compass*)...
- 13    KM:            So what are you trying to do right now?
- 14    Charles:       Well I see, this thing moves (*referring to the compass*), I'm trying to  
15                    show that, um, I'm trying to make, well I could use the protractor,  
16                    I'm just trying to change, show successive change in input.
- 17    KM:            Could you use the Wikki Stix to do that?
- 18    Charles:       Well, actually, yes I can.
- 19    KM:            So, you're trying to show successive changes in what?
- 20    Charles:       In input, which would be the total distance (*marks a distance on a*  
21                    *Wikki Stix mumbling to himself, then marking off successive arc*  
22                    *lengths on the circle*). Ok, so, for every change in total distance, right  
23                    here (*referring to changes in arc length*), he, well, hmmm.
- 24    KM:            What makes you go hmmm?
- 25    Charles:       Because I was thinking the last time I did, this represented right  
26                    (*referring to the top-half of the circle*), hmm.
- 27    KM:            So what's making you go hmm now?
- 28    Charles:       Because it seems as total distance increases (*referring to the arc*  
29                    *length*), the actual change in the height is increasing instead of  
30                    decreasing.
-

Charles first explained that as the total distance increased, the height of the individual also increased (lines 1-3). Next, Charles attempted to identify successive changes of arc length by relating arc length to angle measure (lines 9-12 & 14-16). Charles continued by using a Wikki Stix (a piece of waxed string) to mark equal changes of arc length, which resulted in him questioning his original graph (lines 20-23), and eventually concluding that the changes in height should be increasing rather than decreasing over the first quarter of a revolution (lines 28-30). By using a diagram, Charles was able to describe the correct relationship between total distance and vertical height using amounts of change (MA3), which led to him correcting the graph representing this relationship.

The graph first drawn by Charles was possibly created without reasoning about amounts of changes of height and amounts of changes of total distance. Rather, Charles drew a curved graph, possibly because of an iconic transfer of the shape of the Ferris wheel or an assumption that there was a varying rate of change, and he *then* interpreted the graph relative to amounts of change. His description of the graph was correct in both instances. However, in the first instance, his graph molded his description of the two quantities; in the second instance, his reasoning about amounts of change of the two quantities within the context of the situation drove the construction of the graph.

Charles's actions highlight the role of quantitative reasoning and covariational reasoning in representing the relationship between two quantities. In order to represent the (correct) relationship between two covarying quantities, a student must construct an image of the situation and attributes that are consistent with the instructional intentions. These attributes can then become conceptual objects the students can reason with in a

manner that results in meaningful descriptions and representations. Charles's actions also points out the difficulty of covarying two quantities. Although his descriptions remained attentive to the quantities he was relating (e.g., arc length and vertical distance), he was unable to produce a correct graph (from the researcher's perspective) until he focused on specific intervals of arc length within his image of the situation (e.g., a diagram).

### **Summary of Findings**

Initially, two students were observed holding a loose coordination between angle measure and arc length that enabled them to identify an arc length corresponding to one unit of angle measure for a *particular* circle. The third student did not reason about angle measure relative to arc length beyond attempting to recall a formula.

As the students solved various problems, their conceptions of angle measure continually developed and all three students began reasoning about angle measure in relation to the arc length and circumference of a circle. Specifically, the students conceived of angle measure as the *percentage* of a circle's circumference subtended by the angle, regardless of the size of the circle. As the students continued through the study, this way of reasoning became more dominant. As one example, the students predominantly reasoned about radian measurements conveying the percentage of a circle's circumference subtended by an angle. But, as the researcher questioned the students, they also described radian measurements as a number of radius lengths, or as a fraction of the radius, along a subtended arc length.

Another finding from the exploratory study was the importance of the students distinguishing between and relating a linear measurement of arc length and a



corresponding angle measure. This ability supported the students in reasoning about angle measure corresponding to a circle of any radius. In addition, distinguishing between a linear measurement and angular measurement appears to have aided the students in using multiple units of measurement while describing circular motion. For instance, given a constant angular speed, the students were able to describe how the linear speed would vary for a varying radius.

Lastly, the students' ability to reason about a varying arc length appears to have created a foundation for them to reason about relationships between quantities in the context of circular motion. The students were able to covary an arc length and a vertical distance to construct graphical representations that formalized these relationships. Also, by leveraging a diagram of a problem's context, the students were able to support and modify their graphical representations (their initial graphs were not rooted in this reasoning). In the case of each student, reflecting on a diagram of the situation resulted in their modifying their images of the situations and the mathematical representations of these images.

### **Implications of the Exploratory Study**

Consistent with the findings described in the research literature on trigonometry (Akkoc, 2008; Brown, 2005; Fi, 2003, 2006), the students in the exploratory study initially held underdeveloped conceptions of angle measure. The researcher conjectured that the students involved in this dissertation would hold similar understandings of angle measure upon entering the study. In order to address this issue, a similar pre-interview for the dissertation was used to gain insights into each student's initial conception of angle

measure. Also, a two-day instructional sequence<sup>12</sup> on angle measure was designed based on the findings from the exploratory study and the pre-interviews conducted during the study.

The students' conceptions of angle measure informed the design of the angle measure activities used during the study. For instance, the students' initial method of reasoning about an arc length per unit of angle measure presented obstacles when the students were presented with circles of differing radius lengths. In light of this finding, the instructional activities were altered to include opportunities for the students to reason about circles of multiple sizes. For instance, the activities were changed such that each student was given a different sized circle, and the students were asked to compare their solutions relative to their different sized circles.

Another finding that informed the design of this study was the students' propensity to reason about radian measures conveying the percentage of a circle's circumference subtended by an angle. Each student also reasoned about a radian measure corresponding to the multiplicative relationship between a subtended arc and a radius, but they predominately conceived of radian measures in terms of a percentage of a circle's circumference. This finding led the researcher to design additional tasks that promoted the students conceptualizing the radius as a unit of measure. The researcher also frequently prompted the students in this study to reason about the multiplicative relationship between a subtended arc and the length of a radius.

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<sup>12</sup> The instructional sequence is presented later in this chapter.

The exploratory study also illustrated the importance of a student constructing and refining her or his image of a problem's context in ways that promote reasoning consistent with the instructional goals. As the students solved problems during the exploratory study, they frequently alternated between reasoning about the context of the problem and executing calculations or constructing mathematical representations of relationships (e.g., formulas and graphs). This iterative process resulted in the students refining their image of the problems' contexts and the quantitative relationships within these contexts. The students' actions stress the importance of a researcher or teacher remaining attentive to the mental image a student constructs and the continually evolving nature of this image. Also, this observation highlights the role quantitative reasoning played in the design of this study. A main intention of the study was to gain insights into the mental images the students constructed (and modified), and the relationship of these images to the students' thinking and understandings.

### **Conceptual Analysis and Instructional Design**

The findings gained from the exploratory study, as well as the insights gained from the research literature presented in chapter 2, informed the initial design of the instructional sequence used during this study. The intent of the instructional sequence was to offer the students opportunities to construct meaningful and coherent understandings of trigonometric functions and angle measure in the contexts of the unit circle and right triangles.

The term *coherent* (and variants of this term) is commonly used in mathematics education. Students are expected to develop coherent understandings and curriculum is

expected to promote coherence. Yet, *coherence* and *coherent* remain ill-defined in mathematics education and curriculum development (Thompson, 2008). Recently, the *National Mathematics Advisory Panel* (NMAP) described coherent as, “effective, logical progressions from earlier, less sophisticated topics into later, more sophisticated ones” (2008, p. xvii). As Thompson notes, such a definition can be taken to imply that the NMAP places an emphasis on topics, rather than ideas, meanings, or larger ways of reasoning. In such an approach, it becomes easy to treat topics as segmented and mostly independent goals of learning (e.g., linear functions, then exponential functions, then trigonometric functions), as opposed to identifying larger ways of reasoning (e.g., quantitative and covariational reasoning) that can encompass these topics.

A distinction of “less sophisticated topics” and “more sophisticated” topics is a natural part of curriculum design, but coherence results from the “development of meanings of each and the construction of contextual inter-relationships among them” (Thompson, 2008, p. 47). For instance, placing a lesson on angle measure before an introduction to trigonometric functions is a “logical progression” of topics. However, the mere introduction of angle measure before trigonometric functions does not result in coherence. Rather, coherence is the product of the meanings driving the lesson on angle measure creating a foundation for understanding trigonometric functions. Moreover, coherence should extend beyond sequential topics and include developing ways of reasoning that are advantageous across mathematical topics.

This approach to achieving coherence is much more easily stated than achieved, particularly because curricula coherence cannot be considered independent of a group of

learners. Conceptual analysis (Thompson, 2008) offers a learner-centered tool to help achieve the difficult goal of coherence. Two interrelated uses of conceptual analysis are 1) describing ways of knowing that are immediately *and* developmentally beneficial for learning and 2) analyzing ways of understanding a body of ideas based on describing the coherence between their meanings, where coherence refers to *both* compatibility and support. These two uses of conceptual analysis provide tools for determining instructional goals and the development of curriculum, both of which are elements of this study.

Stressing the importance of considering ways of knowing that are both immediately and developmentally beneficial for learning, instructional goals must be considered relative to their current place in the curriculum, their possible influence on later curriculum goals, and the group of learners that the curriculum applies to. For instance, images of radian measure should be developed that consider the future mathematical experiences to be had by the learner (e.g., the unit circle). This is consistent with Dewey's (1938) call for a "continuity of educative experience." He described that when considering the design of educational material, a designer cannot create materials outside of the experiences of the learner. That is, educational materials should not be based on adult understandings, which is an easy trap to fall into and may often be the dangerous driving force of the "logical progressions" of topics. Rather, materials need to focus on the continual development of connected experiences that promote reflective and developmental thinking. At *every* point in mathematics curriculum a designer or teacher must consider what previous experiences are to be drawn on as well as the future experiences for which one hopes to create foundations. As Dewey stated, "...every

experience both takes up something from those which have gone before and modifies in some way the quality of those which come after” (1938, p. 35).

The following section presents a system of ideas of angle measure and trigonometric functions in order to provide an example of a conceptual analysis. This conceptual analysis drove the design of this study, and namely the design of the instructional sequence outlined later in this chapter. Before continuing, it is noted that this system of ideas is based on the groundwork laid by Thompson (2008) and Thompson, Carlson, and Silverman (2007).

### **Trigonometric Functions and Angle Measure**

What follows is a discussion of a system of ideas and ways of understanding of angle measure and trigonometric functions. The general threads of this discussion are:

- The measure of the openness of an angle can be conceived of in terms of a subtended arc length's fraction, or percentage, of the corresponding circle's circumference, regardless of the unit of angle measure.
- The unit of a radian (or radius lengths) conveys the multiplicative comparison of an arc length and the length of a radius.
- The unit circle results from measuring quantities relative to the radius.
- The sine and cosine functions have an input quantity of angle measure, measured in radians, and an output quantity of a length measured in a number of radius (or hypotenuse) lengths.

Angle measure offers a common foundation for trigonometry that can be leveraged to promote coherence between the various contexts of trigonometry. One could

argue, and argue legitimately, that current approaches to trigonometry are reliant on angle measure. However, this reliance does not necessarily imply that trigonometry is developed on foundations of angle measure, or that angle measure itself is developed in a way that can be leveraged as a foundation to trigonometry (e.g., coherence beyond a logical progression of topics). More pointedly, angle measure can often be found in mathematics curricula, but the measure of an angle (and what this measure is actually of) is often used for a purpose differing from the result of a measuring process or an attempt to develop trigonometric functions.

For instance, in triangle trigonometry, an angle is typically used in a manner such that it exists only somewhere within a triangle of interest; the angle and its measure are both treated as a place on a triangle opposed to the value being a measure of an attribute of the angle (Thompson, 2008). In addition, curriculum often treats trigonometric functions in a special manner relative to triangle trigonometry; curriculum frequently explores *static* relationships between objects of a triangle. Here, the focus is on a result (e.g., finding the side of a specific triangle) rather than a function (e.g., a process relating the varying values of two quantities). This can have the undesired effect of the sine, cosine, or tangent functions becoming only about specific whole triangles and finding values, which requires evaluating the functions. This is in stark contrast to the use of the sine and cosine functions as processes and the openness of an angle as a quantity that can vary causing simultaneous variation in the output of the trigonometric functions.

In addition to the use of angle measure as referencing an object in triangle trigonometry, a majority of the U.S. mathematics curriculum's approach to angle measure

is inconsistent and lacks an explicit focus on the *process* of measuring an angle. Upon the introduction of angles, textbooks frequently describe angle measure as an amount of a rotation or a measurement of a subtended arc. At this time, the degree is *imposed* as the common unit of angle measure and students are immediately presented with a protractor as a tool for measuring an angle. Additionally, students are presented with various geometric objects that are defined to have a specific angle measure (e.g., a straight line is one-hundred and eighty degrees, two perpendicular lines form an angle of ninety degrees). Then, common activities focus on having the students *calculate* angle measures by using properties of supplementary angles, complementary angles, vertical angles, etc. However, the *process* by which an angle's openness is quantified is rarely addressed beyond the *use* of a protractor (that has already been created by someone other than the student) or defining various geometric objects.

The ability to *use* a protractor (e.g., a procedure) is significantly different than understanding the *process* by which an angle-measuring device is created. A student may understand that an angle measure references an arc length or an amount of rotation, but this understanding may not include a mathematical process behind measuring an angle. This lack of understanding can create multiple conceptual hurdles. For instance, in the case that a student solely understands *angle measure is arc length*, if circles of different sizes are used to measure the angle, the magnitude of the arc length varies, possibly resulting in a perturbation in the student's image of angle measure.

Whereas the introduction to angle measures in degrees is frequently vague in terms of the process by which the measurement is based, textbooks frequently describe



angle measures in radians as a measure of an intersected or subtended arc length in a number of radius lengths. In this context, the angle measure is explicitly defined as a number of radius lengths subtended by the rays of an angle. Yet, this approach to teaching angle measure is contradictory to the traditional introduction of angle measure in degrees.

A lack of understanding the process of measuring an angle may also inhibit individuals' ability to conceive of a varying angle measure (a conception necessary in situations of circular motion). When the focus of the sine and cosine functions is on determining the sides of a triangle, angles become static objects with constant measurements opposed to dynamic objects with measurements that can (and do) vary. This opposes the various uses of trigonometry in the dynamic settings of engineering, physics, and other sciences, where the input value of trigonometric functions varies and the use of trigonometric functions is not necessarily about finding sides of specific triangles. Thus, if an individual's image of angle measure consists of references to objects or merely using a protractor, and her or his understanding of trigonometric functions consists of static right triangles, it becomes much to ask for the individual to apply this reasoning to angles and trigonometric functions in dynamic situations.

Developing angle measure as a process based on measuring a subtended arc length's fraction of a circle's circumference can possibly avoid a vagueness of angle measure. In terms of degrees, this approach implies that one degree corresponds to an arc length of a circle that is  $1/360^{\text{th}}$  of the circumference of that circle<sup>13</sup>; in terms of radians,

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<sup>13</sup> The circles referenced are assumed to be centered at the vertex of the angle.

this approach implies that one radian corresponds to an arc length of a circle that is  $1/2\pi^{\text{th}}$  of the circumference of that circle. By developing angle measure in this manner, measuring in degrees and radians are *both* developed as the process of measuring the fraction, or percentage, of a circle's circumference subtended by the angle.

This process is essentially how the construction and use of a protractor works. Measuring in either unit is measuring the same attribute in the same manner, but using a different number of units to divide the attribute being measured (Figure 1). Both units of measure have a fixed total number of units that rotate through any circle centered at the vertex of an angle and measuring the openness of an angle involves determining the fraction of the total circumference of a circle and how many of the total units correspond to this fraction. Thus, if  $x$  degrees or  $r$  radians are subtended by an angle, both measurements correspond to the same fraction or percentage of a circle's circumference (e.g.,  $x/360 = r/2\pi$ ).

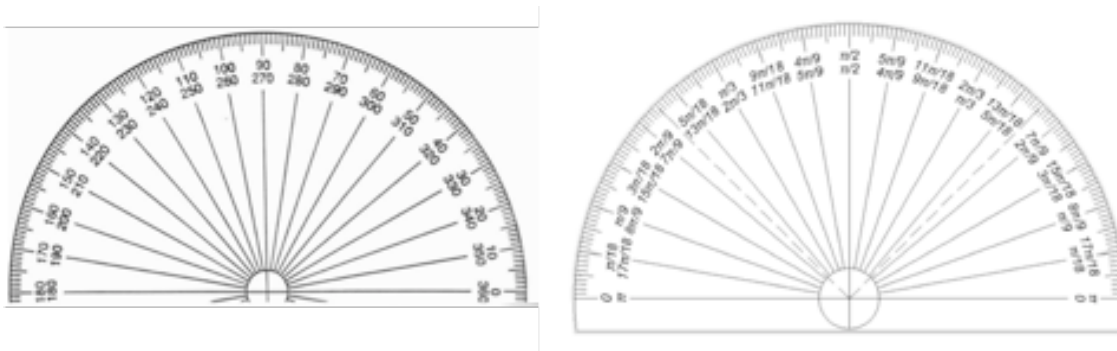


Figure 1. Angle measure and units.

With angle measure presented as an arc length's fraction of a circle's circumference, a varying openness of an angle corresponds to a simultaneous variation in

the fraction of the circumference cut off by the angle. Rather than an angle and the measure of its openness being a static object in a right triangle or a position on a circle, the openness of an angle is a measurable attribute (e.g., a quantity) of the angle that can vary. This understanding is necessary for reasoning about the unit circle and periodic motion. This is one reason for introducing trigonometric functions using the unit circle; it offers a context that can be directly related to angle measure as a fraction of a circle's circumference and leveraged to introduce angle measures that vary.

An important facet of this approach to angle measure is that the approach is not reliant on the radius of the circle that is chosen to measure the angle. Although the linear measurement of the circumference of a circle and corresponding subtended arc length may vary when considering circles of different radii, the percentage of the circumference subtended by the angle, regardless of the circle, remains constant.

The use of a radian as a unit of angle measurement is highly reliant on the image that a circle of any radius can be used to measure the angle. Regardless of the circle used,  $2\pi$  of the corresponding radius lengths rotate through the circumference of the circle; but the linear measure that one radius corresponds to changes depending on the chosen circle. This is especially important when exploring circular motion in physics and distinguishing between angular and linear (tangential) velocities. In these cases, a circular path of a specific radius is often considered. Therefore it is important for a student to be able to apply the use of a radian as a unit of measure relative to a particular circle of interest, although between problems (circles) the linear measurement one radius corresponds to may change.

In addition to an arc length's fraction of a circle's circumference, an angle measure made in radians conveys the multiplicative relationship between a length (e.g., arc length) and the length of a radius, just as any measure conveys a multiplicative relationship between the quantity being measured and the unit of measurement. If an angle has a measure of 2 radians, the length of the arc subtended by the angle is two times as large as the length of a corresponding radius. Or if an angle has a measure of  $\pi/4$  radians, the length of the arc subtended by the angle is  $\pi/4^{\text{th}}$  times as large as the length of a corresponding radius. Note that the focus remains on the arc length subtended by the angle. The angle measure is not on a location on a circle, which may often be inferred from the labeling of the unit circle, or a location in a right triangle.

The relation between a linear measurement of arc length and the corresponding radian measure is also important relative to the use of the unit circle. As Weber (2005) described, it is important to promote that students understand the process by which the unit circle is constructed if the students are to leverage the unit circle as a reasoning tool. The unit circle is often claimed to be a circle with a radius of  $r = 1$ . With a connection between the length of one radius as a unit of measure and a corresponding linear measurement (e.g., inches or centimeters), all circles can be conceived as the unit circle (e.g., a circle with a radius length of one radius). It is the linear measurement that one radius refers to that varies between circles (e.g., the magnitude of the measurement unit). As a consequence, rather than trigonometric functions being related to only a circle of  $r = 1$ , where 1 is a number, trigonometric functions are connected to any circle through the use of the length of a radius of the circle as a unit of measurement (e.g., 1 as a value).

At this time, the cosine and sine functions can be defined as processes that have an input of angle measure, in radians. In the context of the unit circle, the output of the cosine function is the abscissa of the terminus of the arc subtended by the angle and the output of the sine function is the ordinate of the terminus of the arc subtended by the angle, with both measured as a fraction of one radius (the unit of measurement as a quantitative relationship). This definition, along with the ideas presented above, allows the development of the cosine and sine functions coherently in each context (Figures 2-4). In the context of a right triangle, the cosine and sine functions have an input of angle measure, measured in radians, and output a length measured as a fraction of a hypotenuse, where the hypotenuse of a right triangle can be conceived of as the radius of a circle. The outputs of the cosine and sine functions are values (formed by multiplicative comparison) regardless of the context and as the radius of the circle increases (the hypotenuse of the right triangle), the outputs of the cosine and sine functions remain constant due to the similarity between the right triangles. Furthermore, if the radius (length of the hypotenuse) is held constant and the angle measure varies, then the output values of the cosine and sine functions vary accordingly.

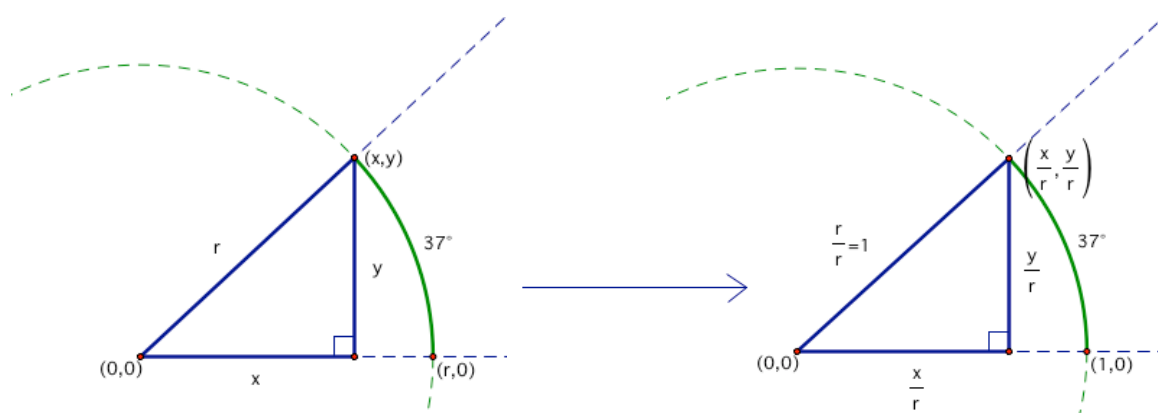


Figure 2. An arc length image of angle measure and unit circle trigonometry.

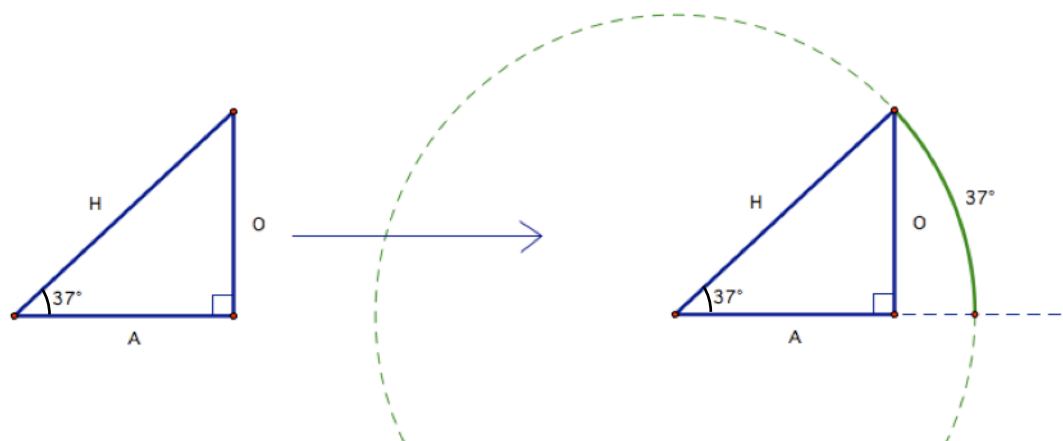


Figure 3. An arc length image of angle measure and triangle trigonometry.

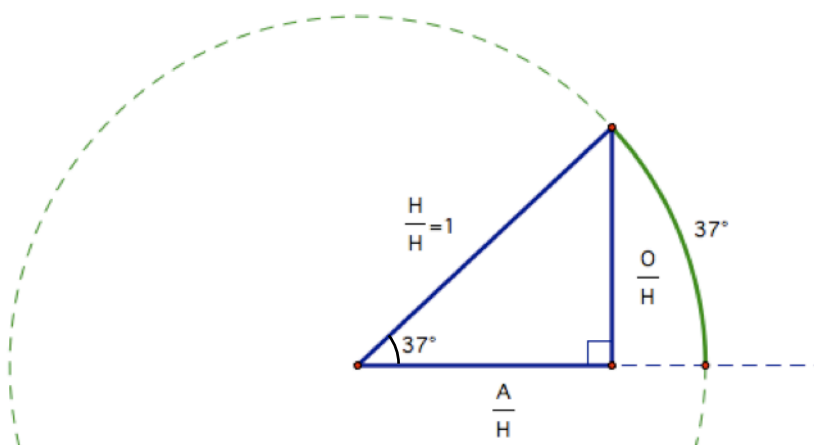


Figure 4. An arc length image of angle measure and the contexts of trigonometry.

Note that the images of the sine and cosine functions described above are intended to differ from the definition of the outputs of the sine and cosine functions as only a coordinate within the unit circle. For example, consider the definition of cosine (not the cosine *function*) as a coordinate given in Axler's precalculus textbook.

The cosine of an angle  $\theta$ , denoted  $\cos\theta$ , is defined to be the first coordinate of the endpoint of the radius of the unit circle that makes an angle of  $\theta$  with the positive horizontal axis. (2009, p. 384)

First, from my interpretation of this definition,  $\theta$  is used to reference the angle (e.g., angle  $\theta$ ) and the measure of an angle (e.g.,  $\cos\theta$ ). Although subtle, this conflation of the use of the symbol  $\theta$  can detract from the symbol representing a *value* (with a unit of measure) to a student. Second, the cosine function is not explicitly defined as a function or a process between two quantities' values in this definition. Rather it is defined as the first coordinate of the endpoint of the radius. Furthermore, this definition does not explicitly identify the length being measured or a unit of measurement. This could have the result

of a student conceiving of the cosine function as merely a number on the unit circle, rather than a value.

The use of the output of the cosine function to represent a value with different meanings becomes especially apparent in the introduction of the cosine function within right triangles by the Axler textbook (2009). This introduction uses the similarity of triangles to conclude that  $\frac{\cos(\theta)}{1} = \frac{a}{c}$ , where  $a$  and  $c$  represent the lengths of the adjacent side and hypotenuse, respectively, of a right triangle with an angle of measure  $\theta$ . In this use of the cosine function,  $\cos(\theta)$  first represents the length of a side and is not described as a process. However, in the simplification to  $\cos(\theta) = a / c$ ,  $\cos(\theta)$  then becomes defined as a ratio. This occurrence of the output of the cosine function representing two different values can be avoided if the cosine function is approached as a function with an output that is a ratio of lengths, as described in the previous conceptual analysis. In such a case, when the cosine function is used in the context of a right triangle, a specific pair of input-output values is being represented.

This study sought to put this conceptual analysis into action by constructing an instructional sequence based on the system of ideas presented in this section. The following section highlights various activities of the instructional sequence by presenting the tasks and the purpose of each task.

### **Instructional Sequence**



The instructional sequence<sup>14</sup> for this study began with activities intended to necessitate that the students reconceptualize angle measure (in both radians and degrees). As revealed in the exploratory study, the students entering the study were expected to have weak understandings of angle measure. The Protractor Problem (Table 2) was first used to engage the students in constructing a meaning for a unit of angle measure through the process of creating a tool to measure an angle.

Table 2
<i>The Protractor Problem (Task 1)</i>
Using the supplies of a Wikki Stix and a ruler, construct a protractor that measures an angle in a number of gips, where 8 gips rotate a circle.

The intentions of The Protractor Problem were to have the students realize the need to partition the circumference of the protractor into equal arc lengths. More specifically, the task intended to promote the students conceiving of a subtended arc length as a measurable attribute related to the measure of an angle. Additionally, each student was given a different sized protractor. This was chosen to promote the students conceiving of a unit of angle measure corresponding to a constant fraction of a circle's circumference, regardless of the size of the circle. It was expected that the students would use an arc length per unit of angle measure approach, with each student determining a different arc length. Thus, by the end of this task (which consisted of creating protractors for multiple units of angle measure), the students were expected to conceive of a unit of

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<sup>14</sup> The entire instructional sequence, including each full problem statement, can be found in Appendix C.

angle measure in terms of a process of measuring the multiplicative relationship between a subtended arc length and the circumference of a circle, where this comparison is constant for a circle of any size.

Following the Protractor Problem, the students were given The Angle Measurement Problem. This activity presented them with an angle that they were to measure using only a compass, Wikki Stix, and a ruler. The intent of this activity was to have the students leverage what they discovered during the The Protractor Problem (e.g., the openness of an angle is measured by determining what fraction of the circle's circumference is subtended by the angle) while continuing to reason about the process of measuring an angle in a way that necessitated the use of a circle. Also, this problem continued to promote the students reasoning about the relationship between a linear measure of an arc length and the corresponding degree measure. The exploratory study revealed that this reasoning ability was critical to understanding angle measure, and particularly reasoning about angle measure in terms of the circumference of any circle centered at the vertex of the angle. The researcher also expected each student to use a different sized circle to measure the angle, which created an opportunity to discuss the implications of this choice.

Table 3

*The Circumference Problem*

Construct a circle using a Wikki Stix as the radius (your group should have Wikki Stix of different lengths). Then, determine how many of your Wikki Stix mark off the circumference of your circle. Compare your result with your classmates. What observations can you make from this comparison? Construct an angle that cuts off one Wikki Stix length of an arc. Compare the openness of the angle with those of your classmates.

The Circumference Problem (Table 3) introduced the radian as a unit of angle measure. By giving the students multiple lengths of Wikki Stix, they were able to conclude that  $2\pi$ , or approximately 6.2-6.5 radius lengths compose the circumference of any circle. That is, the circumference of a circle is always  $2\pi$  times as large as the radius of that circle, or the radius is  $1/(2\pi)$  times as large as the circumference of the corresponding circle ( $C = 2\pi r$  or  $\frac{C}{r} = 2\pi$ ). The characteristic of an angle measure referencing a fixed fraction, or percentage, of a circle's circumference is consistent with how angle measure was defined during the previous activities (e.g., The Protractor Problem); regardless of the radius of the circle used to measure the angle, the same fraction of  $2\pi$  radians will be subtended by an angle with a fixed openness. Also, the same number of radius lengths will rotate along the subtended arc length.

In light of the exploratory study's finding that the students predominantly reasoned about radian measurements as a fraction of a circle's circumference, the

researcher planned to extend The Circumference Problem in order to focus the students on measuring along an arc in a number of radius lengths. In order to accomplish this goal, the problem asked the students to construct angles that measured 2 radians, 3.5 radians, and 7 radians by using their Wikki Stix to measure along the circumference in the appropriate number of radius lengths.

Table 4
<i>The Fan Problem (Sine)</i>
Imagine a bug sitting on the end of a blade of a fan as the blade revolves in a counterclockwise direction. The bug is exactly 3.1 feet from the center of the fan and is at the 3:00 position as the blade begins to turn. Create a graph that shows how the bug's vertical distance above the 9:00 to 3:00 diameter line varies with the total distance the bug travels around the circumference.

The Fan Problem (Table 4) was designed to offer the students an opportunity to reason about a varying angle measure, or arc length, in order to construct the sine function (and the cosine function). The students were prompted to create a graph by covarying (by reasoning about amounts of change and rate of change) the vertical distance of the bug above the horizontal diameter of the fan with the distance traveled by the bug around the circle swept out by its motion. The researcher then formalized this relationship as the sine function,  $f(\theta) = \sin(\theta)$ . Additionally, the problem was stated such that the students were allowed to choose their units of measure for the output quantity. This instructional decision was intended to enable a discussion on the implications of the chosen output unit for fans of differing radius lengths, as well as how

to convert between an output measured in radius lengths to an output measured in the unit presented by the problem (e.g., feet).

The Fan Applet (Figure 5) was also designed to use throughout The Fan Problem. The exploratory study revealed that the students' conceptions of the problems' contexts were highly complex and continually changing structures. Thus, The Fan Applet intended to offer a dynamic diagram of the situation that the students could use throughout the discussion of The Fan Problem. More specifically, as the students described quantities (varying and constant) and relationships between quantities, the researcher prompted them to use the diagram to identify these quantities and how the quantities covary.

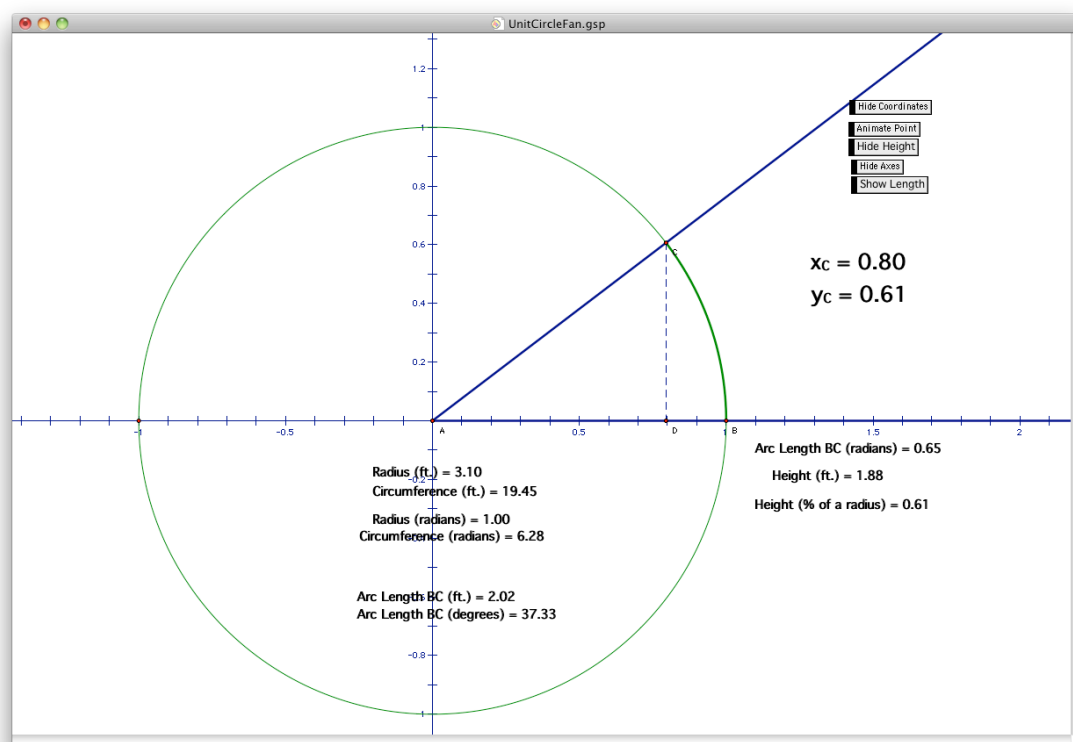


Figure 5. The fan applet.

Table 5

*The Positions on a Circle Problem*

A certain arctic village maintains a circular cross-country ski trail for the enjoyment of its citizens during the winter months. Their trail has a radius of 2 kilometers. A certain skier started at position (2,0) one morning, skiing counterclockwise for 2.2 kilometers, where he paused for a brief rest. Determine the ordered pair that identifies the location where the skier rested.

After The Fan Problem, The Positions on a Circle Problem (Table 5) was designed to further explore the relationship of the sine and cosine functions to positions on circles of various sizes. By offering circles of various sizes, the students had to reason about measuring the relevant quantities in a number of radius lengths to solve the problem correctly. Also, The Fan Problem consisted of mostly indeterminate values. The Positions on a Circle Problem created opportunities for the students to evaluate the sine and cosine functions while continuing to reason about the relationship between two quantities' values. The problem also asked students to generalize the relationship between a position on a circle, an angle swept out from the standard position, and the radius of a circle.

Table 6

*The Finding an Arc Length Problem*

A skier skied on a circular route, starting at the point  $(1,0)$  on the circle, and ending at the point  $(0.951, 0.309)$  on the circle. How many km did she ski?

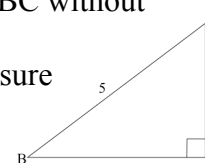
A skier skied on a circular route, starting at the point  $(2.5,0)$  on the circle, and ending at the point  $(2.3775, 0.7725)$  on the circle. How many km did she ski?

Prior to exploring trigonometric functions in a right triangle context, The Finding an Arc Length Problem (Table 6) prompted the students to determine an unknown arc length when given coordinates on various sized circles. This offered the students an opportunity to continue reasoning about measuring quantities relative to the radius, while using the sine and cosine functions to relate quantities' values. Furthermore, this problem presented a situation that necessitated the inverse sine and cosine functions, which also enabled the researcher to see if the students had conceptualized the sine and cosine functions as reversible processes. Lastly, the problem presented coordinates in multiple quadrants in order to raise issues of the domain and range of both the sine and cosine functions, as well as their inverse functions.

Table 7

*The Determining an Output Problem*

Determine the output of the sine and cosine of the measure of angle ABC without measuring the angle. Hint: think of how you would determine the measure of the angle of interest and how the sine function relates to this measurement.



In order to enable coherence between the previous problems and a right triangle context, The Determining and Output Problem (Table 7) created an opportunity for the students to leverage an arc length image of angle measure and the sine and cosine functions as processes relating the values of two quantities. By using the hypotenuse of the right triangle to create the circle (with a radius length equivalent to the hypotenuse), and a subtended arc length, the students could conceive of measuring each leg of the triangle relative to the hypotenuse in order to apply the sine and cosine functions to the right triangle context. After the students used such reasoning for multiple right triangles, they were then asked to generalize their reasoning for generalized values of the sides of a right triangle.

Table 8
<i>The Airplane Problem</i>
A plane leaves the local air force base and travels due east. A radar station 45 miles south of the base tracks the plane and determines that the angle formed by the base, the radar station, and the plane is initially changing by 1.6 degrees per minute. Determine the distance the plane is from the radar station after a number of minutes, $m$ .

To conclude the activities, The Airplane Problem (Table 8) featured a situation that the students could conceive of a right triangle with sides of varying lengths and angles of varying measures. This problem offered a context for the students to apply the outcome of the previous exploration in a case that a right triangle did not have sides of



fixed lengths. Also, the problem enabled the researcher to gain insights into the students' actions when performing symbolic manipulations.

### **Summary of Chapter**

This chapter first presented an overview of the results of an exploratory study into students' conceptions of angle measure. This exploratory study revealed the students holding very limited and fragment understandings of angle measure as they entered the study. However, as the students progressed through the study, they constructed understandings of angle measure that were dominated by reasoning about a subtended arc length's percentage of a circle's circumference. Furthermore, the students were able to leverage their ability to reason about a varying arc length to construct quantitative relationships in the context of circular motion.

The chapter then provided a conceptual analysis of trigonometry and angle measure. The conceptual analysis described a system of ideas that promoted coherence between the contexts of trigonometry and angle measure. Then, to conclude the chapter, a sequence of instructional activities was outlined relative to the intended student learning during each task.

## Chapter 5

### Results Of Zac

This chapter presents an account of Zac's reasoning and problem solving behaviors revealed over the course of the study. This narrative first provides his PCA scores to illustrate his pre- and post-course shift and to situate Zac within the context of the other students enrolled in the precalculus course. This is followed by a characterization of Zac's initial conception of angle measure as assessed in the pre-interview. Following this characterization, Zac's actions during the teaching experiment tasks<sup>15</sup> are presented. The results of the interview sessions are also presented to further illustrate the understandings Zac constructed during the teaching experiment sessions, and to identify the role various reasoning abilities (e.g., quantitative and covariational reasoning) played in his learning. This chapter concludes with a summary and discussion of Zac's progression over the course of the study.

Zac was a full-time student in his early twenties at a large public university in the southwest United States. He was a Bachelor of Arts major with a focus in Ethnomusicology and Audio Technology. Zac completed Calculus I at a different university during the summer previous to the precalculus course. Zac did not intend to take a subsequent mathematics course after completing the precalculus course.

#### **Pre- and Post-Course Assessment**

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<sup>15</sup> The problems displayed in this chapter are shortened versions of the implemented tasks. For the full problems (e.g., diagrams included) that were implemented with the subjects, see Appendix C.

Zac received a ‘B’ for his final course grade. In total, two students from the course received an ‘A’, seven students received a ‘B’, eight students received a ‘C,’ and three students received a failing grade. Zac performed above average relative to the 16 students that completed both the pre- and post-administrations of the PCA exam (Table 9).

Table 9

*Results of the PCA Pre- and Post-test (n=16)*

	Zac	Amy	Judy	Class Average
Pre-test Score	13/25	5/25	15/25	7.31/25
Post-test Score	17/25	10/25	21/25	12.18/25

### **Zac’s Conception of Angle Measure Prior to Instruction**

Prior to the first teaching experiment session, the researcher conducted an interview<sup>16</sup> with Zac to gain insights into his initial conceptions of angle measure. Zac’s reasoning revealed during the pre-interview informed the design of the initial instructional activities.

First, Zac described an angle measure of one degree and an angle measure of thirty-four degrees. Zac explained that a measure of one degree “is a certain measure of distance at a certain point.” Zac then constructed a diagram (Figure 6) and elaborated on his description (Excerpt 3).

#### **Excerpt 3**

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<sup>16</sup> All interview tasks given to Zac are presented in their full form in Appendix D.

1      Zac:    Well, we have it defined (*drawing a horizontal segment and a*  
2                *perpendicular segment to this segment*), ok, that's a really shawdy line,  
3                but we'll deal. That, uh, perpendicular lines (*using his hands to show*  
4                *perpendicular lines*) have a degree of ninety degrees right there, ninety  
5                degrees right there (*drawing right angle symbols*). And a straight line is  
6                technically a degree of a hundred and eighty degrees. And a whole  
7                circle, if you do something like that (*drawing a circle*), that's the whole  
8                three sixty degrees. So, we can stick with the ninety degrees, this section  
9                right here (*drawing an arc outside of drawn circle*), has been divided  
10               into ninety different sections. So that, one of them, which this is not  
11               going to be one of them, but it will work (*drawing line extending right to*  
12               *show one degree*). Like say, say that that distance right there (*drawing*  
13               *segment between two rays*) is one degree. And then saying it has thirty-  
14               four degrees is just thirty-four of those. Like that (*drawing ray at*  
15               *approximately thirty four degrees*), which is definitely not thirty four  
16               degrees, but ya.

17      KM:    Ok. So what do you mean by sections?

18      Zac:    Well, it's been divided up into ninety different areas.

Zac first described perpendicular lines as having ninety degrees between the two lines (lines 3-5). He then continued to describe geometric objects as illustrated by his statement that a straight line is one hundred and eighty degrees and a whole circle is three hundred and sixty degrees (lines 5-8). Zac then alluded to dividing a “section” into

smaller sections corresponding to one degree (line 9-14). He also constructed an arc (lines 9-10), but when asked to explain the “section” he was referring to, he indicated an area (line 18). Thus, it appears that Zac’s initial conception of angle measure consisted of a combination of objects (e.g., properties of lines and circle), areas, and distances between two lines.

Zac identified an arc during his description and referenced breaking up sections, but after this description he admitted that he did not know how to construct or break up the various sections. Thus, his conception of angle measure did not include a systematic way to coordinate the various attributes he identified in order to measure an angle. Soon after this interaction, he further admitted that he had never thought about angle measure deeply previous to this question.



*Figure 6. Zac’s initial angle measure diagram.*

After Zac was given the supplies of a compass, Wikki Stix, and a ruler to measure an angle displayed on the interview protocol, he was unable to measure the angle. Zac also described a protractor, saying, “they have...a [protractor] that’s already designed out, shows you where all the angles are.” Zac’s suggestion of a protractor showing *where*

“the angles are” is consistent with an individual focusing on objects and positions of lines without imagining partitioning a subtended arc length.

Table 10

*The Traversed Arc Problem*

An individual is riding a Ferris wheel that has a radius of 51 feet. On part of a trip around the Ferris wheel, the individual covers an arc-length of 32 feet. How many degrees did the individual rotate?

To conclude the pre-interview, Zac completed The Traversed Arc Problem (Table 10). Zac oriented to the problem by drawing a circle, labeling the given measurement, and calculating the circumference of the circle. Zac then calculated one fourth of the circumference (80.1 feet) and constructed the equation  $\frac{90}{80.1} = \frac{x}{32}$ . Zac subsequently solved for an angle measure of 35.95 degrees and explained his solution (Excerpt 4).

Excerpt 4

- 
- |   |      |   |
|---|------|---|
| 1 | Zac: | Well it's just, if you're given three variables and you just need one more. |
| 2 |      | Well, you, uh, 'cause you're given degrees and feet and degrees and feet.   |
| 3 | KM:  | Ok.   |
| 4 | Zac: | And it just, it gives you three of the four you need. It's a very easy      |
| 5 |      | equation to find a fourth.  |
| 6 | KM:  | Ok, 'cause you're given three of the four, then you know to set up a        |
| 7 |      | proportion?   |
| 8 | Zac: | Ya.   |
-

- 9 KM: Ok, and so, and how do you know how, which way to set up the  
10 proportion?
- 11 Zac: Well you could do it either way. I could do eighty point one over ninety  
12 is thirty two, as long as the top's are both the same unit, their both  
13 degrees, and these are both feet (*writing units by the measurements*).
- 14 KM: Ok, as long as they're both the same.
- 15 Zac: Ya. Well this actually is in a degree, which is why it works if you think  
16 scientific notation wise.
- 17 KM: Ok.
- 18 Zac: 'Cause then these feet cancel each other out so you are just left with a  
19 unit, then the degrees is what's left over.

Zac justified his equation by stating that he had three given variables with one unknown variable (lines 1-2). Zac also alluded to having the appropriate units to construct his equation (lines 12-13). Zac's explanation focused on the *type* of the problem (e.g., three values known, a fourth unknown), as opposed to suggesting that each ratio represented a relationship between an arc length and angle measure. Following this interaction, he was unable to describe a meaning of the ratios as the researcher questioned his solution.

Zac returned to the previous problem and claimed he could solve the problem in the same manner. As opposed to constructing a full circle centered at the vertex of the angle, Zac drew a perpendicular ray to the horizontal ray extending from the vertex of the

given angle. He subsequently constructed a quarter of a circle, calculated the circumference of the entire circle, and determined a quarter of the circumference.

Zac then provided a solution identical to Excerpt 4, which consisted of a focus on matching units between the two numerators and two denominators. The researcher then asked Zac to explain the meaning of one of the ratios. After Zac calculated the value of a ratio, Zac multiplied the third given number by this result. Zac was then unable to provide an explanation beyond the numerical result of the ratio (e.g., not a *value*) and the calculation. The researcher also asked Zac why he chose to use ninety degrees opposed to three hundred and sixty degrees to solve the problem. Zac responded that a “smaller piece” and “small numbers” made the problem “easier.”

### **Summary of Pre-interview**

Zac’s actions during the pre-interview reveal that his conception of angle measure consisted of a loose coordination of objects (e.g., lines and circles) and attributes of these objects (e.g., distances between two lines, areas, and the orientation of two lines). But, his responses did not reveal a process for measuring an angle that consisted of coordinating measurable attributes. Rather, the measurements were pre-defined properties for the various objects. Also, when referencing a protractor, Zac described the protractor as providing the location of angles. These actions imply that Zac had not conceived of a protractor, or angle measure, as identifying the fraction of a circle’s circumference cut off by the angle.

While Zac did not give descriptions that stemmed from a *process* of measuring an angle, he was able to determine an angle measure corresponding to an amount of rotation



(Excerpt 4). Zac's justifications for his solution had a calculational focus independent of the quantities of the problem. Zac claimed that his equation could be used to solve problems consisting of three known values and an unknown fourth value. The researcher prompted Zac to describe one of the ratios composing his equation, which resulted in Zac immediately calculating the ratio rather than providing a quantitative meaning for the ratio. Lastly, as Zac explained an alternative approach to a problem, he focused on the aesthetics of the numbers opposed to the quantities of the problem situation.

### **Teaching Experiment Sessions One Through Four**

The first four teaching experiment sessions consisted of Activities 1-7 (Appendix C). This section outlines the instructional tasks that were implemented during these sessions in the context of the constructions and connections made by Zac. Analysis of Zac's engagement in the instructional activities resulted in a preliminary model of his thinking, which informed the design of his interview tasks. The first two teaching experiment sessions resulted in Zac reconceptualizing angle measure, while the second two classroom sessions led to Zac constructing the sine and cosine functions in the context of circular motion.

#### **Teaching Experiment Session One**

Zac's reasoning during the pre-interview consisted of various geometric objects when providing descriptions of angle measure. Zac's responses did not include coordinating measurable attributes of these objects, or a process of measuring an angle. Thus, the first teaching experiment session focused on identifying the object of an angle (two rays with a common endpoint), the measurable attribute of the angle (the openness),

and a process of measuring the openness of an angle (quantifying the fraction of a circle's circumference subtended by the angle).

***The cannon problem.***

Table 11
<i>The Cannon Problem</i>
Two military historian groups decided to test the range of two different World War I cannons that have different barrel lengths. During their testing both groups noted that the horizontal distance traveled by the projectiles fired from their cannons changed as they tilted their cannon barrels up and down. The two groups wanted to compare the horizontal distances traveled by the projectiles by the two cannons without transporting one cannon to the other. Describe the quantities that one group could measure to convey to the other group how to set up their cannon identical to the other group.

During The Cannon Problem (Table 11), Zac first discussed, “as the angle of the cannon changed, the distance the projectile traveled changed.” Zac subsequently identified a need to measure this angle. He further described the measurable attribute of an angle as, “the curvature between the two lines...I don’t really know how to explain it. I never had to think much about an angle before.” Zac’s difficulty explaining angle measure was consistent with his actions during the pre-interview, and prompted the researcher to use The Protractor Problem (Table 12) to investigate angle measure.

***The protractor problem.***

Table 12

*The Protractor Problem (Task 1)*

Using the supplies of a Wikki Stix and a ruler, construct a protractor that measures an angle in a number of gips, where 8 gips rotate a circle.

The Protractor Problem was intended to support the subjects constructing a *process* of measuring an angle that is reliant on partitioning the circumference of the circle into equal segments of an arc length. Zac first determined a measure of two gips (Excerpt 5).

## Excerpt 5

- 
- |    |      |   |
|----|------|---|
| 1  | Zac: | Well, I already figured out what two gips is, by just dividing it in half.            |
| 2  | KM:  | So dividing what in half?   |
| 3  | Zac: | The protractor. I just drew a line down the middle ( <i>waving hand over</i>          |
| 4  |      | <i>protractor</i> ) and that gives me two gips. And then I just need to figure out    |
| 5  |      | how to find...  |
| 6  | KM:  | So how'd you know how to draw the line?   |
| 7  | Zac: | Uh, I figure out that the diameter is four inches, and just found out                 |
| 8  |      | where two inches is, marked it ( <i>referring to the midpoint of the</i>              |
| 9  |      | <i>diameter</i> ), and found my best two inches this way ( <i>waving pen tip from</i> |
| 10 |      | <i>the bottom to the top of the protractor</i> ) and drew it up.                      |
| 11 | KM:  | OK, so you kind of had to eye-ball that a little bit though, right?                   |
| 12 | Zac: | Ya, but it's a right angle ( <i>attempting to use the protractor on the end of</i>    |
-

13                    *the ruler to verify the right angle).*

14      KM:      No using the protractor.

During this interaction, Zac's method focused on the object, or area, of the protractor to identify a measure of two gips. He drew a line "down the middle" to divide "it in half" (lines 1 & 3-5), and then related this measurement to the geometric object of a right angle (lines 12-13). Zac's mark for the measurement was also based on a visual estimation. The researcher predicted that Zac would encounter difficulties as he attempted to identify other measurements and thus he encouraged Zac to continue while hoping these difficulties would result in Zac reflecting on his solution.

When Zac determined a mark of one gip, he initially attempted to construct two equal "pieces" of the protractor. Zac could not accomplish this goal beyond a visual estimation. After allowing Zac to persist with no success, the researcher asked Zac for a method to verify his original mark of two gips (Excerpt 6).

#### Excerpt 6

1	KM:      Now, that line you drew ( <i>referencing Zac's two gips line</i> ) here...how
2	can we tell if that's in its right location?
3	Zac:      Ya, you get the, measure out the whole thing.
4	KM:      So what do you mean measure out the whole...
5	Zac:      Measure out the perimeter.

Zac's reflection led to him identifying the circumference of the protractor as a measurable attribute that could be used to identify an angle measure (lines 3-5). Zac then

calculated the circumference of the protractor (e.g., half of a circle) and determined half of this amount. At this moment in the teaching experiment session, Zac's actions made a distinct shift to reasoning about angle measure relative to various arc lengths and a circle's circumference.

In order to draw on Zac's reasoning, the researcher drew a diagram of a protractor on the whiteboard and labeled the radius of the protractor as 3.9 centimeters. In order to construct the protractor such that 15 units (quips) rotated through a circle's circumference, Zac suggested the calculation of  $12.2522 / 7.5$  (e.g., half of the circumference divided by half of the total units) and explained the result as the number of centimeters of arc length per quip for that specific circle. When presented with a protractor with a radius of 4.5 centimeters, Zac explained that the construction method remained unchanged, but the numerical values composing the method changed (e.g., 1.885 cm/quip). Zac's explanation implies that his reasoning centered on the quantities of the situation, opposed to the calculations composing his method.

Zac's approach to the problem revealed him coordinating an arc length per unit of angle measure, but Zac was yet to describe a subtended arc length as a *fraction* of the circle's circumference. In order to leverage Zac's approach to constructing a protractor, the researcher posed the ratios of the linear arc length for one unit of angle measure to the total circumference of the corresponding circle (e.g.,  $1.63/24.5044$  and  $1.885/28.2744$ ). Additionally, the subjects were asked to interpret the ratio of  $1/15$  (e.g., one quip relative to fifteen quips).

After the subjects calculated each ratio (approximately 0.067), Zac interpreted this value (Excerpt 7).

Excerpt 7

- 
- |    |      |   |
|----|------|---|
| 1  | KM:  | [Talk about] what each of these operations represented and why those          |
| 2  |      | numbers came out to be the same. So talk to each other about why you          |
| 3  |      | think that might be true.   |
| 4  | Zac: | 'Cause it's just taking the full circumference and then a fifteenth of a full |
| 5  |      | circumference.  |
| 6  | KM:  | So say a little more, so what's going on?                                     |
| 7  | Amy: | You're taking one measurement of one of the fifteen quips...                  |
| 8  | Zac: | Dividing it by the full fifteen quips.  |
| 9  | KM:  | So taking the one and dividing it by the full fifteen quips. Well that's      |
| 10 |      | just the left case, what about the other two [ratios]?                        |
| 11 | Zac: | It's the exact same. You're taking one-fifteenth of the full circumference    |
| 12 |      | and dividing it by the full circumference.                                    |
- 

During this interaction, Zac reasoned that each numerator represented one-fifteenth of the entire circumference (the denominator). These actions imply that Zac conceived of each ratio as a value representing an arc length's fraction of a circle's circumference.

Following this interaction, Zac described 4.1 quips as, "four point one fifteenths [of a circle's circumference]." Zac added that this fractional amount held for any circle centered at the vertex of the angle, which implied that he constructed an understanding of

angle measure corresponding to the angle subtending a constant fraction of any circle's circumference that is centered at the vertex of the angle. Zac then concluded the protractor problem by describing an angle measure of one degree as, “one three-sixtieth of a circle's circumference.”

***The protractor applet.*** The Protractor Applet (Figure 7) was designed to support the subjects continuing to reason about angle measure as the fractional amount of a circle's circumference subtended by the angle. Also, as the terminal ray of an angle is moved or the radius of the circle is changed on the applet, the displayed values change accordingly, which allowed a dynamic investigation of angle measure.

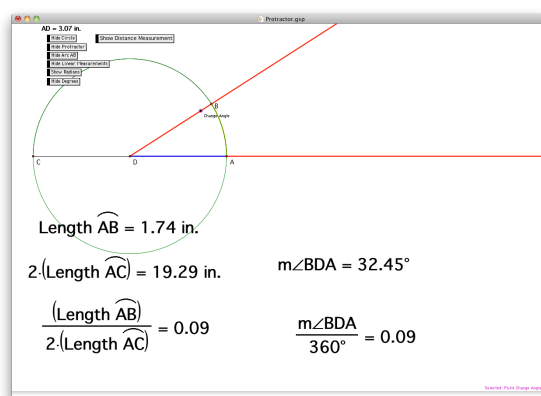


Figure 7. – The protractor applet.

Zac first explained that the *value* of the two displayed ratios represented the “percent of the full circle's circumference” cut off by the angle. Then, Zac claimed that the “along the arc” distance would increase while the radius of a circle was increased; he also explained that ratios remaining constant reflected that the openness of the angle remained constant. These actions reveal Zac continuing to reason about angle measure as

the fraction of a circle's circumference subtended by the angle's rays, with this fractional amount remaining constant for a circle of any radius.

In summary, through his actions of constructing protractors in various units, Zac reasoned about an arc length corresponding to one unit of angle measure (e.g., an arc length per unit image). Zac then reflected on this process and the quantitative meaning of various ratios to construct an understanding of angle measure as a fraction of any circle's circumference cut off by the angle. In each case, Zac reasoned about angle measure corresponding to a subtended arc, where he conceived of measuring "along" this arc.

### **Teaching Experiment Session Two**

The second teaching experiment session returned to The Protractor Applet in order to revisit the outcomes of the previous day. Following this activity, Zac engaged in The Angle Measurement Problem (Table 13), which continued promoting connections between angle measure and the *construction* of a circle. The Circumference Problem (Table 14) then promoted Zac constructing the radian as a unit of angle measure.

***The protractor applet.*** Similar to the previous session, Zac described the two displayed ratios on the applet, explaining, "It's the degrees from point a to point b along the circumference over the full three sixty degrees which gives you the same percentage as the length of, the arc length of point a to point b over the full circumference." This explanation reveals Zac reasoning about measuring *along* an arc, while also reasoning about the arc as a percentage of the circle's circumference.

***The angle measurement problem.***



Table 13
<i>The Angle Measurement Problem</i>
Using the supplies of a Wikki Stix and a ruler, measure the following angles in degrees with only the available supplies. Be prepared to explain your reasoning.

The Angle Measurement Problem offered another opportunity for Zac to reason about the *process* of measuring an angle. Recall that Zac was initially unable to complete this task during the pre-interview, and later reached a solution using a procedural method from a previous problem. During the second teaching experiment session, Zac constructed and measured an arc between the two rays of the angle. Zac then explained that the ratio between the arc length cut off by the angle and the circumference of that circle represented the arc length's percentage of the circle's circumference. He added that the angle measure was the same percentage of 360 degrees.

Zac's actions on this problem reveal that his engagement in the previous activities (e.g., The Protractor Problem) resulted in Zac's conception of angle measure consisting of a quantitative relationship between a subtended arc and the circumference of a circle. Thus, in order to determine the multiplicative relationship between these quantities, Zac constructed a circle with a subtended arc length and then determined the needed values.

***The circumference problem.***

Table 14

*The Circumference Problem*

Construct a circle using a Wikki Stix as the radius (your group should have Wikki Stix of different lengths). Then, determine how many of your Wikki Stix mark off the circumference of your circle. Compare your result with your classmates. What observations can you make from this comparison? Construct an angle that cuts off one Wikki Stix length of an arc. Compare the openness of the angle with those of your classmates.

The Circumference Problem marked the beginning of the activities focused on the radian as a unit of angle measure. In order to leverage Zac's understanding of angle measure as measuring along an arc, this task prompted Zac to construct a circle with a Wikki Stix as the radius of the circle. Zac then determined the number of Wikki Stix along the circumference of his circle, while comparing this to other subjects' results. This resulted in Zac establishing that approximately 6.28 *radius lengths*<sup>17</sup> (or exactly  $2\pi$  radius lengths) rotated through any circle's circumference.

After Zac conceived of a constant number of radius lengths rotating through any circle's circumference, Zac claimed, "[the radius] simplifies a circle, you know, the circumference of a circle is equal to two pi  $r$ , where the radius is the unit, not inches, or anything like that. So it simplifies it a lot, you know, using it as an actual unit." He further described, "Like, you know, one radius, and then the circumference is six point

---

<sup>17</sup> *Radius lengths* is used opposed to *radii* to emphasize that Zac conceived of the radius as a measurable length that could be used as a unit of measure.

two eight radius.” These explanations reveal Zac reasoning about the radius as a unit of measurement for the circumference of a circle. Zac also described the radius as “one radius,” with  $2\pi$  radius *lengths* rotating through the circle’s circumference (e.g.,  $C = 2\pi r$ ). Thus, Zac’s engagement in The Circumference Problem appears to have resulted in his construction of the unit circle (e.g., a circle with a radius of one unit).

Following this problem, Zac described various radian angle measures in terms of rotating along various arcs. For instance, in order to construct the arc length of seven radii after constructing an arc length of six radii, Zac explained, “I just took my Wikki Stix, which is the length of my radius, and moved one more, which gave me seven radii.” Zac’s ability to reason about a number of radius lengths rotating along the circumference of a circle enabled him to easily consider an arc length that was greater than 6.28 radius lengths. Zac also spontaneously described radii measurements of arcs as “relative to the radius” and as a percentage of the radius.

### **Teaching Experiment Session Three**

In order to leverage Zac’s understanding of radian angle measure (e.g., measuring along an arc in a number of radii), The Fan Problem (Table 15) introduced the sine function in the context of circular motion. This task composed the entirety of the third classroom session, and The Fan Applet (Figure 8) was used to support the subjects’ in making sense of the problem situation and identifying the relevant quantities.

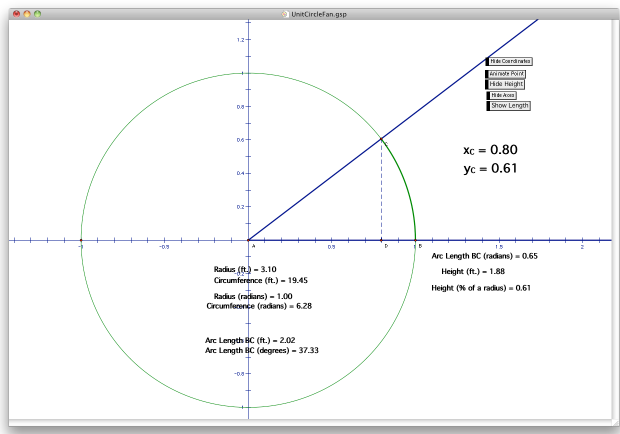


Figure 8. The fan applet.

*The fan problem.*

Table 15

*The Fan Problem (Sine)*

Imagine a bug sitting on the end of a blade of a fan as the blade revolves in a counterclockwise direction. The bug is exactly 3.1 feet from the center of the fan and is at the 3:00 position as the blade begins to turn. Create a graph that shows how the bug’s vertical distance above the 9:00 to 3:00 diameter line varies with the total distance the bug travels around the circumference.

Zac first described that the rotation of the bug formed a circle and that the distance the bug traveled was measurable in a number of radius lengths. Zac then conjectured that the graph would be “like a tangent line...like a sine or cosine,” while mimicking the shape of a sinusoid with the tip of his pen. He also reasoned about the directional variation (MA2) of the vertical distance as the bug traveled around the center of the fan and described that the bug approached a “highest point” and a “lowest point.”

In response to Amy stating that she was measuring the input relative to time, Zac described that he could use any unit of angle measure, further revealing his comfort of relating angle measure to an arc length. All three subjects subsequently generated graphs with an output measured in feet and an input measured in radians. Also, all three graphs perceptually resembled a graph of the sine function.

Zac justified his graph by first describing the directional covariation of the two quantities (MA2). In order to further investigate this covariational relationship, Zac was asked to explain the shape, or curvature, of his graph. Previous to Zac's explanation, the researcher added a graph composed of three linear segments conveying the same directional covariation (Figure 9) in the hopes that Zac would contrast his graph with a linear relationship. Zac immediately identified that a constant rate between the values of the two quantities meant for an equal change of total distance traveled, the vertical distance changes by a constant amount (MA3). Following this description, he claimed that the relationship between the vertical distance and the distance traveled was not linear.

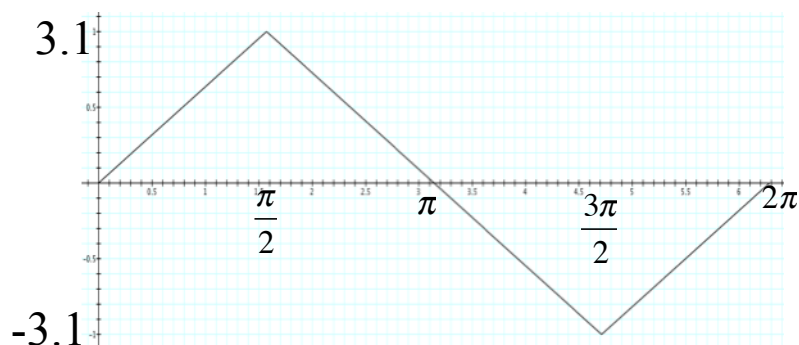


Figure 9. The researcher's proposed graph.

In order to verify that his graph (Figure 10) did not convey a constant rate of change, Zac suggested considering equal changes of distance traveled by the bug. In light of Zac's suggestion, The Fan Applet was used to investigate equal changes of total distance and the corresponding changes in vertical distance for the first fourth of a rotation. This resulted in verifying that the change in vertical distance was decreasing for equal changes of total distance (MA3). After discussing how the graph conveyed this relationship (e.g., concave down), another student (Judy) described the vertical distance as decreasing at an increasing rate for the second quarter of a revolution (MA5). Zac elaborated on Judy's rate of change description and identified that the change in vertical distance was increasing over this quarter of the revolution (MA3). Zac then stated that the vertical distance decreased at a decreasing rate over the third quarter of a revolution and that for equal changes of arc length, "the change in vertical distance is going to get smaller" (MA3 and MA5).

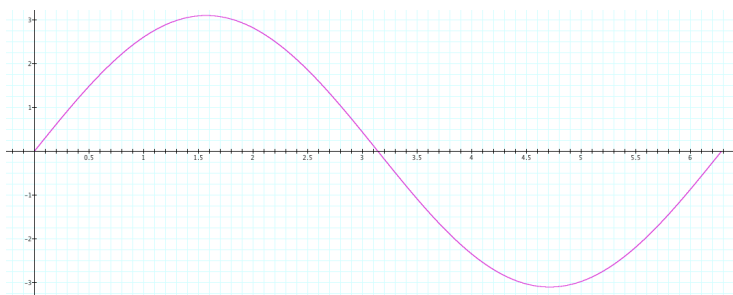


Figure 10. The students' graph.

Zac's actions imply that The Fan Problem offered a situation in which he conceived of both the vertical distance and the *change* in vertical distance as two distinct, measureable, and related quantities. Zac's conception of the situation, possibly aided by The Fan Applet, also enabled him to reason about amounts of change of vertical distance

and how these amounts of change were changing relative to equal changes of arc length (MA3). This reasoning led to Zac constructing a graphical representation rooted in reasoning about the rate of change and amounts of change between two quantities.

In order to formalize Zac's reasoning as the sine function, the researcher prompted the subjects to consider a changing radius relative to the constructed graph. After Judy identified that the graph would change, the researcher presented a graph that was identical in its perceptual features to the subjects' graph except that the maximum and minimum output values were one and negative one, respectively. The subjects were then asked for the output unit of this graph given the requirement that the graph represented the same situation (e.g., a fan of the same radius) as the previous graph.

Zac immediately suggested "the radian" as the output unit and identified the locations where the bug was zero radius lengths or one radius length vertically from the horizontal diameter of the circle. Zac also described that an output measured relative to the radius conveyed the covariational relationship for a circle of any linear radius length. Thus, it appears that Zac's ability to reason about measuring a length relative to the radius enabled him to identify the radius length as a possible unit of measure for a quantity other than an arc length. This resulted in Zac understanding the graph as representing a covariational relationship between an arc length and a vertical distance for a circle of any size.

After the previous interactions, the graph presented by the researcher was formalized as the sine function, or  $f(\theta) = \sin(\theta)$ . Zac then suggested that the graph he produced corresponded to the formula  $g(\theta) = 3.1\sin(\theta)$  and explained that multiplying

the output of the function  $f$  by the radius length in feet resulted in an output in feet. Additionally, Zac provided this description *without* calculating a numerical value of  $\sin(\theta)$ . Hence, Zac's ability to reason about measuring a length as a fraction of the radius while also anticipating the sine function outputting a value measured relative to the radius enabled him to convert an indeterminate output value to another unit of measure.

Zac's engagement in The Fan Problem reveals him constructing a graph rooted in covariational reasoning based on indeterminate values. Zac covaried amounts of change of vertical distance and equal changes of arc length to justify the concavity of his graph. Reasoning about indeterminate values may have promoted his ability to anticipate  $\sin(\theta)$  producing an output representing a fraction of the radius (e.g., a process conception of the sine function). Zac also reasoned that the output value was a constant percentage of the radius regardless of the unit of measure.

#### **Teaching Experiment Session Four**

In an attempt to generate a similar construction of the cosine function, the researcher returned to The Fan Problem (Table 15) during the fourth teaching experiment session. The Positions on a Circle Problem (Table 16) and The Finding an Arc Length Problem (Table 17) were also used during the fourth classroom session to investigate evaluating the sine and cosine functions using numerical values.

***The fan problem.*** Zac's actions during the construction of the cosine function were consistent with those exhibited during the construction of the sine function. When creating the graph of the cosine function, Zac considered equal changes of arc length while comparing changes of the horizontal distance to the right of the vertical diameter



(MA3). This led to Zac constructing a graph rooted in a covariational relationship. The Fan Problem offered Zac a situation in which he could construct quantities and covariational relationships between quantities such that these relationships could be formalized as the sine and cosine functions.

*The positions on a circle problem.*

Table 16
<i>The Positions on a Circle Problem</i>
A certain arctic village maintains a circular cross-country ski trail for the enjoyment of its citizens during the winter months. Their trail has a radius of 2 kilometers. A certain skier started at position (2,0) one morning, skiing counterclockwise for 2.2 kilometers, where he paused for a brief rest. Determine the ordered pair that identifies the location where the skier rested.

Zac oriented to the task by using a diagram to identify the starting position of the skier and the distance traveled by the skier (Figure 11). After determining a solution to the problem, Zac explained, “I found it easier to find the percentage of the radius first,” apparently giving preference to measurements given relative to the radius. Zac then described his solution (Excerpt 8).

Excerpt 8		
1	Zac:	Ok, so first I found out how many radians it took for him to get there
2		( <i>tracing the traversed arc traveled</i> ), which was one point one, ‘cause I
3		took two point two kilometers divided by the radius, two kilometers, and

- 4 got one point one radians (*pointing at the values on the board*  
 5 *throughout his description*). Then I took the cosine of one point one and  
 6 got [point] four five radians, which gave me the horizontal distance  
 7 (*identifying distance on his diagram*). And I took the sine of one point  
 8 one, and uh, got point eight nine radians, which gave me the vertical  
 9 distance (*identifying distance on his diagram*), shown here as a  
 10 percentage of a radius (*pointing to coordinate*), ordered pair in radians,  
 11 percentage of a radius. So I got point four five, point eight nine. And  
 12 then I multiplied those by the two kilometers to get it in kilometers  
 13 (*pointing to coordinate given in kilometers*). I got point nine, one point  
 14 seven eight.
- 15 KM: So why'd you multiply by the two kilometers?
- 16 Zac: Uh, to get the radius. Because it's in a percentage of a radius, you have  
 17 to multiply it by the radius so then I'll get kilometers.

Zac solved the problem by converting the given arc length to a number of radius lengths and finding the corresponding output values of the sine and cosine functions (lines 1-11). Zac's ability to reason about measuring quantities as a fraction of the radius then enabled him to flexibly convert between measurements given relative to the radius and measurements given in kilometers (lines 16-17). Thus, Zac's ability to conceive of measuring quantities relative to the radius enabled him to conceive of the situation such that he could evaluate the sine and cosine functions when given an arc length in a number of kilometers.

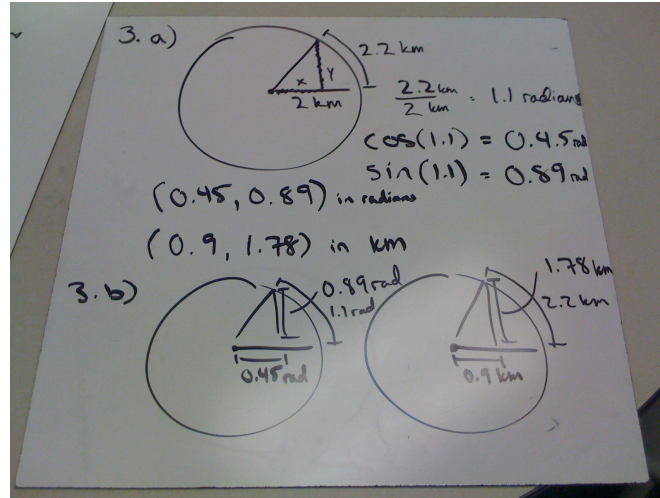


Figure 11. Zac's positions on a circle diagram.

***The finding an arc length problem.***

Table 17

*The Finding an Arc Length Problem*

A skier skied on a circular route, starting at the point  $(1,0)$  on the circle, and ending at the point  $(0.951, 0.309)$  on the circle. How many km did she ski?

A skier skied on a circular route, starting at the point  $(2.5,0)$  on the circle, and ending at the point  $(2.3775, 0.7725)$  on the circle. How many km did she ski?

The first two tasks of The Finding an Arc Length Problem concluded the fourth teaching experiment session. When orienting to the problem, Zac identified, “we are given what we normally find...outputs.” Subsequently, Zac identified the need of an inverse function, which led to the researcher introducing two standard notations for inverse trigonometric functions (e.g.,  $\sin^{-1}(x)$  and  $\arcsin(x)$ ). When presented with the *arcsin* and *arccos* notations, Zac concluded that the functions were named as such because their output represented an angle measure, or *arc* length. As Zac continued

working he remained attentive to the units of measurement for the input and output values of the inverse functions. For instance, on the second task, Zac first converted the given positions to a number of radius lengths in order to use the values as inputs to the inverse functions. This led to Zac correctly completing the task without difficulty.

Zac's actions on this problem imply that his constructed quantitative relationships were not *one-way* processes. That is, Zac reasoned about either quantity as an input or an output of a function. Thus, when given a position on a circle in The Finding an Arc Length Problem, Zac conceived of an arc length (the input to the sine and cosine functions) corresponding to the position. This reasoning enabled Zac to formalize the inverse sine and cosine functions relative to the quantitative relationships he had previously constructed during The Fan Problem. In short, the inverse functions did not require Zac to construct a novel quantitative relationship; Zac had already constructed the quantitative relationship that enabled him to anticipate an arc length corresponding to a vertical or horizontal distance, or vice-versa.

### **Summary of the First Four Teaching Experiment Sessions**

During the pre-interview, Zac referenced various geometric objects (e.g., lines and arcs) corresponding to the measure of an angle, but he did not exhibit reasoning about a process of measuring an angle that consisted of quantitative relationships. Zac's engagement in The Protractor Problem resulted in him engaging in the *process* of measuring an angle. Zac's construction of a protractor and reflection on this process led to him identifying a subtended arc as a measurable attribute. His focus on the subtended arc resulted in him reasoning about measuring *along* the arc and identifying a relationship

between an arc length and a unit of angle measure. Then, by reflecting on various ratios, Zac identified that an angle measure conveys the fractional part of a circle's circumference cut off by the angle. For instance, Zac explained that one degree of angle measure corresponded to "one three-sixtieth of a circle's circumference."

Zac's engagement in The Circumference Problem resulted in Zac constructing an image of measuring along an arc in a number of radius length. As a result of constructing a circle and measuring the circumference of the circle using the radius of the circle, Zac identified that all circles have a radius of "one radius" and a circumference of "six point two eight radius." In other words, Zac's engagement in The Circumference Problem resulted in him *constructing* the unit circle (e.g., a circle of a radius of one).

The Fan Problem then offered a context that Zac could leverage measuring along an arc in a number of radius lengths to construct (within the context of circular motion) the covariational relationships formalized by the sine and cosine functions. Relative to the sine function, Zac reasoned about equal changes of arc length while comparing the corresponding changes of vertical distance. This resulted in both a graph and a formula emerging from his conception of quantitative relationships composed of indeterminate values.

Zac's reasoning about indeterminate values during The Fan Problem parallels a process conception of these functions. For instance, Zac explained multiplying "the output...the percentage of the radius" by the radius "as long as the radius is in feet" in order to convert an output to a number of feet. Zac also altered the formula  $f(\theta) = \sin(\theta)$  without performing calculations on a numerical output. Rather, Zac reasoned that the sine

function had an output relative to the radius, and that multiplying by the radius length in feet resulted in the representation of  $g(\theta) = 3.1\sin(\theta)$ . Zac's reference to the output of the function occurred without calculating or performing an action on a numerical output, which implies he conceptualized the sine function as self-evaluating. Also, Zac's ability to reason about an indeterminate value and the quantitative relationship implied by measuring a quantity relative to the radius enabled him to anticipate converting between two measurement units.

### **Exploratory Teaching Interview One**

After the first four teaching experiment sessions, an exploratory interview (Appendix D) was conducted with Zac in order to further pursue Zac's reasoning revealed in the group setting. Zac's thinking (described above) informed the design of the exploratory interview. As an example, on The Finding an Arc Length Problem, Zac's actions imply that he constructed a quantitative relationship between two quantities that included the ability to reason that for any value of one quantity there was simultaneously a value of the other quantity. Thus, The Ski Trail Problem – Version I (Table 21) was designed such that it consisted of unknown values in both quantities.

The first exploratory teaching interview consisted of problems that focused on both angle measure and trigonometric functions. An analysis of Zac's reasoning relative to problems focused solely on angle measure is first presented. This will be followed by a discussion of Zac's actions on problems involving trigonometric functions.

#### **Angle Measure Interview Observations**

##### ***The arc length problem.***

Table 18

*The Arc Length Problem*

Given that the following angle measurement  $\theta$  is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches.

After reading The Arc Length Problem (Table 18), Zac explained his solution *previous* to executing the solution (Excerpt 9).

## Excerpt 9

---

1	Zac:	Um, ok. So what I plan on doing for this one is converting thirty-five
2		degrees into radians. And a very easy way of doing that is putting thirty-
3		five over three sixty is equal to $x$ over two pi ( <i>writing corresponding</i>
4		<i>equation</i> ). 'Cause three sixty degrees covers the whole circumference of
5		a circle ( <i>mimicking the shape of the circle with the pen tip</i> ) and two pi
6		radians covers the whole circumference of a circle ( <i>mimicking the shape</i>
7		<i>of the circle with the pen tip</i> ). So those two should be equal ( <i>swiping pen</i>
8		<i>across the equality</i> ) and I can just find $x$ . And then with that all I have do
9		is just multiply the answer ( <i>pointing to <math>x</math></i> ) by two inches, two point four
10		inches, and two point nine inches ( <i>pointing to each value in the problem</i>
11		<i>statement</i> ) to get the different arc lengths ( <i>identifying each arc length</i>
12		<i>with his pen tip</i> ) right there, because radians is just a percentage of a
13		radius.

---

Zac first identified a goal of converting the angle measure from degrees to radians, likely giving preference to an angle measure in radians (lines 1-2). After constructing an equation to complete this conversion, Zac explained that both three hundred and sixty degrees and two pi radians “covers the whole circumference of a circle,” while making a motion that mimicked a circle (lines 4-6). Consistent with his actions during the teaching experiment sessions, Zac then described a radian measure conveying a percentage of the radius, while leveraging this understanding to *anticipate* converting the radian measure to the linear measure of each arc length (lines 8-13). These actions further illustrate that Zac’s understanding of a measurement in radians consisted of a multiplicative comparison (e.g., a quantitative relationship) between an arc length and the radius length.

Zac’s reasoning driving his conversion of a degree measure to a radian measure was unclear at this point in the interview. During the pre-interview, Zac described a similar method by referencing that both the numerator and denominator of the ratios needed to have matching units (Excerpt 4). This approach did not require Zac to conceive of each ratio as representing a multiplicative comparison between two quantities’ values. In order to gain insights into Zac’s reasoning, he was asked to further explain his conversion (Excerpt 10).

#### Excerpt 10

- 
- |   |      |  |
|---|------|--|
| 1 | KM:  | So could you explain a little bit more, why that equation works there? |
| 2 |      | How you knew to set that up, why that works?                           |
| 3 | Zac: | Well what you're doing is just technically finding a percentage. Like  |
-



- 4                   thirty-five over three sixty is (*using calculator*), is poi-, or nine point  
 5                   seven percent of the full circumference. 'Cause three hundred sixty  
 6                   degrees (*mimicking the shape of the circle with the pen tip*) in a circle.
- 7       KM:     Ok.
- 8       Zac:     So thirty-five of those degrees equals nine point seven percent of the full  
 9                   thing.
- 

Contrary to his explanation in the pre-interview (Excerpt 4), Zac described a ratio as a percentage of a circle's circumference (lines 3-5). This interaction reveals that the reasoning behind his angle conversion was based on his conception of angle measure as conveying the fraction of a circle's circumference subtended by the angle.

Zac was then asked to explain an angle measure of 0.61 radians (Excerpt 11).

#### Excerpt 11

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- 1       KM:     What does [0.61 radians] mean as an angle measurement?
- 2       Zac:     That means that this arc length right here (*tracing the arc length on the*  
 3                   *smallest circle*) is point six one, or sixty one percent of the radius.
- 4       KM:     Of the radius?
- 5       Zac:     Ya.
- 6       KM:     Ok. Now (*pause*), and then so why times, why'd you do this step here  
 7                   then? With the point six one (*pointing to 0.61(x)*).
- 8       Zac:     Well because I couldn't just leave it as radians, so I have to get it in  
 9                   inches, or you know, an actual distance measure. So knowing that it's
-

10                    sixty one percent, or it's equal to sixty one percent of the radius, all I  
11                    have to do is just multiply it by the radius and I know what it is.

12    KM:        Ok, cool. Ok, so one last question. So we have all these different  
13                    lengthed arcs right (*tracing the three arcs*)?

14    Zac:        Ya.

15    KM:        But yet that angle measurement you know, has the same, the angle  
16                    measurement doesn't change. So why is that, how is it that happens, you  
17                    know why it...

18    Zac:        Well, because it's always the same percent of the circle it's cutting out  
19                    for each different circle.

20    KM:        Ok.

21    Zac:        As long as, you know, you're starting in the very center of the circle  
22                    (*pointing to the center of the circle*), the origin.

23    KM:        Ok. Good. Ok, so what's the same percent there you said? Something's  
24                    always the same percent.

25    Zac:        The, well, the degrees or radians is showing a chunk of the circle being  
26                    cut out, and that's a certain percent of the circle being cut out. It's always  
27                    the same no matter what, as long as you're using that same degree or  
28                    radian length, then you're always going to have that same amount, or  
29                    same percent of the circle, or circumference being cutout no matter what  
30                    size the radius is, or the circumference of the circle is (*making circular*  
31                    *motion with hand*).

---

Zac first traced along an arc length and described this length as sixty one percent of the radius length (lines 2-3), further revealing his ability to reason about an arc measured relative to the radius. Zac also reasoned about the fraction of a circle's circumference cut off by an angle and identified that relative to each circle, the "same percent of the circle" is cut off (lines 18-19). Zac added that the same percentage of circumference, regardless of the radius length, was cut out for the "same degree or radian length" (lines 28-31). These actions reveal that his conception of angle measure as a fraction of a circle's circumference subtended by the angle formed a foundational understanding of angle measure.

In summary, Zac's actions on this problem revealed him reasoning about angle measure as quantifying the fraction, or percentage, of any circle's circumference cut off by the angle. This quantitative relationship enabled Zac to convert an angle measure. Zac also reasoned about measuring along a subtended arc in a fraction of the radius. This reasoning ability resulted in Zac flexibly converting between a linear unit and a number of radians.

***The radian measurements and pi problem.*** The Radian Measurements and Pi Problem prompted Zac to describe angle measures of  $0.5\pi$  radians and 2.2 radians (Excerpt 12). The researcher also chose this problem to gain additional insights into Zac's conception of  $\pi$  relative to the measure of an angle.

#### Excerpt 12

---

1	Zac:     What does it mean for an angle to have a measure of point five pi
---	--

---

- 2 radians? Or two point two radians. Well an angle to have a measure of  
 3 point five pi radians (*draws circle and horizontal radius extending right*  
 4 *from the center of the circle*), or pi halves radians, means it's half of the  
 5 full circle (*drawing vertical radius extending up from the center of the*  
 6 *circle*). You know, one fourth of a full circle. Uh, for two point two  
 7 radians (*drawing circle and horizontal radius extending right from the*  
 8 *center of the circle*), it means that the angle length, cutout, two point  
 9 two, roughly like right there (*drawing radius extending up and to the left*  
 10 *from the center of the circle, drawing arc and labeling it as 2.2 rad*), is  
 11 that length, or the arc length right here (*tracing arc length*) is equal to  
 12 two point two radius lengths.
- 13 KM: Ok. Now how bout on this one (*referring to the angle measure of  $0.5\pi$* ),  
 14 how long's that arc length (*tracing arc length*)?
- 15 Zac: It is equal to pi halves radius lengths.
- 

Zac's orientation to this problem involved him constructing a circle (Figure 12) and reasoning about the length of an arc to construct an angle (lines 1-6). Relative to the first measurement ( $0.5\pi$  radians), Zac identified a fraction of the circle's circumference and described the measurement as "one fourth of a full circle." Zac then reasoned about 2.2 radius lengths rotating through an arc (lines 6-12). Thus, Zac alternated between reasoning about a fraction of a circle's circumference and a number of radius lengths lying along an arc during this interaction. In order to gain further insights into Zac's conception of the measurement of  $0.5\pi$  radians, Zac was asked to explain the

measurement in the same manner as 2.2 radians (lines 14-15). Zac immediately responded that the measurement was “pi halves radius lengths,” and he subsequently approximated this value.

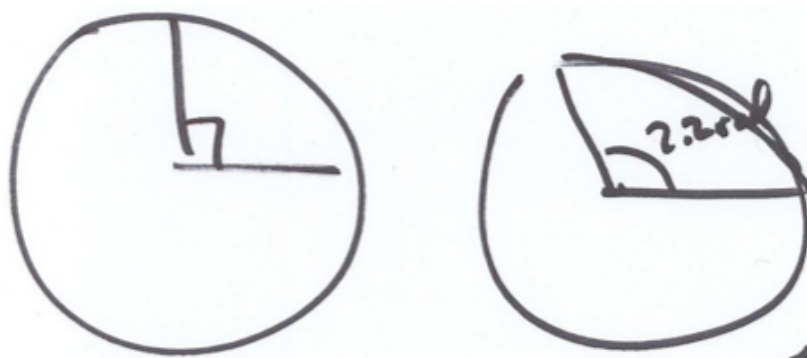
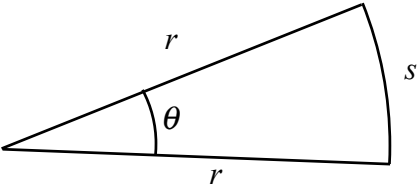


Figure 12. Zac's angle measure diagram.

Zac first described  $0.5\pi$  radians as referencing a fraction of a circle's circumference, but he also described the measurement as the number of radius lengths rotating along a subtended arc. These explanations convey that Zac developed the ability to reason about a radian angle measure relative to a portion of a circle's circumference and relative to a number of radius lengths rotating through an arc length. In both cases, Zac's actions imply his image of angle measure necessitated the construction of a circle centered at the vertex of the angle and measuring a subtended arc length, which was a central process of each instructional task during the teaching experiment sessions.

***The arc problem.***

Table 19
The Arc Problem
Using the following diagram, determine a formula between the measurements $r$ , $\theta$ , and $s$ .



Zac’s understanding of angle measure appeared to be driven by quantitative relationships that consisted of indeterminate measurements, and thus the researcher expected these understandings to form a foundation for Zac’s solution to The Arc Problem<sup>18</sup> (Table 19). After establishing that the angle measure was in radians, Zac constructed a formula representing the relationship between the quantities (Excerpt 13).

Excerpt 13

1	Zac:	Alright. We'll say theta equals radians ( <i>writing <math>\theta = rad</math></i> ), very very
2		simple then. $r$ theta is equal to $s$ ( <i>writing <math>r\theta = s</math></i> ). 'Cause theta is in
3		radians, that means a percentage of the radius. Which would then be
4		equal to this length ( <i>tracing arc length</i> ). So you multiply the percentage
5		of the radius by the radius, you'll get the arc length.

After Zac reasoned about a radian measure as a percentage of the radius (lines 2-4), he explained that multiplying the percentage of the radius length by the radius resulted in the measure of the arc length. Zac’s description implies that his constructed formula

<sup>18</sup> The subjects were not asked to formalize this relationship during a previous teaching experiment session, nor did the researcher provide the formula at any point in the study.

(line 2) stemmed from his ability to reason about an arc length measured as a fraction of the radius (e.g., a number of radians).

Subsequently, the researcher prompted Zac to interpret the formula  $\theta = \frac{s}{r}$ .

Opposed to providing a procedural explanation (e.g., dividing both sides of his previous equation by  $r$ ), Zac explained, “Well this is...a percentage of a radius length...a ratio, that’ll give you a percentage of  $r$ .” Thus, Zac’s conception of measuring a subtended arc length as a percentage of a radius drove his interpretation of the presented formula in addition to his previously constructed formula. Also, Zac showed no difficulty formalizing this relationship between indeterminate values.

### **Summary of Angle Measure Observations**

Zac leveraged his ability to reason about a fraction of a circle’s circumference to perform angle conversions grounded in this quantitative relationship. He also explained that an angle cuts off a constant fraction of the circle’s circumference for a circle of any size. Zac’s understanding of radian measures conveying a quantitative relationship between an arc length and the radius (e.g., measuring along the arc in a number of radius lengths) also generated a foundational way of reasoning. He utilized this understanding to convert between a measurement in radians to a linear measurement of arc length. Furthermore, he reasoned indeterminately about the multiplicative comparison of an arc length and the radius to generate a formula representing this relationship.

In order to leverage these understandings during the novel situations presented in the interview tasks, Zac also oriented to the problems such that he conceived of a situation consisting of these quantities. For instance, on The Radian Measurements and Pi

Problem, Zac first constructed a circle and identified an arc length corresponding to each value. During his solution on The Arc Length Problem – Version I and The Arc Problem, Zac traced arc lengths and referenced various quantities of the situation in ways that implied he conceived of these as measurable attributes of the situation. These orienting actions enabled Zac to construct and reason about quantitative relationships, where these relationships enabled him to plan and anticipate his calculations (e.g., quantitative operations).

### **Trigonometric Function Interview Observations**

#### ***The ferris wheel problem.***

Table 20
<i>The Ferris Wheel Problem</i>
Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full rotation. April boards the Ferris wheel at the bottom and begins a continuous ride on the Ferris wheel. Sketch a graph that relates the total distance traveled by April and her vertical distance from the ground.

In order to investigate Zac's covariational reasoning abilities within a similar context as The Fan Problem, the researcher designed The Ferris Wheel Problem (Table 20). Zac oriented to the problem by constructing a circle and labeling the circle's radius. He also labeled the starting position of the individual using a line to connect this position to the center of the Ferris wheel (Figure 13). While constructing his diagram, Zac wrote "1.2 minutes" and drew a circular arrow, stating that he was identifying how long it took April to complete a full rotation. Zac also traced the circumference of the circle while



verbalizing, “begins a continuous ride on the Ferris wheel.” These actions appear to convey that Zac constructed an image of an object traveling in a circular path, with this path forming a measureable arc length related to an elapsed time.

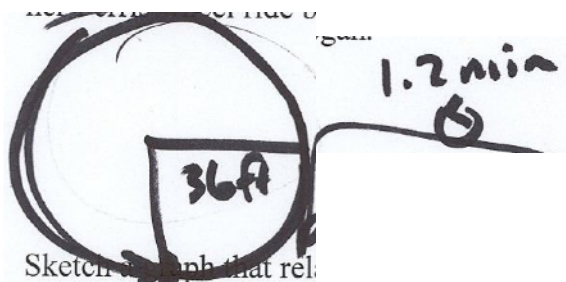


Figure 13. Zac’s diagram of April’s trip.

After Zac determined the circumference of the circle ( $72\pi$  feet), he described, “her vertical distance from the ground,” and traced a portion of the circle beginning at April’s starting position. He proceeded by drawing a larger diagram of the Ferris wheel describing how the two quantities change together as the Ferris wheel rotates previous to he constructed a graph (Excerpt 14).

#### Excerpt 14

---

1	Zac:	Ok. So a really easy way to do this is divide it up into four quadrants
2		<i>(divides the circle into four quadrants using a vertical and horizontal</i>
3		<i>diameter)</i> . 'Cause were here <i>(pointing to starting position)</i> , for every unit
4		the total distance goes <i>(tracing successive equal arc lengths)</i> , the vertical
5		distance is increasing at an increasing rate <i>(writing i.i.)</i> ... Then, uh, once
6		she hits thirty-six feet, halfway up, it's still increasing but at a decreasing
7		rate <i>(tracing successive equal arc lengths, writing i.d.)</i> ... Uh, then when
8		she hits the top, at seventy-two, it's decreasing at an increasing rate

---

- 9                   *(tracing successive equal arc lengths, writing d.i.)*...And then when she  
 10                   hits thirty-six feet again it's still decreasing (*making one long trace*  
 11                   *along the arc length*), but at a decreasing rate (*tracing successive equal*  
 12                   *arc lengths, writing d.d.*).
- 14     KM:     Ok, so in terms of this one, this quadrant (*pointing to the bottom right*  
 15                   *quadrant*), could show me on there how you know it's increasing at an  
 16                   increasing rate? Just show using the diagram...
- 17     Zac:     So like, a, she moves that much there (*tracing an arc length beginning at*  
 18                   *April's starting position*), that much here (*tracing an arc of equal length*  
 19                   *over the last portion of April's path in that quadrant*), uh, the vertical  
 20                   distance there changes by that much (*tracing vertical segment on the*  
 21                   *vertical diameter*), which is really hard to see with this fat marker. And  
 22                   then, uh, the vertical distance here changes by that much (*tracing*  
 23                   *vertical segment from the starting position of the second arc length*),  
 24                   which is a much bigger change.
- 

Similar to his actions on The Fan Problem, Zac utilized his image of the situation (e.g., the diagram) to describe the relationship between the vertical distance (from the ground) and a traversed arc length. Zac identified successive equal changes of arc length for each quadrant of the circle, while describing both the directional behavior of the vertical distance and the rate of change of the vertical distance with respect to a varying distance traveled (MA2 and MA4) (lines 1-12). Furthermore, Zac compared *changes* of

vertical distance corresponding to equal changes of distance traveled (MA4) (lines 17-24).

Immediately after these actions, Zac constructed a graphical representation (Figure 14). *During* his construction of the graph, Zac described the directional change and rate of change of the vertical distance for an increasing arc length (MA2 and MA5). Zac also determined the various arc lengths corresponding to the beginning and end of each quarter of a revolution. Thus, it appears that Zac's graph emerged from his constructed conception of the situation revealed in Excerpt 14.

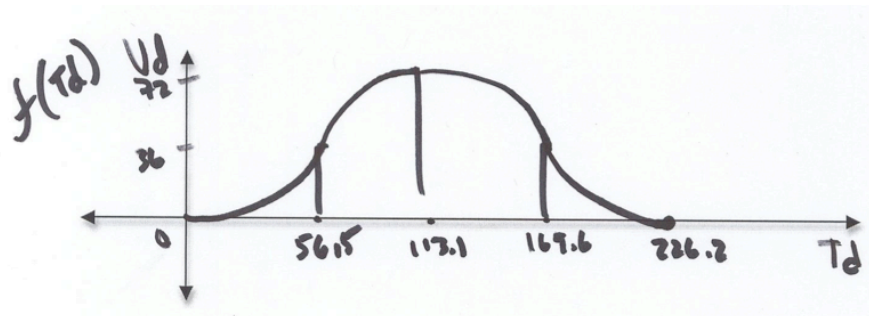


Figure 14. Zac's Ferris wheel graph.

When prompted to create a formula of his graphical representation, Zac described, "...the total distance is the input to get the vertical distance," appearing to identify an input quantity and an output quantity. Zac then rotated his diagram of the situation counterclockwise by ninety degrees and explained, "'cause then I can actually make sine work." He then paused for an extended amount of time and stated he could use the sine function without rotating his diagram (Excerpt 15).

Excerpt 15

---

- 1      Zac:      But I can still technically make it work here just by taking, by making it,  
 2                  uh, the starting point (*pointing to the bottom of the circle*) sixty-nine  
 3                  point six, um, feet around the circle. Or one sixty-nine point six feet  
 4                  around the circle. Or negative fifty-six point five feet around the circle.  
 5                  So it's going backwards (*tracing arc clockwise from the standard*  
 6                  *starting position to the bottom of the circle*).  
 7      KM:      Ok.  
 8      Zac:      I can still get the vertical distance that way. Um, so ya (*pause*). So (*long*  
 9                  *pause*), that means, since I'm doing that, that means whatever the vert-,  
 10                  or total distance is, I have to subtract fifty-six point five from it.  
 11      KM:      Ok.  
 12      Zac:      So let's see, vertical distance (*pause*) is equal to f of total distance  
 14                  (*writing*), which is equal to total distance minus fifty-six point five  
 15                  (*writing*), which will get me there (*pointing to the bottom of the circle*).  
 16                  (*pause*) And how (*inaudible*), divided by thirty-six feet to get me  
 17                  radians. (*pause*) And then I take the sine of that. So sine, so that will  
 18                  give me this one (*referring to first constructed graph, task two*).
- 

Zac constructed the formula  $vd = f(Td) = \sin\left(\frac{(Td - 56.5)}{36ft}\right)$  by the completion of

this interaction. Zac's initial action of rotating his diagram appears to have stemmed from his desire for the starting position of the individual to be at the standard position, which was the only starting position discussed previous to this task. As Zac reflected on the

situation, he identified the position of the rider as measureable from the standard position *along* the circumference of the Ferris wheel (lines 1-6). This image enabled Zac to explain that this length was 56.5 feet less than the individual's distance traveled (lines 8-10). As he continued, Zac converted the total distance to a number of radians (lines 16-18). Thus, Zac conceived of the argument of the trigonometric function as an arc length (measured in radians from the standard position). This resulted in him determining the arc length from the standard position as a function of the traversed arc length.

From the researcher's perspective, Zac's formula was not mathematically consistent with his graph. Although Zac identified the correct argument of the sine function, he was yet to identify a distinct vertical distance or the corresponding unit of measurement. In order to gain additional insights into Zac's conception of his formula, Zac was asked to explain his formula an additional time.

Zac stated that the sine function would output "the vertical distance," paused for a moment, and declared, "[sine] will give me a percentage of the radius length, which I then need to multiply by thirty six." This utterance reveals Zac reasoning about the output of the sine function as a fraction of the radius, which led to him altering his formula to

$$vd = f(Td) = 36 \sin\left(\frac{(Td - 56.5)}{36.ft}\right).$$

In response to Zac not identifying a distinct vertical distance, the researcher asked, "Vertical distance from where?" Zac then identified the vertical distance from the horizontal diameter of the Ferris wheel, which implied that Zac's formula was a mathematically correct reflection of the two quantities he was relating. The problem

statement asked for an output measured from the bottom of the Ferris wheel, but Zac reasoned about the vertical distance above the center of the Ferris wheel.

The researcher then asked Zac to clarify the vertical distance conveyed by his graph. Zac again identified the vertical distance from the horizontal diameter of the Ferris wheel. In an attempt to have Zac notice the inconsistencies in the quantities he was relating, Zac was asked for the “vertical distance” when the individual was at the bottom of the Ferris wheel (Excerpt 16).

#### Excerpt 16

- 
- |    |      |  |
|----|------|--|
| 1  | KM:  | So what should the vertical distance be when you're right here ( <i>pointing</i>   |
| 2  |      | <i>to the bottom of the Ferris wheel</i> )?  |
| 3  | Zac: | Uhhh, zero. Oh, ok, ya, I was thinking about that actually. So you're              |
| 4  |      | gonna have to change the formula a little bit, because you're gonna have           |
| 5  |      | to, add seventy-two, or no, because this problem, from point zero,                 |
| 6  |      | should give you a negative number. Negative thirty-six. I'm right there            |
| 7  |      | ( <i>pointing to the bottom of the Ferris wheel</i> ). So then she would add       |
| 8  |      | thirty-six ( <i>adding thirty-six to his algebraic representations</i> ).          |
| 9  | KM:  | Ok, so why add thirty-six?   |
| 10 | Zac: | ( <i>working</i> ) Because, um, technically right there it should give you zero    |
| 11 |      | ( <i>pointing to the bottom of the Ferris wheel</i> ). Uh, but the problem's going |
| 12 |      | to give you negative thirty-six because the value of sine theta ( <i>tracing</i>   |
| 14 |      | <i>segment from center of the Ferris wheel to the bottom of the Ferris</i>         |
| 15 |      | <i>wheel</i> ) at this point is negative one. Or sine, radians, one radian is      |
-

16                   negative one right there. And since you have to multiply it by the radius,  
 17                   it gives you negative thirty-six.

18     KM:     Ok.

19     Zac:     So then to cancel that out, you just add thirty-six to it. Which would  
 20                   make sense too because then, uh, when you hit this point (*pointing to the*  
 21                   *3 o'clock position*), technically that would be zero, so if you're adding  
 22                   thirty, that means thirty-six feet above the ground. And then at this point  
 23                   (*pointing to 12 o'clock position*), instead of just being thirty-six,  
 24                   because, you know, it's thirty-six from there (*tracing segment from*  
 25                   *center of the Ferris wheel to the top of the Ferris wheel*), it'd be seventy-  
 26                   two (*tracing segment from the bottom to the top of the Ferris wheel*), the  
 27                   whole distance from the ground.

---

Through reflecting on the context of the problem, Zac identified that his original formula resulted in a value of negative thirty-six for an input of zero (lines 5-6), and thus he needed to add thirty-six to this value in order to give the vertical distance from the ground. Zac then contrasted the measurements of the two vertical distances (lines 10-17, 19-27) and how the output of the sine function was related to each measurement. Also, Zac made physical motions identifying the referenced measurements, implying he had constructed an image of the situation that included the two vertical distances and a measureable difference between these two distances. This reasoning enabled Zac to alter

his formula such that it conveyed a relationship consistent with his graphical

representation ( $vd = f(Td) = 36 \sin\left(\frac{(Td - 56.5)}{36ft}\right) + 36$ ).

In summary, Zac reasoned about a varying arc length to identify rates of change and amounts of change of vertical distance for equal changes of arc length (MA3-MA5). Zac then constructed a graph that reflected this quantitative relationship. Also, when constructing the graphical representation, Zac remained focused on the two quantities of the situation and described the rate of change of the vertical distance with respect to the total distance traveled.

Zac's initial formula was inconsistent with his graph (from the researcher's perspective). Yet, Zac's actions indicate that his formula *was* consistent with the situation he was attempting to model at the time. In the case of his graphical representation, Zac related the rider's vertical distance from the ground and the rider's distance traveled around the circle. When describing his formula, Zac identified the rider's vertical distance above the horizontal diameter in relation to the rider's distance traveled. Then, as Zac reflected on his representations relative to the situation, he related various vertical distances and leveraged the radius as a unit of measurement to create a formula consistent with his graphical representation.

***The ski trail problem – version I.***



Table 21

*The Ski Trail Problem – Version I*

An arctic village maintains a circular cross-country ski trail that has a radius of 2.5 kilometers. A skier started skiing from position (0.9665, 1.25) and skied counterclockwise for 12.44 kilometers where he paused for a brief rest. Determine the ordered pair (in both kilometers and percentage of a radius) on the coordinate axes that identifies the location where the skier rested.

The Ski Trail Problem – Version I (Table 21) intended to gain additional insights into Zac's ability to reason about a starting position other than the standard position and inverse trigonometric functions. Recall that on The Finding an Arc Length Problem (Table 17), Zac reasoned that for any given value in one quantity, there simultaneously existed a value in another quantity.

When orienting to the problem, Zac identified the need of determining the arc length from the starting position of the individual to the standard position in order to determine the arc length from the standard position to the ending position of the skier. Zac then determined the arc length (Excerpt 17).

## Excerpt 17

---

1	Zac:	Well we can use sine or cosine, 'cause that gives us both points. So
2		cosine is equal to, ya, cosine theta is equal to point nine six six five
3		percent of the radius. And sine theta is equal to ( <i>pause</i> ) one point two
4		five percent of the radius ( <i>pause</i> ). That doesn't make sense. ( <i>pause</i> )

---

- 5      KM:      So why do you say that doesn't make sense?
- 6      Zac:      (*pause*) Ohh, nevermind. I'm thinking of it as radius equals one, but its
- 7                  radius equals two point five. So it's actually half, and it's not point nine
- 8                  six six five percent of its, uh, you know, a ratio. Ok, so that's easier to
- 9                  think of then. So let's do sine, since it's half. Nice and easy. Sine
- 10                (*working*) theta is equal to let's see (*pause*), one point two five divided
- 11                by the radius. Two point five (*working*). Sine theta is equal to one half,
- 12                or point five. So knowing that, we can do arcsine theta, or arcsine point
- 14                five, and that should give us the radius lengths (*using calculator*). Ok.
- 15                So that equals point five two radius lengths, or radians...that equals that
- 16                length right there (*tracing arc length*).
- 

After identifying a relationship between the given coordinates and the sine and cosine functions (line 1), Zac constructed two equations (lines 2-4) and described a length greater than one radius. Zac assumed the measurements represented a fraction of the radius length rather than a number of kilometers, which caused Zac to further orient to the problem situation (line 6). This further orientation was a result of Zac being perplexed when given a value greater than one radius (lines 3-4). Subsequently, Zac converted each measurement to a fraction of the radius, enabling him to correctly determine the desired arc length using the inverse sine function (lines 12-16).

Zac further oriented to the problem by reading the task and identifying that the problem was asking “for that and that.” As he made this statement, Zac traced both the horizontal and vertical segments representing the coordinate pair for the ending position

of the skier. He then determined the arc length from the standard position to the final position of the skier (Excerpt 18).

Excerpt 18

- 
- 1      Zac:    So I'm just gonna add the one point three to twelve point four four,  
 2                which will get me to that point (*pointing to ending position*). So that's  
 3                thirteen point seven four, ya, kilometers along the circumference  
 4                (*tracing the corresponding arc length*), which will get me to that point  
 5                (*pointing to ending position*). And using that I can determine (*pause*),  
 6                the, uh, horizontal (*tracing horizontal distance*) and vertical distance  
 7                (*tracing vertical distance*) to get to that point.
- 8      KM:    Ok.
- 9      Zac:    So, (*long pause*). So, what do I need to do here? How did I set it up  
 10               before? (*looking back at previous method for determining the arc*  
 11               *length*) Oookk. (*pause*) I'm looking for (*pause*). Alrighty, so, sine of  
 12               theta is equal to thirteen point seven four over two point five (*writing*  
 14               *corresponding equation*), and cosine theta is equal to thirteen point  
 15               seven four over two point five (*writing corresponding equation*). So it's  
 16               going to give the horizontal distance (*pointing to cosine equation*), this  
 17               is giving me the vertical distance (*pointing to sine equation*). So, let's see  
 18               here (*pause*). So thirteen point seven (*using calculator*) four divided by  
 19               two point five (*rewriting equations replacing the ratio with a numerical*  
 20               *value*). So I can take the arc of both of those. So theta, five point five is
-

21 equal to theta (*rewriting equation using inverse notation*). (*using*  
 22 *calculator to calculate sine inverse of 5.5*) Hmm...no real result.

---

Zac first identified the various known and unknown values on his diagram by tracing lengths corresponding to each value (lines 1-7). Then, when attempting to determine the unknown values, Zac reflected (lines 9-11) on his previous solution (Excerpt 17). This resulted in Zac using the unknown values as inputs to the sine and cosine functions (lines 11-20); Zac's previous solution process was implemented to determine an unknown arc length. Zac's execution of his previous procedure resulted in his calculator returning an unexpected result (line 22).

After Zac obtained this result, he reflected on the known values relative to the diagram of the situation. He explained, "What I got here (*referring to the 5.5*) is the total arc length to that point." Zac then explained a revised solution process (Excerpt 19).

#### Excerpt 19

---

1 Zac: I was thinking about it, like, um, this right here (*pointing to previous*  
 2 *solution process*). Instead of finding the arc length, I want the vertical  
 3 distance.  
 4 KM: Ok.  
 5 Zac: So the five point five actually counts as theta. So, that gives me (*pause*),  
 6 uh, (*writing*)  $x$  equals negative seven, or point seven one radians. Which  
 7 is this right here (*tracing vertical distance*), which makes perfect sense,  
 8 'cause it's negative.

---

Zac's description (Excerpt 19) reveals that he previously (Excerpt 18) attempted to apply an earlier computational process to this situation. He did not consider the role of each quantity relative to the input-output process of the sine and cosine functions when applying this previous procedure (Excerpt 18). Rather, he simply replaced the known value from his previous solution with the new known value. Then, after obtaining an unexpected result, Zac's reflection on the situation enabled him to identify his error and determine a correct value (Excerpt 19).

After this task, Zac was asked to determine the coordinate pair for any angle measure of  $\theta$  radians from the standard position. He first identified the ordered pair as  $(\cos(\theta), \sin(\theta))$  and explained, "And if you wanted it in kilometers, or units that the radius is in, you'd just do  $r \cos \theta$  and  $r \sin \theta$ ." Zac also traced vertical and horizontal distances corresponding to the outputs of sine and cosine, respectively. These actions by Zac further illustrate ability to reason about the sine and cosine functions as formalizing a relationship between two quantities. Zac's unprompted conversion to a number of kilometers also exhibits his propensity to reason about the relationship between the radius as a unit of measurement and other linear units of measure.

Overall, Zac's actions on this problem reveal him using his diagram of the situation to identify measurable quantities. Also, he continually reasoned about the sine and cosine functions representing input-output relationships between measures of these quantities (e.g., values). As a result of Zac's ability to reason about either quantity as an input, Zac used inverse sine and cosine functions to determine unknown arc lengths. Zac mistakenly switched the input-output quantities during his solution, which occurred due

to him applying a previous procedure without considering the quantitative meanings of each value. When he obtained an unexpected result, he further oriented to the problem's context and corrected his solution by identifying the input-output role of the two quantities.

***The empire state building problem.***

Table 22
<i>The Empire State Building Problem</i>
While site seeing in New York City, Bob stopped 1000 feet from the Empire State Building and looked up to see the top of the Building. Given that the angle of Bob's site from the ground was 56 degrees, determine the height of the Empire State Building.

The Empire State Building Problem (Table 22) offered insights into Zac's reasoning relative to a context that did not consist of circular motion. The first four teaching experiment sessions did not address a connection between unit circle trigonometry and right triangle trigonometry.



*Figure 15. Zac's initial diagram on The Empire State Building Problem.*

First, Zac oriented to the problem by constructing a diagram of the situation and labeling the given values (Figure 15). Zac then explained, "From the circle, or triangle,

we can determine that cosine of fifty six degrees is equal to one thousand feet (*writing corresponding equation*)...one thousand feet is equal to the radians, because cosine fifty six degrees is determined in radius lengths.” After converting the given angle measure to a number of radians, Zac continued explaining his solution process (Excerpt 20).

#### Excerpt 20

- 
- |    |      |  |
|----|------|--|
| 1  | Zac: | Ok, alright, so. Scratch that, point nine eight. This ( <i>referring to cosine</i>           |
| 2  |      | <i>expression</i> ) is equal to a thousand feet, yada yada yada, back where we               |
| 3  |      | were. Then a thousand feet is equal to the radians times the radius length                   |
| 4  |      | ( <i>writing '(rad)(r)'</i> ), or $r$ .  |
| 5  | KM:  | Ok.  |
| 6  | Zac: | Ok, 'cause the radians is just a percentage of the radius length. Oook. So,                  |
| 7  |      | now what I want to do is figure out what cosine point nine eight is                          |
| 8  |      | actually equal to, and using that I can find out what the radius length is                   |
| 9  |      | ( <i>pointing to r</i> ). So then when I do sine of point nine eight, I already              |
| 10 |      | know what the radius length is, so when I get the answer to that                             |
| 11 |      | ( <i>referring to sine</i> ) all I have to do is multiply by the radius length and I'll      |
| 12 |      | get that part ( <i>identifying the vertical segment on his triangle</i> ).                   |
| 14 | KM:  | Ok, so if you wanna go ahead and do that.  |
| 15 | Zac: | Ya. Ok, so, ( <i>using calculator</i> ) cosine point nine eight is equal to,                 |
| 16 |      | ( <i>writing</i> ) equals point five six radians. And so, all I have to do is ( <i>using</i> |
| 17 |      | <i>calculator</i> ) divide one-thousand by point five six, as shown in this                  |
| 18 |      | equation right here ( <i>referring to '1000=(rad)(r)'</i> ), to isolate the radius           |
-

19 all I have to do is divide it by the radians. (*using calculator*) And I get a  
 20 big number. So that means (*writing*)  $r$  is equal to one seven eight five  
 21 point seven one. So then I do (*writing*) sine point nine eight, (*using*  
 22 *calculator*) and I'm given a radius length, or a percentage of a radius  
 23 length, (*writing*) equal to point eight three. Now all I have to do is  
 24 multiply that (*writing*) by  $r$  and I'll get the length of that side (*pointing to*  
 25 *vertical segment on his triangle*), so, times (*using calculator*) one seven  
 26 eight five point seven one. So that means the length of that side is equal  
 27 to (*writing*) one four eight three point O three feet. Figure definitely not  
 28 drawn to scale.

---

Initially, Zac constructed (Figure 16) a mathematically incorrect equation,  
 $\cos(0.98) = 1000$ . However, Zac described that he needed to determine “what cosine  
 point nine eight is actually equal to,” and explained that  $\cos(0.98)$  represented a fraction  
 of the radius (lines 7-8 & 15-16). Zac’s equation also included a label that emphasized  
 that the value of 1000 represented a number of feet. Thus, Zac’s initial equation of  
 $\cos(0.98) = 1000 \text{ ft}$  appears to have stemmed from him reasoning about the cosine  
 function outputting a horizontal distance that had a measure of 1000 feet or  $\cos(0.98)$   
 radii.



$$\cos(56^\circ) = \frac{1000 \text{ ft}}{r}$$

$$\frac{1000 \text{ ft}}{0.98} = \frac{(r)(r)}{r}$$

$$r = 1785.71$$

$$\frac{56}{360} \times 2\pi = 0.98 \text{ rad}$$

$$\sin(0.98) = \frac{x}{r}$$

$$x = 1483.03 \text{ ft}$$

Figure 16. Zac's work on The Empire State Building Problem.

As Zac continued, he reasoned about measuring in a fraction of the radius to determine the value of the radius and the height of the building. Also, Zac's flexible understanding of the radius as a unit of measurement enabled him to anticipate and perform actions on indeterminate outputs of the sine and cosine functions (lines 6-12). In spite of Zac's paper work (Figure 16) appearing mathematically incorrect, Zac correctly reasoned about measuring multiple quantities relative to the radius in order to obtain a solution to this problem.

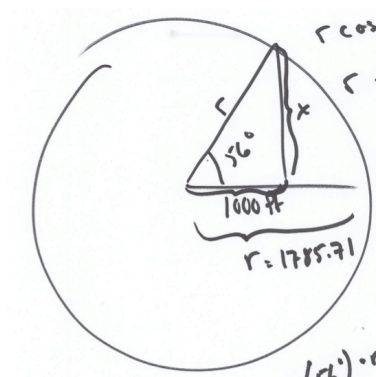


Figure 17. Zac's refined diagram on The Empire State Building Problem.

Up to this point in his solution process, Zac did not observably identify a radius or circle on his diagram. When Zac was asked to further describe his meaning of "the radius," he constructed a circle and then a right triangle in the standard orientation within the circle (Figure 17). After labeling the right triangle with each determined value, Zac

described the hypotenuse of the right triangle as “the radius.” Zac then discussed his choice to construct a circle (Excerpt 21).

Excerpt 21

- 
- |    |      |   |
|----|------|---|
| 1  | KM:  | So what told you to put this in a circle like this? Why did you make that     |
| 2  |      | choice?   |
| 3  | Zac: | Um, to make it easier to understand. Uhhh, ( <i>using calculator</i> ), how I |
| 4  |      | was originally taught was just with triangles. Now that we've started         |
| 5  |      | using circles it makes a whole lot more sense to me.                          |
| 6  | KM:  | So could you say a little bit about why it makes a little more sense now?     |
| 7  | Zac: | Uh, because I always just thought hypotenuse was, you know, just that         |
| 8  |      | side of a triangle. You know, you could use Pythagorean's Theorem to          |
| 9  |      | find out what it was very easily. And now that we've figured out, you         |
| 10 |      | know, now I'm looking at it and seeing it's the radius, it makes a lot        |
| 11 |      | more sense to be able to find, the horizontal and vertical distance           |
| 12 |      | according to the radius ( <i>waving tip of pen across the radius</i> ).       |
- 

Zac's descriptions suggest that he found value in “using circles” to relate an angle measure and another quantity (lines 3-5). Also, Zac's justification for the use of circles was specific; Zac explained that he only considered the hypotenuse as a side of a triangle previous to the precalculus course (lines 7-8). He then explained that his new understanding consisted of the hypotenuse, or radius, as a unit of measurement (lines 9-12).

A right triangle context was not introduced during the teaching experiment sessions previous to this problem. Yet, Zac leveraged his ability to reason about measuring quantities relative to another length in order to determine a correct solution. Zac first conceived of the hypotenuse of the right triangle forming the radius of a circle. He then reasoned about measuring lengths of the right triangle “according to” the hypotenuse. Also, by conceiving of a circle with the radius as a measuring unit, he applied his previously constructed understandings of the sine and cosine functions to determine these lengths. Lastly, Zac expressed a preference in this way of reasoning and conveyed that he found coherence in reasoning about measuring lengths relative to the radius of a circle (e.g., the outputs of the sine and cosine functions), which was the hypotenuse in the case of a right triangle.

### **Summary of Exploratory Teaching Interview One**

Zac’s actions during the interview session imply that The Fan Problem resulted in him constructing understandings of the sine and cosine functions rooted in a quantitative relationship and the unit circle. The Ferris Wheel Problem revealed Zac constructing a diagram before constructing a graph. He then used his image of the situation to construct quantities and subsequently reason about relationships between these quantities. Zac reasoned about the rate of change of a vertical distance with respect to a traversed arc length and he supported this reasoning by identifying equal changes of arc length and comparing corresponding changes of vertical distance (MA3-MA5). These actions led to Zac producing a graph that reflected this reasoning.

Zac's construction of the sine function through exploring circular motion also resulted in Zac having a process conception of the sine (and cosine) function. Zac's use of a diagram to graph the sine function consisted of reasoning about indeterminate values of quantities. Subsequently, when converting between various outputs of the sine function, Zac discussed these conversions *without* performing numerical calculations. Rather, he anticipated the sine function producing an output measured as a fraction of the radius. As another example of Zac's process conception of the sine function, whether given an arc length or vertical distance, Zac conceived of the value of the other quantity without having to calculate this value.

When Zac anticipated converting an output value of the sine function, Zac also leveraged the radius as a unit of measure. On each task during the interview session, Zac conceived of measuring quantities in a number of radius lengths, which enabled him to conceive of each circle as a circle of one radius. He also converted between various units by reasoning about the multiplicative relationship between the radius and a length measured relative to the radius. Zac also admitted that reasoning about the radius as a unit of measure offered coherence between the contexts of a right triangle and a circle. By imagining the hypotenuse of a right triangle as the radius, Zac created coherent conceptions of the trigonometric functions in the two contexts.

Zac incorrectly used the sine function in two instances during the interview. In the case of The Ski Trail Problem – Version I, he exchanged the input and output quantities when attempting to solve for a vertical distance. Relative to The Ferris Wheel Problem, his initial formula was inconsistent with his graphical representation and the intentions of

the problem. In both cases, Zac reflected on the problem situations when checking his solutions, which resulted in him identifying his inconsistencies. Thus, a lack of ability or understanding did not appear to be the cause of Zac's mistakes. Rather, his current conception of the situation was either inconsistent with the problem's intention (e.g., The Ferris Wheel Problem) or not utilized during his solution process (e.g., The Ski Trail Problem – Version I).

Zac's construction and refinement of problem situations, which is an orienting behavior, appears to have formed a critical aspect of Zac's problem solving behaviors. Throughout the interview sessions, his first actions frequently consisted of constructing a diagram of the situation and identifying quantities and values of the problem situations. Also, in problems that did not identify a circle in the problem statement (e.g., The Empire State Building Problem), Zac (mentally) constructed a circle. He also identified various quantities within this circular context and conceived of measuring these quantities relative to the radius of the circle. These conceptions of the problem situations, which consisted of quantitative relationships, enabled Zac to flexibly apply the sine and cosine functions. Zac's propensity to construct, reflect upon, and *continually refine* his image of the contextual situations stresses the important, and highly complex, role that orientation behaviors can play in problem solving. Zac's actions also exhibit that a subject's image of a situation is a continually changing mental structure.

Zac's propensity to engage in his orienting actions may have been promoted by the nature of his engagement in the tasks during the teaching experiment sessions. Both The Protractor Problem and The Circumference Problem led to Zac focusing on the

*process* of measuring an angle through the construction of circles and measurable arc lengths. Also, The Fan Problem resulted in Zac *constructing* the sine and cosine functions by covarying two quantities of circular motion. As a result, constructing quantities and relationships between quantities to solve novel tasks appears to be a central driving force in Zac's reasoning during the first interview.

### **Teaching Experiment Sessions Five and Six**

Two teaching experiment sessions occurred before Zac participated in his second, and last, interview session. During these sessions, the researcher implemented Activities 7-9 (Appendix C) with the subjects. These activities focused on inverse trigonometric functions and right triangle trigonometry. Select observations of Zac's behaviors and responses during these last two classroom sessions follow.

#### **Teaching Experiment Session Five**

Due to limited time during the fourth classroom session, the researcher returned to The Finding an Arc Length Problem (Table 17) during the fifth classroom session. The third task<sup>19</sup> on this problem was used to discuss the domain and range of the sine and inverse sine functions. The fifth classroom session also transitioned into exploring trigonometric functions in right triangle contexts. The Determining and Output Problem (Table 23) was first implemented to promote Zac reflecting on the reasoning he exhibited during The Empire State Building Problem (Table 22).

***The finding an arc length problem.*** After Zac identified the unknown value as an arc length, he used both the inverse sine and cosine functions to determine this value. Zac

---

<sup>19</sup> The third task consisted of a position in the second quadrant, which was intended to generate a discussion of the inverse sine function's output value.

obtained two different values and returned to a diagram of the situation in an attempt to reconcile this unexpected result. Zac's further orientation to the problem situation led to him explaining that the ending position of each arc length (from the standard position) corresponded to the same vertical distance above the horizontal diameter. Zac also explained that the horizontal distance varied from one radius to a negative fraction of the radius over this interval.

In order to leverage Zac's covariational reasoning, the researcher asked Zac to discuss the domain and range of the sine function. Zac first identified that the *sine* function has a range of "one to negative one [radii]." Zac also explained that this variation occurs over an input interval of  $-\pi/2$  to  $\pi/2$  radians. The researcher subsequently formalized this interval as the range of the inverse sine function.

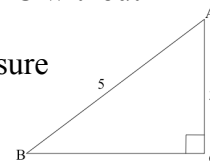
In summary, in an attempt to reconcile the inverse sine and cosine functions outputting two different arc lengths, Zac leveraged the unit circle to explain how the determined output values of the inverse functions related to the corresponding input values of the inverse functions. This led to a need of formalizing the domain and range of the inverse sine function. Zac's ability to covary an arc length and a vertical distance led to him identifying all possible output values of the sine function (e.g., the input values of the inverse sine function) and a corresponding interval of input to the sine function (e.g., the output values of the inverse sine function). This enabled the researcher to formalize these values as the domain and range of the inverse sine function.

***The determining an output problem.***

Table 23

*The Determining an Output Problem*

Determine the output of the sine and cosine of the measure of angle ABC without measuring the angle. Hint: think of how you would determine the measure of the angle of interest and how the sine function relates to this measurement.



Consistent with his actions on The Empire State Building Problem, Zac constructed a circle using the hypotenuse as the radius of the circle. This image of the situation enabled Zac to apply his understanding of the output of the sine function as a vertical length (the side opposite of the angle) measured relative to the radius (the length of the hypotenuse of the right triangle). This led to Zac constructing  $\sin(\theta) = o / h$ .

Relative to determining a similar relationship with the other leg of the right triangle, Zac explained, “The hypotenuse is the radius when using sine and cosine...[the output of cosine is] dividing the measurement  $a$  by the measurement  $h$ ... $a$  is your horizontal distance and  $h$  is your radius.” These actions by Zac further display his ability to conceive of the hypotenuse as the radius of a circle and, consequently, a unit of measure for the sides of a right triangle.

Next, the researcher implemented The Right Triangle Applet (Figure 18). After Judy constructed the equation  $\sin(0.776) = \overline{AC}$ , Zac responded that this equation was correct if the measurement unit of  $\overline{AC}$  is “a percentage of the radius.” Zac also identified that the applet presented the length of the segment in a number of inches. In order to alter Judy’s equation, Zac explained, “multiply that whole function, the sine, by the radius, or



in this case, four inches.” Similar to Zac’s actions on previous problems, he conceived of  $\sin(0.776)$  as a value measured relative to the radius without needing to evaluate this expression. This allowed him to anticipate multiplying this value by the radius length in inches to obtain a measurement in inches.

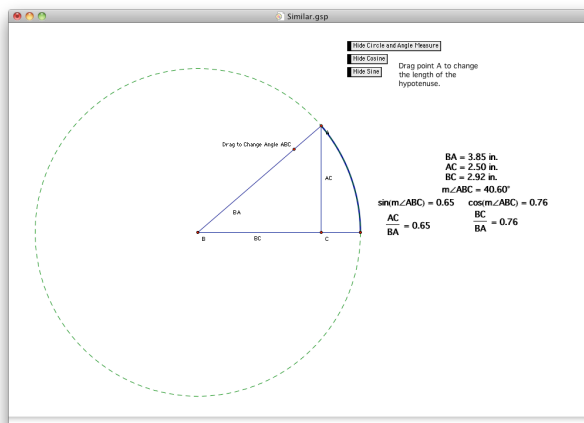


Figure 18. The right triangle applet.

Next, the researcher formalized the tangent function as the ratio of the outputs of the sine and cosine functions. The Right Triangle Applet was then used to discuss the output of the tangent function for a changing angle measure. Zac described, “[The output] is going to get larger...because of the ratio of vertical to horizontal. Because the vertical’s getting bigger as [the angle measure] gets larger, and the horizontal is getting smaller...it’s going to approach infinity.” Zac further described that the horizontal distance approached zero and the vertical distance approached the length of the radius as the angle measure approached ninety degrees.

Zac’s ability to reason about a varying angle measure, a varying horizontal distance, and a varying vertical distance enabled him to reason about the ratio of the vertical and horizontal distance covarying with the angle measure. This reasoning

resulted in Zac concluding that the ratio between the vertical and horizontal distance would approach infinity as the angle measure increased to ninety degrees. His reasoning was then verified using The Right Triangle Applet.

### **Teaching Experiment Session Six**

To complete the teaching experiment sessions, the subjects were given The Airplane Problem (Table 24) in order to gain insights into their reasoning relative to a right triangle context with a varying angle measure.

#### ***The airplane problem.***

Table 24
<i>The Airplane Problem</i>
A plane leaves the local air force base and travels due east. A radar station 45 miles south of the base tracks the plane and determines that the angle formed by the base, the radar station, and the plane is initially changing by 1.6 degrees per minute. Determine the distance the plane is from the radar station after a number of minutes, $m$ .

To begin the task, the subjects used a diagram on the whiteboard. After Judy constructed a circle and a horizontal radius, Zac related the path of the plane to the diagram, stating, “Going straight up is due east” (Figure 19).

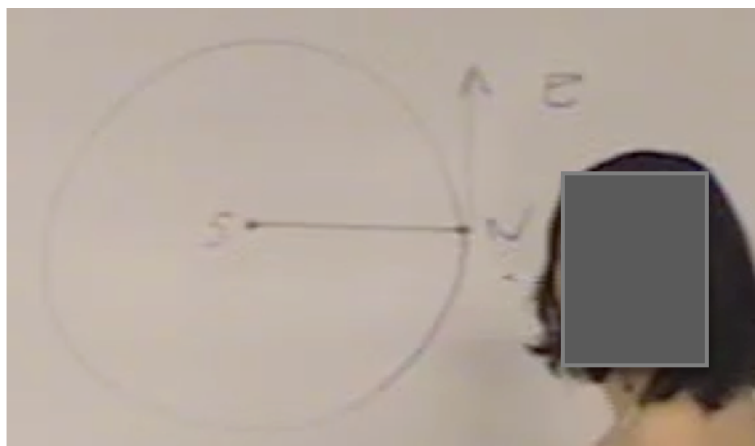


Figure 19. The students' diagram on The Airplane Problem.

Zac then identified the plane's unknown distance from the radar station and suggested using the tangent function to determine this distance. Zac subsequently

provided a solution of  $\tan\left(\frac{1.6m(2\pi)}{360}\right) = \frac{x}{45}$ , which he rewrote as

$x = 45 \tan\left(\frac{1.6m(2\pi)}{360}\right)$ , and Zac justified his original formula by stating tangent as

“opposite over adjacent.”

After determining this plane's distance from the radar station as a function of time, Zac described how to determine the distance of the plane to the base. As he began his explanation, Judy interjected that she used the Pythagorean theorem. Zac acknowledged the use of this theorem, but he was compelled to provide a solution using

the sine function. Zac constructed the formula  $\sin\left(\frac{1.6m(2\pi)}{360}\right) = \frac{45 \tan\left(\frac{1.6m(2\pi)}{360}\right)}{h}$  and

described  $45 \tan\left(\frac{1.6m(2\pi)}{360}\right)$  as the measure (e.g., a value) of the opposite side of the

right triangle. As Zac solved for the length of the hypotenuse, he used  $\theta$  to represent the varying angle measure due to the cumbersome nature of rewriting the value  $\frac{1.6m(2\pi)}{360}$ .

Finally, Zac concluded that  $h = \frac{45}{\cos(\theta)}$  represented the length of the hypotenuse as a

function of the angle measure, which varied with respect to time.

In summary, Zac conceived of The Airplane Problem such that the initial distance of the plane to the radar station formed the radius of a circle. Zac's orientation to the problem also resulted in him conceiving of the plane's path forming a leg of a right triangle such that this leg had a varying length. This image enabled Zac's use of various trigonometric functions to determine various sides of a right triangle. As Zac solved the problem, he continually reasoned about various expressions as representing the measurements of lengths and angles (e.g., *values* of  $45 \tan\left(\frac{1.6m(2\pi)}{360}\right)$  and  $\frac{1.6m(2\pi)}{360}$ ) without needing to calculate numerical values of these expressions. This resulted in Zac manipulating various expressions while maintaining quantitative meanings for the expressions.

### **Summary of the Last Two Teaching Experiment Sessions**

Zac's actions during the last two sessions continued to exhibit Zac reasoning about the sine and cosine functions as representing processes between the measure of two quantities. Additionally, the need to identify the domain and range of the inverse sine function emerged in the context of Zac determining an arc length corresponding to a vertical and horizontal distance. Zac's ability to reason about the covariation of an arc

length and a vertical distance then led to him correctly determining the domain and range of the inverse sine function.

Relative to right triangle contexts, Zac continued reasoning about quantitative relationships and conceiving of the hypotenuse of a right triangle as the radius of a circle. These images created coherence between the unit circle and right triangles such that Zac constructed and manipulated formulas and expressions while maintaining a quantitative meaning for these expressions. For instance, when asked to determine the leg of a right triangle in a number of inches, he explained, “multiply that whole function, the sine, by the radius, or in this case, four inches.” He then added that he could divide the length of the opposite side of the right triangle by the length of the radius to determine the output of the sine function. Zac also gave these descriptions *without* computing specific numerical values. That is, he reasoned about expressions as indeterminate values, which enabled him to construct, manipulate, and interpret expressions while maintaining a quantitative interpretation of these expressions.

### **Exploratory Teaching Interview Two**

The second and final exploratory teaching interview (Appendix D) consisted of two problems focused on angle measure and multiple problems on trigonometric functions. An analysis of Zac’s actions relative to these tasks is provided in this section.

#### **The Adding Two Angle Measures Problem**

Table 25

*The Adding Two Angle Measures Problem*

Determine the measurement (relative and angular) of an angle that has a measurement of 1.5 radians plus  $1.2\pi$  radians. Given a circle with a radius of 3.5 inches, what is the arc-length that corresponds to this angle measurement?

After reading the problem statement, Zac first determined the requested relative measurement (Excerpt 22).

## Excerpt 22

- 
- |    |      |   |
|----|------|---|
| 1  | Zac: | So, one point five plus one point two pi ( <i>writing corresponding</i>                                   |
| 2  |      | <i>expression</i> ), so an easy way to do it, to make it, you know, a percentage,                         |
| 3  |      | just divide both of them by two pi since their in radians ( <i>writing both</i>                           |
| 4  |      | <i>ratios</i> ), you can do that, and that will give you the percentage of how                            |
| 5  |      | many radians it is around the circle.   |
| 6  | KM:  | Ok, so why does that, why does that give you that?  |
| 7  | Zac: | Um, like uh ( <i>pause</i> ), ok. So it's radians, and it takes two pi radians to                         |
| 8  |      | reach the full circle, so two pi is a hundred percent. So by dividing that                                |
| 9  |      | ( <i>referring to the 1.5</i> ) by two pi, you get how much percent of a hundred                          |
| 10 |      | percent it is.  |
| 11 | KM:  | Ok.   |
| 12 | Zac: | Six ( <i>replacing <math>1.2\pi/(2\pi)</math> with 0.6</i> ) , ok. ( <i>using calculator to calculate</i> |
| 13 |      | <i>ratio</i> ) One point five divided by two times pi. You get zero point two                             |
-

- 14 four plus point six (*writing corresponding expression*), which in percent  
 15 is just twenty four percent and sixty percent. Which then equals then  
 16 zero point eight four, or eighty four percent all together.
- 17 KM: Ok.
- 18 Zac: So one point five radians plus one point two pi radians is eighty four  
 19 percent. And it's asking for the angular measurement as well. So all we  
 20 have to do to get the angular measurement is multiply eighty four  
 21 percent times two pi.
- 

Zac first converted each measure to a percentage of a circle's circumference (lines 1-5). Zac justified his conversion to a relative measurement by explaining that  $2\pi$  radians corresponded to reaching the full circle, or 100% of the *circle* (lines 7-10). After calculating each ratio, he explained adding the two relative measurements to obtain a percentage of the circle's circumference. He then identified the radian measure by reasoning that the fraction of  $2\pi$  radians subtended by the angle was the same fraction of the circle's circumference (lines 18-21).

Next, Zac attempted to determine the arc length corresponding to this angle measure (Excerpt 23).

#### Excerpt 23

- 
- 1 Zac: Ok, um, since five two point eight is in radians, one radian is the length  
 2 of one radius, so I just multiply by three point five inches, so five point  
 3 two eight times three point five (*writing corresponding expression*,
-

- 4            *using calculator*). Eighteen point four eight inches.
- 5        KM:    Ok. So what does it mean to have a measurement of, like, one point two
- 6            pi radians?
- 7        Zac:    Um (*pause*), that means, well, pi radians is halfway across the circle.
- 8            This. (*draws a circle with a horizontal diameter*) The full circle, the full
- 9            circumference around the circle is two pi.
- 10       KM:    Ok.
- 11       Zac:    So, uh, and halfway across is pi, so there's, you're taking pi and
- 12            multiplying it by one point two, so you're getting that extra twenty
- 13            percent, there (*making hand motions where a terminal ray would be*).
- 14       KM:    Ok, so you're getting twenty percent of what?
- 15       Zac:    Uh, and extra twenty percent of half the circle.
- 16       KM:    An extra twenty percent of half the circle. When you say half the circle,
- 17            what are you referring to about half the circle?
- 18       Zac:    Well half the circumference (*using pen to trace the circumference*).
- 19       KM:    Half the circumference.
- 20       Zac:    Ya.
- 21       KM:    Ok. Ok, so how bout relative to the radius, the length of the radius.
- 22            What's one point two pi radians mean?
- 23       Zac:    Um, it means it's one point two pi radius lengths.

After Zac reasoned about a number of radians representing a number of radius lengths (lines 1-4), the researcher prompted Zac to explain the meaning of a radian angle



measure (and containing the value of  $\pi$  in the measurement). Consistent with Zac's actions during the previous interview session, he initially described the measurement as a fraction of the circle's circumference. He related the measurement to half of the circle's circumference, or  $\pi$  radians (lines 7-9 and 11-13), and appeared to conceive of this measurement as  $\pi + 20\%\pi$  radians. In response to this explanation, the researcher focused Zac on explaining the measurement relative to the radius of the circle (lines 21-22). Zac immediately responded that the measurement was  $1.2\pi$  radius lengths (line 23). Zac later described, "You get really close to six and a fourth pi radians to get all the way around [the circumference]," further exhibiting his ability to conceive of radius lengths rotating through the circumference of a circle.

Zac's actions on this problem alternated between reasoning about radian measures as a fraction of a circle's circumference (e.g., "half the circle") and as a number of radius lengths along the circumference of a circle. Similar to his actions on previous tasks, his descriptions relative to a circle's circumference were more natural for measurements with the symbol  $\pi$  in the expression. However, when prompted to explain the measurements relative to the radius, Zac reasoned about a number of radius lengths rotating through an arc.

### **The Arc Problem**

The Arc Problem was designed in order to gain additional insights into Zac's ability to reason indeterminately and relate the measure of an angle, the radius, and an arc length. This problem was identical to The Arc Problem (Table 19) given during the previous interview, but due to Zac consistently focusing on quantitative relationships, the

researcher did not expect Zac to attempt to recall a formula without providing a quantitative basis for the formula.

When orienting to the problem, Zac constructed a vertical segment on the diagram (Figure 20), while stating, “That would be sine theta, for that length.”

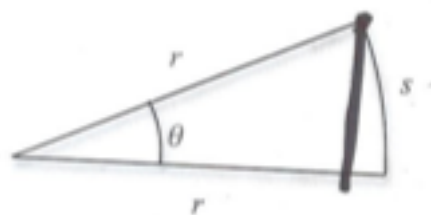


Figure 20. Zac’s vertical segment.

Zac had difficulty continuing and conjectured the formula  $\sin\left(\frac{s}{r}\right) = \theta$ . Upon writing this formula, Zac returned to the diagram and stated, “Ok, what this means is, um, (pause), well (scratches formula out), not sine, just  $s$  over  $r$  equals theta. Don’t know why I was thinking sine.” As he gave this description, he wrote  $\frac{s}{r} = \theta$ , and then described, “And what that gives you is, uh, this arc length divided by the radius, which then gives you a percent of a radius, which is a radian...s relative to  $r$ .”

Zac’s attempt to first apply the sine function may have been a result of the continual focus on trigonometric functions during the recent teaching experiment sessions. Then, when Zac attempted to justify his constructed formula  $\sin\left(\frac{s}{r}\right) = \theta$  relative to the context of the situation, he altered the formula to  $\frac{s}{r} = \theta$ . This formula reflected his conception of a radian measure representing a fraction of the circle’s radius.

Zac's reflection on the situation and his ability to reason about measuring an arc length as a fraction of the radius enabled him to correct an incorrect formula.

To conclude this problem, the researcher asked Zac to explain "the basis for angle measure." Zac explained, "Um it really is the percent of the circumference. Because no matter how big or small the circle is, the percent of the circumference that [the angle] cuts out is always the same." This further illustrates that Zac's understanding of angle measure consisted of the fractional amount of a circle's circumference subtended by the angle formed a foundational understanding of angle measure in addition to his ability to reason about a number of radius lengths rotating through an arc length.

### **The Enemy Approaches Problem**

Table 26
<i>The Enemy Approaches Problem</i>
A castle observation tower is elevated 126 feet above the ground. When an approaching enemy is first noticed, the angle of depression (the angle at which an observer needs to look down) from the observation post was 0.084 radians. How far away is the enemy from the castle? How far away is the enemy from the observer?

On The Enemy Approaches Problem (Table 26), Zac oriented to the problem by choosing to draw a diagram "before we read [part] a at all." As Zac created the diagram, he (incorrectly) labeled the angle formed by the observation tower and the observer's line of sight as the angle of measure 0.084 radians (Figure 21). However, Zac's image was incorrect only from the intent of the problem statement, and thus Zac was allowed to continue with his solution.

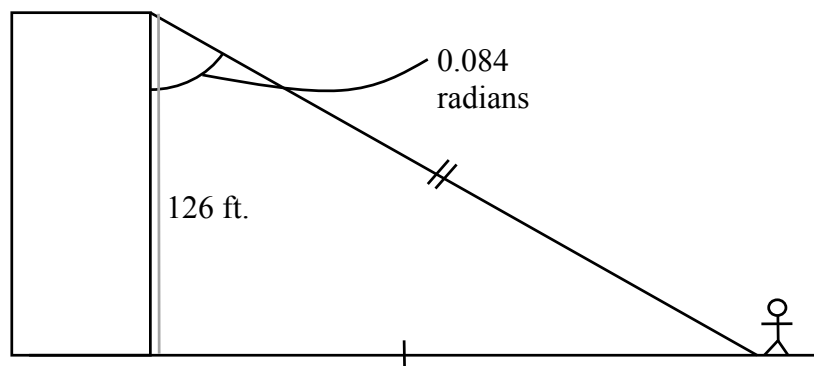


Figure 21. Zac's initial diagram on The Enemy Approaches Problem.

Zac identified the two unknown sides of the triangle referenced by the problem statement and stated, "Let's make it a little easier for me to understand. Thinking of it as a circle (*redrawing right triangle*). We have a right angle, and then we have our point zero eight four radians (*labeling angle measures*).” Zac did not visibly construct a circle on his diagram (Figure 22), but his utterances convey he conceived of the right triangle within a circular context.

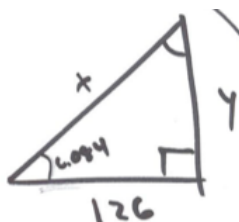


Figure 22. Zac's second diagram on The Enemy Approaches Problem.

Zac then constructed the equation  $\cos(.084) = \frac{126}{x}$  and described, “the cosine function [is] triangularly adjacent over hypotenuse.” After determining that the observer was 126.446 feet from the enemy, he reflected on his diagram, redrew the right triangle (Figure 23), and stated that his first diagram was not to scale (Figure 22).

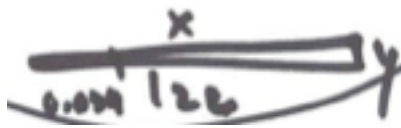


Figure 23. Zac's third diagram on The Enemy Approaches Problem.

Next, Zac explained the tangent function as, “triangularly is opposite over adjacent of the angle we’re talking about.” After constructing the equation

$\tan(0.084) = \frac{y}{126}$ , Zac verbalized, “[10.61 feet] is the length that the enemy is from the castle.”

Zac reflected on his solution and claimed that this distance was alarming because of the enemy’s proximity to the castle. The researcher responded by identifying the intended angle of measure 0.084 radians. Zac then concluded that that he could “just subtract point zero eight four from pi over two” to determine the measure of the angle he used and then apply his previous method to determine the solution of the intended situation.

To conclude the problem, Zac described his action of redrawing the right triangle while alluding to a circle (Excerpt 24).

Excerpt 24

- 
- |   |      |  |
|---|------|--|
| 1 | KM:  | So first of all why can we take a triangle and put it in this orientation? |
| 2 | Zac: | I can take a triangle and flip it however we want.                         |
| 3 | KM:  | Flip it in every...  |
| 4 | Zac: | Ya, it doesn't matter.   |
| 5 | KM:  | So what doesn't matter?  |
-

- 6      Zac:      Um, well how the triangle is oriented. I just did it so we could see, um,  
 7                    in the context of a circle (*drawing a circle centered at the vertex of the*  
 8                    *angle and with the hypotenuse as the radius*).
- 9      KM:      K, so you, say a little bit more about that.
- 10     Zac:      Um, well I assigned a hundred twenty six feet (*pointing to the tower in*  
 11                    *his original diagram*) to my horizontal distance (*tracing length on his*  
 12                    *circle diagram*), and then the distance between the castle and the enemy  
 13                    is the vertical distance, and then the, the diagonal line, or the hypotenuse  
 14                    (*identifying both hypotenuses*), to be my radius.
- 15     KM:      Ok.
- 16     Zac:      So then we can think of it as sine and cosine in that way. Which made it  
 17                    a lot easier.
- 18     KM:      Ok, 'cause sine and cosine in the context of the circle tell us what?
- 19     Zac:      It will tell us the vertical distance and horizontal distance (*tracing both*  
 20                    *distances*) in accordance to the radius (*pointing to the hypotenuse, or*  
 21                    *radius*).

Zac described that his reorientation of the right triangle was an attempt to identify the right triangle in the context of a circle (lines 6-8). This reorientation enabled him to use the hypotenuse of the right triangle as a radius of a circle. Then, Zac conceived of the various legs of the triangle as vertical and horizontal distances in the circle (lines 10-21).

Immediately following this explanation, Zac described that his reorientation made the problem “easier...the vertical distance and horizontal distance in accordance...to the

radius.” This statement implies he preferred reasoning within a circular context and about lengths measured relative to a radius. This preference may have been an implication of Zac’s foundational reasoning ability of measuring lengths as a fraction of the radius. Hence, conceiving of the hypotenuse as the radius of a circle enabled Zac to leverage such reasoning and construct coherence between the two contexts of trigonometry.

In conclusion, Zac first constructed a labeled diagram of the situation, which was consistent with his actions on previous right triangle problems. Zac’s orientation to the problem consisted of him using the hypotenuse to construct a circle with a radius of the same length. This image formed a foundation for his solution by enabling him to leverage measuring vertical and horizontal distances (the two legs of the right triangle) relative to the radius (the hypotenuse). Also, Zac’s conception of the situation led to him reasoning about each trigonometric function as an input-output process. For instance, he mentioned that tangent was “opposite over adjacent...of the angle we’re talking about.”

### **The Tangent Function and Graphing Problem**

Table 27
<i>The Tangent Function and Graphing Problem</i>
How does the function $f(\theta) = \tan(\theta)$ vary as $\theta$ varies from $-\pi/2$ to $\pi/2$ ?

The Tangent Function and Graphing Problem (Table 27) was intended to offer insights into Zac’s ability to reason covariationally relative to the tangent function. Also, the researcher designed the problem such that it prompted Zac to construct a graphical representation of the function, which was not accomplished during the previous sessions.

Zac began the problem by stating, “It goes from negative infinity to infinity.” Zac then explained his reasoning and graphed the function (Excerpt 25).

Excerpt 25

- 
- |    |      |   |
|----|------|---|
| 1  | KM:  | Could you explain? You can draw, if you would like to draw a graph                  |
| 2  |      | too.  |
| 3  | Zac: | Well, um, ok. Actually a circle would be better ( <i>draws a circle with a</i>      |
| 4  |      | <i>crosshair</i> ). Well we'll get to the graph, just understanding the circle will |
| 5  |      | make the graph easier to understand. Ok, so as we talked about in the               |
| 6  |      | last problem, tangent is equal to opposite over adjacent, or vertical               |
| 7  |      | distance over horizontal distance. So ( <i>pause</i> ), when you talk about a       |
| 8  |      | circle, vertical distance over horizontal distance, uh, the points to it is         |
| 9  |      | kinda weird.  |
| 10 | KM:  | So what do you mean by it's kind of weird?  |
| 11 | Zac: | Well at, eventually you're going to have some points that don't ex-, that           |
| 12 |      | aren't defined.   |
| 13 | KM:  | Oh, ok.   |
| 14 | Zac: | Ok, so let's see. Starting here ( <i>pointing to the standard position</i> ), your  |
| 15 |      | vertical distance is zero and your horizontal distance is one radius or             |
| 16 |      | one. So, you know, nice and simple, zero over one. Ok, but you know,                |
| 17 |      | that makes sense. Then over here ( <i>pointing to 9 o'clock position</i> ) it's     |
| 18 |      | negative one zero, which is still zero over negative one, which is zero.            |
| 19 |      | Makes sense. But then when you get to here and to here ( <i>pointing to 12</i>      |
-



20                    *o'clock and 6 o'clock positions*), you get zero, one, and zero, negative  
 21                    one, and that puts one over zero, and then negative one over zero, which  
 22                    isn't defined (*at each position he wrote corresponding coordinates*).

---

This interaction reveals that previous to producing the graph, Zac found value in constructing a circle and describing the tangent function in this context (lines 1-5). Then, after describing the output of the tangent function relative to right triangles and the unit circle (lines 6-9), Zac utilized the circle to identify the output values of tangent for four locations on the circle (lines 14-22). These actions reveal that in order to produce a graph of the tangent function, Zac leveraged his image of the unit circle (a circle with a length of one radius) to construct various vertical and horizontal lengths.

Following this interaction, Zac explained, “The first number [of the coordinate] is the horizontal distance and the second number is the vertical distance...measured in radius length, or percentage of a radius,” which further reveals that Zac conceived of the coordinate values as measurements relative to the radius. Zac also verbalized that as the angle measure approaches  $\pi/2$  radians, “the horizontal distance is getting closer to zero, and [the vertical distance] is getting closer to one...[the output of tangent] approaches infinity.” When constructing his graph (Figure 24), Zac explained, “[As] theta varies from negative pi halves to pi halves...f of theta varies from negative infinity to infinity.” Zac’s ability to covary an angle measure, or arc length, with both a vertical and horizontal distance enabled Zac to directionally covary the angle measure and a ratio between the vertical and horizontal distance (MA2). This reasoning led to him constructing a graphical representation of this covariation.

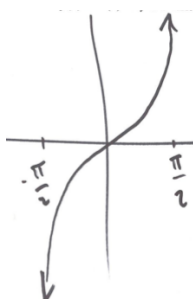


Figure 24. Zac's graph of the tangent function.

### The Ski Trail Problem – Version II

Table 28

#### *The Ski Trail Problem – Version II*

An arctic village maintains a circular cross-country ski trail that has a radius of 2.5 kilometers. A skier started skiing from position  $(-1.76777, -1.76777)$  and skied counterclockwise for 3.927 kilometers where he paused for a brief rest. Determine the ordered pair (in both kilometers and percentage of a radius) on the coordinate axes that identifies the location where the skier rested.

Although similar to The Ski Trail Problem – Version I, The Ski Trail Problem – Version II (Table 28) consisted of a coordinate in the third quadrant. This quadrant was chosen in an attempt to observe Zac's behaviors when neither inverse trigonometric function output represented the desired arc length<sup>20</sup>.

As Zac oriented to the problem, he pointed to the center of the circle, the initial position of the skier, and then the ending position of the skier. Zac then stated, "We know the distance from there to there (*referring to the arc between the initial and resting positions*), we need to get the distance along the circumference to there." While giving

<sup>20</sup> Previous to this problem, Zac did not encounter a position in the third quadrant.

the last part of this description, Zac traced a counterclockwise arc from the standard position to the initial position of the skier.

Next, Zac assumed that the given measurements were in kilometers, stating, “Ya, it has to be ‘cause it’s more than one, so it can’t be radius lengths.” Zac then constructed the equation  $\sin\left(\frac{1.7677}{2.5}\right) = z$ , stating that  $z$  represented the unknown arc length. Zac was perturbed by the result of calculating the arc length. In response to this perturbation, he reflected on the diagram of the situation and stated, “[I’m] doing this backwards...the result of sine is the vertical height. I was thinking the result was arc length. Arc length is the input, not the output.” Zac then corrected his equation to  $\sin(z) = \frac{1.7677}{2.5}$  and used the arcsine function to determine  $z = -0.78536$ . Zac was again perplexed by his answer and verbalized, “that is in radians...and it represents that much arc length (*identifying an arc length*) because if it did represent the whole thing it would equal something bigger than pi.” The arc length Zac identified during this explanation was 0.78536 radians clockwise from the standard position; Zac expressed that this value was also the measure of the counterclockwise arc length from the 9 o’clock position to the starting position of the skier.

Zac then identified that the y-coordinates were equal at the ending position of each arc length he determined, and he attempted to determine the desired arc length using the cosine (and arccosine) function. This resulted in an arc length of 2.35616 radians (Excerpt 26).

- 1      Zac:    I got two point three five six one six. (*pause*) Now I'm starting to get a  
2                    little confused (*long pause*).
- 3      KM:    So what'd you get?
- 4      Zac:    Ok (*long pause*). (*sigh*) I got a positive number. (*pause*) Which I  
5                    shouldn't have gotten.
- 6      KM:    So what'd you end up getting?
- 7      Zac:    I got two point three six.
- 8      KM:    Ok.
- 9      Zac:    Which would be somewhere over here (*identifying position in the*  
10                    *second quadrant*).
- 11     KM:    Ok.
- 12     Zac:    Wait, no no no no no. (*pause*) Horizontal distance. No, ya, nevermind.  
13                    That's right. It's doing the exact same thing again, it's going this way  
14                    (*making clockwise motion from the standard position*). Or, you know,  
15                    'cause uh, x there and there (*identifying positions in the second and third*  
16                    *quadrant*) are the same thing.
- 17     KM:    Ok.
- 18     Zac:    Or cosine there and there (*identifying positions in the second and third*  
19                    *quadrant*) are the same thing.
- 20     KM:    So it's really giving you that arc length (*identifying clockwise arc*  
21                    *length*)?
- 22     Zac:    Ya, it's giving, it's giving the shorter distance.
-

- 23      KM:      Ok.
- 24      Zac:      So actually what we could do is just take two pi and subtract that from it
- 25                      and it will give us the other way.
- 

After expressing further confusion (lines 1-2 & 4-5), Zac used the diagram of the situation to identify a counterclockwise arc from the standard position (lines 9-10). Zac also acknowledged that the same arc length, but clockwise from the standard position, would end at the starting position of the skier and yield the same  $x$ -coordinate (lines 12-16). Finally, Zac identified that the difference between the determined value and  $2\pi$  would result in the appropriate arc length (lines 24-25). These actions reveal that Zac's ability to reason about measuring along an arc in a number of radius lengths enabled him to interpret and relate output values of the inverse cosine function relative to various coordinate positions on a circle.

After computing the arc length counterclockwise from the standard position, Zac determined the coordinates of the skier's resting position (Excerpt 27).

#### Excerpt 27

- 
- 1      Zac:      Oookkk. So two pi (*using calculator*) minus two point three six gives
- 2                      you three point nine two radians. So to get three point nine two seven in
- 3                      radius lengths, so we can add them together. We just divide the three
- 4                      point nine two seven by two point five (*using calculator*) point two
- 5                      seven divided by two point five, and it's one point five seven.
- 6      KM:      So why's dividing that by two point five work?
-

- 7      Zac:      Because I mean it gives us a percentage of a radius, or the radius length  
8                      from there to there along the circumference (*tracing arc length from the*  
9                      *skier's starting position to the resting position*).
- 10     KM:      Ok.
- 11     Zac:      So then we take one point five seven plus three point nine two (*using*  
12                      *calculator*) and that's five point four nine radians to get all the way  
13                      around the circumference to that point (*tracing arc length from the*  
14                      *standard position to the resting position*).
- 15     KM:      Ok.
- 16     Zac:      So knowing that we can then take the sine and cosine of that and find the  
17                      coordinates for that point. So sine of five point five four nine and cosine  
18                      of five point four nine (*writing corresponding expressions*). Ok (*using*  
19                      *calculator*) gives you negative point seven one two, or seven one three  
20                      (*writing value*). And then (*using calculator*) cosine of five point four  
21                      nine, point seven O two (*writing value*). And that's in radians.
- 22     KM:      Ok.
- 23     Zac:      So actually we can say, because it's asking for both, kilometers and  
24                      percentage of a radius, so we have it in one way right now. Point seven  
25                      zero two, um, negative point seven one three (*writing coordinate pair*).
- 26                      That's one way to get it, that's in radians, so percentage of a radius.
- 27     KM:      Ok.
- 28     Zac:      Same thing. Ummm (*pause*). So then all I have to do is multiply both
-

29           these by two point five (*working*) to get them in kilometers. Times two  
 30           point five (*using calculator*). One point seven five four (*writing value*)  
 31           and then we got negative point (*using calculator*) seven one three times  
 32           two point five, negative one point seven eight five four (*writing*  
 33           *coordinate pair*). Negative one point seven eight, and there are two  
 34           different ones. Radians and kilometers.

---

Zac remained attentive to the quantities and the corresponding units of measure throughout the entirety of his solution process. Zac visually traced each referenced arc while describing the corresponding units of measure and the multiplicative relationship between units of measure (lines 1-5, 7-9, & 11-14). Furthermore, Zac's ability to reason about measuring along an arc length appears to have enabled his reasoning about adding two arc lengths (lines 11-14). After determining the appropriate arc length, Zac then used the sine and cosine functions to determine the coordinate pair as a fraction of the radius, or "radians" (lines 16-21). Zac leveraged his ability to reason about a measurement as a fraction of the radius to convert these coordinates to a number of kilometers (lines 28-34).

In summary, Zac continually reflected on his image of the contextual situation to correctly determine the resting position of the skier. Specifically, Zac's ability to reason about measureable arc lengths and measurements given in a number of radians created a foundation for Zac to interpret various values. As Zac determined arc lengths measured in a number of radians, he identified these arc lengths on his diagram while using relationships between arc lengths to determine a desired arc length. For instance, when

Zac determined an arc length of 2.35616 radians, his image of the situation enabled him to reason that this was not the counterclockwise arc length he was attempting to find. He then identified that the arc length corresponded to the appropriate horizontal distance and that the difference between a full revolution ( $2\pi$  radians) and this arc length was equal to the desired arc length. After determining the appropriate arc length, Zac then used the sine and cosine functions to determine the appropriate coordinate position.

### **Summary of Exploratory Teaching Interview Two**

Zac's actions during the second interview were consistent with those he exhibited during the teaching experiment sessions and the previous interview. Relative to angle measure, Zac continued to reason about measuring along subtended arc lengths in a number of radius lengths, or a fraction of the radius. Zac also identified that the same percentage of the circumference is cut off by an angle regardless of the size of the circle. Zac conceptions also necessitated the (mental) construction of a circle when reasoning about angle measure.

Zac also continued to reason about the sine and cosine functions as quantitative relationships between an angle measure and a multiplicative comparison of two lengths. These quantitative relationships were such that Zac reasoned about either quantity as the input or output of the functions. Whether given either value, Zac was able to apply the (inverse) sine and cosine functions to obtain an unknown arc length or multiplicative comparison of two lengths.

Zac's behaviors when orienting to problems also continued to offer insights into his constructed understandings. Zac frequently created and used diagrams throughout the



interview to identify various quantities and measurements of these quantities (e.g., values). He then conceived of relationships between these quantities and anticipated, or planned, calculations based on these relationships (e.g., quantitative operations).

Furthermore, as Zac obtained values, he often reflected on his image of the situation resulting in him refining his diagrams and conceiving of relationships between quantities (e.g., The Ski Trail Problem – Version II). Zac also utilized his image of the unit circle to coordinate the variation of an arc length, vertical distance, and horizontal distance when graphing the tangent function. This enabled Zac to construct a relationship between a varying arc length and the ratio of the vertical distance and horizontal distance when graphing the tangent function.

Zac's orientation to The Enemy Approaches Problem further revealed the coherence he constructed between unit circle trigonometry and right triangle trigonometry. Similar to his actions in previous right triangle contexts, he conceived of the hypotenuse of the right triangle as the radius of a circle. This construction appears to have stemmed from his comfort and flexibility with measuring lengths relative to the radius of a circle. Thus, his ability to conceive of *constructing* a circle using the hypotenuse of the right triangle enabled him to conceive of right triangle contexts in a manner that was consistent with the unit circle. Zac also continued to reason about relationships between an angle measure and a ratio of two lengths.

### **Summary and Discussion of Zac**

Zac's initial conception of angle measure consisted of geometric objects and pre-defined measurements of these objects (e.g., two perpendicular lines have ninety degrees)

opposed to a systematic process of measuring an angle that was based on measurable attributes of these objects. As he solved the various teaching experiment tasks, Zac developed an image of angle measure that required the construction of a circle centered at the vertex of an angle. Zac conceived of the arc length subtended by the angle as a measurable attribute central to the process of measuring the openness of an angle. Furthermore, Zac conceived of measuring the subtended arc length relative to both the circumference and radius of the corresponding circle, where these two measurements were constant for a circle of any radius. Also, the quantitative relationship of a subtended arc length's fraction of a circle's circumference formed a foundation for Zac converting between units of an angle's measure.

Zac's conception of measurements relative to the radius (whether arc lengths or vertical and horizontal distances) consisted of a quantitative relationship between the measured length and the radius. After engaging in The Circumference Problem, Zac reasoned about measuring *along* an arc length in a number of radius lengths and as a percentage of the radius length. This enabled Zac to conceive of any circle as having a radius of one unit and a circumference of  $2\pi$  radius lengths (e.g.,  $C = 2\pi r$ ). Zac also conceived measuring the coordinates on a circle relative to the radius (e.g., the unit circle) when solving The Fan Problem. As a result, Zac was able to construct the unit circle as he encountered circles of various linear radius lengths. His ability to measure quantities in a number of radius lengths also led to him flexibly converting between units of measurement, while often giving preference to measurements relative to the radius.

Zac constructed conceptions of the sine and cosine functions that were grounded in quantitative relationships and entailed the ability to reason about indeterminate values. Relative to the sine function, Zac leveraged his ability to reason about measuring along an arc to covary the vertical distance of a point above the horizontal diameter of a circle and a swept out angle measure. This led to Zac reasoning about rates of change of the vertical distance with respect to a subtended arc length, and he supported this reasoning by comparing amounts of change of vertical distance for equal changes of arc length. Zac's reasoning also consisted of indeterminate values, as opposed to numerical values. This enabled Zac to construct process conceptions of these quantitative relationships that were independent of numerical values.

An implication of Zac's understandings consisting of quantitative relationships was the emergence of mathematical representations and reasoning rooted in these relationships. Zac used mathematical notation and representations (e.g., symbolic functions, variables, and graphs) to formalize and represent quantities' values and the relationships between quantities. As a result, Zac was able to reflect on his solutions in terms of quantities and relationships between quantities. The quantitative structures he constructed provided a foundation for planning, justifying, checking, and correcting his solutions. Thus, the tasks of the teaching experiment and interview sessions appear to have resulted in Zac constructing problem situations that consisted of quantities and relationships between quantities.

Another implication of Zac's thinking and the quantitative focus of the teaching experiment sessions was the nature of Zac's orientation processes during problem

solving. Throughout the study, and particularly during the interview sessions, Zac began solving a problem by constructing a diagram and identifying quantities (with known and unknown values) and relationships between quantities on this diagram. Zac frequently referenced his diagram of the situation when anticipating calculations and interpreting values. For instance, Zac continually returned to his diagram of a situation as he solved a problem in order to refine his image of the situation (e.g., identifying determined values). In this sense, he used his diagram of the situation as a tool of reasoning to make sense of a problem's context and conceptualize the relevant quantities of a situation. This use of a diagram resulted in Zac generating solutions that were grounded in his image of the problem's context. For instance, Zac constructed an image of two covarying quantities that he leveraged to create and refine a graphical representation. The quantitative structures that Zac constructed by reasoning about the context of a problem also formed a basis for Zac checking (e.g., considering determined values relative to his image of the situation) and altering his solutions processes. Zac's checking of his solutions led to him refining his image of the situation if necessary, revealing the dynamic nature of a student's image of a problem situation.

Zac's propensity to construct, use, and reflect on diagrams of a situation also enabled coherence between unit circle and right triangle trigonometries. As Zac encountered right triangle situations, he constructed a circle using the hypotenuse of the right triangle. He subsequently reasoned about measuring legs of the right triangle relative to the hypotenuse, or radius of the circle. This enabled him to leverage his understandings constructed during the unit circle portion of the teaching experiment, such

as angle measure as an arc length and the sine and cosine functions formalizing quantitative relationships.

## Chapter 6

### Results Of Amy

This chapter provides an overview of the reasoning and problem solving behaviors Amy exhibited during the study. First, her PCA scores are provided to illustrate her pre- and post-course shift and situate her within the students from the precalculus course. This is followed by data illustrating the thinking and understandings Amy revealed over the course of the study. In addition to characterizing her reasoning, Amy's problem solving behaviors are discussed in the context of her thinking. This chapter concludes with a summary and discussion of Amy's reasoning and problem solving behaviors.

Amy was a full-time student in her late teens. She was a first-year student and an undeclared major, yet she was planning to be a registered nurse. Amy completed college algebra as a junior in high school and she did not enroll in a mathematics course during her senior year. Thus, upon entering the precalculus course she had not completed a mathematics course in one and a half years. She did not intend to take any additional mathematics courses after completing the precalculus course.

#### **Pre- and Post-Course Assessment**

Amy received a 'C' for her final course grade. In total, two students from the course received an 'A', seven students received a 'B', eight students received a 'C,' and three students received a failing grade. Amy performed below average relative to the 16 students in her class who completed both the pre- and post-administrations of the PCA exam (Table 29).

Table 29

*Results of the PCA Pre- and Post-test (n =16)*

	Zac	Amy	Judy	Class Average
Pre-test Score	13/25	5/25	15/25	7.31/25
Post-test Score	17/25	10/25	21/25	12.18/25

### **Amy's Conception of Angle Measure Prior to Instruction**

Amy's initial conception of angle measure did not include a process for determining the amount of openness between two rays. When presented<sup>21</sup> with an angle and prompted to determine its measure, Amy shaded the space between the two rays and indicated that the angle's measure represented the amount of area between the angle's rays. Then, instead of describing a measurable area, she focused on the geometry of the angle by stating, "I know one eighty's like that (*drawing a line*) and I know ninety is like that (*drawing a right angle*).” This response suggests that her conception of an angle's measure was based on the shape of a geometric object that included the space between two rays, where this conception did not include a process for quantifying the space.

Amy was then unable to measure an angle using a compass, Wikki Stix, and a ruler. When prompted to use these tools to measure her angle, she *immediately* tossed the supplies aside and claimed that she had “no guess” as to how to use the supplies to measure the angle. She then referenced the centimeters on the ruler and described that she

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<sup>21</sup> The interview tasks referenced in this chapter are presented in their full form in Appendix E.

had no idea as to how a number of centimeters related to the measure of an angle.

Consistent with her previous descriptions, Amy's comments and actions suggest that her conception of angle measure did not include a measurement process that consisted of quantifying a subtended arc length.

Table 30
<i>The Traversed Arc Problem</i>
An individual is riding a Ferris wheel that has a radius of 51 feet. On part of a trip around the Ferris wheel, the individual covers an arc-length of 32 feet. How many degrees did the individual rotate?

To conclude the pre-interview, Amy was presented with The Traversed Arc Problem (Table 30). Amy's orientation to this problem consisted of attempting to recall formulas for the radius and diameter of a circle. She first identified the radius as "fifty one feet around...radius is pi times diameter...I was just thinking diameter, is like, pi times the radius squared." As Amy recalled (incorrect) formulas, she also had difficulty identifying the "diameter" and "radius" on a circle. Amy then continued with her initial interpretation of the situation (e.g., the radius as the circumference) (Excerpt 28).

#### Excerpt 28

- 
- |   |      |   |
|---|------|---|
| 1 | KM:  | Well go ahead with the, it's fifty feet around and run with that.   |
| 2 | Amy: | Ok, am I right though? That it's fifty-one feet around? Because I'm |
| 3 |      | gonna screw up the problem if it's not fifty-one feet around.       |
| 4 | KM:  | No, you won't screw up the problem. No, go ahead.                   |
-



5        Amy:    Ok. Um, ok. Covers an arc length of thirty-two feet. So, let's say, that's  
 6                thirty-two feet, something like that. That chunk (*tracing an arc length*) is  
 7                thirty-two feet. How many degrees did the individual travel? Alright,  
 8                well a full circle, I'm just gonna try this, I have no idea if it works. I have  
 9                fifty-one feet. Which in a circle is equal to three hundred and sixty  
 10                degrees (*writing  $51ft = 360^\circ$* ). You're gonna go thirty-two feet, which  
 11                equals  $x$  degrees.

---

At the completion of this interaction, Amy had constructed  $\frac{32\text{ ft} = x}{51\text{ ft} = 360^\circ}$ , which she used as the equation  $\frac{32}{51} = \frac{x}{360}$ . Amy's explanations conveyed that this equation stemmed from a correspondence (exhibited by the use of two equal signs) between the numerators and denominators of the ratios (lines 8-11). Amy also revealed a lack of confidence (lines 2-3) in her ability to solve the problem correctly, and she suggested that she was not sure if her solution was correct (line 8). Amy subsequently determined the equation  $51x = 18720$  through "cross-multiplication" and solved for a correct number of degrees relative to her interpretation of the radius. After this interaction, Amy justified her original equation by explaining that the denominators represented the "full length...full degrees of a circle," and that the numerators represented "what was covered." She remained unable to describe the meaning of the ratios beyond this correspondence across the equality. It appears that Amy's solution approach involved her

matching part and whole measurements in the numerators and denominators opposed to reasoning about a multiplicative relationship.

To conclude the interview, the researcher directed Amy to the task of measuring an angle in order to determine if she could extend her previous solution (Excerpt 29) to this task. Amy did not attempt to use the given supplies and immediately stated that she could not complete the problem. These actions imply that Amy's conception of angle measure did not include *constructing* a circle and measuring a subtended arc in spite of her providing a correct solution to the previous task when given the measurement of an arc length. Also, her approach to the problem revealed that she was reluctant to conjecture how the tools may aid her in measuring the angle.

In summary, Amy's responses during the pre-interview suggest that she conceptualized the measure of an angle as a space (not admitting a measurement process) between two rays. This understanding did not enable her to engage in the *process* of measuring an angle when given the appropriate supplies. Then, when given an arc length, Amy relied on a part to whole correspondence and "cross-multiplying" to determine the measure of an angle. Amy was unable to extend this reasoning to the task of measuring an angle with the given supplies, and she was reluctant to attempt measuring an angle, which may have been related to a lack of confidence in her ability to solve the novel problem. Also, in spite of her correctly solving for an angle measure, Amy did not reason about an angle measure representing a percentage or fractional part of a circle subtended by two rays.

### **Amy's Ways of Thinking During the Instructional Tasks**

When engaging in The Protractor Problem (Table 31), Amy identified an arc along the protractor and claimed that this arc distinguished an area between two rays. When prompted to provide further explanation, Amy stated, “That’s just what I was taught in high school.” Thus, her construction of an arc did not appear to imply the construction of a circle or a measurable arc length. Rather, she used the arc to indicate an area and she remained unable to articulate a measurement process corresponding to the area. Also, she justified her statement by claiming that she was previously taught such an action.

Table 31
<i>The Protractor Problem</i>
Using the supplies of a Wikki Stix and a ruler, construct a protractor that measures an angle in a number of gips, where 8 gips rotate a circle.

As Amy continued solving The Protractor Problem, she referenced “a circle” when describing various angle measures, but she was unable to consistently identify a measurable attribute of a circle. For instance, Amy described one degree as a percentage “of a circle.” However, she could not clarify a measurable attribute of the circle (e.g., an area or circumference of the circle). Amy also referenced a radius as “half the circle,” a diameter as “the full circle,” and a circle’s circumference as “the full circle.” As a result of Amy combining multiple quantities as “a circle,” she had difficulty consistently identifying an arc length or circumference of a circle when reasoning about angle measure. Also, Amy exhibited frustration when the researcher prompted her to describe “a circle” in terms of a measurable attribute. She exhibited a reluctance to identify

distinct quantities of “a circle” and this led her to relying on the other students to provide a solution for the problem.

Table 32

*The Circumference Problem*

Construct a circle using a Wikki Stix as the radius (your group should have Wikki Stix of different lengths). Then, determine how many of your Wikki Stix mark off the circumference of your circle. Compare your result with your classmates. What observations can you make from this comparison? Construct an angle that cuts off one Wikki Stix length of an arc. Compare the openness of the angle with those of your classmates.

On The Circumference Problem (Table 32), Amy exhibited further frustration with the task of constructing a circle consisting of measurable attributes and using a Wikki Stix to measure the circumference. As a result of her frustration, she watched the other students complete the task and her participation consisted of restating the other students’ comments. Additionally, Amy remained reluctant to explain her meanings and calculations. For instance, she was unable to clarify her meaning of “a circle” and began asking for a “pass” when the researcher asked her to contribute or clarify her statements. As another example of Amy’s reluctance to reflect on and make sense of her actions, Amy multiplied 1.5 radians by  $2\pi$  radians to determine the percentage of a circle’s circumference cut off by 1.5 radians. When the researcher asked her to describe her calculation, she responded, “I know what you mean,” and resisted reflecting on her

calculation in terms of the quantities of the situation, which was likely a result of her calculations not being rooted in quantitative relationships.

Amy's resistance to provide conjectures and reflect on the ideas she put forward implies that she lacked confidence in her ability to solve the novel tasks. She also repeatedly watched the other students solve the problems and, as a consequence, she relied on the other students to provide correct solutions. Thus, rather than engaging in and reflecting on her own reasoning, she relied on her interpretations of the ideas and solutions put forward by the other students and the researcher.

Table 33

*The Arc Length Problem*

Given that the following angle measurement  $\theta$  is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches.

Amy's tendency to observe the other students during instructional activities possibly contributed to her reasoning lacking quantitative structures. Opposed to engaging in measurement processes and constructing mental scenes composed of quantitative relationships, she observed the other students performing numerical calculations and other observable actions. This likely inhibited Amy constructing an understanding of angle measure that consisted of quantitative relationships between an arc length, circumference, and the radius of a circle, as she did not engage in the mental processes of measuring or relating these quantities. As an example, on The Arc Length Problem (Table 33) Amy first converted a degree measurement to a number of radians

(0.61 radians) using a recalled formula. She then had difficulty reasoning about a quantitative relationship to solve the problem (Excerpt 29).

Excerpt 29

- 
- 1 Amy: Ya, wait. Do I take the percentage? No. (*looking at the researcher*) Is it  
 2 the percentage divided by the radius or opposite? No, that, no. That's not  
 3 right (*pause*). That's sixty one percent of one radian. And (*pause*). My  
 4 radius is two. Would using cosine and sine be (*pause*) helpful?
- 5 KM: Well let's see. What are you looking for right now? You're looking for...
- 6 Amy: I'm looking for, um, the length of this arc (*tracing the outside arc*  
 7 *length*).
- 8 KM: The length of that arc.
- 9 Amy: Right now I just have it in degrees, but I'm looking for like the actual  
 10 length of it.
- 11 KM: Ok. You have it in degrees, and you also found it in?
- 12 Amy: Radians.
- 13 KM: Radians right. So you said point six one represents sixty one percent of...
- 14 Amy: A single radian.
- 15 KM: A single radian, right. Do you know how long a single radian is? For this  
 16 circle (*tracing the outside circle*)?
- 17 Amy: A single radian for the whole circle? That's (*pause*)...
- 18 KM: Well, so what's it mean to be sixty one percent of a radian?
- 19 Amy: There's, the total number of radians is two pi. So, there's like six point
-

- 20                   two eight radians I think. So that's sixty one percent of one of those  
21                   radians.
- 22    KM:       Of one of those radians, ok.
- 23    Amy:       Ya.
- 24    KM:       And so what is a radian? Just...
- 25    Amy:       Um, in every circle there's, I'm not sure how to describe it. Like, it's the  
26               length around every circle. Like no matter what size the circle is. There's  
27               gonna be six point two eight radians.
- 28    KM:       Radians right. So how long is one radian? Do we know how long, let's  
29               just look at one circle right now.
- 30    Amy:       *(laughs)* Ok.
- 31    KM:       We'll just look at this outside circle.
- 32    Amy:       Alright.
- 33    KM:       Relative to that outside circle do we know how long one radian is?
- 34    Amy:       *(pause)* I actually don't think I know.
- 

Amy first referenced a percentage of “one radian” (line 3), but she was unable to determine each arc length using this value. When prompted to describe “one radian” relative to a specific circle, she explained, “six point two eight radians...in every circle,” appearing to imagine some measurement. However, she was unable to verbalize the length of a single radian (line 34) until the researcher identified the radian as “one radius.” Thus, her conception of a number of radians did not appear to entail her reasoning about a number of *radius lengths* measuring a circle’s circumference. Although

she claimed that 6.28 “radians” rotated through the circumference of the circle (which was discussed during the previous instructional activity), her actions did not indicate that she conceived of “one radian” as the *length* of the corresponding radius.

Amy’s inability to describe “one radian” as the length of the radius may have been the result of her engagement during The Circumference Problem. As previously mentioned, she relied on watching and listening to the other students complete the task, rather than engaging in and reflecting on the process of measuring along a circumference in a number of radius lengths. As a result, Amy’s conception of 6.28 “radians” was possibly a prescribed quality (not a value) of a “full circle” (e.g.,  $Circle = 2\pi$ ) rather than the result of measuring the circumference in a number of radius lengths (e.g.,  $C = 2\pi r$ ).

As a consequence of Amy’s conception of radian measures, she predominantly described radian measures as “out of” 6.28 radians, opposed to a multiplicative relationship between the measure of an arc and the radius (or circumference) of a circle. For instance, consider Amy’s response when asked to describe an angle measure of 2.3 radians relative to the length of a radius (Excerpt 30).

Excerpt 30

- 
- |   |      |   |
|---|------|---|
| 1 | KM:  | So if we look at that arc length compared to the radius, how many times |
| 2 |      | larger is that arc length compared to the radius?                       |
| 3 | Amy: | Compared to th-.  |
| 4 | KM:  | How many times larger is the arc length compared to the radius when I   |
| 5 |      | say it's two point three radians?                                       |
| 6 | Amy: | But we don't have a radius.   |
-



- 7 KM: I know we don't. So does two point...
- 8 Amy: We don't even have a how many times larger it is.
- 9 KM: Ok, so we don't have any times...
- 10 Amy: We just have two point three radians (*laughing*).
- 

When asked to compare an arc length to a radius (lines 4-5), Amy expressed, “we don’t have a radius,” further revealing that she did not conceive of a measurement given in radians implying a number of radius lengths subtended by an angle. Also, it appears Amy found difficulty reasoning about an indeterminate measure of a radius (line 6). As Amy continued, she alluded to the measurement of 2.3 radians being a part of the “full circle.” The researcher asked her to further explain this relationship (Excerpt 31).

Excerpt 31

---

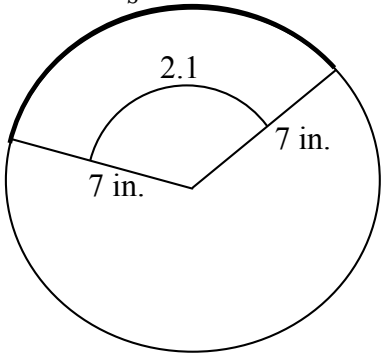
- 1 Amy: Let's say this whole circle (*drawing circle*) is two pi (*writing  $2\pi$  inside*
- 2 *the circle*). For it to have a measure of two point three radians, it's gonna
- 3 be a certain, like, portion of the circle, in terms of radians.
- 4 KM: Ok, so what, do we know what portion that's gonna be?
- 5 Amy: Two point three.
- 6 KM: So two point three's, we'd call that the portion of this?
- 7 Amy: Mm-Hm.
- 8 KM: Ok, then the two pi refers to what about the circle? What about the
- 9 circle...
- 10 Amy: The whole circumference of the circle.
-

- 11 KM: Ok, and that's two pi what?
- 12 Amy: Radians.
- 13 KM: Ok, so if we have two point three radians. Do we know what percent that
- 14 is of the entire circumference?
- 15 Amy: Um, of two point three radians, what percent it is?
- 16 KM: What percent it is of the entire circumference? Can we...
- 17 Amy: Ya. We could figure that out.
- 18 KM: Do you know how we could figure that out?
- 19 Amy: By having the radius. I think. Or we could just figure it out by dividing
- 20 by two pi. Let me see if that works (*using calculator*). I got thirty six
- 21 percent. I don't think that's right.

During this interaction Amy focused on a relationship between a measurement in radians and “the circle.” Similar to Excerpt 29, her references to “two pi” did not appear to stem from her conceptualizing a circle’s circumference as  $2\pi$  *radius lengths*. When asked to describe the portion of the circle that 2.3 radians represented, she repeated the measurement of 2.3 and did not appear to reason about a multiplicative relationship between the portion and the whole (line 5). After the researcher suggested that she find the percentage that the portion of the circle was of the entire circumference (lines 13-14), Amy suggested that she needed a specific radius length (line 19). She then performed a correct calculation to determine the fractional amount of a circle’s circumference (lines 19-21).

Despite Amy determining a correct value, she was not confident in her result. Also, she obtained this result only after continued probing from the researcher and she was subsequently unable to justify her calculation relative to a relationship between two quantities. Similar to The Circumference Problem, she was reluctant to reflect on her calculation relative to the context of the problem after this interaction even when she was told her answer was correct. Amy was able to provide correct answers at times, but it appears that her lack of confidence and unwillingness to reflect on her actions may have led to her inability to create these constructions without continued prompting from the researcher.

Due to Amy conceiving of the measurements as labels of objects rather than a quantitative relationship between an arc length and the length of the radius *or* circumference, her calculations during the angle measure activities were not driven by multiplicative relationships and she was unable to justify her calculations in terms of various measurable attributes. This was further verified when Amy subsequently described  $\pi$  radians as “half the circle,” without being able to describe a number of radius lengths rotating through an arc length. Amy also attempted to recall the location of radian measurements on a circle. For instance, she stated, “I’m trying to think, we have pi over two, pi, and then we have three pi over two down here. And then two pi here. Am I right?” During this explanation, she correctly pointed where these arc measurements are labeled on unit circle. However, when further questioned on these positions, Amy was unable to explain the measurements relative to an arc length being measured in a number of radius lengths.

Table 34	
<i>The Missing Measurement Problem</i>	
Determine the unknown linear measurement of arc-length $s$ cut off by an angle of 2.1 radians.	

In an attempt to have Amy reason about a multiplicative relationship between an arc length and the radius, the researcher presented Amy with The Missing Measurement Problem (Table 34). She first described, “I remember doing this in class.” Yet, Amy did not complete such a problem during the previous instructional activities.

After using the given angle measure to calculate the percentage of the circle’s circumference cut off by the angle (an action the researcher focused Amy on during a previous task), Amy paused for an extended period of time. She then calculated the circumference of the circle and constructed the equation  $\frac{2.1}{2\pi} = \frac{x}{43.9}$ . She followed this by using the equation  $2\pi x = 92.19$  to solve for the unknown measure and subsequently explained her solution (Excerpt 32).

Excerpt 32

1	KM:	Ok, so could you explain to me how you were able to set that up? Why
2		that worked?

3 Amy: Um, well we know that this (*tracing the arc of 2.1 radians*) is two point  
 4 one radians of the entire circle, which is two pi. And we know that the  
 5 full circle (*mimicking the shape of a circle with her pen*) is forty-three  
 6 point nine inches. But we wanna get this particular measurement in  
 7 inches (*tracing unknown arc length*).

8 KM: Ok.

9 Amy: Which is (*pointing to the 2.1 in her original equation*), is gonna be, kind  
 10 of the equivalent to two point one, but in inches.

11 KM: Ok.

12 Amy: So I just set those equal to each other and cross-multiplied.

Amy was unable to use the percentage of the circle's circumference that she calculated previous to this interaction and subsequently relied on part to whole matching (e.g., 2.1 is part of  $2\pi$  and the unknown arc is the same part of the whole circumference) to construct and solve an equation through "cross-multiplying." Thus, it appears that her constructed ratios *did not* reflect a multiplicative relationship between an arc and the circumference. Rather, the ratios reflected a correspondence between part and whole measurements, which was consistent with her inability to use the percentage she calculated.

In response to Amy's part to whole and cross-multiply procedure, the researcher prompted Amy to consider the relationship between the radius of the circle and the arc length (Excerpt 33).

## Excerpt 33

- 
- 1      KM:    If we look at two point one radians, how many times larger than one  
 2                   radius is that?
- 3      Amy:    Two point one.
- 4      KM:    Two point one right. Well what's the radius in this case?
- 5      Amy:    Seven inches.
- 6      KM:    Seven inches right. So, could we use that relationship to find out what  
 7                   that arc length should be?
- 8      Amy:    (*long pause*)
- 9      KM:    If we know this arc length (*tracing arc length*) is two point one times  
 10                  larger than one radius. Could we use that at all to help us out?
- 11     Amy:    Yes. We could just multiply seven by two point one. I like doing things  
 12                  cross-multiplying. It makes more sense to me (*laughing*).
- 

Amy correctly described a multiplicative relationship between an arc length and the radius (line 3) and identified the unknown arc length using this relationship (lines 11-12). This reasoning did not appear to be natural and she expressed that she was more comfortable with “cross-multiplying” (lines 11-12). This interaction reveals that Amy was able to reason about an arc length being so many times as large as a radius, but she did not have confidence in this reasoning compared to “cross-multiplying.”

In an attempt to have Amy continue reasoning about a multiplicative relationship between the arc length and the radius, Amy was asked to complete the next task of The Missing Measurement Problem (which consisted of a given arc length of 14.7 inches and

a radius of 7 inches) without cross-multiplying. Amy immediately determined the circumference of the circle, likely stemming from her part to whole cross-multiplication procedure. She then paused and inquired, “We want to use the same kind of formula...you want me to use that (*pointing to the previous problem*)?” With the researcher’s approval, she continued to solve the problem (Excerpt 34).

Excerpt 34

- 
- |    |      |   |
|----|------|---|
| 1  | Amy: | Um, alright. Well in this case ( <i>pointing to the previous problem</i> ) we had       |
| 2  |      | fourteen point seven inches as the angular measurement. And we have                     |
| 3  |      | seven degrees ( <i>pointing to the previous radius</i> ). So if you divided that        |
| 4  |      | you would get two point one. So ( <i>uses calculator</i> ), so I'm gonna go with        |
| 5  |      | four point zero eight radians.  |
| 6  | KM:  | Ok, so, how'd you determine that again, could you...                                    |
| 7  | Amy: | Well, I looked at this one ( <i>pointing to the previous problem</i> ), and it's        |
| 8  |      | opposite except this is the measure we're trying to find ( <i>pointing to the</i>       |
| 9  |      | <i>unknown measurement</i> ). And if we took the measure that we had                    |
| 10 |      | ( <i>underlining 14.67</i> ) and divided it by the radius ( <i>pointing to previous</i> |
| 11 |      | <i>radius</i> ), we would get the radians. So I just did that.                          |
| 12 | KM:  | So why does that give us the radians? If we take the nine point eight and               |
| 13 |      | divide it by two point four.  |
| 14 | Amy: | Because we're taking the measurement that we have and dividing it by                    |
| 15 |      | the radius. And that should give us the missing angle.                                  |
| 16 | KM:  | Why does that give us the missing angle? What's that, you know when                     |
-

- 17           take nine point eight and divide it by two point four how many of what  
 18           are finding go into what? How many times of what are we finding go  
 19           into what?
- 20    Amy:   We're finding how (*sigh*), ok, let me think about this for a second. (*long*  
 21           *pause*) Sorry, I'm gonna write this down (*writes 4.08*). We're finding,  
 22           how many times two point four goes into nine point eight. Which two  
 23           point four is one radian.
- 24    KM:    Ok.
- 25    Amy:   So that's gonna give us the total number of radians in this, that this is  
 26           (*tracing arc length*).
- 

Amy first reflected on her previous solution and described the *calculation* used to solve this problem (lines 1-5). As she provided this description, she also referred to the number of inches as an angle measure and the measure of the radius as “seven degrees.” Amy then explained that she could perform a similar calculation in order to complete the problem (lines 7-11). These explanations indicate that her solution was driven by a previous procedure (a calculation involving division) and matching the placement of the given measurements relative to this calculation. This was further illustrated when the researcher asked Amy for a quantitative meaning behind her calculation (lines 16-19). Amy correctly reasoned about the number of radius lengths composing the arc length (lines 20-26), but her hesitation and need to reflect on her solution indicated that this quantitative relationship was not *originally* driving her solution.



Just as in Excerpt 33, Excerpt 34 reveals Amy's ability to reason about measuring an arc length in a number of radii, but this reasoning was not natural or spontaneous. Also, Amy's behaviors (e.g., her sigh) imply that she was reluctant to reflect on her solution in the manner the researcher was asking. This may have been a result of her finding more value or more confidence in reasoning about a previous procedure than the quantities of the situation.

Next, Amy was presented with a third task that provided an arc length (13.19 kilometers) and the radius (3 kilometers). In spite of the researcher asking her to use similar reasoning as she exhibited towards the end of Excerpt 34, Amy calculated the circumference of the circle and used "cross-multiplying" to solve for a number of radians. The researcher then asked her to consider a relationship between the arc length and the radius (Excerpt 35).

#### Excerpt 35

- 
- |   |      |   |
|---|------|---|
| 1 | KM:  | How many times larger is thirteen point one nine than three?                        |
| 2 | Amy: | <i>(pause)</i> Than three? Hold on <i>(uses calculator)</i> , four point three nine |
| 3 |      | times larger.   |
| 4 | KM:  | Does that make sense <i>(referring to the previous answer)</i> ?                    |
| 5 | Amy: | Yes.  |
| 6 | KM:  | Why's that make sense?  |
| 7 | Amy: | Because that's the radians and that's what it gives you when you divide             |
| 8 |      | like that. I'm just so used to cross-multiplying everything.                        |
-

Amy was able to reason about a multiplicative relationship between the arc length and the radius (lines 2-3) and state that this was “the radians,” but she continued to emphasize her confidence in “cross-multiplying everything.” Also, she had previously determined the number of radians using her cross-multiplication method, but she found it necessary to calculate the multiplicative relationship between the arc length and the radius. Thus, it appears that the procedure of setting up ratios by using a part to whole correspondence dominated Amy’s reasoning. After obtaining a number of radians using this method, she did not conceptualize this value as the multiplicative relationship between an arc length and the radius. Additionally, her description of “the radians” focused on performing a calculation instead of a quantitative relationship between two quantities (lines 7-8).

Amy’s inability to leverage reasoning about measuring along an arc generated obstacles in her conceiving of an input-output process that is formalized by the trigonometric functions. For instance, Amy’s conception of angle measure as a label of an object or a part of “a circle” presented itself when Amy interpreted the equation  $\cos(\pi) = -1$  (Excerpt 36).

#### Excerpt 36

---

1	Amy: Ok, well, we have cosine pi and you get negative one. So usually when
2	you have, I mean what we've done in the past, like up here, is the
3	positive side ( <i>tracing the top half of a circle</i> ). And down here is like
4	negative ( <i>pointing to the bottom half of a circle</i> ). Like if we do it on like
5	a literal gra-, literal graph. It will, I'm going to far into it, nevermind.

---

- 6 Um, this is like plus one and this will be like negative one (*writing each*  
 7 *value by the corresponding part of the circle*). And then since pi is half a  
 8 circle (*tracing the bottom half of a circle*), when I see cosine pi, to me,  
 9 that means, like the bottom half of the circle, is what it represents.
- 10 KM: So what do we mean by...
- 11 Amy: So negative one, like radius, at the bottom half since...
- 12 KM: So the pi because it's negative now represents the bottom half.
- 13 Amy: Ya, of the circle.
- 14 KM: Ok, and pi because that's half of a circle?
- 15 Amy: Mm-Hm.
- 16 KM: And why the bottom half and not the top half?
- 17 Amy: Because the bottom half is the negative part. Which is what we've, I  
 18 mean, if it's on a coordinate, that's the negative part of the coordinate.
- 19 KM: So why is that the negative part of it? What's negative about it?
- 20 Amy: Because that's the way a coordinate map is done (*laughing*).
- 

Amy first conceived of  $\pi$  as “half a circle,” opposed to representing the measure of an arc in a number of radius lengths. Additionally, she interpreted the number  $-1$  to signify the “bottom half” of a circle rather than the “top half” of a circle. When providing her interpretation of the given equation, Amy also hesitated explaining her reasoning (e.g., “I’m going to far into it.”). After this interaction, Amy further added, “Ok, cosine of pi is just half the circle,” rather than identifying the cosine function as formalizing a relationship between two quantities. These actions imply her reasoning was based on

objects (e.g., half of a circle) opposed to a relationship between measurable attributes of a circle (e.g., an arc length and a horizontal position).

Table 35

*Amy's Ferris Wheel Problem*

Consider a Ferris wheel with a radius of 36 feet. April boards the Ferris wheel at the 3 o'clock position and begins a continuous ride on the Ferris wheel. Sketch a graph that relates the total distance traveled by April and her vertical distance above the horizontal diameter of the Ferris wheel.

Consistent with Amy's conception of "a circle" and her inability to reason about the equation  $\cos(\pi) = -1$  in terms of conveying a relationship between two quantities' values, Amy had difficulty constructing and covarying distinct quantities when reasoning about circular motion. To illustrate her difficulties, consider Amy's actions on Amy's Ferris Wheel Problem (Table 35). Amy first expressed a lack of confidence in her ability to solve the task and uttered, "Oh, I hate the graphs." Amy then oriented to the problem by constructing a circle and identifying April's (the rider) starting position (Excerpt 37).

## Excerpt 37

- 
- |   |   |
|---|---|
| 1 | Amy: The second she gets on the Ferris wheel, she's already thirty six feet             |
| 2 | above ground...So your vertical distance from the origin ( <i>marking the</i>           |
| 3 | <i>center of the circle</i> ) is ( <i>drawing segment to the bottom of the Ferris</i>   |
| 4 | <i>wheel</i> ) thirty six feet ( <i>drawing segment from the origin to the starting</i> |
| 5 | <i>position of April</i> ). Right? Is that right or wrong?                              |
-

- 6 KM: So how are you interpreting vertical distance and total distance?
- 7 Amy: Well you said it's from the origin (*marking the center of the circle*).
- 8 KM: Vertical distance from the origin.
- 9 Amy: Ya, so.
- 10 KM: And where's she starting?
- 11 Amy: She's starting here (*identifying starting position of April*). So technically
- 12 is that our origin? Where you start?
- 

Amy identified *both* a horizontal and vertical segment corresponding to April's "vertical distance from the origin," and (correctly) described each segment as a length of 36 feet (lines 1-5). However, neither segment was consistent with the researcher's intentions of vertical distance, nor did Amy appear to establish a vertical distance. Amy also identified multiple "origin[s]" and her questioning conveyed that she did not identify a unique reference point for the vertical distance (lines 11-12). Throughout this interaction, Amy looked to the researcher for approval of her explanations rather than relying on her own reasoning (lines 5, 7, and 11-12).

After Amy and the researcher discussed a reference point (e.g., a horizontal diameter) for measuring the vertical distance, Amy articulated, "Total, I'm not sure. Just would that be like circumference? Or, total distance as she gradually goes around the circle?" In this case, Amy was confused about the meaning of "total distance," and her explanation conveys that she held the conflicting images of the total circumference and a varying arc length along the circle.

A varying arc length was then established as the “total distance” April had traveled and Amy began to construct a graph of the vertical distance in terms of total distance. Amy once more expressed that the meaning of vertical distance was unclear and she did not attempt to use the diagram to reconcile this lack of clarity. After the researcher used the diagram to identify a specific vertical distance (e.g., from the top of the Ferris wheel to the horizontal diameter), Amy asked, “So vertical distance, it doesn't have to drop down right, it could go up? I just want to make sure it's not like dropping a ball off a building kind of problem.” As she made this statement, she traced the vertical segment both upwards and downwards, continuing to reveal her difficulty conceiving of the relevant quantities. Her reference to a ball situation also implies that she was attempting to relate this situation to a previous problem. The researcher then asked her to identify the length of the vertical segment, to which Amy responded, “Ok, well that's also  $\pi$  over two. That's (*tracing a vertical segment*) 'cause, well, no it's one radius technically. So it's not  $\pi$  over two, it's one.” In this case, Amy referred to the length using a previously discussed angle measure, further exhibiting her tendency to confuse various attributes and measurements. However, she correctly concluded that the length was “one radius.”

Amy subsequently constructed a graph that perceptually resembled the sine function. Yet, Amy was unable to justify the shape of her graph by coordinating two related quantities. Also, she did not attempt to use her diagram to justify her graph. In an attempt to promote Amy constructing and reasoning about quantities, the researcher asked that Amy identify equal changes of arc length on her diagram. Amy completed this

for the first quadrant and (correctly) claimed that the vertical distance was increasing and the change in vertical distance was decreasing over this interval. She was then asked to illustrate the covariation of the two quantities on her diagram (Excerpt 38).

Excerpt 38

- 
- |   |      |   |
|---|------|---|
| 1 | KM:  | So what's representing the change?  |
| 2 | Amy: | This is the change ( <i>shades in areas</i> ).                                      |
| 3 | KM:  | So that area or just that height? 'Cause you're shading the whole thing.            |
| 4 | Amy: | Ok ( <i>pause</i> ). Just the height.   |
| 5 | KM:  | Just the height.  |
| 6 | Amy: | Just the height, ya.  |
| 7 | KM:  | Just that height.   |
| 8 | Amy: | Well I like shading 'em ( <i>continuing to shade areas</i> ). And see it gets less. |
- 

Amy's completed diagram can be found in Figure 25. After shading an area (line 2), Amy hesitated when asked to clarify the change in vertical distance on her diagram (line 4). She then continued to shade in areas and claimed, "see it gets less" (line 8). These actions and explanations, along with the researcher's leading questions, made it unclear whether she had distinguished the change in vertical distance from the areas she identified.



Figure 25. Amy's diagram of Amy's Ferris Wheel Problem.

Amy then explained how her graph reflected the phenomenon she identified on the diagram. Amy first traced the graph while describing, “Ya, whoo, I’m increasing a lot and I’m slowin’ down.” This action implies that Amy reasoned about the shape of the graph to describe the motion of the individual (e.g., less steep means slower). This shape reasoning may explain Amy’s ability to *describe* directional changes and amounts of change of vertical distance while being unable to *illustrate* the relevant quantities on a diagram (Excerpt 38). To gain further insights into her reasoning, Amy was asked to explain how her graph illustrated the covariational relationship of the two quantities (Excerpt 39).

#### Excerpt 39

- 
- |   |   |
|---|---|
| 1 | Amy: Ya, I can do it like, ya, here's one, two, three, four, five ( <i>making marks</i> |
| 2 | <i>along the graph</i> ). And, well, hold on ( <i>drawing vertical lines down from</i>  |
| 3 | <i>her graph</i> ). And, change, change, change, change ( <i>drawing horizontal</i>     |
| 4 | <i>lines between the vertical lines</i> ). See, it's bigger. Like here and here, than   |
-



5 right here (*shading in areas*).

Amy first created marks along the arc of the graph (Figure 26), opposed to the horizontal axis (lines 1-2), which may have been a result of her mimicking her actions from the diagram and focusing on the shape of the graph. Then, when identifying a “change,” she drew a horizontal line and shaded an area (lines 3-5). These actions were consistent with those exhibited by Amy when using the diagram (Figure 25 and Excerpt 38). Following this interaction, the researcher prompted Amy to consider the increments she made along the graph. Amy then stated that she should have placed her initial increments along the horizontal axis of the graph, but she had difficulty explaining this action in terms of a change in a quantity’s value.

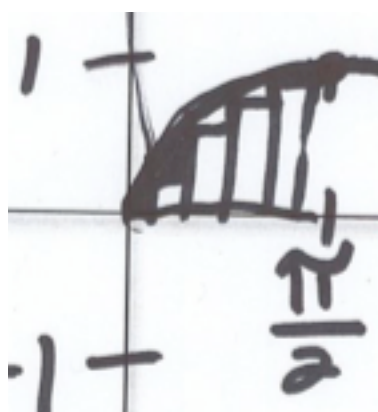


Figure 26. Amy’s graph on Amy’s Ferris Wheel Problem.

Amy’s actions of inconsistently identifying areas and lengths resulted in the researcher prompting her to conduct a similar process for the second quarter of a revolution (Excerpt 40) with the intention of gaining additional insights into her conception of the situation.

## Excerpt 40

- 
- 1      Amy:    Here's one point, here's another point (*marking equal increments along*  
2                    *a horizontal radius and extending vertical segments from the horizontal*  
3                    *radius to the circle*). Equal increments (*singing and marking increments*  
4                    *along the circle*). And there's one. And so we have our change here, our  
5                    change here, our change here, our change here (*drawing horizontal*  
6                    *segments between the vertical segments*). So at first it's like not  
7                    decreasing a lot (*shading areas*). And then it gradually decreases,  
8                    dramatically (*tracing graph*) until we get.
- 9      KM:    And now when you were marking off these equal this...
- 10     Amy:    I decided to do down there (*referring to marks on horizontal radius*).
- 11     KM:    Should you mark them off down there or along the arc? Where should  
12                    you mark those off at?
- 13     Amy:    Hmmm. This one might be along the arc.
- 

Amy first marked “equal increments” along a horizontal radius (lines 1-3), which was possibly a result of previously identifying equal increments on the horizontal axis of the graph. She subsequently identified successive arc lengths from the 12 o'clock position to the 6 o'clock position, where the end of each arc length corresponded to each vertical segment she drew (lines 3-4). Amy also referred to horizontal segments as changes and shaded in areas when describing the change as decreasing (lines 4-8). After the researcher further questioned Amy, she had difficulty describing the initial “equal increments” relative to a measurable attribute of the situation (line 13). These

descriptions reveal that Amy's image of the situation did not include distinguishing and relating quantities consistent with the researcher's intentions. Also, at this point in the task, her tone and approach to the task imply she did not value this reasoning.

Amy's inability to illustrate the covarying quantities may have a result of her repeating actions that were not rooted in reasoning about quantities. Amy appears to have focused on sequences of various behaviors (e.g., segment a horizontal axis, draw vertical lines, draw horizontal lines) without these actions consisting of quantities (e.g., attributes admitting a measurement process) beyond a gross quantification of area.

The researcher subsequently aided Amy's identification of the appropriate quantities on her diagram and asked Amy to covary these quantities over the third quarter of a revolution. Amy used her diagram to construct equal changes of arc length while also constructing vertical segments (Figure 26). Next, she drew horizontal lines and claimed, "I drew something wrong, what am I doing?" After concluding "[I] did it right," and shading areas, Amy verbalized, "So you've got like a lot of decreasing going on and then slowly it kind of bottoms out." Amy could not clarify her meaning of "it," and remained unable to identify a change of vertical distance on her diagram independent of the shaded areas and horizontal segments. Amy's descriptions reveal that she was reasoning about the shape of the circle (e.g., "bottoms out") and previous behaviors (e.g., drawing horizontal lines), rather than reasoning about amounts of change of the relevant quantities.

The researcher then identified two vertical distances and asked Amy to illustrate the change of vertical distance for the change of arc length. After pausing for an extended

period of time, she responded, “It’s (*tracing a horizontal segment*), ya, no, it’s this (*tracing a vertical segment*). It’s right here.” Consistent with previous descriptions, she identified both a vertical and horizontal segment revealing that she had not previously constructed an image of the situation that included changes of vertical distance (from the observer’s perspective), even though her initial verbal descriptions implied otherwise.

After verifying the change in vertical distance, Amy was asked to explain the covariation of the quantities for the last quarter of a revolution, with the researcher emphasizing that she be explicit about the quantities of the situation (Excerpt 41).

Excerpt 41

- 
- |    |      |   |
|----|------|---|
| 1  | KM:  | Ok, so how about the last section, do the same thing. Now I don't want                  |
| 2  |      | you to use words like it, or whatever, I want you talk about total distance             |
| 3  |      | traveled and vertical distance this time.   |
| 4  | Amy: | Ok ( <i>marking equal changes of arc length</i> ). So, we've got our fun little         |
| 5  |      | increments of a certain radius. And these bad boys ( <i>drawing horizontal</i>          |
| 6  |      | <i>segments</i> ) are starting out really small ( <i>shading areas</i> ), and gradually |
| 7  |      | they are increasing.  |
| 8  | KM:  | So what do you mean by they are increasing?   |
| 9  | Amy: | The increments.   |
| 10 | KM:  | What do those represent?  |
| 11 | Amy: | The percentage of a radius increments.  |
| 12 | KM:  | Are...  |
| 13 | Amy: | Are increasing.   |
-

- 14 KM: Ok, and those represent changes in...
- 15 Amy: Vertical distance (*laughing*). And they're starting out small and then
- 16 they're, the changes (*pointing to the areas*) between the distances of
- 17 vertical distances are changing.
- 

Amy was unable to clearly articulate or illustrate changes of vertical distance (lines 4-11) for successive changes of arc length and she did not identify a change of vertical distance on her diagram. Rather, she referenced “bad boys,” drew horizontal segments, shaded areas, and described “increments...percentage of a radius.” These explanations further convey that her actions were not based on covarying changes of vertical distance and changes of arc length that were distinct from other attributes of the situation. Also, Amy’s tone of voice during this interaction suggested that she did not find value in being specific about the quantities of the situation and that she found the researcher’s questions unnecessary or annoying. Her disposition may have been a result of her difficulty engaging in the desired reasoning or not finding value in this reasoning (e.g., constructing and reasoning about quantities).

Amy’s actions during The Ferris Wheel Problem emphasize that a student’s verbal utterances are not necessarily indicators of the reasoning behind these utterances. Amy described changes of vertical distances and arc lengths, but she was unable to consistently identify the relevant quantities on a graph or diagram of the situation. She instead reasoned about the shape of both a circle and a graph when describing the covariational relationship. Also, her explanations and behaviors appeared to mimic her previous behaviors opposed to resting on reasoning about distinct quantities and

relationships between these quantities. Thus, as she did not conceive of the relevant quantities of the situation, she was unable to covary these quantities in order to construct a graph that represented a covariational relationship between two measurable attributes.

Over the course of the study, the researcher attempted to promote Amy constructing measurable quantities consistent with the instructional intentions. For instance, as Amy encountered difficulty solving various tasks, the researcher continually asked Amy to identify lengths and reference points for measuring various lengths. As another example, after Amy verbalized that she did not understand the notion of vertical distance in circular motion, the researcher prompted Amy to discuss and relate measuring the elevation of various cities relative to sea level to measuring a varying vertical height.

During a majority of the attempts to promote Amy constructing images of situations consistent with the instructional intentions, the nature (both in tone and verbiage) of her responses indicated that she did not find value in making these distinctions. In fact, she claimed that the requests made things “more difficult than [they] should be.” To illustrate her difficulty and resistance in identifying distinct quantities, consider Amy’s responses when asked to discuss the sine function (Excerpt 42).

Excerpt 42

---

1	KM:	For the input to sine, the sine function. What are we measuring and what
2		units? For the input to the sine function.
3	Amy:	We're measuring the percentage of a radius.
4	KM:	And what are we measuring? Do you see the distinction there, you're
5		just saying, so I can say I'm measuring in feet, but I'm not telling you

---

- 6                    what I'm measuring. If I just come to you and say I'm measuring in feet,  
7                    you have no clue what I'm measuring right?
- 8     Amy:    (*laughing*) Right.
- 9     KM:     So when you say I'm measuring in a percentage of a radius, you're not  
10               telling me, you're telling me the units you're measuring in, but you're not  
11               telling me what you're measuring. So what are you measuring in  
12               percentage of a radius.
- 13    Amy:    The vertical distance.
- 14    KM:     Ok, is that the input or output to sine?
- 15    Amy:    That's (*pause*), well, that's the output.
- 16    KM:     That's the output, right. What's the input to sine?
- 17    Amy:    (*pause*) Would it be the radians?
- 18    KM:     Ok, that's the units. Now what are you measuring?
- 19    Amy:    (*pause*) (*sigh*) Percentage of a radian.
- 20    KM:     That's still a unit.
- 21    Amy:    (*laughing*) This is killing me.
- 22    KM:     When you're saying radians or whatever, what are you measuring?
- 23    Amy:    (*long pause*) (*sigh*) I don't know. I'm not even sure.
- 

When asked for the quantity she was measuring Amy described a measurement unit (lines 3, 17, and 19). This possibly stemmed from the explicit focus of the teaching experiment sessions on the radius as a unit of measure, leading Amy to conceive of a “percentage of a radian” or “radians” as numbers to be determined rather than the results

of measuring a distance along an arc or segment. Hence, she did not conceive of measurable attributes of a situation distinct from “the radians.” Amy’s tone of voice and casual approach during this interaction also implies that she did not take this distinction seriously. Amy’s disposition may have been a combination of her having difficulty making this cognitive distinction and the obstacles arising due to her inability to make this distinction (e.g., her disposition was a defense mechanism). Immediately following this interaction, she further expressed her struggles by stating, “This is more difficult than it should be.”

As Amy expressed, she had difficulty constructing distinct quantities and reasoning about these quantities throughout the study. Amy frequently reasoned about measurements as references to objects opposed to conveying a quantitative relationship or the result of a measuring *process*. Although she often performed correct *calculations* and provided correct verbal responses at various times, these calculations and responses appeared rooted in recalling procedures rather than quantitative relationships. Then, when prompted to reflect on her solutions relative to a problem’s context, she exhibited discomfort and remained reluctant to engage in such reflection, which inhibited her progress over the course of the study and considering other ways of thinking about the instructional topics. For instance, she expressed that she would better understand the sine function if she knew the formula to calculate the output value. The next section further explores the implications of Amy’s procedural approach to problem solving relative to her understandings of angle measure and trigonometric functions.

### **The Role of Amy’s Problem Solving Behaviors**



As previously illustrated (Excerpts 28, 32, 33, and 35), Amy frequently relied on executing cross-multiplication, which was supported by her reasoning about part to whole correspondences that she identified when orienting to a problem. In order to illustrate Amy's problem solving behaviors in the context of the Multidimensional Problem Solving Framework (e.g., orienting, planning, executing, and checking) provided by Carlson and Bloom (2005), this section begins by discussing her propensity to execute cross-multiplication.

As a first example of Amy's reliance on executing cross-multiplication, consider her actions when she was asked to determine the circumference of a circle given that an arc length of 0.03 inches was 22% of the circle's circumference (Excerpt 43).

#### Excerpt 43

- 
- |    |      |  |
|----|------|--|
| 1  | Amy: | Given that an arc-length of point zero three inches is twenty two percent                          |
| 2  |      | of a circle's circumference, what is the circle's circumference? Alright,                          |
| 3  |      | so I've got an arc length that is point zero three inches. A circle's                              |
| 4  |      | circumference is pi times diameter. Isn't that pi times the diameter?                              |
| 5  | KM:  | Pi times the diameter.   |
| 6  | Amy: | ( <i>writing <math>\pi d</math></i> ) And then, ok, all we have it point three. So we've got point |
| 7  |      | three and the whole circle is pi times the diameter ( <i>writing .03 above</i>                     |
| 8  |      | <i><math>\pi d</math></i> ). Of the circle's circumference. Hmmm. This would be better if I had    |
| 9  |      | a radius. Ok, it's twenty-two percent of the circumference. Twenty-two                             |
| 10 |      | is the result of one hundred ( <i>writing the ratio of 22 to 100</i> ). ( <i>calculating</i>       |
| 11 |      | <i>100 times .03</i> ) I don't know if this is right, I'm just gonna give it a shot. I             |
-

- 12                    get three. Oh, I need an  $x$  somewhere. Hmm.
- 13    KM:        So what'd you do there? You did...
- 14    Amy:       Ya, well I was gonna try and like cross-multiply and everything. But I...
- 15    KM:        So what do you mean by you need an  $x$ ? What are you referring to?
- 16    Amy:       I need like a, something that I'm gonna solve. Which would be the rest,
- 17                   which would be the whole circumference. So, ya, it would be like point
- 18                   three over  $x$  if we wanted to find it in inches (*replacing  $\pi d$  with  $x$* ). So it
- 19                   would be equals 3 (*writing  $22x=3$* ) and that doesn't make sense.
- 20    KM:        So what doesn't...
- 21    Amy:        The whole circle's point one three inches. I mean, the circumference. I
- 22                   don't know if that makes sense.
- 23    KM:        So what do you think? How long was your arc length?
- 24    Amy:        My arc length was point zero three, I just don't like uneven stuff.

Amy's initial orienting behaviors consisted of recalling a formula for the circumference of the circle (lines 3-4) and expressing the need of a radius (lines 8-9) to determine the whole (e.g., the circumference) of a part to whole relationship. After using a part to whole correspondence to construct an equation (lines 6-10), Amy claimed that she needed an " $x$ ...something that I'm gonna solve" before executing her procedure. Her explanations conveyed that her use of the variable  $x$  initially stemmed from representing the measure of a quantity. Rather, her use of  $x$  was to satisfy the need of having an unknown part or whole (e.g., "I need an  $x$  somewhere") in order to "cross-multiply and everything." Lastly, Amy repeatedly stressed that she was not confident in her solution

(lines 4, 11, and 19) and her checking of the answer consisted of not liking “uneven stuff” rather than reflecting on the solution or the quantities of the situation.

These actions reveal Amy’s reasoning was focused on executing the procedure of cross-multiplying, where her efforts consisted of constructing an equation of a form she relied on part to whole matching. Also, the ratios Amy constructed did not have a quantitative meaning beyond this part to whole correspondence. As a result, Amy did not conceive of the ratios as values, which inhibited her ability to check the problem beyond the aesthetics of the result (e.g., “uneven stuff”). As the researcher continued to prompt Amy to explain her reasoning, she was unable to explain her solution beyond alluding to cross-multiplication as the prescribed procedure for the given problem. Then, after the researcher’s probing led her to use the fractional amount of the circle’s circumference without cross-multiplying, she claimed, “I like doing cross-multiplying,” apparently finding confidence and comfort in executing this procedure, which was consistent with her actions in Excerpts 33 and 35.

Amy’s reliance on and confidence in executing cross-multiplication influenced her reasoning and problem solving behaviors throughout the study. For instance, Amy was asked to determine (without cross-multiplying) the measure of an angle that subtended 12% of a circle’s circumference. After Amy found 12% of 360 degrees by using the operation of  $0.12 \cdot 360$ , she explained, “Ya, we could do that, but I trust myself with cross-multiplying more.” Also, after obtaining a proper angle measure using a method other than cross-multiplication, she started to construct an equation to execute cross-multiply in spite of already obtaining the correct answer. The researcher then stated

that Amy should trust her reasoning, to which she responded, “There's no reason to mess up cross-multiplying. It's safer.” These responses illustrate her “trust” in executing cross-multiplying, opposed to being confident in her reasoning about the quantities and relationships of the situation. In turn, her view of cross-multiplication as a prescribed procedure that gave a correct solution became an obstacle for her considering other solutions to problems.

As previously revealed (Excerpts 33-35), Amy was able to reason about a multiplicative relationship between an arc length and the radius. However, after engaging in this reasoning, she explained, “I’m just so used to cross-multiplying everything.” Thus, cross-multiplying dominated her reasoning such that she did not appear to value or have confidence in engaging in novel reasoning patterns or reflecting upon this reasoning.

In addition to Amy’s propensity to execute a cross-multiplication procedure, she frequently attempted to “remember” previously executing procedures and calculations. These actions dominated her orienting behaviors when encountering novel tasks. As an illustration, consider Amy’s initial actions on The Arc Problem (Table 33) (Excerpt 44).

#### Excerpt 44

---

1	Amy:	I'm trying to figure out theta, ok. I remember doing this, I'm just trying
2		to remember how. Ok. Radius. Hmm, and the angle measure's thirty-five
3		degrees. And I'm trying to determine the length ( <i>tracing arc length</i> ), like
4		in inches?
5	KM:	Ya.
6	Amy:	Ok. Alright, let's see. Um. Two inches. And it's thirty-five degrees out of

---

- 7                   three hundred sixty degrees (*writing ratio of 35 to 360*). Hmmm. If two  
 8                   inches (*pause*), no. Hmm, I swear I know how to do this, I just can't  
 9                   remember.
- 10     KM:     Ok.
- 11     Amy:    (*long pause*) (*sigh*) Ok. (*long pause*) Hmm. I'm trying to determine the  
 12               length of the arc. Oh I promise I know how to do this. I can't remember  
 13               though (*long pause*).
- 

Throughout the interaction Amy suggested that she *knew* “how to do” the problem, but that she could not *remember* how to complete the problem (1-9). As the interaction continued, Amy was unable to recall a previous, or prescribed, solution process and stressed that she could not “remember how” to solve the problem (lines 11-13). Amy’s approach to the problem suggests that she equated knowledge and solving the task to remembering a prescribed solution. As a result, Amy then stated she could not solve the problem, opposed to attempting to provide a conjecture or reason about the quantities of the situation (e.g., orienting and planning behaviors).

Amy’s actions throughout the study suggested that she viewed the proposed problems as if a prescribed solution existed for her to “remember” and execute. For instance, on The Ferris Wheel Problem (Table 35), Amy initially stated, “Oh I knew you were gonna ask me this, and I don’t know the answer.” Amy’s response implies that she believed the researcher expected her to recall an answer (a graph) from memory opposed to constructing a graph by reasoning about a covariational relationship. As another example, Amy’s orientation to a right triangle problem consisted of her claiming, “I

know how to do this, I did this last night...I just did this...give me the first step and maybe I can figure it out from there...and we just learned this.” Again, her claims of “I know how to do this” and “give me the first step” emphasize her belief that she was expected to recall a post procedure when orienting to the problem.

As yet another illustration of Amy’s propensity to remember and execute a prescribed procedure or calculation devoid of a quantitative meaning, consider Amy’s attempt to determine the hypotenuse of a right triangle when given an angle measure and the length of the side (670 feet) opposite of the angle measure (Excerpt 45).

Excerpt 45

- 
- |    |   |
|----|---|
| 1  | Amy: Um ( <i>pause</i> ). Doesn't six hundred and seventy divided by this side      |
| 2  | ( <i>pointing to the side adjacent the given angle</i> ) give us the radius? Right? |
| 3  | KM: What do you think?  |
| 4  | Amy: Well I know it does, but.  |
| 5  | KM: Six hundred and seventy divided by what side?                                   |
| 6  | Amy: Right here ( <i>pointing to the side adjacent the given angle</i> ).           |
| 7  | KM: So why do you think that gives us the radius?                                   |
| 8  | Amy: I know it does.  |
| 9  | KM: So why do you know it does?   |
| 10 | Amy: 'Cause that's what is in my notes.   |
- 

After identifying a right triangle and an unknown value, Amy recalled a calculation (lines 1-2) and looked to the researcher for approval. When asked to justify her conjecture, she claimed it was correct because the calculation was in her notes (lines

8-10). In this case, Amy's conjectures focused on calculations not rooted in quantitative relationships, which likely limited her ability to check the correctness of her conjectures. An explanation for her suggesting the calculation of dividing two sides was that the previous instructional activity explored the tangent function within the context of right triangles. Thus, it is likely that Amy recalled the *calculation* of dividing the opposite length by the adjacent length to obtain an answer in a right triangle context.

Amy's reliance on remembering formulas and procedures and executing calculations presented her further difficulties when attempting to reason about the sine function. She claimed, "I don't know exactly what sine does, like I'm sure if I knew what the formula was I'd have more of an understanding what happens." Amy's explanation reveals that she found remembering formulas as "understanding." As a result, Amy felt a need to remember a sequence of calculations in order to understand the sine function, rather than reasoning about the sine function as a process relationship between two quantities

Although the researcher attempted to promote Amy making sense of problem situations (e.g., identifying quantities) and reflecting on her reasoning, Amy was *very* reluctant to participate in such actions and expressed extreme discomfort and a lack of confidence during these instances. Her discomfort was such that during an instructional activity relating right triangle trigonometry to the unit circle, Amy exclaimed in a stern tone "I'm just here to learn...I do not know how to do this so that is why I'm here learning." Amy's response occurred when the researcher asked her to identify the output of the sine function relative to the unit circle, which was a topic of the previous

instructional activities. The nature of Amy's response, in combination with her tendency to watch the other students and the researcher, indicates that she approached learning as watching someone else complete a problem correctly rather than engaging in novel reasoning. Similarly, when the other two students were engaged in problem solving, Amy frequently left the classroom; then, upon her return, she would ask if they had completed the problem and desire the correct answer.

To further illustrate the implications of Amy's disposition to learning mathematics and problem solving, recall that during The Ferris Wheel Problem (Table 35) Amy's initial response was that she did not remember the graph and that her conception of the situation lacked an explicit distinction of quantities. This limited her ability to construct a graph rooted in the covariation of quantities; specifically, she had difficulty supporting the concavity of her graph through reasoning about amounts of change (Excerpts 38-41). Amy's actions when asked to describe the sine function on a later task offered insights into her difficulty in constructing and reasoning about explicit quantities (Excerpt 46).

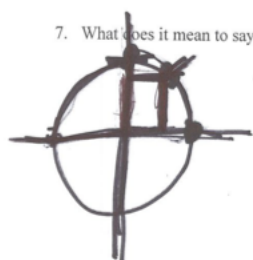
#### Excerpt 46

- 
- |   |      |  |
|---|------|--|
| 1 | Amy: | Sine is the vertical distance. Um, ya.   |
| 2 | KM:  | Can you talk to me, vertical distance from where? You know talk to me                  |
| 3 |      | about what vertical distance you're talking about.                                     |
| 4 | Amy: | Usually we've used like sine and cosine of a circle. So when you have                  |
| 5 |      | like a certain point, it's the vertical distance on that circle, like ( <i>pause</i> ) |
| 6 |      | where it is in terms of the vertical distance.   |
-



- 7 KM: Should we use a diagram maybe? Show me what you're talking about.
- 8 Amy: (*draws a circle with a crosshair*) I've got a point right here (*marking a*  
9 *point on the circle*), it's (*pause*), the distance, like (*tracing a vertical*  
10 *distance to the horizontal diameter*), this distance (*tracing a horizontal*  
11 *distance along horizontal diameter*) right there (*retracing vertical*  
12 *distance*).
- 13 KM: So which distance is the vertical distance?
- 14 Amy: Oh it's so mixed up, um. It screws me up, because you need like the  
15 horizontal distance to get the vertical distance.
- 16 KM: So why do you say that?
- 17 Amy: (*pause*) Because it's a screwed up world. It just, it, like I know what it  
18 means, it just confuses me like in general. 'Cause it doesn't mean what it  
19 says.
- 20 KM: So what do you mean it doesn't mean what it says?
- 21 Amy: Well it says, you know, they're telling me sine's the vertical distance of  
22 something, but I need to find the horizontal distance to find the vertical  
23 distance. That doesn't make any kind of sense.
- 24 KM: So why do you say, what make's you say you need to find the horizontal  
25 distance...
- 26 Amy: 'Cause that's what you told me. (*laughing*) Like you have to find, it's  
27 weird because it looks like you're finding the horizontal distance to get  
28 to the vertical distance.
-

Amy expressed that the sine function represented a vertical distance (line 1). As she was pressed to explain further, she did not use a diagram until the researcher suggested this action. Amy attempted to use the diagram to identify this distance (Figure 27), which resulted in her tracing both horizontal and vertical distances (lines 8-12). She described that the various distances confused her (lines 14-23) and that she needed “to find the horizontal distance to find the vertical distance.” Then, she claimed, “that’s what you told me...it looks like you’re finding the horizontal distance to get the vertical distance.”



*Figure 27.* Amy’s diagram for describing the sine function.

Amy’s difficulties appear to have stemmed from observing the actions of the researcher and other students during the previous instructional activities. During these sessions, a coordinate system was used to measure the horizontal and vertical positions of a point on a circle relative to the origin of the circle. As the researcher and the other students used this coordinate system, they often traced horizontally or vertically to the coordinate axes to determine the appropriate vertical or horizontal positions, respectively. Amy appears to have interpreted these observable behaviors as “finding the horizontal distance to get to the vertical distance,” resulting in her confusing the two quantities.

These actions were consistent with Amy's actions during The Ferris Wheel Problem. As she progressed through The Ferris Wheel Problem (Excerpts 37-41), she was observed identifying multiple segments and an area as changes and she frequently performed actions that she performed for a previous quarter of a revolution. For instance, after identifying "equal increments" along the horizontal axis of a graph, Amy identified equal lengths along a horizontal radius on the Ferris wheel. Amy's repeated actions were not grounded in the relevant quantities of the situation. Similarly, Amy's actions on other tasks that included calculations and procedures were not the result of her reasoning about relationships between quantities. Although she was able to provide correct solutions at times, her resistance to reflect on the quantitative meaning of her procedures resulted in her inability to flexibly solve novel situations without continued prompting from the researcher.

In summary, Amy's problem solving behaviors consisted of attempting to remember a procedure or formula when orienting to a problem and then executing calculations defined by the procedure or formula she remembers. Amy did not appear to construct mental scenes of problem situations that consisted of quantities and relationships between quantities. This may have resulted in Amy's inability to engage in meaningful planning or checking behaviors, as she did not construct quantitative structures with which she could anticipate solutions or interpret her solutions. Rather, remembering a prescribed procedure did not require any planning or checking (beyond the aesthetics of the solution), as she believed the procedure provided a correct solution if applied correctly.

Amy was also reluctant to engage in planning actions such as providing conjectures and considering various avenues of solving a problem without executing calculations. She often made statements such as, “I’m just trying something, don’t ask me what I’m doing. I just want to see if it works,” when she performed calculations and would not consider the meaning of her calculations previous to executing them. Also, when she was asked to explain a quantitative meaning of her solutions, she often made statements like, “This is why I’m not good at this.” Amy also attempted to hide the computations she performed on her calculator and she would clear her calculator when asked for her solutions. These actions imply that Amy lacked confidence in her own mathematical abilities and that her focus was on obtaining correct results opposed to understanding the solution used to obtain these results.

The combination of Amy’s procedural approach to problem solving and lack of confidence in her own mathematical abilities parallels the nature of Amy’s thinking during the study. As illustrated, rather than constructing problem contexts consisting of quantitative structures that she could leverage and reflect upon, her understandings consisted (or prescribed) procedures and calculations, which she attempted to recall when orienting to a problem. These procedures and calculations were not rooted in quantitative relationships, and as a result, her ability to check her solutions relied on the aesthetics of her solution and her confidence in the remembered procedure. For instance, during problems in which she cross-multiplied, Amy remained confident in her solutions. However, in other contexts that did not lend themselves to cross-multiplying or in situations that she was asked to not cross-multiply, she had difficulty engaging in the

reasoning abilities needed to solve the tasks. Also, opposed to engaging in and reflecting on novel reasoning, she cited her notes, provided statements such as, “That’s what I thought we were supposed to do,” and relied on previous observable behaviors (her own behaviors or her interpretation of others behaviors). Hence, her understandings and problem solving behaviors centered on these observable behaviors and numerical calculations that the other students and researcher provided.

### **Summary and Discussion of Amy**

Amy’s solutions frequently revealed that she did not construct and reason about the relevant quantities of a problem’s context. As opposed to conceiving of distinct quantities, Amy often confused quantities, which resulted in her reasoning about inconsistent attributes of a situation. Amy also conceived of measurements as labels for an object, as opposed to measurements representing quantitative relationships and the result of a measurement process. For example, Amy described that “six point two eight” radians were in a circle, but she had difficulty reasoning about measuring the circumference of a circle in a number of *radius lengths* (e.g.,  $C = 2\pi r$ ). Rather, she conceived of the “six point two eight” radians as a “full circle.” This image enabled her to use a part to whole correspondence that supported a cross-multiplication procedure, but this image did not support her reasoning about measuring along the circumference of a circle in a unit length.

Amy’s inability to construct and reason about distinct quantities also inhibited her reasoning about the relationships formalized by the sine and cosine functions. Amy was able to recall a graph of the sine function, but she was unable to support the graph by

reasoning about two covarying *quantities*. Instead, she fluctuated between referencing areas and identifying inconsistent lengths and changes in lengths. Also, she emphasized that she would better understand the sine and cosine functions if she was given a formula to calculate values.

Amy's difficulties and inability to construct quantitative relationships were directly related to her procedural, or formulaic, disposition to mathematics and problem solving (which possibly stemmed from a lack of confidence in her abilities). When orienting to problem situations, she predominantly focused on attempting to remember calculations and procedures, while also having difficulty progressing if she was unable to recall the proper procedure or calculation. Also, when she justified her solutions she often fixated on previous calculations or behaviors, rather than a quantitative meaning behind her calculations and actions. For instance, she justified a division because the calculation was "in [her] notes." As another example, when constructing a graph, Amy mimicked her actions from previous explorations without reasoning about distinct quantities of the situation. Her difficulty in reasoning about the relevant quantities was later revealed to have stemmed from observing and mimicking the behaviors of others (e.g., tracing a horizontal segment to determine a vertical distance).

Her focus on the behaviors of others also consisted of looking to the other students and researcher for approval of her solutions. She exhibited discomfort when pushed to explain her solutions and her responses to the researcher's questioning implied that she was focused on obtaining a correct result, as opposed to understanding solutions in terms of the quantities of a problem. Additionally, Amy showed resistance when

pushed by the researcher to reflect on her own reasoning or the quantitative meaning of her solutions, which conflicted with the problem solving approach of the instructional tasks. Multiple construction activities were used during the teaching experiment sessions, but Amy's explanations and behaviors revealed that she did not find value engaging in these tasks. Rather, she continued to rely on the solutions and correct answers of the other participants in the study. This approach to learning was an obstacle in her engaging in novel reasoning to solve problems and subsequently reflecting on this reasoning.

Her reluctance in engaging in novel reasoning and reflecting on her reasoning may have also been influenced by her mathematical confidence and views of mathematics. She frequently stressed that she did not trust her reasoning and she was often hesitant in putting her thinking on display for the participants of the study. Amy also looked to the researcher for approval of her solutions, rather than checking her solution through reflecting on her own thinking and making sense of her calculations. Amy approached the instructional activities as though her role was to observe other participants correctly solve problems. In turn, she believed these procedures would provide her the necessary tools to obtain correct answers. This led to her referencing previous procedures when solving novel tasks that did not lend themselves to these procedures.

Overall, Amy's actions emphasize the implications of both a procedural approach to problem solving and a lack of confidence in one's ability to engage in novel reasoning. As opposed to orienting to problems and constructing quantitative structures to leverage during problem solving, Amy looked for previous procedures and calculations to perform

in an attempt to obtain a correct result. As a result, her calculations were not based on quantitative relationships. When her procedures did not provide a correct solution, she resisted providing conjectures and relied on prompting from the researcher. Then, after engaging in reasoning consistent with the instructional goals, she was reluctant to reflect on and trust this reasoning, which resulted in her maintaining a focus on calculations and procedures. As a result, her unprompted solutions did not result from her images of quantities and how they were related (beyond her part to whole relationships). Hence, as Amy encountered novel problem contexts, she was unable to carry forward her reasoning.



## Chapter 7

### Results Of Judy

This chapter provides an overview of the reasoning and problem solving behaviors Judy exhibited over the course of the study. Her PCA scores are first provided to illustrate her pre- and post-course shift and to situate her within the students in the precalculus course. This is followed by data illustrating the thinking and understandings Judy revealed over the course of the study. In addition to characterizing her reasoning, a discussion of Judy's problem solving behaviors is presented. To conclude this chapter, a summary and discussion of Judy's progression over the course of the study is provided.

Judy was a full-time student in her mid-twenties. She was a first year biochemistry student at the university. Three years prior to this study, Judy had completed undergraduate degrees in English and Political Science at a different university, although five years had elapsed since taking a mathematics course (College Algebra). She planned to enroll in Calculus I after completing this precalculus course.

#### **Pre- and Post-Course Assessment**

Judy received an 'A' for her final course grade. In total, two students from the course received an 'A', seven students received a 'B', eight students received a 'C,' and three students received a failing grade. Judy's pre- and post-PCA exam scores were above average relative to the 16 students in her class who completed both the PCA pre- and post-administrations (Table 36).

Table 36

*Results of the PCA Pre- and Post-test (n=16)*

	Zac	Amy	Judy	Class Average
Pre-test Score	13/25	5/25	15/25	7.31/25
Post-test Score	17/25	10/25	21/25	12.18/25

### **Judy's Conception of Angle Measure Prior to Instruction**

When prompted<sup>22</sup> to describe an angle measure of one degree, Judy claimed, “A circle has three hundred and sixty degrees and a half circle will have one hundred eighty degrees...an angle has one degree means it's one three hundred sixtieth of a circle.”

When asked to elaborate, Judy responded by saying, “Wow, I don't know how to describe this...I really don't know what an angle is outside of formulas...Like, if you know this was like that, and I had to find this, then it would be a hundred and seventy nine degrees.” During this explanation, Judy drew the supplementary angle to one degree and determined the measure of this angle. Judy's actions reveal that she held a loose coordination of angle measure and a circle, but her descriptions did not reveal her reasoning about a measurement process consisting of measurable attributes of a circle (e.g., quantifying the fraction of a circle's *circumference*).

Judy was then asked to measure an angle using only a compass, Wikki Stix, and a ruler. She initially constructed a circle centered at the vertex of the angle and then determined the circumference of the circle and the arc length subtended by the angle

<sup>22</sup> The interview tasks referenced in this chapter are presented in their full form in Appendix F.

using the Wikki Stix. After she divided the arc length by the circumference and obtained 0.087, Judy explained, “I don’t know...should be eight point seven eight degrees...I don’t think that’s right.” Judy then expressed that she did not understand her calculation and that she could not complete the problem. Judy’s actions on this problem reveal her measuring an arc and attempting to relate angle measure to a subtended arc length and a circle’s circumference. She calculated the fractional amount of the circle’s circumference cut off by the angle (to the observer), but she conceived of this value as a number of degrees. She recognized that this value did not make sense as the measure of the angle, which led to her discarding her calculation without reflecting on the calculation in terms of the quantities of the situation.

Previous to attempting the last task of the interview, Judy explained, “I’m gonna look this up at home...I’m gonna be sitting on the edge of my seat until then...Ohhh, I can’t move on...we just gotta cover it up.” Judy’s utterances reveal that she held a strong desire to understand the previous problem and that she had difficulty moving on without producing an answer that made sense. After covering up the previous problem, Judy immediately stated that she was unable to solve the last task. She also added, “I’m gonna go crazy. I really wanna know!” Judy’s inability to provide solutions that made sense caused her frustration, but she exhibited a strong desire and need to make sense of the problems. The interview concluded with Judy assuring the researcher that she would not research how to solve the tasks before the first teaching experiment session.

In summary, Judy’s conception of angle measure did not include a meaningful measurement process consisting of quantities although she attempted to relate an arc

length and a circumference. Judy calculated the fractional amount of a circle's circumference subtended by an angle, but she did not interpret this value as the fractional part of a circle's circumference subtended by the angle's rays. Thus, she was unable to determine the angle measure using this calculation. Judy also expressed that she did not understand angle measure, but that she had a strong desire to learn this topic. Judy's curiosity and desire to understand angle measure was revealed in her expressing that she would be "on the edge" of her seat until she understood the meaning of an angle measure.

### **Judy's Ways of Thinking During the Instructional Tasks**

When responding to The Protractor Problem (Table 37) during the first teaching experiment session, Judy used a Wikki Stix to measure a circle's circumference. After determining this value, she indicated that dividing the circumference by the number of units corresponding to that length (e.g., 360 degrees for the full circumference) would produce the "amount of distance on the [circumference] for each degree." Judy appeared to view measuring an arc length as a necessary means for *constructing* a tool for measuring an angle. This led to her partitioning the circumference of a circle based on a unit length. Thus, Judy's engagement in The Protractor Problem led to her identifying a subtended arc length as a measurable attribute corresponding to the process of measuring an angle.

Table 37

#### *The Protractor Problem*

Using the supplies of a Wikki Stix and a ruler, construct a protractor that measures an angle in a number of gips, where 8 gips rotate a circle.

Next, Judy was asked to consider the effects of using different radius lengths when creating a protractor with an arc length per unit approach. Judy responded that as the radius length of a circle changed, the arc length subtended by the angle would also change. For instance, she indicated that the arc length would increase for an increasing radius length.

In response to Judy's reasoning, the researcher prompted her to consider various arc lengths that corresponded to one unit of angle measure, where each arc length was from a different sized circle. Judy calculated each arc length, divided by the corresponding circle's circumference, and then concluded that each arc length represented the same fraction of the corresponding circle's circumference. That is, Judy reflected on each calculation and identified that each ratio, or value, represented a fraction of the circle's circumference subtended by an angle of one unit. During the pre-interview, Judy did not reflect on a similar calculation relative to the quantities of the situation. Thus, the focus of the teaching experiment sessions on measuring an arc length and considering multiple radius lengths may have promoted Judy relating angle measure to the fractional amount of any circle's circumference subtended by the angle.

After engaging in The Protractor Problem, Judy transitioned to predominantly reasoning about angle measure as the fractional amount of a circle's circumference cut off by the angle opposed to an arc length per unit relationship. For instance, Judy exhibited this conception when she explained the effects of increasing the openness of an



Table 38

*The Circumference Problem*

Construct a circle using a Wikki Stix as the radius (your group should have Wikki Stix of different lengths). Then, determine how many of your Wikki Stix mark off the circumference of your circle. Compare your result with your classmates. What observations can you make from this comparison? Construct an angle that cuts off one Wikki Stix length of an arc. Compare the openness of the angle with those of your classmates.

After Judy constructed a circle using a compass and a Wikki Stix, she determined the circumference of the circle and divided by the length of the Wikki Stix, or “the radius.” After comparing this result to the other subjects’ outcomes, Judy described that the measurement of a circle’s circumference is always 6.28 “radius.” She also expressed a measure of 1.5 radius lengths by saying, “Well, the circumference of the arc is 1.5 radians.” Judy subsequently determined the linear measurement of an arc corresponding to 1.5 radians and a radius of 3.2 centimeters by “[taking] 3.2 and adding it to half of 3.2.”

Judy’s explanations during The Circumference Problem reveal her reasoning about a measurement in radians corresponding to a number of radius lengths rotating along an arc length (e.g., “the circumference of the arc”). Judy also identified that a constant number of radius lengths rotated through the full circumference of any circle. Furthermore, she converted between a measurement in radians and a linear measurement by imagining a radius length and a fraction of the radius lying along an arc.

Judy's ability to reason about a number of radius lengths rotating through the circumference resulted in Judy also indicating that an angle measure of  $\pi$  radians represented "the amount of radius [lengths] along the circumference. So you would just need to multiply [ $\pi$  radians] by the radius length [to determine the linear measurement]." Then, after returning to The Protractor Applet (Figure 28) with the radian measures displayed, Judy described the ratio of the arc length to the radius as, "A measure of the radian amount passed...on the circumference." Judy's actions reveal that her engagement in The Circumference Problem resulted in her conceiving of a radian measure as a number of radius lengths rotating through the arc length subtended by an angle.

During The Circumference Problem, Judy also spontaneously converted 1.5 radians to a number of degrees by using the equation of  $\frac{1.5}{2\pi} = \frac{x}{360}$ . She reasoned that each ratio represented, "the percentage of the arc in comparison to the circumference." Hence, Judy's understanding of angle measure as a fraction of a circle's circumference created a foundation for her constructing a conversion equation that was *not* discussed previous to this action.

Judy's multiple conceptions of angle measure (e.g., an arc's fraction of a circle's circumference or radius) each consisted of a quantitative relationship stemming from reflecting on a measurement process. After Judy's engagement in The Protractor Problem and The Circumference problem, she leveraged these multiple conceptions of angle measure to solve novel tasks. For instance, consider her responses to the prompts in The Arc Problem (Table 39).



Table 39

*The Arc Length Problem*

Given that the following angle measurement  $\theta$  is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches.

Judy first constructed an equation of the form  $\frac{35}{360} = \frac{x}{2\pi r}$  for each radius and arc length and then explained her solution (Excerpt 47).

## Excerpt 47

- 
- |    |        |   |
|----|--------|---|
| 1  | Judy:  | It's ( <i>referring to each ratio</i> ) the percentage of the, um, of the cutout  |
| 2  |        | length in relationship to the entire circle.                                      |
| 3  | Kevin: | To the entire circle, and what about the entire circle?                           |
| 4  | Judy:  | Oh, I'm sorry, the circumference.   |
| 5  | Kevin: | Ok, so in this case we have these multiple circles with different arc             |
| 6  |        | lengths in terms of inches, but yet they all have the same angle                  |
| 7  |        | measurement. How is it that that happens?   |
| 8  | Judy:  | Oh, I see, um, that works because, um, ( <i>pause</i> ) let's see, oh, um, if you |
| 9  |        | have, you know, your first circle ( <i>making a circle with her hands</i> ) and   |
| 10 |        | you increase the radius ( <i>increasing the size of the circle made with her</i>  |
| 11 |        | <i>hands</i> ) then, um, even though the percentage of the entire circle is the   |
| 12 |        | same, you have to compensate with a larger arc length.                            |
| 13 | Kevin: | Ok, in terms of what units do you need a larger arc length?                       |
-

---

14      Judy:    Inches, in inches.

---

Judy described that each ratio was a value representing the percentage of the circumference cut off by the angle (lines 1-4). Judy also reasoned that for a varying radius length, the arc length increased such that the angle subtended a constant percentage of the circle's circumference (lines 9-13). Judy's understanding of angle measure as the fractional amount of *any* circle's circumference cut off by the angle formed a foundation for her determining various arc lengths, which was revealed by her justifying her solution in terms of the quantities of the situation (e.g., an arc length's fractional amount of a circle's circumference).

When prompted to determine the same arc lengths using a radian angle measure, Judy converted the angle measure in degrees to 0.61 radians using her previously revealed conversion process (e.g., reasoning about a fractional amount of a circle's circumference). She conveyed that, "there are two pi radius lengths that make up a circle's circumference, regardless of the radius length." This explanation reveals Judy continuing to reason about a number of *radius lengths* rotating through any circle's circumference. She then determined each arc length (Excerpt 48).

#### Excerpt 48

---

1	Judy:    Say if you gave me this one ( <i>referring to 0.61 radians</i> ) instead of thirty-
2	five degrees, I would just, um, put that over the entire radians and set it
3	equal to, find the arc length over the entire circle.
4	Kevin:    Ok, good. Now is there any other way you could use that measurement?

---

- 
- 5                    So, this is one option, obviously, setting up that equation. Is there any  
6                    other method you could use using that measurement?
- 7        Judy:    Oh ya! (*laughing*) I always do it the longest way, because it always  
8                    makes sense to me. But you could just really multiply this (*pointing to*  
9                    *the radian measurement*) by the radius length.
- 

Judy first used reasoning (e.g., a fractional amount of a circle's circumference) similar to her previous solution (Excerpt 47) in order to *anticipate* determining the arc lengths (lines 1-3). When describing an alternative solution, she identified an operation of multiplying the number of radians by the radius (lines 7-9). She also expressed that she preferred her previous solution because "it always makes sense to me." This interaction reveals Judy reasoning about angle measure as a subtended arc length's multiplicative relationship with the radius *and* circumference of a circle. Both quantitative relationships formed a foundation for her calculations (e.g., quantitative operations) and enabled her to determine the linear measurement of an arc length. However, she expressed that reasoning about the fractional amount of a circle's circumference was more powerful for her at this point in the instructional sequence.

Despite Judy's claim that reasoning about an arc length's fraction of a circle's circumference made more sense to her, she transitioned to reasoning about a quantitative relationship between an arc length and radius of a circle of any size as she encountered more problems with radian measurements. For instance, after The Arc Length Problem, Judy contrasted a radian angle measure with a linear measurement of an arc on The Inches or Radians Problem (Table 40).

Table 40

*The Inches or Radians Problem*

A student measures the arc-length that an angle cuts off, resulting in 1.7 inches, and claims that the angle has a measure of 1.7 inches. Discuss this student's claim.

Judy oriented to The Inches or Radians Problem by drawing an angle, a circle centered at the vertex of the angle, and tracing an arc length (Figure 29). She then paused for an extended period of time and explained, "I don't think it makes sense...it might make sense if you said one point seven radians and the radius length was one inch. But the angle measure, you don't use a linear unit to describe angle measurement." After claiming that the student's angle measure did not make sense, Judy constructed a smaller circle and explained that an arc length of one point seven inches corresponds to an angle with more openness (Figure 29). In this case, Judy reasoned that conveying an angle measure as a number of inches cut off by the angle does not correspond to a constant openness for a circle of any radius.



Figure 29. Judy's diagram for linear measurements.

Judy continued with her explanation and contrasted a measurement of 1.7 inches with an angle measure of 1.7 radians (Excerpt 49).

## Excerpt 49

- 
- 1       Judy:   [1.7 radians] would mean that on any circle's circumference (*making a*  
 2                   *circle with her hands*), um, that 1.7 times the radius had, um, has been  
 3                   traveled along (*moving finger along an imagined arc*) the circumference.
- 4       Kevin:  Ok.
- 5       Judy:   So, um, I think that makes a lot more sense than that. Because this  
 6                   (*referring to the 1.7 inches*) isn't translatable to any other circle.
- 7       Kevin:  Ok, good, so now I'm gonna throw out and say, uh, the student said the  
 8                   radius he used was 1.5 inches to get the 1.7 inch arc length. So how  
 9                   many radians would that be?
- 10      Judy:   Um (*using calculator*), it would be 1.13 radians.
- 11      Kevin:  Ok, so how'd you find that?
- 12      Judy:   I divided, uh (*laughing*), I divided the arc length by the radian length,  
 13                   which is 1.5 (*labeling radius on diagram*), to see how many times the  
 14                   radius the arc length is.
- 

Judy first conceived of a traversed arc length as so many times the radius length (lines 1-3) and added that the linear measurement was not “translatable” to a circle of a different radius (lines 5-6). Then, after the researcher provided her with a specific radius length (lines 7-9), Judy calculated a number of radians (line 10) and elaborated that this value represented the multiplicative relationship between the arc length and the radius length (lines 12-14). These actions reveal Judy reasoning about a radian measure as conveying a multiplicative relationship between an arc length and the radius for *any*

circle centered at the vertex of the angle. This understanding enabled Judy to calculate a radian measurement using a quantitative operation, while also contrasting this measurement with a linear measurement.

Table 41

*The Radian Measurements and Pi Problem*

What does it mean for an angle to have a measure of  $1.2\pi$  radians? 5.27 radians? How long is the arc subtended by the angle relative to a radius of 3.5 inches?

Judy continued to reason about a relationship between an arc length and the radius when she was subsequently asked (Table 41) to describe angle measurements of 5.27 and  $1.2\pi$  radians. Also, Judy oriented to this problem by first constructing a circle (Figure 30). She then explained the meaning of each angle measure (Excerpt 50).

Excerpt 50

- 
- |   |        |   |
|---|--------|---|
| 1 | Judy:  | It means that if you travel from your starting point here, and you travel       |
| 2 |        | counterclockwise ( <i>tracing arc length</i> ), you travel five point two seven |
| 3 |        | times your radius length along the circumference ( <i>moving her pen tip in</i> |
| 4 |        | <i>the shape of a circle</i> ).   |
| 5 | Kevin: | Ok, what about the one point two pi radians? What's that mean? To have          |
| 6 |        | an angle measure of one point two pi radians?                                   |
| 7 | Judy:  | Um, I guess one point two pi times the radius length, which is, ( <i>using</i>  |
| 8 |        | <i>calculator</i> ) um, about three point seven seven radius lengths.           |
| 9 | Kevin: | Ok, so what role does pi play in that?  |
-

- 
- 10 Judy: Um, just a number.
- 11 Kevin: Just a number, ok. Good.
- 12 Judy: Oh, and then the arc length is, um, I just multiplied the radians times the  
13 radius (*3.5 inches*) length to get the arc length, which is eighteen point  
14 four four five.
- 15 Kevin: Ok, good. So why does that operation work, why does that give you the  
16 arc length?
- 17 Judy: Um, because, uh, I kind of look at the radians like a function almost, so I  
18 always look at it as five point two seven times whatever the radius  
19 length is, is the linear arc length 'cause um, that's how it translates to any  
20 other circle you use. So if it was a larger one where the radius was five,  
21 then I'd multiply it by five instead.
- 

Consistent with Excerpt 49, Judy described that *both* measurements conveyed a multiplicative relationship between a traversed arc length and the radius (lines 1-8). Judy also traced an arc length while giving her description. Her image of traveling *along* an arc may have stemmed from reasoning about angle measure corresponding to the process of measuring *along* a subtended arc length, as well as the various circular motion activities implemented during the study. After discussing an operation relative to the context of the problem, Judy spontaneously described that she interpreted radian measures as a (multiplicative) process (lines 16-20) between an indeterminate radius and arc length of any circle.

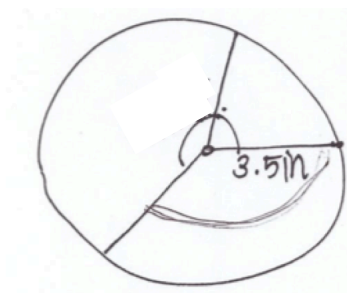


Figure 30. Judy's diagram for angle measures.

Judy's understanding of radian measures as a process between quantities also enabled Judy to formalize a relationship between indeterminate measurements of these quantities. For instance, Judy was asked to construct a formula relating a subtended arc length, the radius of the corresponding circle, and an angle measure. After Judy constructed  $\theta = \frac{s}{r}$ , she described, "Um, because radians is...the number of radius lengths that have passed along the circumference." Judy's reasoning reveals her conceiving of  $\frac{s}{r}$  as a number of radius lengths, which was equivalent to an angle measure of  $\theta$  radians. Thus, Judy's image of a number of radius lengths rotating through an arc length promoted her constructing a formula reflecting this quantitative relationship as a number of radians.



Table 42

*The Rotating Problem*

A Ferris wheel with a radius of 41 feet is rotating at 2.5 full revolutions per minute.

Marcus boards the Ferris wheel for a ride. After 20 seconds, how far has Marcus traveled on the Ferris wheel? Give your answer in a linear measurement (e.g., a number of feet) and a number of radians.

Judy also leveraged reasoning about measuring *along* an arc length in a number of radius lengths when responding to problems consisting of circular motion. Recall that Judy could not solve a problem during the pre-interview that required relating an arc length and an angle measure. When orienting to The Rotating Problem (Table 42), Judy explained, “I just remember this problem from the first time you presented it...I thought I would never understand it.” Judy’s claim reveals her recalling her initial frustrations during the pre-interview and that she initially questioned whether or not she would be able to make sense of angle measure. After making this statement, Judy oriented to the problem by drawing a circle and labeling the radius. Judy also claimed, “I think it’s easier to think of it in radians first,” while identifying that Marcus traveled  $5\pi$  radians per minute.

Judy continued to orient herself to the problem and claimed, “[I’m] trying to think of it in relation to our exercise yesterday...’cause we did a lot of things with time yesterday...I think I have to convert theta into time.” After she could not recall any

specifics from the previous class<sup>24</sup>, she reflected on the context of the problem and expressed, “Let me think a little more about it...I think I way over thought it.”

Next, Judy determined the number of radians rotated per *second* by Marcus and she used this value to determine that Marcus rotated 5.236 radians per 20 seconds. Judy also described that twenty seconds is one third of a minute and that this implied that Marcus rotated one third of  $5\pi$  radians. These actions reveal that after Judy could not recall a previous procedure, she leveraged her ability to reason about a varying arc length to complete the task. This resulted in her reasoning about the rate at which Marcus rotated to calculate the number of radians rotated per one minute, one second, and twenty seconds. Following Judy determining these values, she continued to reason about the context of the problem and multiplied the value of 5.236 by “the radius length to get the linear measurement” of 214.676 feet.

After determining the linear measurement, Judy was asked to explain the effects of increasing the radius of the Ferris wheel to 52 feet while maintaining a speed of 2.5 revolutions per minute (Excerpt 51).

#### Excerpt 51

---

1	Judy:	Um, it'd be, this linear measurement ( <i>identifying previous answer of</i>
2		<i>214.676 feet</i> ), it'd kind of grow in proportion to the new circumference.
3		Um, and the, since the full revolutions per minute, the speed hasn't
4		changed, neither would the radians.

---

<sup>24</sup> During the previous teaching experiment session, the subjects modeled a traversed arc length as a function of time.

- 
- 5 Kevin: So what do you mean by the linear would grow in proportion? Say a  
6 little bit more about what you meant by that.
- 7 Judy: Um, ya, because, since (*pause*), I think it would grow 52 over 41 times  
8 the one measured previously.
- 

Judy's ability to reason about an angle measure corresponding to a circle of any size enabled her to identify that the number of radians rotated by the individual per minute would not change for a varying radius (lines 3-4). Also, Judy reasoned about the proportional relationship between the circumference of a circle (an arc length) and a subtended arc length to conclude that the linear measurement would increase by a factor of  $\frac{52}{41}$  (lines 1-2 & 7-8). In this case, Judy's ability to reason about an angle measure conveying the fraction of any circle's circumference or radius subtended by the angle enabled her to reason about the effect of increasing the radius of a Ferris wheel while maintaining a constant angular speed.

Table 43

*The Fan Problem (Sine)*

Imagine a bug sitting on the end of a blade of a fan as the blade revolves in a counterclockwise direction. The bug is exactly 3.1 feet from the center of the fan and is at the 3:00 position as the blade begins to turn. Create a graph that shows how the bug's vertical distance above the 9:00 to 3:00 diameter line varies with the total distance the bug travels around the circumference.

Judy's ability to reason about a varying arc length and measuring quantities relative to the radius also formed a foundation for her to construct the sine and cosine functions. For instance, during The Fan Problem (Table 43), Judy described that as the bug traveled over the first quarter of a revolution, the distance of the bug above the horizontal diameter was "starting to grow less" as the angle measure approached ninety degrees. Judy then described that the vertical distance was decreasing at an increasing rate over the second quarter of the bug's revolution around the fan (MA5). She also supported her rate of change description by reasoning that the change in vertical distance was "increase[ing], but in a negative way" as the angle measure increased. Judy's reasoning about the covariational relationship between an angle measure and a vertical distance subsequently led to the construction of a graph of the sine function rooted in this reasoning.

Another student, Zac, led much of the discussion during The Fan Problem. Thus, in order to further investigate Judy's reasoning, she was given a similar context during an interview session. Judy's actions when solving The Ferris Wheel Problem (Table 44) offered further insights into her conception of the sine function.

Table 44

*The Ferris Wheel Problem*

Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full rotation. April boards the Ferris wheel at the bottom and begins a continuous ride on the Ferris wheel. Sketch a graph that relates the total distance traveled by April and her vertical distance from the ground.

Judy oriented to The Ferris Wheel Problem by drawing a circle, labeling the radius of the circle, and identifying the starting position of April. *Before* constructing a graph, Judy first determined the formula  $\sin\left(\theta - \frac{\pi}{2}\right)$ . She then returned to orienting to the situation, which resulted in her constructing a vertical segment from the center of the Ferris wheel to April's starting position and underlining *vertical distance from the ground* in the problem statement. This additional orientation led to her altering her formula to  $\sin\left(\theta - \frac{\pi}{2}\right) + 1$ . Also, Judy described that April began at an "angle" of "negative pi halves" from the starting point while tracing an arc between the 3 o'clock position and the starting position of April. These actions reveal that Judy's use of a diagram consisted of her distinguishing between various vertical distances and an angle from the standard position. Also, she conceived of measuring the difference between the two vertical distances as one radius length.

Before constructing her graph, Judy labeled the horizontal and vertical axes with "total distance traveled in radians" and "percent of a radius," respectively. Then, Judy identified the corresponding input and output values for the end of each quarter of a revolution. For instance, she determined, "If my input is zero, my vertical distance is negative one, uh, percent of radius." After identifying these points on the graph, she created a graph (solid curve in Figure 31) by reasoning about the directional covariation of the two quantities (MA2).

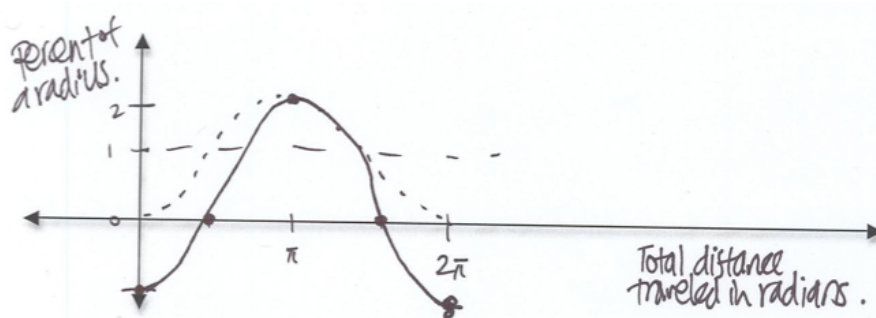


Figure 31. Judy's graph for The Ferris Wheel Problem.

Similar to her initial formula, Judy's graph had an output of the vertical distance above the *center* of the Ferris wheel. After re-reading the problem statement and checking her graph, she stated, "Oh! From the ground...it's just plus the radius." Judy's reflection on the problem statement resulted in her identifying that her graph was inconsistent with the vertical distance formalized by her formula. This resulted in her correcting her graph such that it reflected the proper input-output process (dashed curve in Figure 31). She also continued to reason about measuring the vertical distance relative to the radius when she described increasing the vertical distance by "the radius" and the maximum value as "two-hundred percent of a radius...two radius lengths."

Due to Judy giving only a directional covariation explanation for the shape of her graph, she was asked to reconstruct her graph on a new set of axes and discuss its shape (Excerpt 52).

#### Excerpt 52

- 
- |   |   |
|---|---|
| 1 | Judy: Um, I can definitely try. This is my least favorite question of yours |
| 2 | ( <i>laughing</i> ). Hammering all of the hard questions. Well, I would say |
| 3 | you'd have to compare the rates of change from one part of, uh, the         |
-

- 
- 4 curve (*tracing an arc on the diagram*) to well, let me think (*pause*).
- 5 That's not what I'm trying to get at. (*pause*) Ok. Because from your input
- 6 (*marking positions at 3 o'clock and 12 o'clock, respectively*) zero to pi
- 7 halves, um (*pause*), your (*pause*), change in vertical distance per the
- 8 same input, um, or, per the same change in radians traveled around the
- 9 circumference (*tracing an arc with her finger*), is increasing as you go
- 10 from zero to pi halves (*tracing arc length*). And it is decreasing as you
- 11 go from pi halves to pi (*tracing arc length*). So it's constantly increasing
- 12 or decreasing, rather than at a constant rate. So if you took...
- 13 Kevin: Ok, so ya, go ahead, can you show me on, you just said, let's just focus
- 14 on this section (*3 o'clock to 12 o'clock*).
- 15 Judy: Oh ok, I mean decreasing, sorry.
- 16 Kevin: So what's decreasing?
- 17 Judy: Um, your change in vertical distance. So, if you took, you know, similar,
- 18 um, changes of radius, or radians, along the circumference (*marking*
- 19 *equal changes of arc length*), um. You know, from here to here
- 20 (*identifying change of arc length*) it has a drastic change, but the, as you
- 21 keep on going further along the circumference (*tracing arc length*) the
- 22 change is decreasing. And that's why it has the curve.
- 

Judy expressed that she found justifying the shape of her graph a difficult task (lines 1-2). Judy then returned to the diagram of the situation, rather than the graphical representation, and described amounts of change of vertical distance relative to equal





relationship between the quantities by saying, “From this distance to this distance (*identifying interval on horizontal axis*) you see more steep slope or rate of change...(*tracing graph*) but then it gets closer and closer to a slope of zero.” After this rate of change description (MA5), Judy hesitated as she attempted to identify a change in vertical distance on her graph. After identifying equal changes of arc length along the horizontal axis and pausing for an extended amount of time, she stated, “I think your change in vertical distance is just that,” while correctly identifying a change of vertical distance with an arrow (Figure 33).

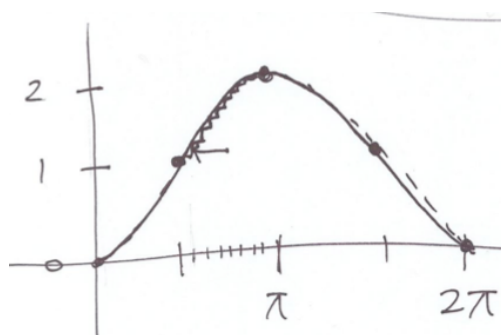


Figure 33. Judy’s refined graph on The Ferris Wheel Problem.

With a change in vertical distance constructed on her graph, Judy then discussed corresponding amounts of change of the two quantities (Excerpt 53).

Excerpt 53

- 
- |   |   |
|---|---|
| 1 | Judy: ( <i>identifying successive changes of vertical distance on the graph</i> ) Um, |
| 2 | they should be getting smaller and smaller.   |
| 3 | Kevin: Ok, and why should they be getting smaller and smaller?                        |
| 4 | Judy: Um, because the change in vertical distance from here to here                   |
| 5 | ( <i>identifying arc length on the diagram</i> ) is getting increasingly smaller      |
-

- 
- 6                   until you get to, until you get to your maximum vertical height.
- 7       Kevin:   So what do you mean by increasingly smaller?
- 8       Judy:    Um (*pause*), oh, because as you progress or as you travel along the
- 9                   circumference (*tracing arc length*), it gets, um, the vertical, change in
- 10                  vertical distance is smaller than the last change in vertical distance.
- 

After identifying changes of vertical distance on her graph, (lines 1-2), she returned to her diagram to verify the relationship between the two quantities (lines 4-6). While reflecting on the diagram, she continued to reason about a varying arc length and the change of vertical distance over this interval on her diagram (MA3) to support her rate of change description (lines 8-10).

Judy's actions suggest that her initial construction of the graph was not rooted in reasoning about *amounts of change* of the two quantities (MA3). Rather, she may have recalled the graph from previous explorations. However, by continuing to reflect on a diagram of the situation, she reasoned about the rate of change of vertical distance with respect to a traversed arc length (MA5) and supported this reasoning by comparing amounts of change of vertical distance with respect to successive changes of arc length (MA3). Judy had difficulty initially identifying a change of vertical distance on her graph, but her understanding of the graph as representing input-output values of a covariational relationship enabled her to relate the graphical representation to a diagram of the situation. As a result, she justified the shape of her graph beyond a rate of change description or a recollection from memory (Excerpt 53).

Judy's solution to The Ferris Wheel Problem reveals that the context of circular motion created a situation that supported her constructing, relating, and formalizing a covariational (quantitative) relationship as the sine function. Throughout her solution, she reasoned about a varying arc length and vertical distances, while also conceiving of measuring these quantities in a number of radius lengths. Also, she reasoned about indeterminate values to anticipate the sine function producing an output in a number of radius lengths (e.g., a process conception of the sine function). Thus, Judy's engagement in The Fan Problem promoted her constructing an understanding of the sine function that was grounded in reasoning about a covariational relationship of two quantities.

When confronted with a right triangle situation, Judy's conception of the sine and cosine functions formalizing a quantitative relationship created a foundation for her to solve the problems correctly. As an example, consider Judy's approach to The Enemy Approaches Problem (Table 45).

Table 45
<i>The Enemy Approaches Problem</i>
A castle observation tower is elevated 126 feet above the ground. When an approaching enemy is first noticed, the angle of depression (the angle at which an observer needs to look down) from the observation post was 0.084 radians. How far away is the enemy from the castle? How far away is the enemy from the observer?

Judy oriented to The Enemy Approaches Problem by drawing a labeled diagram of the situation, identifying two right triangles, and determining the complement of the given angle measure (Figure 34).

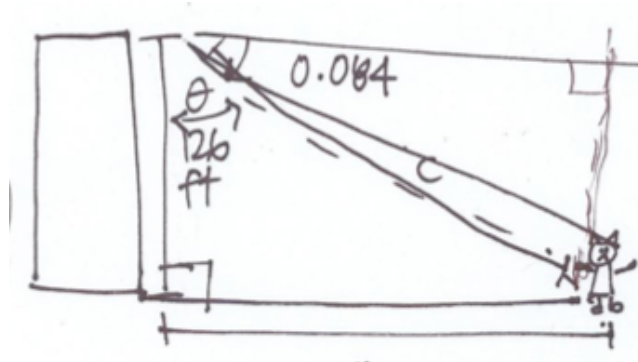


Figure 34. Judy's diagram on The Enemy Approaches Problem.

Judy then identified a “vertical distance,” “horizontal distance,” and “the radius.” She also clarified her meaning of these quantities (Excerpt 54).

#### Excerpt 54

- 
- |    |        |   |
|----|--------|---|
| 1  | Judy:  | If you turn it this way ( <i>rotating Figure 34 ninety degrees</i>                  |
| 2  |        | <i>counterclockwise</i> ) it is the vertical distance because your theta is always, |
| 3  |        | um ( <i>pause</i> ), your, well your vertical distance has to be connected          |
| 4  |        | somewhat to the right triangle. Or right angle of the triangle.                     |
| 5  | Kevin: | Ok.   |
| 6  | Judy:  | And so, when you're solving, it's just going to be that distance ( <i>tracing</i>   |
| 7  |        | <i>vertical segment</i> ).  |
| 8  | Kevin: | So what tells you to turn it that way? Why do you want to turn it that              |
| 9  |        | way?  |
| 10 | Judy:  | Um, I want it to mirror a circle. So, if I, whenever I imagine a circle and         |
| 11 |        | vertical distance, the triangle is always that way ( <i>drawing a right</i>         |
| 12 |        | <i>triangle within a circle</i> ).  |
-

Judy continued orienting to the problem by rotating her diagram such that the angle opened to the right and the height of the observation tower formed the quantity of a vertical distance (lines 1-3). She then claimed that she wanted the diagram to “mirror a circle.” These actions reveal Judy conceiving of the hypotenuse of the right triangle forming the radius of a circle, with the two legs of the right triangle forming measurable horizontal and vertical distances within the context of a circle.



Figure 35. Judy connecting a right triangle to the unit circle.

Judy subsequently discussed, “that’s how [the hypotenuse] becomes the radius length,” when constructing a new diagram of the situation (Figure 35). Judy’s conception of the hypotenuse as the radius *length*, and thus a unit of measurement, resulted in her reasoning about the outputs of the trigonometric functions relative to measurable sides of the right triangle. For instance, she explained, “[The] cosine of the distance traveled in radians is equal to the horizontal distance...divided by the radius length.” Judy’s explanation also reveals her reasoning about an angle measure as traveling *along* an arc length, where she identified that this value was the input to the cosine function.

As Judy encountered additional right triangle contexts throughout the study, she frequently oriented to the task by imagining the hypotenuse as the radius of a circle. This action led to her creating coherence between a right triangle and a unit circle context. As a result, she maintained a focus on measuring quantities relative to a measurable length

(e.g., the radius or hypotenuse), where these values formed the outputs of the trigonometric functions. Also, she maintained an image of angle measure as measuring along an arc length in a number of radius lengths opposed to a label within a right triangle.

It appears that Judy's orientation action of using the hypotenuse of a right triangle to construct a circle created coherence between the two trigonometry settings. Also, this orientation action was context focused, much like Judy's previous problem solving behaviors. Throughout the study, Judy focused on constructing diagrams, labeling known and unknown values, and supporting her solutions by reasoning about relationships between quantities. Judy's problem solving behaviors are explored in the next section relative to her reasoning and actions during the study.

### **The Role of Judy's Problem Solving Behaviors**

Judy expressed a strong desire and need to understand angle measure during the pre-interview in spite of her frustration when she was unable to provide meaningful solutions. As Judy continued through the study, she frequently described her frustration with the topic when orienting to the various tasks. For instance, during The Rotating Problem (Table 42), she expressed, "I don't know, [this problem] just makes me laugh, thinking about when I first saw these things because I thought I would never understand it. I'm glad I got through the initial frustration." This explanation reveals the initial difficulties Judy felt during the study, and may explain the researcher observing limited participation from Judy during the first few teaching experiment sessions.

Although her verbal participation was limited during the first few sessions, Judy was engaged in the problem tasks (e.g., the construction tasks). While she expressed that trigonometry was difficult, she maintained her interest in making sense of her actions. As the study progressed, her participation during the instructional activities increased substantially. During the last interview session and when orienting to a problem, Judy reflected on her attempt to make sense of the various trigonometry activities, as well as the nature of the activities (Excerpt 55).

Excerpt 55

- 
- |    |   |
|----|---|
| 1  | Judy: Trigonometry ( <i>sigh, laughing</i> ).                                       |
| 2  | Kevin: You enjoying trig?   |
| 3  | Judy: ( <i>laughing</i> ) It's challenging.   |
| 4  | Kevin: Well then...   |
| 5  | Judy: Um, I'm kind of glad though that I'm learning it this way. I really had no    |
| 6  | clue what any of it meant. I just knew that there were formulas, pretty             |
| 7  | much.   |
| 8  | Kevin: Good. I'm glad you're enjoying it.   |
| 9  | Judy: Not the word I'd use actually.  |
| 10 | Kevin: ( <i>both laughing</i> ) Well I'm glad you're learning then. I'm glad you're |
| 11 | learning. It may be painful, but I'm glad you're learning.                          |
| 12 | Judy: Ya, this class surprisingly helps me in all my other classes.                 |
| 13 | Kevin: Really?  |
| 14 | Judy: Like every other class it helps me so much.                                   |
-

- 
- 15     Kevin:    How's that? Just...
- 16     Judy:     Um, now when I look at graphs I think about the rate of change a lot
- 17                more in depth. And um, even the way I describe things in class, like my
- 18                other teachers appreciate, you know, that I'm looking at things, you
- 19                know, like "why is it that way?" Everything, so I feel like I understand
- 20                my classes better.
- 

Judy expressed the difficulty she encountered when trying to construct understandings during the trigonometry activities (lines 1 and 3). In spite of these difficulties, she described that she valued the depth at which she was learning and explained she previously did not make sense of her previous experiences beyond the existence of formulas (lines 5-7). Judy then described that reasoning more deeply about rates of change had become a tool for graphing the relationships between quantities (lines 16-17). Also, she acknowledged the depth at which the precalculus course promoted her reflecting on the reasoning behind various procedures and actions. She elaborated that this approach gave her a greater understanding of precalculus *and* her other classes (lines 17-20). Thus, Judy found trigonometry to be a challenging topic, but she also strove to overcome these difficulties and understand trigonometry more deeply than the ability to correctly apply procedures and formulas.

Judy's problem solving behaviors during the study reflected her attitude towards providing meaningful solutions beyond a memorized procedure or formula. During the activities and problems presented during the sessions, Judy spent ample time orienting to the contextual situations of each problem. Her orienting actions frequently consisted of



constructing a diagram and identifying known and unknown measurable attributes (quantities) on the diagram. For instance, during The Inches and Radians Problem (Table 40), Judy first constructed a diagram of the situation and identified the given value on the diagram. As she discussed her reasoning for not using a linear unit to measure the openness of an angle, she continued to reference the quantities of the situation and constructed an additional diagram to contrast the two units of measurement.

In addition to Judy's initial orientation actions, as Judy executed her solutions, she often reflected on her diagrams in order to check her solution and further orient to the context if needed. As an example, during The Ferris Wheel Problem (Table 44), Judy initially created a formula whose output was inconsistent with the intentions of the problem. She then reflected on her diagram and the problem statement, which resulted in her altering the formula to reflect the appropriate vertical distance. Judy also leveraged her diagram of the situation without prompting to verify the covariational relationship conveyed by her graph on The Ferris Wheel Problem (Excerpt 52).

Judy's planning, execution, and checking of her solutions predominantly consisted of reasoning that was grounded in the quantities of the situation, which may have been a product of her meaningful orienting actions. For instance, as Judy anticipated and executed calculations, she described the calculations and resulting values in terms of quantities and relationships between quantities. On The Inches and Radians Problem she explained (Excerpt 49) that she divided an arc length by the radian length and that this operation represented "how many times the radius the arc length is." In this case, her calculation was driven by a quantitative relationship, which also enabled her to describe

the calculation as a quantitative operation. On The Arc Length Problem, Judy reasoned about the fraction of a circle's circumference to calculate an arc length, with her constructed equation consisting of two ratios representing this quantitative relationship (Excerpt 47). In contrast, during the pre-interview Judy divided an arc length by the circumference, but she was unable to make sense of this calculation relative to the context of the problem.

In summary, Judy's focus on the contextual situations and reasoning about quantitative relationships when problem solving promoted her constructing the deeper understandings she desired. Opposed to concentrating on performing correct calculations and memorizing procedures, Judy spent much of the problem solving process orienting to the problem situations. As she constructed quantities and relationships between quantities during this process, this formed a foundation for her to anticipate (plan) and interpret calculations relative to these quantities (e.g., quantitative operations). This also enabled her to maintain a contextual focus when executing her calculations. As a result of her understandings consisting of quantitative relationships, Judy found identifying the relevant quantities of the situation a valuable tool in problem solving.

### **Summary and Discussion of Judy**

Initially, Judy was unable to coordinate a subtended arc length and the circumference of a circle to measure an angle. As a result of her engaging in activities focused on the quantitative meaning of various measurement units, Judy appears to have constructed an understanding of the process for measuring an angle that consisted of quantitative relationships. For instance, after The Protractor Problem, Judy was

frequently observed reasoning about measuring *along* an arc length and describing the measure of an angle as the fractional amount of a circle's circumference subtended by the angle. This quantitative relationship enabled her to fluently convert between various units of angle measure. In fact, Judy leveraged this reasoning to convert between a number of degrees and a number of radians prior to an activity addressing angle conversion.

Relative to radian measurements, Judy also reasoned about measuring along a subtended arc in a number of radius lengths. She conceived of a radian measure as the multiplicative relationship between the subtended arc length and the radius (of any circle), where she referred to this quantitative relationship as a “function” between the length of the radius and the arc length. Judy's ability to conceive of a number of radius lengths rotating along the circumference of a circle also promoted her conception that all circles have a circumference of  $2\pi$  radius *lengths* (e.g.,  $C = 2\pi r$ ). Her actions also conveyed that she imagined traveling continuously along an arc. That is, as she described radian measurements, she imagined traveling along an arc and simultaneously accruing a number of radius lengths.

Judy's propensity of imaging measuring continuously along an arc and constructing circles to describe angle measure may have been a result of the approach of the instructional activities. Judy's actions on both The Circumference Problem and The Protractor Problem consisted of her measuring along an arc length and reflecting on this process to construct units of angle measure. Also, multiple instructional activities resulted in Judy reasoning about a traversed arc length varying with respect to time. As a result,

Judy conceived of a subtended arc length as a measurable attribute of an angle that varied as the openness of the angle was changed.

Judy's ability to reason about a varying arc length also supported her constructing the trigonometric functions of sine and cosine. Similar to the angle measure explorations, The Fan Problem offered a situation for Judy to engage in the construction of the sine and cosine functions. Relative to the sine function, after reasoning about the rate of change of the two relevant quantities (MA5), Judy engaged in reasoning about changes of arc length while constructing and relating changes of vertical distance (MA3). This enabled Judy to justify the concavity of the graph of the sine function. Additionally, due to Judy's conception of the sine function being supported by the context of the unit circle, she was able to consider various starting positions and vertical distances relative to the sine function. For instance, on The Ferris Wheel Problem, Judy created a formula by reasoning about a shift in the starting position, as well as a difference between the vertical distance from the center of the Ferris wheel and a vertical distance from the ground.

Judy's ability to conceive of measuring quantities in a number of radius lengths also supported her applying trigonometric functions to any circular context. As she encountered circles of various linear radius lengths, she conceived of the radius as one radius length (e.g., the unit circle). This enabled her to conceive of vertical and horizontal distances from the center of a circle as measured in a number of radii (e.g., outputs to the sine and cosine function). Additionally, when encountering a right triangle situation, Judy used the hypotenuse of the triangle to construct a circle with a radius length equivalent to the hypotenuse. This supported her continuing to reason about angle measure as a

subtended arc and measuring various lengths (e.g., the legs of the right triangle) relative to the radius.

Judy constructed understandings consistent with the instructional design of the lesson, and she expressed that she found the topic of trigonometry difficult. In spite of this difficulty, she showed high persistence in making sense of the various activities throughout the study. Judy explicitly stated that she appreciated obtaining a deeper understanding of the material than her previous experiences. Rather than focusing on procedures or calculations, Judy sought to understand the reasoning and concepts that formed the foundation to a solution. This attitude was also revealed in Judy's problem solving behaviors.

Judy focused on the contextual situations when problem solving and her actions consisted of reasoning about quantities and relationships between quantities. Judy's approach to problem solving may have been related to her engagement in the instructional activities during the teaching experiment. Judy's actions consisted of various construction processes that she reflected on in order to construct quantitative relationships. As opposed to merely looking to complete each task correctly, she attempted to make sense of her actions relative to the relevant quantities of the situation. This reasoning formed a foundation for Judy's solutions and enabled her to construct a flexible and connected understanding of angle measure and trigonometric functions.

## Chapter 8

### Conclusions

This chapter summarizes the three students' conceptions of angle measure and trigonometric functions. The students' ways of thinking are compared while highlighting reasoning abilities that were revealed to be critical for understanding central concepts of trigonometry. The reasoning abilities that were found to be central to learning ideas of angle measure and the sine function are then summarized in a framework. This chapter also provides a discussion of the students' problem solving dispositions and behaviors. The students' actions revealed that their approaches to problem solving were related to their ability to conceptualize and reason about quantities and their relationships. Suggestions for curriculum and instruction are then provided in the context of this dissertation's findings. Finally, this dissertation concludes by addressing the limitations of this study, as well as this study's implications for future mathematics education research.

#### **Quantitative Reasoning in Trigonometry**

The three students' actions when completing the instructional tasks offered insights into this study's research questions. Specifically, analysis of the data revealed various conceptual obstacles that the students encountered and the reasoning abilities the students used when learning and using ideas of angle measure and the sine and cosine functions. This section outlines these findings by comparing the ways of thinking exhibited by the three students in this study. The students' understandings of angle measure are first described and then the students' conceptions of trigonometric functions

are characterized. The role of quantitative and covariational reasoning is also addressed throughout the characterization of the students' thinking.

Recall that the study's research questions were:

- What understandings of trigonometric functions do students develop during a trigonometry instructional sequence that emphasizes quantitative and covariational reasoning?
- What roles do quantitative reasoning and covariational reasoning play in students developing understandings of trigonometric functions?
- What understandings of the topics foundational to trigonometric functions (e.g., angle measure and the radius as a unit of measure) do students develop during the trigonometry instructional sequence?
- How do understandings of these foundational trigonometry topics influence students' conceptions of trigonometric functions?

### **Students' Conceptions of Angle Measure**

Consistent with research literature on students' and teachers' thinking (Akkoc, 2008; Brown, 2005; Fi, 2003, 2006; Topçu, et al., 2006), and the findings from the exploratory study, all three students held weak understandings of angle measure upon entering the teaching experiment. Each student alluded to a circle or an arc when attempting to describe the meaning of angle measure, but their conceptions consisted of properties of geometric objects (e.g., a line has one hundred and eighty degrees) rather than a measurement *process* involving measurable attributes (e.g., a subtended arc length and a circumference). Due to their inability to reason about the process of measuring an

angle in terms of coordinating quantities, they were unable to measure an angle when given sufficient supplies to accomplish this task. They were also unable to give meaningful explanations of the calculations they performed when trying to solve the pre-interview tasks.

The students' behaviors during the pre-interviews emphasize the implications of an angle measure conception that does not include a measurement process involving quantities and relationships between quantities. The students were able to perform calculations to solve for an angle measure when given the value of an arc length, but they were unable to solve problems that did not provide quantities' values explicitly in the problem statement. The students' preconceptions of angle measure may explain the consistent prior research finding that both students and teachers hold fragmented and underdeveloped conceptions of angle measure (Akkoc, 2008; Brown, 2005; Fi, 2003, 2006; Topçu, et al., 2006), such as their inability to reason about radian measure beyond converting to a number of degrees. These limited understandings may be the result of a conception of angle measure that does not include a measurement process grounded in a quantitative structure.

As the study progressed, Judy and Zac conceptualized angle measure as the fractional amount of any circle's circumference subtended by the angle (given that the circle is centered at the vertex of the angle). Judy and Zac constructed this conception of angle measure when attempting to create a protractor. When completing this task, they identified a subtended arc length as a measurable attribute of an angle's openness. Then, as they reflected on using circles (centered at the vertex of an angle) of various sizes to



identify arc lengths corresponding to one unit of angle measure, they determined that an angle subtended the same percentage of each circle's circumference.

Similar to the students in the exploratory study, Judy and Zac frequently leveraged reasoning about a multiplicative relationship between a subtended arc length and a circle's circumference to solve various angle measure tasks. They were also able to verbalize the meaning of angle measure in terms of this relationship, and spontaneously used this reasoning to convert between units of an angle's measure. As one example, Judy generalized a method for converting a number of radians to a number of degrees, where her method emerged from reasoning about the percentage of a circle's circumference cut off by the angle (e.g.,  $\frac{d}{360} = \frac{r}{2\pi}$ ). Zac also engaged in reasoning consistent with Judy's method when he converted between units of an angle's measure. Thus, it appears that an image of measuring a subtended arc relative to the circumference of the corresponding circle provided a way of reasoning that enabled them to fluently convert between units of angle measure.

Recall that Fi (2003, 2006) found that teachers used a procedural approach for converting between radian measures and degree measures and that these conversions were rooted in a *need* to convert to degree measurements in order to describe radian measurements. Fi noted that the teachers' conceptions of radian measure were dominated by this conversion. Judy and Zac constructed a quantitative conversion between units of an angle's measure, which appeared to help them avoid developing conceptions of radian measure that were dominated by this conversion. In fact, when asked to give explanations

of radian measures, Judy and Zac did not attempt to convert<sup>25</sup> to a number of degrees in order to explain the measure. Rather, Judy and Zac described radian measures as the result of measuring *along* a subtended arc in a number of radius lengths, an image that formed during their engagement in The Circumference Problem.

When completing The Circumference Problem, Judy and Zac identified the need to measure along a subtended arc length and then partition this arc into a number of radius lengths. This image supported their conceptualizing  $2\pi$  radius lengths rotating along *any* circle's circumference as they compared circles of different radius lengths (e.g.,  $C = 2\pi r$ ). By reasoning about a number of radius lengths rotating along a subtended arc for circles of various sizes, Judy and Zac also conceptualized a number of radians as conveying a multiplicative relationship between a subtended arc and a radius length (e.g.,  $\theta = \frac{s}{r}$ ) that was independent of the size of the circle used to measure the angle.

As the study progressed, Judy and Zac primarily reasoned about radian measures as a multiplicative comparison between a subtended arc and the length of a radius. This was contrary to the conceptions of the students in the exploratory study that involved their predominantly reasoning about radian measures in terms of a multiplicative relationship between a subtended arc and the circumference of a circle. Thus, it appears that increasing the focus of measuring arc lengths in a number of radius lengths during

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<sup>25</sup> When given angle measures in degrees, Judy and Zac frequently converted to a number of radians by reasoning about the multiplicative relationship between a subtended arc length and the circumference of the corresponding circle. As the study progressed, both students claimed that they found radian measures more powerful and easier to work with.

the instructional sequence supported Judy and Zac in constructing this conception of radian measure. Furthermore, rather than conceiving of  $\pi$  as a unit or the number 180, as revealed in the research literature (Akkoc, 2008; Fi, 2003, 2006; Tall & Vinner, 1981; Topçu, et al., 2006), Judy and Zac interpreted radian values with the  $\pi$  symbol<sup>26</sup> as a number of radius lengths (e.g., a value).

In contrast to Judy and Zac's thinking, Amy did not reason about angle measure in terms of a process of measuring along a subtended arc in a number of unit lengths; nor did Amy consistently reason about a *multiplicative* relationship between a subtended arc length and the circumference of a circle. Rather, Amy conceptualized angle measures as numerical labels of objects. She consistently expressed that "a circle" was  $2\pi$  radians (e.g., *Circle* =  $2\pi$ ) and her reasoning did not include conceiving of a circle's circumference as measurable in a number of radius lengths (e.g.,  $C = 2\pi r$ ). Amy's understanding of the number  $2\pi$  corresponding to a whole circle (e.g.,  $2\pi$  as a number, not a value) enabled her to reason additively about angle measures and to construct equivalent ratios to solve using cross-multiplication. For Amy, these ratios reflected additive, part (the numerator) to whole (the denominator) reasoning, which did not support her reasoning about a multiplicative relationship between a subtended arc and the circumference of a circle.

Amy's reasoning during the study was dominated by using part to whole relationships to the extent that she resisted reflecting on other meanings of angle measure.

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<sup>26</sup> Similar to the students in the exploratory study, Judy and Zac frequently reasoned about such measures as a fraction of a circle's circumference, but when prompted, they described the measurements in terms of a number of radius lengths.

After Amy reasoned about a multiplicative relationship between an arc length and the radius of a circle in response to prompting from the researcher, she explained that she “trust[ed]” cross-multiplication more. Furthermore, when Amy was prompted to solve tasks and reflect on her solutions without using cross-multiplication, she continually expressed a desire to cross-multiply and resisted making sense of these alternative solutions. As a result, Amy was able to correctly convert between units of angle measure using her part to whole reasoning and cross-multiplication, but she did not conceive of an arc as measurable in a number of radius lengths. This finding is consistent with the observance (Fi, 2003, 2006) of teachers converting between radian measure and degree measure while lacking an understanding of radian measure beyond this conversion. In Amy’s case, she lacked an understanding of radian measure as a quantitative relationship between the radius of a circle and a subtended arc.

In summary, Judy and Zac constructed angle measure conceptions that were rooted in quantitative relationships. After engaging in the instructional activities, they conceptualized processes of measuring an angle that stemmed from reasoning about quantities (e.g., a subtended arc length and the circumference or radius of a circle) and relationships between these quantities. Then, as they encountered novel problems and conceived of the relevant quantities, they leveraged their understandings of angle measure to solve the tasks. On the contrary, Amy did not appear to construct a *process* for measuring an angle that consisted of quantities and relationships between quantities. Rather, she conceived of angle measures as numerical labels of objects, opposed to the result of a process of measuring along an arc in a number of radius lengths. This led to

her relying on using part to whole ratios and cross-multiplication in order to solve problems, while remaining unable to explain her ratios and calculations as multiplicative relationships between quantities. As she encountered novel problems, she was unable to apply her reasoning to these situations without continued prompting from the researcher.

### **Students' Conceptions of Trigonometric Functions**

Consistent with the students' actions during the exploratory study, Judy and Zac leveraged their ability to reason about measuring (or traveling continuously) along an arc in order to imagine a varying subtended arc in the context of circular motion. When presented with The Fan Problem, this reasoning ability formed a foundation for their constructing a covariational relationship and subsequently formalizing this relationship as the sine function. Judy and Zac conceived of the unit circle (e.g., a circle with a radius length of one unit) and then reasoned about an increasing or decreasing rate of change of vertical distance above the horizontal diameter with respect to an angle measure in a number of radians (MA5 of Carlson et al.'s (2002) Covariation Framework). They supported this reasoning by comparing changes of vertical distance for equal changes of angle measure (MA3), which resulted in the graph and the formula of the sine function emerging from this reasoning.

Judy and Zac's use of the unit circle to reason about the sine function is consistent with Weber's (2005) suggestion that the unit circle be developed as a tool of reasoning. Additionally, the context of circular motion supported Judy and Zac constructing quantitative relationships that supported the emergence of the sine function as formalizing the relationship between two covarying quantities. This finding supports the

role that quantitative reasoning (Smith III & Thompson, 2008) and covariational reasoning (M. Carlson, et al., 2002) play in students understanding the sine function.

Judy and Zac also fluently reasoned about the relationship formalized by the sine function in terms of a process between the indeterminate values of two quantities. Specifically, Judy and Zac covaried two magnitudes (e.g., arc lengths and vertical distances) when constructing the sine function. The nature of their reasoning parallels a process conception of function; Judy and Zac were able to anticipate the sine function (or inverse sine function) producing an output and they were able to imagine this value varying for a varying angle measure (or arc length) without calculating numerical values. They were also able to perform operations on this output (e.g., converting to a different unit of measure) without first determining a numerical output value.

Judy and Zac's process conception of the sine function was consistent with their ability to reason about the quantitative relationship conveyed by a measure relative to the radius without performing calculations. Both students were able to anticipate converting between a number of radius lengths and a linear measure without needing to perform the numerical calculations. This reasoning ability stemmed from their understanding of a measure relative to the radius conveying a multiplicative relationship between a length and the length of the radius, where this relationship existed independently of performing numerical calculations.

Whereas Judy and Zac constructed a conception of the sine function that was rooted in covarying changes in an arc length and changes in a vertical distance (e.g., a relationship between two conceived quantities), Amy had difficulty engaging in such

reasoning, nor did she appear to value this reasoning. For instance, she conceived of measurements as referring to objects, which led her to conclude that  $\cos(\pi) = -1$  referenced the bottom half of a circle (e.g.,  $\pi$  as half of a circle, and the negative sign indicating the bottom half).

Reasoning about objects instead of quantities that covary led to Amy encountering difficulty in using a diagram of the unit circle to construct the graph of the sine function. When the researcher prompted Amy to use a diagram of a circle to explore the relationship formalized by the sine function, Amy did not construct distinct quantities on her diagram to covary. Instead, she confused various segments, areas, and arcs. Amy's inability to leverage the various contexts of trigonometry to support her reasoning was consistent with Weber's (2005) finding that the students who experienced the most difficulty had limited understandings of these contexts. Amy's actions reveal that her difficulties stemmed from not conceptualizing the relevant quantities of the situation.

Amy's inability to distinguish between the various quantities appeared to be related to her propensity to observe and mimic the actions of the researcher and her peers. For instance, she recalled observing the other students tracing a horizontal distance to determine a vertical distance on a coordinate system. This led to her having difficulty in distinguishing between the horizontal and vertical lengths. As a result, Amy used the shape of the circle to determine the shape of her graph. She was unable to support this reasoning by covarying amounts of change of a vertical distance and an arc length on her graph and diagram of the situation. Even though she was able to identify a vertical distance, she was not able to support her graph by reasoning about a changing vertical

distance (as distinct from areas) for successive changes of arc length. She also expressed that she did not value reasoning within the problem's context and that she believed it made mathematics more difficult.

Amy's inability (or unwillingness) to imagine quantities and their measures also contributed to her inability to reason about the sine function as a process relating two quantities. For instance, she stated, "I'm sure if I knew what the formula was I'd have more of an understanding what happens." This statement reveals that Amy desired to use a formula to calculate values of the sine function, instead of reasoning about the sine function in terms of a quantitative relationship. Amy's utterances suggest that she relied on an action conception of function, which is consistent with her not engaging in (or valuing) reasoning about quantities and relationships between quantities with indeterminate values. Hence, Amy looked to determine a method for calculating values of the sine function rather than constructing a conception of the sine function as a relationship between two quantities.

The students' conception of the sine and cosine functions, as well as the radius as a unit of measurement, also had implications relative to their understanding of right triangle trigonometry contexts. Both Judy and Zac spontaneously used the hypotenuse of a right triangle to construct a circle with a radius equivalent to the length of the hypotenuse. They then conceived of measuring the sides of the right triangle relative to the hypotenuse, or radius. The image of a common unit of measure between the contexts enabled Judy and Zac to create coherence when using trigonometric functions in a right triangle context or a unit circle context. More pointedly, reasoning about the hypotenuse



as a radius and a unit of measure led to Judy and Zac leveraging the unit circle as a tool of reasoning in both contexts of trigonometry. This finding supports Weber's (2005) suggestion that students may benefit from developing the unit circle as a tool of reasoning.

Contrary to Judy and Zac, Amy's actions when working within a right triangle context consisted of recalling ratios (e.g., opposite over adjacent) and calculations that did not reflect quantitative relationships. In right triangle contexts, she referenced her notes from the previous instructional activities to justify her solutions. As an example, when asked to determine the hypotenuse of a right triangle, she suggested dividing the leg *opposite* an angle by the leg *adjacent* to the angle. In response to the researcher asking her to explain the meaning of this calculation, she argued that the calculation yielded the correct result because it "was in [her] notes." It is important to note that the previous instructional activity explored the tangent function in a right triangle context. In this case, Amy's focus on calculations and procedures led to her recalling a previous session's calculation that did not reflect a quantitative relationship. Additionally, Amy's attempt to recall a previously observed calculation is consistent with her referring to the other students' actions when constructing a graph of the sine function.

Amy frequently attempted to recall procedures and formulas when attempting tasks and she expressed discomfort when the researcher asked her to reflect on her reasoning or engage in reasoning about quantities independent of numerical calculations. In fact, during a teaching experiment session midway through the study, she exclaimed, "I'm just here to learn...I do not know how to do this so that is why I'm here learning."

Her statement was in response to the researcher asking her to recall the two quantities related by the sine function, which was explored during the previous teaching experiment sessions. Amy's inability and resistance to engage in quantitative reasoning inhibited her constructing understandings consistent with the instructional goals. Instead, her understandings consisted of a combination of procedures and calculations that she was unable to apply consistently and flexibly to novel problems. To say more, Amy did not construct measurable attributes of situations or reflect on relationships between these quantities in a way that enabled her to engage in novel reasoning. As a result, when she encountered new problems she could only apply procedures and calculations previously performed and she exhibited no ability to check or justify these actions relative to a quantitative structure.

Whereas Amy's actions focused on procedures and calculations, Judy and Zac engaged in the instructional activities in ways that led to them constructing situations consisting of quantities and relationships between these quantities. The quantitative structures constructed by Judy and Zac formed a foundation that supported and warranted their reasoning in ways consistent with the instructional goals. Then, as they engaged in novel problems, they constructed an image of the situation described by the problem such that they could leverage these understandings. Also, both Judy and Zac emphasized that although they found trigonometry difficult, they valued the reasoning they engaged in during the study. As one example, Zac stated that measuring quantities relative to the radius (or hypotenuse) created coherence between a unit circle context and a right triangle context. Judy claimed that reasoning about the rate of change between two

quantities offered her a tool for reasoning about graphs. She added that she valued exploring the meaning behind her actions and calculations and that she benefited from taking this approach in other classes.

### **Trigonometric Function Framework**

In order to synthesize the findings that emerged from this study, a framework is provided to list a set of reasoning abilities and understandings that can lead to a mature and coherent understanding of trigonometric functions. The Trigonometric Function Framework<sup>27</sup> is composed of reasoning abilities and mental actions (Table 46) that are grounded in the results of this study. Also, the Trigonometric Function Framework parallels and supports the critical ideas and reasoning abilities necessary for trigonometry identified by Thompson (2008).

Each aspect of this framework could manifest itself in a variety of manners relative to student behavior and reasoning. For instance, a student may conceptualize a ratio of lengths in a variety of ways (e.g., as measuring a quantity in units of another quantity, or as a numerical calculation). As another example, a student can reason about the dynamic relationship between angle measure and a length in a variety of manners. Because of the differences that may exist in students' reasoning, the framework is not intended to be exhaustive in describing the numerous subtleties in reasoning that a student may engage in, but instead provide an outline of the various reasoning abilities that can lead to a deep understanding of angle measure and trigonometric functions. This

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<sup>27</sup> Note that the framework is discussed in the context of the sine function.

framework can provide guidance for the design of trigonometry lessons (e.g., instructional goals), as well as future studies into students' thinking in trigonometry.

Table 46

*Trigonometric Function Framework*

<b>Angle Measure</b>	Conceiving of a subtended arc as a measurable attribute of an angle.
	Measuring <i>along</i> a subtended arc in a unit length.
	Constructing a multiplicative relationship between a subtended arc and the circumference of that circle, where this comparison is constant for a circle of any size (that is centered at the vertex of the angle).
	Conceiving of a <i>process</i> of measuring an angle that necessitates the construction of a circle.
	Constructing a multiplicative relationship between a subtended arc length and the length of the radius, where this comparison is constant for a circle of any size (e.g., a radian measure).
	Covarying a subtended arc, or an angle measure, and another quantity (e.g., time).
	Conceiving of an angle swept out by an object traveling on a circular path as a measurable attribute of circular motion.
	Conceiving of measuring the angle swept out by an object by measuring a varying subtended arc length.

Conceiving of a vertical distance<sup>28</sup> as a measurable and varying attribute of circular motion.

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Covarying amounts of change and rates of change of vertical distance with respect to a varying angle measure in terms of indeterminate values (e.g., a process conception of covariation/function).

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**The Sine**

**Function**

Constructing a multiplicative relationship between the vertical

distance and the length of the radius (e.g., constructing the unit circle by measuring all lengths relative to the radius).

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Formalizing the covariational relationship between the vertical distance and angle measure, with both measured relative to the radius, as the sine function.

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Constructing a circle using the hypotenuse of a right triangle.

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Constructing a multiplicative relationship between the legs of the right triangle and the hypotenuse of the right triangle (e.g., the hypotenuse as the unit of measure).

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In order to characterize the students' thinking during this study it was necessary to examine their reasoning patterns (e.g., covariational reasoning) and mental constructs as they engaged with the instructional materials. The findings of this study and the items provided in the Trigonometric Function Framework support that the theories of quantitative reasoning (Smith III & Thompson, 2008; Thompson, 1989) and covariational

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<sup>28</sup> The term vertical distance is used to reference the vertical distance of the object above the horizontal diameter of the circle.

reasoning (M. Carlson, et al., 2002; Saldanha & Thompson, 1998) were effective lenses for describing and comparing the students' mental constructions. As characterized in the framework above, the results illustrate that covariational reasoning provides an accurate characterization of how students reason when constructing meaningful graphs of the sine and cosine functions. In particular, covariational reasoning provides a lens that characterizes the mental processes involved in thinking about how an angle measure and a conceived length (both measured in units of a radius) change together. The results of this study also support the role quantitative reasoning plays in characterizing the mental images students construct when problem solving. In order to determine the nature of the students' thinking during the teaching experiment, it became necessary to build models of their mental images and to consider the role of these mental images in their thinking and reasoning (e.g., covariational reasoning). This analysis revealed the central role of students conceptualizing measurable attributes when reasoning about trigonometric functions and angle measure.

Relative to the theory of quantitative reasoning, this study's findings and the framework above also highlight the importance of students' conceptualizing a measurement *process* and the result of a measurement process as a multiplicative comparison. Both the input and output quantities of trigonometric functions are based on a multiplicative comparison of lengths, and it is therefore important that students construct each of these quantities and a multiplicative comparison between the quantities. This study's findings supports the importance and usefulness of gaining insights into the

mental images students construct and the implications of these mental images (Smith III & Thompson, 2008).

Overall, a student that has the reasoning abilities identified in the framework can use formulas and mathematical representations (e.g., graphs and function notation) as meaningful formalizations of the identified quantitative relationships. Reports have revealed that it is difficult for students to use function notation and language (M. Carlson, 1998). This difficulty can present itself as a large hurdle in representing trigonometric functions (as seen with Amy), as function notation is used in the symbolic representation of the trigonometric functions. Because of this convention, it becomes necessary that students view the notation as representing a relationship between two varying quantities. By first constructing quantitative relationships between indeterminate values, students might be more prepared to reason about  $f(\theta) = \sin(\theta)$  as the notation for a *process* that produces an output value for an input value without having to numerically evaluate the sine function. However, if they have not constructed a quantitative relationship that consists of indeterminate values, it becomes a daunting task for the student to reason about  $f(\theta) = \sin(\theta)$  as formalizing a *process* between two quantities.

This use of function notation within the symbolic rule of the sine function also stresses the importance that students develop the ability to leverage the contexts of trigonometry as tools of reasoning. As Weber revealed (2005), and consistent with the findings from this study, a student's ability to reason about the sine function is highly reliant on his or her ability to construct contextual representations of the quantitative relationships defined by trigonometric functions. The contexts in which trigonometric

functions are used can provide a foundation for reasoning about input-output processes between two quantities. As such, the importance of students constructing meaningful quantitative conceptions in these contexts cannot be over emphasized. The next section explores the implications of the students' propensity to leverage contexts meaningfully (e.g., using a diagram of the unit circle).

### **Implications of the Students' Problem Solving Behaviors**

Over the course of the study, the students' propensity and ability to engage in quantitative reasoning influenced the nature of their problem solving behaviors. For example, students with strong quantitative reasoning abilities were observed to be more effective in orienting to a problem. These relationships offered insights into the various mental actions behind their problem solving behaviors in the context of the Multidimensional Problem Solving Framework provided by Carlson and Bloom (2005) (e.g., the problem solving phases of orienting, planning, executing, and checking).

When orienting to a problem, Amy predominantly attempted to recall a procedure, formula, or calculation from the previous instructional activities. In the instances that Amy constructed a diagram as a strategy for orienting herself to a problem, her focus on the diagram was brief and she did not identify distinct and measurable attributes; nor did Amy return to her diagrams after this initial construction without prompting from the researcher. Thus, Amy's orienting phase was typically brief and did not result in her constructing a meaningful mental image of the problem situation.

If Amy recalled a formula or procedure when orienting to a problem, she immediately executed calculations (not quantitative operations), as opposed to



anticipating or planning future actions in terms of quantities and relationships between these quantities. After Amy executed calculations, she had difficulty checking her solutions. Her checking actions typically consisted of considering the aesthetics of her solution and determining if she “trust[ed]” the procedure she applied. Thus, it appears that Amy did not develop a quantitative structure when orienting to a problem’s context that supported her in checking her solutions.

In the cases when Amy could not recall a formula or procedure to execute, she resisted initiating and considering conjectures, nor did she appear to plan solutions that were not readily apparent. Then, in the instances when the researcher provided prompts to aid her progression, she resisted considering subsequent actions and reflecting on the thinking that resulted from the researcher’s prompts. Also, she continually expressed a need to execute numerical calculations prior to interpreting the meaning of a calculation, regardless of whether the calculation was her idea or posed by the researcher. After she executed a calculation and obtained an answer, she remained satisfied with the solution and did not value reflecting on her calculations. This approach to problem solving implied that Amy strove to obtain a result that satisfied the goal of each problem, but that she did not value understanding her solution beyond obtaining the correct answer. In other words, she did not desire or find value in making sense of her calculations in terms of the quantities of the situation and the relationships these calculations implied.

Amy’s resistance to initiating conjectures and reflecting on her reasoning may have been directly related to a lack of confidence in her mathematical reasoning abilities. She frequently verbalized that she did not trust her reasoning, and her actions during the

instructional activities implied that she valued the other students' solutions more than her solutions. Amy's disposition during the teaching sessions also implied that she believed her role was to watch and listen to the researcher (or other students) provide a solution that would be applied to subsequent tasks. Amy relied on interpreting the actions of the researcher and other students in an attempt to remember procedures and calculations, as opposed to engaging in reasoning that entailed constructing quantities and quantitative relationships. Amy's approach to the group sessions helps explain her inability to provide meaningful solutions beyond applying previously observed procedures and calculations.

While Amy maintained a calculational focus to problem solving and did not appear to value sense making beyond obtaining a procedure to solve a problem, both Judy and Zac focused heavily on the context of a problem when attempting to solve the instructional activities. First, Judy and Zac's orientation behaviors consisted of constructing a diagram and identifying (both known and unknown) values and quantities on the diagram (e.g., tracing an arc length). These orientation behaviors created a foundation that Judy and Zac leveraged to construct quantitative relationships and plan their solutions by *anticipating* the execution of operations. More pointedly, previous to executing calculations, they used diagrams to construct quantitative relationships and reason about these relationships in order to plan a sequence of calculations (e.g., a sequence of quantitative operations). The quantitative structures that resulted from these actions also enabled Judy and Zac to check their solutions in terms of quantitative relationships and the context of the problem. When obtaining an expected or unexpected result, they reflected on their diagrams in order to interpret their solution. This resulted in

Judy and Zac *continually* refining their image of the problem situations as they reflected on and modified their solutions.

Similar to Amy, Judy and Zac expressed that they found trigonometry difficult and frustrating. But, they explained that they valued making sense of their solutions. Rather than focusing on calculations and obtaining correct answers as Amy did, Judy and Zac strived to understand the mathematical ideas that justified their calculations and solutions. On several occasions, both students verbalized that they appreciated the quantitative focus required by the activities and the understandings they had obtained.

The students' approaches to problem solving revealed the role of quantitative reasoning in solving novel problems. Amy's focus on procedures and formulas limited her ability to solve novel problems and engage in reasoning consistent with the instructional goals. On the other hand, Judy and Zac's propensity to reason about the context of a problem resulted in them reflecting on actions consisting of quantities and relationships between quantities. This enabled them to construct understandings and reasoning abilities rooted in quantitative structures. Then, as they encountered novel situations, they leveraged these understandings to solve the various tasks by focusing on the quantities in the problems' contexts and how these quantities were related.

This study offered insights into the various mental actions at play during the problem solving phases identified by Carlson and Bloom (2005). Due to Amy's focus on performing calculations and recalling past procedures, she did not appear to engage in or value constructing a mental image, or scene, of the problem's context. As a result, her approach to solving problems did not involve her conceptualizing quantities and

relationships between quantities. She performed calculations that were not rooted in quantitative relationships and was unable to plan or check her solution independent of calculating numerical values. Contrary to Amy's approach to problem solving, Judy and Zac constructed quantities and relationships between quantities prior to executing calculations and procedures. During the executing stage of solving a problem, their actions were grounded in quantitative relationships that served as a foundation for their creating a robust mental structure of the problem's quantities and their relationships. As a result, this mental structure provided a foundation for their anticipating and checking their solutions. They also frequently reflected on and refined their image of the problem situation as they progressed on a problem. When obtaining an unexpected result, this led to further orienting to the problem situation and constructing an alternative solution grounded in their refined image of the situation.

The instructional activities intended to promote the students engaging in quantitative reasoning in order to solve novel problems, with the hope that the students would reflect on this reasoning and the quantitative structures they constructed. Both Judy and Zac engaged in reasoning consistent with the instructional intentions, which led to them developing meaningful and flexible understandings. However, Amy did not appear to construct quantitative structures to reflect upon when problem solving. Her approach to novel problems emphasizes the importance of curriculum promoting students engaging in meaningful problem solving behaviors. The following section explores suggestions such as this for curriculum and instruction.

### **Suggestions for Curriculum and Instruction**

This study contained an instructional sequence that was designed to promote the students constructing understandings by confronting novel problems that required them to engage in quantitative and covariational reasoning. However, as inferred from the data, the students did not always engage in reasoning consistent with the instructional goals. Specifically, Amy's calculational and procedural approach to solving novel problems did not support her engaging in quantitative and covariational reasoning. This resulted in Amy constructing limited and fragmented understandings of trigonometric functions and angle measure. As such, she was unable to use these ideas to solve novel problems.

Amy's actions reveal that merely offering students contextual problems does not imply that the students will engage in quantitative or covariational reasoning. Thus, it is important that curriculum and instruction promote problem solving behaviors that are consistent with the reasoning students are expected to engage in. If students are to engage in covariational reasoning and quantitative reasoning to construct mathematical understandings, it is necessary that students first construct a problem situation and quantities to reason about. Then, as students continue to solve a problem, they should consider, or anticipate, calculations relative to the quantities of the situation (e.g., quantitative operations (Thompson, 1989)) opposed to immediately performing procedures or using formulas that have no quantitative meaning for the students. By maintaining a quantitative focus when responding to a novel problem, the students are able to build and leverage a quantitative structure to check, correct, and reflect on their solutions. For instance, relative to this study, Judy and Zac were able to construct and

leverage their conception of the unit circle throughout their solutions due to their quantitative understanding of the radius as a unit of measure.

Developing problem solving behaviors consistent with the understandings students are to construct can be applied to any mathematical topic. A student's approach to mathematics and problem solving is influenced by their previous experiences in mathematics courses. Thus, at all levels of schooling, curriculum and instruction should focus on promoting reasoning and problem solving dispositions that are immediately and developmentally beneficial for the learner. As revealed with Amy, approaching problem solving with a procedural and calculational disposition leads to narrow and inflexible reasoning that is not readily adaptable when making sense of novel contexts or new mathematics ideas. The findings of this study suggest that developing robust reasoning patterns and problem solving abilities is complex and requires explicit instructional interventions to do so. As such, it is highly recommended that that curriculum and instruction make concerted efforts to develop and reinforce these reasoning patterns needed for successful problem solving throughout a student's mathematical journey.

Specific to trigonometry, this study also reveals that a student's conceptions of angle measure and the radius as a unit of measure are critical for reasoning about trigonometric functions. Because the sine and cosine functions formalize a quantitative and covariational relationship, it is necessary that a student construct quantities and ways to measure these quantities that support her or his thinking about the trigonometric functions. By constructing understandings of angle measure and the radius as a unit of measure that are rooted in quantitative relationships, the groundwork is laid for a student

to leverage the unit circle to reason about a varying arc length and other quantities measured in terms of the radius. Also, through conceptualizing quantitative relationships within the context of circular motion, a student can create a foundation to reason indeterminately about the input-output processes formalized by the sine and cosine functions. This enables the students to anticipate an input-output process without needing to evaluate the sine or cosine function. In light of these findings, curriculum and instruction should be designed such that students construct understandings of angle measure and the radius as a unit of measure in ways that form foundations for reasoning about trigonometric functions in the various contexts of trigonometry. These foundational understandings can lead to coherent reasoning in the contexts of trigonometry, where the contexts of trigonometry become tools of reasoning for the students.

This study also reveals the importance of students engaging in *processes* of measuring quantities and constructing relationships between quantities. In the case that a student merely observes a peer or the teacher engaging in these processes, the student doing the observing may not develop understandings consisting of quantities and relationships between quantities. Rather, the student may focus on the procedures or calculations, as he or she did not engage in the mental processes of constructing quantities to measure and relate. Thus, it is important that students are given the opportunity to reason through problems and reflect on this reasoning, instead of providing solutions and procedures worked out by another individual to the students.

### **Limitations of the Study**

This study explored the reasoning abilities and understanding of three precalculus students. As each student constructs unique knowledge, the results of this investigation may not apply to all students. Rather, the results of this study provide insights into students' reasoning abilities in trigonometry that can serve as the groundwork for future research. Also, the students of this study were drawn from a reformed precalculus course. The design of the course attempted to promote the students engaging in quantitative and covariational reasoning. Thus, previous to the study, the students experienced a classroom setting that was mostly consistent with the approach of the instructional sequence used in this study. Students that experience a more traditional precalculus course may perform much differently than the students of this study. In such a case, the students may experience difficulty engaging in the instructional activities at a quantitative level, much like Amy's actions during the study.

A second limitation of this study was the group setting in which the students engaged in the instructional activities. This setting made it difficult to capture the entire progress of each student on the instructional activities, as each student's level of participation varied during these activities. An interview setting was used to pose additional tasks to the students in order to test the researcher's models of their mathematics, but their engagement during the initial instructional tasks was not captured at this individualized level.

The study also tracked only what the students did during the instances that they were videotaped. Consistent with Judy's desire to solve a pre-interview problem when the interview concluded, it was assumed that the students considered the instructional



activities when not in the research setting. The study occurred over a five-week period, which likely increased the chance that the students reflected on the activities of the study when outside of the research setting. Additionally, each student mentioned that this study was not their first experience with trigonometry; and for those students that continued on to another math course, it was likely not their last experience with trigonometry. As a result, the results presented in this study explain their thinking and reasoning during the five-week period of the study, but not the entirety of their trigonometric (or mathematical) experiences.

Lastly, this study captured the students' observable behaviors, which were then used to construct and test models of their thinking. However, stemming from the theoretical perspective of this study, these were merely models of the students' mathematics that are not to be interpreted as exact matches of the mental actions driving their behaviors. Thus, the results presented should be read as the researcher's interpretation of the students' understandings and reasoning, where this interpretation was grounded in building and testing models of the students' mathematics, which were inferred from the students' observable behaviors.

### **Directions for Future Research**

This study investigated the role of quantitative and covariational reasoning in precalculus students constructing understandings of angle measure and trigonometric functions. Specifically, this dissertation focused on precalculus students constructing and reasoning about trigonometric functions in the contexts of the unit circle and right triangles. In order to further explore quantitative reasoning in the contexts of

trigonometry, future studies should also explore subsequent topics of trigonometry. For instance, little is known relative to students' understandings of periodicity in the context of trigonometric functions or students' conceptions of polar coordinates, which are reliant on trigonometric functions. As another example, a future study should explore students' abilities to apply trigonometric functions in a periodic, but non-circular context (e.g., wave behavior).

Second, the measurement conceptions held by the students of this study appeared to play a role in their thinking and reasoning abilities. Given students' varying conceptions of measurement (e.g., as labels of objects or as a multiplicative comparison), the role of students' measurement conceptions should be further explored in the context of angle measure and trigonometry.

Third, this dissertation revealed insights into the relationship between quantitative reasoning and students' problem solving behaviors. Due to the importance of students' ability to solve novel problems, the relationship between students' problem solving behaviors and the mental actions driving these behaviors offers a future research path. This includes considering a student's confidence in their ability to make sense of a mathematical topic and their disposition towards learning mathematics and engaging in novel problem solving.

Lastly, a product of this dissertation is the instructional sequence that was used during the study. Yet, someone attempting to use these materials may interpret the intentions of the tasks differently from their design. As research literature on trigonometry reveals (Thompson, et al., 2007), teachers hold very entrenched conceptions

of trigonometry. Also, this study was conducted in a setting of three students, rather than a classroom of twenty or more students. Thus, a study investigating teachers' interpretations of the instructional materials and their implementation of these materials may offer insights into the professional development needed for a teacher to implement research-based materials successfully.

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## APPENDIX A

### COVARIATION FRAMEWORK AND TRIGONOMETRY



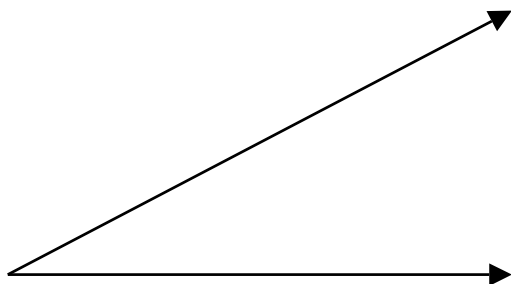
Mental Actions	Description of Mental Actions	Verbal Behaviors Related to <i>General</i> Functions	Verbal Behaviors Related to Trigonometric Functions
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	Verbal indications of coordinating the two variables (e.g., $y$ changes with changes in $x$ )	Verbalizing that the output of $\sin(\theta)$ changes with changes in <i>angle measure</i> , $\theta$ )
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	Verbalizing an awareness of the direction of change of the output while considering changes in input	Verbalizing an awareness of the <i>increasing</i> output values of $\sin(\theta)$ with <i>increasing</i> values of angle measure $\theta$ ( $\theta$ between 0 and $\pi/2$ radians)
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	Verbalizing an awareness of the amount of change of the output while considering changes in the input	Verbalizing that for an angle measure <i>increasing</i> from 0 to $\pi/2$ radians, the output values of $\sin(\theta)$ <i>increases</i> from 0 to 1 length of a radius.
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input	Verbalizing that the average rate of change of the output values of $\sin(\theta)$ with respect to angle measure $\theta$ <i>decreases</i> for successive uniform increments of angle measure $\theta$ between 0 and $\pi/2$ radians.
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)	Verbalizing an awareness that the instantaneous rate of change of the output values of $\sin(\theta)$ with respect to angle measure $\theta$ <i>decreases</i> over the domain of $\theta$ values from 0 to $\pi/2$ radians.

## APPENDIX B

### EXPLORATORY STUDY INTERVIEW TASKS

## Pre-interview Tasks

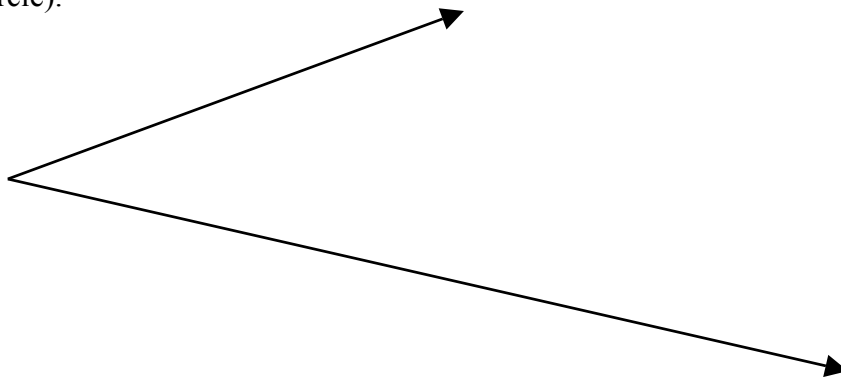
1. An individual is riding a Ferris wheel that has a radius of 51 feet. On part of a trip around the Ferris wheel, the individual covers an arc-length of 32 feet. How many degrees did the individual travel?
2. The following angle has a measurement of 3 units. How could you use arc-length and circumference to determine how many of these units rotate a full circle?



3. What does it mean to say an angle has a measure of one degree? 34 degrees?

## Exploratory Interview Tasks

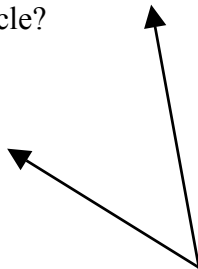
1. What is an angle and what does it mean to measure an angle?
2. Measure the following angle using the available supplies (compass, Wikki Stix, and ruler). Measure the angle in both degrees and quips (recall that 16 quips rotate a circle).



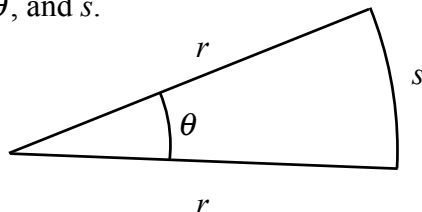
3. Determine the measurement (in degrees) of an angle that has a measurement of 22.3 degrees plus 3.1 quips. Construct this angle using a compass and Wikki Stix.
4. A student measures the arc-length that an angle cuts off and claims that the angle has a measure of 1.7 inches. Discuss how the arc length measurement relates to angle measure. Describe how to reconstruct the circle that was used to measure this angle given that the angle has a measurement of 27 degrees.
5. A Ferris wheel with a radius of 41 feet is rotating at 2.5 full revolutions per minute. Marcus boards the Ferris wheel for a ride. After 20 seconds, how far has Marcus traveled on the Ferris wheel? Give your answer in both a linear measurement (e.g., a number of feet) and a number of degrees.
6. If the arms on the Ferris wheel (described in item 5 above) are extended to 52 feet and the rotating speed of 2.5 full revolutions per minute is maintained i) Would the linear measurement of Marcus' rotation change? ii) Would the number of degrees that Marcos rotated change?
7. While site seeing in New York City, Bob stopped 1000 feet from the Empire State Building and looked up to see the top of the Building. Given that the angle of Bob's site from the ground was 56 degrees, determine the height of the Empire State Building.

#### Post-interview Tasks

1. An individual is riding a Ferris wheel that has a radius of 51 feet. On part of a trip around the Ferris wheel, the individual covers an arc-length of 32 feet. How many degrees did the individual travel? How many radians did the individual travel?
2. The following angle has a measurement of 2.1 units. How could you determine how many of these units rotate a full circle?



3. Using the diagram below, determine an algebraic relationship between the measurements  $r$ ,  $\theta$ , and  $s$ .



4. What fraction of a circle's circumference does an angle that is 4.1 radians minus 2.1 degrees cut off? How many degrees is the resulting angle? How many degrees is the resulting angle? Explain how you would construct the resulting angle using Wikki Stix, a compass, and a ruler.
5. Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full rotation. April boards the Ferris wheel and begins a continuous ride on the Ferris wheel.
- Determine a formula that relates the total distance traveled by April and the time since her Ferris wheel ride began.
  - If the platform to board the Ferris wheel is 8 feet off of the ground, sketch a graph that relates the total distance traveled by April and her vertical distance from the ground.
  - If the platform to board the Ferris wheel is 8 feet off of the ground, sketch a graph that relates the time since beginning the ride to April's vertical distance from the ground.
  - If the speed at which the Ferris wheel rotates is increased, how will your graphs above change? If the speed at which the Ferris wheel rotates is decreased, how will your graphs above change?
  - If the radius of the Ferris wheel is increased, how will your graphs above change? If the radius of the Ferris wheel is decreased, how will your graphs above change?

APPENDIX C

INSTRUCTIONAL SEQUENCE

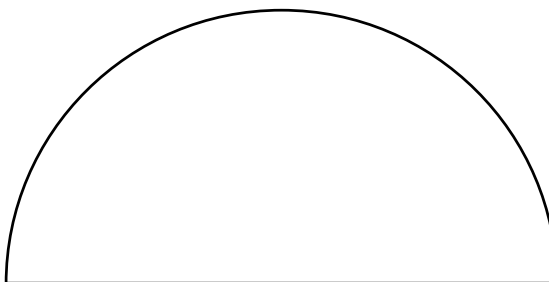
### Activity 1 - The Cannon Problem

Two military historian groups decided to test the range of two different World War I cannons that have different barrel lengths. During their testing both groups noted that the horizontal distance traveled by the projectiles fired from their cannons changed as they tilted their cannon barrels up and down. The two groups wanted to compare the horizontal distances traveled by the projectiles by the two cannons without transporting one cannon to the other. Describe the quantities that one group could measure to convey to the other group how to set up their cannon identical to the other group.

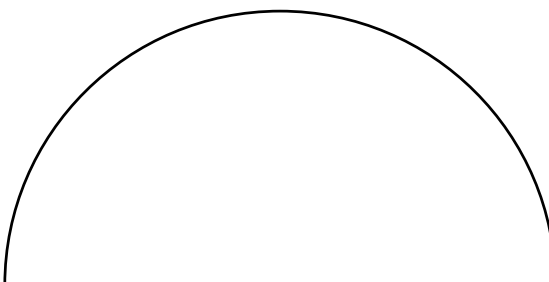
## Activity 2 - The Protractor Problem

Supplies: Wikki Stix, rulers, and unmarked protractors

Imagine that a group of individuals decided to measure an angle in a unit called *gips*, where any circle is 8 gips. Using the available supplies, construct a protractor to measure any angle in gips. Explain your approach to constructing your protractor and say why it works.



A different group of individuals decided they wanted any a circle to have a measure of 15 units. They chose to call these units *quips*. Using the available supplies, construct a protractor that measures any angle in *quips*. Explain your approach to constructing your protractor and say why it works.



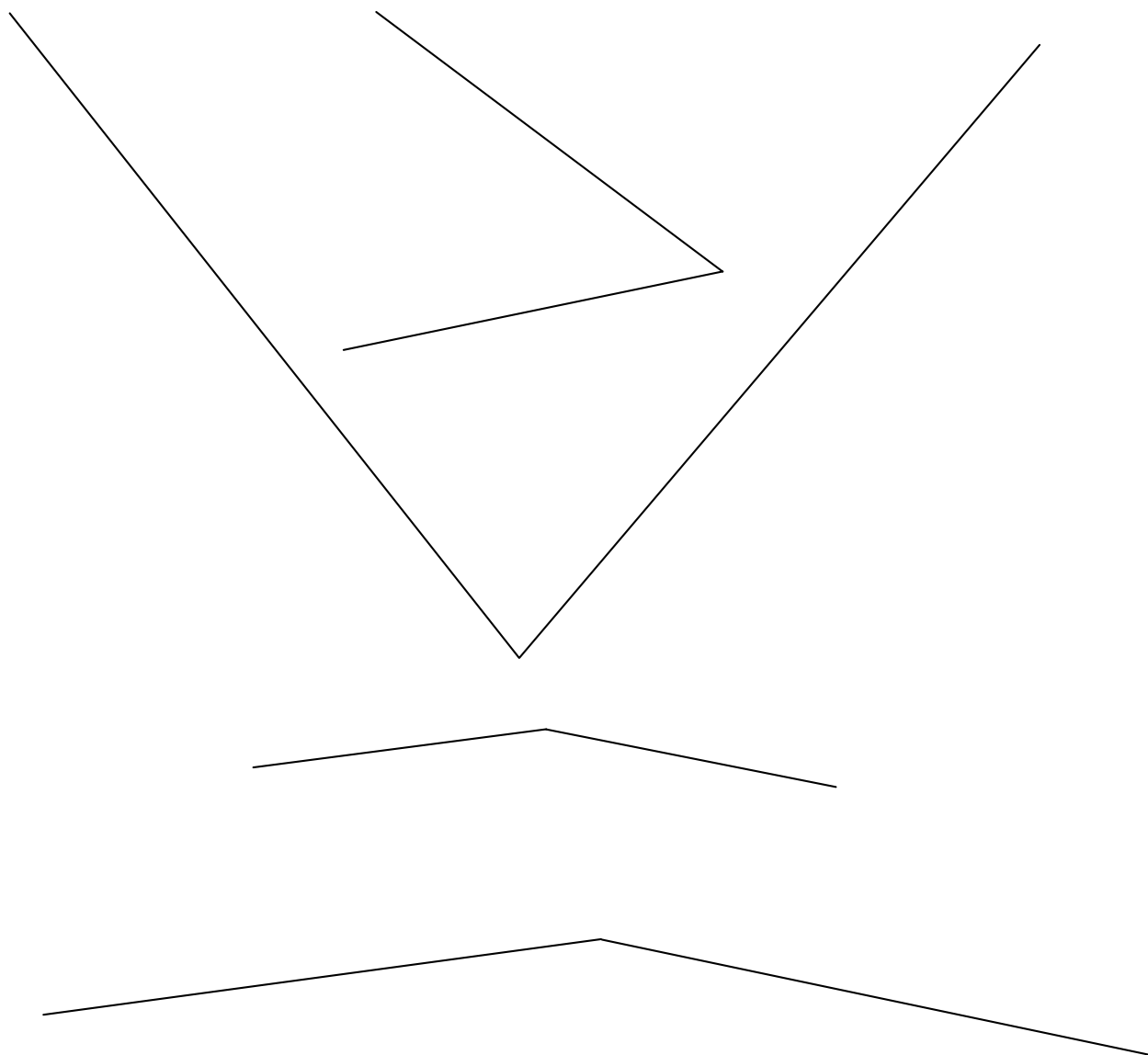
Describe how you might create a protractor to measure the openness of an angle in degrees.



## Activity 3 – The Angle Measurement Problem

Supplies: Wikki Stix, compass, ruler

Measure the following angles in degrees with only the available supplies. Be prepared to explain your reasoning.



#### Activity 4 – The Circumference Problem

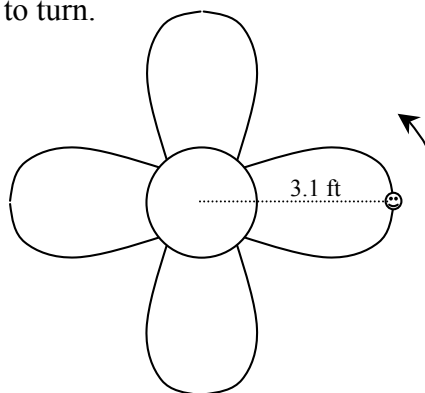
Construct a circle using a Wikki Stix as the radius (your group should have Wikki Stix of different lengths). Then, determine how many of your Wikki Stix mark off the circumference of your circle. Compare your result with your classmates. What observations can you make from this comparison?

Construct an angle that cuts off one Wikki Stix length of an arc. Compare the openness of the angle with those of your classmates.

Construct angles that measure 2.5 radians and 7 radians. Compare the openness of your angles with those of your classmates.

## Activity 5 – The Fan Problem

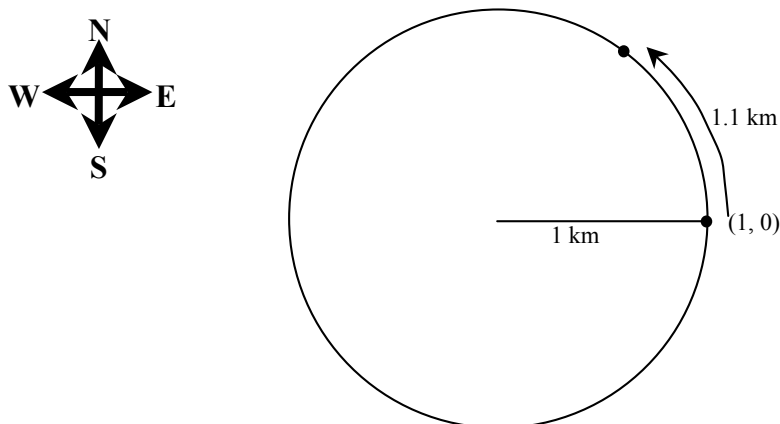
Imagine a bug sitting on the end of a blade of a fan as the blade revolves in a counterclockwise direction. The bug is exactly 3.1 feet from the center of the fan and is at the 3:00 position as the blade begins to turn.



- How does the distance the bug traveled on the fan correspond to angle measure?
- What would it mean to say the bug moved 0.765 radians around the fan? How far around the fan would this be? How many feet would 0.765 radians correspond to?
- Create a graph that shows how the bug's vertical distance above the 9:00 to 3:00 diameter line varies with the total distance the bug travels around the circumference. Label at least 3 points on the graph. Let the vertical distance be positive when the bug is above the 9:00 to 3:00 diameter line, and negative when the bug is below the 9:00 to 3:00 diameter line.
  - Using amounts of change of input and output, explain what your graph conveys about the covariation of the measure of the two quantities.
  - In what units did you measure the bug's total distance traveled and the bug's vertical distance above the 9:00 to 3:00 diameter line? Will your graph change if the radius of the fan is changed? If so, how will your graph change? If no, why not?
- Create a graph that shows how the horizontal distance to the right and left of the 12:00 to 6:00 diameter line varies with the total distance the bug travels around the circumference. Let the horizontal distance be positive when the bug is to the right of the 12:00 to 6:00 diameter line, and negative when the bug is to the left of the 12:00 to 6:00 diameter line.
  - Using amounts of change of input and output, explain what your graph conveys about the covariation of the measure of the two quantities.
  - In what units did you measure the bug's total distance traveled and the bug's horizontal distance to the right of the 9:00 to 3:00 diameter line? Will your graph change if the radius of the fan is changed? If so, how will your graph change? If no, why not?

## Activity 6 – The Positions on a Circle Problem

A certain arctic village maintains a circular cross-country ski trail for the enjoyment of its citizens during the winter months. This trail has a radius of 1 kilometer. A certain skier started at position  $(1,0)$  one morning, skiing counterclockwise for 1.1 kilometers, where he paused for a brief rest.



1. Explain what each of the following ordered pairs on the coordinate axes represents relative to the skier's position on the trail.
  - a.  $(1, 0)$
  - b.  $(0, 1)$
  - c.  $(-1, 0)$
  - d.  $(0, -1)$
2. Why can't the skier's position be represented by the ordered pair  $(1.1, 0)$ ?
3. Determine the ordered pair that identifies the location where the skier rested.
4. What is the meaning of the  $x$ -value of the ordered pair?
5. What is the meaning of the  $y$ -value of the ordered pair?
6. What does the ordered pair  $(\cos(1.1), \sin(1.1))$  represent?

A second arctic village maintains a circular cross-country ski trail for the enjoyment of its citizens during the winter months. Their trail has a radius of 2 kilometers. A certain skier started at position  $(2,0)$  one morning, skiing counterclockwise for 2.2 kilometers, where he paused for a brief rest.

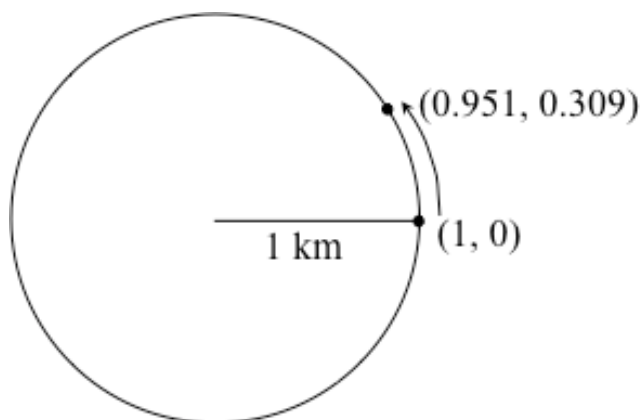
7. Draw a diagram of the situation and determine the ordered pair that identifies the location where the skier rested.

A third arctic village maintains a circular cross-country ski trail for the enjoyment of its citizens during the winter months. Their trail has a radius of 2.5 kilometers. A certain skier started at position  $(2.5, 0)$  one morning, skiing counterclockwise for 2.75 kilometers, where he paused for a brief rest.

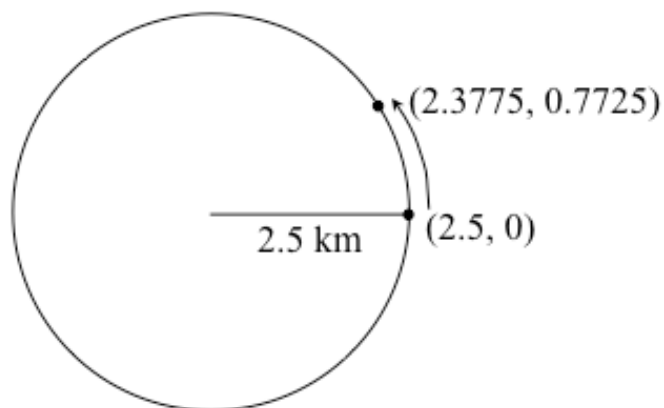
8. Draw a diagram of the situation and determine the ordered pair that identifies the location where the skier rested.
9. In summary, and based on your thinking for questions 3-7, what is the ordered pair for any location on the circle of radius  $r$  at an angle of measure  $\theta$ , where the center of the circle is considered the origin?

## Activity 7 – The Finding and Arc Length Problem

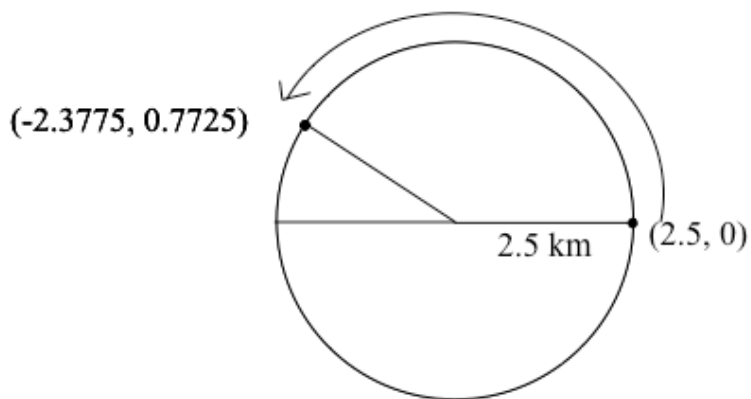
1. A skier skied on a circular route, starting at the point  $(1,0)$  on the circle, and ending at the point  $(0.951, 0.309)$  on the circle. How many km did she ski?



2. A skier skied on a circular route, starting at the point  $(2.5,0)$  on the circle, and ending at the point  $(2.3775, 0.7725)$  on the circle. How many km did she ski?

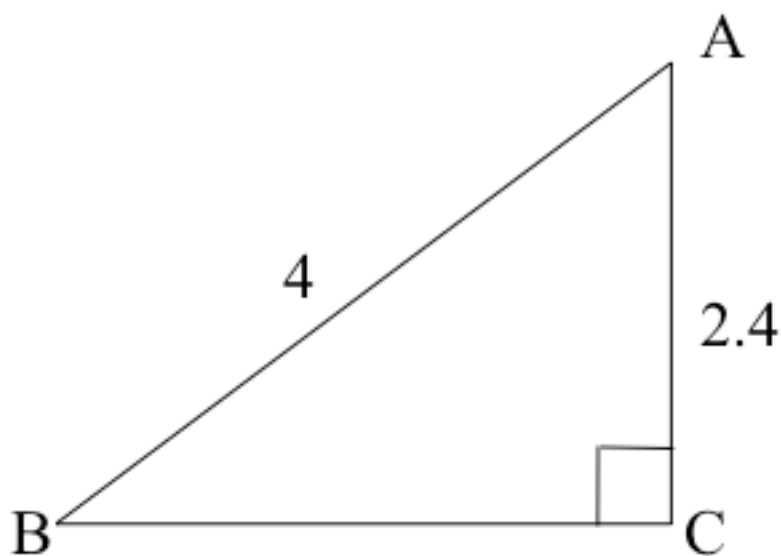
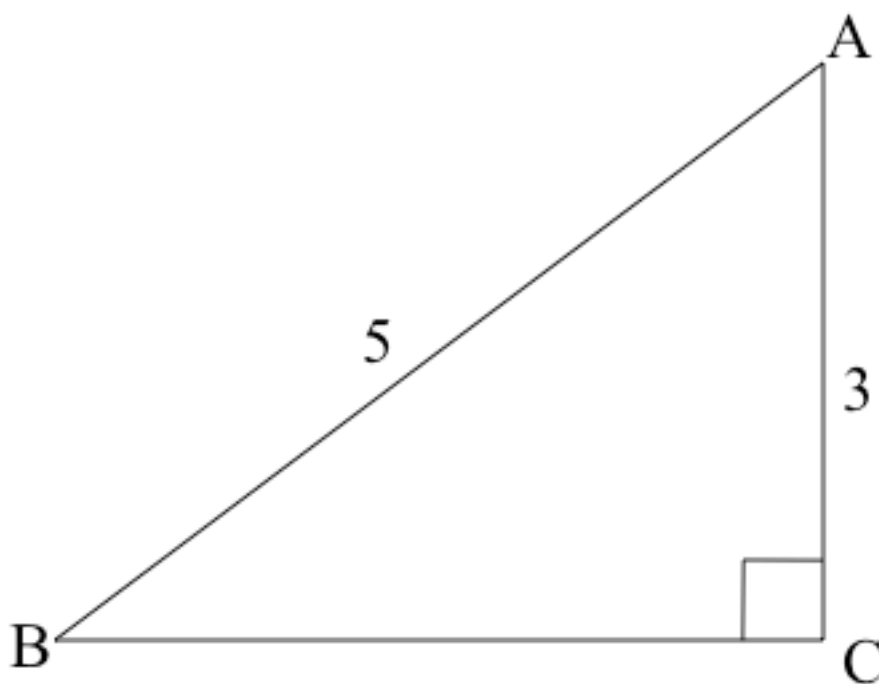


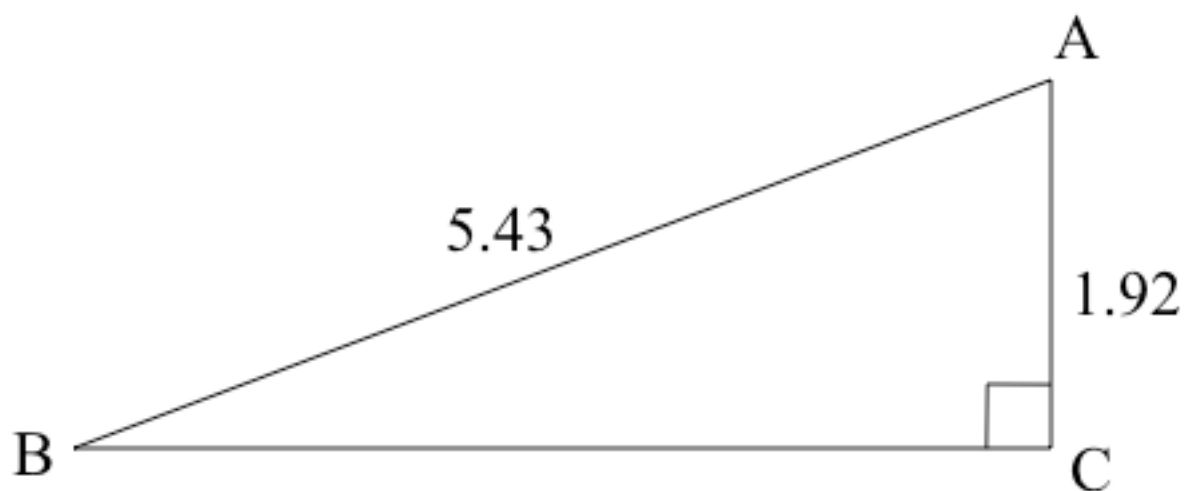
3. A skier skied on a circular route, starting at the point  $(2.5,0)$  on the circle, and ending at the point  $(-2.3775, 0.7725)$  on the circle. How many km did she ski? Calculate this distance using both inverse sine and inverse cosine.



## Activity 8 – The Determining an Output Problem

Determine the output of the sine and cosine of the measure of angle  $ABC$  *without* measuring the angle. *Hint:* think of how you would determine the measure of the angle of interest and how the sine function relates to this measurement.





### Activity 9 – The Airplane Problem

A plane leaves the local air force base and travels due east. A radar station 45 miles south of the base tracks the plane and determines that the angle formed by the base, the radar station, and the plane is initially changing by 1.6 degrees per minute. Determine the distance the plane is from the radar station after a number of minutes,  $m$ .

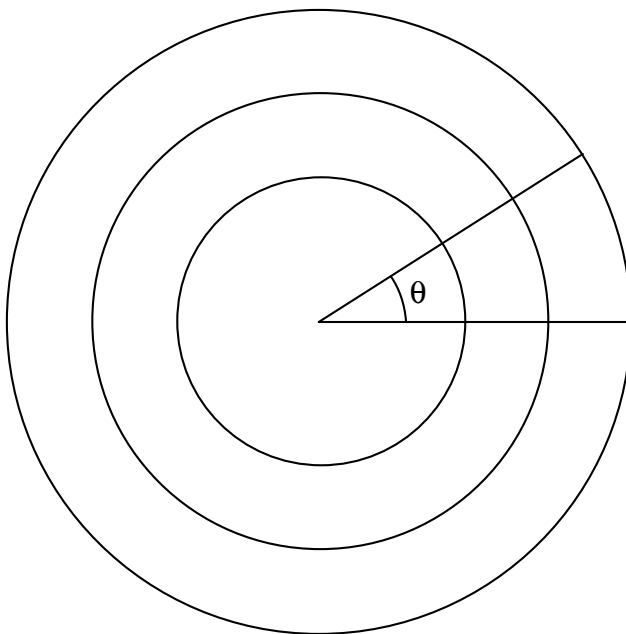


## APPENDIX D

### ZAC'S INTERVIEW TASKS

### The Arc Length Problem

Given that the following angle measurement  $\theta$  is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches. (*Drawing not to scale*)

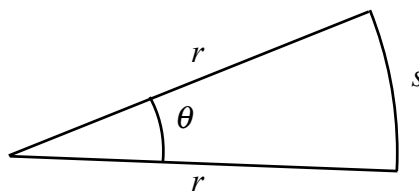


### The Radian Measurements and Pi Problem

What does it mean for an angle to have a measure of  $0.5\pi$  radians? 2.2 radians?

### The Arc Problem

Using the diagram below, determine a formula between the measurements  $r$ ,  $\theta$ , and  $s$ .



### The Ferris Wheel Problem

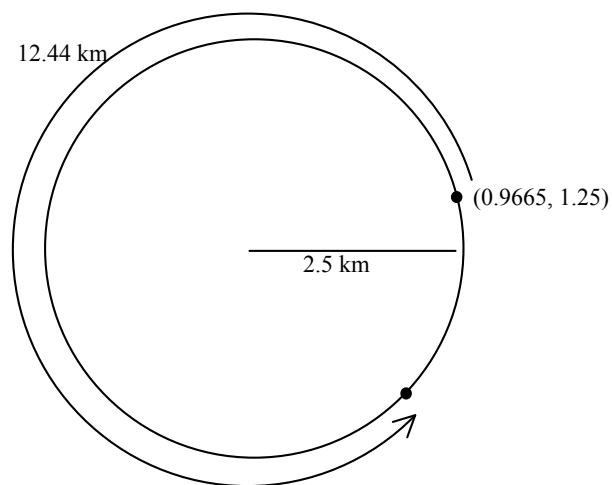
Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full rotation. April boards the Ferris wheel at the bottom and begins a continuous ride on the Ferris wheel.

1. Sketch a graph that relates the total distance traveled by April and her vertical distance from the ground.
2. Sketch a graph that relates the time since beginning the ride to April's vertical distance from the ground.
3. Determine formulas for the functions represented above.

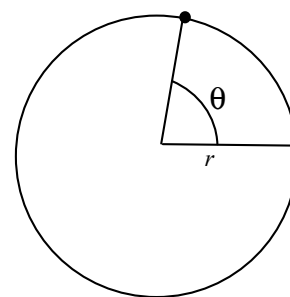
### The Ski Trail Problem – Version I

An arctic village maintains a circular cross-country ski trail that has a radius of 2.5 kilometers. A skier started skiing from position  $(0.9665, 1.25)$  and skied counterclockwise for 12.44 kilometers where he paused for a brief rest.

1. Determine the ordered pair (in both kilometers and percentage of a radius) on the coordinate axes that identifies the location where the skier rested.



2. What is the general form of the ordered pair, in a percentage of a radius, of any coordinate on a circle of radius of  $r$  kilometers that forms an angle of measure  $\theta$  as illustrated below? (Assume that the center of the circle is positioned at the origin of the unit circle.)



3. Determine the coordinates  $(x, y)$ , in kilometers, of a point on a circle of radius  $r$  kilometers in terms of the angle of measure  $\theta$  and the radius  $r$ .

### The Empire State Building Problem

While site seeing in New York City, Bob stopped 1000 feet from the Empire State Building and looked up to see the top of the Building. Given that the angle of Bob's site from the ground was 56 degrees, determine the height of the Empire State Building.

### The Adding Two Angle Measures Problem

Determine the measurement (relative and angular) of an angle that has a measurement of 1.5 radians plus  $1.2\pi$  radians. Given a circle with a radius of 3.5 inches, what is the arc-length that corresponds to this angle measurement?

### The Enemy Approaches Problem

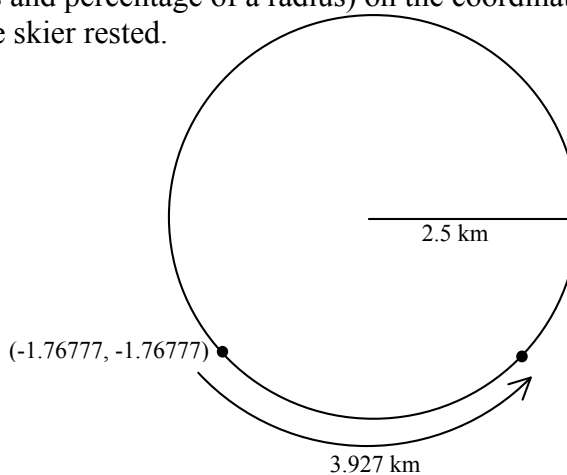
A castle observation tower is elevated 126 feet above the ground. When an approaching enemy is first noticed, the angle of depression (the angle at which an observer needs to look down) from the observation post was 0.084 radians. How far away is the enemy from the castle? How far away is the enemy from the observer?

### The Tangent Function and Graphing Problem

How does the function  $f(\theta) = \tan(\theta)$  vary as  $\theta$  varies from  $-\pi/2$  to  $\pi/2$ ?

### The Ski Trail Problem – Version II

An arctic village maintains a circular cross-country ski trail that has a radius of 2.5 kilometers. A skier started skiing from position  $(-1.76777, -1.76777)$  and skied counterclockwise for 3.927 kilometers where he paused for a brief rest. Determine the ordered pair (in both kilometers and percentage of a radius) on the coordinate axes that identifies the location where the skier rested.

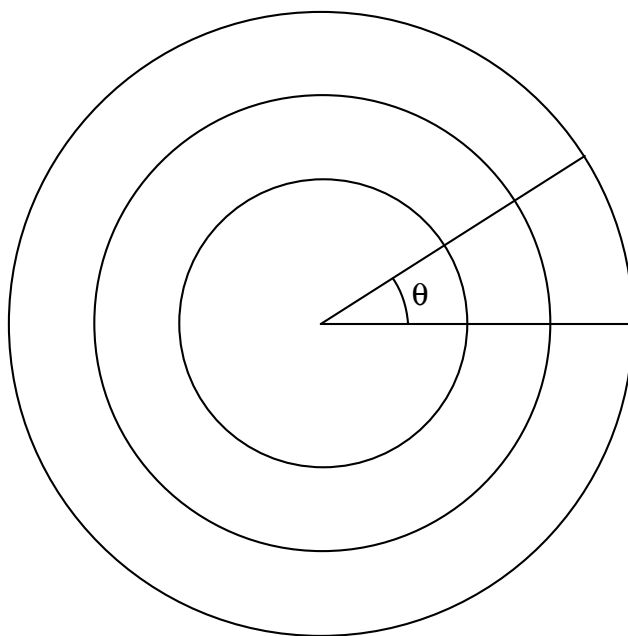


## APPENDIX E

### AMY'S INTERVIEW TASKS

## The Arc Length Problem

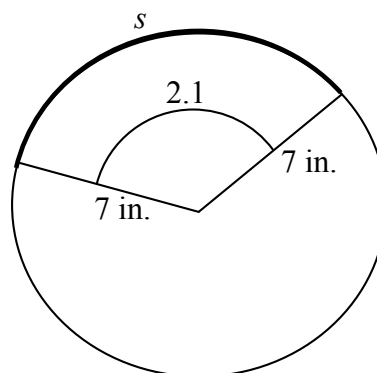
Given that the following angle measurement  $\theta$  is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches. (*Drawing not to scale*)



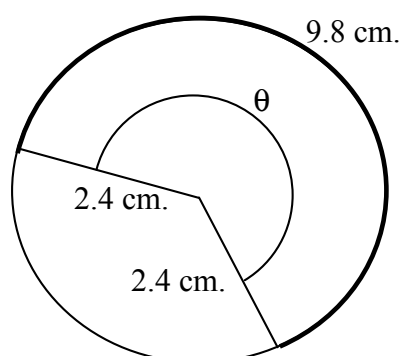
### The Missing Measurement Problem

Answer the following. Consider all angle measures to be given in radians.

1. Determine the missing linear measurement of arc-length cut off by the angle.



2. Determine the missing angular measurement in radians.



3. Determine the measure of an angle that cuts off 13.19 kilometers of arc length on a circle that has a radius of 3 kilometers.



### Amy's Ferris Wheel Problem

Consider a Ferris wheel with a radius of 36 feet. April boards the Ferris wheel at the 3 o'clock position and begins a continuous ride on the Ferris wheel.

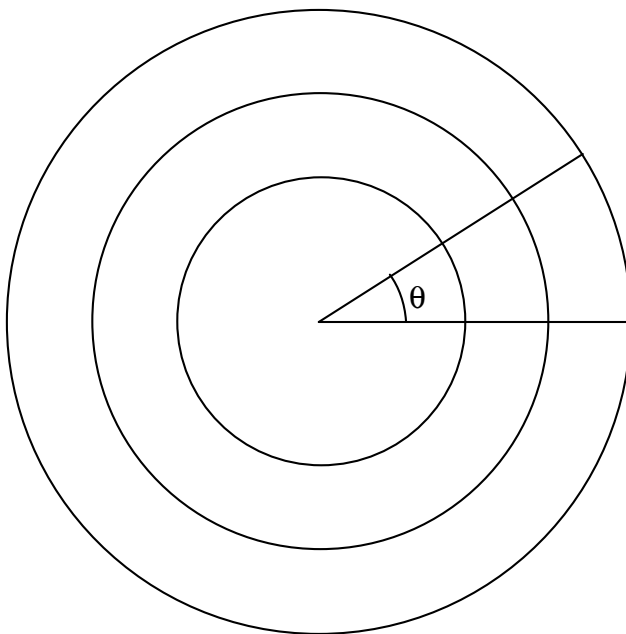
1. Sketch a graph that relates the total distance traveled by April and her vertical distance above the horizontal diameter of the Ferris wheel.

## APPENDIX F

### JUDY'S INTERVIEW TASKS

### The Arc Length Problem

Given that the following angle measurement  $\theta$  is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches. (*Drawing not to scale*)



### The Inches or Radians Problem

A student measures the arc-length that an angle cuts off, resulting in 1.7 inches, and claims that the angle has a measure of 1.7 inches. Discuss this student's claim.

### The Radian Measurements and Pi Problem

What does it mean for an angle to have a measure of  $1.2\pi$  radians? 5.27 radians? How long is the arc subtended by the angle relative to a radius of 3.5 inches?

### The Rotating Problem

A Ferris wheel with a radius of 41 feet is rotating at 2.5 full revolutions per minute. Marcus boards the Ferris wheel for a ride. After 20 seconds, how far has Marcus traveled

on the Ferris wheel? Give your answer in a linear measurement (e.g., a number of feet) and a number of radians.

### The Enemy Approaches Problem

A castle observation tower is elevated 126 feet above the ground. When an approaching enemy is first noticed, the angle of depression (the angle at which an observer needs to look down) from the observation post was 0.084 radians. How far away is the enemy from the castle? How far away is the enemy from the observer?

### The Ferris Wheel Problem

Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full rotation. April boards the Ferris wheel at the bottom and begins a continuous ride on the Ferris wheel.

1. Sketch a graph that relates the total distance traveled by April and her vertical distance from the ground.
2. Sketch a graph that relates the time since beginning the ride to April's vertical distance from the ground.
3. Determine formulas for the functions represented above.

## APPENDIX G

### HUMAN SUBJECTS APPROVAL LETTER

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	Research Compliance Office Office for Research & Sponsored Projects Administration P.O. Box 873503 Tempe, AZ 85287-3503	<b>Phone</b> (480) 965-6788 <b>Facsimile</b> (480) 965-7772
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<b>To:</b>	Marilyn Carlson UC
<b>From:</b>	Mark Roosa, Chair Soc Beh IRB
<b>Date:</b>	02/29/2008
<b>Committee Action:</b>	<b>Exemption Granted</b>
<b>IRB Action Date:</b>	02/29/2008
<b>IRB Protocol #:</b>	0506002513A002
<b>Study Title:</b>	Project Pathways: Overall Project Evaluation Branch

The above-referenced protocol is considered exempt after review by the Institutional Review Board pursuant to Federal regulations, 45 CFR Part 46.101(b)(1) (2) .

This part of the federal regulations requires that the information be recorded by investigators in such a manner that subjects cannot be identified, directly or through identifiers linked to the subjects. It is necessary that the information obtained not be such that if disclosed outside the research, it could reasonably place the subjects at risk of criminal or civil liability, or be damaging to the subjects' financial standing, employability, or reputation.

You should retain a copy of this letter for your records.

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