Students' Ways of Thinking about Two-Variable Functions and Rate of Change in Space

by

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ABSTRACT

This dissertation describes an investigation of four students' ways of thinking about functions of two variables and rate of change of those two-variable functions. Most secondary, introductory algebra, pre-calculus, and first and second semester calculus courses do not require students to think about functions of more than one variable. Yet vector calculus, calculus on manifolds, linear algebra, and differential equations all rest upon the idea of functions of two (or more) variables. This dissertation contributes to understanding productive ways of thinking that can support students in thinking about functions of two or more variables as they describe complex systems with multiple variables interacting.

This dissertation focuses on modeling the way of thinking of four students who participated in a specific instructional sequence designed to explore the limits of their ways of thinking and in turn, develop a robust model that could explain, describe, and predict students' actions relative to specific tasks. The data was collected using a teaching experiment methodology, and the tasks within the teaching experiment leveraged quantitative reasoning and covariation as foundations of students developing a coherent understanding of two-variable functions and their rates of change.

The findings of this study indicated that I could characterize students' ways of thinking about two-variable functions by focusing on their use of novice and/or expert shape thinking, and the students' ways of thinking about rate of change by focusing on their quantitative reasoning. The findings suggested that quantitative and covariational reasoning were foundational to a student's ability to

generalize their understanding of a single-variable function to two or more variables, and their conception of rate of change to rate of change at a point in space. These results created a need to better understand how experts in the field, such as mathematicians and mathematics educators, thinking about multivariable functions and their rates of change.

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TABLE OF CONTENTS

CHAPTER Page

CHAPTER Page

CHAPTER Page

CHAPTER Page

LIST OF TABLES

Table		Page

LIST OF FIGURES

Figure Page

Figure Page

Chapter 1

INTRODUCTION AND STATEMENT OF THE PROBLEM

Physics, engineering, economics and mathematics are concerned with modeling, predicting, and explaining the behavior of complicated systems in terms of quantitative relationships. Because of the complexity of thinking about interactions among many variables simultaneously, we cannot assume that students think about complicated, multi-variable phenomena and functions in the same way that they think about simpler, two-variable systems. The purpose of this dissertation was to gain insight into students' ways of thinking and ways of understanding two-variable functions and rates of change in space as they participated in a teaching experiment focused on the same.

Functions and Modeling

Modeling in science, mathematics and engineering focuses on characterizing change in a system by describing a function that explains and predicts the behavior of that system. However, modeling the interaction of quantities in a system using a function requires particular ways of thinking about function. Thompson (1994b) described two ways of thinking about functions that have very different entailments with regard to ideas students need to understand to have them. The first is an invariant relationship between measures of quantities whose values vary simultaneously. The second is a correspondence between elements of two sets. The first is called a covariation perspective, the second a correspondence perspective. Differences in the meaning of the function concept can result from adopting either a correspondence or covariation view of function.

Both covariation and correspondence definitions of function are crucial to particular areas of mathematics. However, thinking about function by reasoning covariationally is critical for students in algebra, precalculus, and calculus because students use function to construct, represent, and reason about simultaneous change in situations. A student reasoning about function using covariational reasoning also fits naturally with modeling quantitative systems that are described in terms of their change—such as heat transfer. Speed at a moment, for example, is not simply a correspondence between two sets of numbers. Instead, speed at a moment is a measure of a multiplicative attribute derived from a relationship between two quantities, distance traveled and elapsed time, whose values vary simultaneously. A student who thinks about function with correspondence perspective focuses on a relationship between elements of sets, which does not necessarily focus on simultaneously accumulating quantities.

This study took students' covariational reasoning as the basis for a scheme for two-variable functions. As such, I sought to characterize ways of thinking about function in the context of activities focused on thinking about function as covariation to model situations. I recognize that basing the design of this study on covariational reasoning constrains my work mostly to thinking about continuous functions. I anticipate a future teaching experiment will focus on students' thinking about continuous change and rate of change from a function-as-correspondence perspective.

Throughout this dissertation, I refer to covariation, covariational reasoning, and a model for a student's covariational reasoning. These specific wordings are intended to help the reader make sense of when I am talking about the mathematical idea (covariation), the student's conception (covariational reasoning), and a researcher's model of the student's conception (model of covariational reasoning). Attending to these differences in what I mean by covariation is crucial to situating my work within the literature and explaining the theoretical coherence of this study.

Functions and School Mathematics

Tall (1992) called function the most important organizing idea in mathematics because thinking about function as an invariant relationship allows students to represent and reason about changes in quantities. Reasoning in this way is the foundation of derivative, limit, and accumulation in the calculus. Function became an organizing idea in mathematics and science because its development allowed those in mathematics and science to solve previously untenable problems by representing an invariant relationship algebraically and graphically. In this same vein, numerous research studies about teaching and learning function suggest that the concept of function is foundational to both secondary and collegiate level mathematics and must be addressed as such within curricula (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, Oehrtman, & Thompson, 2008; Dreyfus & Eisenberg, 1983; Kleiner, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Tall, 1996).

Because an image of how two quantities change in tandem is foundational to school mathematics, identifying predominant difficulties students have using ways of thinking like covariational reasoning and describing productive ways of thinking about function is crucial to creation of effective instructional sequences. Current research findings in mathematics and science education have largely been about student or teacher thinking about functional relationships between two quantities. These studies have focused on students' actions in the context of tasks related to one-variable functions (Akkoc & Tall, 2003; Alson, 1989; Bakar & Tall, 1992; L. Clement, 2001). While it is possible that ways of thinking about single-variable functions translate to functions with two or more independent variables, until this study, researchers had not yet studied or established these claims.

Students' Understandings of Multivariable Functions

It is rare, in applied sciences, for students to study systems that can be conceived as being modeled by two variables—one dependent and one independent—or as composed of just two co-varying quantities. Rather, most genuine applications of mathematics entail several variables and parameters, and any parameter can be re-conceived as a variable.

Several researchers have studied how mathematics and science students respond when presented with tasks composed of algebraic functions of more than one variable (Martinez-Planell & Trigueros, 2009; Trigueros & Martinez-Planell, 2010), but there is only one study I know of (Yerushalmy, 1997) that has addressed how students identify more than two quantities, relate those quantities in an invariant relationship, and represent and reason about the behavior of the quantities using multivariable functions. This is not a small gap in research knowledge. Vector calculus, differential equations, thermodynamics, and physical chemistry require students to reason about and represent situations involving more than a single dependent and independent variable. However, there are few studies that investigate how students might use functions to reason about and represent invariant relationships among more than two simultaneously varying quantities. The lack of studies in this area means that the field only has anecdotal evidence for claims about teaching and learning of multivariable functions. Given the prevalence of ideas in science and mathematics that require thinking about and representing relationships simultaneously, I identified how students think about functions of two or more variables as an important research agenda.

Research Questions

In response to the need to understand how students reason about two (or more) variable functions, the major research question addressed by this study was,

> What ways of thinking about variables and functions of them do students reveal in a teaching experiment that is focused on construction of multivariable functions as representation of invariant relationship among quantities?

To answer this question, I gained insight into the following supporting research questions,

- If students' ways of thinking about variables and functions of them change, what means of support (e.g. instructional supports, visualization tools, developing understandings) might have facilitated these changes, and in what ways did students use them as support?

- What role do students' understandings of rates of change play in their ability to model dynamic situations with multivariable functions?
- What is the nature of students' quantitative and covariational reasoning in their conception and modeling of a dynamic situation with multiple quantities, and what difference does the nature of their reasoning make for extending function and rate of change from the plane to space?

Implications of Research Questions

Studies related to two-variable functions have characterized what students could or could not do (e.g. find the domain and range of a function of two variables), rather than on their ways of thinking about what a function of two variables *is* (Martinez-Planell & Trigueros, 2009; Trigueros & Martinez-Planell, 2010). It is critical to understand how students think about two-variable functions and to understand how that thinking expresses itself in their attempts to model dynamic situations by stating functional relationships. A study focused on ways of thinking has the potential to contribute to the following areas:

a) understanding how students generalize their ways of thinking about single variable functions to multivariable functions,
b) supporting teachers in helping their students make sense of dynamic situations by modeling them with multivariable functions, and
c) understanding how students conceptualize rate of change of functions in space and in a direction.

I planned to expand on research knowledge in these areas by creating viable models of student thinking about two-variable functions and their rates of change. I intended that creating these models and identifying consistent patterns and constructs within each model would allow me to construct robust explanatory, descriptive and predictive frameworks focused on student thinking about multivariable functions. I believe these steps prepare the foundation for future, large-scale investigations related to multivariable calculus.

The first four chapters of this dissertation focus on the design aspects of the teaching experiment, which include defining relevant constructs from the literature, identifying ways of thinking critical to students coherently understanding functions of two variables, and demonstrating the epistemological coherence of the theoretical frameworks and methodology used to model student thinking. Chapters five through seven contain my analyses of the data from the teaching experiments. In these chapters I discuss my inferences focused on how the students were thinking about functions of two variables and how these models reflect changes in my assumptions and commitments as a result of interactions with students. In chapter eight, I present two frameworks that represent my current thinking about critical ways of thinking about functions of two variables that culminate in my retrospective analysis. By presenting the dissertation in this way, I aim to trace the development of my thinking as a researcher as I gained insight into student thinking about functions of two variables and rate of change.

Chapter 2

THEORETICAL FOUNDATIONS AND LITERATURE REVIEW

In this chapter, I describe *quantitative covariation* and its development in literature. I characterize its most important aspects to show how quantitative covariation addresses some shortcomings of previous work. Overall, I have structured the literature review to do three things. First, the review describes quantitative reasoning and its importance for covariational reasoning. Second, the review characterizes the current literature relevant to identifying important ways of thinking about functions of one variable and the smaller body of literature focused on two-variable functions. Third, the review identifies important ways of thinking for a student to understand rate of change in a coherent way. These foci are intended to prepare the reader to understand the constructs that are the basis for the conceptual analyses in Chapter 3.

Quantitative and Covariational Reasoning

A function of any number of variables is an expression of an invariant relationship between two or more quantities. The most important, and difficult aspect of that definition for students may be thinking about quantities. Quantitative reasoning is a foundation for a coherent understanding of two-variable functions because it allows the student to conceive of two or more quantities in a situation, and think about those quantities as related in a quantitative structure. Quantitative reasoning (Smith III & Thompson, 2008; Thompson, 1989, 1990) plays a vital role in reasoning covariationally, given the description of covariational reasoning as the "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, Carlson, & Thompson, 2008). Carlson et al. (2002). Oehrtman et al. (2008) characterized covariational reasoning as coordinating patterns of change between variables. Thus, it is not only the coordination, but also what the student is coordinating (quantities), that is vital to students' understanding functions of two-variables.

Quantitative reasoning refers to a way of thinking that emphasizes a student's cognitive development of conceptual objects with which they reason about specific mathematical situations (Smith III & Thompson, 2008; Thompson, 1989). More specifically, quantitative reasoning focuses on the mental actions of a student conceiving of a mathematical situation, constructing quantities in that situation, and relating, manipulating, and using those quantities to make a problem situation coherent. Quantities and their perceived interrelationships are the basis of making many mathematical ideas coherent.

As an example, consider the figure below (Figure 1). Suppose that Person A looks at the picture and notes that there are two cars, a marker that says "slow", and some buildings in the background. Person B notices that there is a "space" between the two cars, but does not think any further about that space being a distance that can be measured in a linear unit. Person C attends to the distance between the two cars as a number of feet, the length of the blue car as compared to the silver car, or the fact that the blur in the picture is caused by the car moving some distance in the amount of time that the camera's shutter was open. Person

A’s conception is non-quantitative. Person B’s conception might be called *proto*-quantitative, meaning that this way of thinking is necessary for a quantitative conception of the situation but it is not actually quantitative. Person C has constructed attributes of the situation and imagined them as measurable, whereas the first person just noted details of the picture. In creating a model of the second person’s thinking, I might claim he is thinking quantitatively. If a student needed to understand a general way to express average speed, the student would need to conceive of a change in one quantity (distance that the camera traveled while the shutter was open) in relation to another quantity (the number of seconds that the shutter was open). Conceiving of measuring a change in distance and a change in time necessitates that the student imagine the situation having the attributes of “distance the car traveled while shutter opened” and “amount of time that the shutter was open”.

Figure 1. – Conceiving of the speed of a moving car.

If a student is to construct a quantity, they must construct an attribute in a way that admits a measuring process. Thompson (2011) argued that quantity is not “out there” in the experiential world. A student can reason about a quantity only after he constructs it. Thus, for a student to imagine that a function of two variables is a representation of the invariant relationship among three quantities, the student must construct those quantities, whether it is from an applied context or an abstract mathematical expression.

In a clarification of his original description of quantitative reasoning, Thompson (2011) defined *quantification* as the process of conceptualizing an object and an attribute of the object so the attribute has a unit of measure, which is essential to constructing a quantity. To assign values to attributes of an object, a student must have already constructed the attributes of a situation that she imagines having measures. For example, to imagine assigning a value for a particle’s mass, one must have already conceived of mass as a measurable attribute of the object—which itself entails coming to imagine the same object having an invariant “amount of stuff” in different gravitational fields. Students must see an object’s mass as something that does not change even though its “weight” changes from location to location. Next, a student must imagine an implicit or explicit act of measurement (e.g. a balance beam in which the object is balanced against an object whose mass they take to be a sort of standard for measuring mass), and finally, a value that results from the measurement process. A *value* then, is the numerical result of quantification of a constructed quantity.

Thompson spoke of a quantitative operation as an understanding of two quantities taken to produce a new quantity. A quantitative relationship is directly related to a quantitative operation, and consists of the image of at least three quantities, two or more of which determine the third using a quantitative operation. As an example of a quantitative operation, mass and acceleration of a particle (thought of as quantities) can be taken together to produce the force of the particle. A quantitative relationship consists of imagining mass, acceleration, and force as quantities in a way that mass and acceleration determine force.

Quantitative reasoning is the analysis of a situation in a quantitative structure, which Thompson refers to as a network of quantities and quantitative relationships. If a student is to think about a complicated situation with three quantities and construct quantities and the invariant relationship between those quantities, a dynamic mental image of how those quantities are related is critical. That image positions a student to think about how quantity 1 varies with quantity 2, how quantity 2 varies with quantity 3, and how quantity 1 varies with quantity 3. Understanding these individual quantitative relationships allows a student to construct a function that expresses an invariant relationship of quantity three as a function of quantities one and two, so that its value is determined by the values of the other two.

Quantitative Reasoning as a Foundation for Covariational Reasoning

A number of studies have suggested the importance of using covariational reasoning to think about functions, where one thinks about functions as tracking a relationship between quantities as their values vary simultaneously. However, until recently (Thompson, 2011), there was not an explicit characterization of quantitative reasoning as a foundation for a student's covariational reasoning, and thus, a basis for that student's understanding of function.

Castillo-Garsow (2010) traced the development of researcher's definitions of mathematical covariation and its implications for how they characterized covariational reasoning. I need not repeat his discussion in full, but focus on the most pertinent aspects of his work. He found that there were two parallel developments of covariation, one which focused on successive values of variables (Confrey, 1988), and the other focused on quantitative reasoning and the measurement of an object's properties (Thompson, 1988). To provide a general overview of covariational reasoning, I describe Confrey and Smith's development of covariation to characterize and contrast it with a notion of quantitative covariation that began with Thompson's work.

Confrey & Smith (1995) characterized covariational reasoning as moving between successive values of one variable, and then coordinating this with moving between successive values of another variable. Confrey and Smith (1995) explained that in a covariational reasoning approach, a function is a juxtaposition of two sequences, each of which is generated independently through one's perception of a pattern in a set of data values. In response, Saldanha & Thompson (1998) described covariational reasoning as holding in mind a sustained image of two quantities' magnitudes simultaneously. Saldanha & Thompson's image of covariational reasoning relied on measuring properties of objects, and distinguished simultaneous change from successive change. They spoke of images of successive, coordinated changes in two quantities as an early form of covariation that, if developed, becomes an image of simultaneous change. To contrast these perspectives on covariational reasoning, consider the following table that represents values for the function $f(x)=3x$.

Table 1.

Tabular Representation of a Function.

x	$f(x)$
1	3
4	12
7	21
10	30

Confrey and Smith's definition suggested covariational reasoning consists of paying attention to the numbers in the table, or the landmark values achieved by x and $f(x)$, but not what happens between entries in the table. Because they did not specify imagining what happens between successive values in the table, there is no requirement that the student sees a continuous coupling of values. In contrast, Thompson and Thompson (1992) described covariational reasoning beginning when a student visualizes an accumulation of one quantity in fixed amounts and makes a correspondence to another quantity's accumulation in fixed amounts. Thompson and Thompson (1992) proposed that at the highest level, a student reasoning covariatonally might have an image of quantities' accumulations happening simultaneously and continuously. Thus, if one quantity changes by a factor, the other quantity changes by that same factor.

Saldanha and Thompson expanded on Thompson & Thompson (1992) to describe continuous covariation (Saldanha & Thompson, 1998). They proposed an image of covariational reasoning as one understanding that if two quantities changed in tandem, if either quantity has different values at different instances, it changed from one to another by assuming all intermediate values (Saldanha & Thompson, p. 2). Carlson et al.'s (2002) description of covariational reasoning and accompanying framework built on Saldanha & Thompson's (1998) description of continuous covariation to propose ways of thinking about how those quantities covaried. Carlson suggested that covariational reasoning allows a student to extract increasingly complicated patterns relating x and f(x) from the table of values by ways of thinking the student might use to understand what occurs between those values. She introduced five levels of mental actions to describe the ways in which the students might think about the coupled variation of two variables. These results characterized the importance of students' quantitative and covariational reasoning, but did not explain how quantitative reasoning could be a basis for covariational reasoning.

Thompson recently expanded on his notion of continuous covariation (Thompson, 2011) to propose how a student's construction of quantities and their variation could support an image of those quantities covarying. Thompson introduced the construct of *conceptual time* to propose a plausible scheme of meanings for a student's construction of a quantity that would support an image of that quantity varying with another quantity simultaneously.

Variation, in Thompson's description, results from a student anticipating that a quantity has different values at different moments in time, which could be represented as $x = x(t)$. This variation always occurs over a domain of conceptual time, where each interval of conceptual time can be represented as $(t, t+\varepsilon)$, so that the domain of conceptual time can be represented as $D = \bigcup (t, t+\varepsilon)$

(Thompson, 2011, p. 47). This characterization of variation allows the student to anticipate that the domain of conceptual time is “covered” with these intervals, so that, according to Thompson, the student can imagine that the quantity varies in chunks of conceptual time while understanding that completed chunks can be thought about as the quantity having varied continuously. At the same time, the student can imagine the variation within chunks occurring in even smaller chunks. Thus, a quantity varying over conceptual time can be represented as $x_e = x(t_e)$, where *e* represents intervals of conceptual time of size ε that the student imagines becoming as small as possible.

Thompson’s characterization of variation extends to imagining two quantities covarying, represented as $(x_e, y_e) = (x(t_e), y(t_e))$, where (x_e, y_e) represents an image of uniting two quantities, and then varying them in tandem over intervals of conceptual time (Thompson, 2011, p. 48). This characterization of covariation is similar to a function defined parametrically, which Thompson alluded to in earlier work (Oehrtman, Carlson & Thompson, 2008). If a student has this conception of covariation in mind, it is reasonable to assume they can can think about $(x(t), y(t)) = (t, f(t))$, which unites a function’s independent and dependent quantity into an object. As that object varies over conceptual time, the function is generated.

Saldanha & Thompson (1998), Thompson (2011), Carlson et al. (2002), and Oehrtman, Carlson & Thompson’s (2008) work combines to support a productive image for students thinking about function based on quantitative and covariational reasoning. There is evidence that this approach can support a student's understanding of function in practice. These authors found that covariational reasoning is productive for understanding function as a representation of relationships in dynamic problem situations where multiple quantities are varying simultaneously (Carlson et al., 2002; Carlson, Smith, & Persson, 2003; Saldanha & Thompson, 1998; Thompson, 1994a, 1994b). They found that understanding a function as an invariant relationship between quantities, and imagining the values of those quantities varying simultaneously, is critical to students developing a mature concept of function. They suggested that thinking about functions using covariational reasoning allows students to interpret tables, graphs, and algebraic formulas as various representations of a relationship between two or more quantities. Thus, this dissertation drew on covariational reasoning to support students' development of function, but with explicit attention to the students' quantitative reasoning and use of conceptual time as a basis for that covariational reasoning.

Quantitative Reasoning and Shape Thinking

Given this study's strong focus on graphs of functions, it was important to focus on a construct I drew from my discussions with Patrick Thompson called shape thinking. *Shape thinking* is an association the student makes with a function's graph. For example, a student might associate a function's graph with a particular shape with physical properties, while another student might associate a function's graph with a representation of quantities' values. Thompson first discussed the idea of shape thinking as he was creating case studies of an Algebra

I teacher in 2006, and I sought to expand on his characterization of shape thinking by identifying and explaining understandings that were implicit in shape thinking.

Suppose a student associates a graph with a relationship between two quantities' values simultaneously represented on the coordinate axes. The student's attention to the axes suggests that he thinks about functions' graphs as a direct result of tracking the values of quantities on the axes. This student associates the shape of the graph with a relationship between quantities, and the graph is an emergent construction resulting from tracking and representing those quantities' values simultaneously. The association of a graph with an emergent process was my initial conception of *expert shape thinking*.

Suppose a second student considers a graph as if a member of a class of objects which he associates with a particular function definition. The student might then accurately associate the graph with a type of function, but that association would not entail a quantitative relationship. For example, many students associate a graph that has a "U" shape with the function $f(x)=x^2$. In this association, the student might think about the function's graph as a wire that could be manipulated without a transformation of the relationship between quantities because in the student's understanding, the graph is not associated with quantities. The association of a graph with an object independent of any quantities used to generate it constituted my initial conception of *novice shape thinking*.

Expert shape thinking relies on a student's quantitative reasoning, particularly when focusing on the quantities represented on the axes. I do not claim novice shape thinking is bad, but I do not believe it is productive for characterizing how a student might come to imagine a graph as an emergent representation. Quantitative reasoning entails a way of thinking about situations that is productive for students to construct and reason about functions, and was a foundation for this study. If a student is to think about functions of two variables as representing an invariant relationship among three or more quantities, then identifying quantities, and constructing an image of their relationship to each other is critical to defining and reasoning about the behavior of a two-variable function.

Important Literature and Constructs

This portion of the literature review synthesizes the literature relevant to how students think about functions of two variables and their rates of change that does not fall directly in the purview of quantitative and covariational reasoning. I argue that understanding function is a difficult concept for students, and delineate various frameworks for what it means to understand function. In doing so, I discuss major results related to how students think about functions of one variable to demonstrate how these results might be extended to functions of more than one variable. Lastly, I propose constructs from literature about student thinking relative to functions of one variable that are likely to be related to a students' coherent conception of using two-variable functions and rate of change. In each section, I focus on what work the literature does for this study, delineate what work this study can do to contribute to what previous studies did not address, and describe why I use quantitative and covariational reasoning in place of other well known frameworks.

Students' Difficulties in Developing a Mature Concept of Function

Researchers have found that it is difficult for students to achieve a concept of function that the researchers would call mature, or advanced. Some researchers refer to these students' function concepts as incoherent. Often this incoherence is in the researcher's comparison of the student's thinking to what we might call an advanced concept of function while the student believes his understanding is coherent. Carlson (1998) found that students receiving course grades of A in calculus possessed an understanding of function that did not support the covariant, or interdependent variance of aspects of the situation. Forty-three (43) percent of the A students found f(x+a) by adding a on the end of the expression for f, without referring to $x+a$ as an input for the function (Carlson, 1998). Carlson et al. (2002) found that students had difficulty forming images of a continuously changing rate and could not represent or interpret inflection points of a function. Oehrtman et al. (2008) suggested that students confused an algebraically defined function and a formula, and proposed that many students believed all functions should be tied to a single, algebraic formula.

Other researchers described students' shortcomings in developing a mature concept of function. Researchers described how the students referred to functions as something that fits in set-correspondence diagrams and ordered pairs (Akkoc & Tall, 2003, 2005), equated functions with a set of discrete instructions for what to do with numbers (Bakar & Tall, 1992; Breidenbach et al., 1992), thought about functions as equations (Dogan-Dunlap, 2007), and described inconsistent images of a function across representations (Vinner, 1983, 1992; Vinner & Dreyfus, 1989). Others suggested that students believed a function behaves in a way that looked like the physical situation it was modeling (Dubinsky & Harel, 1992; Monk, 1992a, 1992b; Monk & Nemirovsky, 1994), saw little coherence between graphs, tables, and algebraic forms of the same function (DeMarois, 1996, 1997; DeMarois & Tall, 1996), and had trouble distinguishing between the behavior of the function and the behavior of the function's derivative (M. Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Habre & Abboud, 2006).

Many of these studies focused on the difficulties students had understanding function, but in most cases their focus was on describing how the students' actions or descriptions differed from what the researchers would consider an advanced concept of function, such as covariation or correspondence. However, to characterize the difficulties a student has relative to function, one must characterize his thinking to model the concept of function from the student's perspectives. My interpretation of these studies, based on their transcripts of student talk, suggests that the root causes of these students' shortcomings relative to an advanced concept of function may be related to the students' not conceptualizing functions as invariant relationships between quantities. For example, if a student thinks about the graph of a function as a number of points connected by linear segments, it constrains the student from imagining that every point on the graph of the function satisfies the invariant relationship between the independent and dependent quantity.

In addition, it is not clear that researchers' results extend to how students work with functions of two variables because the tasks used in these studies did not use functions, graphs, tables, or expressions that consisted of two variables. For example, Akkoc & Tall (2003, 2005), Vinner & Dreyfuss (1989) and Vinner (1989) posed tasks where students were asked to decide whether a particular table, expression, or graph could be representations of a function. The tables consisted of two variables x and y, the graphs consisted of points in the x-y plane, and the expressions involved only one variable, x. Others studied how students were able to find the output of a given function in graphical and tabular format (DeMarois, 1996, 1997; DeMarois & Tall, 1996). The task they posed involved tables where the columns were labeled x and $f(\mathrm{x})$, and the graphs were in the two-dimensional x-y plane. Finally, studies that asked students to generate a graph or algebraic expression of a function given a physical model prompted students to generate a function that related an input and an output quantity (Carlson et al., 2002; Monk, 1992a, 1992b). For example, Carlson et al. (2002) posed the bottle problem in which students were given a cross section of a bottle and asked to generate a graph of the height of the water in the bottle as a function of the volume of the water in the bottle.

I do not believe it is appropriate to draw conclusions from these studies about how students think about two-variable functions because the tasks in these studies did not include functions or situations with more than a single input and output variable. My interpretation of the literature suggests to me that that quantitative reasoning may be key in modeling the ways of thinking of students to explain how these ways of thinking lead to the struggles documented by researchers. However, research has not yet established that quantitative reasoning is at the heart of students' documented difficulties, and if it is, these studies' results do not yet extend to student thinking about functions of two variables.

Characterizations of Understanding Function

This section considers the various descriptions researchers have made of what it is entailed in understanding function, which I consider to mean the development of a mature concept of function. I explain how these characterizations of understanding function contributed to this study while also focusing on the ways in which this study was different from previous investigations. I show how thinking about functions as covariation of quantities draws from several researcher's perspectives, and how focusing on covariational and quantitative reasoning support investigating a wide array of issues related to students' ways of thinking about function. It is also important to note that many characterizations of understanding function treat understanding as if it is a categorical, meaning a student has an a particular way of thinking or does not. I have two issues with this approach. First, everyone has an understanding, or way of thinking about function, which leads me to think about understanding as akin to a continuous variable. Second, this way of thinking must be modeled and compared to a "mature" model of function understanding. Thus, I interpret each of these characterizations of understanding function as a model of an epistemic student whose ways of thinking could be considered mature.

Multiple representations of function.

Several studies characterized students being proficient at generating, and relating graphs, tables, and algebraic forms, and understanding those forms of inscription can represent the same thing as an indication of a mature understanding of function (Akkoc & Tall, 2003, 2005; Barnes, 1988; Bestgen, 1980; DeMarois, 1996; DeMarois & Tall, 1996). These researchers asked students to complete tasks using graphs, tables, and analytic representations, and attributed a student's understanding of function to how well they were able to perform tasks within these various milieus. They proposed that students understand function after they have abstracted the commonalities among representations, which I interpret to mean the student understands a table, graph and algebraic definition are representations of a relationship between two or more quantities. For example, Akkoc & Tall (2003), DeMarois (1996) and DeMarois & Tall (1996) suggested that a student does not understand a function such as $f(\mathrm{x}) = 2x$ unless the student thinks about the function in tabular and graphical form as equivalent to the algebraic representation of the function. However, these studies did not explicitly focus on what the students believed that graphs, tables, and algebraic forms represented. This constraint did not allow the researchers to explain what it was that students believed graphs and tables represented. This dissertation built upon, and extended, these investigations by focusing on students' thinking about the extent to which graphs, tables, and algebraic forms could represent invariant relationships between quantities.

APOS theory.

Dubinsky & McDonald (2001) described APOS theory as a theoretical framework that focuses on the processes an individual uses to deal with perceived mathematical problem situations. APOS theory, in their description, arose from the need to extend Piaget's concept of reflecting abstraction to how students learn mathematical ideas in collegiate mathematics. Asiala et al. (1996) modeled the process by which an individual deals with a problem situation into mental actions, processes, objects and schemas, which they referred to as, "APOS Theory".

Asiala et al. (1996) defined an *action* as a transformation of objects perceived as necessitating step by step instructions on how to perform an operation. When an individual repeats, and reflects on an action, I interpret Asiala to mean that the individual can create a *process* in which he can think about the action without stimuli to necessitate the transformation. Asiala et al. described an *object* as constructed by the individual reflecting on the process in a way that allows him to think about the process as an entity on which transformations can act. In Asiala's description, a *schema* consists of the collection of actions, processes and objects a student uses to think about a mathematical concept that the student uses to deal with a perceived mathematical problem.

Dubinsky & Harel (1992) previously used these general descriptions of action, process, object and schema to propose a framework for understanding function. My interpretation of their work suggests that their framework serves as a plausible model of a productive way of thinking about function. The model does not require that all of these conceptions are necessary for a student to reason coherently about function, but together, these constructs do form a descriptive and explanatory model for a mature understanding of function. Dubinsky & Harel (1992) described an action view of function in the following way:

> An action conception of function would involve the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time. A student whose function conception in limited to actions might be able to form the composition of two functions, defined by algebraic expressions, by replacing each occurrence of the variable in one expression by the other expression and the simplifying; however, the students would probably be unable to compose two functions that are defined by tables or graphs. (p. 85)

Students with an action conception of function can think about a function as a specific rule or formula on which the student must imagine performing every action. However, Dubinsky & Harel (1992) assumed the student is working with a function defined symbolically. In their description of action, a student with an action conception can only imagine a single value at a time as input or output, and he conceives of a function as static, leading to seeing a function's graph as a geometric figure (Oehrtman et al., 2008). However, an action conception of function that relies on quantitative reasoning can precede and does not necessitate a symbolic representation of function. Instead, an action results from thinking of two quantities varying in tandem, which does not require a symbolic representation.

A student with an action conception of function, as defined by Dubinsky and Harel, does not imagine the output exists until he has transformed the input. This makes it impossible for the student to think about function as invariance between an independent and dependent quantity because the output does not exist until the input has undergone a transformation. However, a student with an action conception of function who is using quantitative reasoning thinks of the output and input existing simultaneously, which allows them to think about an invariant relationship between those two quantities.

Dubinsky and Harel described a process conception as a more advanced way of thinking about function:

> A process conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. When the subject has a process conception, he or she will be able, for example, to combine it with other processes, or even reverse it. (p. 85)

Students come to have a process view of function once they have repeatedly transformed an input to an output, and abstracted from those actions that a function is a general input-output process. Once the student imagines function as a process, a student can imagine the entire, dynamic input-output process without having to perform each action. Students with a process conception of function also conceive of functions as dynamic, where domain and range are produced by “operating and reflecting on the set of all possible inputs and outputs” (Oehrtman et al., 2008). Oehrtman et al. (2008) identified a process understanding of function as critical to developing a coherent scheme of meanings for function in calculus and more advanced mathematics.

According to Thompson (1994b), when a student possesses a process conception of function and he can imagine the function as being “self-evaluating” over a continuum of inputs, he may begin to think about a function as an object. An object conception of function requires the student to think about function as an object on which other functions can act. Thompson (1994b) stated that, “A representation of the process is sufficient to support their reasoning about it, they can begin to reason formally about functions – they can reason about functions as if they were objects”. A student who thinks about function as an object is able to think about a function as the input to a process and the results of that input undergoing a transformation in the function. As the student solidifies a concept of function as an object, he may begin to reason about operations not only on sets of numbers, but sets of other functions. My study built on students thinking about function as a process and object to inform the models I developed for student thinking about two-variable functions.

Cognitive obstacles.

Sierpinska (1992) and Janvier (1998) proposed that a mature concept of function results from one’s having overcome cognitive obstacles related to understanding function, such as not being able to imagine numerous input and output pairs (Janvier, 1998; Sierpinska, 1992), and the specific understandings required in overcoming those cognitive obstacles. While they recognized that not all learning is impasse driven, they constructed a plausible model for the development of function based on impasses. Sierpinska (1992) and Janvier (1998) reconstructed a possible process of learning a new idea, in this case function, as best described by the qualitative jumps in understanding necessary to achieve an "expert" way of thinking.

The cognitive obstacles that Sierpinska (1992) and Janvier (1998) focused on were: identifying changes, discriminating between number and quantity, and using functions as models of change in a situation. Their primary focus appeared to be on students' identification and representation of relationships between changing quantities, which suggests that quantitative reasoning is critical to a mature concept of function. In addition to the focus on quantitative reasoning, Sierpinska (1992) and Janvier (1998) identified challenges students must overcome in understanding function, but they did not say whether these challenges remain the same as students start to think about two-variable functions. While these challenges were not a direct focus of my study, I think that the type of data I collected contributes to reconstructing their argument for two-variable functions.

Characterization of Research on Multivariable Functions

Recently, several researchers studied how students solve problems involving functions of two variables (Martinez-Planell & Trigueros, 2009; Trigueros & Martinez-Planell, 2007, 2010), and another (Yerushalmy, 1997) characterized how seventh graders could generalize ideas of one-variable functions to two variables. Trigueros hoped to explicate students' understandings of functions of two variables (Martinez-Planell & Trigueros, 2009; Trigueros & Martinez-Planell, 2007) by analyzing students' responses to tasks focused on identifying domain and range. Trigueros & Martinez-Planell (2007) suggested the mental actions required to understand functions of two variables relied on coordinating the Cartesian plane, real numbers, and an intuitive notion of three-space. Building on their previous study, Trigueros and Martinez-Planell (2009; 2010) found that students had difficulty identifying the domain and range of functions of two variables, and attributed this difficulty to a lack of coordination of the schema for space and function. `

Yerushalmy (1997) described activities that she designed for seventh graders that were intended to help them generalize their concepts of one-variable functions to two-variable settings. She sought to gain insight into what understanding students had formed about two-variable functions, and in what manner those understandings had developed from understandings of one-variable functions. She found that students' representation of functions underwent major changes that resulted in their creation of a new symbolic system for representing two-variable functions. However, Yerushalmy did not clarify what the students believed they had generalized as she did not focus on developing a model of the student's thinking based on her inferences in the course of the these interviews. These investigations of students' understanding of two-variable functions focused on what students could or could not do (e.g. find the domain and range of a function of two variables), rather than on their ways of thinking. Because these studies were not focused on describing students' ways of thinking, there is a pressing need for investigations that focus on students' ways of thinking about function of two or more variables.

Students' Thinking about Rate of Change

Students must attend to how quantities are changing with respect to one another to represent dynamic situations with functions. Most calculus textbooks present the idea of rate of change as a property that a function possesses. However, rate of change can itself be thought of as a function whose values at an instant tell you how fast one quantity is changing with respect to another at that instant. Thus, a coherent understanding of rate of change of one quantity with respect to another relies on thinking about function as covariation of quantities and sustaining a process view of function.

Understanding rate of change is foundational to coherent ways of thinking about major ideas in calculus (e.g. accumulation) and differential equations, yet many students possess difficulties reasoning about rate of change (Carlson, Larsen, & Jacobs, 2001; Carlson et al., 2003; Monk, 1987; Rasmussen, 2000; Thompson & Silverman, 2008). Students' difficulties understanding rate of change range from problems interpreting the derivative on a graph (Asiala et al., 1997), focusing on cosmetic features of a graph to talk about constant rate of change of a linear graph (Ellis, 2009), assuming that a function and the rate of change of a function must share specific features (Nemirovsky & Rubin, 1991a, 1991b), and displaying difficulties with activities designed to help them understand the fundamental theorem of calculus (Thompson, 1994a).

Researchers have not only documented students' difficulties understanding rate of change, but have proposed ways of thinking and understandings that explain their difficulties. Carlson et al. (2002) proposed a covariational reasoning framework that centered on ideas of rate of change. Their framework proposed mental actions (ways of thinking) and resulting behaviors involved in a sophisticated understanding of covariation. The five levels of the framework were (1) coordinating changes in one variable on changes in another variable, (2) coordinating the direction of change of one variable with changes in the other variable, (3) coordinating the amount of change of one variable with changes in the other variable, (4) coordinating the average rate of change of the function with uniform increments of change of the input variable and (5) coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function.

Carlson et al. noted that the ability to move fluidly between mental actions 3, 4 and 5 was crucial to a student's mature use of covariational reasoning. This framework did not attribute a specific meaning to variable, but if one thinks about variable as the representation of the result of measuring a quantity's attribute, or as the representation of a quantity's magnitude, the mental actions rely on ideas of quantitative reasoning, and thinking about function behavior using covariational reasoning.

Thompson (1994a) proposed that the difficulties student's displayed in understanding the fundamental theorem of calculus arose from impoverished concepts of rate of change and incoherent images of functional covariation. In that paper, and others, Thompson constructed a hypothesized scheme of meanings and ways of thinking that he believed were foundational to understanding rate of change. Thompson described a rate as a reflectively abstracted constant ratio, where a ratio is the result of comparing two quantities multiplicatively (p. 15). Thompson & Thompson (1992) characterized the ways of thinking that support understanding rate:

> Between ratio and rate are a succession of images and operations. We have identified, in principle, four levels in the development of children's ratio/rate schemes. The first level, ratio, is characterized by children's comparison of two taken-as-unchanging quantities according to the criterion "as many times as". The second level, internalized ratio, is characterized by children's construction of quantities, where the accrual of the quantities occurs additively, but there is no conceptual relationship between ratio of accumulated quantities at iteration x and the ratio of accumulated quantities at iteration $x+i$. The third level, interiorized ratio, is characterized by children's construction of covarying amounts of quantities, where the amounts vary additively but with the anticipation that the ratio of the accumulations does not change. The fourth level, rate, is characterized by children's conception of

> constant ratio variation as being a single quantity – the multiplicative variation of a pair of quantities as a measure of single attribute (Thompson & Thompson, p. 8)

At the first two levels, the student only thinks in terms of static quantities. Whether the student multiplicatively compares the quantities at iteration *x* only (level one) or at iteration *x* and $x+i$ (level two) and might find the quantities in the same ratio, the student possesses no imagery of the quantities accruing in tandem, and thus no relationship between the two ratios. At the third level, the student imagines the quantities accruing simultaneously. At the fourth level, the student no longer needs to focus on the image of two quantities varying in tandem, but imagines the ratio of their measures as an attribute of the comparison of the quantities.

Coinciding with level four, a general scheme for rate entails "coordinated images of respective accumulations of accruals in relation to total accumulations. The coordination is such that the student comes to possess a pre-understanding that the fractional part of any accumulation of accruals of one quantity in relation to its total accumulation is the same as the fraction part of its covariant's accumulation of accruals in relation to its total accumulation" (Thompson, 1994a, p. 237). In others words, given that quantity A and quantity B covaried, if a/b'ths of quantity A has elapsed, then a/b'ths of quantity B has elapsed. Building on a general scheme for rate, Thompson described average rate of change of a quantity as, "if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, then we would end up with the same amount of change in the dependent quantity as actually occurred" (Thompson, 1994a, p. 271). As a result, average rate of change relies on a concept of constant rate of change, from which one constructs a concept of instantaneous rate of change while understanding it cannot exist. Just as quantitative reasoning was foundational to understanding functions, quantitative reasoning is critical to students' understanding rate of change.

Chapter Summary

In this chapter, I have reviewed the literature relevant to how students think about functions of one and two variables. I have described quantitative and covariatonal reasoning, and highlighted how past research findings provide a basis for the design of this study. My synthesis of research relevant to this study represents my thinking prior to carrying out the teaching experiments. The results of this chapter also allowed me to develop a scheme of meanings I believed would be productive for students' understanding of two-variable functions and rate of change. In the next chapter, I use the theoretical constructs from this review to propose a scheme of meanings that constitute a mature understanding of two-variable functions.

Chapter 3

CONCEPTUAL ANALYSIS

This chapter builds upon the constructs of quantitative reasoning, covariational reasoning, and rate of change I described in chapter two. I propose how relevant constructs can be extended to support students in thinking about functions of two variables and their rates of change. I use the frameworks of quantitative and covariational reasoning to explain a scheme of ways of thinking that may comprise a mature understanding of functions of two variables and rate of change. Thus, this chapter is extends the theoretical constructs from chapter two to student thinking about two-variable functions and rate of change in space. The ways of thinking I propose were not only targeted endpoints of my instruction, they also described the ways in which students might need to reason to achieve those endpoints. This mature understanding of two-variable functions and rate of change in space was critical to the design of the study, including data collection and analysis, which I discuss in the fourth chapter.

Conceptual Analysis

Thompson (2008) stated that conceptual analysis is useful in two ways. First, one can generate models of thinking that aid in explaining observed behaviors and actions of students. Second, one can construct ways of thinking that, were a student to have them, might be useful for his or her development of a coherent scheme of meanings that would constitute powerful understandings of a mathematical idea. In the second type of conceptual analysis, the unit of analysis is the epistemic student, which is a student assumed to have certain ways of thinking about a mathematical idea. I used conceptual analysis to develop a scheme of meanings and understandings based on quantitative and covariational reasoning that I believed would comprise a mature understanding of two-variable function in an epistemic student.

Critical Ways of Thinking: Functions of Two Variables

Whether we refer to function as static correspondence between sets or function as a representation of systematic variation between two quantities, a student must reason quantitatively about a situation in a way that allows him to perceive of some regularity or rule governing the relationship between objects in that situation. By reason quantitatively, I mean that the student perceives of a situation in terms of relationships between quantities, which themselves are composed of objects, attributes, and a perception by the student of that attribute as measurable. Quantitative reasoning is also crucial to a student thinking about function without an explicit context because the student can imagine that each quantity in a situation has magnitude. By this, I mean the student recognizes that the quantities have measures in some unit, but the quantities and units do not really matter. What matters is that the student can focus on the magnitudes of the quantity. By focusing on magnitude, the student is able to think about function as a relationship between magnitudes, which allows him to think about functions outside of a specified context.

Using an example mentioned earlier, a particle or piece of matter moving through space does not possess “force”. A force is a result of an accelerating mass, and mass times acceleration is a quantification of force. Mass, for example, is the attribute of an object (such as a particle or piece of matter), which can be thought of as an amount of stuff that does not vary under acceleration. If a student is to construct an invariant relationship or correspondence it is important that he imagine each quantity varying before imagining the simultaneous relationship between the two quantities which forms their functional relationship. In the case of a two-variable function, there must be at least three quantities, two of which the student can relate by a quantitative operation, to form the third and whose values can vary independently of each other.

The student must construct an image of the quantitative relationships between two or more quantities varying individually, and then imagine that the variation in two or more quantities occurs simultaneously to think about the quantities as covarying. In this example, a change in mass and/or acceleration produces a change in the force of a particle. The student must also imagine that this relationship between force, mass, and acceleration can be measured. The student must not only imagine that a relationship between quantities exists, but also imagine that all quantities vary simultaneously, that mass and acceleration can vary independently of each other, and that the relationship between values of force and the values of mass and acceleration is invariant. It is important to note that an invariant relationship need not be linear. The relationship between a side of a square and the square's area is invariant as the area is always the square of the side length. Height as a function of age since birth is an invariant relationship because I had a height at each moment, and my height at that moment will not change.

The ways of thinking I have proposed support a student thinking about function as a process. A student with a process view of function sees a function as a generalized input-output process that maps a set of input values to a set of output values, and is able to imagine running through a continuum of inputs as well as the resulting outputs, as opposed to needing to plug individual numbers into a formula. In the case of the three-variable example of force, a student imagines a domain of values for mass and acceleration, but does not think about only plugging in specific values to "find" the force. Instead, the student relies on his knowledge of the invariant relationship to anticipate how force will vary in tandem with acceleration and mass. In short, the student can imagine the entire process of evaluating the function without having to perform each action (e.g. plug in numbers for mass and acceleration to find force), and the process becomes independent to where the student can focus on the relationship between quantities expressed by the function without needing to consider the specific function to find input and output. For example, a student may think about $f(\mathrm{x}) = 3x$ as a relationship where the dependent quantity's value is always three times as large as the independent quantity's value. The student can imagine a range of outputs that are always three times as large as the corresponding inputs without needing to plug each input into the formula. In this way, the student can think about evaluating the function independently of the formula. I believe this is what Thompson (1993) meant when he said that students reach a phase where they see a function's definition as having become "self-evaluating"—the student anticipates without hesitation that providing the function an input immediately results in an output that he anticipated would be three times as large.

A process view of function suggests a conception of a generalized input-output process, but does not necessarily focus on how input and output vary in tandem. For example, the student may understand that the output is three times as large as the input, but may not be able to imagine the output and input varying simultaneously. The student can build on thinking about function as a process to think about function using covariational reasoning where a student holds in mind an image of how the quantities' measures vary in tandem (Saldanha & Thompson, 1998). Reasoning covariatonally involves thinking about how the value represented by one variable changes with respect to a value represented by another variable. To continue the accelerated particle example, one holds in mind an image of force as a result of a quantitative combination of mass and acceleration (notwithstanding arguments about the possibility of real-world instantaneous speed) so that at every instant there is a force, mass, and acceleration for a particle each of which is a quantity itself. Thus, covariational reasoning about three variables, two of which are independent, rests on thinking quantitatively about a situation.

A student must understand that function notation like $F(t) = m(t)a(t)$ that represents the invariant, co-varying relationship between one dependent and two independent, coupled quantities that vary over conceptual time, while also being a "rule" the student uses to calculate. Up to this point, I have not discussed function in conjunction with a formula because in describing a coherent way of thinking about two-variable functions, a formula does not really represent anything other than an instruction to calculate until the student has achieved a process conception of function. Achieving a process conception of function is non-trivial, and in the next section I describe the ways of thinking that support a student forming a process conception of sin(x).

Developing a Process Conception of Sin(x)

Researchers (Moore, 2010; Thompson, 2008; Thompson, Carlson, & Silverman, 2007) have demonstrated that developing a process conception of sin(x) involves the following ways of thinking:

- Creating a meaning for x as measuring an angle's openness. This entails conceiving of the measure of openness as an arc measured in units of one radius of any circle centered at the angle's vertex.
- Locating the vertical displacement of the arc's terminal point and conceiving of the vertical displacement as measured in unit's of that circle's radius.
- Calling this vertical displacement measured in unit's of the circle's radius the sine of the angle whose measure is x radii.

These ways of thinking then become a process that the student attributes to the symbol sin(x). Given a process conception of function, the student can think about a formula as representing a relationship between the vertical displacement measured in units of the circle's radius and the openness of the angle measured in units of a circle's radius. This provides the foundation on which the student can imagine sin(x) (vertical displacement) and x as varying simultaneously where sin(x) always depends on x. Next, I discuss how thinking about a function sin(x)

as covariation of quantities supports thinking about the construction of a function's graph.

Understanding a Graph of y = *f*(x).

Suppose that a student thinks about the graph of a function as constructed by tracing out a point that simultaneously represents the values of two or more covarying quantities. The following sequence describes a particular understanding of a graph for y = *f*(x) and how that way of thinking extends to functions of two variables. I use *f*(x) = sin(*x*) to illustrate what I mean by thinking about a function as a tracing out of coupled quantities. Note that this part of the conceptual analysis is based on ideas discussed by Pat Thompson within a functions course for practicing teachers (http://pat-thompson.net/MTED2800/SchedFrames.html, November 4).

- Imagine that there are two quantities in a situation, and that *x* and *y* represent their respective values.
- Imagine that the *x*-axis represents all the possible values of a quantity's attribute measured in some unit.
- Imagine that the *y*-axis represents all the possible values of another quantity's attribute measured in some unit.
- Imagine the point that is *y*-units perpendicularly above the value of *x* (represented on the *x*-axis). The coordinates of this point represent the values of the two quantities simultaneously.
- Imagine a graph in two dimensions as the tracing out of that point which simultaneously represents the value of *x* and *y*. (Figure 2)

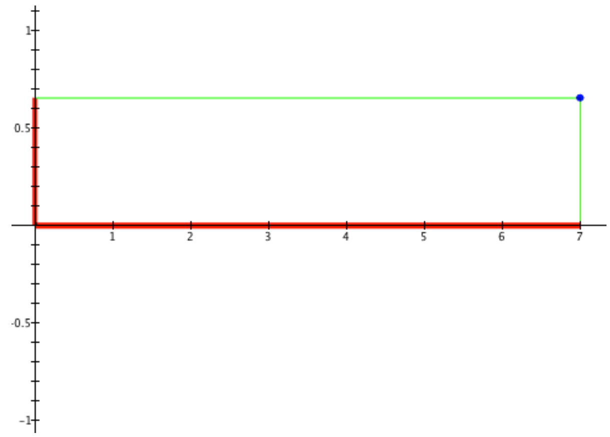

Figure 2. - A point whose coordinates are the values of *x* and *f*(x) simultaneously.

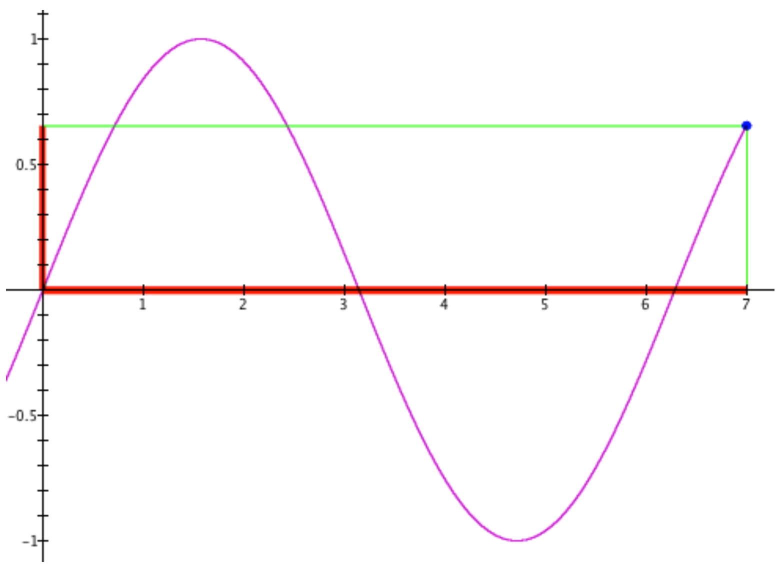

Figure 3. Graph of function *f*(x) = sin(*x*), as a tracing out.

Extension to Two Variable Functions (Parameter Space)

This description of generating a two-variable function in space is based on Patrick Thompson's work in a functions course for practicing teachers, his description of extending ideas of covariation to higher dimensions (Oehrtman et al., 2008), and his most recent work focused on conceptual time. Thompson wrote that the idea of covariation is fundamentally about parametric functions, where thinking about $(x,y)=(x(t),y(t))=(t,f(t))$ supports an image of scanning through values of one variable and tracking the value of another variable. This image supports a student imagining the quantities as coupled. He proposed that this way of thinking could support reasoning about a function defined parametrically, such as $(x,y)=(\sin 10t,\cos 20t), 0\le t\le 1$. He used this example to suggest a way of thinking about curves in space, such as $(x,y,z)=(\sin 10t,\cos 20t,t), 0\le t\le 1$, by imagining that *t* is actually an axis, "coming straight at your eyes" (Oehrtman et al., p. 166). Thompson proposed that this way of thinking about a curve in space could help the student visualize the behavior of $z=f(x,y)$, by thinking about *y* or *x* as a parameter. The graph of the function could then be visualized as generated by a family of functions where *z* is dependent on *x* while *y* varies. I relied on his description of how to think about a curve in space, and our personal discussions, to develop this part of the conceptual analysis.

- Imagine the function $f(x)=a\sin(x)$, where *a* is a parameter value. Think about the graph of *f* as if it is the graph of $f(x)=a\sin(x)$, but drawn *a* units perpendicularly from the sheet of paper on which it seems to appear (that is, think of the sheet of paper is actually a glass plane. The graph is behind the paper when *a* is negative, and it is in front of the paper when *a* is positive.
- Imagine *a* as an axis perpendicular to the flat sheet of paper at the *x-y* origin, and imagine starting with a negative value of *a*, say $f(x)=-2\sin(x)$, pull the function *f*(x) from behind the paper toward you.

As you imagine pulling the graph toward you, imagine that the graph adjusts for the changing value of *a* while also imagining its distance from the *x*-*y* plane being *a* units behind or in front of it.

- Imagine that as you pull the function along the *a* axis, the graph of $f(x) = a\sin(x)$ has "fairy dust" on it, which creates a thin surface as the graph of *f* is pulled along the *a*-axis. Each point on this surface has three coordinates: *x*, *a*, and $f(x)$.

The net effect of this collective of coordinated images and meanings is that one sees a surface being generated as *x* varies to produce the graph of *f* for a particular value of *a* while the value of *a* varies to sweep out a surface generated by the family of graphs generated by *x*, *f*, and *a*.

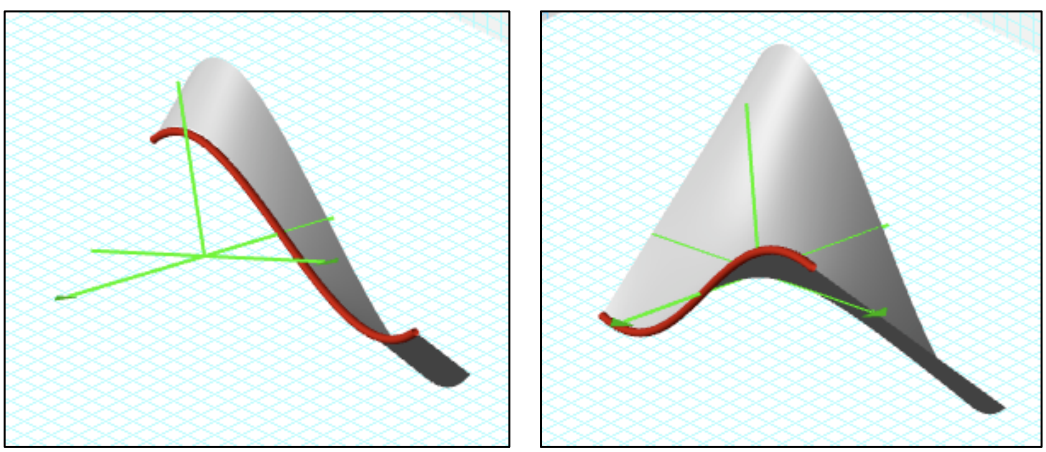

Figure 4. Tracing out f(x,y) = ysin(x) in space.

Critical Ways of Thinking: Rate of Change in Space

Suppose that a student thinks about multivariable functions in the ways described earlier in this chapter. That is to say, the student thinks about function when reasoning about covariation of constructed quantities, when reasoning about quantitative and numerical relationships, and specifically about properties of those relationships (Thompson, 1994b). To think about rate of change in the context of a two-variable function, a student must have a mature understanding of rate. To imagine that a/b represents a value of quantity A measured in units of quantity B, the student must think about the statement a/b=*c* as saying that *a* is composed of *c* b-units. In this way, 16/5 = 3.2 says that 16 is composed of 3.2 units of five, or for every five units of B, there are 3.2 units of A. Put yet another way, the student understands that the statement 16/5 = 3.2 says that 16 is 3.2 times as large as 5.

Instantaneous and Average Rate of Change

To think about instantaneous rate of change as a limit of average rates of change, the student must think about constant rate of change in the following way. Suppose quantity A changes in tandem with quantity B. Quantity A changes at a constant rate with respect to quantity B if, given any change in quantity B and the corresponding change in quantity A, the changes in quantity A and quantity B are in proportional correspondence. In other words, given that quantity A and quantity B covaried, if a/b'ths of quantity A elapses, then a/b'ths of quantity B also elapses . Thus, suppose that a function *f* changes from *f*(a) to *f*(b) as its argument changes from *a* to *b* (Thompson, 1994a). The function's output variable (*y*) changes at a constant rate with respect to its input variable (*x*) if whenever a/b'ths of *b-a* has elapsed, then a/b'ths of *f*(b)-*f*(a) has elapsed.

Though instantaneous rate of change exists mathematically, there is no way to talk about instantaneous rate of change within a real world situation without using approximations. Given that one thinks about rate as the constant accumulation of one quantity in terms of the other, I suggest that for a student to think about instantaneous rate of change, he must first understand that if a function f changes from $f(a)$ to $f(b)$ as its argument changes from a to b, the function's average rate of change over $[a,b]$ is that constant rate of change at which another function must change with respect to its input to produce the same change f produces over the interval $[a,b]$ (Figure 5, adapted from Patrick Thompson's functions course).

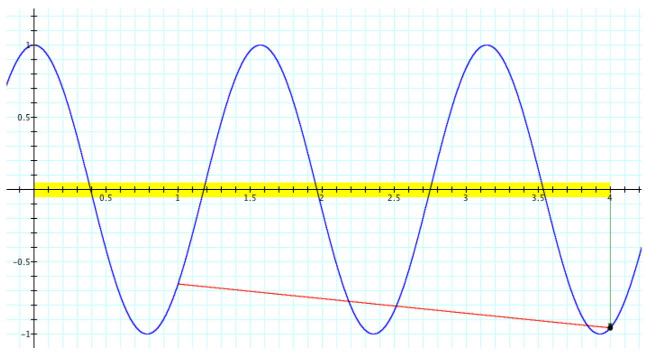

Figure 5. Average rate of change function as constant rate of change.

If a student thinks about average rate of change in this way, by attending to the constant rate of change of covarying, accumulating quantities, then instantaneous rate of change is "unreachable". By unreachable, I mean the student cannot viably say that at a specific instant, quantity A is changing at some *exact* rate with respect to quantity B. Instead, this instantaneous rate of change is the result of a finer and finer approximation, generated by considering average rate of change of quantity A with respect to quantity B over smaller and smaller intervals of change for quantity B. The formula for instantaneous rate of change,

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, is coherent when the student thinks about $f(x+h) - f(x)$ as quantity A, and h as quantity B.

The idea of rate of change of two variable functions is problematic because it depends on direction in space, but builds on the foundation of thinking about rate of change of a one variable function as I just described. With a function of one variable, the idea of change is unproblematic—the argument increases or decreases in value and the function increases or decreases accordingly. But with a function of two variables, the idea of change in the argument is ambiguous. The argument can be thought of as a point in a coordinate plane that can change in an infinite number of directions. It does not simply increase or decrease.

Rate of change of a function generated by the tracing out of three covarying quantities is much like rate of change of a function generated by the sweeping out of two covarying quantities. I suggest that this is because directional rate of change of a two-variable function can be thought of as nearly identical to rate of change of a one-variable function.

Extension to Rate of Change in Space

Most texts state something like the following: The rate of change of $f(x,y)$ in the direction of a vector $u = \langle a,b \rangle$ is

$$f_u'(x,y) = \lim_{d \to 0} \frac{f(x+ad, y+bd) - f(x,y)}{d}$$

which is equivalent to

$$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b\,.$$

It is important to keep in mind that <*a*, *b*> is a vector quantity and is the sum of <*a*, 0> and <0, *b*>. Thus, the statement $D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$ says that the rate of change of the function *f* in the direction <*a, b*> is the vector sum of its rates of change in the directions of its component vectors.

These textbooks refer to the instantaneous rate of change of the function *f* at a fixed point (x_0, y_0), in the direction of *u.* This definition is problematic because it assumes a student understands the necessity and use of a direction vector. It is important that the student thinks about the rate of change of a function as a change in its value relative to a change in its argument, and to understand this in a way that aligns with the way he thinks about the rate of change of a single variable function relative to its argument. To do this, one must imagine picking a point $\mathrm{P}(x_0, y_0, z_0)$ on the surface "at" which he wants to "find" the function's rate of change in a certain direction.

If a student is to understand rate of change of a two variable function using similar meanings to a one variable function, she must consider what it means for a quantity A to change at a constant rate with respect to quantities B and C. Suppose that quantities A, B, and C covaried. Quantity A changed at a constant rate with respect to quantities B and C if for any amount of quantity A elapsed (a/b'ths), a/b'ths of quantity B elapsed, and a/b'ths of quantity C elapsed. Extending this to function notation results from thinking about $f(x,y)$ as quantity A, and *x* and *y* as quantities B and C. This conception of rate of change necessitates the idea of considering a direction of change, as there are an infinite number of directions in which *f*(x,y) changes with respect to *x* and *y*. The direction actually specifies the relationship between x and y. For example, the rate of change of $f(x,y) = 3x + 5y$ from a point (x_0,y_0) will be constant for any changes in x and y only if Δy is a linear function of Δx.

Instantaneous rate of change can be thought of as an average rate of change over an infinitesimal interval. The average rate of change of a quantity C ($f(x,y)$) with respect to quantities A (x) and quantity B (y) in a given direction in space can be thought of as the constant rate at which another quantity D would need to change with respect to quantities A and B to produce the same change as quantity C in the same direction that (x,y) changed. This necessitates that quantity D accrue in a constant proportional relationship with quantity A, and simultaneously accrue in a constant proportional relationship with quantity B. These understandings support thinking that as with functions of one variable, an exact rate of change is a construction of an average rate of change between two "points". The points here are $(x_{0,}y_0, f(x_{0,}y_0))$ and $(x_{1,}y_1, f(x_{1,}y_1))$.

Thus, the average rate of change between those two points is the constant rate at which another function $g(x,y)$ would need to change with respect to x and y over the intervals $[x_0, x_1]$ and $[y_0, y_1]$ to produce the same net change as $f(x,y)$ over those same intervals. The function $g(x,y)$ must change at a constant rate with respect to x and a constant rate with respect to y and those constant rates must remain in an invariant proportion, which necessitates x and y accruing in an invariant proportion as well. An "exact" rate of change then, is a result of considering an average rate of change of $f(x,y)$ over an infinitesimally small interval of $[x_0, x_1]$ and $[y_0, y_1]$, where changes in *x* and *y* also covary in constant proportion to each other.

The denominator for the rate of change of a two-variable function $f'(x,y) = \lim_{h,k \to 0} \frac{f(x+h, y+k) - f(x,y)}{?}$ does not initially make sense because it seems as if *h* and *k* vary independently. Thinking about the rate of change of $f(x,y)$ as above supports thinking that any accrual *d* of either *x* and *y* must be made in constant proportion *b/a*. This proportion *a/b* actually specifies the direction vector to which many calculus books refer.

Thus, rate of change of *f*(x,y) with respect to *x* can be reformulated as $f_u'(x,y) = \lim_{d \to 0} \frac{f(x+h, y+k) - f(x,y)}{d}$, where *ad* = *h* and *bd* = *k* so *h* must be a/b'ths of *k* and *k* must be b/a'ths of *h*. Then, *d* can be thought of in the same way as *h* in the one-variable case, where the student thinks about the derivative as an average rate of change of a function over infinitesimal intervals while realizing that the proportional correspondence between *h* and *k* means they have a linear relationship resulting in approaching the point (x_0, y_0) along a line.

Directional Rate of Change and Path Independence

It is important to note that the conceptual analysis I have described above does not immediately extend to a student thinking about rate of change in many directions simultaneously. However, I believe it is necessary for a student to be able to think about rate of change in at least one direction, which is a focus of this conceptual analysis, if they are to be able to think about rate of change in all directions. I conjectured that to think in all directions, they must think about a single direction as a special case.

Suppose that a student has the ways of thinking I previously described about rate of change in one direction in space. If he is to think about rate of change in all possible directions, I believe the student must first consider when a rate of change exists at a point in space. The student must understand that for $f'(x,y)$ to exist at a point in space, $f_u'(x,y)$ must be equivalent regardless of the direction of the rate of change. This way of thinking provides the foundation for the student to think about path independent rate of change. While path independence was not a primary focus of this study, two students did make progress in thinking about rate of change in all directions in space.

Chapter Summary

The goal of this chapter was to use the theoretical constructs from quantitative reasoning, covariation, and rate of change to establish a coherent scheme of meanings that, were a student to have them, he or she would possess a mature understanding of two-variable functions and a coherent understanding of the concept of rate of change of a two-variable function with respect to its argument. In establishing how I used theoretical constructs to establish specific ways of thinking about functions of two-variables, I created the theoretical foundation for the investigation. The conceptual analyses about two variable functions and rate of change that I presented here drove the design, data collection, and data interpretation for the study, all of which are the focus of the next chapter.

Chapter 4

METHOD

This chapter describes the method I used to construct inferences about and model student thinking during the course of and during analysis of the teaching experiment sessions. First, I discuss the necessity of using a teaching experiment given the constraints of the research questions I posed. I focus on the specific elements of a teaching experiment and how they played a role in creating models of students' thinking about two-variable functions. Next, I explain the data collection and interpretation procedures in the context of a teaching experiment methodology. I then outline a set of tasks for the teaching experiment, and describe how these tasks allowed me to generate and test hypotheses about student thinking while also supporting students in developing the ways of thinking described in the hypothetical learning trajectory in chapter three. These foci allow me to describe how these tasks became didactic objects (Thompson, 2002) for me within the teaching experiment.

Rationale for a Using a Teaching Experiment

One cannot "see" a student's thoughts as they exist in that student's mind without using one's own scheme of meanings. For example, I could observe a student using the words rate of change while calculating a derivative and assume the student is thinking about derivative as measuring a change. This introduces the first constraint in studying student thinking, that models of student thinking are rooted in the observer's schemes.

This constraint requires that the observer must show his interpretations are viable. Viability in models of student thinking can be established by showing that were students to think in ways the observer claims, their behavior would be consistent with observed behavior. Chapter three was my attempt to establish the viability of my schemes as producing worthwhile and achievable goals for students' thinking about functions of two variables.

Constructing a model that explains how students actually think about two-variables functions required a methodology that allowed me to create a valid and reliable model of student thinking. This study built upon Steffe and Thompson's (2000) account of a teaching experiment methodology that focuses on building models of students' ways of thinking about specific mathematical ideas and how those ways of thinking develop in the context of instruction.

Steffe and Thompson (2000) proposed that the purpose of a teaching experiment was to experience students' mathematical learning and reasoning as a first step to constructing models of students' ways of thinking. They emphasized that there are two realities at play within a teaching experiment. The first, students' mathematics, refers to the mathematical reality of the student, which is unknowable to the researcher. Students' mathematics is indicated by what they say and do within a teaching experiment. The second, the mathematics of students, refers to a researcher's models of students' mathematics, or ways of thinking that, were a student to have them, would make the student's actions and utterances sensible for the student. Gaining insight into students' mathematics is critical for creating a viable model of how students understand and think about functions of two-variables.

Given my goal of creating a viable model of students' mathematics relative to functions of two variables, a teaching experiment was most appropriate for this study for three reasons. First, to propose models of students' thinking that might serve as desirable goals of instruction, I needed in some way to construct and test the viability of the model's "learnability" and potential difficulties students might have in developing the ways of thinking that I have proposed. Steffe and Thompson (2000) described a teaching experiment as an opportunity for the researcher to experience constraints of language and actions of students, which expands on the constructivist stance that our understanding of any reality comes by way of experiencing the constraints it places on our actions. Second, a teaching experiment breaks down traditional barriers between researcher and teacher. Students are constantly constructing their own understandings, which I gained insight into during a teaching experiment by using teaching to probe those understandings. Third, a teaching experiment not only allowed for testing and generating hypotheses at the beginning and end of the experiment, but gave me opportunities to generate hypotheses about students' ways of thinking continually throughout the experiment.

Establishing the Viability of a Mathematics of Students

The continuous cycle of task design and testing of hypotheses using new tasks was the basis for establishing viable models of student thinking. By viable, I mean models that provided descriptive and explanatory utility in a variety of situations for a particular student. The continuous cycle of generating and testing hypotheses allowed me to generate not only a possible model of how a student is thinking, but one viable with the students' actions and utterances during the teaching experiment. Establishing a viable model of students' mathematics was a result of going through design and hypothesis testing in a way that allowed me to experience the constraints on my actions due to students' constraints on their thinking that arose from their mathematics.

Steffe & Thompson (2000) identified two types of constraints in a teaching experiment. First, I as the researcher experienced constraints in understanding students' mathematical thinking as a result of the utterances and actions they brought forth in students. This was a necessary result of my own schemes being different from those of the students. Second, students are constrained by the their essential mistakes, which are mistakes created when the demands of the task exceed the capability of their thinking. Essential mistakes remain even in the face of instruction intended to help the student think in a different way. Together, experiencing both types of constraints in the context of generating and testing hypotheses about student understanding allowed me to construct a model that addresses both the student's capabilities and the student's limitations.

In the course of a teaching experiment, I attempted to make sense of students' thinking by hypothesizing ways of thinking that could plausibly lead to the behavior that students exhibited. I tested these hypotheses within teaching episodes by designing tasks and instruction that probed at the constraints discussed above. For example, if I hypothesized that a student was thinking about the graph of a function as a wire and not attending to quantitative relationships that generated the function, I designed a task to elicit how a student was thinking about transforming a graph, which I could model based on their actions and responses to the task. If the student responded in a way that suggested he was thinking about a graph as a wire, I designed a situation to create an essential mistake if the student was in fact thinking about the graph as a wire. However, during each teaching episode, it was critical that I put aside these hypotheses in an attempt to influence students' thinking positively while at the same time respecting the thinking they have. It was after each teaching session that I evaluated the hypothesis that led to the intervention and the revision of learning goals.

If the hypotheses and the models of student thinking generated in an episode were coherent with one another, I designed new tasks in which to test the hypothesis again, but this time, if a hypothesized way of thinking seemed to make a student's actions coherent in a variety of situations, I claimed to have a viable explanation of the mathematics of a student. If the hypotheses and the models of student thinking generated in an episode were not coherent with each other, which I defined as a deviation from the student behaving in the way I anticipated, I developed new hypotheses and created situations in which to evaluate the boundaries of students' thinking. The ability to generate and test hypotheses while experiencing constraints to model students' mathematics relied on systematically collecting data that allowed me to gain these insights.

Data Collection

The following sections focus on my data collection and data interpretation procedures to explain how I employed the theory of a teaching experiment to investigate how students think about two-variable functions. In particular, I describe how I used specific tasks to continuously generate and test hypotheses to create models of students' thinking, and use the hypotheses and reflections generated from data collection to create an initial coding scheme for data interpretation. Lastly, I describe my method for refining, and expanding the initial coding scheme using open and axial coding (Strauss & Corbin, 1998) in tandem with retrospective conceptual analyses.

Subjects, Setting and Logistics

Four students, Jesse, Jori, Brian and Neil participated in the teaching experiments. Jesse and Jori participated in individual teaching experiments, while Brian and Neil participated in a group teaching experiment. The four students were in the process of taking a first semester calculus course that was grounded in quantitative and covariational reasoning with an explicit focus on understanding rate of change in the way I described in chapter three. I initially recruited six students to protect against students dropping the course or becoming unavailable to participate in the study and selected the students based on their willingness to participate in the study, availability to participate at specific dates and times, and their scoring at least an eight on the calculus concept assessment, which was administered at the beginning of the semester.

Jesse was a conscientious student who expressed frustration that some of his mathematics courses in high school had been focused on doing calculations and learning algorithms without making sense of why the calculations and processes were necessary or logical. He enjoyed mathematics because solving problems was both fun and worthwhile. He was unsure of his future plans, but was leaning toward majoring in mathematics and becoming an actuary in the future.

Jori did not enjoy mathematics, which she attributed to numerous negative experiences and perceived failure in high school mathematics and science courses. She described herself as a high-achieving student "overall", but felt mathematics did not reflect her general success in other courses. She attributed her success or failure to sources outside of her control, such as the teacher, the workload of the class, or the general idea that mathematics is confusing.
Jori felt that doing mathematics involved memorizing procedures and examples from the book or from the teacher's presentations.

Neil was a computer science major, and described himself as passionate about mathematics and science yet frustrated because he felt he was supposed to remember so many formulas and methods instead of understanding why they were true. He described his use of graphing calculators and technology as a way to "remedy" the procedures he had forgotten. He felt that it was easy to unload the calculations onto the technology, but was often unsure when it was a calculation he was "offloading" as opposed to an important way of thinking. He described himself as passionate about learning, and said his calculus course was something he looked forward to because he was able to ask questions and debate others so he could make sense of the rules and procedures.

Brian was a construction management major who had returned to school that semester and chosen to take calculus because he was interested in mathematics. He said that when he had an opportunity to make sense of ideas instead of memorizing definitions, he found he loved “doing mathematics”, and had purchased more advanced mathematics textbooks to read. Brian said he had taken only basic mathematics in the past, but after a semester of work in calculus, he described himself as excited to try new approaches in mathematics.

I met with each of these students during tutoring hours at least three times in the weeks leading up to the teaching experiment. The tutoring hours were specifically for the calculus one course in which the students were enrolled, and the students asked questions related to course content and written homework assignments. I asked all four students various questions related to their understanding of function and rate of change, but the questions were not identical as I related them to the homework questions the students brought to tutoring. I also collected their in-class work and homework in the weeks prior to the teaching experiment as background for my preparation. Meeting with the students and observing their classroom participation and interactions prior to the teaching experiment allowed me to become familiar with their particular approaches to problem solving and their relationships with other students in the group, and it allowed the students to become comfortable with me as a tutor.

The teaching experiments occurred over the last two months of the calculus one course and the beginning of the summer session due to time constraints in the student's schedules. The first teaching experiment began after the halfway point of the calculus course that allowed me to work with each student and make initial hypotheses about their thinking with respect to rate of change and graphs of one-variable functions. There were three teaching experiments in total, two one on one teaching experiments and one group teaching experiment involving two students and I. Each teaching experiment occurred over a three-week period, or as the students' schedules permitted. Table 2 provides the meeting dates. Specific tasks and the day on which they took place were variable across the teaching experiments. Thus, a detailed table with tasks and meeting dates is available within each data chapter.

Table 2

Meeting Dates for Teaching Experiments.

Meeting Dates by Teaching Experiment		
Jesse (11 sessions) (3/13/2011-4/4/2011)	Jori (12 sessions) (4/5/2011-4/29/2011)	Brian and Neil (9 sessions) (5/5/2011-5/19/2011)

The students were compensated for their participation. Each teaching episode occurred in a room with overheard cameras to tape both the students' table and the classroom whiteboard. The students were able to use a laptop computer, Graphing Calculator (a program installed on those computers), and had table-sized whiteboards on which they were able to write anything they wished during the sessions. Each session was videotaped on overhead and side view cameras, the screen was recorded using SnapzPro, and the students' written work was recorded on a whiteboard and/or using the Livescribe Echo Smartpen.

Exploratory Teaching Interviews

The first step in creating a model of student's mathematics was constructing a set of initial hypotheses about their ways of thinking prior to the teaching experiment, so that I was able attend to how these ways of thinking persisted or changed in the course of the teaching experiment. To generate these initial inferences, I used their tutoring hours to conduct exploratory teaching interviews. It is critical to note that the exploratory teaching interviews were based on problems the students were working on in tutoring, and thus, were not always the same for each student.

Exploratory teaching interviews extend the clinical interview (J. Clement, 2000; Goldin, 2000) where one seeks to understand students' thinking using task-based questions, to include exploratory teaching where one attempts to help the student extend their current ways of thinking to a desired understanding. These interviews involved the student, an interviewer (me), and an interaction in relation to questions, problems or activities (Clement, 2000, p. 519). The interviews contained problems focused on revealing how students think about ideas of quantity, variation, and function.

The interviews adhered to Goldin's (2000) principles of structured interviews (e.g. minimal heuristics, exploratory questions, choose accessible tasks for subjects) while using the tasks as an opportunity to conduct exploratory teaching about quantitative reasoning, covariation, and rate of change. For example, I posed a bottle problem task, in which I ask the students to describe what is happening to the height and volume of the water in the bottle as I pour an equal amount of water into the bottle every second. I presented the task with certain hypotheses about student thinking, such as the difficulty of thinking about rates of change when the input unit is a generalized unit of volume. I asked the student questions as he or she worked through and reflected on the task to gain insight into his/her thinking, particularly focused on the student's quantitative and covariational reasoning. I used Goldin's principles of structured interviews to elicit language and actions of the student relative to the task, and to experience students' essential mistakes. During the interviews I continually attempted to hypothesize about a student's ways of thinking, and adjust the task, or ask questions to create opportunities for students to reveal essential mistakes.

In the exploratory teaching interviews, I attempted to build on the working model of a student's mathematics to understand what progress the students made given a certain instructional focus. For example, suppose that I established that a student was thinking about creating a graph of height of the water in the bottle as a function of volume of the water in the bottle by constructing a graph that fits the shape of bottle in the problem. An important question was how resilient is the student's way of thinking—the extent to which it is resistant to instruction aimed at changing it. Subsequent interactions moved progressively from probing questions regarding what a point on the graph represented (indeed, even whether the graph has points) to suggestions as to how she might try to think about where graphs come from and what they represent about quantities in the situation.

In asking subsequent questions, I tried to establish whether the student was thinking about a graph as connected to the shape of the given physical situation. In the context of this study, I was interested in understanding what progress the student can make by thinking about a graph as traced out by a point that simultaneously represents the values of two quantities. Once I began to leverage a model of the student's mathematics to help move his/her thinking forward, I was focused on exploratory teaching. In this case, I stepped back and helped the student think about what it means to identify quantities, and how graphs are representations of tracking how quantities vary in a given physical situation or in abstract mathematical notation. I was especially attentive to the ease or difficulty with which a student assimilated my suggestions during the exploratory teaching episodes. If the student easily picked up the way of thinking I proposed, then this suggested that their schemes were better formed than those of a student who struggled to make sense of my suggestion. The nature of the student's actions subsequent to my suggestions often suggested their ways of thinking about the ideas under discussion.

My goal in the exploratory teaching interviews was to make inferences about each student's thinking regarding quantitative reasoning, invariant relationships among quantities, and functions of one variable and their graphs, and to use those models to move the students forward in reasoning about quantities, covariation and rate of change. While quantitative reasoning and rate of change were foundational to the calculus course in which the students are enrolled, I made no assumption that they were part of a student's ways of thinking. By establishing initial inferences about a student's thinking, I was able to adjust and refine tasks in the teaching experiment to allow me to experience students' essential mistakes firsthand. Essential mistakes are mistakes that help the researcher define the boundaries of a model of a student's way of thinking.

Teaching Experiment Sessions

The teaching experiment sessions focused specifically on creating models of student thinking, based on my inferences, while I attempted to help them think in sophisticated ways about functions of two-variables and rate of change based on the conceptual analysis in chapter three. The teaching experiment was based on a set of tasks designed to gain insight into and foster ways of thinking of each student by using exploratory teaching. It was productive not only to ask each student what she was thinking, but to ask the students what sense they were making of what other students are saying in the case of the group teaching experiment, and of what I was saying in the case of the individual teaching experiments. The specific tasks and content of the teaching experiment were fluid because as I experienced specific constraints and generated and tested hypotheses, I found it necessary to create new tasks or adjust the questions in a specific task to establish the viability of a particular hypothesized way of thinking for a student or the group of students. As the teaching experiment sessions occurred at least one day apart, I was able to conduct initial reflections on what happened during each episode, generate and test basic hypotheses, and adjust the next session based on what occurred in the previous session. I recorded this process as text documents before and after each teaching episode to generate data that allowed me to track the development of my own thinking as a researcher.

Reflexivity: Documentation of My Thinking

It was critical that I accounted for my role in what the student said and did and understand that my interpretations of student's behavior and explanations contribute to students' actions, because my actions toward them were predicated on my understandings of what they did. I emphasized earlier that a teaching experiment breaks down the role of researcher and teacher. It also blurs the lines between teacher and student because the student and teacher are co-constructors of the teaching interactions. By this, I mean that both the researcher and the student contribute ways of thinking, actions, and utterances that the other (student or researcher) interprets and responds to in some manner.

Because I brought my own ways of thinking and specific interpretative lens to the teaching experiment, I reflected on how students interpreted my actions and how that interpretation might have played a role in the students' actions. Sometimes, reflecting on one's own actions is complicated, and according to Steffe and Thompson (2000), an outside observer is one way to enhance the researcher's ability to reflect on his or her role in the interactions that emerge. Due to time constraints and schedules of others, I was not able to have an observer for most teaching experiment sessions. This required me to be diligent in documenting the development of my own thinking within the teaching experiment. In order to do this, I prepared a set of hypotheses about the student's actions prior to each teaching session, and my hypotheses were based on my working model of their ways of thinking. Immediately after each teaching experiment session, I watched the recording of the session, did a basic transcription, and reflected on how the student's actions within that session necessitated adjustments or changes in my model of their thinking Thus, I was able to simultaneously track my own thinking and the development of my model for their thinking.

Teaching Experiment Tasks and Hypothetical Learning Trajectory

The Homer Task

The initial tasks consisted of questions designed to reveal the meanings each student attributed to the construction and interpretation of a graphical representation of function. The task used the Homer activity (Figure 6), which is an adaptation of Saldanha & Thompson's (1998) City A – City B task. Trey Cox and Scott Adamson changed City A to Shelbyville and City B to Springfield and the car to Homer because in the Simpson's episodes these cities move all around the map episode to episode. In this version of the activity, built with Geometers Sketchpad (Key Curriculum Press, 2010), Homer moves along a road with two cities on either side. The cities' locations can be manipulated, but their representations, "Distance to Springfield" and "Distance to Shelbyville", remain as the vertical and horizontal segments, respectively. In the Homer adaptation, the user could choose to have the animation reveal 1) Homer and the two cities, 2) One or both segments, each of which represent Homer's distance from Springfield or Shelbyville, 3) A correspondence point which represents his distance from each city simultaneously and 4) A graph created by tracking the correspondence point as Homer's distance from the cities as he moves back and forth on the road.

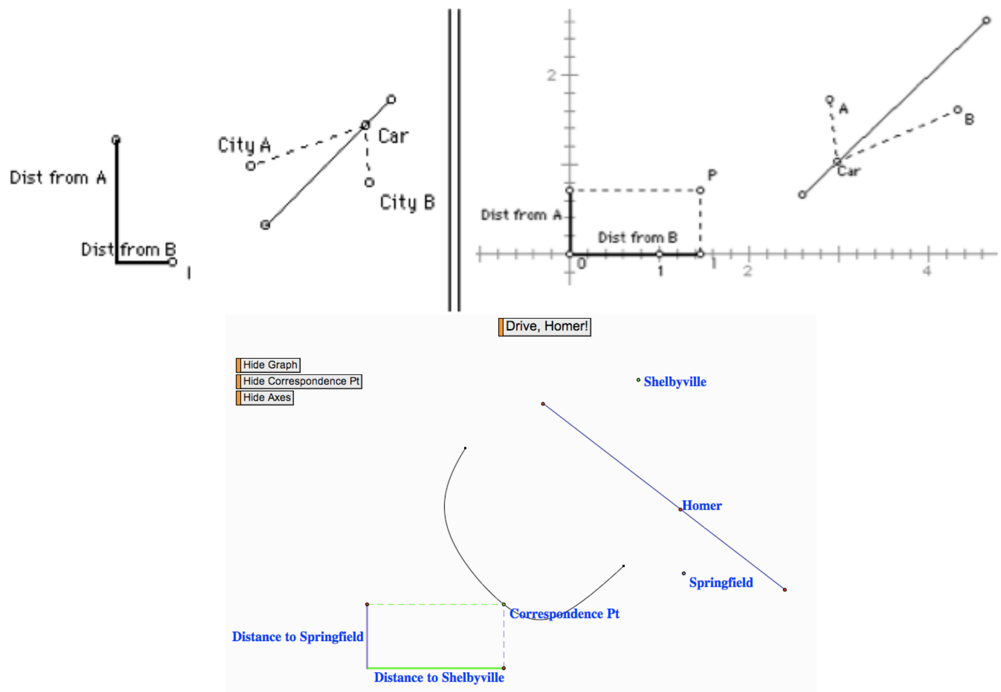

Figure 6. – Homer situation adapted from City A – City B (Saldanha & Thompson, 1998).

Table 3 *Task 1: Initial Questions for the Homer Animation.*
1) Suppose the distance between this fixed point (a stationary object) and your right index finger, horizontal to the fixed point, represents Homer's distance from Springfield. 2) Suppose the distance between this fixed point (a stationary object) and your left index finger, vertical to the fixed point, represents Homer's distance from Shelbyville. 3) As Homer moves along the road, represent how Homer's distance from Springfield is changing using your right index finger. With your left index finger, represent how Homer's distance from Shelbyville is changing. 4) Next, place a finger in the plane so that its vertical distance from the horizontal axis represents Homer's distance from Shelbyville and your finger's horizontal

distance from the vertical axis represented Homer's distance from Springfield simultaneously. Next, move your finger so that its position reflects Homer's distance from each city as he moves along the road.
5) Finally, imagine that your finger that represents both distances has "paint" or "fairy dust" on it. Record what the paint would trace out as Homer moves along the road.

The goal of this activity was to gain insight into the extent to which students were able to think about a graph generated by a sustained image of two quantities varying simultaneously. This sustained image can be generated in the following way. Suppose the student imagines Homer moving along the road over an interval of conceptual time, and that the entire movement back and forth on the road constitutes the domain of conceptual time over which his distance from each city varies. If the student imagines each distance varying as Homer moves back and forth on the road it is plausible the student can create an image of those distances as coupled and varying in tandem over an interval of conceptual time, represented by $(x_e, y_e) = (x(t), y(t))$. My intention was to support an image of a graph as an emergent representation of tracking two quantities' values. While the graph in the Homer situation is a representation of a parametric function, I anticipated it could be easily extended to thinking about a sustained image of x and y covarying in the plane. Covariation in the plane, with $y = f(x)$, would be $(x, y) = \left(x(t), y(t)\right) = (t, f(t))$. However, the most important aspect of this task was to think about a function's graph as an emergent representation of two quantities have covaried simultaneously. If the students initially had difficulty thinking about a graph as generated by a sweeping out of a point that represented two quantities values simultaneously, the goal of the exploratory teaching was to interpret the mistakes they make to understand what ways of thinking may have been preventing them from thinking about a graph in this way.

Table 4

Task 2: Interpretation of Speed in the Homer Animation.

1) Describe the meaning of Homer's speed in this situation. 2) Given that meaning for his speed, what would you need to measure to find his speed? 3) Are there more meanings for speed beyond what you described? If so, state why not. If yes, describe them and how you anticipate measuring that speed. 4) What process would you use to measure Homer's speed as accurately as possible in this animation?

The Homer situation allowed me to ask a student about how they would measure Homer's speed given only the information on the sketchpad. However, I was purposefully vague about the meaning of Homer's speed, as I was interested in what meaning the students would use for speed and what quantities they would measure to track Homer's speed. My goal in asking these questions was to understand how each student was thinking about a rate of change, and if they could generalize that way of thinking in a situation where the independent and dependent quantities are in non-conventional units such as Homer's distance from Shelbyville. Specifically, I intended that they think about Homer's speed, using whatever quantities they determined, as a comparison of changes in two simultaneously varying quantities. These ways of thinking were a focus of their calculus course throughout the semester, but I was interested in understanding the extent to which those ways of thinking were doing work for each student, and if little, to think about how to build on their current ways of thinking to reach a working definition of rate of change.

Car A – Car B

The next task was the Car A- Car B problem (Hackworth, 1994, adapted from Monk, 1992a), which displayed graphs of two cars' speeds from zero to 1.2 minutes after they started from side-by-side positions. The problem depicted two graphs, which the students were told represented the speed of each car (A and B) as a function of time elapsed since the cars started traveling.

Table 5 *Task 3: Interpretation of the Car A – Car B Graph.*
Two cars, Car A and Car B, started from the same point, at the same time, and traveled in the same directions. Their speeds increased, as shown in the graph (Heavy graph is for Car A, light graph is for Car B, so that after 1 minute they were both traveling at 100 mph. Was the distance between the cars increasing or decreasing after 0.8 minutes?

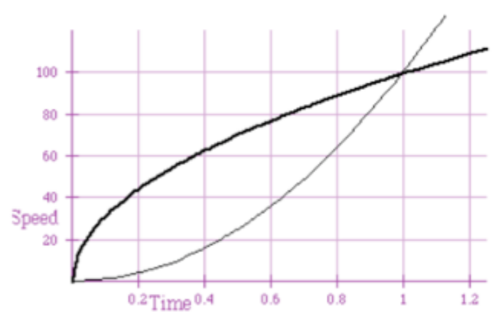

Figure 7. Car A - Car B from Hackworth (1994) adapted from Monk (1992a).

Hackworth (1994) found that the vast majority of college Calculus I students (64 of 90) answered in a pretest before instruction on the derivative that the distance between cars was decreasing because the graphs are getting closer together. Interestingly, 42 out of 57 students (33 students had dropped) answered on a posttest after completing study of the derivative that the distance between cars was decreasing because the graphs were getting closer together.

I modified Hackworth's protocol to first ask students where Car B was in relation to Car A after 0.8 seconds had elapsed, and then asked them Hackworth's question. I used this task to gain insight into what they understood a graph to represent, what they were paying attention to about a graph of a function, and what information they would imagine using to construct the graph of a function. The second focus of this task was to probe the students' understanding of rate of change, specifically comparing the rate of change of the two cars' speed with respect to time and the implication of a car's speed relative to another as to what is happening to the distance between them.

Table 6 *Task 4: Interpretation of Rate of Change in Car A – Car B.*
Given the graph of Car A and Car B, describe what a rate of change of the quantities represented on the graph would represent. How would you measure the rate of change? Indicate this rate of change on the graph.

It was my intention to advance the students' understanding of how they would construct distance time and acceleration time graphs. Specifically, I wanted to extend their ways of thinking to imagine creating the distance-time graph by thinking about the speed-time graph as a rate of change function of the distance-time graph. I also wanted each student to think about creating the acceleration-time graph by thinking about the acceleration-time graph as a rate of change function of the speed-time graph. In conducting the exploratory teaching interview, I focused on having the students compare the similarities and differences of the Car A – Car B situation with the Homer task. I wanted to help them think about a rate of change being a measure of average rate of change regardless of the units of the dependent and independent quantity.

The Difference Function

Table 7 *Task 5: Construction of Graph of the Difference Function.*
1) Given the following two functions, construct the graph and explain the behavior of the functions $f(x)=x^3$ and $g(x)=2x$. 2) As x increases from 0 to 1, why does $f(x)=x^3$ behave in the way you have indicated? 3) As x increases how do $f(x)$ and $g(x)$ vary? 4) Construct a graph of $h(x)=x^3-2x$, and explain why it behaves in the way it does based on their descriptions of the behavior of $f(x)$ and $g(x)$.

The fifth task used an activity designed to help students think about polynomial functions as sums of monomial functions (Dugdale et al., 1992), with an explicit focus on quantitative reasoning. Dugdale created this task as a way to support students reasoning about the behavior of a polynomial as dependent on the behavior of the monomial functions that comprised it. I used this task not only to support student thinking, but also to gain insight into how quantitative reasoning, or lack thereof, could support or constrain an image of the polynomial function as dependent on its monomial parts. The purpose of having students think about polynomial functions as sums of monomial functions was to help them explain the qualitative behavior of a graph by focusing on quantities represented by x and $f(x)$ in complicated settings. Lastly, I had students reflect on how these tasks and the graphs each student generated were the same, and how they were different from the Homer activity. The purpose of this reflection was to understand whether students were able to think about the graphs in these tasks and Homer situation as generated by keeping track of the values of coupled quantities represented by a point.

The Box Problem

The Box problem task, as I conceived of it for this study, was based on the work of Marilyn Carlson's Pathways to Calculus course. In the box problem, the student must imagine making square cutouts from a rectangular piece of paper and imagine folding up the sides to create an open box.

Table 8 *Task 6: Construction of the Box.*
Suppose you are a computer programmer for a shipping company, and you are writing a program that will allow workers to print a shipping label that provides the volume of a box. The box is constructed from a rectangular piece of cardboard that is 33 by 24 inches, and is constructed by making cutouts from each corner of the cardboard. You need to construct a function so that when a worker inputs a cutout length, the function determines the resulting volume of the box and prints it on a shipping label.

Volume as a function of cutout length.

The first part of the task required students to generate a function that related the volume of the box to the length of the cutout. The question was phrased so that students attended to the constraints of the cutouts to be made from the box, particularly whether or not the cutouts need to be square and have equal lengths. Previous interviews had indicated that constructing a function that models volume as a function of cutout length required that the student construct the box as a quantitative structure, that is, construct an image of the box how cutting out a square from each corner enables one to construct a box, and how quantities length, width, and height are related to each other.

Length of the box as a parameter.

Understanding in what ways each student thought about the effect of a parameter on the behavior of the function was key to helping them build the ways of thinking necessary to think about the parameter as a third axis. I asked them to think about the value of the parameter (length of the sheet of cardboard) as represented on a third axis perpendicular to the x-y coordinate plane.

Table 9

Task 7: Length of Box as a Parameter.

1) Describe the behavior of the function $V(\mathrm{x})$ as the length of the original cardboard takes on different values such as 30, 25, 20, 10, and 5 inches. 2) Describe how changing the value of the length affects the formula for volume as a function of cutout length and compare the behavior of the graph to the behavior of the function when the original cardboard length was 33 inches. 3) Imagine that starting at a very small value of the cardboard length, the graph of volume as a function of cutout length has fairy dust attached to it so it keeps a record of where the graph has swept out. Construct what this sweeping out would look like and describe what the sweeping out represents.

I intended that this task help me understand if their ways of thinking about the behavior of the function for different parameter values helped them anticipate a surface being traced out by the graph of the function, and accurately sketch that surface. I used this subtask to understand students' models for the behavior of functions of one variable based on varying parameter values and how those models allowed them to imagine a surface being swept out in space (Figure 8, created by Patrick Thompson).

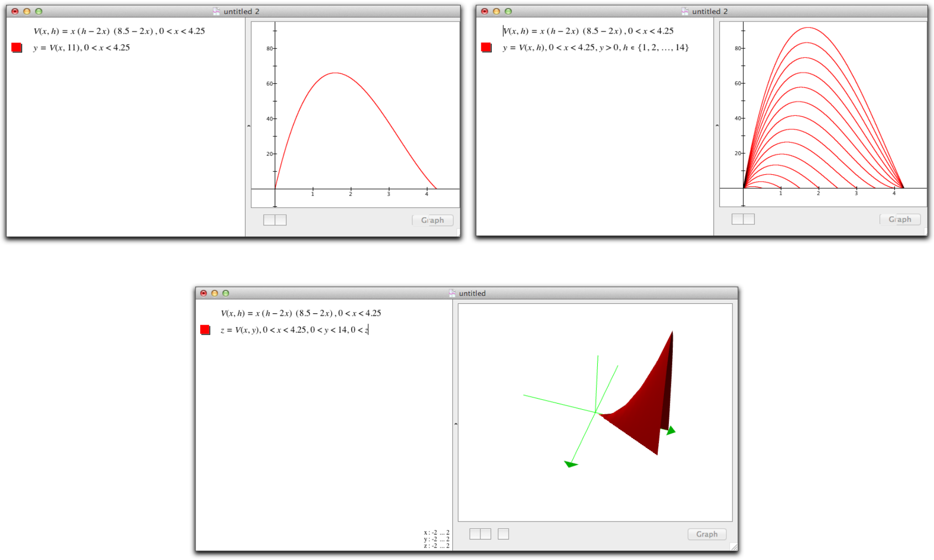

Figure 8. *Sweeping out in the Box problem situation.*

Length of the box as a variable.

In part two, I focused on understanding students' models for anticipating the effect of a parameter on the behavior of the function. Part 3 focused on helping students think about the parameter as a second variable, and having the students discuss what it would mean to think about the length of the cardboard as a variable instead of a parameter. Specifically, I used this task to gain insight about how they represent the cardboard length as a variable instead of a parameter, and to understand if the students used the same ways of thinking to account for *a* as a variable as they accounted for it as a parameter.

Table 10 *Task 8: Length of Box as a Variable.*
1) Suppose you define *a* as variable. What does it mean to define *a* in this way? 2) What do you see as the difference between a parameter and variable in how you are using it in the Box problem? 3) What distinguishes a parameter and a variable, if anything, in your understanding?

At this point in the teaching experiment, I had students reflect on the similarities and differences they saw in thinking about functions of one and two variables. I provided specific questions asking them to compare the Homer situation to the box problem situation, with particular attention to how they thought about generating the graph of each function. I used these questions to understand if the students were thinking about the graph in each situation as generated by a tracing out of a point in the Homer task and then a two dimensional graph in the box problem.

Rate of change of volume with respect to cutout length.

In the last part of the task I asked students to focus on rate of change of volume with respect to cutout length. I had them think about what the rate of change of volume with respect to cutout length represents when volume is a function only one variable, cutout length.

Table 11 *Task 9: Rate of Change in the Box Problem.*
1) Describe how the volume of the box changes with respect to the length of the cutout. 2) Construct a means to measure the rate of change, and interpret what a measurement of the rate of change means in this context. 3) Suppose the volume of the box is a function of two variables, base area of the box and cutout length of the box. Graph that function in Graphing Calculator. 4) Suppose we display a point on the surface of the graph. What does rate of change mean at that point? 5) How would you measure the rate of change at that point given the meaning you described?

Students had talked about rate of change extensively during their calculus course, but I used this opportunity to understand what meanings the students attributed to the phrase “rate of change”. I then focused on their meanings for rate of change of volume with respect to cutout length where volume is now a function of two variables, base area of the box and cutout length. I expected that the students would puzzle over what it means to find a rate of change. When the students constructed a graphical representation of the function in space, I constructed a point on the graph, and asked them to interpret the meaning of rate of change in the context of that graph at that point. I anticipated a key to this task for students would be revisiting the definitions the students constructed for constant, average, and instantaneous rate of change earlier in the teaching experiment, and thinking about how those definitions fit with or did not fully handle rate of change of a function of two variables.

I expected that students would immediately focus on the difficulty of “calculating” the rate of change, so I asked them to think about how/if rate of change has a different meaning in this context, and how one would represent that rate of change. I used this task to gain insight into students’ models for rate of change and how they imagined those models generalizing to account for a rate of change of a function with two independent variables. I expected that it would be key for me to discuss representing a rate of change given a specific relationship between the two input quantities, much like indicating a direction vector for the derivative. As a result of the exploratory teaching around this task, I intended the students think about the rate of change of a function in space as representing the rate of change in a specific direction in space. I intended that that this way of thinking result from understanding that considering rate of change in a direction allows the student to represent the rate of change of the dependent quantity with respect to two independent quantities simultaneously.

The Drivercost Function

Situation two followed a similar structure of tasks as the Box task to allow for generating and testing hypotheses about student thinking that I developed from the Box task. This set of tasks revisited, and then extended a problem developed by Patrick Thompson. The problem was related to optimizing total costs that the students had worked on in their calculus course during the semester.

Table 12

Task 10: Description of the Drivercost Function.

The CEO of a trucking company is considering instructions to give his drivers regarding how fast they should drive on long-haul trips. For budgeting purposes, the CEO needs to be able to anticipate a truck's travel costs. Define a function that gives how much the CEO pays a driver to drive 400 miles at a constant speed of v mi/hr assuming the driver earns \$25/hr.

This worksheet originally occurred during a portion of their calculus course where students were revisiting the idea of function as a model for a situation. Later in the semester, they worked on developing an understanding of rate of change and revisited this problem to interpret the meaning of rate of change in the total cost situation. They were provided a Drivercost function (Table 12), and a function r(x), which is the Drivercost function's average rate of change over an interval of driver speeds of length h. They were asked to answer the following questions,

1. Interpret $r(\mathrm{x}) = \dfrac{f(x+h)-f(x)}{h}$ (where h is fixed while x varies) in terms of the situation being modeled, the meaning of the primary function, and any value of h that you specify.
2. Give h a reasonable value for the situation being modeled.
3. Evaluate $r(\mathrm{x})$ with several different inputs. Interpret each output in relation to the input. Speak to trends and implications of what you observe for the situation being modeled.
4. Graph $\mathrm{y} = r(\mathrm{x})$ at an appropriate scale. Interpret the graph. That is, say what its point's represent and information that you can get from the graph.

Drivercost as a function of speed.

Table 13 *Task 11: Parameterizing the Drivercost Function.*
1) Construct a graph of the Drivercost function and anticipate the behavior of the graph of Drivercost function for Triplength parameter values 10, 100, 200, 400, 1000, and 5000. Do you anticipate any patterns in the behavior of the graph? 2) How is your thinking about this situation the same, and how is it different from thinking about length of the cardboard as a parameter in the Box problem?

The students had already determined in their homework that Drivercost is a function of speed (v), where $\mathrm{drivercost}(v) = 25(\frac{400}{v})$ where 25 is the driver cost per hour, and 400 is the trip length. I planned to have the students focus on how changing the value of Triplength affects the formula for Drivercost as a function of speed as well as the behavior of the graph as compared to the behavior of the graph when the Triplength was fixed at 400 miles. By asking them to reflect on the similarities and differences between the Drivercost function and the Box problem, I hoped to elicit responses that would provide insight to their thinking about how quantities' values affect the behavior of a function.

Drivercost as a function of Triplength and speed.

This task built on part one by having students think about the Triplength parameter as a second variable, and eliciting from students their ideas about how considering Triplength as a variable was the same or different from thinking about Triplength as a parameter. As with situation one, I used this subtask to understand if the students used the same ways of thinking to account for Triplength as a variable as they accounted for Triplength as a parameter. I also asked them to reflect on how thinking about Triplength as a parameter or variable was the same as or different from thinking about the length of the cardboard as a parameter or variable.

Rate of change of Drivercost.

At the time of this study, the students had spent extensive time constructing and thinking about the rate of change function in their calculus course. In this situation, I asked the students to refer to the function Drivercost, where $\text{drivercost}(v) = 25(400/v)$, and answer questions one and four from their previous worksheet. I anticipated that the students would have difficulty with these questions even though they were focused on heavily in class. I anticipated the difficulties would arise from a multitude of ways of thinking about the meaning of rate of change, and I used this question to gain insight into students' understanding of rate of change.

Table 14   *Task 12: Rate of Change of Drivercost and Triplength.*
1) Describe the meaning of the rate of change function in the driver cost situation, and describe how the graph of the rate of change function conveys that meaning.   2) Explore how changing the parameter values for Triplength affects the behavior of the rate of change function. What do you anticipate will happen? What do you notice about what happened, and did it fit with your conjectures? Why or why not?   3) Interpret the meaning of the rate of change function for the different parameter values of Triplength. Does the meaning change?

I wanted to understand if thinking about the behavior of the rate of change of the Drivercost function as they thought about the behavior of the Drivercost function for parameter values of Triplength did work for how they thought about rate of change. I attempted to gain insight into this question by asking the students to interpret the meaning of the rate of change function for the different parameter values of Triplength. In doing so, I wanted the students to consider how the meaning of rate of change of Drivercost stayed the same or changed as Triplength varied.

I wanted them to interpret the meaning of a point on the surface of this rate of change function after they had constructed the graph of the function in Graphing Calculator as a means to revisit the questions they answered in situation one about the meaning of a rate of change of a function that represented a surface in space.

Table 15 *Task 13: Rate of Change of Drivercost as Two-Variable Function.*
1) Using Graphing Calculator, graph Drivercost as a function of Triplength and speed. What does a point on the graph represent? 2) Identify a point on the graph of the function, and describe the meaning of rate of change at that point given the Drivercost context. 3) How would you measure the rate of change at this point? 4) Describe a more general process for measuring a rate of change at any point on the graph of the function?

In this subtask, I built on part three by having the students imagine the rate of change of Drivercost where Drivercost was a function of Triplength and speed. I wanted the students to think about how various parameter values for Triplength affected the behavior of the rate of change function for Drivercost, and used those explorations to think about tracing out the rate of change of driver cost where Triplength was now a variable. I anticipated that posing this question while revisiting ideas discussed in the Box problem would allow me to gain insight into the meaning they might make of rate of change of a two-variable function, as well as the ways of thinking they use to develop a rate of change function for a given function of two variables.

Specified Quantities to Using Variables Problems

This task intended to engender student thinking about functions that did not model a specified context. The reason for doing this was to understand how the ways of thinking the students developed in the Box problem and Drivercost function task could support or constrain them in their ability to reason about functions of two variables without an underlying context to give the variables meaning. I also anticipated using this situation to test hypotheses I developed about the students' ways of thinking in the Box problem and Drivercost function, and that moving to a non-contextual situation would allow me to experience their essential mistakes and further consider what sources might be leading to the mistakes they made.

Behavior of *t*(x,a) = a*h*(x).

Part one revisited the function *h*(x) from the pre-interview task based on Dugdale et al.'s (1992) work. I asked the students, without first specifying parameter values for *a*, to think about the behavior of the function *t*(x,a) = a*h*(x).[1] I did not ask them to think about the behavior of the function a*h*(x) for parameter values of *a*, as I wanted to see if thinking about *t*(x,a) as a sweeping out of *h*(x) over a second variable *a* was a useful, or sensible way of thinking for the students. Once they had constructed a graph and explained the behavior of the function *t*(x,a), I asked each student to explain how they thought about generating the graph, and if they thought about *a* in the same way as cardboard length and Triplength in situations one and two. I anticipated that asking these questions would allow me to understand if the students attributed common and coherent ways of thinking through the three situations.

Rate of change of t(x,a) with respect to changes in *a*.

In part two, I asked the students to determine a function that represented the rate of change of *t*(x,a) with respect to changes in *a* and a function that determined the rate of change of *t*(x,a) with respect to x. I did not pose intermediary ways of thinking as I did in situations one and two as I hoped to

[1] Other ways to symbolize what I called *t*(*x, a*) would be $t_a(x)$ or $t(x,y)\big|_{y=a}$.

understand how the ways of thinking I focused on in the first two situations were useful for the students given a situation without specified quantities. Once they had generated a rate of change function for $t(x,a)$ with respect to x or a, I asked them to explain their reasoning, and to compare those ways of thinking to how they approached the rate of change tasks in situations one and two. I anticipated that by asking a question requiring reflection on all three situations, I could elicit responses that revealed each student's way of thinking about rate of change of two-variable functions.

Post Teaching Experiment Interview Tasks

The post-interview tasks were given the day after a teaching experiment ended, and were structured as clinical interviews. I used these interviews as a way to further test my hypotheses about how students were thinking about functions of two variables and the rates of change of those functions. I designed the post-experiment tasks toward the end of a teaching experiment, and constructed the tasks to test the hypotheses I had developed after reflection on the teaching experiment as a whole. For example, Jori's post interview tasks were focused on her reflection about her understanding of rate of change. Jesse's focused on thinking about graphical representation of three-variable functions because the ways of thinking I believed the students reached in the teaching experiments were different. In each teaching experiment, the tasks remained the same, but small adjustments were made to the type of questions asked based on the previous teaching experiments. I describe these adjustments in the data chapters.

Analytical Method: Conceptual and Retrospective Analyses

The data corpus consisted of video and audio of each interview and teaching experiment session, recordings of my verbal and written reflections before and after each teaching session, periodic reflections by the students participating in the study, and written work of the students. This section focuses on how I used the data to systematically construct models of each student's mathematics. The construction relies partly on systematic collection of data and analytical procedures used to develop grounded theory (Strauss & Corbin, 1998).

The second use of conceptual analysis mentioned by Thompson (2008) is to propose models of student thinking that help to make distinctions in students' ways and means of operating, and to explain how these ways of and means of operating persisted or changed during the teaching experiment. Retrospective analysis consists of using the collected data to make a developmental analysis of students' mathematical constructions. This process involved viewing videotape and reading reflections to recognize an interaction as already having been experienced to make interpretations and hypotheses about students' thinking that I did not have in the moment of the teaching experiment. As I made interpretations and hypotheses about my students' understandings using retrospective analysis, I used conceptual analysis to continually generate and revise a scheme of meanings that would have made what the students said and did coherent for them. While this conceptual analysis initially focused on observations I made during the teaching experiment, the retrospective analysis allowed me to revise and expand on the ways of thinking I attributed to each student during the teaching sessions.

Using retrospective and conceptual analysis required developing a systematic coding scheme for the data. I used note-taking and basic transcription as a first level of data analysis to generate a coding scheme for the entire data corpus.

The first level of analysis was note taking, which occurred between each teaching experiment episode. I viewed all video and audio of the interviews and teaching sessions and created a narrative of the students' actions and responses from the study. The second level of analysis consisted of taking observational, methodological, and theoretical notes, which also occurred before and after each teaching episode during the three teaching experiments. Observational notes consisted of recordings like "this issue with graphing a function has happened before", "this way of drawing a graph might be important", or "this way of constructing a graph is something new for the student". Methodological notes consisted of observations about instructions given to the students, or issues that were important to the methodology and conduct of the study. Examples are, "I asked the question in a way that could have been confusing to the student", or "I assumed too readily that the student shared my understanding of the task. In subsequent sessions, I was alert to the possibility that we did not possess the same understanding." Theoretical notes contained conjectures that explained students' behavior and verbal descriptions. Examples are, "The student seems to be thinking about a function as connected points, which would explain why she said she needs to plot points to graph the function", and "The student seems to be thinking about the graph as a wire, which explains why he describes the graph shrinking and expanding without reference to quantities."

The note taking process and construction of a narrative facilitated my abstraction of potential ways of thinking that the students exhibited over the course of the teaching experiment. I documented these potential ways of thinking and utterances to suggest ways of thinking that made these behaviors and utterances sensible for the individual student. I looked for patterns in the theoretical and observational notes, which led to the development of a coding scheme for the video data.

I used StudioCode (Sportstec, 2009) to code videos and notes. The coding process centered on theorizing about ways of thinking that explained categories of student behavior by using a combination of open and axial coding (Strauss & Corbin, 1998). Open coding consisted of a process by which I analyzed the data collected in the interviews and teaching experiment sessions for instances of a student verbalizing or constructing a concept identified from the note taking process (e.g., graph of a function as a shape). Once I identified a major phenomenon (e.g. graph of function as a shape or an object), I used axial coding to identify the dimensions along which this category varies.

For example, one subject in my pilot study appeared to be thinking about a graph as a "U" shape. Axial coding explicated the properties involved in thinking about a graph as a shape. This involved systematically documenting what indicated one was thinking about a graph as a shape, as opposed to other ways of thinking. After identifying the major phenomena (e.g., graph as a shape; graph as covariation of quantities), I coded video data to reflect instances of the interviews and teaching sessions that reflect these phenomena. The coding of video data served two purposes. It was a way to create objective counts of various coded instances. But more importantly, the use of Studiocode to code video instances allowed me to generate videos that contain all instances falling under a particular code, which supported further conceptual analysis of ways of thinking that were allied with the behavior that the code marks.

Coding of the video used the major phenomena generated from open and axial coding to help me evaluate the usefulness of a particular explanatory or descriptive construct. For example, in my pilot study, I initially generated categories such as graph as covariation, graph as connected points, and graph as a shape. After I used these constructs as code buttons, I found that what I had referred to as graphs as connected points did not provide a great deal of explanatory or descriptive power for explaining how a student thought about graphing functions of one and two variables. If a particular construct did not do enough work for generating models of student thinking, I reflected on how thinking about the construct in a different way could provide more explanatory power for the ways of thinking I elicited from students during the teaching episodes.

Chapter Summary

This chapter explicated the method of collecting data and the theoretical framework for interpreting it to create models of students' mathematics. I explained how I used the theoretical constructs of a teaching experiment to study student thinking about functions of two-variables. I described a sequence of tasks that help me to generate and test hypotheses about students' understanding of functions and rate of change. I suggested how this sequence of tasks could support students developing a mature concept of two-variable functions as I outlined in chapter three. Finally, I demonstrated how my data collection and analytical method processes fit within the purview of my theoretical perspective, which I hope demonstrates an epistemological coherence of this study.

Chapter 5

JESSE'S THINKING

This chapter describes how Jesse, a high-achieving student enrolled in calculus one, thought about graphs of a function while viewing an animation of independent and dependent quantities varying in tandem. Jesse was able to imagine a point that swept out a graph of a one-variable function. He was also able to imagine how the graph of a one-variable function swept out a graph of a two-variable function. His attention to the axes seemed to allow him to think about graphs as a representation of quantitative relationships.

In Part I, I show how Jesse was able to visualize graphs of two-variable functions from various perspectives (z-x, z-y, y-x) by thinking about a variable as a parameter and focusing on the relationship between the two quantities' values that were represented on the axes in that perspective. In this way, he was able to anticipate, spatially, the behavior of a function from multiple perspectives and reflect on how its three-dimensional representation was generated by the relationships between quantities' values. My inferences Jesse's thinking suggest that his visualization abilities allowed him to reason about how a surface was generated, as well as to generate a three-variable graph from a two-variable function defined algebraically.

Part II focuses on Jesse's understanding of rate of change. His way of thinking centered on the meaning of rate of change as a ratio of changes in two quantities. He calculated the rate of change by using a measurement method called *the calculus triangle.* The calculus triangle is a mnemonic for thinking about measuring rate of change that Patrick Thompson has used for many years in his courses and a teaching experiment in 2006 and was recently expanded upon to characterize the images Thompson intended the calculus triangle to engender in students (Weber, Tallman, Byerley, & Thompson, in press). The central issues with which Jesse grappled were: (1) what it means for one quantity to change with respect to two other quantities, (2) how to systematically "combine" rates of change from each "perspective" in which he imagined using calculus triangles, and (3) deciding what quantity's change to make small. Jesse visualized simultaneous calculus triangles in the *z*-*x* and *z*-*y* perspective, which involved looking at a rate of change in a single direction by considering the *x*-*y* perspective and treating *z* as a parameter value. This way of thinking prepared him to think about a directional derivative by noticing that he could not add a "little bit" to each independent quantity independently without generating an infinite number of possibilities for the rate of change. Jesse believed it was necessary to add a little bit to one quantity, which he conjectured would automatically add a little bit to another quantity. His method required that he make only one change in a quantity small—the other quantity became small because it was dependent on the first. As a result, Jesse generated the rate of change functions for directional derivative and partial derivatives, and reached a point where he was able to consider rate of change in more than one direction.

Background

Jesse was high achieving in school, particularly in mathematics and science. He attributed his past success to the teachers he had, to his desire to do well in his courses, and his desire to understand the material so that he could teach it to other students in his job as a high school mathematics tutor. Jesse said that he generally understood the material well in all his courses and strove to make sense of how ideas in a course fit together. He expressed frustration that some of his mathematics courses in high school had been focused on doing calculations and learning algorithms without making sense of why the calculations and algorithms were necessary or logical. He enjoyed mathematics because he found solving problems both fun and worthwhile. He often went above and beyond what the class was doing to understand why things worked in both mathematics and science.

Jesse was enrolled in first semester calculus while he participated in the teaching experiment. I observed him in tutoring sessions throughout the semester making sense of ideas like rate of change. He asked questions during class about how mathematical concepts were related to other ideas from the course. When he was not satisfied with his understanding, Jesse worked individually and with tutors until he was satisfied.

Initial Inferences about Jesse's Ways of Thinking

I worked with Jesse ten times in tutoring sessions during the seven weeks prior to the teaching experiment. His calculus instructor focused on quantitative reasoning to support students' thinking about functions and their rates of change. During the three weeks prior to Jesse's teaching experiment I began taking detailed notes on our interactions during tutoring. I extracted a few general observations from these notes, which became working assumptions I intended to test systematically.

I noted that Jesse was attentive to describing relationships between quantities in applied problem contexts. He defined an algebraic function or created a graph to represent relationships between the quantities. Consistent with the focus of the calculus course, Jesse attended to quantities in the problem, their unit of measure, and on relationships between quantities' values. Whether working with an algebraically defined function or a function's graphical representation, he focused on describing how the input and output quantities were related. For example, in describing the behavior of the function $f(x) = 6x$, he said that the output is always six times as large as the input. I anticipated that these ways of thinking would support him in making progress toward thinking about the behavior of two-variable functions.

The teaching experiment with Jesse occurred over a period of three weeks. The activities were based on the conceptual analysis and tasks presented in chapters three and four. The table below shows the dates on which the activities occurred. My interactions with Jesse centered around two themes: the behavior of two-variable functions and how to think about how fast a two-variable function is changing. These topics did not arise sequentially in the teaching experiment, but for ease of description of my hypotheses, I present them separately.

Table 16

Task Sequence for Jesse's Teaching Experiment.

Task Description	**Date (Day)**
Homer: Correspondence Points and Graphs	3/13/2011 (1)
Homer: Discussing the Meaning of Speed	3/13/2011 (1)
Car A – Car B: Interpreting Meaning of the Graph	3/15/2011 (2)
Car A – Car B: Measuring Rate of Change Using the Graph	3/15/2011 (2)
Generating Graphs of $f(x)=x^3, g(x)=2x$	3/17/2011 (3)
Generating Graph of $h(x)=x^3-2x$	3/17/2011 (3)
Anticipating Behavior of $m(x)=a(x^3-2x)$	3/17/2011 (3)
Treating *a* as a Third Axis	3/17/2011 (3)
Box Problem: Box Construction and Treating Parameters as Variable	3/21/2011 (4) 3/23/2011 (5)
Box Problem: Interpreting the Meaning of Rate of Change at a Point in Space	3/23/2011 (5)
Box Problem: Measuring the Rate of Change at a Point in Space	3/23/2011 (5) 3/25/2011 (6)
Drivercost Problem: Parameters as Variables and Rate of Change	3/25/2011 (6)
Generalized Rate of Change Function and Direction	3/27/2011 (7) 3/29/2011 (8)
Generalized Functions: Construction and Interpretation of Graphs	4/1/2011 (9) 4/3/2011 (10)
Working with Strange Functions (More than 2 Variables)	4/4/2011 (11)

Part I: Jesse's Ways of Thinking about Two-Variable Functions

Part I describes the sequence of tasks and development and testing of my hypotheses about Jesse's thinking about the behavior of two-variable functions. Our discussions about behavior of two-variable functions and rate of change were not sequential, but I present them as such to provide more clarity about the development of my model for Jesse's thinking. Table 16 outlines the days around which I based the analyses and excerpts in Part I.

Day 1 – The Homer Activity

Based on my initial assessment of Jesse's thinking during tutoring hours, I predicted that, in the Homer situation, Jesse would construct Homer's distance from each of the cities as quantities and understand that as one distance varied the other distance varied simultaneously. I also predicted, based on his approach to thinking about graphs as generated by relationships between quantities, that understanding a function's graph as a sweeping out of a correspondence point would fit with his understanding of representing a function's behavior.

I introduced the Homer situation within Geometer's Sketchpad (Figure 9), and asked Jesse to explain what the diagram represented. He described how Homer was traveling along a straight road and he imagined that there were segments that measured Homer's distance from Springfield and Shelbyville. I asked Jesse to describe how he anticipated Homer's distance from each city would vary as Homer moved along the road to understand whether Jesse was attending to the distances varying simultaneously.

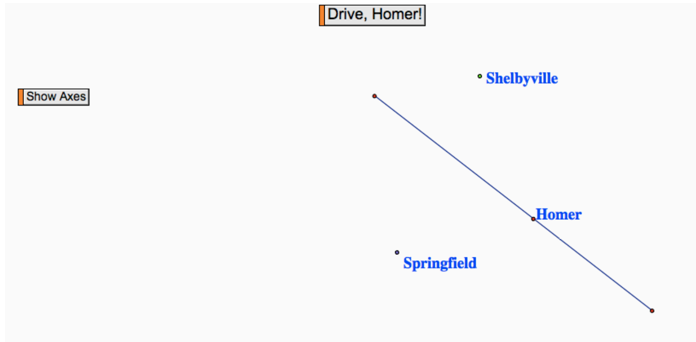

Figure 9. The Homer situation in Geometer's Sketchpad.

Excerpt 1 (Day 1 of TE, 00:00:25)

EW: As I animate Homer to move up and down the road, what will happen to the two distances you mentioned?

Jesse: He, mmm, goes this way [demonstrates Homer moving from the top end of the road to the bottom], and his distance from Springfield increases and then decreases as I move the animation.

EW: What happens to his distance from Shelbyville as he moves?

Jesse: Well, his distance from Shelbyville also increases and then decreases after he gets as close to Shelbyville as he can, which occurs when he is perpendicular to Shelbyville from the road. I'm not sure exactly how far he is from either of the cities, but one segment is longer than the other.

EW: When he is travels over this segment of the road, what is happening to his distance from Springfield?

Jesse: Well, when he is closest to one city, you can compare how close he is to another city by the length of the other segment. Even though you can't measure it, you can compare the lengths of the segments to tell which distance is greater. You sort of can always compare the distances just by eyeballing it, so at anytime, I can tell which city he is further away from.

When I animated Homer to move along the road, Jesse pointed to the screen as he described how Homer's distance from Springfield varied (Excerpt 1, lines 3-5). His description of Homer's distance increasing and decreasing precisely matched with the animation, and supported the idea that he was tracking a distance. Jesse indicated that he was tracking Homer's distance from a city when he discussed minimizing Homer's distance from Shelbyville (Excerpt 1, lines 7-9). Subsequently, he described that while he could not say how far Homer was from either of the cities exactly, he could imagine measuring those distances using an ordinal comparison of the lengths of the segments he was imagining (Excerpt 1, lines 9-11). This indicated to me he was thinking about these distances as simultaneously varying quantities. Jesse made it clear that he was imagining these quantities existing simultaneously when he said, "When he is closest to one city, you can compare how close he is to another city by the length of the other segment" (Excerpt 1, lines 14-15). His response suggested that he not only imagined measuring the quantities ordinally, but that he could imagine comparing the quantities' magnitudes wherever Homer was located on the road.

I asked Jesse to track Homer's distance from Shelbyville on the horizontal axis, and Homer's distance from Shelbyville on the vertical axis. I asked him to predict the behavior of those segments as Homer moved on the road because I believed that Jesse's success at imagining the segments varying simultaneously would indicate that he was thinking about the two distances' magnitudes varying simultaneously.

Excerpt 2 (Day 1 of TE, 00:00:36)

EW: As I animate Homer to move up and down the road, can you describe how his distance from each city is varying and use the horizontal axis for his distance from Shelbyville and vertical axis for Springfield?

Jesse: Sure, well, at the beginning [Homer at the top of the road] and he begins, then his distance from both cities is decreasing, until he reaches his minimum distance from, hmm, Shelbyville. After he gets there, his distance from Shelbyville becomes larger, and his distance from Springfield continues to get smaller, until he reaches his minimum distance from Springfield, then he hits the end of the road, turns around, and basically the opposite of what just occurred happens.

EW: It seemed when you were talking that you were concentrating really hard on the screen, what were you focusing on?

Jesse: Well, it's hard because as he is moving, both distances are always changing so it is tough to track both of them on the axes at the same time. It's like if I forget one of them, it makes it easier, but it's not correct because I have to track both distances always.

Jesse's response (Excerpt 2, lines 4-10) suggested he was thinking about Homer's distance from each of the cities varying in tandem (Excerpt 2, lines 4-10). His language in lines 5-6, where he described the distance from both cities decreasing, and line 8-9, where he described the distance from Shelbyville increasing as the distance from Springfield decreasing, supported that he was imagining the two quantities varying simultaneously. As he made those descriptions, he said that both distances were always changing, and described the necessity of tracking both distances always (Excerpt 2, lines 13-16). Jesse's description of each distance changing suggested he imagined them varying over intervals of conceptual time. His described need for tracking both distances simultaneously suggested to me that Jesse thought about the two distances varying over the same intervals of conceptual time and had conceived of them as covarying. It appeared that he had difficulty imagining how the quantities could not be coupled, even as he struggled to verbalize their covariation.

In the following minutes, I asked Jesse to represent Homer's distance from Springfield using a vertical segment, extending upward from the point on the *x*-axis that represented Homer's distance from Shelbyville. This was the basis for discussing the construction of the correspondence point that I introduced. The correspondence point is the point whose coordinates correspond to the two distances simultaneously. Jesse said that the correspondence point was created by "programming", where one "inputted" the distance from Shelbyville, which tells the point "how far to move to the right of the origin", and a distance from

Springfield, which tells the point “how far to move above the point which represents Homer from Shelbyville”. I asked him whether it mattered in what order the points were “programmed”. He noted that the correspondence point needed to remain above the point that represented Homer’s distance from Shelbyville, but in “reverse programming”, the point would “always be horizontal to the point of Homer’s distance from Springfield”.

I anticipated, in the moment, that Jesse would visualize the behavior of the point as dependent on two quantities’ values, Homer’s distance from Shelbyville and Homer’s distance from Springfield. I believed this would support him in thinking about the graph of a function as a tracing out of the “programmed” correspondence point.

Excerpt 3 (Day 1 of TE, 00:01:09)

EW: So, if we imagine this correspondence point, can you tell me how it behaves? [Correspondence point not displayed in GSP]

Jesse: Well, basically I just program the point so that it always represents the two distances. It knows how far to be horizontally by distance to Shelbyville and how far to be vertically by distance to Springfield.

EW: When I hit animate, I’d like you to imagine that the point [not shown in GSP] leaves behind a trail, and to talk about how you think that trail might appear.

Jesse: Well, it would leave behind a trail that sort of opens facing up, which makes sense with my description about that representing programming the two distances in at all times. The opening up and to the right comes from both distances increasing at the same time.

EW: As we think about graphs, like a function $f(x) = 2x$, how is that the same and different from this situation?

Jesse: Well basically the graph is made the same way. You program two values in, $2x$ and x, and track the point as x changes, which is kind of like making Homer move I guess.

Jesse's description suggested to me he thought about the behavior of the correspondence point as the result of programming the point to represent Homer's distance from each city (Excerpt 3, lines 3-5). I asked Jesse to imagine, without actually seeing a correspondence point, that the point would leave behind a trail to understand how he believed the correspondence point would behave. The trail he anticipated was similar to that which the animation would produce if GSP actually generated it (Figure 10).

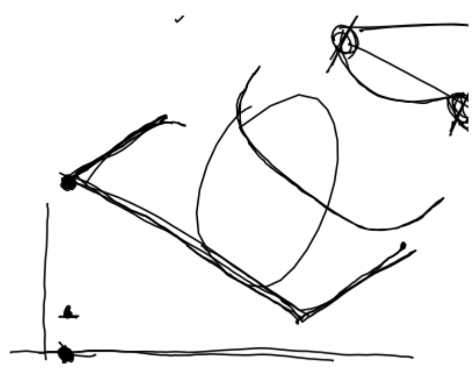

Figure 10. Jesse's anticipated trace of the correspondence point.

He described how the trail opened facing up (Excerpt 3, lines 11-12). While Jesse was describing an opening up he was also attending to how the correspondence point's programming made the shape sensible by describing the opening up as the result of both distances increasing simultaneously (Excerpt 3, lines 11-13). I was convinced that he was able to think about a graph as generated by a correspondence point, and that his idea of "programming" could extend beyond the Homer situation. I asked him to compare the Homer situation to a simple linear function (Excerpt 3, lines 16-17), and he described how he could program the correspondence point using x and $f(\mathrm{x})$ (Excerpt 3, lines 15-17).

Jesse's responses suggested to me, in the moment, that he was thinking about a graph of a function as generated by tracking and representing a relationship between quantities (in the case of the Homer situation) and a relationship between variables which represented the values of quantities (in the case of $f(\mathrm{x}) = 2x$). That he could imagine each situation entailing programming a correspondence point made these two situations similar to Jesse. This suggested to me that the correspondence point was a useful scheme for tracking quantities' magnitudes simultaneously. I believed that his way of thinking about graphs prior to this episode fit with thinking about a graph as a sweeping out of a correspondence point. These episodes also suggested that Jesse's *attention to the axes* and his attention to which quantities' values were represented on those axes supported his anticipation of the behavior of functions and supported his ability to reflect upon the graph he generated. His construction of a graph as an emergent representation resulting from tracking quantities' values supports this claim.

Following Day 1, I reflected about how Jesse was able to think about a function's graph as a sweeping out of a correspondence point. In this case, I intended that Jesse think about the correspondence point representing two functions of a single variable simultaneously, where each function represented Homer's distance from a city. Thus, Jesse appeared to be tracking $(x(t), y(t))$, a parametric function. I believed that this notion could be easily extended to thinking about $(x(t), y(t)) = (t, f(t))$ in the one-variable function case. I anticipated that in both cases Jesse would associate the graph as an emergent construction dependent on how the correspondence point was "programmed". I anticipated that thinking about programming a correspondence point while attending to the quantities' values that were represented on each axis would support him in describing the behavior of algebraically defined functions.

Day 3 – The Difference Function

Day 2 of the teaching experiment led to discussions about rate of change, which I discuss in Part II. At the beginning of Day 3, I asked Jesse to anticipate, through a verbal description of their covariation, the behavior of two functions, $f(x) = x^3$ and $g(x) = 2x$. Next, I asked him to represent this verbal covariation using a graph. He explained that when he thought about a graph of a function, he tried to recall the appearance of the graph in his graphing calculator. He explained he could think about programming a correspondence point if the function was not familiar to him. However, he said he would plot a bunch of points, and connect them by making the intervals of change in the independent quantity very small.

Jesse explained that the graphs of *f*(x) and *g*(x) could be produced by correspondence points that use (x, *f*(x)) and (x, *g*(x)) to "know how to behave". He described how those functions were relatively simple, but if they were more complex, he would need to think about the behavior of the function using smaller intervals of input to "understand what happens between the points that I decide to plot". He described how x and *f*(x) were similar to Homer's distance from Springfield and Shelbyville. In this excerpt, I asked him how he would think about the behavior of a "difference of the two functions".

Excerpt 4 (Day 3, 00:00:11)

EW: When you try to anticipate the behavior of the function $f(x) = x^3 - 2x$ what do you pay attention to?

Jesse: Well, I know about programming the correspondence point, which is still strange for me, but now I'm programming the difference between two other correspondence points, x^3 and $2x$ and the new point tracks x and the difference between those two values [Creates Figure 11].

EW: So what kind of graph do you anticipate would be produced?

Jesse: Let me think here [lengthy pause]. Well at first I thought it would be like adding the two functions, but it's the difference. I think it would basically behave like x cubed the entire way because that term is much bigger than $2x$. [Graphs the function]. Wow, well, I was somewhat right, but between here and here [-1 to 1 on the x-axis], it behaves more like $-2x$. I guess when x is really small, oh wait! When x is less than one, cubing it makes it smaller and smaller, so 2*x* dominates the cube term there.

EW: Good description! You seem to be thinking a lot about correspondence points. If we had a new function, which involved multiplying the difference function by a parameter *a*, how would you think about the graph of that function, would it affect the correspondence point?

Jesse: Well, the horizontal value stays the same, and the *a* would stretch the function vertically. Basically it would make the humps of the function stretch longer if *a* is positive and more than one. The correspondence point would just be, umm, just the vertical programming would change.

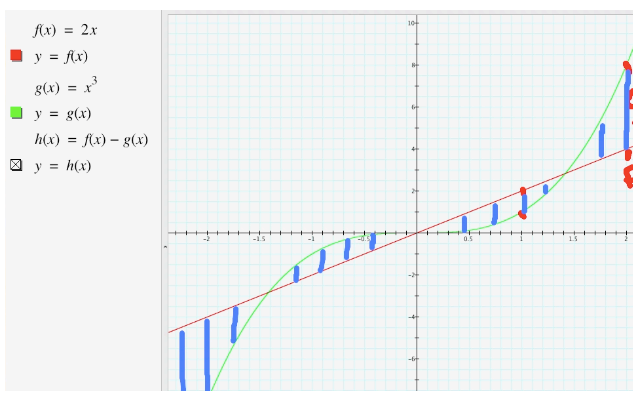

Figure 11. Jesse's diagram of how the values for $f(x) - g(x)$ were generated.

Jesse was not able to graph the function $f(x) - g(x)$ initially. As a result, he focused on how to program the correspondence point. He used the correspondence point to simultaneously tracking the difference "between two other correspondence points" (Excerpt 4, lines 3-6, see Figure 11). Jesse drew the vertical bars in Figure 11 using a screen marker program. I believed, in the moment, that Jesse was thinking about tracking the difference between two functions because he said, “the new point tracks x and the difference between those two values”, while simultaneously making the vertical bars in Figure 11. Jesse appeared to think about a correspondence point as a general way to track two quantities’ values. In this case, the length of the vertical bar represented one value.

Jesse was initially confused about the function’s graph because the shape he anticipated was not what he generated using Graphing Calculator (Excerpt 4, lines 8-12). However, his attention to how the correspondence point was programmed supported him in reasoning that the cubic term dominated the linear term everywhere except in the interval [-1,1] (Excerpt 4, lines 12-15). Jesse then thought about multiplying $f(x)-g(x)$ by a parameter, and reasoned it would only shift the correspondence point vertically. While he used the word “shift”, his gestures indicated he was thinking about a “stretch” of the graph (Excerpt 4, lines 22-24).

My interpretation at that moment was that he programmed the correspondence point with variables that represented unknown quantities’ values, such as x and f(x). He used this same way of thinking to generate the graph for a difference function f(x) – g(x), and anticipated the effect of multiplying the difference function by a parameter by thinking about its effect on programming the correspondence point. When Jesse was thinking about the behavior of a function, he appeared to focus on understanding how to simultaneously represent the covariation of two quantities' values using coordinate axes. At the same time, Jesse was not able to initially predict a graph of the difference function, which indicated his programming scheme was limited by a focus on comparing the numerical result of the cubic and linear term. His assumption that the cubic term would always dominate the linear term constrained him from imagining that a different relationship could be possible in the interval [-1,1] until he graphed the function in Graphing Calculator. Thus, even though he could imagine programming a correspondence point, his thinking limited him from a sustained image of two quantities varying, which I believed could constrain him from using the correspondence point as a scheme for generating graphs of functions.

Day 3 – Sweeping Out a One-Variable Function

During a ten minute break in the Day 3, I predicted that Jesse's way of thinking about a graph of a function as being generated by a correspondence point would fit with thinking about a one-variable function in a plane generating a surface in space. I planned to use his thinking about the parameter as representing the distance in front of or behind the laptop screen to support a way of thinking about a third axis along which a surface was swept out in space. After the break, Jesse and I discussed thinking of a as a parameter that represented a distance in front of or in back of the laptop screen on which the function $h(\mathrm{x}) = \mathrm{a}(f(\mathrm{x}) - g(\mathrm{x}))$ was displayed for a given value of a.

Excerpt 5 (Day 3 of TE, 00:00:38)

EW: So, like we talked about, let's say *a* represents the distance of the function in front of or in back of the laptop screen. If we get you this piece of paper [rips off strip of paper], can you anticipate how the function will behave if this strip of paper traces out its path?

Jesse: Well, like I said, if *a* is positive and getting larger, the function's humps stretch vertically and upward, and if *a* is negative and getting more negative, the function's humps stretch vertically but downward.

EW: Okay, and how do you think that would look using this piece of paper, what would that produce?

Jesse: I kind of think it would fill in a shape, like a windshield wiper sort of, and it would cloud over because the function is stretching vertically in both directions as the value of *a* changes.

EW: Could you say a little more about what you are imagining?

Jesse: Yeah, well, this function is going to sweep up and down, and after awhile, will fill in the window I am seeing (Figure 12).

EW: Okay, let's take a look at this animation, and treat *a* like another axis [animates a function to demonstrate sweeping out along a parameter value in three dimensions using Graphing Calculator; see Figure 13]

Jesse: Weird! It's like I have to consider depth, which I can sort of do with the strip of paper, but I've never thought about that before.

Jesse's windshield wiper metaphor (Figure 12), which he described as "filling in" the window, suggested he was thinking about the family of graphs moving within the *x*-*y* plane instead of thinking of it as a graph in space (Excerpt 5, lines 10-12). I suspected, in the moment, that Jesse had difficulty imagining a single variable function sweeping out along a third axis because he believed a sweeping out occurs within the plane. Given this assumption, I displayed an animation of $f(x) = \sin(ax)$ as the value of *a* varied, without displaying its definition, to demonstrate the type of visualization I intended to support (Figure 13). As he watched the animation, Jesse found it strange that he had to "consider depth" (Excerpt 5, lines 19-20).

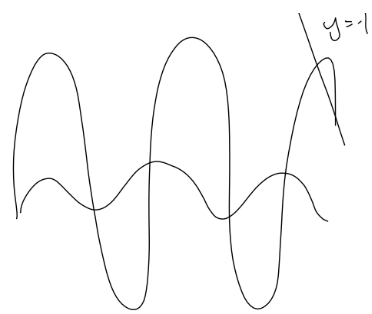

Figure 12. Jesse's windshield wiper effect of the parameter *a*.

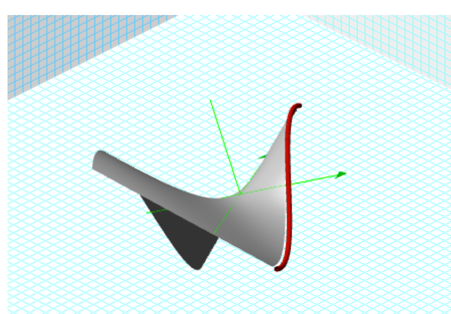

Figure 13. Using Graphing Calculator to support thinking with depth.

Jesse went on to say that thinking with depth was strange because he had never thought about a graph in space. He noted that using depth created new ways to look at functions represented in space because they could be viewed from multiple perspectives. In Excerpt 6, he described his meaning for thinking with depth and defined his meaning for “perspective”.

Excerpt 6 (Day 3 of TE, 00:01:05)

EW: We’ve talked a little about thinking with depth and it being important for thinking about how to generate the surfaces in space.

Jesse: Yeah, well, at first it was weird for me to think about that, but after I figured out what I needed to do, like treat something as a parameter, it wasn’t that hard. So with our first function, I treated the parameter like the extending axis. It’s kind of like you do two correspondence things, one is the point that makes the 2-D graph, the other is the graph as the correspondence thing, making the 3-D graph.

EW: You mentioned also you had another way of making the same 3-D surface, how?

Jesse: Well, I was looking at some things, and if we call the output z [term introduced by Jesse], we kind of created the z-x view before, where a was like the y-variable. We could look at the z-y view too, where x is the parameter and y is the variable. This would be like a z-y view. You can kind of look at 3-D things from a lot of ways, you can just change what you are calling a variable and what you are calling a parameter.

EW: That's really interesting. So could you apply this to our original function?

Jesse: Yeah, just make $x^3 - 2x$ the parameter and *a* into the *y*-variable. Then you will just have a linear function from the *z*-*y* view that moves to create the same surface. I know the function will give me, umm, the same surface, because in the *z*-*x* and *z*-*y*, I eventually treat both *x* and *y* as a variable in both views, but just make that view easier to think about by making one of them a parameter.

EW: So, in your mind, what's the difference between a variable and a parameter?

Jesse: A variable, you just let it represent all the values, but the parameter is a snapshot of values of the variable, like in class, which you can control.

Jesse's responses suggested to me, in the moment, that he noticed a similarity between sweeping out a point and sweeping out a function by sliding a plane through space. In the first case, a point swept out to generate the graph of a one-variable function, and in the second case, the graph of that one-variable function swept out along a third axis to generate the graph of a two-variable function (Excerpt 6, lines 3-8). The ease with which he designed a way to relate one and two variable functions' graphs being created suggested that he imagined them as created by a "generating object". The generating object for one-variable functions' graphs was a point and the generating object for a two-variable function was the graph of a function in a plane, where the plane is determined by the value of one of the variables and the associated graph generated by letting the other variable vary.

During the last twenty minutes of Day 3, Jesse used his notion of sweeping out to address what he called multiple perspectives. He referred to the z-x view, in which he treated *z* as a function of *x* that swept out along the y-axis where *y* is treated like a parameter. In the z-y view, he treated *z* as a function of *y* that swept out along the *x*-axis where *x* was treated like a parameter (Excerpt 6, lines 11-16). He visualized the function's graph being generated from multiple perspectives to explain how a cubic function graphed from the *z-x* perspective could sweep along the *y*-axis to generate the same surface as a linear function graphed from the *z-y* perspective and swept out along the *x*-axis (Excerpt 6, lines 19-21). He believed both processes would generate the same surface because "eventually I treat both *x* and *y* like a variable in both views". Jesse was beginning to conjecture that a function generated by different ways to sweep through a domain still generates equivalent surfaces.

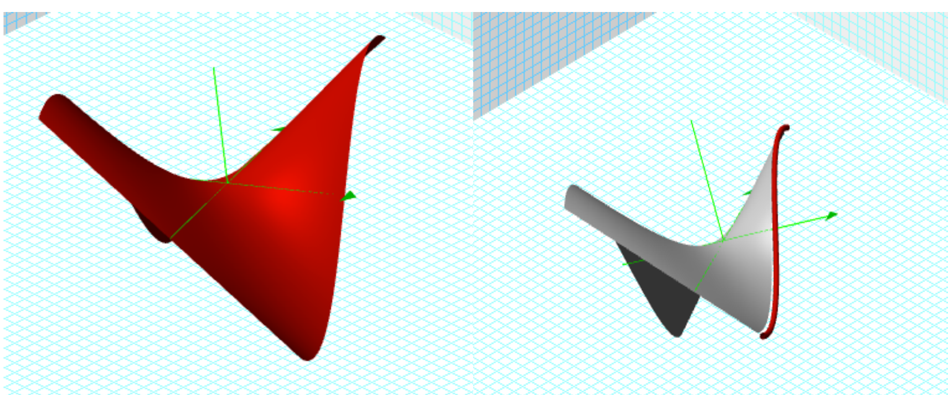

Figure 14. The student sees a surface in space as the result of a tracing out of a function in the plane.

My reflections on Day 3 indicated Jesse's way of thinking about generating a one-variable function's graph using a *generating point* supported him in thinking about generating a two-variable function's graph using a *generating* graph of a function in the plane (Figure 14). I had not anticipated his initial thinking that the family of graphs stayed in the x-y plane. However, I noticed that his method of sweeping a curve through space as he varied the value of one of the variables constituted a kind of "advanced shape thinking". I conjectured that his advanced shape thinking allowed him to imagine the function's graph was generated from a sweeping out along a third axis as a trail of the function's values restricted to a certain domain.

Days 4 and 5: The Box Problem

Days 4 and 5 were short sessions. I used them to evaluate my working hypotheses about Jesse's shape thinking and his use of perspectives. I anticipated that in the box problem he would continue thinking about functions' graphs from multiple perspectives. I also believed he would need to relate the quantities in the box problem using a functional relationship to imagine generating a surface in space.

Excerpt 7 (Day 4 of TE, 00:00:06)

EW: Suppose you are a box designer, and you must program a computer to
determine the volume of an open box, which has no top to it. This
open box is created from making cutouts from a rectangular sheet of

paper. Jesse, could you tell me how you would think about making such a box, as well as how to represent the volume of the box?

Jesse: Well, first, the formula is just length times width times height. So I would need to make the cutouts, I'll call the cutout the long way *e* [motions along the longer side of the sheet of paper] and the short way *f* [motions along the shorter side of the sheet of paper]. From what I am imagining, I would subtract the e from the length two times and the f from the width two times, and maybe the height of the box is half of *e* times *f*?

EW: Say a little more if you can.

Jesse: Well, somehow the cutouts have to make the height of the box. Wait! I guess *e* and *f* always need to be the same or the box would not be a real box, at least not like the problem wants it to be. The height would not be the same all around otherwise.

EW: How would you determine the height of the box?

Jesse: Well, by imagining folding up the box, it [pointing at the height on his diagram] would have to be the same as e and f because they are the same now. So the formula would be $(l-2e)(w-2e)(e)$ where *e* is my cutout.

EW: Formula for what?

Jesse: Volume of the box.

The problem statement provided values for the length and width of the original paper, but Jesse's use of an unknown length and width suggested he was attempting to solve the most general case. He believed that the area of the base and the height of the box determined the volume of the box. He dismissed his initial conjecture that he would have cutout lengths from both the length of the paper and the width of the paper (Excerpt 7, lines 6-12). He did this when he imagined that the height of the box would need to be uniform - necessitating equal size, square cutouts (Excerpt 7, lines 15-17). Jesse's anticipation of how the box was created from a rectangular sheet of paper allowed him to determine the volume of the box given the cutout length. He identified cutout length, width and height of the box, and volume of the box as quantities, and created an image of the situation in which all of the quantities' values were related (Excerpt 7, lines 19-22).

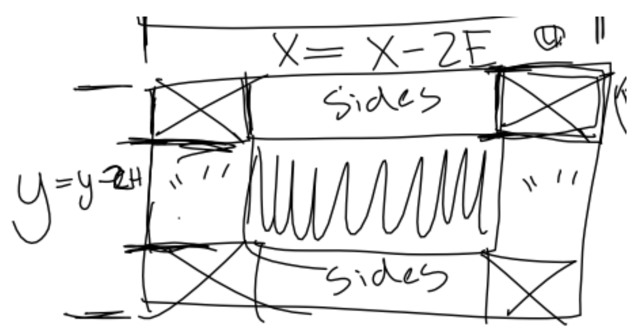

Figure 15. Jesse's sketch of the quantities in the box problem.

In Excerpt 8, I asked Jesse to treat the original length of the paper as a parameter with a function for the volume of the box, *V*. I anticipated that because he had defined a function for the volume of the box, he would see similarity in generating the graph of V(x) where original length was a parameter and the difference function $h(x) = a(f(x)-g(x))$.

Excerpt 8 (Day 4 of TE, 00:00:25)

EW: Now, we just considered a graph of volume of the box as a function of the cutout length made from the sheet of paper. If we made the length of the sheet of paper a parameter, and varied the parameter, how do you think that would affect the behavior of the function for the volume of the box?

Jesse: Well, hmm. I think that would affect only the output, or the volume of the box, so you would see the hump increase or decrease (Figure 16). Weird, I’m mostly right, but when the parameter for length [points to the length of the original sheet of paper] gets less than 12 when the width is 12, the zeroes of the graph change because the length constrains the volume as it is less than the width (Figure 17).

EW: I’d like you to think about this like we did in the last example. If we imagine the parameter sweeping out a surface, how do you think this function [for volume of the box] might behave?

Jesse: Well, again, I would treat the volume like the z-axis, the cutout like the *x*-axis and the like the *y*-axis. I could imagine the function from two ways, either treating *x* [the cutout length] like a parameter and the length of the original sheet like a variable or *y* like a parameter and *x* like a variable. Either way, I could sort of imagine the tracing out happening, but this function is a little weird so I would want to use GC to do that instead of just guessing.

EW: So, you talked about referring to these views as perspectives, what perspective would those be?

Jesse: Well, one would be the z-x and one would be the z-y, like the volume-cutout perspective and the volume-length perspective. In both cases, I'd guess you see humps that open downward, which probably comes from some interaction of all the variables.

Jesse anticipated that varying the length of the original paper would only affect the volume of the box (Excerpt 8, lines 6-8), which would make the hump (a feature of the graph he noted), move vertically (Excerpt 8, lines 7-8). I asked him to anticipate the behavior of the volume function where the length of the paper was represented on a third axis. He used his notion of perspectives to think about the graph. He focused on volume-cutout, which was like the z-x perspective, and volume-length, which was like the z-y perspective.

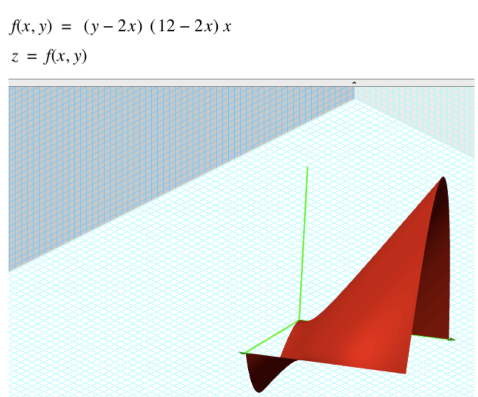

Figure 16. Jesse's graph of the two-variable box problem function.

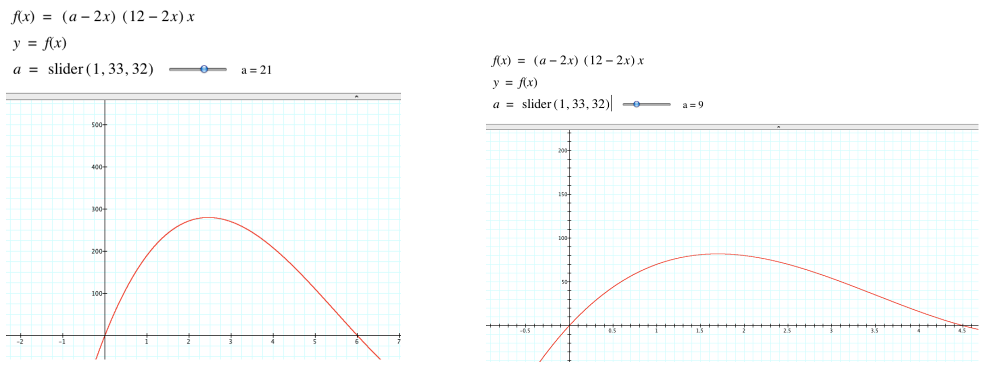

Figure 17. Jesse's screen as he described the zeroes changing.

Jesse visualized the function's graph in space from each perspective, but I believed he was not attending to generating the function by using level curves, or the x-y perspective. This resulted in his guessing at how the perspectives fit together (Excerpt 8, lines 26-27), which he referred to as an interaction of variables. By interaction, I believed he meant he was trying to anticipate how the z-x and z-y perspectives "fit" together as objects to create a surface in space.

In Excerpt 9, I explored my conjectures about his use of perspective by asking him to anticipate the graph of the volume function if the *y*-axis represented the area of the base of the box. I asked him to place a function of *x* on the *y* axis (area) to gain insight into how he would think about the effect of changing the quantity represented on an axis on the surface in space. I intended to promote a focus on the quantities represented on the axes as critical for interpreting the "shape" of a surface in space.

Excerpt 9 (Day 5 of TE, 00:00:04)

EW: You just described the graph of the function where the length of the original sheet of paper was represented on the third axis. Suppose that you treated the third axis as representing the area of the base of the box. How would that affect the graph of the function?

Jesse: Well, let's see, so I am still keeping volume as like the z variable, cutout length is the *x* variable, and now base area is the *y* variable. Well, isn't *z* just coming from multiplying *x* and *y*? Yeah, just multiplication I think. Okay, well, look at the z-x perspective, then *y* is a parameter, so we just sweep out a linear function, same thing from the z-y perspective. Sweep out two linear functions, and then sort of bring their behavior together.

EW: What do you mean, bring their behavior together?

Jesse: Well, if I look overhead, I have to think about how to combine the z-x and z-y perspectives, this is kind of like the x-y look.

EW: Can you think about the x-y perspective like you do the others?

Jesse: Mmm, process of elimination, *z* becomes the parameter then?

EW: Sure, let's work with that.

Jesse: Okay, well, then you are going to be sweeping out things like $y = 1/x$

Jesse focused on generating the function from both the z-x and z-y perspectives when thinking about the effect of changing the quantity represented on the y-axis. He accurately described a linear function tracing out a surface, from both the z-x and z-y perspectives (Excerpt 9, lines 5-11), which suggested to me, in the moment, that he was focusing on the surface as a representation of a relationship between quantities. I believed this because he was unsurprised by the difference in the surface (compared to when *y* was cardboard length) when he graphed it in GC. I was also surprised that he used level curves by treating *z* as a parameter value because he had focused on "eyeballing" how the z-x and z-y perspectives fit together up to that point (Excerpt 9, lines 13-14, 18).

Jesse's work on the box problem during Days 4 and 5 suggested to me that his use of perspective relied on his associating a graph with an image of how it was generated using covariation. Whenever he thought about a surface, he imagined that surface constructed by a one-variable function sweeping out in a plane. This image is similar to visualizing a vertical cross-section moving along an axis. In the case of the z-x perspective, he imagined thinking about the variable *y* as a parameter *a*, which he could move in increments along the third axis. This way of thinking allowed him not only to imagine a continuous sweeping out of cross-sections, but also helped him interpret what those cross sections represented. Thus, at that time, I believed an association of a surface in space with a process of covariation used to construct it could characterize expert shape thinking.

Days 9 and 10 - Non-Quantitative Setting

The first iteration of the teaching experiment sequence intermixed ideas of function with rate of change tasks. During Days 6, 7 and 8 we focused on discussing rate of change. This was due to the original task trajectory. We resumed talking about the graphs of two-variable functions on Days 9 and 10. I present days 9 and 10 here, and focus on Days 6, 7 and 8 in part two of this chapter.

Having introduced the x-y perspective in Day 5, I believed entering Day 9 that Jesse would combine the x-y, z-x and z-y perspectives to create an “accurate” representation of the function. I anticipated his use of perspectives to visualize the construction of a surface in space would allow him to 1) generate a surface in space given an algebraically defined two-variable function and 2) given a surface in space, identify an algebraically defined two variable function that the surface represented. I focused on these hypotheses with my questions in Days 9 and 10.

Excerpt 10 (Day 9 of TE, 00:00:09)

EW: So far, we have been working with functions that apply directly to situations, but like we do in class, a lot of functions don’t depend on a situation. Suppose we have the function $f(x,y)=x^2y^2$ could you talk about how you think about it?

Jesse: Sure, well it is not a lot different than how I think about some other functions, except its not like in the context of a box. I know by how it is defined that two variables vary in different directions, which makes a 3-D surface that I can think of. Going back to perspectives, hmm, let me calculate this here… I would look at the z-x and z-y perspective, and here you would have a parameter times y squared in z-x and a parameter times x squared in z-y, so you would see parabolas from each perspective that sweep out like we’ve been talking about.

EW: You mentioned earlier today that you aren't sure how to combine these views?

Jesse: Well, just looking at two perspectives isn't telling me exactly what the function looks like in space because I sort of have to guess at combining shapes. I don't know what to do overhead.

EW: When you say overhead, what do you mean?

Jesse: Well, my guess is to treat the output like a parameter value, then put y as a function of *x*, kind of like implicit functions that we talked about in class, that would make an x-y perspective.

EW: If you did that for this function, what do you think you would see from the x-y perspective?

Jesse: Well, it would be a function like $f(x) = a/x^2$, so if I graph that [graphs *f* using Graphing Calculator] would produce a diamond like thing that sweeps out. Now, that (Figure 18) fits better with the function I see when I graph the actual *f*(x,y) function.

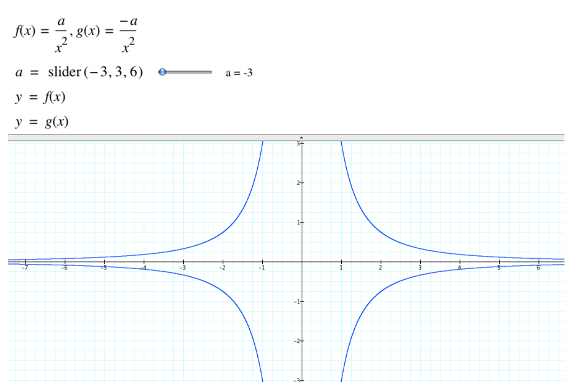

Figure 18. Jesse's graph of x-y cross sections projected in the plane.

Jesse found it natural to think about unspecified quantities as he had when the quantities were specified, as in the box problem (Excerpt 10, lines 5-7). He imagined generating the graph from the z-x and z-y perspectives, which would produce "parabolas from each perspective that sweep out" (Excerpt 10, lines 11-12). When he began imagining how to "combine" these perspectives, he described his difficulty with imagining what the graph would look like from "overhead". He relied on the level curves projected into the x-y plane. He defined the generating function $f(x) = a / x^2$, which he believed would sweep out a "diamond like thing" above the x-y plane, which appeared to fit with the surface he generated (Figure 19) in Graphing Calculator (Excerpt 10, lines 24-27). This excerpt suggests that Jesse was beginning to think about the graph of a two-variable function by generating it from what he would call "side" perspectives (z-x and z-y), and an "overhead" perspective (x-y), which is like a level curve projected in the x-y plane. In the moment, I wrote that Jesse appeared to think about the x-y perspective as a way to generate a more accurate graph because he could "check" how the z-x and z-y perspectives interacted. I understood him to mean that he was not guessing at how "parabolas" from z-x and z-y perspectives combined. Instead, he described the interaction of those parabolas from each perspective by projecting x-y cross sections in the plane.

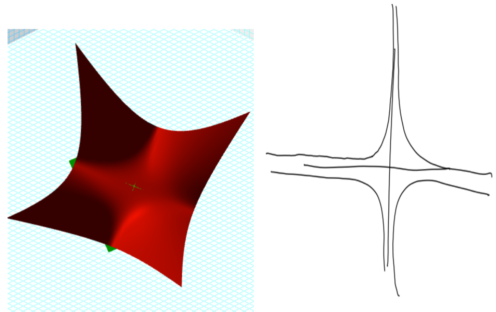

Figure 19. The “diamond like thing” Jesse predicted and observed in GC.

Day 10: Algebraic Definitions from 3-D Graphs

On Day 10, I graphed a two-variable function without showing Jesse its algebraic definition. I wanted to understand if his thinking about perspectives, now including x-y, could help him propose an algebraically defined function that “fit” with the graph I showed to Jesse.

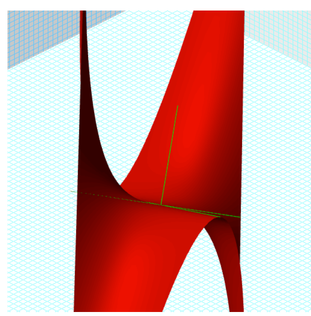

Figure 20. The surface of the function as displayed to Jesse.

Excerpt 11 (Day 10 of TE, 00:00:45)

EW: Suppose we are considering this function that I have graphed in GC. How might you go about figuring out the actual definition of the function? [Reveals Figure 20].

Jesse: Hmm, this is pretty hard, more thinking backwards type of stuff. I guess I would have to decide which perspective I am looking at. So if this is z-y, it looks like it could have been generated by a moving parabola. Then in z-x, well, it looks linear in some places, so maybe produced by a moving linear function. But, wait, on top the function looks a little strange, so I would have to do some more exploring. With all of these, I would have to be able to look at points and check they fit with a parabolic function, or linear I think.

EW: So, based on what you have said, how might you define the function?

Jesse: My best guess is like ay^2 from z-y and bx from z-x, so a parameter times xy^2

Jesse defined the z-y plane to fix an orientation on the graph (Excerpt 11, line 6), and conjectured that perspective was generated from a moving (or sweeping out) parabola. Once he defined the z-y plane, this created the z-x plane, which he believed was produced by a moving (or sweeping out) linear function. While Jesse attended to the shapes associated with the graph in each perspective (Excerpt 11, lines 7-10), he held in mind that the graph actually represented a linear or quadratic relationship between quantities. I believed that Jesse was able to describe the algebraic form of the two-variable function $f(x,y)=xy^2$ (Excerpt 11, lines 13-14) because he was thinking about one-variable functions in the plane generating the surface from the z-x and z-y perspective.

Part I Conclusions

I began the teaching experiment with a naïve notion of importance of shape thinking. However, my analyses revealed that expert shape thinking was at the heart of my model for Jesse's ways of thinking about 2-variable functions. In this section, I describe the foundations of my model for his thinking that resulted from both my intra teaching experiment and retrospective analyses.

Expert shape thinking.

It is important to note that shape thinking is an association the student has when thinking about a graph. Thus, shape thinking as a construct is not "good" or "bad". Expert shape thinking consists of associating a function's graph as a emergent representation of quantities having covaried, where those quantities' values are represented on the axes in Cartesian coordinates. When Jesse, an expert shape thinker, attended to the shape of a graph, such as describing a "hump" or a "valley", he appeared to associate those features as the result of tracking a multiplicative object composed of two or more quantities' values. His understanding that the graph was created by the quantities' values represented on the axes displayed a critical component to expert shape thinking, attention to the axes. By focusing on the axes and variables' variation on them, Jesse was able to imagine how the graph of a given two-variable function had been generated using a coupling of the quantities in the situation.

Generating the domain of the function.

My analyses suggested that Jesse's expert shape thinking supported his development of visualizing graphs of one-variable functions having been swept out (producing a completed graph) and then those completed graphs being swept out to produce surfaces in space from various "perspectives". When Jesse referred to thinking about a perspective, he imagined a sweeping out occurring. By sweeping out, I mean that he imagined a two-variable function as a one-variable function by treating one of the two variables as a parameter and imagined that parameter iterating through a set of values, which produced the sweeping out. Jesse's understanding of a function in the plane sweeping out a surface in space suggested he was using expert shape thinking. Jesse saw the sweeping out as the tracing out of a relationship between variables representing quantities' values. He described how he thought about both one and two-variable functions' graphs as the result of programming "a generating object". Jesse's meaning of "programming" seemed to be that he envisioned putting into his imagined object the power to covary its coordinates on its own. By "programming" the object, Jesse seemed to be saying, "and I can imagine this object behaving like it should without me having to think of what it should do". Programming the object thus resulted from attending directly to the quantities values represented on the axes, which is a component of expert shape thinking.

This instructional sequence used functions whose graphs can be envisioned by the sweeping out of a plane that contains a graph. This way of thinking, in effect, generates the *x-y* plane by sweeping a line through it parallel to one of the axes. But there are many ways of generating the domain. Another way is to imagine pre-images of level curves—the set of points (x,y) in the *x-y* plane that map into $z = c$. For example, the graph of $f(x,y) = \cos(x^2 + 3y^2)$ is most easily imagined by thinking of generating the *x-y* plane by varying c in

$x^2 + 3y^2 = c,\ c \geq 0$. Jesse constructed the domain of the function $f(x,y) = x^2 y^2$ by considering pre-images of level curves. However, my analyses revealed that Jesse's way of thinking about functions' surfaces being generated by a sweeping of a plane that contained a graph was productive for the types of functions in this instructional sequence.

Part II: Jesse's Ways of Thinking about Rate of Change

Part two of this chapter focuses on Jesse's ways of thinking about rate of change in space. Table 17 provides an outline of tasks and dates, on which I base my analyses in Part II. I drew a number of excerpts for Parts I and II from the same day, but the time demarcations show when each discussion occurred.

Table 17

Jesse's Rate of Change Task Sequence.

Task Description	Date (Day)
Car A – Car B: Interpreting Meaning of the Graph	3/15/2011 (2)
Car A – Car B: Measuring Rate of Change Using the Graph	3/15/2011 (2)
Treating *a* as a Third Axis	3/17/2011 (3)
Box Problem: Box Construction and Treating Parameters as Variable	3/21/2011 (4)
Box Problem: Interpreting the Meaning of Rate of Change at a Point in Space	3/23/2011 (5)
Box Problem: Measuring the Rate of Change at a Point in Space	3/23/2011 (5) 3/25/2011 (6)
Drivercost Problem: Parameters as Variables and Rate of Change	3/25/2011 (6)
Generalized Rate of Change Function and Direction	3/27/2011 (7) 3/29/2011 (8)

My interactions with Jesse in tutoring suggested that he thought about rate of change as tracking the value of a quotient that compared an change in a function's input to the change in the function's output. During an exam review, he said originally he thought rate of change as another way to talk about the slope of a function, and did not understand why the instructor focused on talking about rate of change as more than just the slope of a function. He said he had an "ah-ha" moment when he realized rate of change was a function that tracked how "fast one quantity was changing with respect to the other". Jesse appeared to think about rate of change as the multiplicative comparison of changes in the input and output of the function. I anticipated that this way of thinking could be naturally extended to think about a rate of change in space.

Key Terms and Definitions

This teaching experiment focused on rate of change, but used terms with which the reader may not be familiar. In the following excerpts, Jesse describes functions expressed in *open* and *closed form*. A function is expressed in closed form when it is defined succinctly in terms of familiar functions, or algebraic operations on familiar functions. The function *f* defined as $f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}$ is in open form, whereas *f* defined as $f(x)=e^x$ is in closed form. You will see Jesse referring to rate of change functions in open and closed form. The function r_f defined as $r_f(x)=\frac{f(x+h)-f(x)}{h}$, where *x* varies and *h* is fixed, is in open form. This function, first used by Tall (1986), gives the average rate of change of *f* over the interval $[x, x+h]$ for all values of x in f's domain. r_f approximates the derivative of f for suitably small values of h. This simultaneous variation of x and r_f is illustrated by the *sliding calculus triangle*, to which Jesse referred to numerous times in our sessions (Figure 21). It must be noted that the calculus triangle is a mnemonic for thinking about measuring rate of change that Patrick Thompson has used for many years in both teaching and research. Though we recently submitted a paper about calculus triangles (Weber et al., in press) to characterize the images Thompson intended the calculus triangle to engender in students, Thompson originally conceived of the term *calculus triangle* and its associated images.

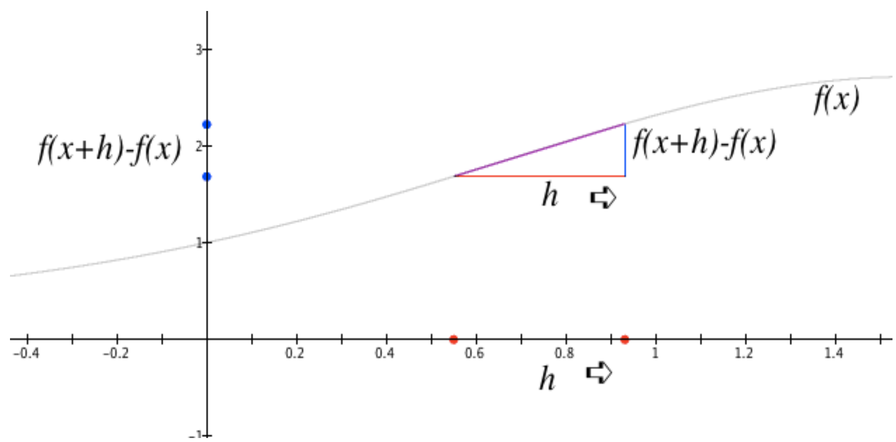

Figure 21. The calculus triangle slides through the domain of the function (from Weber, Tallman, Byerley, and Thompson, in press).

In order to generate outputs for the average rate of change function, one measures the quantity $f(x+h) - f(x)$ in units of h and systematically associates this output value with x, the left endpoint of the interval $[x, x + h]$. Accordingly, a point on the rate of change function can be interpreted as $(x, f(x+h) - f(x)$ units of

h) (see Figure 21). Then, as x varies throughout the domain of the function, this point traces out the rate of change function $r_f(x)$ (Figure 22).

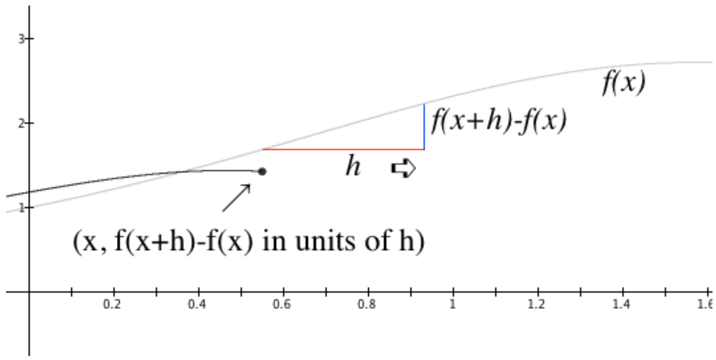

Figure 22. The sliding calculus triangle generates the average rate of change function over intervals of size h.

Lastly, when Jesse referred to the *closed form rate of change*, he was describing the "exact" rate of change function in which h is indistinguishable from zero. Closed form rate of change often resulted from using "rules of derivatives" or "going through the open form calculations and making h essentially zero", according to Jesse.

Day 2 - Car A-Car B Activity

The episodes focusing on Jesse's ways of thinking about rate of change were intermixed with tasks related to graphs of function. The Car A-Car B activity (described in chapter three, Figure 23) occurred on Day 2. I anticipated that Jesse would focus on a correspondence point that was "programmed" using the car's speed and the amount of time elapsed. I conjectured that Jesse would focus on tracking the quotient of the change in the car's speed over a certain interval of time, and would repeat that process over equivalent intervals of elapsed time to describe the car's acceleration as increasing or decreasing. Also, given Jesse's focus on using open form rate of change, I thought that he would describe that a more accurate rate of change of the car's speed with respect to time could be generated by making *h* very small.

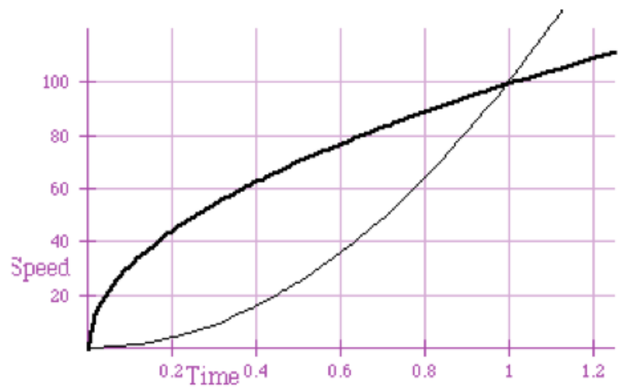

Figure 23. The graphs of Car A and Car B's speed as a function of time.

Excerpt 12 (Day 2 of TE, 00:00:07)

EW: So if we look at the graph of Car A and Car B (Figure 23), where do you imagine they are after one hour in relation to each other?

Jesse: They look like they are at the same location. The lines seem to intersect there. Hmm, we have speed, and time, oh duh, stupid, it's speed and time not distance so the top car is further than the other one.

EW: So Car A is further along than Car B you think. What can you say about their rates of change?

Jesse: Well, this is a rate of change graph because we are talking about speed. So if you zoomed in like we did in class, you would probably see a bunch of connected linear segments representing average speed over small intervals.

EW: What would it mean if they were linear segments, as you said?
Jesse: It would be a bunch of constant speeds, coming from average speed, over an interval of h, which is really small probably. In theory, we wouldn't see the segments if h were zero, but in that case, we would not have a graph.

Jesse first interpreted the graph as a distance-time graph, and hence believed the cars were at the same location because the graphs crossed after one hour (Excerpt 12, line 4). He caught his error when he reread the labels on the axes, and noted one car must be in front of the other after one hour (Excerpt 12, lines 4-5). I inferred that his realization resulted from thinking that the two cars started at the same place and, after starting, one car always went faster than the other so they could not be in the same location.

Jesse described the graph as tracking average speed (Excerpt 12, line 10). He anticipated zooming in on the graph to see connected linear segments representing the average speed of the car over small time intervals (Excerpt 12, lines 9-11). Jesse's response suggested that he was thinking about speed as a rate, which itself was created by comparing a change in distance with a change in time. As a result of this comparison of changes, Jesse emphasized the graph *must* be linear segments because speed could not be measured without constructing average speeds of the car over small intervals of time. In order for the graph to be made of average speeds over small intervals, the graph must represent a step function. The linear segments would come from the distance-time graph and graphing average speed over those intervals.

In Excerpt 13, Jesse described measuring the rate of change of speed with respect to time. He focused on using a "consistent *h*" (i.e. fixed change in *x*) to measure the rate of change in the domain of the given function. He imagined using the open form rate of change with a small, but fixed value of *h* to approximate the rate of change function, because making *h* zero in the context of a graph was impossible (Excerpt 12, line 16). I believed that Jesse's description of a consistent *h* allowed him to compare differences in the change in the output. I believed, in the moment, that using a consistent *h* was much like using the calculus triangle to think about the rate of change function. In order to test these conjectures, I asked Jesse to compare the rates of change of Car A and Car B.

Excerpt 13 (Day 2 of TE, 00:00:28)

EW: Earlier I was hoping we could compare the rates of change of the cars.

Jesse: Well, the rate of change of the information here would be the acceleration of the cars, not the speed, but you would just give h a number, probably like .01, then starting from zero seconds, take the difference in speed every .01 seconds and divide that by .01. It's pretty much the open form rate of change we have talked a lot about in class, sort of like the calculus triangle where you use a bunch of them.

EW: Could you say more about the calculus triangle and the open form?

Jesse: Well, from what I understand, you basically fix a slider value of *h*, then take successive differences. In the calculus triangle, you fix the length of the bottom leg, then let it slide through whatever you are trying to find the rate of change of, or for I guess. I'm more comfortable with using the open form rate of change though.

EW: So if the open form rate of change produced a value of 3.45 for an *h*-value of .01, what would that mean?

Jesse: Well, the constant rate of change of speed would be 3.45 over that small interval, so every time you add something to *x*, the y value goes up 3.45 times that much, theoretically.

EW: Theoretically?

Jesse: Well, it doesn't actually change constantly usually, it's just an average rate of change that is like a constant rate when calculated.

Jesse anticipated he would need to fix the change in time to compare the rate of change of the each car's speed with respect to time (Excerpt 13, lines 3-4), which would allow him to compare the difference in speed every time .01 seconds elapsed (Excerpt 13, lines 4-5). He believed fixing *h* was related to calculus triangles and the open form rate of change function (Excerpt 13, lines 5-6). His description of using a bunch of them (Excerpt 13, line 7) and letting the calculus triangle slide (Excerpt 13, lines 10-12) support that he was thinking about many, small, calculus triangles existing in the domain of the function (Figure 24).

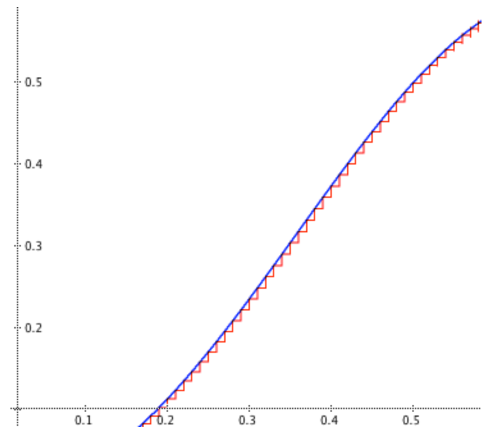

Figure 24. There is a calculus triangle at *every* point on a function's graph.

Jesse said he was more comfortable with the open form rate of change (Excerpt 13, lines 12-13). Our interactions suggested he was more comfortable with the "meaning" of the open form rate of change. When the open form rate of change returned a value of 3.45, with an h of .01, he demonstrated he understood that around that small interval of the function's domain, the change in the output is 3.45 times as large as the change in input (Excerpt 13, lines 16-18).

Jesse's responses in the Car A – Car B activity suggested to me that his way of thinking about rate of change focused on measuring and comparing differences in a dependent quantity's value by fixing incremental changes in the independent quantity's value. While he said he was "more comfortable" with the open form to track the value of the rate of change, he was attending to changes in quantity's values in such a way that he was using a calculus triangle. By thinking about a very small value of h, and the calculus triangle sliding through the domain of the function, Jesse attended to the rate of change as a function that measured how fast one quantity was changing with respect to another quantity at every point in the domain of the function, and he appeared to do so multiplicatively. His responses in Excerpt 13 (lines 20-21) supported that he was thinking about open rate of change and calculus triangles as measuring average rate of change. His interpretation of the graph of the Car A- Car B graph also supported that as he thought about average rate of change as a constant rate of change.

Day 3 - Sweeping Out a One-Variable Function

The Car A-Car B (Day 2) episode occurred prior to generating the surface of a two-variable function (Day 3). I used Day 3 to test two conjectures: 1) I believed Jesse was thinking about open form rate of change as more “reliable” because it used the original function in its calculations, 2) I anticipated it would be critical for Jesse to think about f(x,y) changing with respect to both x and y, so that he could imagine partial rates of change using calculus triangles, which could support developing directional rate of change. After we discussed the sweeping out of a one-variable function, I asked Jesse to describe how he would think about the rate of change of the difference function $f(x) = x^3 - 2x$.

Excerpt 14 (Day 3 of TE, 00:00:22)

Jesse: If I go back to the function we just talked about sweeping out and thinking with depth. I could use the open form on it, with a small value of h which would graph a rate of change with linear segments, or I could calculate the closed form rate of change, three x squared minus two, using rules.

EW: Could you say more about open and closed form?

Jesse: Well, open form is more directly tied to the function because it depends on the original function every time you calculate a value, but closed form is like an open form for when *h* is zero, which it can't actually be. Like I said, umm, closed, no, open form has linear segments, but the closed form is exact, it won't have those linear segments because it doesn't use an average.

EW: So why use both?

Jesse: You don't always know the rule to calculate the rate of change function, or sometimes its just impossible. If you make *h* small enough, it gets super close to the closed form anyway, so it's a good approximation. The closed form doesn't really mean the same thing as open form, which stands for a constant rate. Closed form kind of just is, it just is the value, not an approximation.

Jesse described the rate of change of *h*(x) by using open or closed form rate of change. He believed that open form rate of change was directly constructed from the function itself, but could not measure an "exact" rate, while the closed form rate of change represented the value of the open form if *h* was indistinguishable from zero (Excerpt 14, lines 7-12). Jesse appeared to think that the open form change was applicable to all functions with which he was familiar, while closed form rate of change was sometimes intractable because a "rule" was not available or the algebra was too complex (Excerpt 14, lines 14-15) to calculate a rate. Jesse's responses indicated to me, in the moment, that he was thinking about the values of open and closed form rates of change as different when *h* is large, but equivalent when *h* becomes extremely small. However, he thought about the meaning of open and closed form rate of change as different because closed form was exact, while open form was an average rate of change over an interval of the function's domain.

Minutes after this discussion, I asked Jesse to describe what it would mean to describe a rate of change of a function represented in space (as we had just generated a function's graph in space using graphing calculator). I wanted to understand how my model for Jesse's ways of thinking might extend to the notion of rate of change at a point, or over an interval, in space. This excerpt comes from the end of Day 3.

Excerpt 15 (Day 3 of TE, 00:00:55)

EW: So you just worked on starting to think about functions using depth, and created some ideas of how to think about surfaces in space. If you look at the surface in space we just generated, how do you think rate of change is the same or different from a function of one variable?

Jesse: Well it's actually pretty strange, because now you have to account for an output and two variables, so it's like an output is changing with respect to two things.

EW: Suppose we had defined a two variable rate of change function, whatever that might be, and its output was 1.23. What would that 1.23 mean?

Jesse: Oh man, well, it would probably be a constant rate of *z* with respect to *x* and *y*, which is pretty weird, because how can it be changing with respect to both variables at once? My first guess is that we are going to have another open form rate of change function. I guess, maybe, you have two rates of change, one from z-x and one from z-y [referring to perspectives], and you add them together in some way that might make sense.

Jesse's was thinking about the dependent variable changing with respect to two independent variables simultaneously (Excerpt 15, lines 5-7). I believed, in the moment, that he was struggling with what a rate of change in space would mean. In response, I asked him to interpret the value of a theoretical rate of change function in space (Excerpt 15, lines 5-7). His description of a constant rate of *z* with respect to *x* and *y* (Excerpt 15, lines 11-13) appeared to be consistent with his meaning for rate of change of a one-variable function. His reliance on what he knew about rate of change in one variable helped him describe two rates of change occurring from the z-x and z-y perspectives, notions which he had just started developing (Excerpt 15, lines 15-17). Jesse was struggling to think about how a number could represent how fast a quantity was changing with respect to two different independent variables. Yet his description of two rates of change occurring simultaneously in different planes suggested that thinking about partial rates of change, and thus, rate of change in a direction, could be tenable.

Between Day 3 and 4: Jesse's Reflections

Minutes after our Day 3 discussion ended, Jesse said that he had some ideas about defining a two-variable open form rate of change function. I encouraged him to submit his responses to me prior to our next session, Day 4. I asked him to consider the following three questions for each of his "ideas".

a) Why do you think this function is a possibility for the rate of change function?

b) Suppose that the result of the calculation in your conjecture was 243. What would 243 represent?

c) Are there any issues with your conjecture that make you question its accuracy?

The following is what Jesse sent by email, prior to our meeting on Day 4.

The following descriptions were written by Jesse. He produced four "conjectures", and answered the three questions for each conjecture. I discuss my interpretation of each of his responses, and how I planned to use them in Day 4 of the teaching experiment.

1) $r_V(x,y) = \dfrac{V(x+h, y+h) - V(x,y)}{h}$

a) The reason I think this could be a possibility for graphing the 3 dimensional representation of *V*(x,y) is because when we want to graph the rate of change of a 2 dimensional function the form is $[V(x+h) - V(x)]/h$ because you want to see the change in *V*(x) over the interval of *x* to *x*+*h* over a unit of *h*. However when we are representing the rate of change in 3 dimensions the reason this function could be a possibility is because we're not only looking at the change that happens to *V*(x,y) from *x* to *x*+*h* but also the change that is happening to *V*(x,y) from *y* to *y*+*h* Another way of expressing why I think we should try to account for the change in y over an interval of *y* to *y*+*h* is because there is change happening not only the x-z axis but on the y-z axis.

b) If the answer to this calculation was 243 it would be saying that over an interval of *x* to *x*+*h* and an interval of *y* to *y*+*h* there would be an average rate of change of 243. So since we're talking about volume in a box it would mean that if an imaginary box increased its volume by 243 from *x* to *x*+*h* and *y* to *y*+*h* that it would account for the same amount of change over the given interval.

c) The thing about this function that I'm not sure about is trying to also take into account the change in *y* over the interval of h and the affects it has on *V*(x,y) , I feel that I need to represent this change in *y* on the graph but I'm not sure that this is the way to do so.

I believed Jesse's first conjecture indicated he was thinking about comparing a change in the dependent variable that resulted from adding *h* to both *x* and *y*. However, it was not clear what he imagined *h* to represent, and did not explain why he anticipated adding the same value of *h* to *x* and *y*.

2. $r_V(x,y) = \dfrac{V(x+h,y) - V(x,y)}{h}$

a) The reason I think this function is a possibility is because maybe in accounting for the rate of change over the interval of *x* to *x*+*h* and being given a certain value of *y* maybe this will also show the change in y on the graph. Or for example when we were dealing with the rate of change of this function when it was two dimensional we did not have to account for the rate of change of the parameter a, so maybe the three dimensional graph will act the same and account for the change in *y* by itself.

b) If the result of this function was 243 then it would be saying that over the interval of *x* to *x+h* they're an average rate of change of 243. This also is assuming that by showing the rate of change from *x* to *x+h* it also accounts for the change in *y*.

c) Since in this function we don't try to account for change in *y* from *y* to *y+h* we are assuming that its already going to be shown since it needs to be accounted for in a 3D graph. This is definitely something I'm not sure about but at the same time the exact reason why I think it's a possible solution.

I believed his second conjecture suggested he was thinking about varying *x* while fixing the value of *y*, creating a rate of change in the z-x perspective. In this case, the value of *h* he was thinking about was the same as if he had been describing a one-variable rate of change function.

3. $r_V(x,y) = \dfrac{V(x+h)-V(x)}{h}$

a) I think the reason that I suggested this function as a possibility is because I felt it was saying the same thing as the function from problem 2. However I honestly don't think this would work since *V*(x) uses a parameter a to find the volume where *V*(x,y) uses y in place of a. Even though these two functions are solving for volume they are not defined in the same terms. I don't trust this one anymore.

4. $r_V(x,y) = \dfrac{V(x+h,y+w) - V(x,y)}{hw}$

a) The reason I thought that this could be a possibility is because much like the function from problem one this would account for the change happening in x and for the change in y. The only difference here is that the input x in this function stands for a the cut out length which can be in feet or inches or something along those lines that also means that h must be in the same units of measurement. So the reason for adding w to y instead of h is because maybe y isn't a value that represents something in the same units of measurement as x and h.

b) Just like in problem one this would mean that in the average rate of change over the intervals of *x* to x+h and *y* to y+w would be 243.

c) After thinking about question b for this function I think my biggest area of concern is adding *w*. I wouldn't know how to determine what w should be, should I give w the same value as h? Not only that but what would 243 from question b really represent when I have *x* over an interval of one value and x increasing by an interval of another value, I wouldn't know if it was the combined increase of both the change in *x* and *y* or something else.

After doing this exercise I'm thinking that my second idea still seems like the best solution to me. The reason for this is because I think it has the best way of representing the change happening over *x* and in *y*, since it would be solving the rate of change in much the same way we did for the two-dimensional graph and still account for the value of *y*.

I believed Jesse's fourth conjecture was almost identical to his first, except he imagined adding different amounts to x and y, as indicated by his use of h and w instead of simply h. However, Jesse did not explain his rationale for using the denominator hw.

Jesse's three rate of change conjectures (I considered his response to the third to indicate he did not find it valid) indicated that he was struggling with how to think about a quantity's value changing with respect to the value of two other quantities. I believed that each of Jesse's conjectures could be referenced to support him in thinking about a rate of change in a direction in space. Together, his conjectures indicated to me that he was struggling with 1) how to combine or use rates of change in the z-x and z-y direction and 2) how to define the denominator of the rate of change function so it would fit with the meaning he created for a two-variable rate of change function. I used the context of the box problem to explore these conjectures and to build supports for helping Jesse think about rate of change in a direction in space.

Days 4-5: Box Problem and Rate of Change

I anticipated that because of his rate of change conjectures, it would be key to support Jesse in thinking about 1) partial rates of change as occurring simultaneously in the domain of the two-variable function, and 2) imagining making the changes in quantities represented on the x and y axes small in tandem in order to approximate the rate of change of the volume of the box with respect to cutout length and base area. I believed that these ways of thinking would help him imagine rate of change in a direction, in which the changes in *x* and *y* occur simultaneously and depend on each other. Thus, this session focused on supporting Jesse in conceptualizing the two-changes problem. I began Day 4 by asking Jesse to reflect on the conjectures he had developed about the rate of change of a two-variable function.

Excerpt 16 (Day 4 of TE, 00:00:03)

EW: I'd like to focus on some of the conjectures that you worked on after our last meeting. You said you've had a chance to think more about it.

Jesse: Yeah, well, I really liked conjecture 2, and now I think it's important because it tells me about the rate of change from a specific perspective. It is like holding y-constant while adding a little bit to *x*, so it is kind of like making a calculus triangle from the z-x perspective, if that makes sense?

EW: I think I am following, but can you say a little more?

Jesse: Yeah, well this was for z-x, if I wanted to think about the rate of change in the z-y perspective, I would fix x, then add a little bit to y, and just adjust the function a little bit. Then I have the two rates of change that I am looking for, one for z-x and one for z-y. It's kind of like making calculus triangles in each perspective, but I'm still working on combining the rate of change functions somehow.

EW: I do think I am following now, I just want to clarify, when you use *h* in the two rate of change functions, are they the same *h*?

Jesse: I hadn't thought about it, but I suppose they don't need to be because you have control over picking their value.

Jesse indicated he was thinking about conjecture two as a perspective dependent rate of change (Excerpt 16, lines 3-6). By perspective dependent, I mean he was thinking about the z-x perspective by considering the rate of change of *z* with respect to *x* while fixing *y,* and the rate of change of *z* with respect to *y* while fixing *x*. He adjusted his conjecture to account for the rate of change in the z-y perspective (Excerpt 16, lines 9-12). He imagined that perspective dependent rates of change could be represented by a calculus triangle in each of the z-x and z-y planes (Excerpt 16, lines 10-14), but said he was unsure how to "combine" the rates of change (Excerpt 16, lines 17-18). His response suggested two things to me as we were talking. First, he was focused on the calculation of rate of change, without first focusing on the meaning of the rate of change that the calculation would represent. It appeared that he was attempting to accommodate a calculation in space, not accommodate a meaning that could support a rate of change function Second, when describing two partial rates of change, he saw the change in *x* and *y* as independent parameter values. By thinking about these as independent parameter values and focusing only on accommodating a calculation, the two-changes problem was no initially evident to Jesse.

During a five-minute break in Day 4, I wrote that Jesse was attempting to calculate an "overall" rate of change. I believed that he would not have occasion to think about rate of change in a direction without thinking about the x-y perspective to bring about the two-changes problem. I anticipated considering projections of the function in the x-y plane would allow Jesse to consider the independence of the changes in *x* and *y*, and to consider how directional rate of change could create a fixed relationship between *x* and *y.* The imagery I intended to support was a calculus triangle sliding in a direction in space. Excerpt 17 followed a discussion between about why he believed the two values for rate of change, returned by using the calculus triangle in each perspective, needed to be combined additively or multiplicatively. His responses focused on the idea of finding a “net change”. Day 4 ended abruptly due to unexpected time constraints on the interview room. We resumed our discussion on Day 5.

Excerpt 17 (Day 5 of TE, 00:00:04)

EW: You are pretty focused on how to combine these two rate of change functions, before we do any combining, what do they mean individually?

Jesse: Well, they are a rate of change for a fixed y, with the z-x open form, and a rate of change for a fixed x, with the z-y open form. They kind of fix one of the variables as a parameter, and focus on a one variable rate of change function. I want to get a net change, coming from z-x and z-y.

EW: If you have that information, how are you thinking so that a combined rate of change function becomes important?

Jesse: Well, in space, you can move in directions other than where one of the variables stays constant, like in a loop, or in a straight line. I need a more general function that accounts for not fixing one of the variables. Ugh, I guess I'm just guessing at how to come up with the function again.

EW: You mentioned moving in other directions, and your 4th conjecture was about how to come up with one open form rate of change with and *h* and *w*. What adjustments might you make to that conjecture? Let's work with that.

Jesse: For one, I don't know exactly what to make small, like what is the *h* for two variables because there are two *h*'s here.

EW: Suppose you thought about walking out in a direction from a point in space, like where the change in *y* is twice as large as the change in *x*.

Jesse: Well, I would have an h_1 and an h_2, but h_2 would be twice as big as that h_1. This is kind of like looking from overhead! I just look at changes in *y* and changes in *x*. This is kind of like combining the calculus triangles in space in a way. Then as I make either *h* small, the other one gets small too.

During our discussion in Excerpt 17, I believed Jesse was focused on a "net rate of change" (Excerpt 17, lines 7-8), but this kept him from thinking about the relationship between changes in the *x* and *y*. I inferred that the net rate of change was a way to verbalize his anticipation that he would need to add rates of change from two perspectives. Thus, I believed that to Jesse, “net” meant, “additively combined”. I decided to build on his fourth rate of change conjecture, in which he had created parameter values two represent the change in *x* and the change in *y*. His description of not knowing which change to make small (Excerpt 17, lines 20-21), suggested to me that he was thinking about the changes in *x* and *y* as unrelated. He used my suggestion of walking from a point in space as a way of “combining” (Excerpt 17, line 26) calculus triangles. This appeared to help him think about a way to find a “net” change. At the end of Day 5, he was focused on how to relate h_1 and h_2 (the changes in *x* and *y*), which appeared to solve his dilemma by fixing a relationship between the changes in *x* and *y*.

During Day 6, we had the same conversations about the Drivercost problem as with the box problem. Jesse reached a point where the context appeared irrelevant to him, and on Day 7, we focused on developing a general, open form rate of change for a function of two variables. At the beginning of Day 7, I asked Jesse to reflect on his original rate of change conjectures. He noted that his second conjecture represented a direction in space, just one in which there was no change in *y* (z-x perspective) and no change in *x* (z-y perspective). He said that his most accurate idea had been the fourth rate of change conjecture because it accounted for unequal changes in the values of *x* and *y*. His new open form rate of change conjectures was $r_{f(ab)}(x,y)=\dfrac{f(x+h_1,y+\frac{b}{a}h_1)-f(x,y)}{?}$. He described how he could define h_2 in terms of h_1 by considering a direction in space from the x-y perspective. He attributed his confusion to whether he should include one or both changes in x and y in the denominator.

Excerpt 18 (Day 7 of TE, 00:00:09)

EW: So how would you define the denominator now, based on what we have talked about? What's hard about it?

Jesse: The denominator was confusing to me at first because I wanted whatever was in the denominator to become small, and I knew two small changes needed to be included. But if I make, umm, the changes depend on each other, then if one of them gets small, the other one gets small as well. I think I could just put one of them in the denominator, maybe h_2, then define h_2 in terms of h_1, then it's like a one variable function that I know how to handle. I guess this is like making and combining two calculus triangles, to make one in a certain direction.

EW: Okay, it looks like you have defined the function, now in the context of the box problem, what would a particular value of the function mean?

Jesse: The output would be the constant rate of change of the volume with respect to cutout length and base of the box, where those two are in a relationship

Jesse's confusion centered on how he could think about changes in both x and y (Excerpt 18, lines 3-6). It was critical for Jesse to imagine the simultaneous variation of h_1 and h_2 and the invariant relationship between them. Thinking about the changes depending on each other (Excerpt 18, line 6), allowed him to imagine that as one change became small, so did the other (Excerpt 18, lines 6-7). Anticipating the dependence of the values on each other allowed Jesse to construct the denominator in the open form rate of change function,

$r_{f(ab)}(x,y)=\dfrac{f(x+h_1,y+\frac{b}{a}h_1)-f(x,y)}{?}$. He concluded that h_1 alone was a sufficient denominator because the change in h_1 implicitly required a change in h_2 to which the change in *f*(x,y) was compared.

He described the meaning of the open form rate of change function as a constant rate of change of the volume with respect to two quantities, which are "in a relationship" (Excerpt 18, lines 14-16). This suggested that he had resolved his earlier confusion about a dependent quantity changing with respect to two quantities simultaneously by thinking about the two independent quantities in a fixed relationship.

Day 8 - Closed Form Rate of Change

Given that the closed form of rate of change in his calculus course coincided with rules for differentiation, I constructed a way in which to help him use his idea of perspectives of functions to define a rule of differentiation. In the following excerpt, I asked Jesse to reflect on his open form rate of change function, and to discuss how he would think about closed form rate of change for a function of two variables.

Excerpt 19 (Day 8 of TE, 00:00:05)

EW: If you recall, we talked quite a bit about open and closed form rate of change. You mentioned this rate of change function is an example of open form?

Jesse: Yep, this is open form, because we are dealing with calculus triangles. The closed form requires some sort of rule. I'm not sure how to deal with it.

EW: Let's use your way of thinking about taking perspectives on functions, if we consider the z-x perspective, for a function like $f(x, y) = x^2y^2$, how would you think about the rate of change?

Jesse: Well, from z-x, *y* is like a parameter, or we fix *y*, so we can just use the rules we've learned in class, it's like a one variable derivative function, then I can do that for the z-y perspective too. Once I have both of them, I'd probably add them together.

EW: How would you check if the closed form was accurate?

Jesse: I would make either h_1 or h_2, small in the open form rate of change.

Jesse anticipated he would need a "rule" for the rate of change (lines 5-6). At my mentioning of thinking with perspective (Excerpt 19, lines 7-9), Jesse immediately described how thinking about the z-x perspective requires treating *y* like a parameter value (Excerpt 19, lines 10-11). In his calculus course, Jesse had constructed the rule for differentiating a one-variable function with a parameter. By thinking about a perspective, Jesse made the two-variable function "similar" to a one-variable function with a parameter (Excerpt 19, lines 12-13). He anticipated he could use his rules for differentiation to generate two closed form rate of change functions and could check their accuracy (as well as his rule of differentiation) by making h_1 small in the open form rate of change (Excerpt 19, line 15). He was continuing to think about open form rate of change as an approximation to closed form rate of change.

I asked him to take me through this "process" for $f(x,y)=x^2y^2$. Jesse said the rate of change function from the z-x perspective would be $r_f(x)=2ax$ and the rate of change function from the z-y perspective would be $r_f(y)=2ay$. His first conjecture was to add $r_f(x)$ and $r_f(y)$ because "we are adding together components, I think I've done something like this in physics in the past". He noted his open form rate of change function required a direction, but a direction was not present in his initial conjecture about the closed form rate of change. He attributed this inconsistency to open form rate of change "depending" on the original function all of the time, while closed form depends on a rule. He decided on a direction in which the change in *y* was equal to the change in *x*.

He graphed both the open form and closed form rate of change functions, and noted when he made h_1 very small, the graphs "appeared to get closer together" (Figure 25). He noted that for both directions, the open and closed form became "identical". Jesse was unsure why this occurred, because he constructed the rate of change as direction dependent. However, in two different directions, his closed form rate of change functions appeared identical.

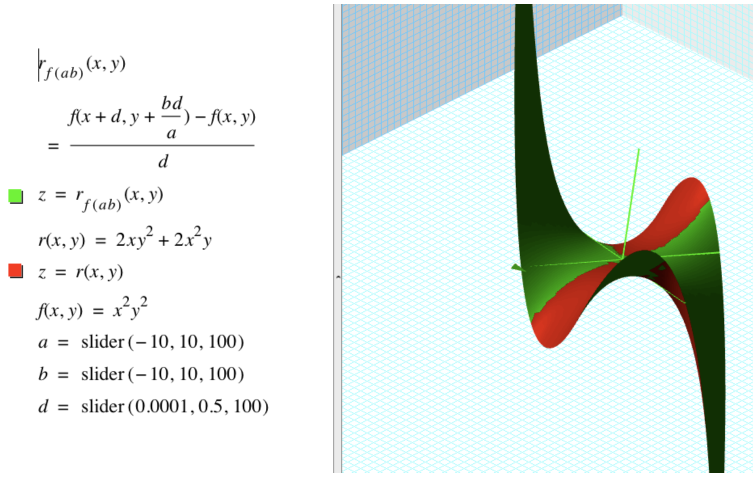

Figure 25. Jesse's comparison of open and closed form rate of change.

Excerpt 20 (Day 8 of TE, 00:00:24)

EW: I noticed that you seem to be getting frustrated, or a little confused here. Can you tell me what you are thinking?

Jesse: Yeah, well, rate of change is direction dependent, at least that is how I think about it, so the closed form rate of change should differ by what direction I am considering. I'm think you need to sort of weight the closed rate of change by what direction you are considering to make the closed forms different, but I'm not quite sure.

EW: Can you say more about what you mean by weight?

Jesse: Yeah, well, hmm. What do I mean? I mean that if the change in y is twice as big as the change in x, the rate of change from one perspective is going to have more of an influence on the overall rate of change than the rate of change from another perspective. It sort of comes back to my idea of net change, I just am not sure how to set it up right now.

Jesse's was struggling to think about a "net" derivative (Excerpt 20, lines 12-13), but was imagining this net derivative as composed of the rate of change from two directions. He appeared to be imagining the overall rate of change (Excerpt 20, lines 10-11) as the result of adding multiple rates of change, each weighted by some influence factor that depended on the relationship between the change in *x* and the change in *y*. I believed that Jesse was dancing around the idea of total derivative, but based on my model of his thinking, a great deal more support was required to develop a meaningful and coherent notion of path dependent and independent derivatives.

$$
\begin{aligned}
f(x,y) &= x^2 y^2 \\
f_x &= 2xy^2 (a) \\
f_y &= 2yx^2 (b) \\
r_{f(ab)}(x,y) &= f_x a + f_y b
\end{aligned}
$$

Figure 26. Jesse's prediction of a closed form rate of change for *f*(x,y).

Part II Conclusions

My model for Jesse's understanding of rate of change evolved based on gaining insight into his understanding of open form rate of change, his meaning for constant rate of change, and his understanding of a calculus triangle as a measure of a rate of change function. My model of his thinking also developed from specific instructional supports such as using his way of thinking about perspectives in visualizing functions of two variables to help him imagine calculus triangles in each perspective.

Meaning for rate of change.

Jesse described the meaning of average rate of change of quantity A with respect to quantity B as the constant rate of change of quantity A with respect to quantity B required to produce the same change in quantity A over the equivalent change in quantity B. I believe that his use of open form and closed form rate of change allowed him to think about average rate of change as the dependent quantity in the open form rate of change function. He imagined open form rate of change (which implicitly was associated with average rate of change) as an approximation to closed form rate of change (which he associated with an exact, or instantaneous rate of change) for small values of the change in the independent quantity.

Understanding of the calculus triangle.

Jesse's use of a fixed value of h to compare differences in the dependent quantity suggested that he was already implicitly using the calculus triangle. Jesse began to think about the calculus triangle and open form rate of change as equivalent ideas. For example, he noted that the calculus triangle was a graphical representation of the quantities in the open form rate of change function. At the same time, his use of a calculus triangle allowed him to think about three ideas that were not apparent in his initial open form rate of change function. First, he imagined the calculus triangle sliding through the domain of the function, which resulted in tracking the value of the rate of change. Second, he imagined that an infinite number of very small calculus triangles existed in the domain of the function. Third, he was able to illustrate the idea of fixing h as a parameter value and by making h small, the value returned by comparing the lengths of the legs of the calculus triangle would approach the "exact" rate of change of the function as a point.

Extending rate of change to two variables.

Jesse's focus on rate of change using calculus triangles became clear when he initially grappled with the meaning of the rate of change at a point in space. His initial attempts to describe rate of change in space combined his understanding of average rate of change, his notion of calculus triangles, and his way of thinking about visualizing two variable functions from different perspectives.

He used his understanding of perspective and calculus triangles to imagine perspective dependent calculus triangles that would measure the rate of change of z with respect to x, or z with respect to y, and attempted to combine these values to define a rate of change at a point in space. I used directional rate of change as a way to help him resolve his imagery of simultaneous calculus triangles in the z-x and z-y perspective with independent changes in the dependent quantities x and y. By thinking about a calculus triangle in a direction in space, Jesse was able to define an open form rate of change function, which he believed measured the average rate of change of a dependent variable with respect to two independent variables.

Total derivative.

Jesse's development of "overall rate of change" suggested that when thinking about rate of change at a point in space, he was dancing around the idea of thinking about the path to the point (x, y). I believe that it would have required numerous extra sessions to problematize the issue of directional and total derivative to think about directional derivative as path dependent. However, I do anticipate that given these extra sessions, Jesse's ways of thinking could support thinking about the function *f* at (x,y) as differentiable when all the directional derivatives at (x,y) lie in a plane, but the time constraints of this teaching experiment did not allow for this development to take place.

Chapter Summary

In this chapter, I have described the evolution of my inferences for Jesse's ways of thinking about two-variable functions and their rates of change. While I examined the ways of thinking about visualizing functions separately from thinking about rate of change, Jesse's ways of thinking about these ideas were developing simultaneously, as in the case of thinking about z-x, z-y and x-y perspectives. I demonstrated the development of inferences for his thinking by describing my hypotheses, a description of key excerpts, and my reflections from each teaching experiment session. In doing so, I described expert shape thinking, perspective thinking, and calculus triangles as key understandings for Jesse's ways of thinking.

Chapter 6

JORI'S THINKING

In this chapter, I describe Jori's thinking. Jori was a student enrolled in first semester calculus, as she participated in a teaching experiment focused on functions of two variables and their rates of change. I describe her association of a graph with a malleable object and her attempt to associate a set of symbols with a memorized image of the graph from her calculator. I argue that she focused on features of a graph as an object in space, which constrained her in imagining a function's graph as an emergent representation of quantities having covaried. Lastly, I describe how Jori's focus on rate of change as a calculation associated with the slant of a graph constrained her from thinking about rate of change as a comparison between changes.

Background

Jori "was not a fan of mathematics", which she attributed to difficulty in high school mathematics and science courses. She described herself as a high-achieving student "overall, just not in mathematics". She attributed her success or failure to sources outside of her control, such as the teacher, the workload of the class, or the general idea that mathematics was confusing. She noted her experiences in her calculus course were frustrating because she was required to think about the meaning of an answer, but her intent when solving a problem was to come up with the correct calculation. She believed she was not really "doing mathematics" because she was pressed to describe how she was thinking about new ideas rather than how to do calculations.

Jori scored near the mean on the calculus readiness exam administered at the beginning of the semester. She was enrolled in first semester calculus when she participated in the teaching experiment, and in that course, she had average to below average scores on exams. She often “punted” away clarifying questions from the instructor or other classmates. For example, in class she described the rate of change as represented by a straight line, a number of her group members suggested she meant constant rate of change. She described that she saw no difference between what she said and what they said, and dismissed their appeals.

Initial Inferences about Jori’s Ways of Thinking

I worked with Jori five times during tutoring in the two weeks before the teaching experiment. When she defined a function, she described a set of values for quantities in the problem and attempted to define a function that fit only a static set of values. For example, when she was working on defining the distance of the bottom of a ladder away from a wall as a function of time given that the ladder started out three feet from the wall and was sliding at 2 feet/sec, she created a set of points (1,2), (2,4), and described them as time, and how far the ladder was away from the wall. She then plotted these points, and tried to define a function whose graph would fit through all of the points she had generated. Her approach suggested that she read problems by focusing on the numbers in the problem statement. Her focus on constructing graphs by connecting points suggested she was not attending to the graph as a representation of the relationship between quantities’ values as they vary. Because of these behaviors, I believed that Jori thought about graphs of one-variable functions as wire-like objects in a plane.

Jori's teaching experiment began after Jesse's was completed, and used the same tasks. However, I adjusted the sequence of the tasks. I moved the rate of change tasks later in the teaching experiment so as to provide a more cohesive focus for my generation and evaluation of hypotheses. As a result, day-to-day the focus of our discussions was on material covered at most three days before. The breakdown of tasks and the day on which they took place is in Table 18.

Table 18

Task Sequence for Jori's Teaching Experiment.

Task Description	**Date (Day)**
Homer: Correspondence Points and Graphs	4/5/2011 (1) (10am)
Homer: Moving Cities and Varying Animation	4/5/2011 (2) (5pm)
Generating Graphs of $f(x)=x^3, g(x)=2x$	4/7/2011 (3)
Generating Graph of $h(x)=x^3-2x$	4/7/2011 (3)
Anticipating Behavior of $m(x)=a(x^3-2x)$	4/7/2011 (3) 4/9/2011 (4)
Treating *a* as a Third Axis	4/9/2011 (4)
Box Problem: Box Construction and Treating Parameters as Variable	4/12/2011 (5)
Generalized Functions: Construction and Interpretation of Graphs	4/14/2011 (6)
Car A – Car B: Interpreting Meaning of the Graph	4/16/2011 (7)
Car A – Car B: Measuring Rate of Change Using the Graph	4/17/2011 (8)

Box Problem: Measuring the Rate of Change at a Point in Space	4/20/2011 (9) 4/23/2011 (10)
Drivercost Problem: Parameters as Variables and Rate of Change	4/25/2011 (11)
Generalized Rate of Change Function and Direction	4/25/2011 (11) 4/27/2011 (12)
Working with Strange Functions (More than 2 variables)	4/29/2011 (12)

Part I: Jori's Ways of Thinking about Two-Variable Functions

Part I of this chapter focuses on the development of a model for Jori's thinking about the behavior of two-variable functions. This part of the teaching experiment comprised Days 1 to 6.

Days 1 and 2 – The Homer Activity

Based on my initial assessment of Jori's thinking, I anticipated that the Homer situation would present two perturbations to her thinking. First, the construction and representation of quantities' magnitudes on the axes would be problematic because she was accustomed to focusing on numbers. Second, imagining a correspondence point that tracked these two quantities' magnitudes simultaneously would not fit with her general conception of a graph as a path that connects a discrete set of points. I anticipated that her conception of a graph would rely on the graph "looking like" the situation. In the Homer situation, this could be a road and two paths emanating from Homer.

On Day 1, Jori mimicked Homer's motion when asked to track Homer's distance from Springfield and Shelbyville. She noted that she focused on Homer's motion, not his distance from the cities, and was unsure why his distance from each city was important. The following excerpt occurred after my discussion with her about how to focus on Homer's distance from each city as she tried representing Homer's distance from each city using a vertical and horizontal "bar" in Geometer's Sketchpad.

Excerpt 21 (Day 1 of TE, 00:00:24)

Jori: Well, when I am trying to pay attention to both of his distances at the same time, it is really hard. I understand how to represent both distances, Shelbyville on the horizontal and Springfield on the vertical, but I'm having trouble trying to track both of them.

EW: Can you tell me what you are having the most trouble with?

Jori: I guess I'm not quite sure, but when I was paying attention to each distance individually, I could just make sure my finger was moving like Homer was moving on the road, kind of keeping the same speed and tracking his distance, but now my fingers can't both imitate Homer's motion at the same time, it's just frustrating, I'm not sure, uh…

EW: Are you trying to imitate Homer's movement, or trying to track the distance from each city?

Jori: I guess I am trying to do both of them. I kind of lose track of the distances when I am trying to keep up with Homer's motion. [As the animation moves, she appears to focus on the distances exclusively]. Uh, well, it is a lot harder, but I feel like I am being more accurate when I am not trying to keep up with Homer's motion at the same time.

EW: Okay, so what are you trying to pay attention to now? [as Homer is moving on road in the animation]

Jori: Well, it seemed like I want two distances, so I have to find Homer’s distance from the two cities at the same time. At first I thought I would have to imitate his motion, but the distances make more sense. They aren’t the same.

Once she focused on Homer’s distance from each city as a quantity, Jori was able to imagine each distance varying individually, but found it problematic to track both of the distances simultaneously (Excerpt 21, lines 4, 6-7). She attributed this difficulty to the importance of mimicking Homer’s motion (Excerpt 21, lines 7-11). Even as she attempted to track a distance, and imagined those distances having values simultaneously (Excerpt 21, lines 21-24), Jori was imitating the motion in the animation (Excerpt 21, lines 14-15). I anticipated, in the moment, that her inattention to quantities would foster a way of thinking about a correspondence point as a feature of the Homer animation rather than a representation of the quantities in it.

After introducing the notion of a correspondence point, I asked Jori to tell me what she understood about the behavior of the point. Her responses suggested to me that she felt “in control” of the correspondence point at every instant. By this, I mean she felt the point could be manipulated without affecting Homer’s location on the road. The following excerpt occurred after discussion about the correspondence point as a representation of Homer’s distance from each city.

Excerpt 22 (Day 1 of TE, 00:00:45)

EW: You were struggling with the correspondence point idea earlier, but pretty quickly picked up imitating its behavior.

Jori: Yeah, once I realized it did not need to imitate Homer, it was easier. You asked me to think about it leaving a trail behind [sketches the trail she anticipates]. I think this [pointing to Figure 27], would be what it left behind. It tracks where the correspondence point has been, and sort of shares some things that look like the road and the cities' locations.

EW: So what do you pay attention to decide if the path is accurate?

Jori: Well, it has to make sense with the situation, so it has to imitate some of Homer's motion no matter what. A graph tracks a situation, so sometimes it will look like that situation.

EW: Well, let's look at the trail that the sketch produces. [Reveals computer sketch, Figure 28]. What do you think?

Jori: It sort of opens upward, like I thought. The rest is weird, but I could probably account for it if I was more exact in my measurements.

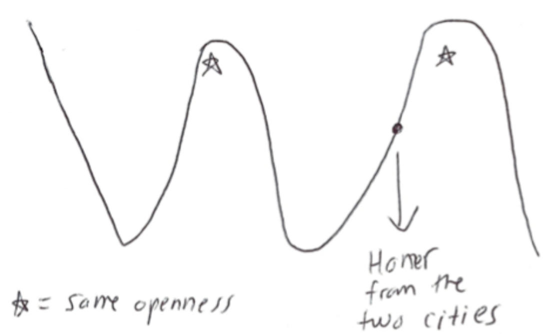

Figure 27. Jori's anticipated tracing out of the correspondence point.

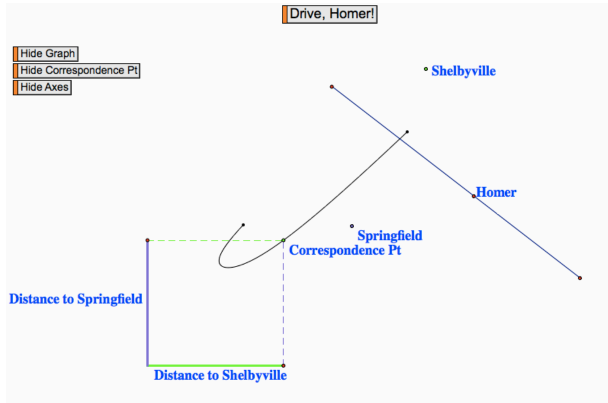

Figure 28. Computer generated graph of Homer's distance from two cities.

Jori indicated that the correspondence point was not representative of Homer's motion, but its path would be related to Homer's location on the road. This concept of mimicking fit with her concept of a graph as "looking like" the situation it represented (Excerpt 22, lines 9-11). Jori appeared to be convinced that the correspondence point would rely on the features (e.g. road) of the situation. She may have been trying to express quantities created by measuring aspects of those features, but her thinking was very muddled. Her description of the graph "opening upward" and her inability to describe why it opened upward (Excerpt 22, line 15) was evidence that she may have been thinking about a graph as an object in space.

Jori's focus on Homer's movement along the road had two consequences. First, she focused on recreating features of the animation in her graph. Second, her focus on Homer's movement likely led to her inattention to the Homer's distance from the two cities as quantities. Her description of the correspondence point as recreating Homer's motion suggested that she was not attending to the values represented on the axes. Thus, after the morning teaching session (Day 1) I believed that she was not thinking about the correspondence point as a representation of two quantities' values. I thought she focused completely in the quadrants. For example, when describing her graph as opening upward (Excerpt 22, line 14) she was focused on describing the shape she saw, rather than thinking about how the shape was related to the quantities in the Homer context.

At the beginning of the afternoon session (Day 2), I tested these hypotheses by asking Jori 1) to anticipate how the graph would change if Shelbyville and Springfield were moved and 2) to describe the location of the cities given a graph that represented Homer's distance from each city simultaneously. I anticipated that if she focused only on the shape of the graph, she would imagine that a transformation of the quantities in the Homer animation as unrelated to the graph.

I exchanged the location of Springfield and Shelbyville in the animation in the first minutes of Day 2. I intended to understand how Jori would think about the effect of this exchange on Homer's distance from each city in the animation. Jori said that the graph would not change, because "all of the objects are in the same place, the names are just different in this case" (Figure 29). Her response indicated that she was focused on the location of objects in the animation. I believed, in the moment, she thought because the graph was accurate an exchange of the cities did not affect the graph. This indicated to me that she was not focused on what each axis represented, which constrained her from imagining the graph as a representation of quantities having covaried.

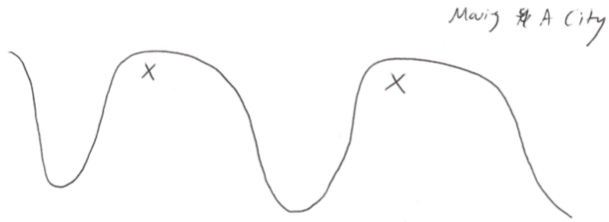

Figure 29. Jori's anticipated graph after I exchanged the positions of the two cities.

Excerpt 22 was the first indication of what I called a *focus on the quadrants*. When Jori imagined graphing a function or interpreting an already constructed graph, she appeared to think about the features of the object in space as just features. The features were not emergent or imbued with quantitative properties. As a result, Jori was unable to the functional relationships that these "features" represented because they were not representations of quantities. For example, Jori believed the graph would remain invariant under an exchange of cities. She appeared to associate a single graph with the Homer situation, and because the shape of the graph was not the output of covariational reasoning, this graph did not need to transform even though the cities locations were exchanged.

I then gave Jori an already-generated graph (Figure 30) and asked her to say where the cities might have been located to generate this graph, in anticipation that if she could not imagine the graph's shape as a result of the distances having covaried, she would describe the graph as a roadmap of how Homer moved in the animation. Our discussion is given in Excerpt 23.

Excerpt 23 (Day 2 of TE, 00:00:12)

EW: Okay, so we are going to kind of work in reverse here. Given this graph (Figure 30), could you tell me where the cities might be located in the animation?

Jori: Okay, I'm going to assume everything works the same way. The first thing I notice is that the lines of the graph hit the axes here, so that might mean Homer turns around at those points, maybe turns around from the cities? Well, but at the same time, the graph I'm seeing here opens up and is kind of square like, not smooth like the other ones, so, umm, I would guess that because there is straightness, well, he's maybe traveling at a constant rate to or from each one of them?

EW: What do you mean by them?

Jori: The cities. The linearity of the graph means he's going at a constant speed, I think, or that he is traveling in a straight-line path.

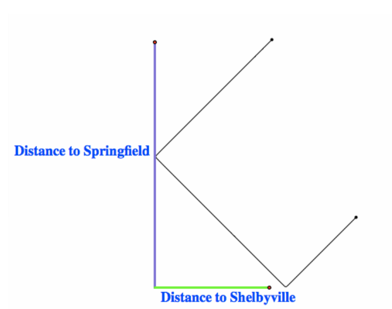

Figure 30. The graph Jori was given to determine the location of the cities.

Jori's responses indicated that she had engaged in novice shape thinking. It seemed that the associated the image of a straight line with "constant speed" and that a sharp corner meant "turn around". Because she was focused on meanings implied to her by the graph's shape she was not attending to the axes or how distances from cities were represented on the axes and reflected in the correspondence point's position. This association is supported by her description of graphs not as a relationship between quantities' values, but as conveying meanings unrelated to the situation (e.g., "straight means constant speed", Excerpt 23, lines 12-13).

In my reflection on Days 1 and 2, I noted that Jori's lack of quantitative reasoning differentiated my model of her thinking from Jesse's. Jori appeared to associate the graph with an object that had physical features, but those features were non-quantitative. I anticipated that Jori's lack of attention to how graphs as an emergent representation of covariation and her focus on the graphs as shapes would lead to her associating a surface in space with an object, where that object was independent of the quantities I intended her to use to generate the surface.

Day 3 - The Difference Function

I had not anticipated the degree to which Jori would focus on the shape of graphs and struggle with imagining their construction. I believed it would be very difficult to observe a major shift in her thinking about the graphs of functions because of her insistence on focusing on the graph as an object in space with properties independent of the function that generated it. Given this issue and Jori's time constraints, I believed it would be powerful to characterize how her ways of thinking constrained her in thinking about two-variable functions. Thus, I moved forward in the activity sequence.

I anticipated that generating a difference function where the algebraic forms of functions were given would reveal her association of the graph as an object situated in the quadrants, independent of quantities that generated it. As such, I believed that she would imagine the difference function as a physical combination of the graphs and their features. The following excerpt occurred at the beginning of Day 3, in which we discussed how to think about generating the graphs of the functions $f(x) = x^3$ and $g(x) = 2x$.

Excerpt 24 (Day 3 of TE, 00:00:05)

EW: If we are generating the functions x^3 and $2x$, how are you thinking?

Jori: Well, I am trying to think to when I have used my graphing calculator. I know 2*x* is a straight line because it has a constant slope, but x^3 is difficult because I can't remember how the function looks. I think it is sort of wavy.

EW: How do you approach generating a new function if you haven't seen it before?

Jori: Generally, umm, I just use a graphing calculator, then, and then and try to memorize the shape of the graph, kind of the contours I see. Then I can figure out what other similar algebraic functions look like, and see if they imitate the same shape.

EW: So how would you go about looking at the cubic term? Or should I say, how does a calculator know how to graph that function?

Jori: Well, I remember the wave sort of, but I'd have to ugh, well, plug in a bunch of numbers and then connect the points as best as I can. I'd probably try to connect a wave, because that's what I remember.

EW: Okay, well, remember our discussion about the correspondence point, how would that come into play here?

Jori: I'm not sure, I'm not sure how to track the things I need to track.

Jori "generated" graphs by trying to recall their shape from a graphing calculator (Excerpt 24, line 2). She appeared to memorize a graph's shape and contours and compared those shapes to general classes of algebraic functions defined by their shape (Excerpt 24, lines 10-11). Thus, I believed to Jori, an equivalence class, or family, of functions was completely reliant on the shape of the graph, independent of imagining that graph as an emergent representation of two quantities having covaried. It appeared to me, in the moment, that Jori imagined the calculator running a process in which it plugged in a bunch of numbers, then "connected a wave" that went through each point (Excerpt 24, line 16). I believed, while working with her in Day 3, that she was trying to connect points *in space* rather than points in space created by quantities' values represented on the axes.

I asked her how she would think about generating a difference function composed of the two functions she had just graphed using Graphing Calculator. I anticipated that she would focus on 1) generating the function's graph by attempting to imagine what it would "look like" and plot points and try to guess as to how the points were connected, and 2) combining the shape of the two graphs into a graph that shared features of each individual graph.

Excerpt 25 (Day 3 of TE, 00:00:24)

EW: So, given your graphs of the two functions, let's try to create a new function, which is the difference between these two, $x^3 - 2x$.

Jori: Okay, oh, great. Ugh, I'm having trouble doing this because I'm not sure about x^3, but I graphed it in GC, so I know what it looks like. If I can graph both functions together, I might have a better idea.

EW: Can you say any more? What might you do with the two graphs?

Jori: You said a difference function, right? Yeah, okay, well, if it is a difference, I'd imagine it goes through the middle of the space between the two functions [draws linear function crossing through open space between two functions, see Figure 31].

EW: I'd like you to use GC to compare the graph you generated.

Jori: Okay, umm, [graphs difference function], wow. I didn't expect those humps to be there, they are taller than both of the functions. I figured a difference would mean the functions sort of meet in the middle, but GC says the humps on the function actually get bigger.

EW: Okay, I just want to finish on this question, if we defined a new function, where we multiply a parameter by $x^3 - 2x$, how would the function behave?

Jori: In class, the y values would only change, so that would make the humps taller and skinnier I think, if the parameter gets bigger like you seem to be asking.

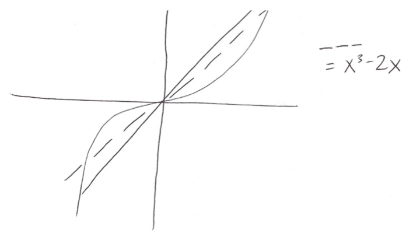

Figure 31. Jori's anticipated graph of the difference function.

Jori focused on what each graph "looked like" as if she were describing an object in space. She wanted to have both graphs displayed simultaneously to see what a function composed of the two might look like (Excerpt 25, lines 4-5). She believed the difference function would lie between the graphs of the two original functions (Excerpt 25, lines 8-9). Her way of thinking about the "difference function" appeared to rely on splitting the difference between objects, which I referred to as "meeting in the middle" (Excerpt 25, lines 14-15). I allowed her to graph the function (Excerpt 25, lines 12-13), and she was surprised by the "humps" in the function's graph because the graph was "taller" than either of the original functions (Excerpt 25, lines 12-14). This excerpt was also the first case where she began distinguishing between what she thought, and what she believed GC (Graphing Calculator), to be telling her (Excerpt 25, lines 16-17). She attributed the difference in what she anticipated and what GC produced as the result of an error in Graphing Calculator.

Even though she attributed an error to Graphing Calculator, she relied on it to depict the behavior of a function novel to her. I believed that even if she possessed a graphing calculator to graph a function, her understanding of how it was generated could be thought of as "calculator magic", she was not even sure the calculator was not full of errors. I thought her need for the graphing calculator to see the "shape" of an object perpetuated itself. She could not imagine graphs of novel functions because she did not have a general scheme for how a graph was generated.

Day 4 - Sweeping Out a One-Variable Function

I predicted that thinking about sweeping out a function's graph to generate a surface in space would not fit with Jori's association with a graph as independent of quantitative relationships. I predicted that if given a surface in space, her inattention to the axes would lead her to focus on features of the surface, much like a topographical map. At the same time, I anticipated that developing the idea of *perspective* would support Jori in imagining the surface of the graph in space from multiple perspectives, but that she would focus on the shape of the surface or the appearance of the graph from multiple perspectives. To evaluate these hypotheses, Day 4 focused on imagining a surface being swept out in space.

At the beginning of Day 4, Jori described the effect of a on $h(x) = a(f(x) - g(x))$ "vertically affecting the difference function we had before. The a only affects the output value, because the x stays the same, the a has no effect on the x is what I am trying to say".

I asked Jori to think about a as the distance in front of or in back of the laptop screen. Jori appeared to visualize this effect of a as the graph sliding along another axis. In the following excerpt, I asked her to describe the tracing out of the function's graph to produce a surface in space.

Excerpt 26 (Day 4 of TE, 00:00:13)

EW: In the Homer activity, we talked about tracing out a function from a point. In this case, I want you to imagine we are sweeping out a function along a third axis, which is given by *a*, and represents the distance in front of or in back of the laptop screen.

Jori: Do you want me to use this strip of paper? It might help. [interviewer nods approval]. Okay, well, as *a* changes, the humps of this function are going to get larger as *a* gets more negative and more positive, kind of creating tunnels that open [Draws Figure 32]. This piece of paper is kind of like my starting graph, and then adjusts when *a* changes.

EW: Okay, could you say a little more, where are these humps coming from, what makes them appear?

Jori: The tracing out you said, described, it's what happens when you follow and track where the strip of paper has been.

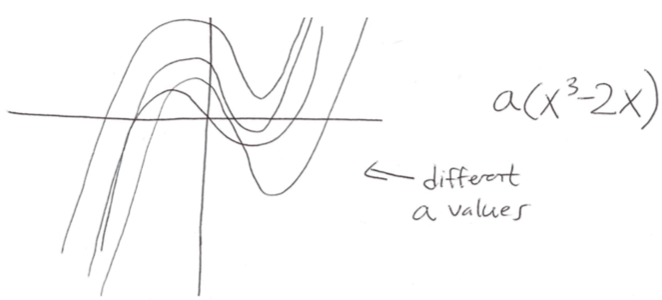

Figure 32. Jori's sketch of the effect of *a* on the difference function.

Jori responses indicated to me that she was imagining the effect of *a* on the shape of the function's graph (Excerpt 26, line 6-7). Her description of "tunnels that open up", suggested she imagined a surface being produced as the function varied with values of *a* (Excerpt 26, lines 7-9). I believed she was thinking about the piece of paper moving through space, staying perpendicular to the *a* axis, and thinking that the graph changed within the piece of paper, which created a surface. Then, the “tunnels” and “humps” became larger as the paper’s distance from the x-y plane increased.

Excerpt 27 (Day 4 of TE, 00:00:34)

EW: So when you are considering the sweeping out we just talked about, what are you visualizing?

Jori: I’m kind of imagining watching it happen from the side, if I watched it, umm, directly [pause], I don’t think I could see 3-D because I wouldn’t have much depth.

EW: Can you describe again how this function is generated?

Jori: Well, I treat the output like the upward axis, and *a* kind of like another variable, but like a parameter at the same time. Then I change the parameter and track the function, where it leaves a trail behind, here sort of tunnelish looking. In 3-D, I think it is normal say up is “z”.

EW: Okay, I’d like to think about this as the z-x view, where we are considering *z* as a function of *x*, and treating *y* like a parameter value to imagine it sweeping out. This might seem strange, but given this, what might a z-y view be?

Jori: Well, offhand I’d treat *x* like a parameter, and *y* like a variable, kind of watching the function tracing out happen from a different viewpoint. I still think it would create the same shape overall, same features, same ups and downs and curves.

EW: What would be the generating function in the z-y perspective?

Jori: It would be a parameter times a variable, but I have no idea what the graph would look like. I'm not sure how I would check it in GC, but it looks like a straight function from z-y [Sketches Figure 33].

Jori's description of watching the sweeping out occur from the "side" suggested she was comfortable thinking with depth. Jori described the sweeping out of a function by thinking about *a* as "like another variable", but like a parameter at the same time (Excerpt 27, lines 7-9). Her description suggested that she was thinking about *a* as a variable, and she found it useful to think of *a* as a variable, even though it could be used like a parameter. Her characterization of the z-y perspective revealed that she viewed perspective as a way to view a surface in space (Excerpt 27, lines 15-17), but had difficulty imagining how that surface was constructed (Excerpt 27, lines 21-22).

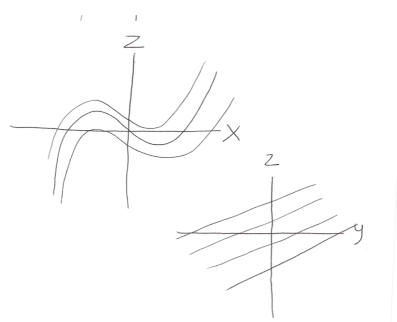

Figure 33. Jori's depiction of viewing two perspectives.

I believed Jori had accommodated her ways of thinking about functions of one variable to fit with a surface in space. When she was able to think about a sweeping out occurring, her description of the resulting surface in space focused on its physical features (i.e. ups, downs and curves). This was much like Jesse's descriptions, but I believe that she associated the ups, downs and curves as the result of sweeping out, but what was swept out was a wire independent of covariational reasoning. This does not suggest that her thinking was "bad", but does support that even as she and Jesse used the same words to describe the surface, only Jesse imagined that surface imbued with quantities' values. I used the box problem context on Day 5 to evaluate these hypotheses.

Between Day 4 and 5 - Jori's Reflections on Parameter

I believed it was critical to establish Jori's distinction between a parameter and a variable. I asked her to write a short reflection on this topic between Days 4 and 5. Prior to Day 5, I asked Jori to describe her use of "parameter" and "variable". Jori described a parameter as "a letter that represents a number, like we talked about in class, kind of like a variable except you are in control of it and can change its value, where a variable is something you are not really in control of, our instructor said it's like representing a whole bunch of values at the same time. I think they are the same because a parameter and variable represent numbers, but are different because one represents a whole bunch of numbers and values while the other only represents one".

Her response indicated to me that she imagined a variable representing a continuum of values, while a parameter represented a single value, which she could specify. It appeared that for her to imagine the sweeping out occurring, she needed to imagine that she was changing a parameter value. I anticipated that treating *a* like a variable meant she imagined the sweeping out having already occurred over a continuum of values, each of which could have been represented as a parameter.

Day 5 - The Box Problem

Entering Day 5, I believed Jori was focused on the graph as an object situated in the quadrants, which along with having difficulty imagining a function being swept out, restricted her from thinking about the function that graph represented. I believed that these ways of thinking applied to both her understanding of one-variable functions, and the sweeping out of a one-variable function to a surface in space. I intended to gain insight into how she constructed an image of the volume of the box as a function of the cutout length, and how that image would constrain her conception of the graph of the volume function. I was also interested in understanding her distinction between a variable and a parameter, and how that supported her in visualizing a surface in space. In the following excerpt, I asked Jori to describe the volume of the box as a function of the cutout length.

Excerpt 28 (Day 5 of TE, 00:00:04)

EW: Suppose you are a box designer, and you must program a computer to determine the volume of an open box, which has no top to it. This open box is created from making cutouts from a rectangular sheet of paper. Jori, could you tell me how you would think about making such a box, as well as how to represent the volume of the box?

Jori: I finally got an easy one! The function would be volume equals length minus two *x* times width minus two *x* times *x*, where *x* [creates Figure 34] is the cutout you make from the sheet of paper, right?

EW: That seemed pretty easy for you, what made it that way?

Jori: Well, the box to have uniform height means the cutouts have to be the same because the cutouts are the same as the height. Then from length and width, you take away two cutout lengths to make the box, then fit that into the formula for volume.

EW: Good! What do you imagine would happen to the volume if the cutout started increasing from a small value to a larger one?

Jori: Not sure, I would test it out, but probably the box volume keeps getting smaller the whole time because there is more cutout. Can I put this function in graphing calculator? That's messed up, well, I guess I was wrong. The volume gets bigger for a while, then smaller.

EW: When you say for a while, what do you mean?

Jori: Umm, well, look, the volume goes up, then it goes down again, so it is getting bigger for a while, but I was mostly right.

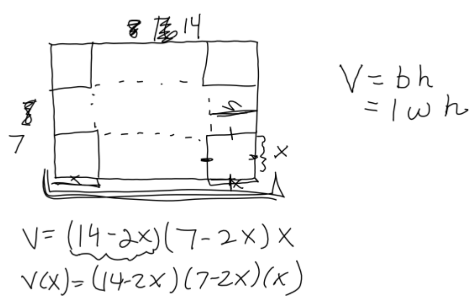

Figure 34. Jori's image of the construction of the box.

Jori's described the volume of the box (Excerpt 28, lines 6-8) as a function of the cutout length. Her anticipation of how the sheet of paper would fold up into a box allowed her to distinguish between the length and width of the box and the length and width of the original paper (Excerpt 28, lines 10-13). Her understanding that the box height would be uniform required all cutouts to be the same length, and the measure of the length of the cutout was equivalent to the height of the box (Excerpt 28, lines 10-11).

Jori also imagined the cutouts becoming larger, which made the length and width of the box smaller, but she did not account for the increase in box height resulting from the increasing cutout length (Excerpt 28, lines 18-19). When she inputted the function into graphing calculator, she was surprised that the graph showed the volume going up, and down again (Excerpt 28, line 21), but immediately punted away this issue, saying she was mostly right (Excerpt 28, line 22). As Jori followed the path of the graph, she focused on the "hump" that the function produced without explaining how the hump fit with the function she constructed.

Two minutes after the discussion in Excerpt 28, Jori described how the graph of the function was really hard to think about because it was a third degree polynomial once the function was expanded. She indicated that once she had a general idea of what a third degree term does to a function's graph, she might be able to "have a decent idea of what the graph looks like and why". Her response suggests to me now that if she knew the shape of a graph and she could imagine how the shape was affected by changing the parameter, she could operate like Jesse in generating a surface. It was also plausible that her reliance on a graph's shape revealed itself when she had difficulty graphing a function with which she was not familiar.

In the following excerpt, I asked Jori to imagine that the base of the box was represented by a parameter value. I anticipated that based on her initial description of volume going up then down, she would continue to associate the graph with an object independent of quantities represented by the function. I intended to gain insight into how creating a “strange” situation (where *a* is a function of *x*) would affect her interpretation and construction of the graph.

Excerpt 29 (Day 5 of TE, 00:00:29)

EW: When you talked last time, you had also described the formula for volume as base multiplied by height?

Jori: Yeah, well, base comes from length times width.

EW: Okay, let’s suppose we treat the base like a parameter value.

Jori: Okay, so you want me to do that sweeping out with the function again? Again, I think the function is only going to stretch vertically when I increase the parameter value, and shrink vertically when I decrease it, so the path it would leave behind would be an open tunnel, which might get bigger or narrower, I'm not sure, I’d have to do the math to test out which one it really is.

EW: Why don’t you test it out in GC, and talk me through it?

Jori: Okay, well, let me make this a parameter, okay, Weird! I have no idea what’s going on there, but I’ll look in a second. If I do this as a two variable function, I just replace *a* with *y*, and define *f*(x,y), and I kind of think there must be a tunnel [looking at Figure 35].

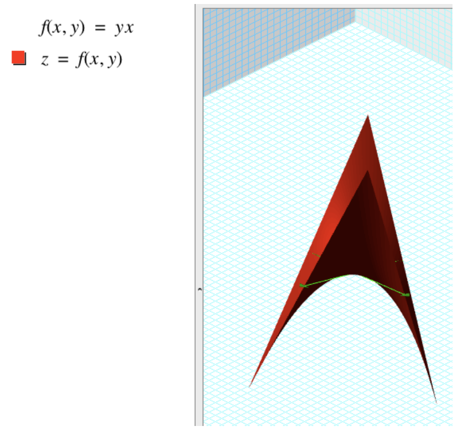

Figure 35. The tunnel Jori viewed in the Box problem graph.

I inferred that Jori was thinking about a parameter value as a third axis, along which the sweeping out of the one-variable function took place. Her description of the effect of the parameter (Excerpt 29, lines 7-8), and of the shape of the path left behind (Excerpt 29, lines 8-9), suggested to me that Jori was using facets of both expert and novice shape thinking. She could associate the surface in space with the effect of the parameter value on that surface (expert), while describing features of the graph without explicit attention to the quantitative relationship that it represented (novice). Her actions also suggested that she was thinking about graphing $f(x,y)=xy$, not $f(x,y)=x*g(x)$ where $y=g(x)$. She appeared to be using a parameter as a way to think about generating the surface of a function. Jori provided evidence for this claim when she replaced *a* with *y* (Excerpt 29, line 14), and in defining and graphing *f*(x,y), anticipated a "tunnel" (Excerpt 29, line 15) appearing without describing a tracing out of a function in a plane that could create the tunnel.

Jori's responses suggested to me that her distinction between parameter and variable went further than "numbers". She used a parameter as a way to imagine sliding a one-variable function in a plane along a third axis, where the parameter values represented specific snapshots of that function. Imagining the snapshots being displayed simultaneously for every value on the third axis fit with Jori's use of a variable because she anticipated a variable representing all possible parameter values, creating a surface without going through the process of tracing out.

Excerpt 29 suggested to me that novice and expert shape thinking needed to account for how a student *generates a graph* and *interprets a graph*. I now believe that characterizing shape thinking an association the student has with a graph accomplishes this purpose. Jori was able to describe how the surface in space was generated, but when she focused on interpreting that same surface, she was focused on the graph as an object with physical properties. In both cases, she displayed some sophisticated methods of constructing the surface in space. However, she appeared to associate the surface with an object situated in the quadrants. An expert shape thinker, I believed then, should be able to imagine a graph being generated, and given a graph already generated, all the while imagining that graph as an emergent representation of two or more quantities' covariation.

My hypotheses about Jori's novice shape thinking and her distinction between parameter and variable were generated from this excerpt as well as my reflections from numerous teaching episodes. In Excerpt 30, I intended to gain insight into Jori's thinking about graphs of two-variable functions, as well as how she would interpret and use the notion of perspective. I intended to understand if

Jori's use of perspective could support her in imagining a surface in space and its generating functions in the context of the Box problem (Figure 36).

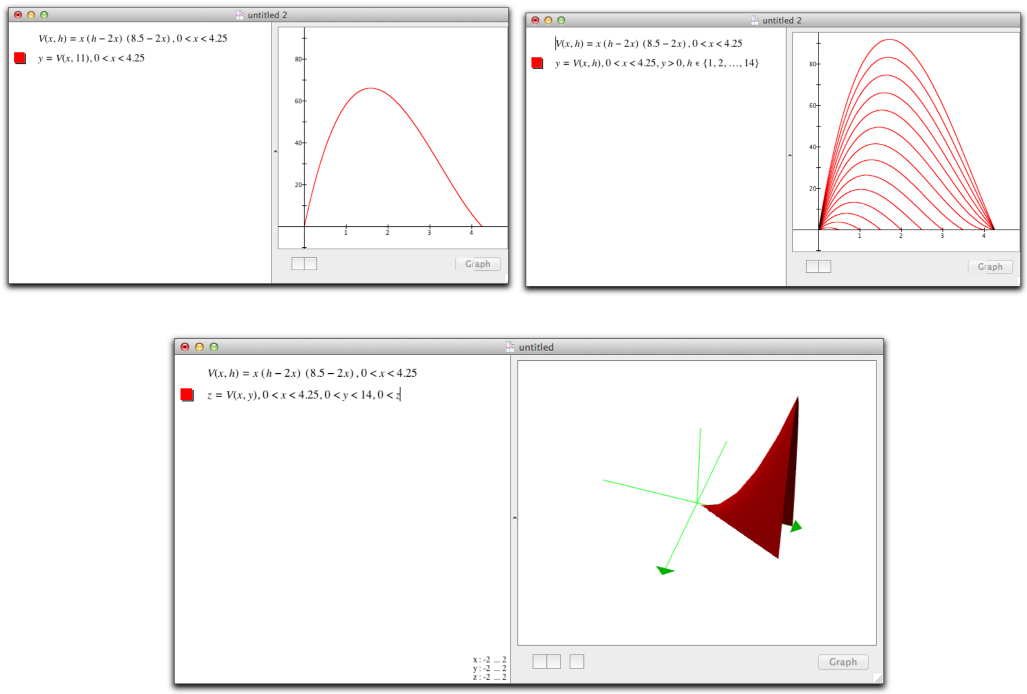

Figure 36. The Box problem graph's generating functions.

Excerpt 30 (Day 5 of TE, 00:00:55)

EW: We talked earlier about thinking about the z-x and z-y perspectives for functions. Can we do the same thing here? [Reveals Figure 37]

Jori: Sure, but it's more like volume-cutout and volume-length perspectives because we know what the things are we are measuring.

EW: You mentioned earlier the perspectives help you look at the function in different ways?

Jori: Yeah, well, for volume-cutout, I just treat length like a parameter, and volume-base, treat cutout like a parameter. I figure, I can do this and have cutout-base view too where volume is a parameter. But I can't visualize the exact sweeping out until I graph it. I mean, I know how it gets created, but I can't see it until I look at it in graphing calculator, either on the computer or in an actual calculator. [Looks at Figure 37].

EW: What about the process you just described? Does that help you think about how the function was created?

Jori: No, not really, because I really need to get an idea of what the surface looks like, its' kind of been like looking at maps for me, and imagining I'm looking at it from the East or the West. I sort of have to see the whole before I can see each view and part. Sometimes the parameter thing helps, but it's also hard to visualize.

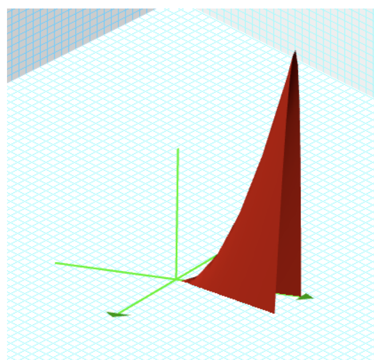

Figure 37. Graph shown to Jori in discussion about perpsectives.

Jori said she could imagine "how it gets created", referring to the surface in space (Excerpt 30, line 11). I believed she was saying "how" while imagining the process of generating a surface in space by sweeping out a function in the plane. Her description of volume-cutout and volume-length perspectives (Excerpt 30, lines 3-4) suggested she imagined parameterizing paper length and cutout length, respectively to produce the sweeping out. However, Jori's description of a looking at a map (Excerpt 30, line 16) suggested that even as she imagined a perspective as a way to generate a function's surface, she was quick to fall back on shapes in the quadrants. It seems that in both cases her thinking could have been non-quantitative. Even as she was describing the sweeping out, the surface that resulted was not necessarily imbued with quantities' values.

Day 6 - Functions in Space

To evaluate the viability of my hypotheses about Jori's focus on a graph as an object in the quadrants, I asked her to do two things. First, given the algebraic form of a two-variable function, construct its graph in space. Second, given the graph of an unknown two-variable function in space, describe a possible algebraic form for the function. I used the first activity because I anticipated that thinking about generating a surface given a two-variable function was not possible without using a graphing calculator. I anticipated she would rely on identifying the shape of the object without imagining the surface being generated from multiple perspectives. I used the second activity because I believed that interpreting a graph without the given algebraic form of the function, and asking her to create the generating function could not be done without her attending to the axes. I began the session by asking her to describe the behavior of the function $f(x,y)=x^2y^2$, without being able to graph the function in Graphing Calculator.

Excerpt 31 (Day 6 of TE, 00:00:04)

EW: We've been mostly, uh, working with applied situations, so I want to work with something you might see more often for functions. Let's say we have a function *x* squared times *y* squared. Can you take me through how you would think about this function?

Jori: It has squares in it, which make me think parabolas right away.

EW: Where do these parabolas come from?

Jori: The algebra says they are there I think. If you have a square, something is probably a parabola.

EW: Could you describe how you might think about generating the function from each perspective, before we get to graphing it?

Jori: I said, that's too hard, I'm not sure.

EW: Okay, from z-x, you said *y* is like a parameter. If that is the case, what kind of function do we have?

Jori: *a* times *x* squared? Yeah, so if *a* is one, or wait, any value, it's a parabola. Oh, same for the other perspective! They sort of combine then, right?

EW: What do you anticipate?

Jori: I need to look overhead somehow, but without graphing the function, that seems impossible.

EW: You mentioned earlier treating the output like a parameter, would that help?

Jori: Maybe, then it would be the x-y view, yeah! Then I could tell how the two other views sort of work together.

Jori's description of *f*(x,y) as involving parabolas because "it has squares in it" (Excerpt 31, line 5), and her description of the parabolas coming from the "algebra" (Excerpt 31, line 7), suggested to me that she was attempting to recall a shape that she associated with a particular set of symbols. Even in her description of the z-x and z-y views, she focused on connecting a symbolic representation she had associated with a particular shape (Excerpt 31, line 14). Her description of the perspectives combining (Excerpt 31, line 15), and using the x-y view to see how the other views combine (Excerpt 31, lines 22-23), indicated to me, in the moment, that she was describing an already generated graph of a two-variable function, rather than imagining its construction from multiple perspectives.

During a break in Day 6, I anticipated it would be problematic for Jori to analyze a graph of a two-variable function to determine the algebraic form of the generating function. I believed that unless she was able to identify shapes that readily fit with specific classes of functions she already knew, the situation would be untenable for her. For example, if she noted a feature of the graph that looked linear, she would conjecture that the function definition must include a linear term.

Excerpt 32 (Day 6 of TE, 00:00:37)

EW: For this last one, I'm going to do a little changeup. I'm giving you this graph [the graph was of $f(x,y)=x^2y$], but not telling you the function. Could you think about how to determine what the function is?

Jori: Ugh, no, this is like reverse thinking like integration, and I'm bad at that too.

EW: Just give it your best shot.

Jori: Alright, well, from this side it (Figure 35) looks like a line, and the other it looks like a curve, maybe a parabola? I would guess the equation has a curve and a parabola in it then, right?

EW: As you are looking at this graph, what is your strategy?

Jori: Well, I know what certain functions look like, like their general shape, linear and quadratic, so I look for those. If it is a weird shape it's hard for me to think about where it came from.

EW: It seems like you are saying if you haven't seen it before, it's hard to imagine how the graph was made by a function?

Jori: Yeah, I kind of rely on my graphing calculator for that, so it's really hard to just come up with a function, especially one the has to describe a whole surface.

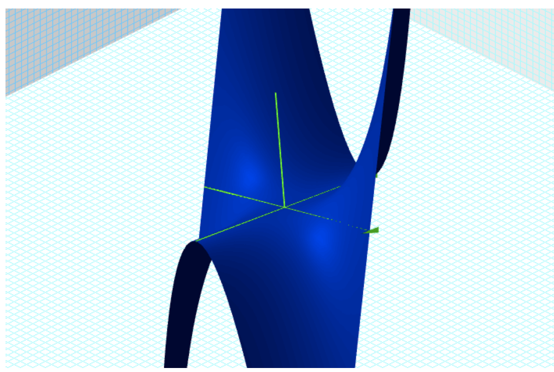

Figure 38. The surface in space Jori observed in Graphing Calculator.

It seemed that Jori saw a line from one perspective ($z = ay$) and a parabola from another perspective ($z = bx^2$) (Excerpt 32, lines 8-10). This led her to predict a squared term and a linear term in the function definition. Jori described connecting shapes of graphs to specific functions (Excerpt 32, lines 12-13), and noted she would rely on her graphing calculator to help her (Excerpt 32, line 17). I interpreted her to mean that she needed help imagining what object would be generated when a particular function as input. Her responses indicated to me, in the moment, that her definition of a function would need to fit with features of the surface that she recognized. Jori noted that she knew what linear and quadratic functions looked like, but that almost any other shape was hard for her to connect to a specific type of function. Her description of the graph as an object in this context supported my hypothesis that she thought about a function was just a set of symbols associated with a particular shape situated in the quadrants.

Part I Conclusions

My inferences about Jori's ways of thinking occurred in the context of already having conducted a teaching experiment with Jesse, which affected my instructional supports and the ways in which I wanted to engender ways of thinking about two-variable functions. My analyses revealed a number of constructs that became aspects of the more general concept of novice shape thinking.

Novice shape thinking.

Novice shape thinking occurs when the student associates a graph with an object in space, where that object is not an emergent construction. For example, Jori could imagine the process of how a shape was constructed in space, but often did not think about the resulting surface as the result of covariation. In my reflections, I wrote that Jori appeared to associate a graph with a *malleable object*. By malleable, I mean that she appeared to think about a wire (one-variable) or topographic map (two-variable) when she looked at a graph. A key component of her focus on a graph as a malleable object was her *focus on the quadrants* in the plane and in space. When Jori imagined graphing a function or interpreting an already constructed function, she was focused on the features of the object in space, without reference or attention to the functional relationships that these "features" represented. As a result, the graph was not an emergent construction that represented quantities having covaried.

The graphing calculator

Jori referenced using a graphing calculator thirty-nine times, each of which was in the context of "having the graphing calculator do that" for her. The majority of the time she did not know how to use a graphing calculator when she offloaded a particular task to it. The graphing calculator was a way to "punt" away her difficulties. For example, when I asked her how she would use a graphing calculator to help her visualize a two-variable function, she initially said she did not quite know how to program it, she just knew what she wanted it to do overall. This supports the notion that Jori could describe the general sweeping out process to create a surface in space, but sometimes imagined that process independently of imagining quantities covarying. I believed that she saw the graphing calculator as a tool to solve everything she did not understand. In particular, when she did not know what the graph looked like given a particular function, she could use the calculator to the graph an object in space.

Part II: Jori's Ways of Thinking about Rate of Change

The second part of this chapter focuses on Jori's ways of thinking about rate of change, and how those ways of thinking supported and constrained her in making sense of the rate of change of two-variable functions. My observations of Jori's class work, homework, and interactions during tutoring hours suggested to me that she equated rate of change with calculating slope, often independent of thinking about a relationship between two quantities. She described rate of change as the same "idea" as a slope because the calculations were identical. She also described the calculation of rate of change as the calculation of "something about a straight line". Her responses suggested that the meaning of an idea was equivalent to its calculation. Thus, she interpreted the value of rate of change as a feature of a straight line. I anticipated that her way of thinking about slope constrained her from thinking about rate of change as a function. For example, Jori described open form rate of change as slope and "just a fancy method that could be replaced by using a rule". The outline of days and particular activities discussed in Part II are in Table 19.

Table 19

Jori's Rate of Change Task Sequence.

Task Description	Date (Day)
Car A – Car B: Interpreting Meaning of the Graph Constant and Average Speed Activity	4/16/2011 (7)
Car A – Car B: Measuring Rate of Change Using the Graph	4/17/2011 (8)
Box Problem: Measuring the Rate of Change at a Point in Space	4/20/2011 (9) 4/23/2011 (10)
Parameters as Variables and Rate of Change	4/25/2011 (11)
Generalized Rate of Change Function and Direction	4/25/2011 (11) 4/27/2011 (12)

Day 7 - Car A-Car B Activity

I intended to use the Car A – Car B activity to understand how Jori's meanings for slope and rate of change were connected to ideas from her calculus course, such as open form rate of change and calculus triangles. I anticipated that she would find these ideas equivalent to the method for calculating them because of her focus on calculations. In the following excerpt, I asked Jori to interpret the meaning of the graphs in Car A – Car B crossing after one hour.

Excerpt 33 (Day 7 of TE, 00:00:10)

EW: In the Car A- Car B situation I just described, and here is the graph again, could you tell me what information the graph is providing?

Jori: The graph seems to give a location of the car as it travels, oh wait, a speed of the car as a travels over time. It looks like the cars meet each other after one hour has elapsed because the graphs cross.

EW: What does it mean for the graphs to cross in this context?

Jori: It means, um, they are meeting each other, wait, no, they have the same speed, I'm not sure if they are crossing each other's path at that time.

EW: Could you describe Car A's rate of change?

Jori: Well, you can tell the rate of change, which we know here is acceleration because we already have speed, is like the slope. The slope is getting less and less [points to point on graph after 0.2 hrs, 0.5 hrs], so the car is decelerating.

EW: When you say the slope is getting less and less, what do you mean?

Jori: The straight line along the curve would be getting more horizontal.

EW: What does that mean about the car's acceleration?

Jori: It means it is getting less, the slope is getting less, calculating it as a change in speed over a change in time. I think more horizontal means less, yeah [Constructs Figure 39].

EW: I noticed you calculated something over here, can you tell me what it is?

Jori: Yeah, I found a change in speed, 10 miles per hour, over a change in time, and that told me the slope.

EW: What do you understand that number to mean?

Jori: I'm not sure what you mean, it is just a slope, it tells you how steep the graph is at that time. Wait, well I guess in class, we've talked about average, so I think this is like the average acceleration of the car, the acceleration it had the majority of the time.

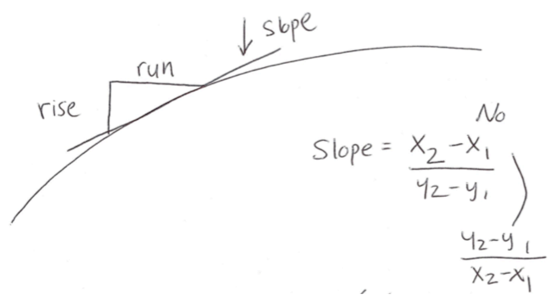

Figure 39. Jori's description of calculating rate of change in Car A – Car B.

Jori interpreted the axes correctly (speed and time), but she was still unsure if the intersection of the two graphs represented the cars having traveled the same distance in one hour (Excerpt 33, lines 7-9). It seems that Jori had a strong predisposition to see the cars' graphs as depicting trajectories instead of traces of correspondence points generated by covarying time and speed. As such, she had difficulty shedding the idea that the car's paths crossed where the graphs crossed.

I then asked Jori to describe Car A's rate of change, which she interpreted as acceleration (Excerpt 33, line 13), where she represented acceleration with a slope (Excerpt 33, line 12). Her gestures indicated that she was imagining slope at two points, after 0.2 hours and after 0.5 hours. Her description of the slope "getting less and less" (Excerpt 33, line 13) because of a "straight line along the curve getting more horizontal" (Excerpt 33, lines 18-20) suggested she was focused on a feature of the graph (a straight line). She indicated that she thought about the value of slope as a measure of steepness of a graph (Excerpt 33, lines 26-27), which she recalled should be associated with the word "average", or a property that a quantity possessed most of the time (Excerpt 33, line 29).

I believed, in the moment, that Jori was thinking about rate of change as an equivalent to slope, where slope meant the "slant" of a graph. Thus, I believed rate of change was simply a feature of an object in her thinking. Her description of the rate of change as the acceleration the car had the "majority of the time" (Excerpt 33, line 29) suggested to me that she was not thinking about measuring a ratio of changes in two quantities. Given Jori's lack of focus on rate of change as a comparison of changes, I inserted an extra activity into Day 7 that allowed me to better understand her way of thinking about steepness. I intended to draw her focus to the ratio of changes in two quantities.

Day 7 - Constant and Average Speed Activity

I posed the following situation to Jori: Suppose you have your car on cruise control, and as you pass mile marker 110, a truck passes you. Assume that you keep your cruise control set to the same speed, and you pass the truck at mile marker 117, six minutes later. I asked her to describe everything she could about the situation.

Excerpt 34 (Day 7 of TE, 00:00:35)

EW: What can you tell me about what is going on in this situation?

Jori: Umm, well, you basically are driving and you are going at the same speed the whole time, and the truck must speed up and slow down again because he passes you, and then you pass him about seven miles later.

EW: Could you say more about what it means to travel at the same speed the whole time?

Jori: Well, for any amount of distance, if you go that distance again it will take the same amount of time.

EW: Okay, if it takes you 55 seconds to travel one mile, how long would it take to travel 1.25 miles?

Jori: Well, I need to do some multiplication, but, it would just be 55 seconds plus a fourth of 55 seconds.

EW: Interesting, how did you get to that point?

Jori: Well, he always takes 55 seconds to go a mile, so a quarter mile will take a fourth of that because he is on a constant speed.

EW: And what about how far he would go in 123 seconds?

Jori: Well, divide 55 into 123, that many miles.

Her way of thinking about constant speed always taking the same amount of time to travel a particular distance (Excerpt 34, lines 8-9) supported her in thinking about how long it would take to travel some fraction of one mile (Excerpt 34, lines 12-13). Her understanding of constant speed can be described as if one travels *x* miles in *y* seconds, then in some fraction (a/b) of *x* miles, one travels (a/b) of *y* seconds, and vice versa. I anticipated that thinking about constant rate of change in this way would allow Jori to think about average rate of change as a comparison of changes. In the next excerpt, I asked Jori to think about the average speed of the car and the truck from the example I posed to her, and describe how she would measure average speed.

Excerpt 35 (Day 7 of TE, 00:00:58)

EW: So, what did you come up with in regard to the average speed of the car and the truck?

Jori: Well, at first I figured the average speed of the car is the speed he goes most of the time, which in this case is all the time, but then I remembered talking in class about average speed, and I know somehow I need to bring in a change in distance and time, which I guess doesn't work with speed most of the time.

EW: How do you see it not fitting?

Jori: Well, so, I was thinking if you just have a change in distance and change in time, the truck could have been going all sorts of crazy speeds between the miles, but his average speed is the same as yours because of the same distance same time, and he wasn't going the same speed as you most of the time, err, or didn't have to be I think.

EW: So, I want to say this, and have you explain what you hear me saying.

Average speed over a given time interval is like the constant speed another object would need to travel to go the same distance in the same amount of time. Think for a minute, and tell me what you heard.

Jori: Well, that makes sense, because all you know is the two changes, time and distance, and for your car going the same speed the whole time, the changes were the same as the truck. My most of the time thing is more conventional, not the mathy use of average I guess.

EW: Okay, and if you sketched a graph of the car's distance with respect to time and the truck's distance with respect to time between the mile markers, what do you anticipate?

Jori: Well, one would be curvy, the truck's, and the other would be straight.

EW: When you say curvy and straight, what are you paying attention to?

Jori: The car goes the same distance every minute, for example, so kind of like slope, you go up and over the same each time, so it's straight, but the truck doesn't need to do that, as along as he starts and ends at the same points.

Jori initially said average speed was the speed traveled "most of the time" (Excerpt 35, lines 3-5), but recalled a discussion from her calculus course about average speed resulting from changes in distance and time (Excerpt 35, lines 6-7). She described the car and truck as having the same average speed as the total changes in distance and time were equivalent for each vehicle (Excerpt 35, lines 9-13). She agreed with my description of average speed as a constant speed because "all you know is the two changes, time and distance" (Excerpt 35, lines

18-21), which she believed were the same for the car and truck. She described her definition of average speed as “conventional”, and mine as “mathy”. It seems that in earlier phases of this task, Jori’s “conventional” meaning is the one upon which she relied.

I believed, in the moment, that it was critical for Jori to represent average and constant rate of change. Her description of the graph’s of the car and trucks’ distances with respect to time as “curvy” and “straight” (Excerpt 35, line 25) suggested she was thinking in terms of shapes. Her description of the straight line as generated by going up and over the same each time (Excerpt 35, line 28), but I believed she was focusing on the slope without imagining measuring the rate of change of the car’s distance with respect to time. I asked Jori to construct a function that would measure the “exact” speed at which the car was traveling, in anticipation that the issue of exact speed would allow us to talk about average rate of change over very small intervals of input.

Excerpt 36 (Day 7 of TE, 00:01:10)

1 Jori: So, if I wanted to make that function to get exact speed, I know I
2 would have to look at a really small change in time, like 1 second, or
3 maybe even less than that, and how far I went in that small time.
4 EW: Okay, so how would you create a function that would do that for you?
5 Jori: Well, let’s see, oh, like slope? Umm, this is just like the open form
6 from class!
7 EW: Say more.

Jori: Well, I just have $f(x+h) - f(x)$ on top, and h on the bottom, and I have always thought of it like slope, except with function notation.

EW: Okay, how would that be related to exact speed?

Jori: You could just make h really small, that's like making time small, then the top tells you how far you went, and then you divide.

EW: Does that tell me an exact speed?

Jori: Almost exact, I guess. For .0001, it's pretty much exact.

EW: But, is it exact?

Jori: No, I guess. It's not exact, it's just an average, oh constant speed, but it's pretty much exact, we can't really see that small of a change, it's too quick.

Jori's thought about exact speed as a change in distance over a very small change in time required to travel that distance (Excerpt 36, lines 1-3). She assumed that exact speed could be measured by making the change in time small enough. She recognized this situation as appropriate for "open form from class" (Excerpt 36, lines 5-6), which indicated that she *could* see open form rate of change comparing the changes in two quantities, although this is not the meaning that came first to mind for her. However, Jori also appeared to think about open form as slope (Excerpt 36, lines 8-9). Jori anticipated that to measure exact speed, h (change in time) must be small. She suggested if h was .0001, the speed would be "pretty much" exact (Excerpt 36, line 14). Jori eventually said that she was thinking about open form rate of change as equivalent to an average, and in turn, constant rate of change (Excerpt 36, lines 16-17).

At the conclusion of Day 7, I suspected that Jori's preferred meaning for rate of change was steepness of the line segment between two points. When interpreting rate of change in the context of a graph, Jori rarely referred to measuring the change in one quantity in terms of the other. It appeared her primary way of thinking about slope was a degree of slant between two points.

Day 8 - Jori's Reflection on Rate of Change

I used Day 8 to gain insight into how Jori was thinking about average rate of change, open form, and calculus triangles. I believed that it was important for me to understand how she related those representations in order to understand how she would think about rate of change in space.

Excerpt 37 (Day 8 of TE, 00:00:12)

EW: So, after the last session in which we talked about the meaning of average and constant speed, what do you understand now that you weren't so comfortable with before?

Jori: I guess, umm, I see how average speed is like a second trip sort of idea that produces a same change in distance during a same change in time, and it can happen over really small intervals to get more exact.

EW: How is that related to your overall understanding of rate of change?

Jori: It's interesting, because I see it is kind of related, it is almost like open form rate of change, but not like closed form. It also seems like the calculus triangle thing we work with in class, but I'm not sure exactly how. I know in open form and the calculus triangle, like the trip we talked about, you can make the input very small, a slider value of h, to calculate a pretty accurate rate of change. You can sort of pull out the calculus triangle on a graph to do this when you need a rate of change value.

EW: You also mentioned open form?

Jori: Yeah, open form is kind of like the calculation behind the calculus triangle, it gives you, like an average speed, or rate of another trip in the trip example you and I talked about earlier. This is all different from closed form, which is like an exact rate of change, a perfect one that does not involve an h. So basically, thinking graphically, it tells you the degree of slant of the graph, which is related to speed, I think.

Jori used a "second trip" to think about average speed, which led to her focus on changes in distance and in time (Excerpt 37, lines 4-5). She also indicated rate of change could be thought of over "really small intervals" (Excerpt 37, line 6), which suggested she was thinking about average rate of change as an approximation. I believed, in the moment, that she was thinking about open form as the calculation behind the calculus triangle (Excerpt 37, line 17-18). However, I believed that the meaning she attributed to what I called rate of change had little to do with rate. By this, I mean that she was able to use open form rate of change and a calculus triangle, but was unsure how to interpret the result of that calculation. She indicated she saw the measurement as "speed" (line 22), but was unsure as to how it related to speed.

Jori seemed to be making shifts in the moment, such as developing meaning for average speed and its measurement, but was also falling back on her tendency to calculate without interpretation. I anticipated that Jori would focus on drawing similarities between the calculation of rate of change in the plane and rate of change in space. Thus, I anticipated asking her to describe the meaning of rate of change in space would provide insight into what she believed the calculations of rate of change represented. Based on her descriptions, I believed she would describe rate of change in space as a degree of slant at a point.

Day 9 - Sweeping Out a One-Variable Function

Day 9 focused on rate of change in space. I anticipated that Jori would think about rate of change at a point in space as a degree of slant between two points, but representing the correct changes would produce host of issues. I believed she would be able to visualize a calculus triangle in the z-x and z-y perspectives, but I anticipated her focus on calculations and not on meaning of rate of change would make it difficult to see a direction in space as important. I began our discussion by asking Jori about the sweeping out of the function $h(\mathrm{x}) = a(f(\mathrm{x})\text{-}g(\mathrm{x}))$.

Excerpt 38 (Day 9 of TE, 00:00:06)

EW: In the function where we talked about thinking with depth and sweeping out a new surface, how would you approach talking about rate of change? What might be the same or different from just a one variable function?

Jori: Well, man, this is pretty weird to think about. It’s like *z* is changing with *x* and *y*, so it can’t just be a simple description. My first guess, though I have no idea if I am right, is that, hmm, I don’t know.

EW: Assume we had a rate of change for functions of two-variables, and it returned a number like 3.45. How would you interpret the meaning of the 3.45?

Jori: I'd say it was an average change, constant rate of change, but I'm not sure of what with respect to what, especially when *x* and *y* are both involved. It might be like a combination of the rates from the z-x and z-y perspectives we just talked about, like two rates in different directions that you need to combine I think. Then you can make it like one variable thing. I'd probably just sum the two rates of change, one for z-x and one for z-y.

Jori's description indicated she was imagining *z* changing with respect to *x*, and *z* changing with respect to *y* (Excerpt 38, lines 5-6). Because I anticipated she was thinking about rate of change as a combination of degrees of slant, I asked her to interpret a rate of change value. She described the value as an average, constant change (Excerpt 38, line 11). However, she was unsure of how to describe *z* changing with respect to *x* and *y* (Excerpt 38, line 13). She proposed combining (Excerpt 38, line 15) the perspectives' rates of change because she wanted the calculation to be similar to what she was comfortable with in functions of one-variable. After this excerpt, she made two conjectures about rate of change in space (Figure 40). I asked her to solidify her conjectures and discuss the strengths and weaknesses she saw in each rate of change function.

I anticipated she was focused on combining the rate of change in each direction to account for being in space. I believed that the purpose of her combination was to make the structure of the two-variable rate of change function similar to the two dimensional case.

$$r_1(x, y) = \frac{V(x+h) - V(x) + V(y+h) - V(y)}{h}$$

$$r_2(x, y) = \frac{V(x+h, y+h) - V(x, y)}{h+h}$$

Figure 40. Jori’s conjectures about a two-variable rate of change function.

Excerpt 39 (Day 9 of TE, 00:00:37)

EW: Let’s look at the conjectures you developed. You mentioned looking at rates of change in each perspective. Would it help to think about calculus triangles in each perspective?

Jori: Sure, umm, I think that’s what I am doing in my first conjecture here, I kind of have a z-x and z-y calculus triangle, and I am adding them together. I sort of want to take the calculus triangles and make a calculus wedge of cheese (Figure 41), like a three-dimensional object made up of the two triangles, but I don’t know how that might work.

EW: Alright, can you say more about where your calculus triangles are in your first conjecture, because you said you had them?

Jori: Yeah, well, I took the rate of change for a function of *x* and a function of *y* and basically combined them.

EW: What do you anticipate that your first rate of change function will do?

Jori: It takes a rate of change for *x* and *y*, and combines them to make a total rate of change.

EW: Alright, could you say more about your second conjecture?

Jori: Yep, this one was harder, but basically I was thinking, there is a two variable function, so we have to use both variables, so I figured you have to add *h* to each *x* and *y*, so you get a change on top that does what the first conjecture did, but in a shorter calculation, then you get the change on the bottom, which comes from adding *h* two times.

EW: Okay, in your second conjecture I noticed you are using *h* for the two views that you are adding together. Are they the same *h*?

Jori: Yep. These are the same, but, mmm, I'd guess they don't need to be because you can pick any value you want for each of them, huh? I guess for the one calculus triangle in each perspective, the little bit you add could be whatever you wanted.

Jori's description of imagining a *single* calculus triangle in the z-x and z-y perspectives (Excerpt 39, lines 4-6), and combining those triangles to make a calculus wedge (Excerpt 39, line 7, see Figure 40) suggested to me she was attempting to combine features of two one-variable functions from the z-x and z-y perspectives. She attempted to mimic the structure of the one-variable open form rate of change function in both conjectures, and noted the difference between the two was primarily in whether the function *V* should include *x* and *y* simultaneously. Jori's second conjecture did allowed her to consider that the change in *x* and *y* need not be equivalent (Excerpt 39, lines 24-27).

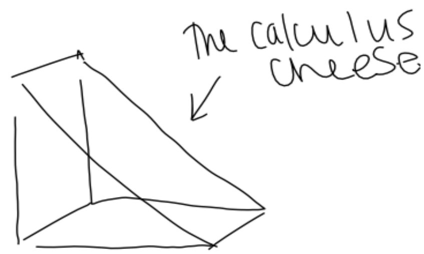

Figure 41. Jori's drawing of the calculus cheese.

In my reflection after Day 9, I surmised that Jori thought about rate of change as a feature of a graph that could be combined with other features by simple calculations. Jori's model for understanding rate of change at a point in space was to consider how to perform an operation on the rate of change value from the z-x and z-y perspective. Essentially, she imagined a slope from each perspective to produce a total slope at a point in space. Her focus on making the two-variable rate of change function structurally equivalent to the one-variable rate of change function created many issues when she attempted to combine two rates of change. The calculus cheese appeared to be an ad hoc way of accounting for rate of change in two directions. It seems that once she constructed the calculus cheese, she believed it was plausible to add the two rates of change as it fit with her need to "combine".

Day 10 - The Box Problem

On Day 10, I intended to evaluate the viability of describing Jori's thinking about rate of change of a two-variable function as the result of combining the degree of slant between two points, each obtained using the z-x and z-y perspective. I also intended to affect this way of thinking by asking her to construct calculus triangles from each perspective using function notation. By asking her to construct the calculus triangle in the z-x and z-y perspectives, I intended Jori would think about the comparison of the "legs" of the triangles as representing partial rates of change. I also anticipated that Jori's novice shape thinking and understanding of graphs as objects was foundational to her notion of slope and rate of change and would make any major shifts in her thinking momentary.

Excerpt 40 (Day 10 of TE, 00:00:07)

EW: You've had a little while to look over the candidate rate of change models that you came up with before, how are you thinking about them now?

Jori: I realized that what I was doing before would not quite work, the calculus triangle from each perspective would have to include a consistent y in the z-x view or consistent x in the z-y view, or I mean, I have to account for that variable. So I would use $f(x+h,y) - f(x,y)$ for z-x and $f(x,y+h)$ for the z-y view numerator [Sketches Figure 42]. I hadn't accounted for it being two variables before.

EW: Okay, great, that's really insightful. So could you say more about what you mean by consistent?

Jori: Well, by consistent, you have to keep the variable steady, so you can look at a normal calculus triangle, so I could give *y* a value of three, then be looking in the z-x perspective, like here [points to Figure 42].

EW: Also, remember when we talked at the end of the last session about adding a little bit to *x* and a little bit to *y*? Could you recap?

Jori: Well, I knew we had brought up the issue, where the *h* values in my second rate of change idea didn't need to be the same. I also thought more about the adding the little bit to *x* and adding the little bit to *y*. I think you can add whatever you want to each of them, which makes coming up with a rate of change a little difficult because you have so many choices for those values.

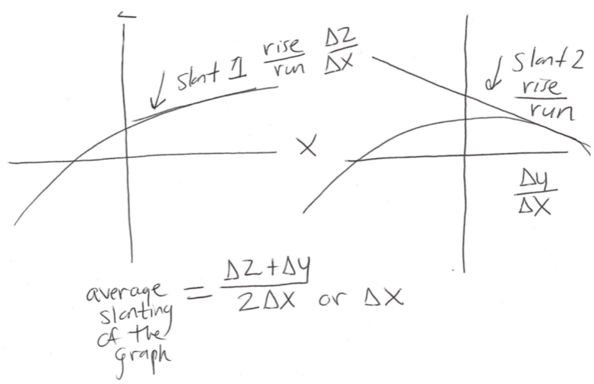

Figure 42. Jori's depiction of slope in the z-x and z-y perpsectives.

Jori's shift from both rate of change conjectures resulted from a method to consider a calculus triangle representing change in one-variable while working within a two-variable function. Her description of a "consistent" variable (Excerpt 40, lines 5-6, 12-14), suggested to me, in the moment, that she was visualizing a cross section of the function's graph from a particular perspective to consider a calculus triangle from both the z-x and z-y perspectives. In creating calculus triangles by treating variables as "consistent", Jori was able to explain how the changes in the independent variable in each calculus triangle were not dependent on each other (Excerpt 40, lines 17-19), which she believed was problematic because there were so many possible changes to use (Excerpt 40, lines 21-22). Jori's description of a calculus triangle in both the z-x and z-y perspectives, and her problematizing of the changes in *x* and *y*, suggested that she could understand the need for a direction in space. Day 10 ended due to an unexpected class conflict for Jori, and we picked up with discussion on Day 11.

Excerpt 41 (Day 11 of TE, 00:00:03)

EW: You said earlier you could add a little bit to *y* and a little bit to *x* in the rate of change function. What issues did that bring up?

Jori: Well, it seems like there'd be no way to get a consistent rate of change if you are wildly picking those two values all the time, but like I said before, I don't see any way to deal with that.

EW: We've looked at these calculus triangles from the z-x and z-y perspective, how could the x-y perspective help us?

Jori: I can see what overhead direction I'm looking? I don't know…

EW: Sure, let's build on that, suppose you are going in a direction where the change in y is twice as large as the change in x.

Jori: Well, that would follow this line [indicates a linear trend from a point in space] h_2 would be twice as big as h_1 then, I think.

EW: What if you made one of them small, how does that affect the other?

Jori: That makes it… oh! It makes it small too, like just by doing one thing you end up doing two.

EW: Okay, so what would happen if you made one change small, how would that affect the other change?

Jori: Well, it would get small slower because the one change will always be two times as much the other, but eventually it could be pretty small.

EW: So how would this be related to the calculus triangles we were talking about earlier? What would be the same and what might be different?

Jori: You are kind of bringing the triangles together, but not quite in a wedge, more like into a direction.

When I suggested that Jori consider the x-y perspective *y* (lines 6-7), she described an "overhead direction" (Excerpt 41, line 8). She used her fingers to indicate many possible directions emanating from a point in space being viewed from the x-y perspective. I described a direction where the change in *y* was twice as large as the change in *x* (Excerpt 41, lines 9-10). Jori interpreted this as a specific direction emanating from the point in the x-y plane, making h_2 (the change in *y*) twice as large as h_1 (the change in *x*). Jori's response to my question about how h_1 and h_2 behaved when one of them became small (Excerpt 41, lines 13,16-17), suggested that she had an image of the changes in *x* and *y* becoming small simultaneously (Excerpt 41, lines 14-15). Jori also imagined that fixing a proportional relationship between the changes in *x* and *y* was similar to describing a calculus triangle in a direction (Excerpt 41, lines 22-23).

I asked Jori to consider how she might adjust her two rate of change conjectures in light of the discussions we had about rate of change in a direction to understand. I anticipated that while Jori had problematized that rate of change could have a direction, the focus on direction was less important to her than abstracting the structure of the one-variable rate of change function.

Excerpt 42 (Day 11 of TE, 00:00:31)

Jori: Alrighty, well, going back to my conjectures, I like number 2, but I need to specify an h_1 and an h_2, but there is also the denominator issue.

EW: What issues are you seeing?

Jori: I don't think I can have both of them down there, because I just want to make one of the two values small, the h-values.

EW: Well, what happens if you just make one of them small?

Jori: Oh yeah, then the other gets small as well, sort of by force. I think I can just leave one of them in the denominator then, and put the numerator in terms of h_1 because I can define one change in terms of the other when I know the ratio of their changes, like a direction.

EW: Okay, great, so if you have defined the function, let's say you get an output from your function of 2.95. How would you interpret that number?

Jori: Like it has sort of been all along, it is like the total slope or slant at that point, looking in a direction.

EW: Could you imagine it as related to average or constant rate of change?

Jori: Umm, well, slope is like a constant rate of change, so you found the constant rate of change of the graph.

Jori's solution to the denominator issue was to include just one of the *h* values. This decision was dependent on her imagining that as h_1 decreased and became small, h_2 was forced (Excerpt 42, line 6) to become small as well. By imagining that h_1 depended on h_2, Jori was able to write h_1 in terms of h_2 by using a direction. Even as she attempted to make the structure of the two-variable rate of change function like the one-variable rate of change function, she was able to problematize that change in space has a direction. Jori developed an accurate rate of change function, but her description of the output of the function as measuring a total slope or slant (Excerpt 42, line 15), suggested she saw the function as a method of combining two partial rates of change, where each rate of change was a slant between two points. However, it was not clear in what sense she understood rates to be combined. She associated constant rate of change with slope (Excerpt 42, line 18), but noted the output of the rate of change function was the constant rate of change of the graph (Excerpt 42, line 19). Thus, while she developed important ways of thinking about the open form rate of change function, she thought about rate of change in a superficial way.

Part II Conclusions

Jori developed sophisticated ways of thinking about how to calculate rate of change, even as she had difficulty interpreting the meaning of rate. She appeared to make inroads in thinking about the meaning of rate, but was handicapped by having "graph is a shape" filter into her thinking. Even when she was thinking about graph as a shape, and rate of change as a measure of a slope on that shape, she made progress in thinking about rate of change in space.

Jori focused on the making the calculation of a two-variable rate of change function structurally similar to the one-variable case. Her construction of the calculus cheese was a way to illustrate her focus on combining rates of change in the z-x and z-y plane. Once she determined she needed to “combine” rates of change, and addition was the most viable operation to do so, she developed conjectures about a two-variable, open form rate of change function. She attempted to mimic the structure of the one-variable rate of change function in performing this addition. Even as she focused on drawing structural similarity between one and two-variable rate of change functions, she broached the two changes problem in two-variable rate of change. She was able to accommodate this problem without focusing on the meaning of rate of change at a point in space because she recognized there were two rates. She eventually grappled with the issues of imagining h_1 and h_2 varying in tandem from considering a direction in the x-y plane, which allowed her to develop an accurate two-variable open form rate of change function, her interpretation of the function’s output did not change. She thought about the output of the rate of change function as a slope.

Chapter Summary

In this chapter, I illustrated how Jori’s attention to a graph as an object in the quadrants, sometimes without reference to what the object represented, led me to hypothesize, within the teaching experiment, that she used novice shape thinking. I described how Jori’s understanding of graphs as shapes and attention to slope and calculations affected the development of ways of thinking that I intended to support in the teaching episodes. While she relied heavily on calculations and focused on graphs as shapes in the quadrants, she did think with some sophistication within the teaching episodes. However, her thinking often gravitated back to shapes and slopes.

Chapter 7

BRIAN AND NEIL'S THINKING

This chapter describes Neil and Brian's ways of thinking. Neil and Brian were enrolled in the same calculus course as Jesse and Jori. They participated together in this teaching experiment, which changed the one on one dynamic of the previous teaching experiments. The modified dynamic allowed me to observe their conversations and the ideas they developed in the course of those conversations. Thus, the basis for the development of my models of their thinking included conversations between Neil and Brian as well as their interactions with me.

In this chapter, I describe the how Brian and Neil made the programming of a correspondence point the central issue of the teaching experiment. They used a correspondence point to think about generating graphs of one and two-variable functions. They also used the idea of programming to think about how to generate a rate of change function in one and two variables. I describe the development of their use of a correspondence point, and argue that it provided coherence for their ways of thinking about the functions tasks and rate of change tasks.

Background

Brian and Neil worked with each other throughout the semester in calculus, and their facility at discussing ideas with each other while focusing on ways of thinking contributed to their engagement with the teaching experiment tasks. Neil was a computer science major, and described himself as passionate about mathematics. He had taken advanced mathematics in high school and at community college, but because he was returning to school after a break, had not remembered a lot of the “rules and procedures” he needed in order to be successful. He described his use of graphing calculators and technology as a way to “remedy” the procedures he had forgotten. He felt that it was easy to unload the calculations onto the technology. He looked forward to his calculus course because of the heavy use of technology. Brian was a construction management major who had returned to school that semester and chosen to take calculus because he was interested in mathematics. He was typically a detached student who was not interested in participating or making sense of ideas, but said his experience in calculus that semester had fundamentally changed how he thought about mathematics.

Initial Inferences about Ways of Thinking

I started interacting with Brian and Neil when I began the first teaching experiment with Jesse, and met with them as a group six times prior to the beginning of the teaching experiment. My observations and exploratory teaching interviews with them took place until the end of Jori’s teaching experiment. As I created initial inferences about their ways of their thinking, I found that I interpreted what they were saying and doing through the lenses I was developing in the ongoing teaching experiments.

Brian could think about a graph of a function as a shape, but with the understanding that the shape moved in response to a transformation of the relationship the function represented. For example, he described a quadratic function as “tilting upward”. He described how the tilting was generated by the output becoming so many times larger than the input as the rate of change increased. Brian made little distinction between the graph of a rate of change function and the graph of a "normal" function. He said, "you can think of any graph of a function as a rate of change function, no matter what it is".

Neil focused on properties of graphs, such as slant and openness. For example, in describing an exponential function, he said the "graph goes up quickly", but could not differentiate its behavior from a quadratic function whose graph was similar. I anticipated his attention to the graph as an object could constrain him in imagining a generating point for functions of one variable and a generating function for functions of two-variables. Neil also was able to quickly calculate a rate of change given a function definition, but he interpreted the rate of change as a slope, which was not always connected with how fast a quantity was changing with respect to another. During my interactions with them, I noted that I was using facets of expert and novice shape thinking. Thus, unlike with Jesse, the idea that someone could use aspects of both types of reasoning was part of my initial hypotheses about their thinking.

Part I: Brian and Neil's Ways of Thinking about One and Two-Variable Functions

The teaching experiment began after my reflections on Jori's teaching experiment were completed. The sequence of activities followed the same outline as my teaching experiment with Jori, but the timeline was compressed due to time constraints of Neil's summer work. As a result, the sessions were longer, but the same activities were covered (Table 20). Note the first two activities took place on the same day (5/5/2011), but were 5 hours apart. I conducted analyses between the sessions, and thus, counted them as distinct days.

Table 20

Task Sequence for Brian and Neil's Teaching Experiment.

Task Description	**Date (Day)**
Homer: Correspondence Points and Graphs	5/5/2011 (1) (10am)
Homer: Moving Cities and Varying Animation	5/5/2011 (2) (3pm)
Generating Graphs of $f(x)=x^3, g(x)=2x$	5/7/2011 (3)
Generating Graph of $h(x)=x^3-2x$	5/7/2011 (3)
Anticipating Behavior of $m(x)=a(x^3-2x)$	5/7/2011 (3) 5/9/2011 (4)
Treating *a* as a Third Axis	5/9/2011 (4)
Box Problem: Box Construction and Treating Parameters as Variable	5/12/2011 (5)
Generalized Functions: Construction and Interpretation of Graphs	5/12/2011 (5)
Car A – Car B: Interpreting Meaning of the Graph	5/13/2011 (6)
Car A – Car B: Measuring Rate of Change Using the Graph	5/13/2011 (6)
Generalized Rate of Change Function and Direction	5/15/2011 (7) 5/17/2011 (8)
Working with Strange Functions (More than 2 Variables)	5/19/2011 (9)

Day 1 – The Homer Activity

Brian and Neil agreed that as Homer was moving back and forth on the road, his distance from each city was "changing at the same time". They illustrated the two distances varying by using their index fingers and a fixed reference point created by a coin (Figure 43). As Homer moved back and forth on the road, their motions indicated they accurately tracked how the distances were varying. They also noted that they were in agreement about how to move their fingers to track the two distances. Neil clicked the show correspondence point before I had introduced it. I described the correspondence point as a representation of Homer's distance from Springfield and Shelbyville simultaneously.

I asked Brian and Neil to describe their understanding of my description of the correspondence point. They agreed that the correspondence point would "always keep track of both distances, one on the y-axis and one on the x-axis". Brian described correspondence point as the result of going "over and then up", where the "over" was given by a distance from Shelbyville and the "up" was given by the distance from Springfield. The following excerpt occurred after I asked Brian and Neil to anticipate the correspondence point leaving behind a trail as Homer moved on the road. Both students noted that what they were doing was creating a graph (Figure 44), but their ways of thinking about the graph's construction were different.

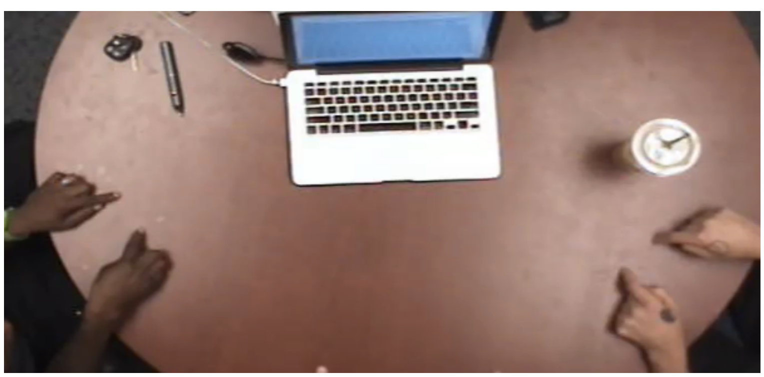

Figure 43. Brian and Neil track Homer's distance from the two cities.

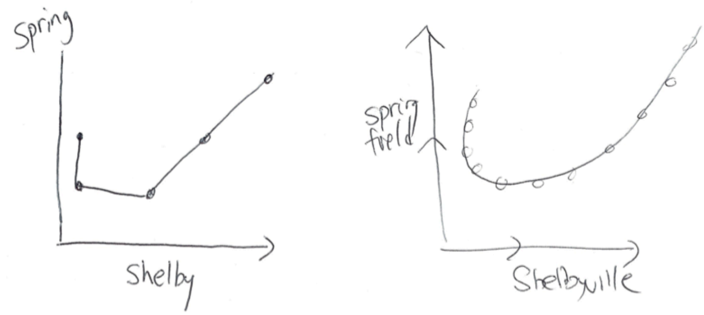

Figure 44. Neil (left) and Brian's (right) anticipated graph in the Homer situation.

Excerpt 43 (Day 1 of TE, 00:00:36)

EW: Okay, so can each of you tell me how you are thinking about your graph?

Neil: Sure, I can. Well, so what I did was I, well, when I make a graph I take a few highlight points, which are kind of like fixed points that I can use. So when Homer was moving on the road, there were a few places where I was pretty sure I knew how far he was from both cities, so I made that a bold point. Then I connected those bold points with lines, sort of like estimating how he traveled, er, how his distances changed I guess as he moved on the road. I sort of filled in the trend I noticed from the points for estimation.

EW: Okay, Brian, what did you hear Neil say, and how is your graph the same or different in your mind?

Brian: Well, I sort of also had the idea of taking highlight points, because it's hard to keep track of every point, but I think if you did keep track of every point, my graph would fit better because it is what you would get from making a huge number of highlight points, a smoother curve I think, where you get both distances all the time.

EW: Neil, what do you think about what Brian said?

Neil: Well, I guess I don't see the difference, we are both guessing about how the points we chose are connected, both following the trend that we see.

EW: Alright, I noticed both of you also had talked about programming a bit before?

Brian: Yeah, well, we agreed that the correspondence point is programmed to keep track of the two distances, kind of feed the program the two distances, except I think Neil was going for getting a highlight point every so often, where I was thinking about how to illustrate that there is always a highlight point.

Neil: See, I don't see why there needs to be a whole bunch of highlight points? I think my estimates do a good enough job.

Neil and Brian's graphs suggested to me, in the moment, that they were thinking about interpolating between correspondence points in different ways. Neil appeared to think about linear interpolation, but Brian took progressively finer approximations (shorter changes in Homer's distance on the road) to construct his graph. I believe he took these finer approximations by imagining that the quantities had measures simultaneously over infinitely small intervals. Neil's described highlight points as fixed points that represented Homer's distance from each of the cities simultaneously (Excerpt 43, lines 4-7). However, I believed he was not imagining highlight points existing between the highlight points he chose. The reason for this is not clear. It is plausible that he was unable to construct more highlight points, but it is also possible he found it sufficient to trace out a rough approximation of the correspondence point's graph. At that time, I believed he was not imagining the quantities covarying *within* the intervals, whose endpoints were defined by the highlight points and that he interpolated by fitting a shape between highlight points plotted. This suggested he was guessing at the graph by using an shape approximation (Excerpt 43, lines 8-10). I believed he imagined the ideal graph being smooth, but the smoothness was the result of taking more highlight points, but not an infinite amount.

Brian also used highlight points to construct his graph, but he was thinking about interpolating between these highlight points with more highlight points (Excerpt 43, line 13). Brian anticipated that interpolating between the highlight points would generate a smoother curve (Excerpt 43, lines 16-17). I believed his notion of a smooth curve came from imagining every point on that curve representing Homer's distance from each city simultaneously. Neil's image of smoothness came from generating a large number of linear segments. I believed that Brian saw his graph as different from Neil's because he was imagining an infinite number of correspondence points so that there was a highlight point everywhere on the graph (Excerpt 43, lines 27-28). It appeared that Neil saw no difference between the two graphs because he thought both graphs followed the trend of the highlight points (Excerpt 43, lines 19-21).

I believed, in the moment, that their differences in thinking about the graph arose from Brian attending to a graph as a representation of the continuous covariation of Homer's distance from Springfield and Shelbyville. I thought that Neil constructed a graph from "choppy covariation", by which I mean he imagined the two quantities values' existing simultaneously, but he was only able to measure them at the endpoints of intervals of conceptual time, not within those intervals. Thus, he constructed a finite number of highlight points.

I hypothesized, within the session, that Neil was thinking about a correspondence point only when he produced highlight points. Thus, I asked Neil questions about graph between the highlight points to understand if he saw a need need for a continuous tracking of the correspondence point.

Excerpt 44 (Day 2 of TE, 00:00:21)

EW: Okay, Neil and Brian, back to it. I want to go back to a comment Brian made about the difference in how your graphs were constructed. He talked about keep track of the distances all the time, not just at highlight points.

Neil: Well, why would need more than the highlight points, isn't that just enough to make your graph?

Brian: Hold on, I'm gonna jump in here. I think where we differ is that the graph has to be all correspondence points. Like, in your straight lines in your original graph, I don't think those are correspondence points, right?

Neil: Well, no, but I don't see why they need to be at all, it just sort of fills in between the information that we already know.

EW: Look at it this way, if I point to a highlight point, what information do I have?

Brian: You get two exact distances.

EW: What if I point to any place on the straight line you have here?

Neil: You just get an estimate of the two distances I think, because I just guessed at how to fill them in, but that seems good enough.

EW: Okay, so when you make a graph of another function, like $f(\mathrm{x}) = 2x$, is an approximation good enough?

Neil: Well, not, because there you just multiply by two, but if it is really complex function that behaves weird, when I am making the graph I have to estimate between highlight points, I can't be totally accurate (Figure 45).

EW: Okay, so when you are estimating, what do you mean?

Neil: I'm following the general trend of how to connect the highlight points.

Neil believed a finite number of highlight points were sufficient to construct a graph (Excerpt 44, lines 5-6), and his conjecture fit with his thinking that a graph was an approximation (Excerpt 44, lines 17-18). Brian believed Neil was thinking about a graph as plotting of a finite number of correspondence points followed by "estimation" between those points (Excerpt 44, lines 21-23, 26). At the conclusion of Day 2, I hypothesized that Brian was thinking about a graph as representing the simultaneous variation of two quantities' values over a continuous interval of time. I thought Neil was thinking about a graph as a shape that connected discrete correspondence points.

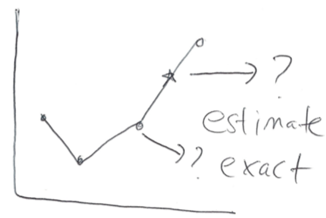

Figure 45. Neil's depiction of exact and estimated values on his Homer graph.

Day 3 – The Difference Function

I used the difference function activity on Day 3 to evaluate my hypotheses about Brian's thinking about a graph as an infinite number of highlight points and Neil's understanding of a finite number of highlight points. I anticipated that it would be critical to use Brian's descriptions of a graph to help Neil develop an image of tracking the two quantities' values continuously. Thus, I intended to encourage discussion between Neil and Brian about the similarities and differences they perceived in the construction and appearance of each other's graphs.

Neil and Brian graphed $f(x) = 2x$ and $g(x) = x^3$ individually on sheets of paper, and then shared them on a whiteboard which they worked on collectively (Figure 46). Their graphs of $g(x)$ appeared to be similar, and Neil believed that he and Brian had come to the same result (graph). I asked them why I saw no highlight points in their graphs, yet they had both identified a set of highlight points in their graph of the Homer situation. Neil said he already knew the graph of the function *g*, and thus did not need to go through constructing the graph. Brian said his graph was an infinite set of correspondence points.

I hypothesized, in the moment, that while their graphs were visually similar, the way in which they imagined constructing them was not. I believed that Neil saw no need to use a correspondence point if he already knew the "shape" of the graph because the graph was already constructed. However, I believed he still thought about a graph he knew as a shape that fit a number of highlight points.

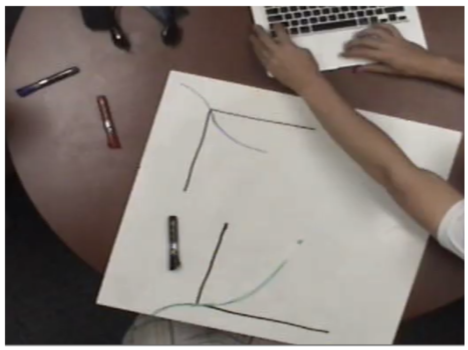

Figure 46. Brian and Neil's graphs of $g(x) = x^3$.

In the following excerpt, Brian and Neil discussed generating the graph of the function h(x) = f(x) – g(x). I anticipated that they would not be able to fall back on already having memorized the appearance of the function, which would allow me to gain insight into their understanding of a graph's construction.

Excerpt 45 (Day 3 of TE, 00:00:21)

EW: So, suppose you want to construct the graph of the difference function, *h*(x), talk to each other about how you would approach it.

Neil: Brian, we kind of like to use the programming idea, so let's program the input and the output. So, tell me if you disagree, but the input, kind of like Shelbyville is *x* and the output, kind of like Springfield, is *h*(x)?

Brian: Yeah, I agree with that, makes sense.

Neil: Okay, so now like in our graphs in Homer, we can make some highlight points. So, we can plug in 1, 3, 5, 7 for *x* and find *h*(x).

Brian: Okay, but I want to know where we are going with this, like I wouldn't necessarily do this with a whole bunch of separate points, this one seems easier in intervals. We sort of always think continuously, so I would guess the function behaves like *x* cubed when *x* gets large, and like -2*x* when *x* is pretty small.

Neil: Okay, but I'll get the same thing if we do it my way as well, right? So, if I plug in these points (Figure 47). Now, I just graph them and do estimation between the points, and our graphs should be pretty much the same.

Brian: Okay, well, let's do these on separate sheets of paper.I don't think our graphs look the same!

Neil: I think they are, just a little off because of estimation, it seems like we are ending up with the same product.

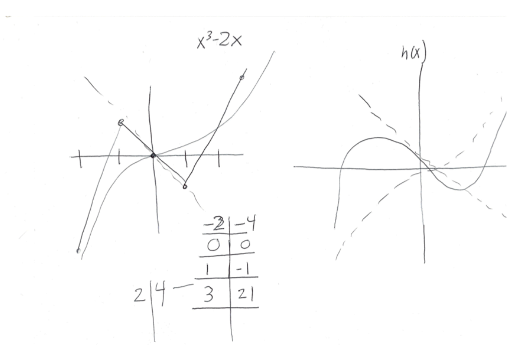

Figure 47. Neil (left) and Brian's (right) graph of the difference function.

Neil thought he and Brian agreed on how to program the correspondence point (Excerpt 45, lines 3-5). However, Neil's work suggested to me that he was finding highlight points and estimating the graph between those points (Excerpt 45, lines 7-8), while Brian thought about the graph as an infinite number of correspondence points, the behavior of which would depend on the value of x, f(x) and g(x) (Excerpt 45, lines 9-13). Neil believed that the graphs (products) were about the same (Excerpt 45, lines 20-21), and attributed differences in the graphs to estimation. His description of similarity appeared to be based on the graphs possessing similar shapes.

I believed Neil was thinking about a graph as a shape that fit a number of points resulting from choppy covariation. By this, I mean that he imagined the highlight points representing two values simultaneously, but estimation (interpolation) was necessary because there were only a finite number of highlight points. I was not yet convinced that Neil thought about a familiar function's graph without imagining highlight points.

I anticipated that asking Neil and Brian to construct the graph of a function on the whiteboard without lifting their marker would help me gain insight into their understanding of a graph as a representation of two quantities having covaried, or lack thereof. In addition, I hypothesized this activity would allow me to gain insight into what Neil *did* understand about a graph because he would be constrained from plotting points and doing estimation.

Excerpt 46 (Day 3 of TE, 00:00:42)

EW: So, I asked you to construct the graph of the function without lifting the pen from the paper. Neil, what did you come to?

Neil: I couldn't really generate a perfect function, I kind of had to calculate in my head the points I knew I needed to go through, sort of like looking ahead, but it was hard because I couldn't mark them down so I had to think about where they would be and follow the trend through the highlight points to make the graph.

Brian: It's weird, I can kind of see what you are saying, but I don't do it the same way. I kind of just considered the graphs of the two functions, so I could think about *m*(x) like a difference function as before, then focused on which intervals of x would correspond to where $2x^2$ or $-4x$ would weight more (Figure 48).

EW: Neil, what do you think about what Brian said?

Neil: Well, again, we sort of are doing the same thing, making the graph, just going about it differently. I make some points and follow the trend of those points, and Brian just seems to decide on the trend before making any points, kind of like skipping a step.

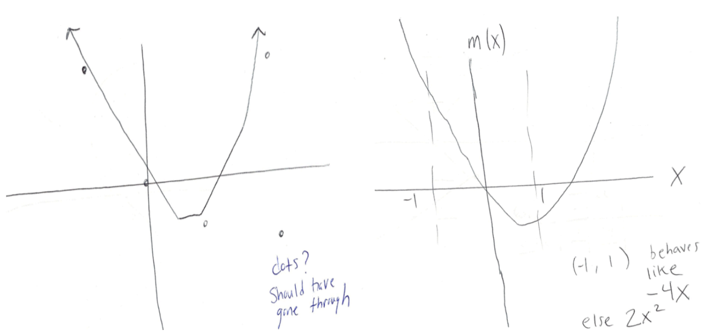

Figure 48. Neil (left) and Brian's (right) graph created without lifting the marker.

Neil appeared to be creating both correspondence and non-correspondence points while keeping his marker on the whiteboard (Excerpt 46, lines 3-7). His graph relied on both plotted points and another set of points that created a path between those plotted points. However, the two types of points did not represent the same quantities. I anticipated it would be critical for Neil to think about a graph as created by a point that always represented a correspondence because he did not have an image of two quantities' having values over a continuum, which allowed him to think about "filling in" between the highlight points.

Brian generated the difference function graph by focusing on the behavior of more familiar functions (Excerpt 46, lines 8-12). Brian's description fit with his process of programming a correspondence point and tracing it out continuously. In this case, he was programming the correspondence point to represent a difference of two familiar functions. I anticipated his image of continuous covariation would allow him to interpret a given graph of a function by imagining it as generated by a correspondence point having been traced out.

My analysis of Day 3 suggested that Neil was thinking about a graph as a shape, situated in the plane, that fit a number of highlight points. In contrast, I believed Brian was thinking about a graph as created by a correspondence point traced out continuously in the domain of a function. I described Neil's thinking as similar to novice shape thinking and Brian's thinking as similar to expert shape thinking. However, I believed that Neil was at a point where his thinking could become more advanced. I thought Neil was at a point where thinking about a correspondence point creating a function's graph continuously would rely on his having an image of the simultaneous variation of quantities. I hypothesized that if he thought in this way, he potentially could develop and image of the correspondence point continuously tracing out a graph.

Day 4 - Sweeping Out a One-Variable Function

I evaluated my hypotheses in the sweeping out activity by 1) understanding in what ways Brian and Neil thought about a graph as the result of a continuous tracing out and 2) documenting how they responded to thinking about a sweeping out of a function in the plane and its relationship to a surface in space. I anticipated that an image of continuous tracing out fit with Brian's understanding, and that I could build on his notion of a correspondence point to think about a graph being swept out in three dimensions. I anticipated that Neil's way of thinking about highlight points would constrain him from thinking about a continuous sweeping out of a graph in the plane.

Brian and Neil agreed that defining a new function $m(\text{x}) = \text{a}h(\text{x})$ would affect only the output values of the function. Brian described the effect of *a* as a "vertical stretch" of the graph as the outputs changed and Neil anticipated that *a* would be the factor by which the *y*-value of the "significant, highlight points would move upward". In both cases, they visualized a transformation in the dependent quantity, but Neil appeared to believe the transformation only affected the highlight points.

Excerpt 47 (Day 4 of TE, 00:00:15)

Brian: So, here's how I think about *a*. It only affects the *h*(x), so the
function's output becomes *a* times as large. It's also kind of like

making the difference between the two functions which created h(x) a times as large. I think the graph would reflect that, where you shift, well, program the correspondence point to be five times as large vertically, then trace out.

Neil: Yeah, I see what you are saying, where the a value affects the output value for the function, it definitely doesn't change the x. You sort of anticipated it ahead of time, kind of made a plan, where I just think about creating points for h(x) then the y-value gets multiplied by a, so you move those points up, and the slope of the graph between the points you chose gets more pronounced.

Brian: What do you mean by the slope gets more pronounced?

Neil: Um, slope of the lines connecting the major points could get steeper or less steep, they probably wouldn't keep the same level of steepness.

Brian: Okay, well I wasn't really talking about steepness, I was imagining all the points moving up, not a few points which you seem to be indicating with your hands.

Neil: Well, moving a few points takes the rest of the graph with it because the graph has to fit the trend of the points.

Brian's description of the effect of a on every correspondence point allowed him to visualize the entire graph stretching vertically by a factor of a (Excerpt 47, lines 1-3). He anticipated that a would affect the dependent quantity, which allowed him to imagine programming a correspondence point with x and ah(x) as values (Excerpt 47, lines 4-6). Neil appeared to visualize the vertical stretch occurring only for the highlight points, but eventually all points moved because the transformation of the highlight points dragged the other points with it. (Excerpt 47, lines 8-9). I believed he thought about the graph bending to accommodate the isolated points that stretched vertically (Excerpt 47, lines 10-12). Neil was imagining the graph between the highlight points as a shape with physical properties which needed to fit the trend (Excerpt 47, lines 19-20).

After this exchange, Brian asked Neil what would happen if he could use more highlight points. Neil said the graph would just be laid over them [highlight points] and become "better fitting". His description of better fitting indicated he imagined the graph becoming a better approximation, but it was not clear to what the graph became "better fitting". I believed he thought that more highlight points (e.g. adding 5) was a better approximation to a graph composed of an infinite number of highlight points.

Neil assumed that he and Brian were constructing graphs in the same way because the graphs appeared to share the same features. Brian clearly saw a difference in his and Neil's thinking, which he characterized as thinking about how graphs were made in separate ways. While they both used the notion of a correspondence point to generate their graphs, Neil imagined that a correspondence point existing only where it was calculationally convenient, while Brian attended to the correspondence point sweeping out continuously to generate a function's graph. It seems this difference could have been attributed to their goals. Neil believed a finite number of correspondence points in tandem with estimation were sufficient to create the same graphs that Brian generated.

Midway through Day 4, I asked Brian and Neil to think about the value of *a* as representing the distance in front of (for positive values of *a*) or in back of (for negative values of *a*) of the current function's graph. Neil commented that *a* = -3 created the function's graph, and also situated it 3 units behind the computer screen. Brian noted that as *a* increased from positive to negative numbers, the function's graph would travel through the computer screen.

Excerpt 48 (Day 4 of TE, 00:00:39)

EW: Right, so let's pick up, we just talked about how to think about *a*, what did you guys hear me say?

Neil: I'll take a shot, okay Brian? Yeah, well basically we are thinking about *a* corresponding to another axis, but this one comes out of the laptop screen, so it kind of represents depth where the zero depth is the laptop screen.

Brian: I agree with that, it turns into thinking mostly in space.

EW: Can you say more about thinking in space, each of you?

Brian: Mind if I take this first? [Neil nods] Now, what you do is kind of think about stringing together a point that generates the graph in the plane from a sweeping out, and then we move that plane, corresponding to a value of *a*, and that traces out a surface. It's not totally right, but we kind of have a correspondence graph consisting of correspondence points (Figure 49).

EW: Neil, what did you hear Brian say, what do you think?

Neil: I'm a little confused about all the sweeping out, but I can see how he might be relating it to the Homer situation again. Again, I kind of focus on making highlight points, except these are highlight graphs. So I would generate the graph for values of *a*, umm, like -2, 1, 3, 5, then fit the general trend between those graphs in space.

EW: So what is going on between those values of *a*?

Neil: I'm not sure because I picked those values of *a*, I'd have to pick all those values of *a*, which isn't possible. So I pick the highlights and estimate between them, sort of like I did without the third axis.

EW: Brian?

Brian: I think it is possible to account for all values of *a*, just by sweeping out it covers all of the values.

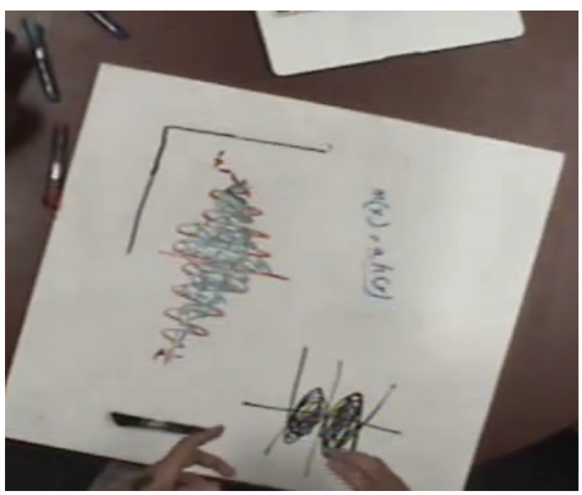

Figure 49. Neil and Brian's depictions of viewing *h*(x) sweeping out.

Neil and Brian's words and gestures indicated that they were thinking about an object being created in three dimensions (Excerpt 48, lines 1-7). Brian referred to this generating graph as a correspondence graph, likely trying to draw a parallel between sweeping out in space and sweeping out a one-variable function in the plane (Excerpt 48, lines 12-13). Neil indicated a highlight graph was the result of fitting together graphs in the plane, corresponding to integer values of *a* (Excerpt 48, lines 19-20). He believed that he could pick graphs between those highlights graphs, but that would require picking "all" the values of *a*, which he saw as cumbersome. (Excerpt 48, lines 22-24). I believed, in the moment, that Neil thought that he would have to make a new graph for every possible value of *a*, which would have required an infinite number of graphs. Brian visualized that the process of sweeping out a one-variable function in a plane would account for all values of *a* (Excerpt 48, lines 26-27).

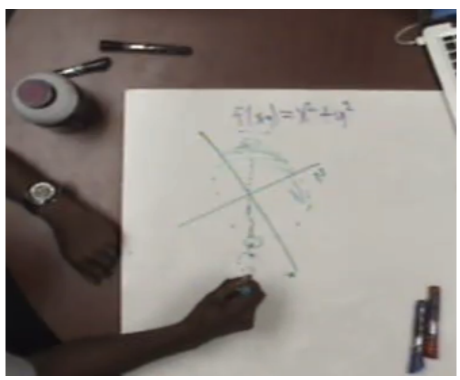

Figure 50. Neil's illustration of cross sections in space at integer values of *a*.

Neil appeared to imagine a process of "taking a snapshot" at every integer value of *a* and he connected those graphs by interpolation, or trend fitting. I characterized his thinking as imagining approximate planar regions, which fit with his process of estimation for a graph of a one-variable function. Neil seemed to imagine that the relationship between quantities' values only existed at the tick marks, often corresponding to integer values, but at the same time, he could imagine that he needed to make an infinite number of graphs at values of a to create a surface.

To communicate what he was seeing "in his mind", Neil defined a two-variable function and attempted to graph it on the whiteboard (Figure 50). Brian joined in, and together, they constructed a set of axes, consisting of x, a, and z, and noted that a was similar to y. The convention of z was their own, but they said it was natural to pick z already having x and y. They were able to generate an accurate sketch of the function, but Brian believed they were not constructing the graph the same way because Neil was only estimating. Neil continued to believe if they generated the same graph, then their method of constructing the graph must be closely related.

I began Day 4 with the goal of gaining insight into how Neil and Brian thought about the continuous tracing out of a function by a correspondence point and correspondence graph in the plane. Each time Brian referred to a, he described it as a continuous interval from which he could pick a plane to describe the function's behavior. Neil used a as a parameter without imagining it as a variable which simultaneously represented all possible values of a as Brian had. It seems that Neil did not see a need for thinking about what he was doing as a special case of generating all possible functions over an interval of values for a. In contrast, I believed that Brian visualized the function sweeping out continuously along the a axis.

Day 5 - Generating Functions from Perspectives

Day 5 focused on Neil and Brian generating functions from various perspectives, which they had initially introduced when constructing the *z* and *a* axes in Day 4. In the following excerpt, Neil and Brian exchanged thoughts about the role of *a* in generating a two-variable function.

Excerpt 49 (Day 5 of TE, 00:00:10)

EW: Brian, I noticed that you commented about *a* being like another variable. Could you say more?

Brian: Yeah, basically I was thinking that because I am tracing out along another axis, and for all the values, *a* is kind of like another variable, where my height above the axes depends on both *x* and *a*.

Neil: I see what Brian is saying, our output, which seems to be the, what, *z* axis? That depends on the value of *x* and *a*. I guess I would call them both variables, containing the highlight points.

Brian: Well, I can’t get past this highlight point thing, I think of everything as a highlight point, like, uh, you are describing them.

EW: Okay, so, how would you define the function?

Brian: Well, I think *f*(x,a) because they are both variables. [Neil nods].

EW: Okay, so take me through the process you would use to generate the graph of a function of two variables then.

Brian: I think I’d do the programming thing overall, but I noticed in GC that it is normal to use *z*, *x* and *y*, so I’ll call *y* like *a*, and *z* is like the output of the function. So I can either look at z and x for fixing values of *y*, like a parameter, or z and y, for fixing values of *x*, where *x* is like *a* then. I'm guessing the graph you get is smooth, sort of like all the correspondence points but now we have to program each point with three pieces of information.

Neil: Hold on, I'm not sure what you mean all the correspondence points. I'm imagining that we take some highlight functions from the z-x perspective and z-y, then sort of estimate the general trend between them to build up the graph.

Brian: Why would we need to estimate, when we know the function so we can be exact?

Neil: It would take infinitely long, you'd have to make all the points individually, and estimation is better. You can sort of lay the surface accurately then.

I interpreted Brian's description of a parameter value to mean he imagined an infinite number of highlight points, which he called correspondence points (Excerpt 49, lines 9-10, 19). I believed that he imagined that highlight points resulted from programming the correspondence points with three pieces of information (Excerpt 49, line 21), which he indicated were *x*, *a*, and *f*(x,y). I continued to hypothesize that Brian thought a function's graph could not exist without correspondence points, and the correspondence points could not exist without relationships between dependent and independent quantities (specified or unspecified). This created the necessity for him thinking about a function's graph as a representation of a relationship between quantities which he visualized by using perspectives.

Neil agreed with Brian's use of the z-x and z-y perspectives (Excerpt 49, lines 24-25), but believed it was necessary, and sufficient, to use estimation (Excerpt 49, lines 24-25) because making all the points individually would take forever (Excerpt 49, lines 28-30). Neil's responses suggested to me that he was thinking about plugging in and graphing every correspondence point individually, and he used highlight points and estimation to cope with the impossible task of graphing an infinite number of points. However, even if he had thought about a graph as an infinite number of correspondence points, he still imagined creating the graph as like laying the surface (Excerpt 49, lines 29-30). I believed, in the moment, that Neil thought a function's graph, in the plane or in space, was a covering laid over a number of highlight points or highlight graphs.

Just after the discussion in Excerpt 49 (Day 5), Neil noted that he was trying to deal with an information overload from trying to graph a million points. He said that in both functions of one and two variables, the graphs must be estimated by the general trends of highlight points to construct the graph. Brian responded that there were an infinite amount of points on a graph, but he did not need to generate them all. Instead, he described the programming of a correspondence point to "operate" on each point. It appeared that Neil thought about the correspondence point as a program he would need to run for every (x,y) pair, while Brian was thinking about the correspondence point as a repeatable process with general instructions for representing the output resulting from each (x,y) pair.

I believed that Neil's solution was to limit the number of times he had to type information into a program by choosing the types of points he would program. Neil did not use his programming metaphor to describe the "estimation" portion of his graphs. As a result, he was not thinking about those areas of the graphs as having been programmed. In constrast I believed that Brian imagined every point as the result of a programmed correspondence point.

Day 5 - Interpreting the Graph of a Two-Variable Function

After a short break in Day 5, I provided Brian and Neil with the graph of a two-variable function and asked them to describe a possible function that the graph could represent. I anticipated that Brian would focus on how each perspective of the graph was programmed to generate the surface, while Neil would see the surface as a covering laid over highlight points.

Excerpt 50 (Day 5 of TE, 00:01:01)

EW: Okay, so what we have here is a graph of a function in space (Figure 51, right). I'd like you to discuss how you think it was created, as in what function could have generated it, based on your understanding.

Neil: Okay, since we were talking about perspectives, so I think of this graph as kind of a surface laid over a bunch of exact points, and it fills in between to get some estimation.

Brian: But that can't help us figure out what function's graph we have.

Neil: Sure it can, you'd just need to figure out a commonality between those points, and kind of view the general trend of the overlaid graph. So, for example, if the y is twice as much as the x in the points, we could have part of our function be $y = 2x$.

Brian: I don't understand, you'd basically be guessing then, right?

Neil: No, but it would be hard, I don't see another way really at all I guess.

Brian: What about perspectives, like we've been talking about? I think we could pick a z-x and z-y perspective, and also an x-y, where the variable not included becomes the parameter and look at the graph to see what function might have made that perspective?

Neil: Won't that just make us estimate a point from a general trend anyway?

Brian: No, it seems like you want to look at the trend of the whole surface, which I think is pretty much impossible, we have to consider the z-x, z-y and x-y, and see if we can get them to agree on some kind of function.

EW: Neil, can you say more about how you think your and Brian's approaches agree, and how they compare to what we've talked about so far?

Neil: Okay, no offense, but I think you guys are talking about a fancier way of doing things than we need to. It's just a bunch of points and estimation is good enough, guessing the general trend. I think my highlight points or functions approach ends up with the same thing.

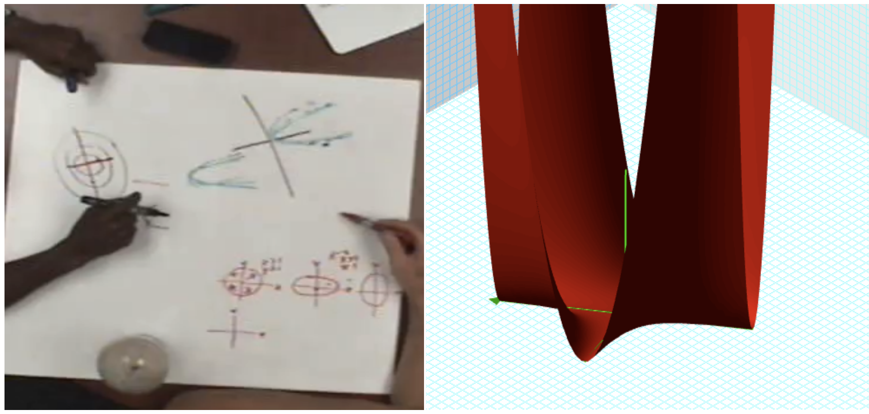

Figure 51. Brian and Neil's sketches of cross sections of the function (right).

Brian and Neil viewed the graph of $f(x,y) = 2x^2y$ (Figure 51, right). Neil's description of perspectives (Excerpt 50, lines 4-6) indicated to me that he imagined perspectives were a way to generate a set of exact points (highlight points) and conduct estimation between those points. As I had anticipated, Neil believed the function that generated the graph could be defined by the highlight points (Excerpt 50, lines 8-10). Brian interpreted this method as guessing (Excerpt 50, line 12), but Neil saw no other way to complete the task I had described (Excerpt 50, line 13). Neil was baffled at how to move forward and I hypothesized it was because he saw no way of determining a function that fit all of the highlight points.

Brian described his use of perspectives to imagine how the tracing out occurred, which he anticipated would provide insight into the perspective dependent functions (Excerpt 50, lines 14-18). Neil interpreted my and Brian's responses as equivalent to his interpretation of a graph (Excerpt 50, lines 25-28). I did not believe Neil was dismissing the perspective approach that Brian believed in, but he saw perspectives as another "trick". He saw no reason why Brian's approach would end with a more "correct" result (a graph) than his own. Neil likely believed it was sufficient to produce a "good enough" graph, using his process of estimation. It is plausible he saw Brian's process as unnecessarily complicated.

When Brian saw a graph in space, it appeared that he saw an infinite number of correspondence points. These points could be generated from the z-x, z-y or x-y perspectives. Thus, he focused on reverse engineering his method of constructing function's graphs via a "program". He did this to deduce the program that generated the given graph. Neil also attempted to reverse engineer from his construction of a graph, but I believed because his process for constructing a graph relied on picking highlight points and doing inexact estimation, he saw no way to reverse the process given an already constructed graph in space.

Part I Conclusions

Brian's ways of thinking about functions.

During Day 1, I noted that Brian was focused on tracking quantities' values as they varied. He understood that as long as Homer, Springfield and Shelbyville existed, Homer's distance from Shelbyville and Springfield existed. Then, as Homer moved on the road, the distances varied simultaneously. While I cannot attribute Brian's thinking about a graph solely to one activity in this teaching experiment, it appeared the Homer activity allowed Brian to think about a function's graph as created by the sweeping out of a correspondence point, which simultaneously represented two values. This way of thinking not only allowed him to imagine how a graph was constructed, but helped him thinking about every graph as generated by an infinite number of correspondence points.

My model for his thinking indicated that he began thinking about "programming the correspondence point" to move in a particular way by tracking the independent and dependent variables. He imagined sweeping out a one variable function in a plane from multiple perspectives, each generating the same surface in space. He interpreted the surface of a function's graph as an infinite number of correspondence points, each programmed with three pieces of information. Brian's understanding of how a surface in space was constructed then helped him hypothesize about what function could represent a "mystery" surface in space.

Neil's ways of thinking about functions.

The Homer activity suggested to me that Neil was thinking about using highlight points as a way to estimate the correct shape of a function's graph. I described his way of thinking as choppy covariation, in which Neil was imagining highlight points existing, programmed by the value of the independent and dependent variable, but only at discrete points. By discrete, I mean that he only tracked the quantities' values as certain instances (e.g. specific locations of Homer on the road) to produce "exact measurements", and estimated the behavior of the graph elsewhere. I believed that Neil did not imagine the quantities' values varying simultaneously over those "breaks". He believed the best way to approximate the graph was to use linear interpolation. However, it remained unclear what he was attempting to approximate.

I thought Neil was thinking about a graph as a wire or a mat overlaid on a set of highlight points. While he appeared to program the highlight points in the same way that Brian did, he did not imagine tracking the independent and dependent values over a continuous interval of each quantity. Neil was so intent on the end product (the graph) appearing to be the same as Brian's that any difference in how the function's graph was constructed was unimportant to him.

Part II: Brian and Neil's Ways of Thinking about Rate of Change

Table 21

Brian and Neil's Rate of Change Task Sequence.

Task Description	**Date (Day)**
Car A – Car B: Interpreting Meaning of the Graph	5/13/2011 (6)
Car A – Car B: Measuring Rate of Change Using the Graph	5/13/2011 (6)
Generalized Rate of Change Function and Direction	5/15/2011 (7) 5/17/2011 (8)

I anticipated that Neil and Brian would rely on the notion of a calculus triangle to measure rate of change because of their use of it in class and during tutoring sessions. I also believed that Neil and Brian's interpretation of the meaning of rate of change relied on thinking about average rate of change as a constant rate of change of "another car going the same distance in the same amount of time as the original car, but with its cruise control on". This was the primary analogy they used during our pre-teaching experiment sessions.

Days 6 and 7 - Car A-Car B and Rate of Change

I anticipated that because Neil and Brian relied on programming a correspondence point or correspondence graph as a way to describe graphs of one and two variable functions, they would continue to use programming as a way to think about the behavior of a rate of change function.

Neil and Brian interpreted the crossing of graphs of Car A and Car B as an indication that they were traveling the same speed after one minute of time had elapsed. Brian and Neil both noted that the cars had identical average accelerations because the change in speed and change in time were equivalent for both cars. Neil said that “this is our usual way of thinking about average rate of change, if the changes in the top are the same as the changes in the bottom, then that means the average rates of change are the same.” Brian agreed with Neil, but that the rates of change were not the same between those two points in the domain.

Excerpt 51 (Day 6 of TE, 00:00:13)

EW: Alright, so you two seem to be focused on the rate of change of the function graphed here. Can you say more about the rate of change?

Brian: Yeah, well, I guess I was gonna say that you program an input, like where you want to know the rate of change, then use the open form rate of change as the output. But we have talked about using a correspondence point, so I want to go at it that way, so the two things the point needs is an x and $r_f(x)$

EW: Okay, and what is $r_f(x)$?

Brian: It is the open form rate of change, so for some fixed value of h. So in class, I would say if h is .05 then the value for the open form is like a constant rate of change, same distance, same time sort of thing.

EW: Neil? Did you want to add anything?

Neil: Yeah, I agree with the constant rate of change, that's the output of the open form rate of change function. I can also see why we are programming the point, kind of tracking the rate of change.

EW: So how does the correspondence point relate to the rate of change function?

Brian: I was thinking it would make its graph, so you continuously track the values x and $r_f(x)$, and that makes a rate of change [makes Figure 52].

Neil: Oh man, why continuous again? I figured we just took a measurement every h units, or every now and then like before, then we can make highlight points and estimate between them with a constant rate of change.

Brian: Why would we do that though, rate of change is a function, so it has to be made of an infinite number of correspondence points.

Neil: No, this comes back to the same disagreement we had before I think.

Brian: But estimation doesn't work! Rate of change is a continuous function.

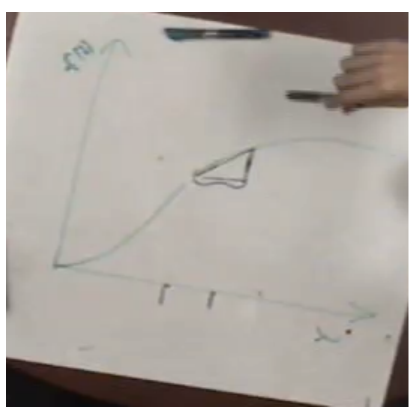

Figure 52. Brian's use of a calculus triangle to represent rate of change.

Brian used a calculus triangle to represent the quantities he was measuring in the open form rate of change (Figure 52). At the same time, he considered how to program a correspondence point to represent the rate of change function. His description of the rate of change as continuous (Excerpt 51, lines 3-5), indicated that he imagined "programming" the rate of change function as he would in the Homer activity. He programmed the correspondence point with x and the open form rate of change function $r_f(x)$ (Figure 53), and imagined these values simultaneously varying (Excerpt 51, lines 18-19). I believed, in the moment, that Neil was thinking about highlight points of the function being available only when the independent variable increased in increments of h (Excerpt 51, lines 20-21). Neil's comment that he would estimate between the highlight points using a constant rate of change (Excerpt 51, lines 22-23) may have been his connection between linear estimation and constant rate of change as a linear function.

$$r_f(x) = \frac{f(x+h) - f(x)}{h}$$

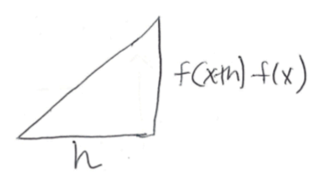

Figure 53. Open form rate of change and Brian's original calculus triangle.

Brian believed that h had no effect on the selection of highlight points every h units because to him, h specified the interval over which the rate of change was being measured. Neil insisted that every rate of change function comes from estimation, and he indicated that if we were able to zoom in on the graphs in the Car A- Car B situation, there would be linear segments connecting highlight points.

Neil's use of highlight points fit with his thinking about the shape of the graph as more important than the correspondence points. Because he knew that a linear function's graph was associated with constant rate of change, he was able to justify his estimation technique between highlight points. I anticipated that for Neil to make any shift from thinking "every *h* units", he would need to encounter the graph of a rate of change function that did not fit with his anticipation of linear, connected segments.

Day 7 – Programming a Rate of Change Function

At the beginning of Day 7, I used Graphing Calculator to allow Brian and Neil to explore their conjectures about how to construct a rate of change function's graph. Brian and Neil worked on programming the rate of change correspondence point, and debated about the smoothness of the resulting graph.

Excerpt 52 (Day 7 of TE, 00:00:12)

EW: Right, so let's get to this one. You two seem to be at a crossroads about how to generate the function, so I thought we would do it in GC.

Neil: Can I pick a function? How about a cubic function, those are good.

EW: Alright, so I want to program everything like you told me to before, where this point will represent x and $r_f(x)$ (Figure 54), and you gave me the definition of open form before. Now, if we traced the correspondence point out, it would produce a graph, each of you tell me about the graph.

Neil: I'll start, I basically think it is going to be a bunch of linearly connected segments between hinge, er, highlight points, linear slope between. Because GC takes certain measurements because it can't take every possible one, and does a best guess between.

Brian: No surprise, I guess, I disagree, I think the graph is going to be smooth, because GC does a correspondence like thing, and keeps track of them over super small intervals to make a smooth looking graph.

EW: Alright, so if I increase the value of *x*, the correspondence point moves here [points to computer screen]. Now, if I graph the function, here it is, what do you notice?

Neil: I guess it looks pretty smooth, but I don't think it should. This is weird, I don't see the highlight points I was expecting, but maybe there are a bunch of them.

Brian: Hey Neil, try this, I think we actually are along the same lines, just imagine you have a huge number of highlight points.

Neil: But that couldn't work because at some point you have to estimate between them, it's too many to do, but maybe GC did it your way here.

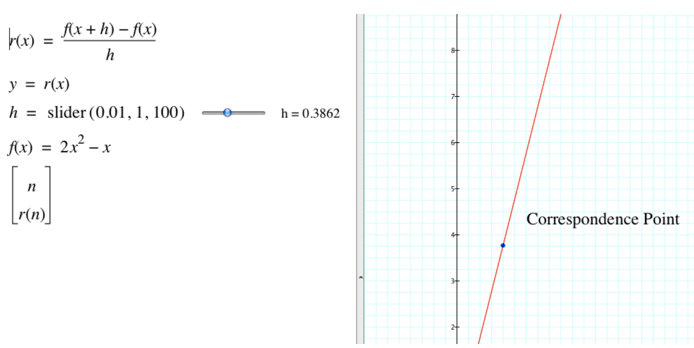

Figure 54. The basis of discussion about programming a rate function.

Brian focused on generating a graph of the rate of change function while understanding the *x* and $r_f(x)$ were simultaneously varying. This insight helped

Brian anticipate that the rate of change function's graph would be smooth (Excerpt 52, lines 13-15). Neil's attention to the highlight points made the smoothness of the rate of change function puzzling for him because he had anticipated a function's graph to be a number of linearly connected segments (Excerpt 52, lines 9-12). Neil's anticipation that there could be a large number of highlight points (Excerpt 52, lines 20-21), but could not imagine actually generating them (Excerpt 52, lines 24-25).

I turned the conversation to the continuous tracing out of a correspondence point so there were highlight points everywhere. I created a Graphing Calculator document that initially displayed only "highlight points", which Neil had calculated (Figure 55, purple points). Neil believed the function would connect those points using "linear lines" because the general trend was linear. Brian programmed a correspondence point (Figure 55, blue dot), which he said would trace over the highlight points and every point between. I ran the animation, and Neil confirmed that Brian's assertion was accurate. Neil said, "Man, I think I sort of see what is going on here, GC is using a whole bunch of highlight points without really showing them." Neil insisted that there was still estimation going on between the highlight points, but could not describe where the estimation was being generated.

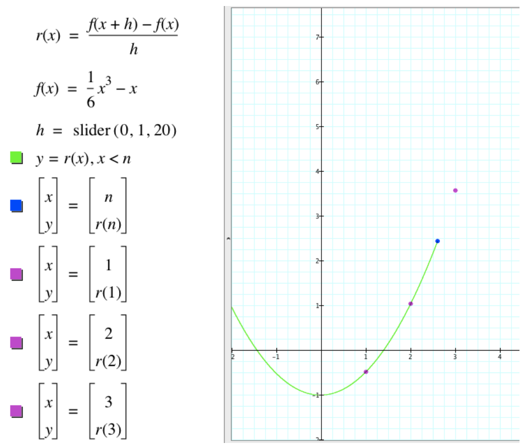

Figure 55. Representing the rate of change function as a tracing out.

After Day 7, I believed that Neil attributed the difference in his thinking and what he saw in the animation to an estimation process used by Graphing Calculator. Brian saw his way of thinking about programming a point using a rate of change function as identical to that of Graphing Calculator. Moving forward, I attempted to use the notions of estimation and programming to gain insight into how they would interpret and measure rate of change in space.

Day 8 - Interpretation of Rate of Change in Space

Day 8 began with a discussion about the meaning of rate of change at a point in space. The previous sessions had focused largely on "how to graph" the rate of change function. The following excerpt centered on how Neil and Brian would interpret the meaning of rate of change at a point in space.

Excerpt 53 (Day 8 of TE, 00:00:02)

EW: How would you think about rate of change at a point in space?

Brian: My first though is that it has multiple rates of change, kind of like sitting on a hill, depending where you look, the steepness, slope at that point can be different.

Neil: I agree with that, I thought about the kind of example too, or just sitting on the surface we swept out, maybe we can use the z-x and z-y rates of change?

Brian: Yeah! Umm, let's see though, if we want to make a rate of change function and then graph it in space, we need to figure out a way to program it, and do a sweeping out.

Neil: What about just plugging in *x*, *y* and the rate of change? Oh, I guess we don't know the rate of change yet, so my point thing wouldn't work.

Brian: Alright, for z-x, it is kind of like, okay let's back up, let's say we are at a point (a,b,c) in space. Then for z-x, we fix *y* at *b*, then do the normal rate of change except it has to be two-variables.

Neil: Yeah, that makes sense, so we need an *h*, maybe like *f*(x+h,b)-*f*(x,y), then divided by *h*, and then for z-y, we just say y+h and fix *x* at *a*?

Brian: Yeah, let's go with that.

EW: Okay, so where do you want to go next then, what is your plan?

Brian: We need the two rates of change to make an overall rate of change function, then we can graph it by doing the sweeping out I think.

Neil: I'd rather just draw the two calculus triangles that I am imagining each in a perspective.

Brian's description of rate of change (Excerpt 53, lines 2-3), indicated he was thinking about rate of change in a direction "at" a point on the surface of the function's graph. Brian's suggestion of considering multiple rates of change from a perspective (Excerpt 53, lines 14-16) led to Neil's sketching of perspective dependent calculus triangles (Figure 56). Their descriptions of multiple rates of change, as well as specific rate of change functions for z-x and z-y perspectives, indicated they were imagining rate of change occurring in at least two directions. I intended to understand if they imagined the rates of change occurring simultaneously.

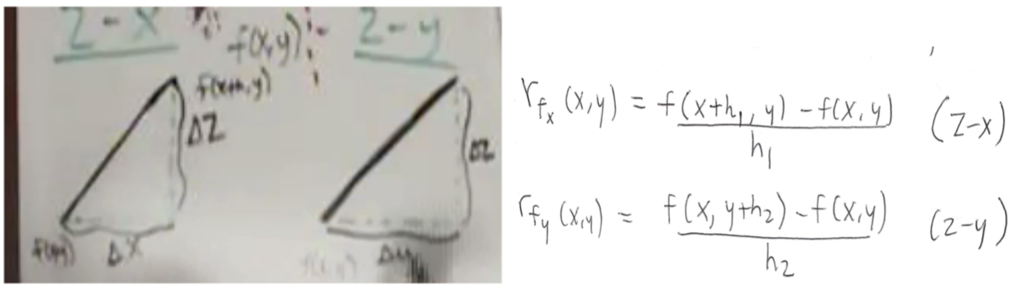

Figure 56. Calculus triangles from the z-x and z-y perspectives.

Brian described the z-x perspective rate of change as the average rate of change of *f* with respect to *x* while holding *y* constant, and z-y as the average rate of change of *f* with respect to *y* while holding *x* constant. Neil agreed, noting that he was trying to express the holding constant idea with his function definitions (Figure 56). They reached an impasse when they were discussing how to combine the rates of change resulting from each perspective. I believed, in the moment, that this impasse was the result of trying to describe the meaning of a quantity changing with respect to two independent quantities. I thought that if they had an image of the two rates of change occurring simultaneously, they could consider rate of change might have a direction.

In the following excerpt, I asked Neil and Brian to expand on their description of their perspective dependent calculus triangles, in particular their use of h_1 and h_2 in the denominators of the open form rate of change functions. I anticipated that thinking about the relationship of h_1 and h_2 would be critical to their creating a need for considering rate of change in a direction.

Excerpt 54 (Day 8 of TE, 00:00:16)

EW: So I noticed that you guys constructed your calculus triangles in each perspective and labeled the *h*'s as different. Can you say more?

Neil: Yeah, well basically the *h*'s are independent, so they don't have to be the same, but I guess they could be.

Brian: I was thinking about this more last night, and had a sort of moment. We talked earlier about being on a hill, or walking on a function, for example, and the rate of change depended on the direction you were facing, I think the same thing applies here.

EW: What do you mean by a direction?

Brian: Okay, so let's say we head from a point, or we went directly Northeast, and imagine we are doing this from overhead, it was looking at the x-y perspective that made me think of this.

EW: Okay, so if you were heading Northeast, what's the significance of that?

Brian: Then we know how the two *h*'s are related to each other, because most of the time you don't go in a direction of just north, west, east, whatever, you head in some combination.

Neil: Okay, not sure I am following, but basically if we head Northeast, the change in *x* and change in *y* would be equal, a different direction, change in x could be twice as big as change in y, which are like the h's right?

Brian: Yeah, let me make an illustration here. Then we can just call the numerator a change in z, $f(x+h_1, y+h_2) - f(x,y)$ either h value that we want to.

Brian introduced direction as a way to account for all possible rates of change in space (Excerpt 54, lines 10-12). I believed Brian had an image of the relationship between h_1 and h_2 to define a direction (Excerpt 54, lines 15-17). Brian's key insight was that *any* direction was more general than considering only z-x and z-y perspectives. I believed, in the moment, that a key to Brian and Neil imagining direction as important was their image of standing on a point of a function's graph in space, and imagining that the rate of change might depend on the direction in which they turned. They described how thinking about the relationship between h_1 and h_2 would allow them to simplify their functions because h_1 and h_2 depended on each other. Neil and Brian conjectured that only one of the values needed to be included in the function as h_1 and h_2 were dependent on each other.

Neil and Brian continued to work on developing a two-variable open form rate of change function, and they agreed that the numerator represented a change in the output, represented by $f(x+h_1, y+h_2) - f(x,y)$, where either h_1 or h_2 was written in terms of the other h-value. However, both Neil and Brian questioned how they have a single denominator that represented a change in x and a change in y. Even though they saw that h_1 and h_2 depended on each other, that dependence did not immediately resolve their issue of what change to represent in the denominator. I believed that this was because they were focused on trying to represent a single change in the denominator while they understood that there were changes in both *x* and *y*.

Excerpt 55 (Day 8 of TE, 00:00:49)

EW: So, what is your denominator in the function, your conjecture?

Brian: I was thinking either *h*-value, whichever you have in the numerator.

Neil: But doesn't that kind of just delete it, it goes away?

Brian: No, you just define it in terms of h_1, if you are talking about h_2.

EW: So if one *h* is in the denominator, and we make that value small, what happens to the other *h* value??

Brian: Well, oh yeah, it becomes small as well, it's not like the *h*'s have the same value, but they can end up getting so small it doesn't matter.

Neil: Ah, I see what you mean. So like in class we talk about making changes so small, because we have an equation to relate the two *h*'s, then when one gets really small, the other one has to get smaller too? I was thinking multiply in the denominator I guess, but that doesn't need to happen, you just need one of them to become small because they are related, the *h*'s I mean, so then both do.

Brian: So, I guess that's our overall rate function for two variables. Now we can sort of program the points to do the graph.

Brian's insight that the changes in *x* and *y* became smaller in tandem allowed him to conjecture that using a single parameter in the denominator

(Excerpt 55, lines 7-8) was acceptable (Excerpt 55, line 2). Neil appeared to focus on deleting one of the parameters, but Brian's insight allowed him to think about the equation they had specified between h_1 and h_2. By imagining progressively smaller values for h_1, he found that h_2 became smaller as well given the proportional relationship specified by choosing a direction in space (see Figure 57).

$$r_f(x,y) = f(x+h_1, y+h_1 d) - f(x,y) \,/\, h_1$$

direction = d d= fixed $\frac{\Delta y}{\Delta x}$ $h_1 = \Delta x$
$h_2 = \Delta y$

Figure 57. Brian and Neil's open form two variable rate of change function.

Day 8 - Existence of Rate of Change at a Point in Space

Brian and Neil developed a two-variable rate of change function in Excerpt 55 (Figure 57). Brian focused on how to think about generating the graph of the rate of change function by a sweeping out. Brian did not think he could assume rate of change existed at a point in space. He described how if one were standing on a hill, the rate of change at that point would be different in most directions. As a result, graphing a rate of change at a point would be impossible. In the following excerpt, Brian and Neil expanded on this discussion.

Excerpt 56 (Day 8 of TE, 00:01:21)

EW: So now you have come up with a rate of change function for a two-variable function, I know you wanted to talk about graphing the rate of change function.

Brian: Yeah, so I was thinking about how to do this, and basically you would follow the same process, z-x, z-y perspectives, so for z-x, you would fix *y*, basically treating the function like a one variable rate of change, and sweep out along different *y* values. Do this from z-y too, whichever is more convenient.

Neil: I know we disagree, but I would pick some points, so an *x*, a *y*, and find the rate of change at that point I think. I could estimate between the points again. Of course I'd have to pick a direction, but where do I graph that part?

Brian: I just thought of a huge problem, so suppose we are graphing a rate of change at a point (x,y, $r_f(x,y)$). What if the rate of change isn't the same in every direction?

Neil: That's the problem I was thinking of, where do we graph direction? How can we show that I guess?

Brian: My first thought is that the rate of change just can't exist, in a rate of change graph (x,y, $r_f(x,y)$) unless the rate of change is the same in every direction right?

Neil: Yeah, I agree, so when you make the *h* values really small, to graph a rate of change point, the rates of change in every direction would need to be the same.

EW: Okay, let's work with that, so suppose the rate of change at a point existed, let's say it was 5.6. What does that 5.6 mean?

Neil: It still comes from an open form rate of change, like we had, so it is an estimate of average rate of change.

Brian: Same here, but I don't know how to phrase it.

Neil: Something like the average rate of change of *f* with respect to *x* and *y*, that's my best guess.

Brian: Yep, and if we made *h* really small, we could probably find the closed form rate of change too.

Brian and Neil's insights about graphing a value for rate of change at a point allowed them to conjecture that the rate of change could not be graphed unless the rate of change at a point was the same in every direction (Excerpt 56, lines 18-20). I believed they thought graphing a rate of change function required an infinite number of perspectives, defined by a direction, of which z-x, z-y and x-y were special cases. Brian and Neil's consideration of the z-x, z-y and x-y perspectives allowed them to think about those perspectives as special cases (Figure 58). I believed that Brian and Neil's insight that those three perspectives were special cases of an infinite number of perspectives allowed them to problematize rate of change at a point in space in ways that Jesse and Jori did not.

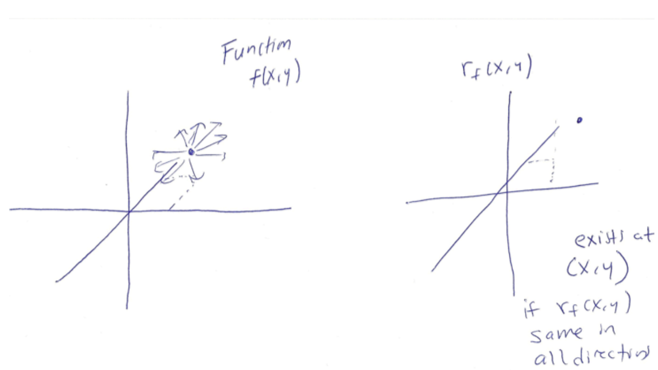

Figure 58. Brian's illustration of rate of change at a point in space.

Part II Conclusions

Brian and Neil problematized how to interpret and calculate a rate of change at a point in space because they were concerned with defining rate of change at a point in space. Neil's construction of "simultaneous calculus triangles" and Brian's reflection on them as a way to represent partial rates of change indicated to me that they were imagining change occurring in different directions in space. The x-y perspective allowed them to think about relating the *simultaneous* changes in x and y by considering a direction, and supported their construction of an open form rate of change function. By attempting to generate the graph of the rate of change function, Brian, and then Neil, were able to problematize the existence of rate of change at a point in space. They came to understand that to "graph a point from the rate of change function", the rates of change would have to be invariant under direction. However, for this to be true, the function would need to be constant.

Chapter Summary

In this chapter, I described the development of my inferences about the thinking of two calculus students, Brian and Neil, who participated in a group teaching experiment focused on functions of two-variables and their rates of change. I hypothesized that Brian thought about a function's graph as an infinite number of correspondence points. I believed that Neil thought about a function's graph as a finite number highlight points with estimations between them. Lastly, I showed how their use of programming a correspondence point supported their interpretation and construction of a rate of change in a direction.

Chapter 8

CONCLUSIONS

In this chapter, I describe the results of my retrospective analyses. These analyses considered the data from the three teaching experiments after the teaching experiments had concluded. I intend to accomplish two things in this chapter. First, I describe how my work contributes to, and builds on prior work of the field with regard to what we know about student thinking about functions and rate of change. Second, I characterize how my thinking has developed in regard to the literature and method with which I framed my study. The goal of structuring the chapter in this way is to provide the reader with an understanding of important research questions to consider going forward, and how the development of my thinking has prepared me to gain insight into those research questions.

Contributions

My analyses made at least three contributions to mathematics education. First, my analyses built on Thompson's description of shape thinking to describe a tentative framework for a student's association with a function's graph. Second, my analyses emphasized the importance of quantitative covariation for thinking about rate of change, and showed how quantitative rate of change can be extended from functions of one to functions of two variables. Third, my retrospective analyses identified a number of surprising responses from students, each of which has implications for future studies relative to student thinking about multivariable functions and their rates of change. In the following sections, I explore each of these contributions in greater detail.

Contribution 1: Types of Shape Thinking

Shape thinking is an association a student has with a function's graph, and shape thinking took on two forms in my study: *novice* shape thinking and *expert* shape thinking. Novice shape thinkers might associate a shape with a particular algebraic form (e.g., anything that looks like a "U" is a parabola and hence goes with "x^2", and *vice versa*). It is no more sophisticated than associating a name with a face. The student does not think about the graph as a representation of quantities having covaried. Expert shape thinking is thinking about a graph emerging from the covariation of two or more variables. Novice shape thinkers essentially ignore variation on axes and focus on the shape in the plane because they do not see the graph as an emergent construction. Expert shape thinkers are drawn naturally to think about variables on axes and how the variables' values interact because they associate the graph as an emergent representation of covariation. In other words, the graph cannot exist independently of the quantitative relationships it represents. In the remaining sections I describe expert and novice shape thinking more completely and relate a hypothesized developmental trajectory in relation to the teaching experiment sessions.

Expert shape thinking.

Expert shape thinking entails thinking about the graph of a function as the result of simultaneously tracking the values of two or more quantities' values. This characterization of expert shape thinking relies on Thompson (2011) and Saldanha & Thompson's (1998) description of quantitative covariation in which the student imagines that if two quantities covary, as one varies over an interval of conceptual time, the other quantity varies over that same interval.

Jesse and Brian used tracing out and sweeping out as processes to imagine constructing graphs of one and two-variable functions. Embedded in their image of tracing out or sweeping out was an understanding that the resulting graph was a representation of a relationship between two or more quantities. Implicit in that way of thinking is that any feature of the graph, such as a shape, is a representation of the behavior of two or more quantities. I believe Jesse and Brian saw the surface in space as a representation because they could imagine sweeping out a point in the plane to create a graph in the plane, which then swept out to generate a surface in space, and each process depended on the variation of quantities. For example, they imagined the sweeping out as a result of varying a quantity, and then imagined tracking that sweeping out to create a representation of the function. Implicit in this imagery is the notion that the definition of the function provides the quantities' values that are to be tracked. Expert shape thinking also entails other ways of thinking: smooth continuous variation (Castillo-Garsow, 2010; Saldanha & Thompson, 1998; Thompson, 2011), covariation, and process conception of formulas.

In this study, students whom I characterized as expert shape thinkers associated graphs with the correspondence point and graph in a plane that moved perpendicularly to an axis. Thus, they associated the graph with the process used to construct it. They imagined the resulting graph as composed entirely of correspondence points. Expert shape thinkers might describe the graph as an object in space, but they are also able to explain how that object is generated by the relationship between the dependent and independent quantities' values as they vary. This image relies on the student imagining on simultaneous, continuous covariation characterized by Thompson (2011) and Saldanha & Thompson (1998) that allows the student to imagine how the graph was constructed.

Novice shape thinking.

Novice shape thinking entails associating a graph with an object in space, but is not an emergent representation of quantities. For example, Jori thought about classes of functions (e.g. cubic, linear) by associating them with a particular shape. She did not always understand the graph as emerging from the function definition, and even when she appeared to do so, she fell back on her focus on shapes. Novice shape thinkers imagine the graph as an object situated in the coordinate plane without reference to quantities' values represented on the axes.

By imagining the graph as an object with properties independent of the quantities in the function, the novice shape thinker focuses on properties like "the graph goes up sharply", without understanding how the definition of the function and the scale of the axes create that feature of the graph. This also allows novice shape thinkers to imagine changing the function definition and the graph of the function independently of each other. For example, a vertical stretch of the graph would not necessarily be accompanied by a change in the function definition, nor would a change in the function definition necessarily require a change in the graph. For example, Jori imagined that the graph in the Homer situation would not be affected by exchanging the location of the two cities. I believe this was a result of her not imagining how the quantities were to be represented on vertical and horizontal axes. As a result, Jori was not attentive to how exchanging the city's locations would necessitate a reconstruction, if not a transformation, of the prior graph.

When constructing the graph of a function, a novice shape thinker might be unable to generate a graph unless the function fits an exemplar for which the student has an image. This image might be the result of inputting a function into graphing calculator numerous times. For example, Jori struggled to generate the graph of the difference function because she was not sure how to incorporate the shape of the two functions $f(x)=x^3, g(x)=2x$ in the new graph. In general, I believe that novice shape thinkers think about a graph by associating it with a familiar shape, which might then be associated with a particular function. However, this association turns the novice shape thinker's attention from visualizing how the graph could be an emergent representation. Thinking about the graph in this way makes imagining the effect of a function transformation on the graph, or the effect of a transformation of the graph on the function unrelated. By this, I mean the student does not necessarily see a transformation in the graph as a transformation of the function, and vice versa.

Transitions between novice and expert shape thinking.

My description of Jesse, Jori, Brian and Neil as novice or expert shape thinkers was mostly categorical, and the focus of my study was not to move a student from novice to expert shape thinker. As a result, I cannot define a transition stage between novice and expert shape thinking, but I can use my data to suggest the mechanisms by which I hypothesize this transition might occur.

My analyses characterized Jesse and Brian as mostly expert shape thinkers, and Neil and Jori displaying actions consistent with both novice and expert shape thinking. Jesse and Brian could use expert shape thinking to explain that the shapes were a representation between quantities. Jori, and to a lesser degree, Neil, relied on their association of graphs with familiar shapes, and then, familiar function definitions. Both briefly explained a graph as a representation of a function's behavior, but when confronted with a graph, with which they were not familiar, focused on shapes again. Based on these observations, I hypothesize that a transition in shape thinking might occur in either direction. The transition from expert shape thinker to novice shape thinker is implicit in being an expert shape thinker. The student can focus on shapes, but when necessary, can think about how those shapes are generated. I believe the transition from novice to expert shape thinker requires the student to accommodate a new way of thinking about a function's graph being generated.

A preliminary framework for shape thinking.

My analyses suggest that shape thinking can be thought of in terms of a developmental trajectory. This preliminary developmental trajectory can be thought of as the beginning steps toward an explanatory framework composed of constructs that I developed directly from my models of student thinking in the teaching experiments. Within each column, I describe a way of thinking and actions that could be evidence of a student thinking in that way.

Table 22.

A Tentative Framework for Shape Thinking.

Novice Shape Thinking	**Possible Transitions from Novice to Expert**	**Expert Shape Thinking**
Attention to the quadrants without regard to the coordinate axes on which quantities' values are represented.	Attention to quadrants, but beginning to focus on the axes as related to the object in the quadrants.	A function's graph as a representation of the quantities' values represented by the Cartesian coordinates.
A function's graph as an object, but not imbued with quantitative properties, and thus, not an emergent representation.	A function's graph as a curve that fits a finite number of plotted points, but not an infinite number of points.	A function's graph as a representation of a functional relationship using an infinite number of correspondence points.
A function's graph as a wire that can be manipulated independently of the function definition.	A function's graph can be manipulated, but because it becomes a different graph, a new function is required to fit that particular graph.	A function's graph and its definition change simultaneously because they are representations of the same relationship between quantities.
A function definition as a set of symbols associated with a particular shape that the student calls the graph.	A function definition as a way to generate a finite number of points with which the student interpolates to create the function's graph.	A function definition as a way to track two or more quantities' values to generate a representation of the function.
Perspectives (z-x, z-y, x-y) are ways of looking at an already generated graph that the student thinks about as an object situated in the quadrants.	Generates a function's graph from various perspectives, but does not understand that these perspectives generate equivalent surfaces in space.	Perspectives (z-x, z-y, x-y) as ways of imagining the two-variable function's graph being generated while understanding each perspective will generate equivalent surfaces in space.
A two-variable function's graph as a topographic map with a focus on its properties as if describing terrain.	A two-variable function's graph as a map, where the features of that map are related to the specific function with which the student associates the graph.	A two-variable function's graph as a map, where the features of the map are a result of representing the relationship between quantities.

It is important to note that the definitions of expert and novice shape thinking are based on only the students in my study. I do not claim that expert shape thinking explains how "experts", such as mathematicians and scientists, think about a graph of a function. Nor do I claim that most beginning students think about a graph as a novice shape thinker. These constructs are an attempt to explain and describe the students' ways of thinking from this study, but do not encompass all possible ways of thinking about a graph. Thus, the framework is an initial step. It is plausible that there could be "expert-expert shape thinking", in which one thinks about a graph by focusing on its topological properties. If one focuses on the topology of the surface, they focus entirely on the geometric properties of that surface and the axes become irrelevant. However, because I did not provide evidence that suggests a focus on topological properties as a plausible hypothesis, it is not appropriate to include in this framework.

Contribution 2: Ways of Thinking about Rate of Change

My analyses, and my understanding of literature about quantitative reasoning, covariational reasoning, and rate of change, suggest two ways to characterize students' ways of thinking about rate of change. 1) Non-quantitative rate of change where the student focuses on rate of change as a number without reference to a comparison between quantities as they are varying, and 2) Quantitative rate of change as covariation describing how fast a quantity is changing with respect to one (or two) quantities. In this section, I describe each of these characterizations and explain how they rely on, and contribute to the notion of quantitative covariation and rate of change I described in Chapter Two

(Castillo-Garsow, 2010; Saldanha & Thompson, 1998; Thompson, 2011; Carlson et al., 2002).

Non-quantitative rate of change.

Students who think about rate of change without reference quantities might, 1) Understand that to find a rate of change, they must identify changes in two or more quantities and display some sophistication in how to algebraically or graphically represent these changes, and 2) Interpret the resulting calculation or ratio of changes without reference to the quantities change from which the calculation resulted. This does not mean the students do not have an image for variation. I believe they may have an image of each quantity having varied. They might imagine that these quantities vary over conceptual time, so they imagine observing a change in the quantities' values over sub-intervals of conceptual time. I believe their way of thinking breaks down because they may not have an image of quantitative covariation, in which they imagine that two, or more, quantities vary in tandem over that interval of conceptual time (Saldanha & Thompson (1998); Thompson (2011)). Suppose that the student imagines the changes in two or more quantities occurring independently of each other. A comparison of those changes would then be a comparison of independent quantities, which might not have a sensible interpretation. The student might wonder why it would be useful to compare a change in z and a change in x if those two quantities varied independently. However, if the student possessed an image of z and x covarying, it is plausible that a comparison of the changes in z and x could describe how fast the quantities were changing with respect to one another.

Jori used a tangent line, a single calculus triangle in the plane and slope to represent her process for calculating a rate of change. I believe that she was comfortable using these to "find" rate of change because she knew they accounted for changes that were necessary to produce a rate of change. She knew she needed to find changes to measure a rate of change, but she did not imagine those changes to have occurred simultaneously. Thus, when I asked her to interpret a result of the rate of change function, she rarely made references to how fast quantities were changing in tandem because she did not originally imagine them changing in tandem. At the same time, she sometimes displayed sophisticated ways of thinking, particularly in the average speed activity. In the moment, she thought about a comparison of distance and time, and interpreted the resulting calculation as a rate. My analyses suggest that she drifted between focusing on the calculation and trying to interpret the meaning of the calculation. However, when trying to accommodate an interpretation of rate of change, she often relied only on the accuracy of the calculation. Jori appeared to use calculational reasoning, which led her to rely on the structure of the function, rather than the quantities used to construct the rate of change function. Her calculational reasoning allowed her to make significant progress in developing a rate of change function, but simultaneously constrained her in interpreting the rate of change as a measure of covariation.

Rate of change as a measure of covariation.

My characterizations of students' thinking about rate of change as a measure of covariation relies on Saldanha & Thompson (1998), Carlson et al.

(2002), and Thompson's (2011) description of quantitative covariation, conceptual time, and framework for explaining how students think about the behavior of quantities with intervals of conceptual time.

My analyses suggest that students who think about rate of change as a measure of covariation must have an image that two or more quantities varied in tandem. They must understand that if quantity 1 varied over a sub-interval of conceptual time, then quantity 2 and quantity varied over that same sub-interval of time. The ways in which an image of quantitative covariation supports interpretation of a ratio of changes has appeared in Thompson (2011) and Carlson's (2002) work. However, I do think it is important to contrast rate of change as a measure of quantitative covariation with non-quantitative rate of change.

One who thinks about rate of change as a measure of covariation *must* have an image of the quantities changing in tandem or having changed in tandem for the ratio of two changes in the quantities to convey any meaning about how fast one was changing with respect to the other. This covariation is generated by 1) simultaneously tracking two or more changes in quantities and 2) understanding that the tracking of the changes in quantities measures how "fast" the dependent quantity is changing with respect to one or more independent quantities. A student with a focus on non-quantitative rate of change may identify and create the same ratio as the student thinking about rate of change as a measure of covariation, but their image of the quantities changing independently of each other make an interpretation of a ratio as "how fast" nonsensical.

Both Jesse and Brian described how they would program the correspondence point to trace out the rate of change function by tracking the independent value *x* and the dependent value determined by the open form rate of change function. I believe they thought about rate of change as a comparison of changes in two quantities, represented on the axes, which resulted in a quantification of how fast the quantities were changing with respect to one another. As a result, measuring a rate of change in space necessitated that Brian and Jesse create a method to measure the change in the dependent quantity with respect to changes in two or more independent quantities. Jesse and Brian appeared to think about the ratio of changes as a method of quantifying "how fast". Jori appeared to think about the identification of changes in quantities and placing them into a ratio as the goal in and of itself.

Rate of change in space.

Jesse and Brian problematized how to measure how fast a dependent quantity was changing with respect to two independent quantities. They quickly identified the two-change problem, and devised a relationship between those changes to determine an open form rate of change function. Their eventual construction of simultaneous calculus triangles was a way for them to represent how fast the dependent quantity was changing with respect to each independent quantity. They wanted to make the changes in the independent quantities small in order to "more accurately" measure the rate of change at a point in space. I believe they did this because they were focused on describing how fast *z* was changing with respect to *x* and *y* because they had an image of these quantities varying simultaneously, not just because they made an independent quantity's change small for rate of change in the plane.

Jori tried to assimilate rate of change in space. It appeared she had two options. First, try to abstract a meaning, which was already problematic for her. Second, focus on the measuring process, with which she felt confident. She created an image of the calculus cheese, which to her represented a combination of two rates of change, which she had already calculated. She was not focused on devising a quantitative operation to describe how fast a quantity was changing with respect to another. Thus, addition fit with her need to combine rates of change. Eventually, she displayed sophistication in using perspectives and problematizing the two-change issue, in an attempt to fit her structure for rate of change in the plane. She determined a viable open form rate of change function, but was unable to interpret its meaning. These actions suggest that her goal was to identify the changes in quantities to produce a ratio. It was not her goal to use that ratio to quantify a relationship between quantities.

Directional rate of change.

Students differed in their understanding of a need for direction in space. Jori reached an impasse in calculating the rate of change when there were two different parameter values, h_1 and h_2 because she had no meaning for the denominator, except that she needed one. The two h-values in the denominator did not fit with her established pattern. It was not until that end of the teaching experiment that I believe Jori considered that change had a direction. This was important for finishing her calculations and making the denominator structurally equivalent to the one-variable rate of change function, but the idea of direction of change to Jori was very different than it was for Jesse. Jori seemed to think that the change had a direction, which was happening in space, but this change had no connection to the axes. Even when Jori considered that change had a direction, she did not have an image of z varying in tandem with x and y in that direction. Thus, her construction of the rate of change function was all about creating a ratio with changes, but not about measuring "how fast".

Jesse clearly saw that the changes were happening in the x-y plane, and that they needed to account for this change on the x and y axes. Jesse described a direction as a way to think about the dependent quantity changing with respect to an independent quantity by fixing the relationship between the changes in the two independent quantities. Like Jori, he saw rate of change in a direction as a way to resolve the denominator issue. However, I believe he understood that the denominator implicitly represented the changes in both quantities as changing h_1 necessitated a variation in h_2, while Jori saw this as a simplification of her rate of change calculation. When Jesse realized change had a direction, he imagined z varying in tandem with x and y in that direction, and which x and y covaried in a relationship defined by a vector in that direction.

My characterization of non-quantitative rate of change and rate of change as a measure of covariation use the ideas of conceptual time and quantitative covariation proposed by Thompson (2011), Saldanha & Thompson (1998), Castillo-Garsow (2010), and Carlson et al.'s (2002) work. My results suggest that rate of change as a measure of how fast a quantity changes with respect to two or more quantities relies on a student's image of the quantities varying, in tandem, over infinitely small units of conceptual time. Should the student not imagine the quantities varying in tandem, it is plausible that they are like Jori. They can make shifts in the moment about rate of change as a measure of how fast quantities are changing, but fall back on the identifying changes to construct a ratio without understanding what that ratio measures. In Jori's case, her decision to use ratio likely resulted from thinking a rate of change required a calculation using a ratio.

Framework for rate of change.

There is not a clear transition from thinking about rate of change non-quantitatively to thinking about it as a measure of how fast one quantity is changing with respect to another. As with shape thinking, I believe it can be described as a continuum of ways of thinking. However, The students in my study did not make transitions to a point where I can propose a full continuum. Thus, I propose a framework for thinking about rate of change that ranges from non-quantitative rate of change to rate of change as covariation, and describe how those ways of thinking about rate of change affect a student's interpretation of rate of change in space.

Table 23.

A Framework for Ways of Thinking about Rate of Change.

	Non-Quantitative Rate of Change	**Rate of Change as Covariation**
Meaning for Rate of Change	A calculation that integrates changes in variables.	How fast one quantity is changing with respect to another.

Interpretation of a Measurement of Rate of Change	A number that results from dividing the changes. Sometimes interpreted as a slant or slope.	A number that represents an approximation of how fast one quantity is changing with respect to another quantity.
Way of Thinking about Rate of Change in Space	A structurally similar calculation to rate of change in the plane. Might continue to describe as a slant or a slope.	How fast a dependent quantity is changing with respect to two independent quantities.
Implications for Thinking about Directional Rate of Change	Consider directional rate of change as a solving the "what to put in the denominator" question.	Consider directional rate of change as a way to systematically vary the changes in *x* and *y* simultaneously.
Implications for Thinking about Path Independence	Student might not be aware of the difference as a result of approaching a point along different paths.	Problematize path independence by noticing the dependent quantity changes at different rates with respect to the two independent quantities.

Contribution 3: Documentation of Surprising Insights

Some student responses within the teaching experiments were unsurprising. By "unsurprising", I mean that their responses fell reasonably close to the expectations I had in mind for student behavior when I was constructing the conceptual analysis. For example, Jesse's use of sweeping out to imagine a graph as an emergent representation of covariation fit with the ways of thinking I intended to engender. However, each of the students developed a number of results that I did not originally expect or intend to engender. In this section, I describe the most surprising student insights, and how these insights suggest important research agendas about student thinking in multivariable calculus.

Perspectives and sweeping out.

Jesse and Jori's use of perspective surprised me because of the distinct ways in which they were able to use sweeping out. Jesse's characterization of sweeping out from a perspective surprised me because he appeared to be able to imagine generating a surface by flexibly switching between parameters and variables. His use of perspective appeared to be a way to talk about cross sections, but it was also an indication that he saw the graph as an emergent representation of covariation. I attempted to engender the same use of perspective in Jori, and she quickly adapted the process of sweeping out a surface. I was surprised by her proficiency at describing the process for generating a surface in space, but came to believe that her sweeping out process was non-quantitative. Thus, Jesse and Jori both were proficient at sweeping out. In Jesse's case, the surface was an emergent representation of covariation. In Jori's case, the surface was emergent, but not imbued with quantitative properties. My conceptual analysis did not explicitly attend to the possibility that one could imagine sweeping out without imagining the surface as a representation of covariation. This insight suggests that in future studies, I must be cognizant of both the process the student is using to generate a function's graph and what they understand that process to represent.

The calculus cheese and calculational reasoning.

Jori's construction of the calculus cheese surprised me because it allowed her to consider a non-quantitative combination of rates of change. The *calculus cheese* was her way of dealing with the perceived difficulty of measuring rate of change at a point in space. She noted that the cheese, which looked like a wedge in her illustration, represented the combination of two calculus triangles. She visualized two calculus triangles, one each from the z-x and z-y perspectives, combining to form a wedge. This geometrical object constituted her representation of measuring a rate of change at a point in space. I also think that the combination of the calculus triangles, each of which implicitly measured a rate of change, suggested to Jori it was necessary to add two separate rates of change in measuring rate of change in space. Given this geometrical approach to rate of change, and her accompanying calculational reasoning, I was surprised that she was able to construct a viable measurement of rate of change in space. She was able to characterize the "two-change" problem in addition to finding a relationship between the changes in x and y. These results suggest that in future studies, I must be attentive to, and prepared for engendering productive ways of thinking in students who use calculational reasoning. My conceptual analysis in Chapter 3 did not build on or attend to calculational reasoning, but Jori's responses suggest it would be productive to characterize how calculational reasoning constrains a students' ability to reasoning quantitatively.

Highlight points, estimation and goals.

Brian and Neil's discussions about highlight points surprised me because of the possible explanations for what was occurring. During the teaching experiment, I believed that Neil could not visualize an infinite number of "highlight points" (correspondence points), and used linear interpolation between a finite number of highlight points to estimate the path the graph might assume. Neil saw no need to use Brian's process of sweeping out with an infinite number of correspondence points. I now believe Neil thought his process for generating a graph was sufficient. Neil's major goal appeared to be to generate a graph that mimicked Brian's. Thus, it may not have been that he could not imagine an infinite number of highlight points, but that Neil saw no reason to do so when estimation produced a reasonable approximation to Brian's graph. The possibility that Neil was not constrained by the limits of his thinking, but by his goals for that task, suggests that my conceptual analysis and tasks must be attentive to what the student intends to achieve with a setting. I perceived Neil's emphasis on a finite number of highlight points as a constraint of his thinking, while he plausibly could have thought in that way if he found it necessary. This result suggests that my conceptual analyses must not only attend to a plausible way of thinking a student might develop, but also should characterize why a student would find it useful to think in a particular way over other methods.

***The* rate of change at a point in space.**

Brian, and later, Neil's, emphasis on defining *the* rate of change in space surprised me because they were attending to invariance of rate of change under direction, which requires a constant function. They were able to not only consider rate of change in a direction in space, but a rate of change in all directions. As they began considering rate of change in all directions, Brian broached the issue of defining *the* rate of change. This exchange occurred near the end of the teaching session, and I was not prepared to make use of his insight in the moment. However, I believe following up on his comment could have sparked an interesting discussion about the importance of approaching the point in space from different paths. It also could have led to discussions about how to represent the rate of change function in space. For example, suppose Brian and Neil had proposed that the rate of change function in space could not be graphed unless rate of change was path independent. However, suppose the rate of change function is not invariant under direction (as is almost always the case). How then, might they have represented direction in their construction of the rate of change function? I cannot answer this question with data, but I predict that following up on this discussion with students in the future could generate interesting characterizations of thinking about path dependence and independence.

Role of tasks in eliciting surprising responses.

I was also surprised at the degree to which the tasks I used were able to elicit utterances and actions that were productive in characterizing student thinking. The tasks, coupled with my use of them as didactic objects, allowed the students and I to use them as conversation pieces. For example, most of the students continually referenced a correspondence point and sweeping out through their entire teaching experiment. Brian and Neil made a correspondence point and graph the central issue in thinking about two-variable functions and their rates of change. Jesse and Jori continually referenced sweeping out and perspectives in their teaching experiments. The tasks allowed the students to use things like correspondence points and sweeping out to create coherence between tasks using particular ways of thinking. In turn, I was able to document how the students applied initial concepts in the teaching experiment to approach complex issues about two-variable functions and their rates of change. I think that the tasks I used provide an initial idea to the field of how one might gain insight into students' ways of thinking about concepts in multivariable calculus.

In summary, I believe this dissertation has made three major contributions to studying student thinking about two-variable functions and their rates of change. First, I have characterized some facets of shape thinking, and the implications of those ways of thinking for thinking about function behavior. Second, I have described how calculational and quantitative reasoning lead to different entailments for the meaning and construction of a function to measure rate of change in space. Third, I have proposed how some surprising student insights suggest important avenues of research in multivariable calculus. Finally, I have shown how each of these contributions rely on the theoretical foundations on which I situated this study.

Development of My Thinking

The teaching experiments and my subsequent analyses not only have made contributions to the knowledge base of the field, but have affected my thinking. The first four chapters of this dissertation described my thinking relative to the instructional sequence and conceptual analysis prior to the teaching experiments, and the data analysis chapters focused on my thinking during and between teaching sessions. I have shown that as my thinking developed throughout the teaching experiments and the analyses, my interpretation of students' actions and the resulting models of their thinking were developing simultaneously. In this section, I consider my understanding of the literature and its constructs, my thinking with regard to methodological considerations, and my thinking about analytical method.

Literature and Constructs

Quantitative and covariational reasoning were the foundation for conceptual analysis in chapter three, and that conceptual analysis laid the groundwork for the didactic objects I used in the teaching sessions. My understanding of the literature about student thinking about function and rate of change supported that these frameworks were critical to a student's coherent understanding of function, but it was not until I reflected on the teaching sessions and analyzed data in creating models of student thinking that I operationalized the importance of these constructs in my own work.

The students' development of "perspective" to visualize function's graphs emphasized the importance of quantitative reasoning. Only the expert shape thinkers thought about a perspective as a method to generate a function's graph. The novice shape thinkers developed perspective as a way to "look at" the graph as an object that was already constructed. Thus, a student's quantitative reasoning manifested itself in how they interpreted perspective. In general, I consider expert shape thinking to be reliant on a high-degree of quantitative reasoning, while novice shape thinking is reliant on non-quantitative reasoning, such as a focus on graphs as physical objects.

My interpretation of covariational reasoning drew from Saldanha & Thompson's (1998) definition where one understands that two quantities are varying in tandem and each has a value at all points in conceptual time. Their work captured a great deal of the ways of thinking I was attempting to characterize in the teaching experiments. My analyses about rate of change led me to revisit Carlson's (2002) framework for covariational reasoning, which focused on how students think about how two variables vary in tandem. Carlson's (2002) work used Saldanha & Thompson's (1998) image of simultaneous, continuous variation to explain how the students in their study imagined quantities changing in tandem using five levels of mental actions. Saldanha & Thompson's (1998) description of covariational reasoning helped characterize the students' thinking about rate of change because it allowed me to explain how an image of two or more quantities having covaried is necessary to interpret a rate in the way I described in my conceptual analyses.

My framework for rate of change also built upon Saldanha & Thompson's (1998) description of simultaneous, continuous covariation, and Thompson's (2011) descriptions of conceptual time. I now believe that it is not possible to have an image of rate without an image of quantities covarying or already having covaried. Though these descriptions were present in Thompson's work about rate of change, I did not fully understand why until I attempted to characterize Jori's thinking about rate of change.

In addition to Thompson's work, Carlson's (2002) framework helps characterize the actions of my students within the study. For example, open and closed form rate of change and calculus triangles are ways to support students in thinking in ways consistent with MA4 and MA5 in that they help the student imagine how one quantity changes with respect to one or more quantities over intervals of conceptual time. My results also identify areas in which Carlson's (2002) framework could be extended to account for functions of two variables. For example, it may be possible to extend her framework to include a description of how a student reasons about average and instantaneous rate of change in a direction. In this case, not only does z change with respect to x and y, but x and y are in an invariant relationship. By nature of what her covariational reasoning framework intended to model, Carlson did not consider how students think about a dependent variable changing at a rate with respect to two or more independent variables.

I situated this study within the broad research literature on function. At the time I designed this study, I was working under the impression that these studies more or less were evaluating a well-defined idea of students' work and thinking about functions. In short, my understanding of the literature was that an understanding of function could be construed as categorical. A student understands function, or does not. This approach led to my initial development of frameworks that included only expert and novice shape thinking without accounting for plausible developmental models between the two. However, I now recognize that having a possible developmental model allows one to characterize the student's thinking on a continuum.

A continuum is more powerful for capturing the nuances in students' thinking because it allows for the possibility that a student is part of more than a single category. For example, a novice shape thinker is not thinking about function in any classical definition because he is attending to the graph as an object associated with a particular set of symbols. They attribute the graph's construction to a graphing calculator without understanding how the graphing calculator generates the graph of the function. This insight came as a result of a focus on modeling students' ways of thinking about function, where the most interesting and complex ways of thinking to model were when the student's actions indicated they did not think about function in a typical sense of the word.

As a result of this insight, my reading of the literature relative to function now focuses on three major ideas. First, does the study define what an understanding of function encompasses? Second, does the study attempt to explain what the student thinks about a function, regardless of whether that way of thinking fits the study's definition of an understanding? Third, does the study identify a plausible model by which the student's understanding of function could develop?

I believe my work contributes to exploring and clarifying what ways of thinking about function are productive for students, as well as describing the ways of thinking that are not productive at the moment, but nonetheless require consideration if we desire to support students in developing the productive ways of thinking. In general, the results of this dissertation are evidence of the importance of quantitative and covariational reasoning.

Methodological Considerations

Initially, I conceptualized a teaching experiment as a way to experience constraints of a student in creating a model of a student's mathematics. While I understood that the researcher was a necessary part of the teaching experiment, I now understand that without documenting the researcher's thinking it is nearly impossible to trace the development of a construct within a model of student thinking. In my case, I was not often able to have a witness present during my teaching experiment sessions, and at best had discussions with others who watched videotapes of the sessions. This constraint required that I document my thinking before each session, including what I anticipated the student would do and say based on my continuously evolving model of his or her thinking and reflect on my hypotheses in the refinement of my model immediately after each teaching session. In this way, I was able to track the development of each construct in developing a model of my own thinking. This strict attention to modeling my own thinking allowed me to make my reflections part of the data corpus. Finding ways to attend to reflexivity by integrating the researcher's thinking with observations of a witness could delineate how the researcher's thinking contributed to each aspect of that model.

Prior to the teaching experiments, I saw microgenetic analysis as a technique of data collection for documenting students' actions. I now think that microgenetic analysis can be a key component of developing a robust, viable, and precise model of student thinking. Siegler & Crowley (1991) described the microgenetic method as a method of studying change with three properties. First, observations span the period from the initiation of a change to the end of a change, marked by the stability of a system under study. Second, the density of observations is high relative to the rate of change of the phenomenon. In short, the rate of change of number of observations with respect to time increases if one anticipates the system to be at a point of a critical change. Third, observed behavior undergoes trial-by-trial analysis with the goal of attributing causal agents to particular aspects of change in a system. (Siegler & Crowley, p. 606).

The microgenetic method allows one to specify the density of observations over a period of time based on the type of change and type of complex system the researcher is attempting to characterize. I believe the correct question in attempting to characterize change is not how many observations over how much time, but over what period of time to increase the density of observations. For example, suppose that a researcher is attempting to create a mental model of a student's thinking as he or she participates in a two-week long instructional sequence such as in this study. The researcher believes that the major shifts in student's thinking will occur on Days 1, 4 and 9 based on analysis of the instructional sequence. Thus, the researcher might increase the density of observations (i.e. number of documented actions, verbal cues, or gestures) on Days 1, 4 and 9 relative to the other days in the instructional sequence. This does not mean that the researcher takes fewer observations between these major days, but it could suggest particularly useful days for a witness to be present. If a witness is not able to present, these particular days, over which the greatest change was anticipated to have occurred, could constitute a sample of video for various witnesses to watch and discuss with the researcher.

Suppose a researcher has already conducted a number of teaching experiments focused on a particular idea as I have in this study. I could hypothesize when the student experiences the greatest constraint in their thinking or undergoes the greatest amount of change in their actions and descriptions of mathematical ideas. This approach could allow a researcher to systematically focus on evaluating a particular aspect of a model of student thinking much like a focused clinical interview. For these reasons, I believe the microgenetic method can provide a useful complement to teaching experiment methodology in the development of more precise and viable models of student thinking.

Future Directions

I have described how my thinking changed during the course of conducting the teaching experiments and analyzing my data. My thinking has also changed with regard to how I believe my work contributes to the field of mathematics education. I think that the construct of shape thinking and my detailed description of its facets and levels allow a more thorough characterization of what students think about functions and their representations rather than what they cannot do with tasks involving functions. I also think shape thinking is a first attempt at describing what distinguishes the thinking of experts in the field from those just learning about functions and graphs of functions. Thinking about shape thinking as a continuum provides a preliminary framework on which subsequent research can design tasks and instructional sequences in modeling student thinking about functions.

I think that my descriptions of rate of change are an example of the implications of shape thinking, and allow for further exploration into the implications of thinking about rate of change non-quantitatively. I also think that this study creates a base from which I can explore student thinking about total derivative. In general, I see my results as both complementing and providing specific examples of the importance of quantitative and covariational reasoning in the context of thinking about functions and their rates of change. I am also interested in understanding how Thompson, Castillo-Garsow, and Carlson's notions of conceptual time, smooth continuous variation, and a covariational reasoning framework can build on the characterizations of student thinking I have developed in this dissertation.

This study focused on generating models of student thinking of the four students in the study. As I mentioned earlier, the utility and viability of the frameworks and the constructs that compose them cannot yet extend beyond the students who participated in this particular instructional sequence. Thus, I have designed subsequent studies to further develop shape thinking and its many components. I plan to focus on working with graduate students, research mathematicians and mathematics educators to define a working limit for the developmental trajectory in which I have situated shape thinking. I also anticipate working with younger students as well as students entering college to define beginning points for the developmental trajectory. While working with mathematicians and mathematics educators, I plan to understand the ways of thinking required to understand total derivative, which will allow me to design subsequent teaching experiments in which I model the ways of thinking of students as they think about rate of change in many directions. I believe that pursuing these avenues of study will allow me to begin building this work into a larger research program focused on multivariable functions and rate of change.

Chapter Summary

In this chapter, I characterized the contributions I believe my work has made to the field of mathematics education. In doing so, I described shape thinking as a developmental trajectory composed of novice and expert shape thinking. I described particular ways of thinking and actions associated with those ways of thinking for each level of the framework. Next, I proposed a framework for students' understanding of rate of change. I described this framework as a continuum of ways of thinking, and situated students from the teaching experiment session along this continuum. In this chapter, I also characterized how my thinking changed with regard to the literature and methodology from the study. Lastly, I considered how the development of my thinking and the frameworks I proposed suggest some interesting directions for research.

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APPENDIX A

HUMAN SUBJECTS APPROVAL LETTER

ARIZONA STATE UNIVERSITY
RESEARCH AND ECONOMIC AFFAIRS

Office of Research Integrity and Assurance

To:	Marilyn Carlson UC
From:	Mark Roosa, Chair Soc Beh IRB
Date:	12/01/2009
Committee Action:	**Exemption Granted**
IRB Action Date:	12/01/2009
IRB Protocol #:	0911004581
Study Title:	Students' Understanding of Multivariable Functions

The above-referenced protocol is considered exempt after review by the Institutional Review Board pursuant to Federal regulations, 45 CFR Part 46.101(b)(1) .

This part of the federal regulations requires that the information be recorded by investigators in such a manner that subjects cannot be identified, directly or through identifiers linked to the subjects. It is necessary that the information obtained not be such that if disclosed outside the research, it could reasonably place the subjects at risk of criminal or civil liability, or be damaging to the subjects' financial standing, employability, or reputation.

You should retain a copy of this letter for your records.