Mathematical Knowledge for Teaching: Exploring a

Teacher's Sources of Effectiveness

by

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ABSTRACT

This study contributes to the ongoing discussion of Mathematical
Knowledge for Teaching (MKT). It investigates the case of Rico, a high school
mathematics teacher who had become known to his colleagues and his students as
a superbly effective mathematics teacher. His students not only developed
excellent mathematical skills, they also developed deep understanding of the
mathematics they learned. Moreover, Rico redesigned his curricula and
instruction completely so that they provided a means of support for his students to
learn mathematics the way he intended. The purpose of this study was to
understand the sources of Rico's effectiveness.

The data for this study was generated in three phases. Phase I included
videos of Rico's lessons during one semester of an Algebra II course, post-lesson
reflections, and Rico's self-constructed instructional materials. An analysis of
Phase I data led to Phase II, which consisted of eight extensive stimulated-
reflection interviews with Rico. Phase III consisted of a conceptual analysis of the
prior phases with the aim of creating models of Rico's mathematical conceptions,
his conceptions of his students' mathematical understandings, and his images of
instruction and instructional design.

Findings revealed that Rico had developed profound personal
understandings, grounded in quantitative reasoning, of the mathematics that he
taught, and profound pedagogical understandings that supported these very same
ways of thinking in his students.

Rico's redesign was driven by three factors: (1) the particular way in which Rico himself understood the mathematics he taught, (2) his reflective awareness of those ways of thinking, and (3) his ability to envision what students might learn from different instructional approaches. Rico always considered what someone might already need to understand in order to understand "this" in the way he was thinking of it, and how understanding "this" might help students understand related ideas or methods. Rico's continual reflection on the mathematics he knew so as to make it more coherent, and his continual orientation to imagining how these meanings might work for students' learning, made Rico's mathematics become a mathematics of students—impacting how he assessed his practice and engaging him in a continual process of developing MKT

It is with deepest love and admiration that I dedicate this dissertation to my

mother, Maru Ramírez.

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CHAPTER 1—STATEMENT OF THE PROBLEM

It is important that mathematics education research develop a deep understanding of the complexities of teaching and the ways in which teachers manage them. With this, we can plan and act judiciously in helping teachers and preservice teachers to improve their practice and in turn provide better opportunities for student learning.

First, what does it mean to be an effective teacher? In spite of the depth of research in the field, there is no consensus as to the definition of an effective mathematics teacher. Franke, Kazemi, & Battey (2007) assert that: "what constitutes good teaching is consistently controversial and will remain controversial" (p.226). Other researchers affirm that the question: "what makes mathematics teachers effective?" does not have an obvious or easy answer (Hiebert & Grouws, 2007). This lack of agreement is in part because the idea of effective mathematics teaching rests implicit on researchers' and teachers' notions of learning (Kirshner, 2008), the philosophical underpinnings of the research (Wilson, Cooney, & Stinson, 2005), and the disparity of views about teaching entailed by cultural differences (Bryan, Wang, Perry, Wong, & Cai, 2007; Pang, 2009).

Researchers have used different lenses to better understand what is entailed in diverse views of effective teaching. Kirshner (2008) points to distinct metaphors of learning (e.g. learning as habituation, learning as construction, learning as enculturation) as a way to frame discussions of what "good teaching"

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means. Ball's (1991) review of research on teaching and the role that subject matter knowledge plays in defining teacher effectiveness revealed that views of effective mathematics teaching are often tacit in the research paradigms under which the studies take place. Other researchers call attention to cultural differences in views of effective mathematics teaching. For example, in a study of what teachers from different countries believe effective mathematics teaching to be, Cai et al. (2007) found that teachers from Australia and US had more to say about teacher's enthusiasm, rapport with the students, and classroom management as characteristics of an effective mathematics teacher, than teachers from Mainland China and Hong Kong. Instead, the latter teachers focused on how well teachers prepare and present a lesson and their ability to provide clear explanation of the points covered in the lesson. Teachers from these countries did not mention classroom management at all (Bryan, et al., 2007).

Moreover, the study of effective mathematics teaching necessarily entails a simplification of teaching and its relationships to student learning. In constructing a model for studying teaching and learning, researchers need to decide to put more emphasis on some aspects of the model, discarding others (Fenstermacher & Richardson, 2005; L. Shulman, 1986; A. G. Thompson & Thompson, 1996). Hiebert and Grows (2007) point out that the ways in which teaching influences students' learning does not have a trivial answer and "documenting particular features of teaching that are consistently effective for

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students' learning has proven to be one of the great research challenges of education" (p.371).

Aware of the different ways in which effective teaching can be appraised, Fenstermacher and Richardson (2005) propose three models that provide a lens to differentiate between successful, good, and quality teaching. For them, successful teaching is one that produces learning, regardless of the method being used. In this sense, effectiveness in teaching should be measured by student learning.[1] Good teaching is "teaching that comports with morally defensible and rationally sound principles of instructional practice" (p.189). In this case, what the teacher knows and the methods that she uses are central in considering whether a teacher's actions constitute good teaching. According to Fenstermacher and Richardson, the lens that researchers use to value good teaching can be viewed as learner-dependent or learner-sensitive depending on the role that student learning plays in the researchers' judgments about the quality of a teacher's instruction. For example, a researcher displays a learner-sensitive perspective if she defines an effective teacher as one who adjusts her actions based on what she interprets the students' thinking to be. On the other hand, a researcher displays a learner-dependent perspective by taking into consideration measures of student learning—such as tests or interviews—when defining effective mathematics instruction. Finally, Fenstermacher and Richardson argue that quality teaching

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[1] Usually, student learning has been measured by students' achievement in standardized tests.

goes beyond the combination of good and successful teaching, because "learning does not depend only on good teaching". Quality teaching refers to: "what we are most likely to obtain when there is willingness and effort on the part of the learner, a supportive social surround, ample opportunity to learn, and good practices employed by the teacher" (p.191). Each view, whether it refers to good teaching, successful teaching or quality teaching, entails the use of different methods to investigate the phenomenon of teaching.

For this study, I follow a learner-sensitive perspective[2] to investigate what constitutes effective mathematics teaching—good teaching in the sense of Fenstermacher and Richardson (2005). This means that instead of trying to document whether the teacher is successful in relation to his students' learning, the main focus of analysis is on understanding the teacher's actions and the systems of meanings and values that give rise to them. In other words, my goal with this study is not to provide evidence of the teacher's effectiveness in teaching mathematics. Instead, I am trying to understand the ways in which the teacher understands the mathematics that he teaches so as to support his design of instruction. To do so, this study investigates the case of Rico, a high school mathematics teacher who had become known to his colleagues and his school's students' as a superbly effective mathematics teacher. His students not only

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[2] A consequence of selecting a learner-sensitive perspective to the investigation of what Fenstermacher and Richardson (2005) would refer as *good teaching*, is that I analyzed videotaped lessons of the teacher's instruction (among other artifacts). However, I did not follow individual student learning in this study.

developed excellent mathematical skills, they also developed deep understanding of the mathematics they learned. Moreover, Rico redesigned his curricula for his Algebra II honors course—which is the main focus of analysis for this study—and his instruction completely so that they provided a means of support for both him and his students to learn mathematics the way he intended.

The purpose of this study is to understand the sources of Rico's effectiveness by exploring Rico's thinking. By Rico's thinking I mean the ways in which Rico understands the mathematics that he teaches and the ways in which he frames the learning goals, material support and instruction for his Algebra II course in relation to his students' learning. In this sense, a focus on the teacher's thinking can produce a model composed of interacting parts, in which no element acts in isolation. Components of the model refer to the way in which the teacher envisions the curriculum to be taught (whether driven by sections of the book or by ideas and ways of thinking), the teacher's mathematical knowledge for teaching (MKT) (as a way in which the teacher transforms his own mathematical understandings into pedagogical actions), and attention to the students' mathematics from a developmental perspective.

By adopting this perspective for investigating what constitutes effective mathematics teaching, this study contributes to research in providing characterizations of the teacher's mathematics. This is relevant because recent discussions on teacher knowledge and its impact on effective mathematics

teaching have focused on teachers' mathematical performance abilities and not on the teacher's ways of thinking that can support effective teaching.

The research questions that this study investigates are the following:

- *What are mathematical ideas and ways of thinking that Rico envisions for his students as suggested by his design of instruction?*

- *What are the mathematical understandings that support Rico's pedagogical actions and design of instruction?*

- *What are Rico's conceptions of his students' mathematics?*

- *In what ways do the above express themselves in Rico's teaching?*

CHAPTER 2—LITERATURE REVIEW

Introduction

In what follows, I present excerpts of an interview[3] with Rico that will

allow me to better explain the reasons why this case is valuable. The excerpts

illustrate aspects of Rico's mathematical knowledge as it pertains to teaching

Algebra II that lie outside the scope of most past research on teachers' MKT, and

hence as I highlight important aspects of Rico's thinking, I outline the research

literature that I drew upon to investigate his MKT. This also provides an

opportunity to state the reasons why some research literature that one might think

is important for investigating Rico's effectiveness is not useful to investigate the

complexities in Rico's thinking.

The interview took place in March of 2006, eight months after Rico's

initial contact with his professional development program. Of note, Rico

described the reasons why he changed the content of a unit on systems of linear

equations and about his motives for changes in the way he taught it. He described

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[3] This interview took place in the context of a research and professional
development project called Teachers Promoting Change Collaboratively (TPCC)
for which my advisor, Dr. Patrick Thompson, was the P.I. and I was a research
assistant. It was in the context of analyzing the data from the project that I first
met Rico, who at the time was a participant in the project. In Chapter 5, I will
expand on Rico's background as a teacher.

the new ways in which he conceived what he wanted his students to learn and how he adjusted his curriculum and teaching accordingly.[4]

Excerpt 1. Why Rico changed his approach of Systems of Equations (March, 2006).

Interviewer: Last fall you prepared a special unit on systems of equations. Why did you do that, and how did it go?

Rico: Of all the concepts that I teach, writing equations of lines and solving systems of equations has just driven me crazy. Because no matter how many times I explained it to students and all…even one year it got to the point, because there was these series of questions that they will miss all in their final exams and they didn't do well on the AIMS questions. I was just trying to get them...this year I will make sure that they nail these questions. And, so every week or two, just as a warm-up I had them review solving systems of equations, writing an equation. Every time they saw them, it was like, "I've never seen this before in my life". And, you know, I did [this] every couple of weeks. And as I began to look ahead at what we were going to do for that particular lesson [applications section of solving systems of equations], I realized that our textbook doesn't prepare

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[4] While Rico's reflections were about teaching systems of linear equations for his Algebra I course, this is not the only unit he eventually redesigned. Specifically, 15 months after this interview he completely rearranged the standard curriculum for his Algebra II course, which is the focus of main interest for this study. An outline of how Rico modified the standard curriculum in Algebra II will be provided in Chapter 5.

them for that kind of thinking. That really what it does is, and I think this is pretty typical of the way the American textbooks and curriculum works; it goes through several sections where it says, "here you graph two lines and you find the intersection point" and then, "here's algebraically how you can do a substitution method", and "here's how you do addition and subtraction method". And then, they put it into a word problem and expect them [the students] to understand what the slope means, how to write equations out of the word problem, and do like four or five things and put all of them in context without ever having told them to do that. Even though they sometimes put the problems in a word problem, in these earlier sections, it was always: "here's 20 drill problems of doing this without context", and now, after they are already getting bored with this, "here's the word problem that's kind of going to make you think". And they are done. They just quit and they don't know how to do it. They don't know how to write equations.

In Excerpt 1, Rico spoke of his extraordinary effort to help students remember procedures for solving systems of linear equations and his frustration that, despite these measures, students remembered very little. He also spoke of his insight as to why these measures were ineffective—the textbook never prepared them for the kind of reasoning that applying these methods presumes.

One explanation as to why his students acted as if they had never seen such problems before even after actually having seen many is that they did not

develop productive ways of thinking about such situations. In other words, they did not learn to conceive of described situations in terms of quantities, relationships among them, and to express this conception in algebraic expressions (Ellis, 2007; Lobato & Siebert, 2002; Smith & Thompson, 2008; Patrick W. Thompson, 1990, 1993, 1995). To do so, as Rico pointed out, students would need to experience instruction and tasks that were designed to help them do just that. But the textbook chapter presented one procedure after another, having students practice applying those procedures without any context leaving the word problems until the end of the section. Thus, the textbook did not support Rico in fostering such learning nor did it provide students the support they needed to develop those ways of thinking.

Excerpt 2: Rico's rationale for the changes he made (March, 2006).

Rico: So, even before the systems of equations chapter, I looked at the way that I was teaching writing equations of lines. I really focused on the meaning behind 'slope' [rate of change]; this way, when I got them to the chapter they have already a little bit deeper understanding of that. And then, I re-did the way that I taught the chapter entirely. I started off day one, with a big word problem that I just asked them to think through and give me answers for. And I didn't care how they did it. Different students knew more about using the graphing calculator, the technology, or some students just plugged points until they found something, but I wanted them to work through the word problem. And then we looked at all the different

solutions that they created. And I kind of used that as an idea as we kept going. So as we got into the early sections, every day I would give them a big word problem to start with. And I would focus their discussion on creating a table of values to describe what is going on and how can we look...“now, when these values are equal what does that really mean”, “now we are looking at writing equations for these and what do the equations mean”, “what does the slope mean, the vertical intercept”…So then, they were really building their understanding for the procedures out of the contextual problems. And I did that every single day as I was building substitution method, showing them…having them explain to me why would they replace y in one equation with what it equals. Then when I would give them those abstract problems with no context, that was always at the end. “Ok, now you know what all those mean, so you don't have to know what x and y represent; you can still solve it the same way”. And, students were telling me: “this is interesting”, “this is easy”…We ended up doing a project. They went out and they found something that they enjoy doing and they found something that they could write a system of equations. They came together and they worked in groups. They did a presentation on how to find the solutions and what it meant for the context and no problem. In the test I gave them harder problems, than I did in the past. Even when students were missing the algebra on it, they still were

showing that they knew what they were doing and how to solve the

system...I have never experienced that before. It was amazing.

As with the Excerpt 1 of Rico's interview, there are some aspects that

deserve attention. First, Rico did not think of the systems of equations chapter

isolated from the rest of the material that would need to be presented in the

following months. Instead, he anticipated that, if students were to think about

systems of equations in powerful ways, then there were other ideas that students

needed to develop previously, such as 'slope' (Excerpt 2). One cannot discern

from Rico's interview what he meant by "the meaning behind slope", but the

instructional materials he created for his Algebra II course suggest that the idea of

constant rate of change between two varying quantities was key to his

development of a meaning for slope. From this point of view, solving systems of

equations fits into a larger context in which students are provided with

opportunities to think about quantities and how they are related, instead of

meaningless application of procedures, that do not necessarily convey anything

about the relationships between the quantities involved in the problem.

The second aspect that can be highlighted from Excerpt 2 is that Rico's

understandings of systems of equations allowed him to envision differences in

what students might learn from the two different instructional sequences (the one

from the textbook, as mentioned above, and the one that he created based on the

ways of thinking about systems of equations that he wanted the students to

develop). He anticipated that the symbolic procedures, which the textbook taught

without meaning, would emerge meaningfully and naturally, under his guidance, from the reasoning that students developed as they created approaches to solving problems that were grounded in their understandings of the situations (Excerpt 2). Furthermore, he had an image of how he would carry out instruction following the two approaches (e.g. the problems he would pose, the conversations that he would have with students). He was able to imagine what students would learn by following the traditional approach, in comparison to what they might learn from his redesign of the curriculum and instruction. Finally, he spoke about evaluating the efficacy of his new approach by asking students to solve the kinds of problems that the textbook posed and by asking them to find situations within their everyday lives for which systems of equations might be applicable.

In summary, the case of Rico is worth studying because his mathematical and pedagogical understandings are such that his teaching is attuned with what has been described as *a conceptual orientation to mathematics teaching* (Silverman & Thompson, 2008; A. G. Thompson, Philipp, Thompson, & Boyd, 1994; A. G. Thompson & Thompson, 1996; Patrick W. Thompson & Thompson, 1994). That is, his actions seem to have been driven by:

- An image of a *system of ideas* and *ways of thinking* that he intends the students to develop; (A. G. Thompson & Thompson, 1996)
- Constructed models of the variety of ways students *may* understand the content (decentering);
- An image of how someone else might come to think of the mathematical

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idea in a similar way;

- An image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea;

- An image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas (Silverman & Thompson, 2008).

In the pages that follow, I present a brief review of the literature in mathematics education research regarding issues of MKT that were highlighted from Rico's interview—his ability to envision what students might learn from different instructional approaches and his command of the mathematics with regard to how students might experience it more coherently. Whether a study contributes to a focus on teachers' ways of thinking mathematically or ways of thinking about their students' mathematics is my primary criteria for deciding whether to build upon it in my study. Though not all of the research reviewed here is useful for investigating teachers' mathematical ways of thinking and their instructional decision making, I do review several major programs of research as I make the case for why I do not take them as foundational. As such, I first explore current research in effective mathematics teaching. Then, I present a synthesis of different frameworks for *mathematical knowledge for teaching*.

Effective Mathematics Teaching

The idea of effective mathematics teaching is an elusive concept that defies a simple characterization (Pang, 2009). Nevertheless, it has been present in the research literature for decades. In the previous chapter, I raised some of the issues that are related to the investigation of what constitutes effective mathematics teaching—that researchers' and teachers' conceptions of effective mathematics teaching are influenced by their notions of learning, their philosophical underpinnings, and their cultural identity, among others.

In this section I elaborate on some aspects that have received attention in prior research—attending to student thinking, mathematics that students experience, and teacher knowledge—and that researchers have identified as factors that influence effective mathematics teaching. I emphasize that I don't mean that these are the only aspects that have been considered as influential in the ongoing discussion on what constitutes effective mathematics teaching. Rather, my focus is on those aspects that provide support to my selection of the framework that I used to study Rico's thinking.

Attending to student thinking. Attending to student thinking has been considered an important factor that contributes to effective mathematics teaching (Chamberlin, 2005; Fennema, et al., 1996; M. L. Franke & Kazemi, 2001; Hill, Ball, & Schilling, 2008; McClain, 2002; A. G. Thompson & Thompson, 1996; Patrick W. Thompson, 1994a). McClain (2002) states that, "research on effective teaching often characterizes the teacher's classroom decision-making process as

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informed by the mathematical agenda but constantly being revised and modified in action based on students' contributions" (p.217). For researchers, a model of what it means to attend to student thinking would be more or less similar to this: attending to student thinking entails a constant interaction between the learning goals (which need to be stated in cognitive terms and thus attending to a theory of learning), the instructional material (the activities that are designed and re-designed as the teacher interacts with the students) and the actual instruction that occurs. This interaction should also be informed by a constant assessment of what the teacher interprets that students understand based on their engagement in instruction. This assessment derives from holding conversations with the students and "listen[ing] for cues as to what sense students have made of what was said or done, including asking for students' interpretations of it" (Patrick W. Thompson, 2002, p. 193).

This elaborated model of what it means to attend to students' thinking does not necessarily reflect what happens in classrooms and thus, what teachers mean by attending to student thinking. On the one hand, Franke, Kazemi, and Battey (2007) state that "most U.S. mathematics classrooms maintain an initiation-response-evaluation (IRE) interaction pattern, where the evaluation move on the part of the teacher focuses on students' answers rather than the strategies they use to arrive at them. The teacher assumes responsibility for solving the problem while student participation often involves providing the next step in the procedure" (p.229). Teachers engage students in procedure-bound

discourses, asking for right answers rather than students' reasoning behind those answers (Spillane & Zeulli, 1999; A. G. Thompson, et al., 1994). In this sense, attending to what students say serves to pace instruction rather than to shape it (Wilson, et al., 2005).

On the other hand, the notions of student learning from which teachers operate do not often reflect what researchers intend, often graduating from teacher education without an appreciation of how children come to know mathematics or ways in which discourse invites mathematical thinking (Grouws & Shultz, 1996). Researchers at the elementary level, specifically the Cognitively Guided Instruction (CGI) project, have embarked in efforts to "help teachers build relationships between an explicit research-based model of children's thinking and their own students' thinking by encouraging reflection on how the model can be interpreted in light of their own students and classrooms." (Fennema, et al., 1996, p. 405) While these researchers claim that there exists a "model of children's thinking [that] is extremely robust" (ibid., p. 408) for teachers to compare their actual students' thinking to such model and adapt instruction accordingly, at a high school level this is not the case. Thus, teachers need to create those models themselves when they develop instruction. To do so, teachers need to reflect on the "students' thinking en route to developing a richer understanding of the students' mathematics (Steffe, 1994) at play … [or ways in which] one might help a student who understands the mathematics in a certain way develop a deeper understanding" (Silverman, 2005, pp. 35-36).

The mathematics that students experience in their courses according to cross-cultural studies. Current research on effective mathematics teaching has adopted a cross-cultural perspective. The rationale behind this perspective, as Hiebert et al. (2005) notes, is "because teaching is such a common activity, one embedded within a culture, it can be difficult to notice common features, especially those that are most widely shared. Contrasts with less familiar methods used in other countries make one's own methods more visible and open for inspection" (p.112). Insights regarding teachers' beliefs of good practice, and the mathematics that students are taught, are some of the salient aspects of this perspective to the study of effective teaching.

Regarding teachers' beliefs of good practice, Cai and colleagues (2007) examined mathematics teachers' beliefs on effective teaching from a cross-cultural perspective. Their study sheds light on what teachers from China, Hong Kong, the US, and Australia value as important characteristics of effective mathematics teachers. For example, teachers from Australia and the US valued teacher's enthusiasm and rapport with the students, while teachers from China and Hong Kong did not. Teachers from China and Hong Kong focused on how well the teacher prepares and presents a lesson and the teacher's ability to provide clear explanations (Bryan, et al., 2007). Another important difference that Bryan et al. found is that US teachers considered classroom management as a characteristic of an effective mathematics teacher, whereas teachers from other countries did not mention classroom management at all (ibid.).

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The second aspect that cross-cultural studies have uncovered refers to the mathematics that the students in different countries experience in their classrooms. For example, drawing from the results of the video component of the Third International Mathematics Science Study (TIMSS), Stigler and Hiebert (1999) and Stigler et al. (1999) developed an analysis of eighth-grade mathematics teaching using probability samples from three countries, Germany, Japan, and the United States. Results revealed that students in the U.S. experience mathematics as a set of procedures for solving problems, where students are expected to develop skills, rather than make sense of situations and ideas and solve problems based thereon. On another study that also draws from TIMSS, Schmidt et al. (Schmidt, Houang, & Cogan, 2002; Schmidt, Wang, & McKnight, 2005) compared the US curriculum to the curriculum from high-achieving countries. The authors found that US curriculum introduces a wide variety of topics in each grade, becoming highly repetitive across grades, including content that is not very demanding and lacking coherence[5].

In summary, results from cross-cultural studies, together with teachers' attending to student thinking highlight the crucial role that teachers play for effective mathematics teaching. Teachers' beliefs about mathematics and instruction, as well as the curriculum, are essential for providing students with

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[5] For this study, I follow Thompson's (2008) idea of coherence of a curriculum, which states that, "coherence is more than just a sequence from less sophisticated to more sophisticated topics. Coherence of a curriculum (intended, implemented, or experienced) depends upon the fit of meanings developed in it"(p.49).

opportunities to learn mathematics that strives for developing coherent meanings and problem solving.

Teacher knowledge and MKT. The idea that teachers' mathematical knowledge is essential for effective teaching is widely accepted. Despite this, the ways in which researchers have come to conceptualize teacher knowledge and the ways in which they have tried to measure its impact in teaching has varied greatly. In her review of the literature of research on teaching, Ball (1991) highlighted how teachers' subject matter knowledge "figured, faded, and reappeared as key influence on the teaching of mathematics" (p.1).

In the 1960s, researchers approached the study of effective teaching by exploring the link between measurable teacher characteristics (e.g. years of teaching experience, credits in mathematics, having a major or minor in mathematics) associated with student achievement in mathematics. Overall, no single teacher characteristic as a representative of teacher knowledge was found to be significantly correlated with student learning (Deborah L. Ball, 1991).

Later, teachers' generic behaviors (e.g. pacing, questioning, clarity) were incorporated to research studies with the goal of trying to find what would better predict student learning. Teachers' subject matter knowledge faded into the background of the studies serving only as context of the studies. Mixed and inconclusive results led researchers to reject the notion that observable teacher behavior was the full story; as such, researchers developed new ways of studying and representing teacher thinking. The notion of effective mathematics teaching

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shifted into that of expertise (e.g., Leinhardt, 1989; Leinhardt & Smith, 1985), where the most effective teachers were the experts; which in contrast to novices, experts developed lessons with rich agendas, consistent but flexible lesson structures, and gave explanations that met the goals of clarifying concepts and procedures (Deborah L. Ball, 1991).

Teachers' mathematical knowledge started to regain attention and, eventually, was again seen as essential to the characterization of effective mathematics instruction. In 1986, Shulman introduced the idea of PCK to address the issue that content knowledge is not sufficient for teachers to be successful (nor are general pedagogical principles of "good teaching"). For Shulman, PCK consisted of the ways of making subject matter comprehensible to others (representations of ideas, analogies, illustrations, examples, explanations, and demonstrations), identifying which topics are most difficult to understand, and students' conceptions and pre-conceptions.

For the past two decades, researchers in mathematics education have built upon and extended Shulman's idea of PCK and have developed different conceptualizations of mathematical knowledge for teaching—mathematical knowledge that is specifically useful in teaching mathematics (Deborah L. Ball, Thames, & Phelps, 2008; Hill, et al., 2008; Silverman & Thompson, 2008). Grounded in these different conceptualizations of MKT, frameworks have emerged from researchers' attempts to understand and characterize, and in some cases, even measure what they see as MKT.

Researchers' conceptualizations of MKT differ in many ways; at the core of these differences is how researchers think about knowledge. The first difference is whether MKT can be interpreted from an objectivist point of view— that is, as knowledge that is independent from the knower. As a consequence of this perspective, the focus of investigation becomes what there is to know to become an effective mathematics teacher. On the other hand, MKT can be seen in an active sense, meaning that knowledge is dependent on the knower. In this sense, the question becomes: in what ways does the teacher need to understand the mathematics so that she can become an effective teacher?

The second difference between conceptualizations is whether MKT refers to 'knowing in the moment' or if it refers to 'knowledge that drives what the teacher does in the moment'. The notion of MKT as 'knowing in the moment' builds from the idea of PCK. Ball and Bass (2000) define PCK as the knowledge base that bundles mathematical knowledge, knowledge of learning, and pedagogy. These bundles help the teacher anticipate what students might have trouble learning, and they entail having ready alternative models or explanations to mediate those difficulties. In addition to those previously acquired models, the teacher needs to be able to engage in what Ball and Bass call a "real time problem solving". This problem solving involves attending, interpreting, deciding, and making moves as teachers look at student work, choose a text to read, design a task or moderate a discussion (Deborah L. Ball & Bass, 2000). In a similar vein, Mason and Spence (1999) state that, "Shulman's forms of knowledge are

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supposed to equip the effective practitioner to act, but knowing-to act when the moment comes requires more than having accumulated knowledge-about" (p.139). The authors further explain that knowing-to act is when relevant knowledge comes to the fore so that it can be acted upon.

The excerpts from Rico's interview show that the ways in which he understood the mathematics that he teaches goes far beyond what can be observed in the moment of teaching. He had an understanding of systems of equations that allowed him to envision what students were going to learn from the two approaches he contemplated. Furthermore, he had an image of how he would carry out instruction in both approaches. This is not MKT in the moment. Instead, this is MKT that guided what Rico did in the moment. Silverman and Thompson (2008) refer to this MKT as neither "bundled" in advance, nor as just being able to "act in the moment". Instead, they speak of MKT as entailing the ability to discern the network of mathematical ideas into which particular pedagogical choices will thrust the subsequent instruction and as the body of mathematical understandings that allow a teacher to act in these ways spontaneously.

In summary, teachers' MKT has become central to the discussion of effective mathematics teaching. But the frameworks that have emerged in this arena are substantially different. In what follows I present an overview of two of these frameworks: Ball and colleagues' and Silverman and Thompson's. Then, I will discuss specific aspects of the frameworks that lead to my framework choice.

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Ball and colleagues' framework. Ball and colleagues have been working to characterize and define the construct of MKT for more than fifteen years and their work has followed two clearly identified lines of research. As explained by Ball, Thames and Phelps (2008), in their first project the research group took an empirical approach to understand the content knowledge needed for teaching mathematics. They focused on "the work teachers do in their teaching" (ibid., , p. 390). In what they call a related line of work, the research group also embarked in developing survey measures of content knowledge for teaching mathematics. It is by integrating these two lines of research that they have proposed a refinement to Shulman's model of PCK, which they call: a practice-based theory of content knowledge for teaching.

Ball and colleagues, building from the seminal work of Shulman on pedagogical content knowledge (L. Shulman, 1986; Lee Shulman, 1987) partition MKT into two main categories: subject matter knowledge and pedagogical content knowledge (Figure 1). Subject matter knowledge is further partitioned into three sub-domains: *common content knowledge* (CCK), *specialized content knowledge* (SCK), and knowledge at the mathematical horizon. They describe CCK as the "knowledge that is used in the work of teaching in ways *in common with* how it is used in many other professions or occupations that also use mathematics" (Hill, et al., 2008, p. 377). As they define it, this knowledge helps someone perform, say, a subtraction computation without mistake. SCK is defined as "the mathematical knowledge that allows teachers to engage in

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particular *teaching* tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Ball et al., 2005; cited in Hill, et al., 2008, p. 377). This knowledge would help the teacher identify when a student has made a mistake in a computation, validate an alternative method, provide an appropriate representation to an example based on the students' level of understanding. Finally, the third sub-domain, knowledge at the mathematical horizon refers to knowledge of "how mathematical topics are related over the span of mathematics included in the curriculum" (Deborah L. Ball, et al., 2008, p. 403).
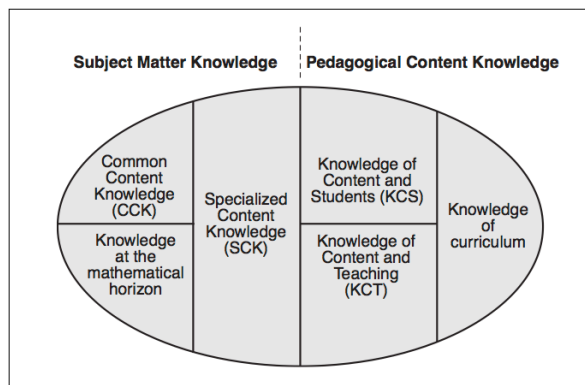
Figure 1 MKT according to Hill et al. (2008).

The second category of MKT is associated with Shulman's notion of pedagogical content knowledge (PCK). It is partitioned into the three sub-domains: *knowledge of content and students* (KCS), *knowledge of content and teaching* (KCT), and knowledge of curriculum. The first sub-domain of PCK, KCS relates knowing about students and knowing about mathematics. This sub-domain refers to teachers' understanding of how students learn particular content

(common errors students make, ways in which previous knowledge can support learning of other topics, etc.). The authors further state that it "is separable from knowledge of teaching moves—for example, how best to build on student mathematical thinking or how to remedy student errors." (Hill, et al., 2008, p. 378). KCT, the second sub-domain of PCK, combines knowing about teaching and knowing about mathematics. This knowledge informs teachers' decision-making during planning and instruction. The last sub-domain, which Ball and colleagues include, but clarify that needs to be refined, refers to Shulman's knowledge of curriculum, "the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (L. Shulman, 1986, p. 10).

Ball and colleagues are interested in the mathematics that teachers use in teaching, not in teachers *per se*. In other words, the framework characterizes knowledge as a discipline and not as someone's understandings about the ideas of the discipline. This way of defining MKT is useful, as Silverman and Thompson (2008) point out, as a way to share strategies for teaching, which includes an awareness of the typical misconceptions that students have when taught particular topics. However, this approach is non-cognitive in that there are no teacher's cognitive processes involved. As such, this approach is not conducive in explaining aspects of Rico's thinking, specifically the reasons why he decided to

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change his unit on systems of equations in the manner he did, why he re-structured his teaching as he did, and ways in which he might have developed new knowledge. Nor does the Ball et al. framework help me explain the ways Rico thought of the re-structured content and instruction impacting his students learning. To do this, I need a framework that addresses teachers' thinking and reasoning, and transformations in that thinking and reasoning.

Other researchers have built upon Ball and colleagues' work on MKT. Two of these groups are Ferrini-Mundy et al. (2005) and Adler and Davis (2006). In the case of Ferrini-Mundy et al. (2005), they developed a conceptual framework for understanding and assessing teachers' mathematical knowledge relative to algebra instruction; with a specific emphasis on algebraic expressions and equations along with linear relationships. The authors' main goal is to use the framework as a template for organizing the knowledge needed for or used in the teaching of algebra and to lay the groundwork for empirical research that probes the connections between teachers' mathematical knowledge and student outcomes in algebra. Adler and Davis (2006) from the QUANTUM[6] project, are concerned with "the mathematics (how much and what kind) the middle school and senior school teachers need to know and know how to use in order to teach mathematics successfully" and "how, and in what ways, programs that prepare and support mathematics teachers can/do provide opportunities for learning" (p.271).

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[6] QUANTUM is the name given to a research and development project on quality mathematical education for teachers in South Africa.

Both groups have built from Ball and Bass' notion of decompression or unpacking in order to deconstruct one's own mathematical knowledge into less polished and final form, where elemental components are accessible and visible (Deborah L. Ball & Bass, 2000, p. 98). This assumes that the ways in which the teachers know mathematics will allow them to unpack what they know to benefit the learners. But if a teacher's understandings are such that she holds incoherent and isolated schemes, she will be unable to unpack her understanding in meaningful ways. On the other hand, there are teachers whose understandings go beyond learned procedures and do understand the mathematics in personally powerful ways, but this does not mean that they will immediately be able to unpack their understandings in ways that can actually help someone else to learn. We need ways to describe how this mechanism of "unpacking" works.

Silverman and Thompson's framework. Silverman and Thompson (2008) propose a framework for studying the development of mathematical knowledge for teaching. Their goal is to investigate the ways in which the teachers understand the mathematics that they teach rather than coming to characterize the mathematics that becomes visible in teaching. The reason behind this approach is because "teachers' personal understandings of the mathematical ideas that they teach is what constitutes the most direct source for what they intend students to learn and what they know about ways these ideas can develop" (Liu, 2005, p. 1). Thus, "if teachers' conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and

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procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students" (Patrick W. Thompson, p. 416).

Silverman and Thompson's framework builds upon Simon's idea of *key developmental understanding* (Simon, 2002, 2006), Silverman and Thompson's idea of *key pedagogical understanding* (Silverman, 2005; Silverman & Thompson, 2008; A. G. Thompson & Thompson, 1996; Patrick W. Thompson, Carlson, & Silverman, 2007; Patrick W. Thompson & Thompson, 1994), and Piaget's idea of reflective abstraction (J. Piaget, 2001; Simon, Tzur, Heinz, & Kinzel, 2004; Patrick W. Thompson, 1985). A person has developed a *key developmental understanding* (KDU) when they construct a scheme of meanings that proves central for understanding a broad swath of mathematical ideas and methods. The scheme of meanings comprised by number naming, counting, adding, subtracting, and whole number numeration is a KDU in arithmetic.

KDUs need not be as the example of arithmetic KDU given above, and they are not mutually exclusive. As schemes, KDUs can participate in other KDUs. A KDU is about personal knowledge. People can have a KDU and be unaware of it. They are aware that "things make sense" and they are aware that they can make connections. A teacher with a KDU could be a good student of the mathematics they teach without expressing that KDU in his or her instructional actions. When a teacher becomes reflectively aware of a KDU and realizes that

her students would benefit from "thinking this way", she is developing a *key pedagogical understanding* (KPU). A teacher has a fully developed KPU when she has a KDU, is reflectively aware of it, intends that her students have it too, and has built a mini-instructional theory about how she can support them attaining it. Put another way, a teacher has developed knowledge that supports conceptual teaching of a particular mathematical topic when he or she:

- Has developed a KDU within which that topic exists. For example, an understanding of slope as the constant rate of change between two varying quantities.

- Has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas. For example, an approach to systems of equations, systems of linear inequalities, and linear programming, all of which are build upon an understanding of variation, covariation, and constant rate of change between two varying quantities.

- Has constructed models of the variety of ways students *might* understand the content (decentering). This is, the teacher puts herself "in the place of *a student* and attempt[s] to examine the operations that a student would need and the constraints the student would have to operate under to (logically) behave as the prospective teacher wishes a student to do" (Silverman & Thompson, 2008, p. 508).

- Has an image of how someone else might come to think of the

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mathematical idea in a similar way;

- Has an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea. Silverman and Thompson (2008) emphasize that, "students' participation in conversations about their mathematical activity (including reasoning, interpreting, and meaning-making) is essential for their developing rich, connected mathematical understandings" (p.507). Therefore, it is in the context of instruction that supports reflective conversations (Cobb, Boufi, McClain, & Whitenack, 1997) that students are most likely to develop KDUs.

In summary, this framework for MKT provides the potential for explicating the mechanism by which the teacher transforms her personal mathematical understandings into pedagogical actions. Thus, Rico's thinking is best explored by applying Silveman and Thompson's framework. In addition, their focus on KDUs becomes a constant reminder that it is not what is listed in the curriculum, but how the teacher conceives what is listed in it that becomes the source of her pedagogical actions.

CHAPTER 3—BACKGROUND THEORIES

In this chapter, I explain the theoretical perspectives that guided the study. The first section involves a discussion about a theory of knowing and its implications to the study of MKT. The second section focuses on a theory of quantitative reasoning as a way to explain how people come to make sense of quantitative situations. This discussion provides a point of reference to characterize my descriptions of Rico's mathematical and pedagogical understandings.

A Theory of Knowing and its Implications to the Study

Piaget (1971) developed a theory that he called genetic epistemology to provide explanations to what knowledge consists of and how knowledge develops (Campbell, 2001). Glasersfeld (1995) further elaborated Piaget's genetic epistemology (1971) into a theory of knowing that is commonly referred to as radical constructivism. Radical constructivism follows two basic principles:

1) Knowledge is not passively received either through the senses or by way of communication; knowledge is actively built up by the cognizing subject.

2) The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability; cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality. (Glasersfeld, 1995, p. 51)

To adopt radical constructivism as background theory implies that there is a set of commitments and constraints to how I frame the problem to be studied, the phenomena that need to be explained and the actual explanations that I can provide (Patrick W. Thompson, 2002). For example, the main objective of this study is to develop insight into Rico's personal and pedagogical understandings of the mathematics that he teaches, as well as his images of his students' mathematics. To do so, my starting assumptions are that:

- MKT is not a thing-in-itself that can be divided into categories;
- The teacher does not have access to a mathematics that is independent of his ways of knowing; and, I (the researcher) do not have direct access to the teacher's mathematics.

The most we can do is to construct models of another person's mathematics. That is, the observer (either the researcher or the teacher) develops explanations of the ways and means by which another person operates mathematically, by frequent interactions in which one observes the other's actions. Such models are considered viable as long as they are not contradicted by what the other says or does (Steffe, 2007). As such, claims about a teacher's MKT is a second-order model: "[a model] observers construct of the subject's knowledge in order to explain [the observer's] observations (i.e., their experience) of the subject's states and activities" (Steffe et al., 1983, p. xvi, cited in Silverman & Thompson, 2008, p. 8). Furthermore, to adopt a radical constructivist perspective to the investigation of Rico's MKT, allows me to move away from what there is to

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know about mathematics that are used in teaching, to explore ways in which the teacher understands the mathematics and thus transforms his own understandings into pedagogical actions. It is then important to clarify what I will mean by understanding, and to describe a mechanism by which the teacher can transform his current mathematical understandings into MKT.

Understanding

Thompson & Saldanha (2003) define "to understand" as "to assimilate to a scheme," relying on Piaget's notion of assimilation (Glasersfeld, 1995; J. Piaget, 2001). Assimilation is described as "[coming] about when a cognizing organism fits an experience within a conceptual structure it already has" (Glasersfeld, 1995; cited in Silverman, 2005, p. 7). To understand, then, results from a person's interpreting an experience by assigning meanings according to a web of connections the person builds over time—either through the person's own interpretations of settings or through interactions with others (Patrick W. Thompson & Saldanha, 2003). This way of thinking about knowing and understanding is highly compatible with Dewey's notion of thinking (Dewey, 1910) and with Piaget's notion of assimilation (J. Piaget, 1950, 1968a, 1968b, 1976, 1980). Montangero and Maurice-Naville quote Piaget as saying:

> Assimilating an object to a scheme involves giving one or several meanings to this object, and it is this conferring of meanings that implies a more or less complete system of inferences, even when it is simply a question of verifying a fact. In short, we could say that an assimilation is an association accompanied by inference. (Johnckheere, Mandelbrot, & Piaget, p. 59; quoted inMontagero & Maurice-Naville, 1997, p. 72)

To describe Rico's understandings from this perspective requires, as Thompson and Saldanha note, addressing two sides of the assimilation: (1) what I (the researcher) see as the thing that Rico attempts to understand, and (2) the scheme of operations that constitutes Rico's actual understandings (Patrick W. Thompson & Saldanha, 2003).

Simon et al. (2004) points out that this way of explicating the process of assimilation seems to imply a vicious cycle, or what has been called the learning paradox. This is, if "in order to experience a new concept… one must already have that concept available to organize that experience… [then,] how can learning of new conceptions be explained without attributing to the learners prior assimilatory conceptions that are as advanced as those to be learned?" (ibid., p. 310). Piaget proposed the mechanism of reflective abstraction to explain how new knowledge is generated from the person's current conceptions.

Reflective Abstraction

The models of Rico's mathematical knowledge for teaching not only posit what that knowledge is, but also look for mechanisms by which he created it. Silverman and Thompson address the issue of a teacher's creation of a KPU—a teacher having a key developmental understanding, which the teacher takes as a target way of thinking for students to have, along with a mini-theory about how she can act instructionally to support students gaining it. Regarding how a teacher creates a KPU, Piaget (2001) proposed reflective abstraction as a way to explain how new, more advanced conceptions develop from the subject's current

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conceptions. Thus, reflective abstraction is a key concept in Silverman and Thompson's (2008) framework; it is the mechanism by which a teacher develops MKT. That is, it is by reflective abstraction that the teacher develops powerful personal understandings of the mathematics he wants the students to learn and the pedagogical understandings that will guide the teacher's instructional actions.

For Piaget, all new knowledge presupposes an abstraction, but not all abstractions are the same (Glasersfeld, 1995). Piaget distinguished between 'empirical', 'pseudo-empirical', and 'reflective' abstractions; and although reflective abstractions are the basis for explaining MKT, I will provide a brief description of the other two types: empirical and pseudo-empirical abstractions.

The first kind of abstraction is called empirical abstraction and it "ranges over physical objects or the material aspects of one's own action (such as movements, pushes, and the like)" (J. Piaget, 2001, p. 29). To engage in empirical abstraction does not mean a pure "read-off" of data from the environment. In order to abstract any property, "the knowing subject must already be using instruments of assimilation (meanings and acts of putting into relation) that depend on sensorimotor or conceptual schemes. And such schemes are constructed in advance by the subject, not furnished by the object" (ibid., p. 30). Piaget referred to those abstractions in which "the knowing subject cannot carry out some constructions (which later on will become purely deductive) without relying constantly on their observable results (cf., using the abacus for the first numerical operations)" as pseudo-empirical abstractions (J. Piaget, 2001, p. 31).

He further explained that, "while the results are read off from material objects, as is the case with empirical abstraction, the observed properties are actually introduced into these objects by the activities of the subject" (ibid., p. 31). An example of pseudo-empirical abstraction is when a student solves several related problems, and creates a rule based on a pattern in the answers he obtained. While the student might have been engaged meaningfully in solving the problems, his abstraction of the rule was pseudo-empirical because it was not made from reflecting on the operations of his reasoning. Instead, the abstraction of the rule was made from the products of his reasoning.

The third kind of abstraction is reflective abstraction. This mechanism "ranges over the subject's cognitive activities (schemes or coordinations of actions, operations, cognitive structures, etc.)…[and] separates out certain characteristics of those cognitive activities and uses them for other ends (new adaptations, new problems, etc.)"(J. Piaget, 2001, p. 30). It does so, in two phases. In the first phase, the mechanism of reflective abstraction "transposes onto a higher plane what it borrows from the lower level (for instance, in conceptualizing an action). We call this transfer or projection a réfléchissement" (ibid., p. 30). In the second phase, "it must therefore reconstruct on the new level B what was taken from the previous level A, or establish a relationship between the elements extracted from A and those already situated in B. This reorganization… will be called a reflection [réflexion]" (ibid., p. 30). Finally, when reflecting abstraction is applied to the products of reflected abstraction, then

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the subject is involved in the process of reflective thinking or metareflection. It is this mechanism of reflective abstraction that will allow me to investigate ways in which Rico might transform his current KDUs (understandings at a level A) into KPUs (understandings at a level B).

Quantitative Reasoning

It is well documented that students' experience with mathematics in the U.S. is often based on meaningless manipulation of symbols and concrete strategies for solving problems. But having students engage in problem solving of real-world problems is not by itself a solution. It is the manner in which the student interacts with the situation that matters. For example, "a student could attend to number patterns extracted from a real-world situation and be engaged in number pattern reasoning alone. Similarly, a student could examine relationships between quantities in a highly unrealistic, abstract, or imaginary situation and still be engaged in quantitative reasoning" (Ellis, 2007, p. 441).

Quantitative reasoning emphasizes operating with quantities and their relationships. Thompson defines quantities as "attributes of objects or phenomena that are measurable; it is our capacity to measure them –whether we have carried out those measurements or not that makes them quantities (Patrick W. Thompson, 1989, 1993, 1994a)" (Smith & Thompson, 2007, p. 101). It is important to notice that a quantity is a conceptual entity that is related to how the person conceives of the situation rather than the situation itself. Thompson further defines quantitative operations as "the conceptual operations one uses to *imagine* a situation and to

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reason about a situation—often independently of any numerical calculations"

(Thompson, 1995, p.12). For example, I can imagine the heights of two objects. I

can also imagine comparing those two heights, without knowing their specific

measures, and creating a new quantity that refers to the difference between the

two original heights. It is by a process of *quantification* (Patrick W. Thompson,

1990) that I would assign numerical values to these heights, "but knowing their

measures does not add to conceptualizing the comparison. Rather, knowing their

measures simply adds information about the comparison" (Smith & Thompson,

2008).

Throughout the course, Rico invested considerable effort into helping

students make sense of word problems. Such efforts involved engaging students

in defining variables, explaining relationships between quantities, interpreting and

developing algebraic expressions based on the context of the problems, etc. That

is, Rico's instruction focused on reasoning with quantities. Therefore,

Thompson's framework of Quantitative Reasoning (Smith & Thompson, 2008;

Patrick W. Thompson, 1990, 1993) was an ideally suited theory for examining

Rico's KDUs and KPUs in the context of his Algebra II course.

CHAPTER 4—METHODOLOGY

The goal of this chapter is to explain the methods that I employed for the analysis of the data generated in this study. To do so, this chapter is divided into two sections. In the first section, I provide a brief description of the analytical methods that I used: a) conceptual analysis, b) grounded theory and retrospective analysis of classroom videos, and c) stimulated-reflection interviews. In the second section of the chapter, I provide a description of the data and the analytical procedures that I followed to analyze the data.

Analytical Methods

Conceptual analysis. In light of the background theories stated in the previous chapter, I do not have direct access to another person's thinking. Yet, I can construct models—conceptual systems held by the modeler which provide explanations of the phenomenon of interest of knowing that can help me think about how others might understand particular ideas (Patrick W. Thompson, 1982). Glasersfeld (1995) proposed an analytical method called conceptual analysis to address the issue of constructing such conceptual systems. The method's aim is "to describe conceptual operations that, were people to have them, might result in them thinking the way they evidently do" (Patrick W. Thompson & Saldanha, 2000, p. 4). This method can be used in different ways. One of these ways is to generate descriptions/explanations of a person's understandings. In coming to better understand Rico's MKT that supports his teaching of Algebra II, I constructed models of Rico's mathematical and pedagogical understandings, for

example, by answering questions such as: what KDUs and KPUs do I attribute to Rico based on his instructional design of linear functions?

A second way in which conceptual analysis can be used is for devising ways of understanding systems of mathematical ideas, without necessarily referring to a particular person. I used conceptual analysis from this second perspective as I described the mathematical ideas and ways of thinking that a person might have developed as he or she experienced instruction based on the instructional materials and conversations held in Rico's class. In this case, I am not referring to a particular person's ways of understanding such ideas; rather, I talk about hypothetical ways of thinking—what Piaget (1970) and Thompson and Saldanha (2000) call *the epistemic subject*. For example, a student might come to think of $x_2 - x_1$ as either a subtraction of two numbers or as a new quantity that is the result of comparing two quantity's magnitudes. In the first case, the person comes to think of a number. In the latter, that is what is suggested by Rico's self-constructed instructional material, the person comes to think about a quantity.

Grounded theory and longitudinal analyses of videorecordings. Grounded theory (Strauss & Corbin, 1998) as a way to build claims that are grounded in the data seemed an appropriate analytical method to validate the models that I constructed about Rico's understandings and the mathematical ideas and ways of thinking conveyed by his Algebra II course. Rather than theory coming first and researchers trying to find data to fit or disconfirm the theory, a

grounded theory is inductively derived from studying the phenomenon it represents.

Cobb and Whitenack (1996) propose an analytical method that is consistent with grounded theory that is aimed at analyzing large sets of data such as classroom videorecordings and transcripts. In general, this analytical method proposes to analyze the data in chronological order and on an episode-by-episode basis. As episodes are analyzed, initial conjectures are made and constantly revised when analyzing subsequent episodes. Then, "the chains of inferences and conjectures" become data for further analysis in a process the authors call "zigzagging between conjectures and refutations." (Cobb & Whitenack, 1996, p. 224) The ultimate goal is the search of more stable categories that will evolve into explanatory constructs. This analytical method became useful for the study for two main reasons. First, the analytical method provides guidance in cases where the researcher deals with a large amount of data—which was the case of this study (videotaped lessons, post-lesson reflections, instructional material, and stimulated-reflection interviews). Second, the method provided a way to validate my claims. In this sense, I was able formulate hypotheses by analyzing the data from Rico's Algebra II course to test by interviewing Rico. Then, further hypotheses emerged and others were refuted from analyzing the interviews. This process led to an additional analysis of the data from the Algebra II course in order to further test and validate the resulting models; which became more stable over time.

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Stimulated-reflection interviews. Stimulated-recall has been used extensively in educational research, nursing and counseling. This technique consists of an introspection procedure in which videotaped passages of behavior are replayed to individuals to stimulate recall of their concurrent cognitive activity (Lyle, 2003).

For this study, I used a similar technique that I refer to as stimulated-reflection interviews. This is similar to stimulated-recall in that I used artifacts from Rico's teaching—instructional material, the textbook, the agenda of the course, and videotaped passages of the lessons—to further investigate Rico's understandings by stimulating his recollection and theorizations of the events. The difference between this technique and stimulated-recall interviews is that my focus of investigation was not to reconstruct the teacher's thinking in the moment of teaching. Rather, I used the above-mentioned artifacts as "things to talk about" to stimulate Rico's reflection on his understandings (of his students' mathematics, his design of instruction, and his mathematical understandings) in the moment of the interviews.

These interviews were open ended. Although, I had certain topics and questions in mind, I let Rico guide the conversations. Usually, I presented him with artifacts from his Algebra II course—either instructional material or videotaped episodes from his lessons—and then asked him some specifics about the artifacts; letting Rico guide the rest of the conversation as he discussed aspects of his practice, his rationale behind designing or using particular tasks, his

anticipations of students' understandings, and his theorizations about the results of using particular tasks with the students.

It is important to mention that, as I present the results of the analysis I place Rico's comments in time. I do so, not to imply agency of change; rather, it was Rico who referred back to different periods of time during the stimulated-reflection interviews. He did so to share his insights about what he used to think of his practice before he joined his professional development program, before and after the Algebra II course, and at the moment of the interviews. By placing Rico's comments in time, I am not making claims regarding what made him change; instead, I just report what he said.

Finally, another important aspect that needed consideration was that Rico's thinking clearly evolved in the time when he taught the course and the time in which the stimulated-reflections took place. The actions that he would take in the moment of the interviews were better informed than what they actually were when he taught the course. This data provided interesting opportunities to explore ways in which Rico's thinking changed over time.

Data and Analytical Procedure

The data collection and analysis for this study took place in three different phases. In each phase, new data was generated and analyzed to make way for the next phase. In what follows, I describe the type of data that was generated and the analytical methods that I followed for the analysis in each of the three phases of the study.

Phase I—Rico's Algebra II course. The data for the first phase was collected when Rico taught the Algebra II course (academic year 2007-2008). This data includes:

- Videos of Rico's lessons recorded during the first semester of the academic year (60 lessons x 50 minutes approx. = 50 hours).

- Structured post-lesson reflections. Immediately after each lesson, a set of five questions were asked by the videographer[7] to Rico with the goal of helping him reflect on what he had previously taught. Each post-lesson reflection was also videotaped (60 sessions x 5 minutes approx. = 5 hours). The questions that were asked during each post-lesson reflection are listed below:

1. Do you think your lesson was successful?

2. Did you follow your original plan or did you make adjustments during instruction?

3. Do you think that your students were truly involved in today's lesson?

4. Give an example of a question or questions you would ask to find out how students have learned what you taught

5. Have your ideas changed for next lesson of this class?

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[7] I was not the videographer in the data collection for Phase I of the study. A videographer assigned by the research project videotaped Rico's Algebra II course and the post-lesson reflections.

- Rico's self-constructed instructional materials. These materials include the agenda for the course, as well as all the worksheets and exams that Rico developed for the entire academic year.

At the time, Rico was teaching different sections of the same course. Only one of those sections (the first one of the day) was videotaped and immediately after the lesson the post-lesson reflection took place. This means that, Rico developed his post-lesson reflections based on teaching the lesson to that particular group of students; however he might have generated his later insights about the Algebra II course, based not only on that first section that he taught, but from teaching that same lesson and refining it over and over again, throughout the day.

The Phase I data analysis consisted of an initial review of the videotaped lessons, post-lesson reflections and instructional material with the goal of gaining an understanding of what the course was about. This preliminary analysis of the data allowed me to develop hypotheses about the mathematical understandings and ways of thinking that Rico might have had envisioned for his students to learn. As I watched the videos and looked at the instructional material, I asked the following questions:

- What are the mathematical ideas addressed in the worksheet/lesson?
- How does Rico treat those mathematical ideas so that they prepare students to learn ideas to be addressed in future lessons?
- How does Rico structure his interactions with students?

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- Are there any unique and interesting features in the lessons?

- Do I need Rico to provide clarification about something in particular regarding the lesson/worksheet; including the learning goals that he set out for students?

One problem I faced regarding the data was that there were no videotaped lessons for the first two weeks of the course, so I looked ahead to Phase II with three main goals in mind. First, I planned to ask Rico about his overall organization of the course to better understand the learning goals he had established for his students. Second, I planned to ask Rico to provide details about what he did in the first two weeks of the course and why. Since I didn't have video for those first two weeks, I used the instructional material that he developed to generate talking points to discuss his practice. Finally, my third goal was to further investigate Rico's understandings of his students' mathematics regarding variable, rate of change, and functions; as it had become evident from the first approach to the data that Rico had paid special attention to those ideas in the Algebra II course.

Phase II—Stimulated-reflection interviews. The data collection for Phase II of the study took place during the months of March through April 2010. During this period, I held a total of eight stimulated-reflection interviews with Rico.

As a result of the Phase 1 analysis, I selected a preliminary list of 'things to talk about' with Rico during the interviews. It was preliminary in that I kept

modifying the topics of discussion as a result of my ongoing analysis of the stimulated-reflection interviews. Although I had certain topics and questions prepared in advance, I let Rico's spontaneous recollection of events guide the conversations. For example, there were occasions in which as he was talking about specifics of a worksheet, he would remember general aspects of the course.

The initial interview provided information regarding Rico's rationale behind his general organization of the course, his insights as to why he redesigned the entire curriculum of his Algebra II course, and how he used the textbook. The remaining interviews were geared toward investigating more specific aspects about Rico's learning goals and pedagogical actions associated with them as they related to specific instructional sequences throughout the course. As I mentioned before, since I didn't have video for the first three weeks of the course, we spent more time than what we did with other parts of the course—three out of the eight stimulated-reflection interviews—discussing the instructional sequences on Unit 1 (constant rate of change). In addition, it was during the first weeks of the course, that Rico set the stage for what they were going to talk about during the course and the ways in which they were going to engage with the mathematical ideas. In the other interviews—from the fifth to the eighth interview—we discussed Unit 2 (non-constant rate of change, average rate of change), function transformations, and other aspects of Rico's general organization of the course; such as his expectations for student engagement and assessments.

The stimulated-reflection interviews were videotaped, transcribed, and open coded in a line-by-line analysis (Strauss & Corbin, 1998). Themes regarding Rico's KPUs and KDUs started to emerge from this analysis. For example, two types of categories emerged from the analysis of the data: 1) categories pertaining to KDUs (variable, rate of change, and function) and 2) categories pertaining Rico's KPUs (e.g. coherence, meaningful operations). I then went back to the data generated in Phase I (videotaped lessons, post-lesson reflections and instructional material) to further test and refine the emerging models of the mathematics conveyed by Rico's course, Rico's pedagogical understandings, and his understandings of his students' mathematics. This process led to Phase III of the study.

Phase III—Narrative of the course and Rico's MKT. Phase III consisted of a conceptual analysis of the prior phases with the aim of creating models of Rico's mathematical conceptions, his conceptions of his students' mathematical understandings, and his images of instruction and instructional design. To do so, I first developed the narrative of the course (Chapter 6), focusing on the progression of three main ideas: variable, rate of change, and functions. I then used the narrative of the course as data to delve into what allowed Rico to create the course that he created and to interact with students in the way he did. As a result of this analysis, I developed an explanatory model of Rico's MKT (Chapter 7).

In the remaining chapters of the dissertation, I present the findings of the study. Chapter 5 presents an overview of who is Rico—his personal and professional background—and an overview of the Algebra II course. Chapter 6 develops a narrative of the progression of ideas in Rico's Algebra II course. In Chapter 7, I discuss Rico's mathematics and his mathematics of students. Finally, I present the concluding comments in Chapter 8.

CHAPTER 5—OVERVIEW

Who is Rico?

  Personal and professional background. Rico was a mathematics teacher

at a high school located in the suburbs of a major metropolitan area in the

southwestern United States. He held an undergraduate degree in secondary

education with an emphasis in history and held a master's degree in teaching

secondary education. Rico had always enjoyed mathematics as well. While he

was in college, Rico was a mathematics tutor for four years, which influenced his

decision to apply for a job teaching mathematics, instead of history, after

graduation.

  As a student, Rico ranked at the top of his high school class. In college, he

graduated with a 4.0 GPA[8]. Regarding his experience with mathematics, he was

always enthusiastic and persistent about the subject. Rico explained that when

faced with a problem in his mathematics courses, he would work until he figured

out the solution, regardless of the time spent working on the problem.

  When he taught the Algebra II course (academic year 2007-2008) that

relates to this study, he was in his fifth year of teaching high school mathematics

and in his third year as a participant in a professional development and research

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[8] GPA stands for Grade Point Average. The GPA refers to the average grade
earned by a student. It is the result of dividing the grade points earned by the
number of credits attempted and it is measured on a scale from 0 to 4.

project called Teachers Promoting Change Collaboratively (TPCC)[9]. As a

participant in the TPCC, Rico met with other colleagues in his high school in

weekly Reflecting on Practice Sessions (RPS) guided by a facilitator. The purpose

of a RPS was for participants to discuss matters of instructional practice,

curriculum, and student learning. Rico also took two courses on a functions

approach to a unified secondary curriculum as part of his participation in the

research project.

Although Rico had always strived to be a good teacher, after joining the

TPCC in 2005, he came to think differently about what this meant. By the time of

this study, he was in the second year of trying a new approach to his Algebra II

course.[10] As he explained in RI#1[11] (March 9, 2010), Rico had come to realize

that his past way of teaching was not oriented to students' thinking. He further

explained that, previously, his teaching consisted almost entirely of direct

instruction with nearly all homework and assessments coming from the textbook

and its supplemental materials.

Rico said he had realized, as he tried to revamp his curriculum, that his

Algebra II textbook (and textbooks in general) conveyed a message to students

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[9] NSF-funded research project (P.I. Dr. Patrick Thompson). Project aimed at creating a model for a Professional Learning Community (PLC) to assist secondary pre-calculus teachers in providing high quality instruction.

[10] Rico first tried the new approach to teaching the Algebra II course during the academic year 2006-2007.

[11] RI#1 denotes Stimulated-Reflection Interview 1. I use analogous notation to reference other reflective interviews.

about what it means to think mathematically that differed greatly from the message he now wanted to convey. The textbook's goal was to teach students how to perform efficient calculations and algorithms. It made no effort to explain lines of reasoning that led to formulas or why particular formulas worked the way they did. In addition, Rico felt the textbook's content was compartmentalized—the contents of one chapter were exclusive to that chapter only; subsequent chapters never referred to the contents of previous chapters, even though, the mathematical ideas were closely related. Thus, students received the message that whatever formulas they learned in one section of the textbook were particular to that section, and that there was no need for them to use or remember those formulas once they had taken the exam for that specific section of the textbook.

Rico did not like the textbook's approach because when he followed it, students ended up experiencing completely disconnected lessons throughout the year. This made it very hard for them to remember what they had learned in any usable form. As he explained in RI#1 (March 9, 2010), by fall 2007, Rico had decided to do the opposite of the textbook's treatment. The approach that Rico had adopted after joining the TPCC was to incorporate more opportunities for his students to develop and apply their own reasoning and for him to listen to—and learn from—his students. Rico did not want students to merely learn rules and apply them without having developed an understanding of why they worked or the line of reasoning that gave rise to them. Therefore, Rico felt he had to write his own instructional material to support the lessons he wanted to teach.

Rico's goals for the course. In addition to writing his own instructional material, Rico set new goals for his Algebra II course. One of his main goals for the course, as he noted in RI#1 (March 9, 2010), was for students to develop meaningful ways of thinking about formulas and to incorporate those ways of thinking into their mathematical problem solving.

Another goal Rico held for his students was that they be excited about learning—that they actually be curious about why a formula works the way it does, be proud after using their powers of reasoning to solve problems, and to strive for the personal satisfaction of understanding the mathematical ideas being discussed in class. Rico intended to first help students develop ways of reasoning about problems and then let formulas emerge as a way of generalizing such reasoning. In order to attain this goal, he believed there were two key aspects that he needed to consider. First, he needed to provide students with real world applications that could serve as a basis for developing their reasoning. Second, he needed to reinforce that his students learn to communicate their thinking and listen to their classmates' attempts to communicate their understandings, too.

Finally, another goal that Rico held back in 2007 was to create a course that became coherent in his students' minds. He intended to accomplish that by establishing a few key ideas—e.g., rate of change, functions, graphs—and build upon them throughout the course. Rico was aware that students' prior mathematical learning would not help them in this regard, so he also intended to

provide opportunities for them to rethink and revise their previously developed understandings by making meanings explicit topics of discussion.

Overview of Rico's Algebra II Course

Rico's perceived constraints. Rico recognized several constraints on his redesign of Algebra II (Fall 2007) that determined his overall organization of the course. Two in particular were, one, the school district had a mandated timeline for topic coverage and two, students district-wide would sit for standardized end-of-semester examinations. Rico arranged the topics of instruction in accordance with the district's timeline to ensure that his students were prepared for the examinations. However, Rico treated the mathematical ideas in the course in a substantially different way than was standard for the school district.

According to Rico, the integrated mathematics curriculum, which is the curriculum adopted by the school district, resembles a spiral. The Algebra II course is the third course of a series in which students visit and revisit the same topics with the goal of gaining more sophisticated understandings each year. This means that by the time students get to Algebra II, they have already seen in the previous two courses topics such as linear functions, systems of equations, and polynomial functions. However, students are expected to learn new concepts related to these topics such as end behavior in polynomial functions or new methods for solving problems (e.g., the matrix equation method in systems of equations).

Since students had already seen many of the topics in the Algebra II course in the previous two courses, Rico faced yet another challenge to his redesign of the course. That is, students' lack of meaningful mathematical experiences. On the one hand, Rico anticipated that students might not think about the topics in the ways he wanted them to. For example, Rico expected that they would think of linear functions as just lines and not as functional relationships, or that they would know how to use the methods for solving systems of equations, but not necessarily understand why the methods worked. On the other hand, Rico felt that he did not have the time to go back and re-teach all the ideas that the students should have already learned by the time they get to Algebra II. Therefore, he came up with a strategy for addressing the students' lack of meaningful experiences in their previous courses along with the time constraint. His strategy was to intervene in students' way of thinking about the previously learned topics by asking them questions that students felt they should be able to answer, but they kept finding that they couldn't. He did this to help students become aware that they needed to refine their meanings and, at the same time, to provide them with opportunities to expand upon those meanings. However, this was a challenging task for Rico, as he repeatedly expressed some frustration in the post-lesson reflections. It usually took him more time, than he had anticipated, to go over the ideas in the homework; making it hard to keep up to speed with all the topics he had to cover—as mandated by the District's timeline—and what Rico considered important for his students to learn.

Overall organization of the course. The Algebra II course consisted of 8 units and each unit was divided into instructional sequences.

Table 1 shows the instructional sequences in each unit and the number of weeks Rico spent teaching each unit.

Table 1

Overall Organization of Rico's Algebra II Course

Unit (Duration)	Topics
Unit 1 Semester 1 (5 weeks)	Linear Functions, Systems of Equations, Function Notation, Piecewise Functions, Linear Inequalities, and Linear Programming
Unit 2 (6 ½ weeks)	Introduction to Unit 2, Direct Variation, Accelerating Objects, Area and Quadratic Functions, Cubic Functions, General Polynomial Functions, Radical Functions, Rational Functions, Composite Functions, and Combining Functions
Unit 3 (4 weeks)	Introduction to Patterns, Sequences, Categorizing Sequences, Arithmetic and Geometric Sequences, Finite Arithmetic Series, Finite Geometric Series, Sigma Notation, and Infinite Series
Unit 4 (2 ½ weeks)	Coordinate Proofs, Angles in Polygons, Circular Geometry, and Inscribed and Circumscribed Polygons
Unit 5 Semester 2 (6 weeks)	Exponential Function, Fractional Exponents, Using e in Exponential Functions, and Logarithms
Unit 6 (4 weeks)	Inverse Functions, Logistic Functions, Translations, Stretches and Compressions, Graphing Stretches and Translations, Stretches and Reflections, and Multiple Transformations
Unit 7 (8 weeks)	Trigonometry, Modeling with Sine & Cosine Functions, The tangent function, Polar Coordinates, Polar Functions, Vectors, Parametric Equations, Law of Sines and Law of Cosines
Unit 8 (1 week)	Distributions, Standard Deviation, Normal Distribution, and Z-Score

In addition to the instructional sequences, there were individual lessons incorporated throughout the course that did not address specific mathematical

ideas, but they were geared toward providing students with opportunities to improve their communication skills when expressing their mathematical thinking. For example, at the end of Unit 1, Rico gave his students a worksheet that contained a range of written answers to a problem that they, as a class, had already studied in Unit 1. Students had to rank the answers depending on what they considered that a *good answer* should include. Rico took the class discussion about the worksheet as an opportunity to discuss with the students the importance of communicating their mathematical meanings; and to provide students with a reference point in constructing their own answers to the problems in the course.

Homework assignments and structure of the lessons. The class met five times per week for approximately 50 minutes. Often, Rico handed out a new worksheet to the students at the end of the lesson for them to work on at home. The following day, Rico would give the students the first ten minutes of the lesson (sometimes more, if necessary) to discuss their homework answers amongst each other. Rico used that time to go around the classroom and gauge the effort that students put into the homework in order to determine the issues that needed to be addressed in a whole class discussion. Sometimes, the discussion about the homework naturally led to the main ideas that Rico wanted to address during the lesson. Other times, Rico would present a new contextual problem that would serve as a basis for talking about the mathematical ideas he wanted to address.

The homework assignments were a central part of Rico's lessons. He was not familiar with the term "didactic object" —a thing to talk about that was designed to support reflective discourse about mathematical ideas and ways of thinking. However, Rico designed the homework assignments with similar intentions. That is, Rico explained in the reflective interviews that when he had designed the course, his goal with the worksheets was not just to get the students to practice with similar problems that they had learned in class, but also to take the discussion from class one step further —an extension of and expansion upon the class discourse. It is important to clarify that the objective of the problems was to teach students to explain their reasoning rather than merely computing the solutions. Rico constructed the assignments with the expectation that the students' engagement with the problems could generate a range of interpretations. He did this with the goal of promoting mathematical discussions between the students and their partners and as a class.

The time that the class spent working on each worksheet varied depending on whether the students experience difficulties with the worksheets or not. Sometimes, if Rico noticed that students were experiencing some trouble with specific ideas addressed in the worksheets, he would let the students spend more time working on the problems in groups. Other times, he would prepare new tasks, related to the ideas with which students were struggling, to help him address those difficulties during class discussion.

The problems that Rico included in the worksheets came from different sources. Some of the problems were modifications of the contextual problems included in the textbook. Other problems came from newspapers. Rico explained that if he read something on the newspaper that he could relate somehow to what they were talking about in class then he would use it. Finally, he also included contextual problems from other subjects—for example, Physics and Chemistry— and addressed the mathematical aspects of those problems. In the post-lesson reflections, Rico commented that he liked to use contextual problems that students could find interesting and that students could relate to them. Therefore, Rico carefully elaborated the contextual problems he used in class and he seemed to have fun using them with his students.

Rico's expectations for student engagement. Rico held two main expectations for his students' engagement with the course. First, as aforementioned, Rico considered that students' participation in mathematical discussions was crucial for extending and refining their mathematical meanings. Rico structured the lessons to ensure that every student in the class could engage in a mathematical discussion at least once during the lesson, either with their partners or as a class. Along with their participation in mathematical discussions, Rico expected the students to make an effort to provide explanations based on reasoning and not only on rote memorization of facts and formulas.

Second, Rico expected students to continuously work hard to make sense of the ideas discussed in the lesson. In exchange, he made an agreement with his

students to design every lesson to relate to the meanings that supported the mathematical ideas addressed in it. Rico made the commitment to not ask the students to work on something if they did not have access to that meaning, as this would only undermine the students' confidence in their own reasoning.

Throughout the first semester of the course, getting students to engage with the course in the way Rico intended them to, was not an easy task. For example, as students considered many of the topics as review—even if Rico was addressing those topics from a different perspective and in a more sophisticated way—then the students would sometimes provide memorized facts as answers and have a hard time explaining what those facts really meant. Also, Rico commented in a post-lesson reflection that there were some students that he knew they needed tutoring because they were having trouble with the material from the course; however they were not seeking the appropriate help from him.

Rico further explained in RI#5 (March 30, 2010) that the second semester of the course felt different than the first semester in that, students were now used to Rico's expectations about their engagement in the course. Also, the mathematical ideas in the second semester were new to the students, that is, they had not seen them in their previous mathematics courses; in consequence, students had to rely on what they had learned in the Algebra II course to make sense of those ideas.

CHAPTER 6—PROGRESSION OF MATHEMATICAL IDEAS

Introduction

In this chapter I will provide details about the progression of mathematical ideas in Rico's Algebra II course. I will focus on three key mathematical ideas—variable, rate of change, and function—and how Rico treated them in instruction. These ideas played a very important role in the course. Although I use them as separate themes to guide my narrative of the course, it is the way Rico treated them in his instruction to build upon one another and create a system of meanings that better explains what the course was about. Then, in Chapter 7, I will take the course as data to further explore Rico's MKT.

Before I move on to the main theme of this chapter, I first provide some context for Rico's motives in redesigning the course and the strategies he employed in developing a course that could become a coherent system of meanings in the students' minds.

I emphasize that in this case I am only reporting Rico's insights as to how he thought he had come to think different about his practice. I did not have access to Rico's thinking, instruction, and instructional design previous to the Algebra II course so as to make claims regarding how he changed. Instead, I share Rico's insights with the goal of portraying to the reader how reflective Rico was about the impact his instruction had on his students' mathematics.

Rico's motives for redesigning the course. In RI#1 (March 9, 2010), Rico explained that early in his involvement with the TPCC project, Rico was

required to develop a teaching experiment as part of his final project for the Functions I course[12]. Rico chose function transformations as a topic for the assignment. When he tried to incorporate what he had learned from the teaching experiment into his Algebra II course, he realized that his past way of teaching function transformations did not help students develop powerful ways of thinking about functions.

The content Rico used to teach prior to his involvement in the TPCC, as he noted in RI#1 (March 9, 2010), was strictly from a graphical point of view—consisting of identifying the shape or formula of the parent function and then identifying or applying the transformations in order (i.e. reflections, stretches, and translations). Students were provided with rules to follow with no attached meaning to them other than to identify some numbers from a formula and make sure the points were located on the right spot. Rico also noted in RI#1 (March 9, 2010) that his instruction prior to 2005 did not provide students with ways of thinking that would allow them to reason their way to a solution. It also had the problem that it was not about functions; that is, according to Rico, "As I vary the argument, what happens to the outputs of the function" (Rico, RI#5—March 30, 2010).

Rico decided that students would be better served if the chapter's focus shifted to comparing inputs and outputs of functions. For example, he wanted his

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[12] Functions I, was the first of a series of courses that the teachers took as part of their involvement in the TPCC.

students to think that they would define a new function *g(x)* using the outputs of some other function, as in $g(x) = f(x-2) + 1$. That is, he wanted the students to define new functions based on the outputs of previously defined functions, regardless of the "shape" of their graph or how the formula of the parent function looked. In order to understand function transformations in this way, Rico imagined that the students had to ponder questions such as "What happens to the *argument* of *f* as the value of *x* varies?" and "What are all the calculations in the expression doing to the outputs of *f(x)*?" in order to obtain the outputs for *g(x)*.

Rico further realized that while this new chapter might be more beneficial for his students' future thinking in mathematics courses, there was nothing in the course that would prepare them to think about functions in the way that was needed in the chapter. In Excerpt 3, Rico relates such insights.

Excerpt 3. Rico's realization that he did not prepare students' thinking (RI#1—
March 09, 2010).

Rico: Nothing that I did in the chapters leading up to function transformations

prepared students to think about functions in that way. None of the rules

and the formulas, and anything that we teach in the first semester,

prepared students to really focus on the relationships between the outputs

of these two functions [f(x) and g(x)] in the way that I wanted them.

After 2005, Rico stopped thinking about the function transformations chapter as an isolated chapter in the course. Instead, he anticipated that if students were to think about function transformations and functions in general in powerful

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ways, then they would need to experience instruction aimed at developing the kinds of thinking that Rico had in mind. As a consequence, Rico ended up redesigning the entire curriculum for his Algebra II course.

Rico's strategies for developing a coherent system of meanings. By 2007, Rico had realized that his previous instructional approach did not orient students to thinking about functions as a relationship between covarying quantities. In addition, the textbook's compartmentalized presentation of topics did not support his attempt to redesign the course. Therefore, Rico came up with a set of strategies that guided his organization of the new course.

First, he anchored the course in a few mathematical ideas that would provide students with tools for reasoning about functions. At the core of these ideas were a dynamic view of variable, a conception of rate of change as deeply embedded in proportional reasoning, and a conception of function as a relationship between covarying quantities.

Second, Rico made sure, especially at the beginning of the course, to provide students with opportunities to rethink, extend, and expand upon their previously developed meanings of mathematical concepts such as variable, slope, and linear function. He did this because he anticipated that students might not think about these ideas in ways that would help them understand the meanings he wanted to convey in the Algebra II course. For example, some of the meanings that Rico expected students to have based on their previous mathematical experiences, were:

- Variable —A letter in an equation

- Slope —The steepness of a line or a number that is the result of plugging numbers into the equation $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and simplifying the result.

- Linear function —a line. That is, the result of plotting two points and connecting them; or, the result of finding an equation, which is the result of applying a set of memorized steps.

These meanings are problematic in at least two ways. First, if someone holds these as her foundational meanings, then there is nothing about them that encourages thinking about variation, and thus, thinking about functions as a covariation of quantities. Second, someone holding these meanings might be able to solve problems that require them to "Find the equation of the line passing through the points (5, 10) and (8, 4)". However, there is nothing about these meanings that extends to other ideas such as "Find the equation of the line passing through the point (5,10) with a rate of change of -2". In other words, the material that comes later in the course does not build upon these meanings.

For Rico, the development of ideas and ways of thinking had become a long-term process and it was not confined to particular instructional sequences. As he redesigned the lessons, he constantly tried to connect what the students learned before to what they were currently learning. At the same time, Rico prepared students for what they were going to learn later in the course. For example, function transformations and inverse functions were two instructional sequences that, according to the District's timeline, pertained to the second

semester of the course. However, throughout the first semester, Rico included questions in the lessons that were intended to build toward both ideas, so that by the time they got to those topics, instruction would involve formalizing ways of thinking about functions that they had been practicing all along.

Together, Rico's curriculum design strategies were aimed at developing a course that could become a coherent system of meanings in students' minds. That is, the meanings the class established throughout the course remained consistent and built upon each other. I refer to the class and not to Rico, because as he noted in the stimulated-reflection interviews, when teaching the course, he emphasized that the students should take part in creating the definitions they used in class.

Variables as Quantities

Rico, without expressing any personal knowledge of a formal theory of quantitative reasoning (e.g.,Smith & Thompson, 2008; Patrick W. Thompson, 1989, 1993, 1994a), seemed to have a strong intuitive grasp of the importance that his students reason quantitatively. For example, he constantly oriented students to pay attention to the quantity whose values a variable represented. Every time a variable came into play, Rico put special emphasis on the variable's definition and its units. Also, he stressed that in a problem, a variable is free to vary —it can assume any of a range of values for which the variable is defined.

Allowing a variable to assume any of a range of values over a specified interval was relevant in two ways. First, it allowed Rico to hold discussions regarding a variable, say *x*, as it continuously varied throughout an interval. This

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dynamic view of variable is key to developing a covariation view of function

(Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, Carlson, & Thompson,

2008; Patrick W. Thompson, 1994b) and is compatible with the idea of function

that Rico intended to promote in class. Second, it allowed him to focus his

instruction not only on specific values of the variable, say $x = a$, but also on what

he called "changes in variables" —the change from some value of one variable to

another value of the same variable. For example, suppose that $x = h$ for some

value h, then $x - h$ represents the change in x away from h. Rico did not explicitly

refer to "changes in variables" as defining new quantities. However, he treated

them in instruction as if they were. Rico's notion of "changes in variables" is

compatible with Thompson's (1994a) idea of creating a *difference* —a new

quantity that is the result of comparing two already conceived quantities

additively. As Smith and Thompson (2008) point out, a difference is not the result

of subtracting. In this sense, $x - h$ does not need to assume any particular value for

someone to come to conceptualize $x - h$. And it is in this sense that Rico treated

changes in variables in instruction.

In the following sections, I first describe Rico's strategy for orienting

students to paying attention to quantities. Then, I describe the two cases in which

Rico paid attention to *differences* ($x - h$ and $x_2 - x_1$) in instruction, and the

purpose these conceptualizations served in the progression of ideas.

Orienting students to pay attention to quantities. Back in 2007, one

major theme that Rico wanted to develop throughout the academic year was that

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all the functions in the course represented a relationship between two variables. And that it was by allowing the variables to vary throughout an interval and analyzing the changes in one variable in relation to changes in the other that someone could understand the relationship that is, the function. Furthermore, this analysis of a function's behavior could be done regardless of the operations that were performed on one variable to obtain the other. In other words, this way of analyzing a function's behavior could be applied to any function in the course. Thus, one of Rico's main goals for students was to orient them to thinking about functions as composed of variables —as opposed to thinking for example, that a quadratic function is a *u-shaped* figure.

When Rico redesigned the course, he anticipated two main issues regarding students' previous experience with variables. First, he expected that for students variable might mean "a letter in an equation" which did not convey the dynamic view of variable that he wanted to promote in class. Second, due to students' previous experiences in mathematics courses, they might be used to working with variables without ever questioning what the variables in a problem stood for. To deal with these issues and help students develop a more sophisticated conceptualization of variable, Rico included questions in the homework assignments that provided the students with opportunities to examine and extend their meaning of variable (Example 1). He also included contextual situations that oriented students to paying careful attention to the variables' definitions (Example 2) and the units in which they were measured (Example 3).

In what follows, I present Examples 1-3 from Rico's instructional material. In each example, I present the contextual problem, a proposed solution that resembles[13] how Rico approached instruction regarding the task, and a brief discussion about the purpose that the task served in Rico's instruction.

Example 1. First assignment of the course: state, test, and refine students' definition of variable.

Figure 2 shows a series of four questions that Rico included in the first homework assignment in the course. Rico designed these questions with the goal of getting students to state, test, and refine their meaning of *variable*.

3. How would you define the term *variable*?

4. According to <u>your</u> definition (meaning, read your definition carefully and use **your definition**, not other prior knowledge), identify each *variable* in the formula $y = mx + b$.

5. Reflect on the following definition:
     <u>*Parameter*</u>: a constant in an equation that can be varied to yield a family of similar curves

   a. Identify any *parameters* in the function $y = mx + b$.

   b. Explain the concept of *parameter* in your own words. Include a discussion of the differences between a parameter and a variable (for example, why is "*b*" a parameter and not a variable?).

6. Consider your responses to Exercises 3 through 5. Rewrite your definition of *variable* if necessary to clarify your ideas.

Figure 2. Refining the meaning of variable (Homework Assignment—August 08, 2007).

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[13] The tasks that I present in this section are from the first two weeks of the course, for which I do not have videotapes of the lessons. I rely on Rico's accounts of how he used the tasks in instruction. I searched for evidence in written material and videotaped lessons from later instructional sequences to test whether his recollection of events fit with his actual approach to instruction regarding quantities. What I present as solutions fit both his recollection of events and his instruction during the time he taught the course.

As shown in Figure 2, Question 3 asked the students to state their definition of variable. Then, in Question 4, they had to test their definition by identifying the variables in the equation $y = mx + b$. Rico had anticipated that a common response to Question 3 would be something along the lines of "a variable is a letter in an equation". Then, strictly speaking, y, m, x, and b should be considered variables in the equation of a linear function. Question 5 introduced the definition of parameter to allow students to differentiate between the two notions.

Rico designed the task with two goals in mind. First, he wanted to help students recognize the important difference between parameters and variables in an equation. Even though letters can represent both within a problem, variables are free to take any of a range of values for which the problem is defined, whereas parameters are fixed —although they can vary from problem to problem. Second, he also wanted to convey the message that the course required students' commitment to continually engage in trying to make sense of the mathematical ideas discussed in class and that students should take part in creating what he called "powerful definitions," which are definitions that can actually be used as tools for thinking about mathematical problems and communicating one's meanings.

According to Rico's recollection of events, he was puzzled by the fact that students were able to differentiate parameter from variables in $y = mx + b$. Still, students' definitions of variable did not account for such differentiation. He

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further explained that if he were to teach the course again and assign this task to students, instead of assigning it as homework, he would have students test each other's definitions. He would do this with the purpose of helping students realize that their definitions did not clearly and effectively communicate their meanings.

Rico did not expect students to reconsider their meanings by working exclusively on one task. However, it was by helping students become aware of their meanings, and allowing them to test whether these fit with what was conveyed in class, that he could help them start engaging in learning previously seen ideas in new ways. I refer to previously seen ideas, because almost all the material in Unit 1 (e.g. linear functions, systems of equations) served as review. Thus, at the time it felt like a real challenge for Rico to help students realize there was more to know about the topics than the memorized facts and procedures of solving problems learned in the previous courses.

Example 2. The role of contextual problems for reasoning with quantities.

Figure 3 presents a contextual problem included in the first assignment of the instructional sequence on Systems of Equations —the second sequence in Unit 1. The main goal of the assignment was for students to "draw the mathematics out of the context" (Rico's words, RI#1 —March 09, 2010) in order to set up, solve and interpret the system of equations associated with the problem.

An engineer is designing a new platform for tall buildings so that window washers, repairmen, painters, etc. can move up and down the side of the building. He needs to attach either rope or metal chain to the platform, which will hang from a system of pulleys attached to a support structure on top of the building.

This is a sketch of this situation:

Support Structure

Rope or chain →

Platform

Note: The rope or chain is unwound from this location to lower the platform (in other words, the excess chain or rope is not dangling from the support structure or stored on the platform).
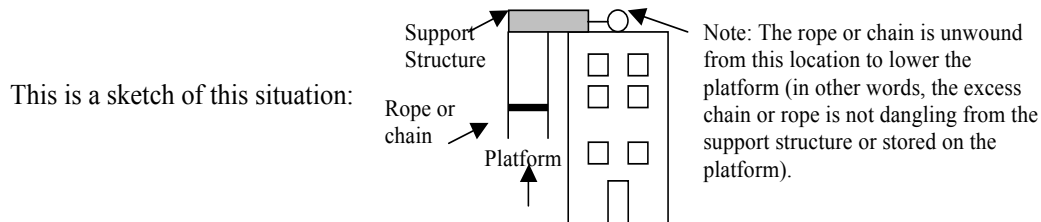
In order to attach the ropes securely and use the proper pulleys, the engineer must attach 50 pounds of additional equipment to the platform, which already weighs 200 pounds. Using the chain, however, will only require attaching 15 pounds of additional equipment. The platform is designed to carry workmen and supplies weighing 500 pounds.

Figure 3. The cantilever problem—Part 1 (Homework Assignment—August 13,2007).

The problem describes a setting in which an engineer designs a new platform that can be moved up and down the side of a building. The engineer can use either rope or chain to hold the platform to a support structure situated on the top of the building. A series of questions and more information is introduced throughout the task to help students construct linear functions that represent the total weight (in pounds) that the support structure must hold as a function of the distance (in number of feet) that the platform is lowered.

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1. What is the total weight that must be supported (so far) if he plans to use rope (don't forget the workmen and their supplies)?
Answer: The total weight that must be supported (so far) if he uses rope is 750 pounds.

2. What is the total weight that must be supported (so far) if he plans to use chain (don't forget the workmen and their supplies)?
Answer: The total weight that must be supported (so far) if he uses chain is 715 pounds.

The chain weighs slightly more than the rope per linear foot. A foot of rope weighs 0.56 pounds. A foot of chain weighs 0.65 pounds. The platform needs <u>four ropes or four chains</u> to hold it.

3. As the platform is lowered what happens to the total weight that must be supported from the top of the building (the support structure)? Why?
Answer: As the platform is lowered, the weight that must be supported from the support structure increases, because more rope or chain is needed to support the platform.

4. For each foot that the platform is lowered, how much weight is added to the total weight that the support structure must hold if (remember that there are four ropes or chains):
    a. rope is used
    Answer: $4(.56) = 2.24$ pounds for each foot that the platform is lowered
    b. chain is used
    Answer: $4(.65) = 2.6$ pounds for each foot that the platform is lowered

Figure 4. The cantilever problem—Part 2 (Homework Assignment—August 13, 2007).

The purpose of the first set of questions (Question 1-4 in Figure 4) was to help students conceptualize the problem. In this case, merely looking at the problem to search for key words and numbers —a strategy commonly taught to students for setting up systems of equations —does not work in successfully setting up the equations that describe the problem (Question 5 in Figure 5 in page 76). First, students have to put together the information that is provided within the context of the given problem.

As Rico recalled in the stimulated-reflection interviews, his use of real world applications was different from what is done in many textbooks. It is common that textbooks present the application problems at the end of the section, expecting students to apply what they have learned. Usually, students find these problems complicated and unrelated to what they have practiced throughout the section. In Rico's view, real world applications served a different purpose (Excerpt 4).

Excerpt 4. Rico's use of real world applications (RI#1—March 09,2010).

Rico: If you can successfully frame your discussions around interesting real world situations and you can pose natural questions that might come up if someone starts thinking about this real world situation, it engages the students a little bit more. Students can apply some reasoning that otherwise they wouldn't apply. It gives the students some extra tools to attack the problem.

For Rico, the use of contextual problems served as a tool for helping students conceive of situations mathematically. Conceiving situations in this way allowed students to deal with and think about much more than just numbers or

symbols void of meaning.[14] In this case, the students had to engage in operating

with quantities.

5. Write a function that represents the weight the support structure must hold if:
   a. rope is used
   Answer: $y = 750 + 2.24x$
   b. chain is used
   Answer: $y = 715 + 2.6x$

6. In Exercise 5, what do your variables stand for (include units)?
   x: number of feet (ft) the platform is lowered
   y: number of pounds (lb) the structure must hold

7. What are the rates of change for each of the linear functions defined in Exercise 5? What does each rate of change mean in the context of this situation?
   Answer: If rope is used, the rate of change is $2.24lb / ft$ and it means that whatever the number of feet the platform is lowered, the weight (in pounds) the support structure must hold increases by 2.24 times as much. If chain is used, then the rate of change is $2.6lb / ft$. It means that whatever the number of feet the platform is lowered, the weight (in pounds) the support structure must hold increases by 2.6 times as much.

8. How much weight would be added to the amount the support structure must hold if the platform was held up by rope and the platform was lowered by 0.75 feet?
   Answer: $(2.24lb / ft) \times (.75 ft) = 1.68lb$

9. Identify two pairs of coordinate points that are true for the function representing the weight of the platform held by chains. What does each coordinate point mean in the context of the situation?
   Answer: $(3, 722.8)$, if the platform is lowered by 3ft, then the weight the support structure must hold is 722.8 lb. And, $(15, 754)$ if the platform is lowered by 15 ft, then the weight the support structure must hold is 754 lb.

Figure 5 The cantilever problem—Part 3 (Homework Assignment—August 13,

2007).

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[14] Rico argued in RI#3 (March 23, 2010) that in his experience, it has been very common that students know how to use the techniques for solving systems of equations without knowing why the techniques work. During the time he taught the course, Rico spent most of the instructional sequence on systems of equations —which was meant as a review —going back to the techniques (substitution, graphical, and elimination methods) that students learned in previous courses to develop explanations about the reasoning that gives rise to each of the mathematical techniques.

The second set of questions (Question 5-7 in Figure 5) led students to set up the system of equations (Question 5), define the variables (Question 6), and discuss what the rate of change meant within the context of the situation (Question 7). Other questions, (e.g. Questions 8 and 9 in Figure 5) further explored the meaning behind the equations.

Paying attention to variables' definitions also involved identifying the units in which they were measured. Rico did not provide an explanation as to why he thought the units were such an important part of defining variables. However, when the idea of rate of change came into play, paying attention to the units was essential to developing explanations about what the rate of change meant in the context of the situation.[15]

---

[15] I will further discuss this point in the section on rate of change. In general, Rico's objective was that, when they discussed the idea of constant rate of change, as in $m = \dfrac{2}{3}$, they do so by focusing on the relationship between the variables, "whatever the change in one variable, the other changes by $\dfrac{2}{3}$ as much"—instead of focusing on the numerical operation 2 divided by 3.

16. Consider only the platform held by chain (write its weight function here: $y = 715 + 2.6x$). The engineer knows that rope stretches slightly as weight pulls down on it. He estimates that the actual length of the rope will be 3% longer than it was originally measured. He creates the function $y = 1.03x$, where x is the original length of the rope (in feet) and y is the stretched length of the rope (in feet).

Find the solution of the system involving the function $y = 1.03x$ and the function you wrote in the space above. What does the solution mean in this context? Explain the significance of your answer in terms of systems of equations in general.

Figure 6 The cantilever problem—Part 4 (Homework Assignment—August 13,

2007).

Finally, Question 16[16] (Figure 6) proposed setting up and solving a new

system of equations. Rico designed the question in anticipation of students'

possible responses to it in one of three ways (Table 2):

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[16] I do not discuss Questions 10-15 here. These questions further explore the context and ask students to solve the system of equations. I include these questions in the Appendix.

Table 2

Rico's Anticipated Student Answers to Question 16 of the Cantilever Problem

Student's Answer	Rico's Explanation for the Student's Answer
Student (1): The answer is $x = -455.41$, $y = -469.07$	The student uses one of the three different methods for solving systems of equations (elimination, substitution, or graphing method). This student just provides an answer.
Student (2): The answer does not make sense because in this context we can't have a negative number	The student solves the system and then realizes that negative numbers do not make sense in the context of the problem.
Student (3): There is no point in setting up the system, even though we are using the same notation, the variables stand for completely different quantities, and thus it does not make sense to set up the system.	The student realizes that even though the same notation was used to represent the two functions, the variables in the two functions describe different quantities. It did not make sense to set up the system.

Rico's goal with Question 16 was to lead a class discussion addressing each of the answers he anticipated the students to give (Table 2). By having students compare their answers to the ones discussed in class, Rico wanted students to realize that they should continuously engage in trying to make sense of the situations presented in class; and one way to do so was by paying attention to the variables' definitions.

Example 3. Units of measurement.

The following contextual problem (Figure 7) was included in the quiz on piecewise functions —which was the third instructional sequence in Unit 1. I present the solution to the problem drawing on ideas of constant rate of change and linear functions, which I explore in a later section of this chapter. This

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example further explores ways in which Rico oriented students to pay attention to carefully defining variables, including the units in which they are measured.

> **John pulls up to a stoplight in his car. He sits at the stoplight for 40 seconds before the light turns green. John then begins to accelerate, increasing his speed by 12 mph each second. Four seconds after the light turns green, another car swerves in front of John's car and John hits the brakes. Over the next 2 seconds, John's speed decreases by a total of 30 mph at a constant rate. He maintains his new speed for 3 seconds.**
>
> 1. One student says that the variables for this piecewise function should be:
>
> t is the time since John first stopped at the light
>
> $s(t)$ is John's speed
>
>   What is wrong with these variable definitions?
>
> 2. Rewrite the variable definitions to clarify them.
>
> 3. Write a piecewise function $s(t)$ for the situation using your variable definitions.
>
> 4. Find s(45.5) and describe what it means in the context of the situation.

Figure 7. Determining John's speed—the context (Piecewise functions quiz—August 28, 2007).

The task (Figure 7) presents a contextual situation that can be modeled with a piecewise function defined in four parts. In Question 1, students are provided with a set of variables for the function and they are asked to determine what is wrong with the variable definitions. Both variable definitions are missing the units in which they are measured. According to the context, the variable t (the time since John first stopped at the light) should be measured in seconds and *s(t)* (John's speed) should be measured in miles per hour.

Figure 8 illustrates the solution to the problem. For each part of the function, I provide a brief explanation of the ways of thinking that Rico expected students to use when solving problems like this one.
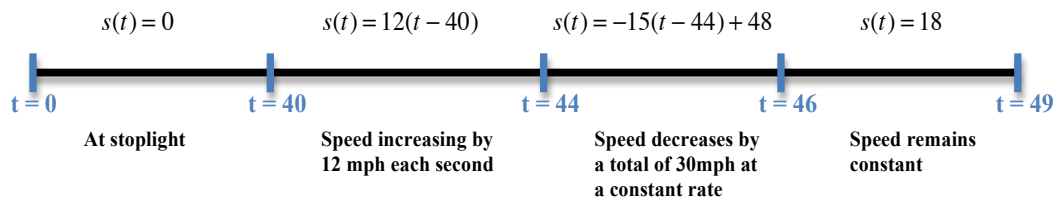
| $s(t) = 0$ | $s(t) = 12(t - 40)$ | $s(t) = -15(t - 44) + 48$ | $s(t) = 18$ |

| t = 0 | t = 40 | t = 44 | t = 46 | t = 49 |
| At stoplight | Speed increasing by 12 mph each second | Speed decreases by a total of 30mph at a constant rate | Speed remains constant |

Figure 8 Determining John's speed—the solution (Piecewise functions quiz—August 28, 2007).

Section 1: During the first 40 seconds, John sits at the stoplight, thus his speed does not change. The function that describes this situation is $s(t) = 0$. This function definition only works for values of t between 0 and 40 seconds. Outside this domain, the function does not describe the situation.

Section 2: During the next 4 seconds, John's speed increases 12mph each second. This function only works for values of t between 40 and 44 seconds. Outside this domain, this function does not represent the situation. The function $s(t) = 12(t - 40)$ models this part of the trip. Note that 12 represents that the rate of change (in mph/s) of John's speed (in mph) in relation to the time elapsed (in seconds) since he first arrived to the stoplight. Note that we use the expression $t - 40$ —instead of defining a new variable for this part of the problem, because we want to keep our control variable defined the same for all the four parts of the situation.

Section 3: Note that by the end of the previous portion of the trip, John's speed was 48 mph. In this section of the trip, John's speed decreases by 30 mph over a period of 2 seconds, or 15mph/s. This function only works for values of t between 44 and 46 seconds. Outside this domain, this function does not represent the situation. $s(t) = -15(t-44) + 48$ models this part of the trip.

Section 4: Once John's speed reaches 18 mph it remains the same for the next 3 seconds. Thus, the function that defines this portion is $s(t) = 18$ and it describes the situation for when t is between 46 and 49 seconds. The resulting function is:

$$s(t) = \begin{cases} 0 & \text{if} \quad 0 < t \leq 40 \\ 12(t-40) & \text{if} \quad 40 < t \leq 44 \\ -15(t-44) + 48 & \text{if} \quad 44 < t \leq 46 \\ 18 & \text{if} \quad 46 < t \leq 49 \end{cases}$$

Finally, Question 4 (Figure 7) asks students to find $s(45.5)$ and describe what it means in the context of the situation. We substitute $t = 45.5$ in the third part of the function and obtain that $s(45.5) = 25.5$, which represents John's speed 41.5 seconds after he first stopped at the stoplight.
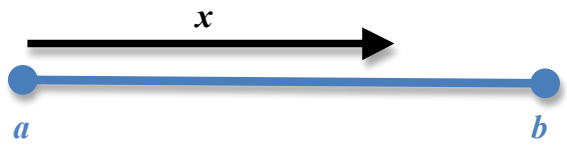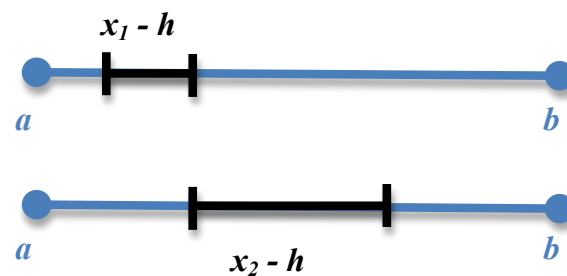
Creating *differences* by operating on variables. In Rico's redesign of the Algebra II course, the idea of *differences* became central in two ways. First, $x - h$ (the change in x away from a reference point h) was a way for him to build toward the idea of function transformations. Second, $y_2 - y_1$ (the change in y) and $x_2 - x_1$ (the change in x) as defining quantities, rather than numbers, were key to understanding rate of change —another key idea of the course. In what follows, I

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provide a brief explanation of the images conveyed by Rico's instruction

regarding the two cases, $x - h$ (Case 1) and $x_2 - x_1$ (Case 2).

Case 1: Change in x away from a reference value h, $x - h$. Rico

emphasized that h be thought of as an arbitrarily selected value of the variable x.

Thus, $x - h$, the "change in x away from the reference value h", represented a new

varying quantity, whose values were determined as follows:

Table 3

Image of $x - h$ Conveyed by Rico's Instruction

Suppose x is a variable defined over an interval $[a,b]$	
Select $x = h$ as a reference point	
$x - h$ is a new quantity whose magnitude varies according to x's current value.	

Example 3 in page 79, from the previous section (Orienting Students to

Pay Attention to Quantities) can be used to illustrate how Rico encouraged

students to operate with differences. For example, the problem asks students to

construct a piecewise function that describes John's speed $s(t)$ as a function of the time elapsed (in seconds) t since he first arrived to a stoplight. Section 2 describes what happens to John's speed after sitting still at the stoplight for 40 seconds. In this section, his speed increases 12mph every second. Section 2 of the piecewise function that models the situation is $s(t) = 12(t - 40)$ —for t between 40 and 44 seconds. This means that whatever value of t that I input into the function, I will first calculate the change in t away from the reference value of 40. John's speed is increasing by 12 mph every second, but this happens until 40 seconds after John arrives at the stoplight. The function $s(t) = 12(t - 40)$ can be thought of as a transformation of $s(t) = 12t$ (John's speed increasing 12mph every second). We just adjusted the definition of the function so that it can reflect the fact that we are not measuring the time since John first started moving.
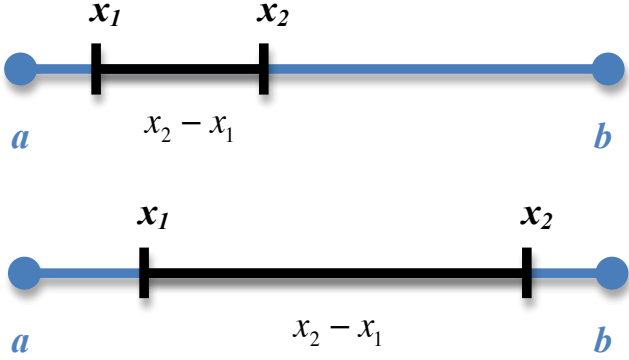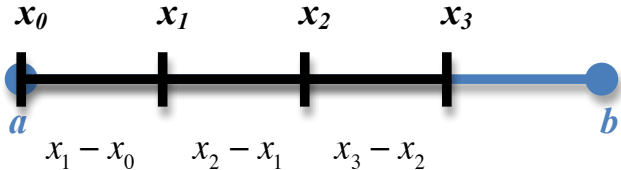
Case 2: Change in x, $x_2 - x_1$. Rico also treated $x_2 - x_1$ as if it represented a quantity generated by additively comparing two different magnitudes of x — rather than just the result of subtracting numbers. Depending on the context in which $x_2 - x_1$ was used, it could either represent a varying quantity or a constant quantity. On the one hand, $x_2 - x_1$ was a varying quantity in that values of x (x_1 and x_2) could be selected as close or far apart as the domain of x allowed.[17] On

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[17] According to Rico, the idea that x can change by either a very large amount, say 1,000,000 units, or a very small amount, say $\dfrac{1}{1,000,000}$ of a unit, constituted a key aspect of defining the idea of constant rate of change, as we will see later in this chapter in the section on constant rate of change.

the other hand, the difference between two values of x could be held constant in order to generate subsequent intervals of equal size ($x_j - x_i$) as x varied throughout a continuum —for example, in analyzing a function's behavior by analyzing the average rate of change. The following table (Table 4) illustrates both cases.

Table 4

Image of $x_2 - x_1$ Conveyed by Rico's Instruction

Suppose x is a variable defined over an interval $[a,b]$	x a ———————— b
$x_2 - x_1$ as a varying quantity.	x_1 x_2 a $x_2 - x_1$ b x_1 x_2 a $x_2 - x_1$ b
'constant bits' generating equally spaced intervals of size $x_1 - x_0$	x_0 x_1 x_2 x_3 a b $x_1 - x_0$ $\quad$ $x_2 - x_1$ $\quad$ $x_3 - x_2$

Summary

The notion of variable as representing more than just "a letter in an equation" was key in Rico's redesign of the Algebra II course. For example, he included opportunities for students to utilize their previously developed meaning of variable and refine it. He also developed contextual situations for which students had to carefully define variables (and their units), as it was a means for students to make sense of the contextual situations presented in class.

In summary, Rico's efforts were geared toward helping students re-conceive variables as representing quantities' magnitudes as they vary. Then, he further used quantities to develop other ideas in the course, such as rate of change, which I explain in the following section.

Rate of Change

As mentioned in the introduction to this chapter, one of Rico's main goals for the redesigned Algebra II course was to establish a few key mathematical ideas upon which new ideas could be continuously built. In the previous section I explained the ways in which Rico oriented students to thinking about variables as representing quantities' values. In this section I will proceed to describe how he treated the idea of rate of change in instruction. It is important to clarify that, even though I talk about Rico developing the idea of rate of change after having developed the idea of variable, he actually addressed both from the beginning of the course. Thus, students were encouraged to extend their meanings of variables as they worked with ideas of constant rate of change, and vice versa.

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At the beginning of the course, Rico had anticipated that students'

previous understandings about the idea of slope might be iconically related to a

graph or to obtaining a numerical value as a result of plugging numbers into the

formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Neither of these two ways of thinking about m conveyed to

students anything about how one variable changes in relation to another. Thus,

throughout Unit 1, Rico decided to include opportunities in instruction for

students to reconsider their meaning of m, that is, for it to represent the *constant*

rate of change between varying quantities, rather than simply being slope.

Rico encouraged students to think about m as quantifying the relative

changes between two variables, as $y_2 - y_1 = m(x_2 - x_1)$ or $\Delta y = m \Delta x$.[18] That is,

define the two variables x and y and compare the relative changes between them.

If the changes in one variable remain *proportional* to the changes in the other,

then there is a *constant rate of change* between them. When the variable x

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[18] I use this notation for presentation purposes—Rico did not use it in class.
Instead, he presented the idea as follows: (change in y)= rate of change*(change
in x). Rico presented the idea of rate of change in this manner so that students
concentrate on the meanings and not be distracted by the symbols—which, for
students, sometimes are void of meaning (Rico, RI#1 —March 09, 2010). Rico
further explained that regarding notation, he followed one of two strategies
depending on the situation. For example, if introducing the notation (e.g. function
notation) was a way to avoid students' misconceptions—as when thinking about
why the substitution method works for solving systems of equations—then he
would try to introduce the notation as early as possible. However, if the notation
(e.g. sigma notation) was only a way to represent the ideas in a compact way, but
did not provide any additional tools for thinking about the concept, then he would
not introduce notation until students felt comfortable thinking about the ideas.

changes by any amount, then the variable y changes by m times as much, whether x changes by a very small amount or by a very large amount.

For Rico, orienting students to think about constant rate of change in this way served two main purposes. First, it was a powerful way for thinking about linear functions. Second, if students first developed the idea of *constant rate of change* in the context of linear functions, then the idea of *average rate of change* would emerge later in the course and become an effective tool with which to think about functions and analyze their behavior within other contexts —for example, in comparing what happens to the changes in the outputs of the function as successive equally spaced intervals are selected for the input variable, regardless of the function that is being analyzed.

In what follows, I first explain how Rico drew upon the idea of constant rate of change within the context of linear functions. I will then describe Rico's use of average rate of change as a tool for analyzing functions.

Constant rate of change as a way of thinking about linear functions. In the context of the instructional sequence on linear functions, Rico drew upon the idea of constant rate of change as an occasion to provide students with new ways of thinking about a previously learned procedure that had proved to be challenging for them—finding the equation of the line given two points. According to Rico (RI#2 —March 11, 2010), in the "traditional" curriculum, teachers only teach the standard form ($y = mx + b$) of the equation of the line when they work with linear functions. He further explained that prior to

redesigning the course, he had noted that students experienced great difficulty in finding the equation of a line—regardless of the many practice problems he would assign to students or the many times he would explain the process in instruction.

After 2005, Rico came to realize that students were more inclined to memorize a procedure for solving this type of problem. Applying a memorized procedure required that they remember all the steps that were involved in the process of solving a problem. In his view, students had a hard time finding the equation because they did not understand the logical reasoning in performing the various steps of the mathematical procedure applied in solving the problem. The following example (Solution 1) illustrates the different steps that students had to follow in order to successfully find the equation of the line that crosses through (5,10) and (8,4).

Solution 1. (Memorized Steps).

Step 1: Start with the formula for obtaining the slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Step 2: Substitute numbers into the formula and get the result $m = -2$.

Step 3: Substitute for m in $y = mx + b$. It becomes $y = -2x + b$.

Step 4: Substitute for x and y using one of the points provided, for example (5,10)

Step 5: Solve for b in the equation $10 = -2(5) + b$.

Step 6: Finally, substitute $b = 20$ in $y = -2x + b$.

In his redesign of the course, Rico decided to introduce the point-slope

formula $y = m(x - h) + k$ as a way to provide students with a different means to

addressing the problem of finding the equation of the line that passes through two

points. In Rico's view, using this representation would allow students to focus on

changes in variables—rather than just values—more easily than with the standard

form. In addition, by using $\Delta y = m\Delta x$ as the guiding idea behind the method of

finding the equation of the line, there was no need for students to memorize any

steps. Rico's learning goals for his students now went beyond having them

successfully obtain the equation of the line. He now stressed that all the

calculations had a certain meaning and that students could and should develop

those meanings for themselves. Solution 2 below, emulates the way in which Rico

encouraged students to think about the problem in his new approach to teaching

linear functions.

Solution 2. (Reasoning).

Given the two points (5,10) and (8,4), we note that as x changes from 5 to

8, it changes by 3, and as x changes by 3, y changes by -6.[19] If we say that this is a

constant rate of change (this is a given because we want to find the equation of the

line), then y is changing by twice as much as x, whatever the change in x. But y

decreases as x increases, so it changes by negative two times as much as x is. So,

---

[19] Rico commented in RI#3 (March 23, 2010) that he purposely tried not to use
the formula: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. This prevented students from going back to their
previous ways of plugging numbers into the formula without paying attention to
the reasoning behind it.

the constant rate of change is -2, and I can substitute this information in the equation $y = mx + b$ to get $y = -2x + b$. Now, if I think of the equation of the line as $y = m(x - 0) + b$, then b is the value of y when $x = 0$. In other words, the point $(0,b)$ is my reference or starting point. So, let's take one of the given points (5,10). If I think of x changing from 5 to 0, then x changes by -5. y will change by -2 times as much, or 10. y's current value is 10 and it changes by 10 as x changes from 5 to 0. This means that y changes from 10 to 20 as x changes from 5 to 0, so b (or the value of y when x is 0) must be 20.

In Solution 2, the idea of $\Delta y = m\Delta x$ was used twice in finding the equation of the line. First, it was used to determine the actual value of the constant rate of change. Then, it was used to find the reference point $(0,b)$ given that the value of m was already known. In RI#2 (March 11, 2010), Rico explained that whether students followed this line of reasoning every time they answered a similar problem was not the issue. The idea was that if at some point they could not remember the steps in the procedure, they still had a way to reason their way through the problem.

One important aspect of using the point-slope formula as opposed to using the standard formula is that for any given linear function, there can exist infinitely many ways to represent the function algebraically, depending on the choice of the reference point (h,k). Rico addressed this idea in instruction by presenting two different algebraic formulas: $y = -2(x - 4) + 3$ and $y = -2(x - 6) - 1$. He asked students to substitute different values for x and obtain their corresponding y-

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values. He then used this as an opportunity to point out that, even though the

formulas "looked" different, both represented the same function, the reason being

that all the inputs for each of them were associated with exactly the same outputs,

thus defining the same relationship between the variables (Videotaped Lesson—

August 27, 2007).[20] Oehrtman, Carlson and Thompson (2008) also raise this

point: "Any means of defining the same relation is the same function. That is, a

function is not tied to specific computations or rules that define how to determine

the output from a given input" (p.159).[21]

Finally, another way in which Rico used the idea of $\Delta y = m\Delta x$ in

instruction was by conveying the message to students that the definitions that they

used should be "powerful" (Rico, RI#2—March 11, 2010). For example, he noted

in the reflective interviews that students in their previous courses had been

successful in answering questions in class by just providing memorized facts as an

answer. For instance, a typical answer to the question "What does it mean for a

line to have a constant rate of change of zero?" is "it's a horizontal line". In the

redesigned Algebra II, Rico encouraged students to use the definitions stated in

class as tools for thinking about problems and communicating their meanings. For

example, for "What does it mean for a line to have a constant rate of change of

---

[20] In a 'traditional' approach to teaching function transformations the choice of (h,k) always leads to defining a new function. However, not all choices for (h,k) end up defining a new function. And this is an issue that is simply not raised.

[21] I further explore this idea and its implications in the section on functions, which will appear later in this chapter.

zero?" Rico held a discussion in class using the idea of $\Delta y = m\Delta x$ as a way to answer the question. The class concluded that a constant rate of change of zero, $m = 0$, implies that whatever the change in x, m times the change in x will be zero, $0\Delta x = 0$. Thus, y, which is what gives rise to the horizontal line, will not change (Videotaped lesson—September 10, 2007).

Throughout the first weeks of the course, regardless of the context—whether working with systems of equations, linear inequalities or linear programming—Rico continually revisited the notion of constant rate of change. For example, he did this by orienting students to pay attention to the quantities and their units, as in Example 2—the cantilever problem— described in the previous section (Orienting Students to Pay Attention to Quantities). He also, later, used the idea of constant rate of change in linear programming as a way to justify why the maximum occurred in specific corners of the feasible region (Videotaped Lesson—September 4, 2007). Finally, Rico drew from the idea of constant rate of change to build toward the idea of average rate of change, and thus nonlinear functions—which do not follow a constant rate of change (Videotaped Lessons—September 11-13, 2007).

Average rate of change. Building on the idea of constant rate of change, Rico encouraged students to use the idea of average rate of change as a way to analyze the behavior of nonlinear functions. In the introductory assignment associated with Unit 2, Rico pointed out that "Not all functions in math are linear (in fact, few really are). But we can apply some similar reasoning to nonlinear

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functions in order to examine their behavior" (Homework Assignment—

September 10, 2007). During the following lessons (September 11-12, 2007),

Rico held discussions with the class regarding ideas such as: "what causes a

function's graph to curve?", "what does it mean for a function not to have a

constant rate of change?", "why would someone use a linear approach to analyze

a nonlinear function?", etc.

In general, the image of average rate of change that Rico promoted in class

can be explained as follows: suppose f is a nonlinear function. And we want to

analyze the function's behavior over some interval, say $[x_1, x_2]$. Since f is a

nonlinear function, we know that the rate of change of the function might have

varied within the interval. We can calculate the total change in x: $x_2 - x_1$. We can

also calculate the total change in $f(x)$: $f(x_2) - f(x_1)$. In order to describe the

function's behavior, we can compare it to that of a linear function that crosses

through the endpoints, $(x_1, f(x_1))$ and $(x_2, f(x_2))$. If the function f had followed a

constant rate of change over the interval, then the output would have changed by

the same amount as the original function did over $[x_1, x_2]$.[22]

The paragraph above describes the idea of average rate of change applied

to a specific interval. In class, Rico promoted that students use average rate of

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[22] In class, Rico addressed the idea of average rate of change in both contextual
and non-contextual problems. One example of a contextual problem was
determining the average speed of a car that had been accelerating over a certain
period of time. The class concluded that the average speed was as if the car had
traveled at a constant speed the entire time covering the same distance in the same
amount of time (Videotaped lesson—September 18, 2007).

change as a way of thinking for analyzing a function's behavior by selecting

equally spaced subintervals for either x or $f(x)$, and comparing the relative

changes to the other variable (Table 5).

Table 5

Image of Average Rate of Change Conveyed by Rico's Instruction

Case 1: "As I keep changing x by the same amount, *f(x)* is changing by more and more for the same change in x".	Case 2: "As I keep changing *f(x)* by the same amount, x is changing by less and less each time for the same change in *f(x)*".
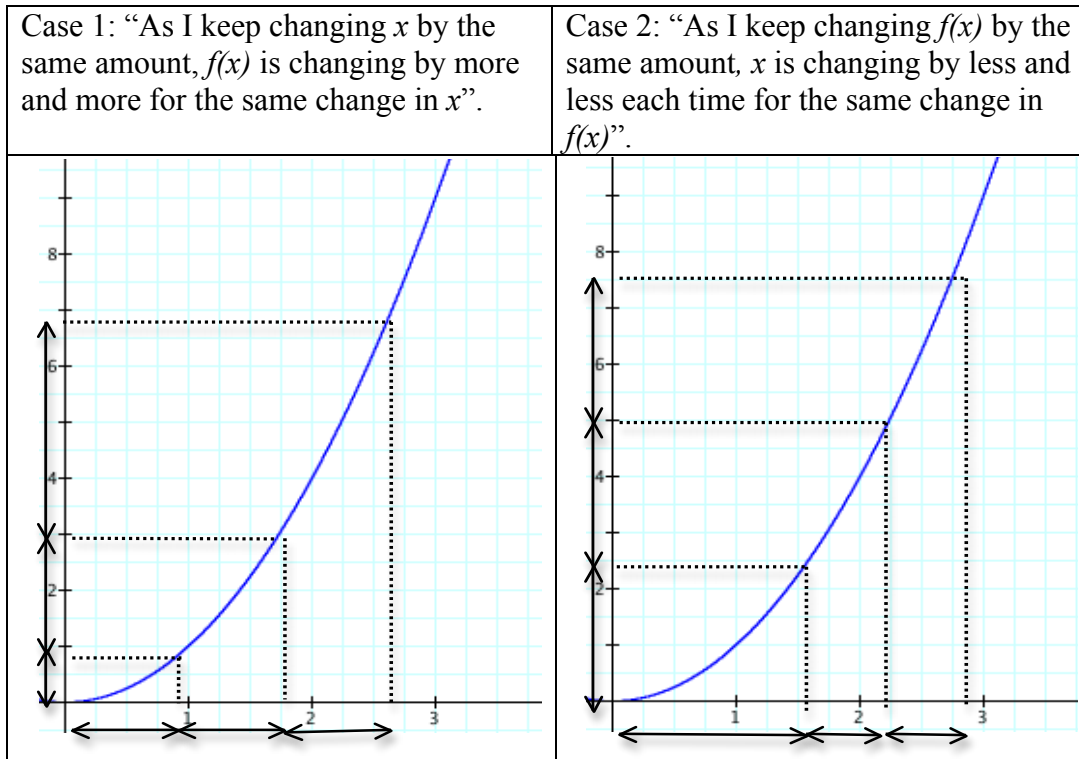
In what follows, I present an example from Rico's videotaped lessons. The

example is from the first lesson of the instructional sequence on radical functions

(October 4-15, 2007). I use this example to illustrate Rico's use of average rate of

change as a tool to analyze a function's behavior.

Example 4. Using average rate of change to think about an inverse relationship

between ($f(x) = \sqrt{x}$ vs. $f(x) = x^2$) (Videotaped Lesson—October 4, 2007).

The homework assignment previous to this lesson was an exploration.[23]

Rico used the exploration as an opportunity to discuss the idea of average rate of change in a contextual problem, and as a way to introduce radical functions. His goal of the lesson was to have students understand specific characteristics of the function $f(x) = \sqrt{x}$ that made it different from other functions in the course. In other words, he wanted students to analyze the effect the square root had in the relationship between the variables.

Rico started the discussion by writing $y = \sqrt{x}$ on the board. He then asked students to create a table of values for each, noting that the relationship was not defined for negative numbers. Then, he wrote on the board a table of values, such asTable 6 below, and started a discussion about the changes in the output relative to the changes in the input. Excerpt 5 outlines the conversation.

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[23] The previous lesson discussed ladders and rulers. This exploration gave students an opportunity to work in groups. The context was a 12-foot ladder leaning against a wall in an almost vertical position. Students were asked to analyze the relationship between the distance the ladder moved down the wall in relation to the distance the ladder was pulled away from the wall. The resulting function that modeled the situation was: $y = \sqrt{144 - x^2}$

Table 6

Table of Values for $y = \sqrt{x}$

x	y
0	0
1	1
2	1.4

Excerpt 5. Exploring changes in the output relative to changes in the input for

$y = \sqrt{x}$ (Videotaped Lesson—October 4, 2007).

Rico: Already in the table what do we know about this function?

Students: It's not linear.

Rico: S1, how come?

S1:   Because there's not a constant rate of change.

Rico: So, from 0 to 1 we saw a change by 1 in the values of the function. From

        1 to 2, we saw the function changed by…

Students: .41

Rico: .41, so it's not linear. We also notice what's happening to the changes, at

        least so far.

S1:   They are getting smaller.

Rico: The relative changes here, the amount y changes for the same change in x, appears to be decreasing.[24] Now, whether that continues or not, we have to see. We can keep going 3,4, and so on [he writes numbers on the board as the students dictate the values to him] [...] Does that pattern seem to be continuing? So, why is this happening? Why is y changing less and less for the same change in x? What about the relationship of the numbers is causing it? How could you explain it to somebody else?

At this point in the course, the class had already established a way to test whether a function was linear or not, by inspecting a table of values for the function. In this particular conversation, Rico made the choice of selecting x to change by one unit each time. However, he did not do so in other instances, in an effort to orient students' attention to the relative changes between the variables, and not just on the changes in the output variable—which can be promoted unintentionally if x is selected to always change by 1-unit increments.

In Excerpt 6, Rico continues the previous conversation. He went to the board and wrote $y = x^2$ next to the table of values for $y = \sqrt{x}$ that he had constructed before.

Excerpt 6. Comparing $y = \sqrt{x}$ and $y = x^2$ (Videotaped Lesson—October 4, 2007).

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[24] Despite's Rico's attempts to redirect students' attention to focus on the relative changes of one variable with respect to the other, it seems that the students kept paying attention to only one of the variables. I will later present Rico's insights regarding this type of problem.

Rico: Okay, so when we looked at this function $y = x^2$, and we talked about as x

changes by the same amount, the y values continued to change more and

more as x increased… starting at zero and x is increasing, right?[25] So x

changes by the same amount, y changes by a larger and larger amount

each time. Could we also describe what's happening to this function [

$y = x^2$] by changing y by the same amount each time? And talking about

the change in x required such that y would change by the same amount

each time? So, what would be true if we wanted to change y by, say 2 each

time, what would be true about the amount that x would have to change?

Talk to the person next to you. [The students break in groups and talk to

each other for approx. 3 minutes. Then, they go back to the group

discussion.]

Rico: So, what would be true about the change in x for us to fill out a table that

looks like this [table of values for which y changes by 2-unit increments]

for $y = x^2$? The x values would be getting larger?

S2:   Yes.

Rico: What else?

S3:   It's getting smaller.

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[25] Rico was referring to a problem with which they had worked on in a previous
homework assignment, and which they analyzed in a similar way as he describes
in the excerpt. The context referred to the distance (in feet) an object travels when
starting from rest and falling for t seconds: $d(t) = 16t^2$. The domain starts at $t = 0$.

Rico: You have to decrease your x values? So, I go from 10 to 9 to 8 to 7? You mean like that?... The x values are getting larger. But, what else do we know?

S4: x changes by a smaller amount each time?

Rico: x changes by a smaller amount each time? You guys agree with that? So, if when we change x by the same amount each time, y changes by larger and larger amounts. If we turn that around when we talk about, if y is changing by the same amount, that x would be changing by less and less and less, in order to make that true. Well, let's see if we can kind of apply this "backwards thinking" over to $y = \sqrt{x}$. For y to change by the same amount each time, let's just choose 1... for y to change by 1... from 0 to 1, from 1 to 2, from 2 to 3, from 3 to 4 and so forth. What has to happen to x? x has to increase but…

S5: x has to increase slow.

Rico: What do you mean by increase slow?... What has to happen to the change in x each time for y to change by the same amount for $y = \sqrt{x}$? …As y goes from 0 to 1 how much did x change by?

Students: 1

Rico: As y goes from 1 to 2 how much did x have to change by?

Students: 3

Rico: 3? As y goes from 2 to 3 how much did x have to change by?

Students: 5

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Rico: And for y to go from 3 to 4, x would have to change by 7. So if you are

thinking about your perfect squares as kind of benchmark points as you

are going, you have to change x by larger and larger amounts in order to

continue to get the same change in y, right? [See part a) from Table 7 as a

reference] And so, that means that for equal changes in x then the change

in y is getting smaller, and smaller, and smaller [see part b) from Table 7

as a reference].

S4: Is that because the perfect squares keep getting farther and farther apart?

Rico: Well, yeah, basically... if you are thinking of those as your benchmark

points, then yeah, in the x values.

Table 7

Analyzing the Behavior of $y = \sqrt{x}$ by Setting Equal Increments for One Variable

and Comparing the Relative Changes on the Other Variable

Part a) 1-unit increments in y	Part b) 1-unit increments in x

Rico: And really there is a relationship between $y = \sqrt{x}$ and $y = x^2$, right?

Because in this function [he points to the table of values for $y = \sqrt{x}$],

when you think about the perfect squares over here [he points to the

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following values (1,1), (4,2), (9,3)] wouldn't those appear as your y

values? Whereas over here [as he speaks, he constructs a table of values

for $y = x^2$, including the same points he highlighted before, but the inputs

and outputs are reversed, see Figure 9] they are your x values and the

number that's being squared is your x values here [$y = x^2$], but it is your y

value in this relationship $y = \sqrt{x}$. […] So, when x changes by constant

amounts for $y = x^2$ and the output values change by more and more and

more each time ... then if you reverse that relationship and you want your

output values here [$y = \sqrt{x}$] to be changing by the same amount those x

values would have to be changing more and more. Or we can go back to

our thinking of changing x by the same amount and the output y would be

changing by less and less [for $y = \sqrt{x}$]. So then, what does the graph of

$y = \sqrt{x}$ would look like to support this understanding of the behavior of

the function? Sketch what you think it looks like just real quick on your

paper and then graph it in your calculator. But, sketch it first. How does it

have to look like to support the behavior we just discussed?

Figure 9. Comparing the table of values for $y = \sqrt{x}$ *and* $y = x^2$.

As the students constructed their graphs, Rico went around the classroom looking at students' graphs. Finally, he went back to the board and sketched the graph of the function, pointing out that the graph did support the function's behavior that was discussed before —as *x* changes by the same amount each time, *y* changes by less and less.

In the post-lesson reflection, Rico explained that his choice to bring in $y = x^2$ as a way to analyze $y = \sqrt{x}$ had been intended for students to start noticing the relationship between the two functions. He had decided to discuss the idea of the perfect squares as possible benchmarks as a way to bring to students' attention to the fact that the input and output variables were reversed for $y = x^2$ and $y = \sqrt{x}$. Rico noted that for him, "to understand the behavior of a square root function was

to understand what has to happen for x for all those perfect squares —x has to change by more and more, to hit those perfect square points, as y changes by the same amount" (Rico, Post-Lesson Reflection 20071004).

Rico further explained that the idea of setting equal changes for the input and analyzing the relative changes in the output, and vice versa, was a way of thinking that they, as a class, had already talked about in another context.[26] This time, he had brought it up with the goal of making the link between $y = x^2$ and $y = \sqrt{x}$.[27]

Summary

Throughout Unit 1 of his redesigned Algebra II course, Rico encouraged students to use the idea of constant rate of change as a way to think about linear functions. He did so, by encouraging students to extend their meaning of slope, and provide new meanings to procedures (e.g. find the equation of a line that crosses through two points) and facts (e.g. if $m = 0$, then the line is horizontal) that they had learned in previous courses. He then drew on constant rate of change to develop the idea of average rate of change as a tool for analyzing a function's behavior.

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[26] Rico was referring to the instructional sequence on accelerating objects —Unit 2.

[27] Rico further noted in the post-lesson reflection (20071004) that making the link between $y = x^2$ and $y = \sqrt{x}$, was a way to "plant the seed" (Rico's words) for students, so that they could keep building on the idea of an inverse relationship between the variables. Later, he would formalize it in the instructional sequence on inverse functions. I come back to this idea in the section on Functions.

In RI# 6 (April 13, 2010), Rico commented that when he taught the course in 2007, his goal had been to get students to develop a general understanding of average rate of change. In the introductory lessons to Unit 2, there was a discussion in which the class concluded that selecting smaller and smaller intervals led to an increase in the accuracy of describing a function's behavior (Videotaped Lesson—September 12, 2007). And, in some cases, Rico selected the intervals to be so small that comparing subsequent intervals might be perceived as running through a continuum (Videotaped Lesson—October 04, 2007). However, Rico had made the conscious decision not to formalize the idea of rate of change to the extent of dealing with the idea of instantaneous rate of change, which is the limit of taking smaller subintervals for calculating the average rate of change.

Rico further explained in RI#6 (April 13, 2010) that if he were to teach the Algebra II course again, this would be the section of the course where he would make the most adjustments to instruction. In Excerpt 7 he shares his insights.

Excerpt 7. Rico's insights about his instruction on average rate of change (RI#6—April 13, 2010).

Rico: [This second way through the redesigned course] I was still experimenting and trying to see what would work with my students. I still wasn't completely satisfied with how this went [in the previous academic year] because with all those things said about how to analyze the functions, I don't think I did enough to help the students appreciate what average rate of change meant and really what it was that they were finding. And I think

that had I made a larger effort to do that, I could've leveraged that to assist

their understanding of other functions and not have it be so, sort of...not...

inexact about the way we were talking about this…. We sort of talked

about that, if you keep changing x by the same amount, here's how y

changes, we compared the actual changes in the y's. But we didn't really

compare average rates of change and what would that mean for different

intervals[28] […] But again, it took me going through and trying what I did

in this unit to realize that I didn't do a good job with average rate of

change. I think that if I had… it just kept feeling like I was a little bit

limited by the choice that I made to not, I guess, to not make it so

important that the students understand specifically what the average rate of

change is.

Function

In the introduction to this chapter, I explained some of the reasons that led

Rico to redesign the curriculum for his Algebra II course. Among these reasons

was, after 2005, Rico came to realize that his previous instructional approach to

function transformations—and the course in general—did not orient his students

to developing powerful understandings about functions. Specifically, in the

function transformations chapter, as he used to teach it, Rico had students practice

graphing complicated transformations throughout the chapter. Sometimes, these

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[28] I interpret Rico to mean that he was not encouraging students to compare the
slopes of the secant lines for different intervals.

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transformations, as he recalled back in RI# 4 (March 25, 2010), would take students more than five minutes to graph by hand. And in the end, as he notes in the following excerpt (Excerpt 8), he did not orient his students to thinking about functions in the ways he now wanted them to.

Excerpt 8. Rico's insights about his approach to teaching function transformations prior to redesigning the course (RI#4—March 25, 2010).

Rico: [Rico's approach to teaching function transformations prior to his redesign] There was no connection between what was happening in the function itself and what the graph was actually representing […] To me, there is nothing about meaning in that at all. Nor was there even anything about functions. […] As I vary my argument what is happening to the outputs of the function? That discussion was completely absent from anything going on in function transformations.

There are two main ideas that I want to highlight from Excerpt 8. First, Rico came to realize that students' interpretation of the tasks he assigned about constructing the graphs of the transformed functions was different from his own. That is, while he assigned tasks with the idea that students construct graphs of functions, the tasks for them did not necessarily involve thinking about a functional relationship at all. Second, Rico also came to realize that his previous approach to teaching the topic did not provide the means for students to develop an understanding of function as a relationship between covarying quantities.

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In the previous sections of this chapter I explained that, in his redesign of the course, Rico encouraged students to pay attention to defining variables and to think of them as varying. He also oriented students to thinking of rate of change as a tool for comparing the relative changes between the variables that conform a functional relationship. In this section, I expand upon two ideas. First, I describe ways in which Rico oriented students to build a sustained image of two quantities as their values $(x, f(x))$ vary. Then, I describe what Rico did in instruction building toward inverse functions. I do this with the goal of providing a sense to the reader of the ways in which he encouraged students to build an image of function as an invariant relationship between varying quantities.

Building a sustained image of two quantities as their values vary. A common instruction Rico gave in the homework assignments was to find the input value for a given value of the output. Rico encouraged students to solve the problem either algebraically or graphically. While this type of question frequently appears in most Algebra II courses, it is asked generally only in a section on solving equations. In Rico's case, he asked his students to find the input value for a given value of the output in many varied settings to reinforce the need for students to pay attention to the meaning behind each operation that they performed as they worked their way to a solution.

The following examples demonstrates the line of reasoning that Rico promoted for his students to follow for evaluating a function for a given value of the input (Solution 3) and finding the value of the input for a given value of the

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output (Solution 4).[29] Both examples are drawn from the instructional sequence

on linear functions.

Solution 3. Rico's approach to answering, "How the function calculates the

output value for a given input value?"

Consider the linear function $y = -2(x-5)+10$. When you input a value of

6, how is the function calculating what the output value will be?

First, we put in 6 for x: $y = -2(6-5)+10$ and we get 1 inside the

parenthesis $y = -2(1)+10$. This means that the change in x away from 5 is 1. So x

increased by 1, right from where we had our arbitrary starting value. Next, we

multiply -2 times 1, which is -2. This value now represents the change in y away

from its starting value of 3. If y decreased by 2 units from its starting value of 3,

this means that the actual value of y is 1.

Solution 4. Rico's approach to answering: How does the function calculate the

input for a given output value?

Consider the linear function $y = -2(x-5)+10$. If the value of y is 7, what

is the value of x?

First, we can rewrite the equation as $y - 10 = -2(x-5)$. y's current value

is 7, and it changed away from its original value of 10. 7-10 represents the change

in y away from 10, which in this case is -3. This means that the value of y

decreased by 3 units $-3 = -2(x-5)$. We know that the constant rate of change is

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[29] Both examples draw from the idea of constant rate of change to explain the meaning behind the calculations.

-2. This means that whatever the change in x, y changes -2 times as much. In addition, if y changed by -3, then the change in x was $\dfrac{1}{m}$ (or $\dfrac{1}{-2}$) times as large as the change in y. So the change in x was $\left(\dfrac{1}{-2}\right) \times (-3)$ or $\dfrac{3}{2}$. Finally, we know that x's reference value was 5 and it changed by $\dfrac{3}{2}$. This means that x's current value

is $5 + \dfrac{3}{2} = \dfrac{13}{2}$.

As noted before, Rico did not expect that every time the students solved similar problems, they did so by following each of the steps outlined above. However, Rico felt that knowing what each operation meant was as important as obtaining a result. Thus, he encouraged students to practice this way of thinking in class. He even included similar questions in the quizzes.

Later in the course, Rico encouraged students to graph the function $f(x)$ in their calculators and use the function's graph or table of values to estimate x. This method was suitable for cases where solving the problem algebraically was not viable, for example, with polynomial functions.

Excerpt 9 illustrates the method described above. The excerpt is from a videotaped lesson (September 28, 2007) in Unit 2. In the previous class meeting, Rico posed a contextual problem that could be modeled with a cubic function. He expected the students to answer a series of questions (e.g. finding the appropriate formula for the function, determining the domain and range, graphing the function, explaining the meaning of the function in terms of the context of the

situation) as homework. The next day, Rico allotted some time for students to discuss their answers with each other. Then, he started a class discussion in order to highlight some aspects about the homework. It is in the context of the class discussion that Excerpt 9 takes place.

The prior lesson's contextual problem was about an artist (Amy) who was selling sculptures. As part of the assignment, students had to come up with a function that modeled Amy's income in dollars *f(n)* as a function of *n*, the number of replicas that she could create and sell. The function that modeled the situation was the following: $f(n) = -.3n^3 + 1500$.

Excerpt 9. How many replicas did she create if you knew Amy's income was $36,704.70? (Videotaped Lesson—September 28, 2007).

Rico: How many replicas did she [Amy] create if we knew her income was $36704.70, assuming that our model is perfect in terms of what it predicts? So, what do we do with this? [Rico writes on the board:

$36704.70 = 1500n - n^3$]

S1: Same as we did last time: plug in 36704.70. y equals that, and then, find where the line of intersect goes.

Rico: So, S1 is saying we should solve this with a graph. Can we solve it algebraically? [Some students say "yes" and others say "no"]

Rico: Well, with quadratics your technique is to set the equation equal to zero, so you can employ either your factoring approach or the quadratic formula. Do we know any formulas that work with cubic functions?

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Students: No.

Rico: And, if we subtract, we end up with [he writes $3n^3 - 1500n + 36704.70$

on the board as he speaks]. Can we factor that down?

Students: No.

Rico: Not very easily. Your best approach here is going to be to use a graph, just

as S1 said. We are going to plug in our original function [Rico shows the

calculator's screen on the overhead and graphs the following functions on

the same screen: $f(n) = 1500n - .3n^3$ and $g(n) = 36704.70$. Figure 10

shows what is shown on the screen.

Figure 10. Graphs of $f(n) = 1500n - .3n^3$ and $g(n) = 36704.70$.

Rico: Here's our income function… What does the horizontal line represent?

S2: $y = 36704$

Rico: 36704.70 dollars. So, what you want to figure out is when Amy's income [tracing the graph of $f(n)$ with his finger as he speaks]... When is her income equal to 36704.70 dollars [tracing the graph of $g(n)$ with his finger as he speaks]? So we find the intersection of those two functions and we get 29.72 sculptures... and let's check the other one, because there are two intersection points here... and we get 51 sculptures. Now, assuming that our model is perfectly accurate, how many sculptures did she sell?

S3: 51

Rico: Did she really sell the 29.72 sculptures?

S3: We need to cut it down.

Rico: Well, we really don't even think about that in terms of fractions of sculpture, right? So really the answer to this one… We really want to choose 51 sculptures. The other answer doesn't seem to make sense in this context. And again, this is assuming that our price function... if she really followed the price function30 on how to price the items, then this is a perfect model for her income.

Rico oriented students to paying attention to all the operations[31] that they performed to the input in order to obtain the output and vice versa. That is, Rico

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[30] The function was stated in the contextual problem. It refers to the price $p(n)$ that collectors are willing to pay (in dollars) as a function of the number of replicas that Amy creates, $p(n) = 1500 - 0.3n^2$.

[31] I must clarify, as I did before, that in this case I am referring to operating with quantities, not numbers.

also encouraged students to find the value of the input for a given value of the output. In this sense, Rico was promoting to his students the development of a sustained image of the two quantities involved in the functional relationship.

In what follows, I explain what Rico did in instruction as he built toward the idea of inverse functions. It is in this context that a sustained image of the two quantities involved in the functional relationship was key to the students' understanding. Specifically, Rico introduced the idea of a logarithmic function as the function that reversed the relationship between the input and the output variables for an exponential function.

The instructional sequence on inverse functions was a sequence in the second semester of the course. However, in his redesign of Algebra II, Rico decided to include questions throughout the course that built toward a notion of an inverse relationship between the variables. For example, in the section on average rate of change, I presented excerpts from the introductory lesson to the instructional sequence on radical functions. In that particular lesson, Rico used $f(x) = x^2$ as a resource to further explore $f(x) = \sqrt{x}$. He did so by orienting students to analyze the relationship between the inputs and outputs of the functions and what happened to changes in one variable relative to changes in the other variable for the two functions.

Later in the course, after having defined and worked with exponential functions, Rico drew from reversing the relationship between the variables —just

as he did before with $f(x) = x^2$ and $f(x) = \sqrt{x}$—to define logarithmic functions

as follows:

> Consider the following: $4^x = 10$. Without graphing the function and using
> Trace or the Table, we can't answer the question "What exponent on 4
> gives us 10?" (Although we do know the answer is somewhere between 1
> and 2). We have a function whose input, x, is the exponent and whose
> output, 10, is the result of using that exponent on a base 4 ($4^x = 10$). A
> _logarithmic_ function, however, reverses this. It takes the result (10) of
> raising 4 to some exponent and uses this result (10) as its input. The value
> the function returns is the exponent necessary on 4 to get 10. Since we are
> working with an exponential of base 4, the logarithmic function would be
> called a "base 4 logarithm" or "log base 4". So if you want to know the
> answer to the question "What exponent on 4 gives us 10?" the answer
> would be "The log base 4 of 10" (which again means the output of the log
> function, which is the exponent you are looking for, and will be some
> number between 1 and 2). (Rico, Homework Assignment 20080124)

There are three key aspects of the previous excerpt that I would like to

highlight. First, Rico's choice of sequencing the ideas (exponential, logarithmic,

and inverse functions) was different from the textbook's sequencing. Second, this

equation $4^x = 10$ in the context of exponential functions, was about finding the

input x such that $f(x) = 10$. Third, Rico introduced logarithmic functions by

attending to the *meaning* of the relationship between the quantities of an

exponential function.

Regarding the first aspect, Rico's sequencing of ideas regarding

exponential functions, logarithmic functions, and inverse functions was different

from the textbook's approach. The textbook introduces the topic in this order: 1)

exponential functions, 2) inverse functions, and 3) logarithmic functions. In

Rico's case, he reversed the order between inverse functions and logarithmic

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functions. I do not consider whether one approach is better than the other in terms of which topic should go first. Instead, I point to what Rico did in the course prior to the instructional sequences on logarithmic functions and inverse functions that was aimed at preparing students' thinking to get to a point in which reversing the order of the topics was intended to help students develop more powerful meanings about exponential functions, inverse functions, and logarithmic functions (and about functions, in general).

In the lessons prior to the homework assignment from which the excerpt above was taken, the class had worked with exponential functions of the form $f(x) = ab^x$ where a represents the initial value and b denotes the growth factor. Also, as I had explained before, the question of finding the input—in this case x—for a given output of the function $f(x) = 10$, was already established as a natural question to ask about functions, including exponential functions. Up to this point in the course, students' methods for answering the question of what input will produce a given output included creating a table of values or a graph of the function, rather than employing the algebraic method.

Together, these two examples of defining functions that are inverses of each other ($f(x) = x^2$ and $f(x) = \sqrt{x}$, and $f(x) = 4^x$ and $f(x) = \log_4(x)$)[32] point

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[32] I also refer the reader to a previous example from the section on constant rate of change. In that example, Rico's instruction brought to students' attention that even though the formulas $y = -2(x-4)+3$ and $y = -2(x-6)-1$ looked different, both represented the same function, the reason being that all the inputs for each of the formulas were associated with exactly the same outputs, thus defining the same relationship between the variables.

to an important characteristic of how Rico treated functions in instruction: what made a function be a function was not specific features of the graph (e.g. a *u-shaped* graph) or specific features of the formula (e.g. the one with $\sqrt{\ }$ or $\log$). Instead, he constantly oriented students to paying attention to the relationship between quantities, regardless of the representation with which they were working.

Throughout the course, Rico kept orienting conversations so that students, as a group, talked about aspects of the functional relationships rather than features of the representations per se. He did so in many ways. For example, he always asked students to plot a point and explain its meaning in the context of the situation. He held discussions with students contrasting features of the phenomena that they modeled vs. features of the graphs used to represent the phenomena. Finally, as I showed in the introductory lesson to radical functions, Rico first discussed the properties of the function with the students and then he asked them to sketch the graph according to what they had just discussed. He encouraged students to sketch the graph according to their interpretation and then compare their graphs to that provided by their calculator.

In his new Algebra II, Rico encouraged students to use different representations for functions. However, for Rico, the ultimate goal was no longer that his students move flexibly from one representation to another, as was his goal when he previously asked students to construct graphs from algebraic formulas. Instead, his goal was that students come to conceptualize functions in such a way

that regardless of the representation they were using at the moment, they were still

thinking about a function—an invariant relationship between covarying

quantities.

The relationship between exponential and logarithmic functions that we have studied in great depth recently is a great example of inverse functions. Consider the functions $f(x) = 5^x$ and $g(x) = \log_5(x)$.

For $f(x)$:

x	y
-3	1/125
-2	1/25
-1	1/5
0	1
1	5
2	25
3	125
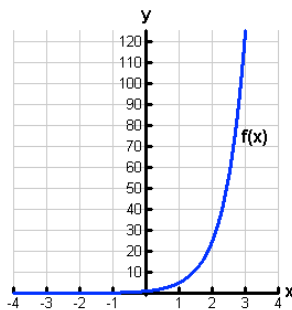
For $g(x)$

x	y
1/125	-3
1/25	-2
1/5	-1
1	0
5	1
25	2
125	3
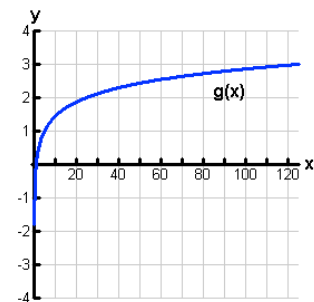
The relationship between these two functions should be clear to us – the input and output values show the same relationships, but the variables are switched. In one function, the exponent is in the input and the result of raising 5 to that exponent is the output, in the other function the result of raising 5 to an exponent is the input and the corresponding exponent is the output. The functions f and g given above are **inverses** of each other. An **inverse function** is a function that shows the same relationship as another function but reverses the inputs and outputs (so what used to be x is now y and what used to be y is now x).

You'll notice in this case that the inverse of f (which is g) is also a function since each input value has a single unique output. This may not always be the case. Other examples of inverses include $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ or $f(x) = x^2$ and $g(x) = \sqrt{x}$ (although we have to be careful with this second inverse pair – we'll get to that).

Figure 11 Introduction to Inverse Functions (Homework Assignment--February 8, 2008).

This is not to say that Rico did not value using multiple representations of

functions. In fact, in the definition shown in Figure 11 from the introductory

homework pertaining to the instructional sequence on inverse functions (February

8, 2008), Rico introduced the idea of an inverse function by means of four

different representations (algebraic, graphic, table of values, and a written

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description).[33] The difference now, however, was that, what Rico oriented

students to pay attention to was what remained the same *across* representations —

the functional relationship between the variables. In other words, the idea of

function that Rico now envisioned for students was such that, "the core concept of

'function' is not *represented* by any of what are commonly called the multiple

representations of function, but instead our making connections among

representational activities produces a subjective sense of invariance" (Patrick W.

Thompson, 1994b, p. 23).

A brief comparison between Rico's curriculum design and the

textbook's presentation of topics. According to Rico, other colleagues (and

himself in the past) used to follow the textbook much more closely than he did in

his redesign of the course. He stated in RI#1 (March 9, 2010) that he used the

textbook in his redesign as a source of contextual problems, but he never used the

follow up questions because, to him, those questions were just "procedures with

words around them". To further understand what Rico meant by this, I present a

brief comparison between the textbook's approach and Rico's approach to

teaching inverse functions.

Contrasting Rico's curriculum design with the curriculum as suggested by

the textbook, opens up new insights about Rico's design of instruction at different

levels (from particular tasks, to their overall treatment of the mathematics). To

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[33] In the next section of this chapter, I expand on Rico's treatment of inverse
functions.

further expand on those insights, I would like to bring into the discussion,

Simon's (2007) contrast between a teacher with perception-based perspective and

a teacher with a conception-based perspective to teaching. A key difference

between both perspectives (as I understand them) is the teacher's philosophy that

guides his selection (or design) of tasks to use in instruction. I use the word

philosophy, because it seems that in appearance either teacher, informed by 20

years of reform, would select tasks that revolve around real world applications, or

promote that students work in groups, or even both teachers would expect to hold

whole group conversations in which attending to students' strategies behind their

solutions are valued and encouraged. However, a teacher with a perception-based

perspective might fail to provide substantiated reasons as to what purpose these

"in vogue" activities might serve in students coming to learn the intended ideas

behind the tasks. In contrast, a teacher with a conception-based perspective about

his practice, would be able to develop a personal theory of student learning that

would inform his practice. In this sense, holding whole group conversations

allows the teacher to make better sense of how the students are interpreting

instruction, rather than, "let's engage in a whole group discussion in which you,

students, participate in trying to guess what I have in mind". Now, going back to

the contrast between the textbook and Rico's curriculum design, I stretch[34]

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[34] I use "stretch" because I'm extending a researcher's lens to analyze a teacher's practice into an author's intended ideas as conveyed by the textbook. However, I am the first one to understand how hard it is to convey one's ideas in written form. So I do not make claims about the author's perspective. I do make claims about the author's conveyed message.

Simon's (2007) two perspectives (conception- and perception-based) to contrast the two approaches. I take the idea of inverse function as an example. I first describe very briefly the textbook's approach, and then I expand on what I discussed in the previous sections of this chapter regarding Rico's treatment of inverse functions.

Textbook's approach to inverse functions. The instructional sequence starts by providing a contextual problem. The contextual problem involves a skating track that is .4 km long. The number of laps a skater needs to skate depends on the distance of the race. Students are asked to fill in a table that relates the distance in km, and the number of laps.

Speed Skating Races	
Distance (km)	Number of laps
10	?
5	?
3	?
?	3.75
?	2.5
?	1.25

Figure 12. Table that relates the number of laps to the distance (in km) of the race. Source: (Rubenstein, Craine, & Butts, 2002).

Then, the problem asks students to write a rule that describes the number of laps skated as a function of the distance of the race and the rule that describes

the distance of a race as a function of the number of laps skated. Finally, the students are asked to graph both relationships on the same set of axes.

Then, the textbook provides a definition: "Two functions f and g are inverse functions if $g(b) = a$ whenever $f(a) = b$" (Rubenstein, et al., 2002, p. 297) and shows students that the graph of the inverse is the reflection of the graph of the function. Also, students are directed to be careful not to confound the exponent -1 in $f^{-1}(x)$ with the reciprocal of the function $\dfrac{1}{f(x)}$. In addition, a procedure for finding an inverse function by reflecting the original function over the line $y = x$ is provided, noting that students should use the vertical line test whenever they reflect the graph, so that they can make sure that the reflected graph represents a function. Finally, a method for finding the inverse function algebraically is provided. The section is followed by practice problems.

Comments to the textbook's approach to inverse functions.

1. The definition of an inverse function is provided as a mere fact and it is not used in any other context—not even to justify the algebraic procedure for finding an inverse function.

2. The image of an inverse function conveyed by the textbook—with careful attention to notation and to outline the different steps in the procedures to determine inverse functions either graphically or algebraically—promotes at most a pseudo-empirical abstraction of what inverse functions are.

3. The following section in the textbook introduces logarithmic functions as the inverse function of an exponential, but based on the previous image,

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one can imagine that the logarithmic function $f(x) = \log_5(x)$ becomes a 'switched graph of $f(x) = 5^x$'.

Comments to Rico's approach to inverse functions.

1. As I explained in previous sections, Rico was 'planting seeds' towards building the idea of inverse function since the first semester of the course.

2. Rico introduced inverse functions, by means of comparing functions that had the characteristic that the relationship between the quantities was 'reversed'.[35]

3. Drawing from function composition (which was an instructional sequence in the first semester), Rico then asked students to compose the graphs of $f(x) = 5^x$ and $g(x) = \log_5(x)$ (Figure 11) and further explore what it means for two functions to be inverse of each other. (Figure 13)

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[35] Someone might argue that by introducing the contextual problem, the textbook was doing the same. However, a student might also interpret the relationship between the numbers in the table (Figure 1) as letting the .4 float around the table without paying attention to the actual relationship between the variables: distance traveled (in km) vs. the number of laps.

Function Composition and Inverse Functions

      Remember function composition? Composing functions involves taking the output of a function and using it as the input of another function. Use the graphs above (and the corresponding tables), **create the graph of each composed function give below**.
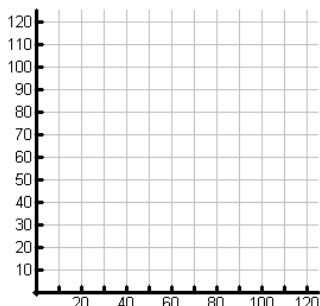
$$f(g(x)) \qquad\qquad\qquad\qquad g(f(x))$$
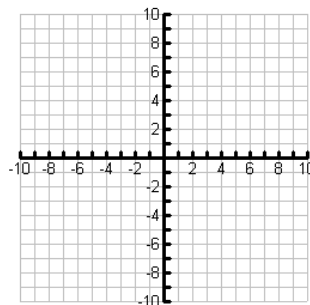
What happened when you composed the functions?

Why did this happen, and do you think it will happen every time you have two functions that are inverses of each other? Explain.

Now, try algebraically composing the functions (create the formula for the composed function, then simplify). Remember that $f(x) = 5^x$ and $g(x) = \log_5(x)$.

Find the formula for $f(g(x))$.   Find the formula for $g(f(x))$.

Do the formulas support what you found in the graphs? Explain.

In the functions $f(g(x))$ and $g(f(x))$, does "x" mean the same thing? Explain.

Figure 13 Continuation of Introduction to Inverse Functions (Homework

Assignment—February 8,2008).

4.   Then, Rico provided a list of functions ($f(x) = 0.5x^3$, $f(x) = x^2$,

     $f(x) = \dfrac{2}{3}x - 4$, and $f(x) = 2^x$) for the students to create the table of

     values, graph the function and its inverse according to the table of value,

     determine whether the relationships described a function or not, and to

     create formulas for the inverse functions. Once the students had followed

     the same process for different functions. Rico asked the students to graph

all functions and their inverses on a set of axes and to see if they noted

something interesting about the graphs. He further asked students to graph

$y = x$. (See Figure 14 for an example)
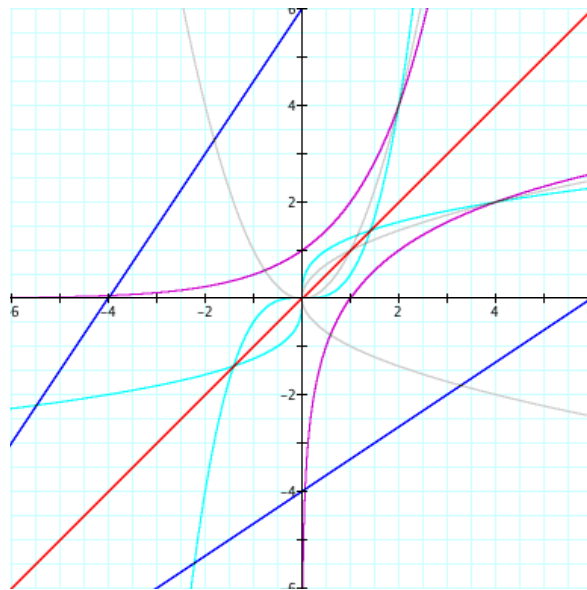
Figure 14 Graphs of $f(x) = 0.5x^3$, $f(x) = x^2$, $f(x) = 2^x$ and $f(x) = \frac{2}{3}x - 4$, and

their inverse relationships (not all functions).

In summary, from a general perspective, Rico was paying attention to the

long-term development of ideas and ways of thinking, such as the development of

the idea of function as a covariation of quantities. From a fine-grained perspective

he paid careful attention to the logical development of every method, formula,

operation within formula, etc. that emerged in the discussions throughout the

course.

A student experiencing instruction about inverse functions as treated by the textbook would probably develop substantially different understandings (about inverse functions, functions in general, where methods and algorithms come from, etc) than a student in Rico's Algebra II. Coming back to Simon's perspectives, even though the textbook starts the discussions with contextual problems, and deals with multiple representations of functions, etc. it addresses the mathematical ideas from a perception-based perspective. Whereas, I would argue, based on the multiple examples that I've tried to construct from his design of instruction, that Rico is an example of a teacher with a conception-based perspective towards his instructional practice.

I referred to inverse functions as just an example, but if we were to zoom in into other parts of Rico's curriculum design, we would be able to appreciate a similar pattern: his careful attention to the integrity of the mathematical ideas and the development of tasks and discussions around them that attend to the logical necessity of someone who does not already understand the ideas, to develop meaningful and powerful understandings of them.

Rico's course aimed at helping students develop long-term ways of thinking and ideas. This aspect of his MKT cannot be described by attending to particular lessons. It has to do with long-term learning goals for his students. However, if we were to discuss Rico's curriculum design only from a general view, then we would fail to truly describe the many ways in which he provided his students with multiple opportunities to engage in an intellectually challenging

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and fun Algebra II course. Thus, a balance of the two views (a general view of Rico's overall curriculum design and fine-grained analyses of his lessons) is what better helps to characterize Rico's MKT.

Summary

In this chapter, I discussed ways in which Rico treated important mathematical ideas in his redesign of the Algebra II course. Specifically, I addressed Rico's ways of thinking about variable, rate of change, and function. Through various examples from Rico's instructional material, videotaped lessons, and reflective interviews, I highlighted ways in which Rico oriented his students to paying attention to variables as quantities, to developing an idea of rate of change as a way of quantifying the relative changes between two quantities, and to function as defining an invariant relationship between covarying quantities.

In the following chapter, I will use Rico's Algebra II as data in order to further explore Rico's MKT. I now delve into what allowed Rico to create the course that he created and to interact with students in the way he did.

CHAPTER 7—RICO'S MATHEMATICAL KNOWLEDGE FOR TEACHING

Introduction

In Chapter II, I proposed Silverman and Thompson's (2008) framework for MKT as a way to describe how a person transforms his mathematical understandings into pedagogical actions. According to the framework, a teacher has developed a KDU when he constructs a scheme of meanings that proves central for understanding a wide range of other ideas and methods. However, it is until the teacher becomes reflectively aware of the KDU and realizes that students would benefit from thinking in similar ways, that he is developing a KPU. A teacher has a fully developed KPU when he has further developed a mini-instructional theory about how he can support students in building the KDU.

In the previous chapter I provided details regarding the progression of ideas in Rico's Algebra II. I divided my descriptions of the progression of ideas into themes according to three of the key mathematical ideas in which Rico anchored the Algebra II course: variable, rate of change, and function. For each of these ideas, I provided details about the images that Rico promoted in instruction by drawing upon instructional material, videotaped lessons, and excerpts from the stimulated-reflection interviews. In this chapter, I use the images conveyed by Rico's instruction to further explore his MKT that supported his redesign and teaching of the Algebra II course. To do so, this chapter is divided into three sections. In the first section, I describe the KDUs that I attribute to Rico, as they relate to the Algebra II course. The second section provides explanations

regarding what allowed Rico to envision an alternative approach to teaching the Algebra II course. In the third section, I discuss Rico's KPUs that supported his instructional design of the course. And in the last section of this chapter, I comment on other aspects of Rico's MKT, such as how he envisioned student engagement and how he assess his practice.

Rico's KDUs

In what follows, I provide explanations of the KDUs that I attribute to Rico in relation to his instructional design and teaching of the Algebra II course. I focus on the schemes of meanings that Rico operated on specifically addressing his notion of variable, rate of change, and function—key ideas from which the rest of the course emerged.

- Rico was fluent in conceiving of and operating with quantities. For Rico, a variable represented a quantity's magnitudes.

- Rico imagined variables moving smoothly. This means that, if x represents a quantity's magnitudes, then as x varies throughout its domain, it does so by taking on all the possible values along a continuum.

- Rico's understanding of constant rate of change was deeply embedded in proportional reasoning. For him, m represented a rate[36](P. W. Thompson & Thompson, 1992)—a quantity that is the result of a multiplicative comparison between two other quantities. For example, suppose that $m = \frac{1}{3}$ then, for Rico:

  o Whatever the change in x, y changes m or $\frac{1}{3}$ times as much as the change in x.

  o Whatever the change in y, x changes by $\frac{1}{m}$ or 3 times as much as the change in y.

- In thinking of average rate of change for analyzing the rate of change of a function, Rico could imagine secant lines of a fixed length running all the way through either the domain or range of the function, in order to describe how the values of one variable changed in relation to values of the other variable. In addition, Rico could vary the length

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[36] It is not always the case that m is conceived as a rate. Suppose, for example, that someone holds either of the two following images of $m = \frac{1}{3}$: "for every unit that y changes, x changes by *3* units" or "for every unit that y changes, x changes by $\frac{1}{3}$ of a unit". In both cases, the person is coordinating the changes in one quantity to changes in another quantity. However, in both cases, the changes are fixed by 1-unit increments. It might be possible that the person holding either of the previous images present some difficulties in imagining the proportional relationship between x and y being maintained if the change in either of the variables is .001 or 1,000, 000.

of the secant lines to the point of making their length almost 0,

allowing him to describe the behavior of the function as if he were

referring to instantaneous rate of change.

- Rico seemed to have an understanding of function as an invariant

  relationship between covarying quantities. It was an invariant

  relationship in that, regardless of the representation being used, what

  defines a function is the relationship between the quantities and not the

  representation per se.

- Rico imagined the quantities defining a function as existing

  simultaneously, $(x, f(x))$. That is, for Rico, if $x = a$ then, he could find

  $f(a)$; and vice versa, if he knew $f(a)$, he could also find a. This

  image is opposed to an image in which if $x = a$ then, one can find

  $f(a)$, but the question about finding x for specific values of $f(x)$ is

  never asked. In the latter case, $f(x)$ only exists as a consequence of

  knowing specific values of x.

- Rico's view of the domain and range of a function was compatible

  with how Oehrtman, Carlson and Thompson (2008) define someone's

  understanding of domain and range from a process view of function:

  "Domain and range are produced by operating and reflection on the set

  of all possible inputs and outputs" (p. 36); as opposed to conceiving

  them as either the solution to an algebra problem (e.g. the denominator

  cannot be zero) or as a property of a graph.

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- Rico conceived of graphs of functions as composed of points, each point representing the simultaneous state of the two quantities in the functional relationship (Saldanha & Thompson, 1998). In addition, Rico could differentiate between the attributes of a physical situation and the attributes of the graph of the function that modeled the situation.

- More specifically, in envisioning function transformations, Rico did not think of $f(x) = x + 2$ as the result of adding two units to the algebraic expression of the function $f(x) = x$, so that, in order to calculate an output of the function there is one more operation to perform, 'adding a two'. Nor did Rico think of function transformations as shifting the graph of the function 'two units up'. Instead, Rico performed operations on functions. He envisioned a new function $g(x)$, which could be described in terms of the function $f(x)$ as follows: $g(x) = f(x) + 2$. 'Adding the two' was an operation performed on all the outputs of a previously defined function and not on single input values.

- Rico did not conceive of inverse functions as the result of an algebra problem, in which one has to solve for x and interchange the literals in the problem. Instead, he conceived of inverse functions as "the reversal of a process that defines a mapping from a set of output values

to a set of input values" (Oehrtman, Carlson, and Thompson, 2008, p.36).

Envisioning an alternative approach to teaching the Algebra II course

In Chapter 5, I explained that, according to Rico's report, he had always enjoyed mathematics and that, for him, mathematics was about reasoning his way through novel problems. Specifically, Rico demonstrated a strong personal understanding in the two courses that he took as part of his participation in the TPCC project.

Rico's understandings of the mathematical ideas in Algebra II seemed to have supported his curriculum redesign. Despite Rico's powerful understandings, his instructional approach prior to joining the TPCC, according to both his own narrative and sample materials that he shared, did not aim at helping students develop powerful understandings such as the ones he himself held. It seems that it was not until he became reflectively aware of his own understandings that he was able to envision a new instructional approach for the course. By becoming reflectively aware I mean that Rico became able to 'see' his own understandings and take them as a point of reference for comparison (or objects for reflection) in relation to his hypotheses of how someone else (his students) might come to learn the mathematical ideas of the course.

In addition, it seems that Rico also became reflectively aware that his students' mathematical realities were different from his own. Meaning that in envisioning pedagogical actions to help students develop the ways of thinking he

held as learning goals, he also needed to attend to his hypotheses of where the students were coming from and how they might end up interpreting his pedagogical actions, leading him to develop mini-theories about how he could best support his students as he redesigned the course.

As a result of trying to come up with new ways to help his students develop powerful was of thinking, Rico came to realize that if students experienced instruction from a traditional approach—consistent with the textbook, other colleagues, and Rico's previous instruction—the students were likely to develop misconceptions and might not develop powerful understandings of the course material. According to Rico, (RI#1, March 9,2010), this was the case with the function transformations chapter that he used to teach prior to joining the TPCC. His previous approach reinforced students' impoverished ways of thinking about graphs and did not help them develop an understanding of functions as a relationship between covarying quantities.

In revamping the course, Rico came to realize that for students to develop ways of thinking about functions that were compatible with what he had in mind, he would need to begin establishing student thinking from the beginning of the academic year. In order to do so, Rico anchored the course in a few mathematical ideas that would provide students with tools for reasoning about functions. Rico's curriculum was not about topics or sections from the book anymore, which led students to develop isolated and disconnected meanings, but instead it was about helping students develop ways of thinking that allowed them to internalize the

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curriculum as a coherent system of ideas. As a system, meanings were interconnected and built on each other.

Rico's KPUs

Variable. Rico anticipated that students' previous understanding of variable might be based on the idea that a variable is a letter in an equation. Rico realized that he had to provide opportunities for students to extend their previously developed meanings, so that they could come to imagine variables as taking on any range of values through a certain domain for which the variable was defined. However, at the beginning of the course, Rico did not emphasize variables varying smoothly. He did this as he talked about average rate of change. All throughout Unit 1, Rico did emphasize variables assuming many different values, but it seems—from what can be interpreted from videotaped lessons and instructional material—that variables varying smoothly, came into play until the class discussed ideas of average rate of change in Unit 2. Rico's emphasis in Unit 1 seems to have been in students becoming aware that variables could take on any of a wide range of values, and that those values were not always integers.

Rico designed contextual problems as tools for helping students conceive of situations mathematically. He constantly oriented students to pay attention to the quantity whose values a variable represented. Every time a variable came into play, Rico put special emphasis on the variable's definition and its units. Also, he stressed that in a problem, a variable is free to vary —it can assume any of a range of values for which the variable is defined.

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Furthermore, Rico anticipated that if students could come to think about variables in the ways that he now intended for them, then, they would be encouraged to think about constant rate of change as the result of operating with quantities rather than numbers. Also, he was laying the groundwork so that students could develop a dynamic view of variable, which at the same time would empower students to thinking about functions as composed of covarying quantities.

Rate of change. Rico anticipated that students' previous understandings of slope were tied to an iconic image of a line. In Rico's view, this way of thinking about slope was not about quantifying the relative changes between variables. It was just about finding a number *m*, the result of dividing two numbers. Rico encouraged students to pay attention to the proportional relationship between the variables. Furthermore, Rico seemed to be aware of the problems that might cause to students to consider whole-unit changes, so he emphasized that, whatever the change in one of the variables, the other changed by *m* times that amount.

According to the new ways of thinking about functions that Rico wanted his students to develop, the idea of constant rate of change (with linear functions) and later average rate of change (with non-linear functions) could become a powerful way to orient students to analyzing a function's behavior by means of assessing how the values of the variables changed in relation to one another. For Rico, this way of thinking about rate of change provided students with a new tool for analyzing a function's behavior, regardless of the type of function, without

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having to memorize a set of parent functions provided by the teacher, as he had done in the past.

Function as an invariant relationship between covarying quantities. Rico's goal regarding the concept of function was to set up a way of thinking about functions that remained coherent throughout the course, regardless of the specific operations that defined each type of function. Different operations defined different functions, but in essence, all the functions in the course represented a relationship between covarying quantities, and that relationship remained the same regardless of the representation that was used to describe the relationship.

In order to develop such a sophisticated view of function, Rico promoted that his students develop an algebraic reasoning based on reasoning about quantities. That is, Rico oriented students to think of algebraic expressions as tools for reasoning about contextual situations. As students manipulated those algebraic expressions, they were encouraged to reflect on the operations they performed, shifting away from mere manipulation of symbols. In Rico's view, his previous instructional approach and the textbook's presentation of topics encouraged students to memorize endless sets of algorithms and formulas (e.g. the algorithm for finding the equation of the line that passes through two points) and students were not equipped with the ways of thinking and reasoning that gave rise to those formulas and algorithms. Rico's efforts were geared toward helping

students first develop a way of reasoning about the problems, and then, let the formulas emerge as a way of generalizing such reasoning.

It was this way of thinking about functions that empowered students to come to think about function transformations—defining new functions in terms of previously defined functions—and inverse functions—reversing the relationship between the variables—in more powerful ways.

Other Aspects of Rico's MKT

Rico's pedagogical actions were geared toward helping students develop powerful meanings of variable, rate of change, and function as supporting all the different topics in the course, instead of treating each topic as isolated from the rest. By setting these few key ideas and building upon them, Rico's new message to students was, "you are not always learning something new; we are just advancing your reasoning a little bit" (Rico—March 9, 2010). In addition, Rico came up with new ways to engage his students in the course. They had to become aware that their role in a mathematics course changed, too. They now had to engage in making sense of what others said and to become adept at communicating their meanings. Furthermore, Rico considered that if he could successfully frame his discussions around interesting practical situations and pose natural questions that might come up if someone started to think about the problems, then it was a way to engage students in applying some reasoning that they otherwise would not apply. Instead of having students successfully apply

techniques to get to a solution, Rico now intended for students to actually practice thinking mathematically about situations.

As Rico tried new ways of teaching the ideas of the course, he sometimes felt constrained by the time he had throughout the year to cover all the topics of the District's mandated timeline. However, he also found it worth to spend more time working on what he considered the foundational ideas of the redesigned course, finding himself at times pleasantly surprised with what students were now able to do.

Finally, Rico established a new way for assessing student learning, and in turn, he developed a means of evaluating his own instruction. He did so in two ways. First, he structured his exams in ways that differed considerably from his previous exams and the District's final examination for students. For example, in the past, he used to give a considerable weight in the exams to questions in which students were asked standard problems, such as, 'find the equation of the line that passes through two points'. Whether the student used memorized facts to get to a solution or reasoned her way through the problem in order to obtain one, or if the student was able to solve contextual problems regarding linear functions, was not part of the final examinations nor Rico's assessments.

In contrast, the philosophy Rico followed to write the assessments that he used with the redesigned curriculum was very different. Although he included problems similar to the one described above, about finding the equation of the line, usually those problems came at the end of the exam. In their place, Rico

included contextual problems for which students had to set up the function that modeled the situation. They also were asked to define the variables paying close attention to the units in which they were measured. Students were asked to explain the meaning behind each part of the algebraic expressions they used. Rico valued his students learning to communicate their meanings and explain their thinking behind the operations they performed, rather than merely coming up with a solution without ever providing their rationale behind it. In a sense, Rico now evaluated student understanding rather than just right answers—which may or may not be indicative of student understanding.

It was in providing feedback to students in class conversations and in the assessments that Rico was able to develop certain theories for himself about the ways in which students might have understood certain ideas in the course and how his pedagogical actions might have led students to think in unanticipated ways.[37] In other words, by reflecting on his own instructional practice, Rico devised a way for generating new MKT.

Summary

Rico's re-design of his Algebra II course seems to have been driven by his ability to envision what students might learn from different instructional approaches and by his strong command of the mathematics with respect to how students might experience the course material more coherently. In his curriculum

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[37] As an example, I refer the reader to the section in the previous chapter about average rate of change.

redesign, Rico *always* considered (1) what someone might need to understand already in order to understand "this" in the way he was thinking of it, and (2) how would understanding "this" in the way he was thinking of it help students understand related ideas or methods. In a sense, Rico's constant reflection on the mathematics he knew so as to make it more coherent, and his continual orientation to how these meanings and ways of understanding might work for students' learning of mathematics, made Rico's mathematics *become* a mathematics of students. In addition, by listening to his students communicate their meanings, and developing hypotheses about the sense the students made from instruction, Rico now had developed a new way to developing MKT from his practice.

CHAPTER 8—CONCLUDING COMMENTS

Rico, a high school mathematics teacher, with only a few years of teaching experience when he joined the TPCC, made the decision to undertake the arduous task of redesigning the entire curriculum for his Algebra II course. As a result of his efforts, Rico's students began to appreciate their newly found interest in mathematics, and other colleagues began to notice that Rico's students were well prepared for their courses.

In attempting to answer the question *"what are the sources of Rico's effectiveness?"* my objective was to better understand what about Rico's mathematical understandings (Rico's mathematics) and his understandings of his students' mathematics (Rico's mathematics of students) enabled him to teach, according to Rico, from a different perspective from what he had experienced in the past. I emphasize that I do not claim that Rico's approach to teaching Algebra II is not necessarily the "right way" to do it. In fact, Rico was the first person to recognize his teaching as part of an ongoing process of improvement. By assessing student understanding of the mathematical ideas presented in the course, Rico was able to consistently evaluate the effectiveness of his instructional approach. In this sense, Rico has engaged in a process of learning from his own practice (Simon, 2007).

I must also clarify that this was not a study about teacher change. At the time that Rico taught the Algebra II course that is the subject of this study, he was in the second year of his innovative   curriculum. The purpose of this study was to

explore Rico's strategic ways of thinking that enabled him to develop a different approach to teaching the course. Specifically, I attempted to gain insight into Rico's MKT by asking the following research questions:

- *What are the mathematical ideas and ways of thinking that Rico envisions for his students, as suggested by his design of instruction?*

- *What are the mathematical understandings that support Rico's pedagogical actions and design of instruction?*

- *What are Rico's conceptions of his students' mathematics?*

- *In what ways do the above express themselves in Rico's teaching?*

The data for this study was generated in three phases. The data derived for Phase 1 included videotaped lessons from the first semester of Rico's Algebra II course during academic year 2007-2008; semi-structured post-lesson reflections; and Rico's self-constructed instructional material for the entire academic year.

A preliminary analysis of the data allowed me to develop specific hypotheses regarding the mathematical concepts and ways of thinking that Rico envisioned for his students to learn. From this first approach to data analysis, it was evident that Rico paid special attention to mathematical concepts such as variable, rate of change, and functions. One problem that I faced in evaluating the data was that I did not have videotapes of the first two weeks of the course. Therefore, I looked ahead to the stimulated-reflection interviews with three primary objectives in mind. First, I planned to ask Rico to address his overall organization of the course to better understand the learning goals that he had

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established for his students. Second, I planned to ask Rico to provide details regarding the steps he took in the first two weeks of the course and why. Since I did not have video for those first two weeks, I used the instructional material that he developed to generate talking points in order to discuss his practice. My final objective for the stimulated-reflection interviews was to further investigate Rico's understandings of his students' mathematics regarding variable, rate of change, and functions.

Phase II of data collection took place during the months of March through April 2010. During this period, I conducted a total of eight stimulated-reflection interviews with Rico. The interviews were videotaped, transcribed, and analyzed. Finally, Phase III consisted of a conceptual analysis of the prior phases, with the primary objective of creating models of Rico's mathematical conceptions, his perceptions of students' mathematical knowledge, and his images of instruction and instructional design.

The findings revealed that Rico's schemes of meanings regarding quantities, variation, constant rate of change, average rate of change, and functions supported him in developing a course aimed at helping his students develop powerful understandings. However, other studies have demonstrated (e.g., Silverman, 2005; A. G. Thompson & Thompson, 1996) that developing powerful personal knowledge regarding mathematics that one teaches does not necessarily extend to enabling students to grasp the same level of understanding.

With the education reform efforts that have evolved over the past two decades, it is anticipated that Rico's Algebra II course redesign was not an isolated incident, yet this appears to be the case. I will not provide explanations regarding why this might be the case, since this is not within the scope of this study. However, I will emphasize key areas of Rico's MKT that are likely to provide insights to teacher educators and researchers in developing new resources to support other teachers in their development of MKT.

As I discussed in the previous chapter, it appears that the interplay between two aspects of Rico's MKT supported him in his curriculum design and teaching of Algebra II course. Specifically, he was reflectively aware of his mathematical knowledge; and also, his deep concern for his students' mathematical understandings. For example, Rico began to understand that students' previous experiences in mathematics courses did not provide a conceptual basis for building new ideas in Algebra II in the way he now intended. Thus, Rico included numerous opportunities in his lessons for students to rethink key ideas (e.g. variable, rate of change, function).

Another important aspect of Rico's instruction, which was also reflective of his MKT, was the myriad of connections among ideas that he persistently kept in mind while designing his units and conducting his lessons. He repeatedly included hints of connections to come—or as he called it, "planting those seeds" in students' thinking—that he anticipated leveraging later during instruction.

Finally, Rico's instructional design suggests that he was trying to encourage his students to become reflectively aware of their own knowledge to develop new methods of thinking. He also developed strategies to help students acquire the means that were necessary to make sense of the formulas and methods that they had memorized in the past but did not fully comprehend. Along with these efforts, Rico also developed new means of assessing whether students were making sense of the lessons that they learned. The key to this assessment was that students had to become adept at communicating their mathematical meanings by talking to each other, in group discussions, and in writing. Although he did not articulate it as such, Rico appeared to have developed a personal theory of student learning that was facilitated by the interaction between his anticipations of students' reactions to the homework assignments and students' actual reactions.

Rico's case illustrates the potential for viewing MKT from a point of view that pays close attention to the teacher's understandings, with the ultimate objective of uncovering mechanisms by which we can help teachers to become reflectively aware of their own understandings in favor of advancing their instructional design and student learning. These observations resemble similar efforts at the elementary level (Simon, 2007; Simon & Tzur, 1999). At the secondary level, Silverman and Thompson's (2008) framework for MKT provides a tool for exploring how teachers' personal understanding is transformed into pedagogical understanding, and how they might engage in further development of their MKT based on their practice.

To conclude, earlier I referred to Fenstermacher and Richardson's (2005) differentiation among successful, good, and quality teaching. I further explained that for this study I would adopt a perspective of good teaching from a learner-sensitive perspective rather than a learner-dependent perspective. In this sense, although I was made aware of student participation in whole group conversations, I did not possess access to further explore the sense that individual students made of the ideas provided in the course. Thus, I was unable to present a comprehensive assessment of the effects of Rico's instructional approach on student learning. I view this as both a limitation to this study and at the same time, an opportunity for further research into this area.

Regarding the quality of instruction (Fenstermacher & Richardson, 2005), efforts undertaken by individual teachers are most likely to fail if other teachers at all school levels do not engage in similar efforts. By the time students were enrolled in Rico's Algebra II course, they had not only developed schemes of meanings that might be incompatible with what Rico wanted them to learn in the course, but at the same time, students had already developed theories regarding their roles in a mathematics classroom, making it difficult for them to realize their own level of responsibility for their own learning. Rico invested a significant amount of time and effort in providing students with opportunities to extend their meanings and, at the same time, to enable them to engage in the course in a different and more effective manner. From Rico's point of view, the effort was worthwhile because by the second semester of the course, the students were

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already familiar with his expectations regarding their involvement in the course. The learning goals that Rico had established for his students at the beginning of the course were gradually being met. However, the question remains in regards to what might have happened with those same students if after one year they went back to a traditionally taught mathematics course. It is hoped that their newly discovered enjoyment of mathematics and mathematical thinking might have been powerful enough so that they didn't go back to old habits of mind in which striving to make sense was not a norm.

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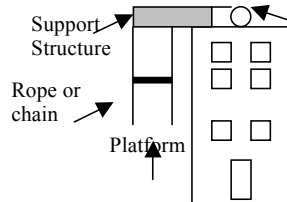
APPENDIX A

THE CANTILEVER PROBLEM

Systems of Equations
Honors Math Topics 5

An engineer is designing a new platform for tall buildings so that window washers, repairmen, painters, etc. can move up and down the side of the building. He needs to attach either rope or metal chain to the platform, which will hang from a system of pulleys attached to a support structure on top of the building.

This is a sketch of this situation:

Note: The rope or chain is unwound from this location to lower the platform (in other words, the excess chain or rope is not dangling from the support structure or stored on the platform).

In order to attach the ropes securely and use the proper pulleys, the engineer must attach 50 pounds of additional equipment to the platform, which already weighs 200 pounds. Using the chain, however, will only require attaching 15 pounds of additional equipment. The platform is designed to carry workmen and supplies weighing 500 pounds.

1. What is the total weight that must be supported (so far) if he plans to use rope (don't forget the workmen and their supplies)?

2. What is the total weight that must be supported (so far) if he plans to use chain (don't forget the workmen and their supplies)?

> *By the way…the support structure in this exercise is an example of a* **cantilever** *– a horizontal beam that is fixed at one end and supports weight at the other end.*

The chain weighs slightly more than the rope per linear foot. A foot of rope weighs 0.56 pounds. A foot of chain weighs 0.65 pounds. The platform needs <u>four ropes or four chains</u> to hold it.

3. As the platform is lowered what happens to the total weight that must be supported from the top of the building (the support structure)? Why?

4. For each foot that the platform is lowered, how much weight is added to the total weight that the support structure must hold if (remember that there are four ropes or chains):
   a. rope is used

   b. chain is used

5. Write a function that represents the weight the support structure must hold if:
   a. rope is used

   b. chain is used

6. In Exercise 5, what do your variables stand for (include units)?

7. What are the rates of change for each of the linear functions defined in Exercise 5? What does each rate of change mean in the context of this situation?

8. How much weight would be added to the amount the support structure must hold if the platform was held up by rope and the platform was lowered by 0.75 feet?

9. Identify two pairs of coordinate points that are true for the function representing the weight of the platform held by chains. What does each coordinate point mean in the context of this situation?

10. What is the solution to this system? What does it represent in this situation (be as descriptive as possible)?

11. What is true about the weights the support structure must hold when the platform is lowered less than the distance described in Exercise 10?

12. What is true about the weights the support structure must hold when the platform is lowered more than the distance described in Exercise 10?

13. The engineer knows that the support structure can only safely hold 1150 pounds. How far down the side of the building can the platform be safely lowered if:

    a. rope is used

    b. chain is used

14. What information must the engineer know (or find out) in order to determine if his platform design will work safely?

15. Are there any other considerations that might come into play for the engineer as he decides whether to use rope or chain?

16. Consider only the platform held by chain (write its weight function here: _____). The engineer knows that rope stretches slightly as weight pulls down on it. He estimates that the actual length of the rope will be 3% longer than it was originally measured. He creates the function $y = 1.03x$, where x is the original length of the rope (in feet) and y is the stretched length of the rope (in feet).
    Find the solution of the system involving the function $y = 1.03x$ and the function you wrote in the space above. What does the solution mean in this context? Explain the significance of your answer in terms of systems of equations in general.