Undergraduate Students' Development of Covariational Reasoning

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Abstract

Previous research discusses students' difficulties in grasping an operational understanding of covariational reasoning. In this study, I interviewed four undergraduate students in calculus and pre-calculus classes to determine their ways of thinking when working on an animated covariation problem. With previous studies in mind and with the use of technology, I devised an interview method, which I structured using multiple phases of pre-planned support. With these interviews, I gathered information about two main aspects about students' thinking: how students think when attempting to reason covariationally and which of the identified ways of thinking are most propitious for the development of an understanding of covariational reasoning. I will discuss how, based on interview data, one of the five identified ways of thinking about covariational reasoning is highly propitious, while the other four are somewhat less propitious.

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Statement of the Problem

In instructional settings, merely offering definitions of covariation is insufficient for helping students to develop the conceptual operations that reasoning covariationally demands. To best equip educators with methods that can assist students to construct invariant relationships between quantities in dynamic situations, an in-depth study on how students think when trying to develop these relationships, as well as an analysis of which ways of thinking are most propitious for learning, is essential. Furthermore, since instructional settings vary among different teachers and students, controlling for the support offered to students is significant to evaluating the propitiousness of their thinking alone, excluding factors related to productive teaching approaches. With this problem in mind, the present research will attempt to answer the following question:

Which of students' ways of thinking about reasoning covariationally are propitious for developing an understanding of covariation with minimal instructional support?

Literature Review

Previous mathematics education research on how students think about solving problems has given centrality to the concept images that they possess (P. W. Thompson, 1994a, 1994b; P. W. Thompson & Thompson, 1992). In his discussion of "Students, Functions, and the Undergraduate Curriculum," Thompson, building upon the work of Piaget and Kosslyn, explains how the merging of "visual representations, mental pictures, experiences and impressions evoked by the concept name" constitutes what one would call an individual's concept image (1994b, p. 4). Equipped with concept images, a construct first developed by David Tall and Shlomo Vinner (Tall & Vinner, 1981), students form the basis for developing an understanding of how to reason about related problems or concept definitions by constantly constructing images with or without an overt awareness of the construction process. Mathematics education researchers emphasize the importance of deeply-rooted images in producing meaningful connections in students to a variety of topics present in current curricula, including rate, function, covariation, magnitude, algebra, calculus, and differential equations (Nunez, 2006; Siegler & Richards, 1979; A. G. Thompson & Thompson, 1996; P. W. Thompson, 1994a, 1994b; P. W. Thompson & Thompson, 1992, 1994). To take one study as an example, Thompson (1994a) conducted a teaching experiment with senior and graduate mathematics students that illustrated how underdeveloped images of covariation and multiplicatively-constructed quantities can inhibit a student's capacity to understand the Fundamental Theorem of Calculus. With this notion of image transfer in mind, several authors studying covariation argue that a student's understanding of covariational reasoning, specifically, can be taken to develop from that student's concept images (Carlson et. al., 2002; Castillo-Garsow, 2012; Moore, Paoletti, & Musgrave, 2013; P. W. Thompson, 2011).

Before attempting to delve into the images that researchers find essential or propitious for developing covariational reasoning, I will first examine what it means to reason covariationally and what further implications this way of reasoning offers for students' learning across topical boundaries. Moore et. al. provide the following definition that synthesizes the ideas of Carlson, Saldanha, and Thompson: "Covariational reasoning entails the mental actions involved in conceiving two quantities as varying in tandem" (Carlson et. al., 2002; Moore et al., 2013, p. 462). Taking a step back in building these mental processes, it is essential to possess a foundation for conceiving of just one quantity's value varying.

Thompson characterizes the conception of a quantity of varying measure by a capacity to *anticipate* how the quantity's measure holds different values at different instances in time (P. W.

Thompson, 2011). He further explains this "anticipation" as implicitly thinking of the value of a quantity "varying within an interval in chunks, but with the immediate and persistent realization that any chunk of completed variation can be re-conceived as entailing continuous variation" (2011, p. 47). This description of how one conceives of variation resonates with Carlos Castillo-Garsow's analysis of "smooth" and "chunky" quantitative reasoning, which I will discuss in more detail later in this section. To conceive of the variation of two or more quantities in tandem, and consistent with Moore's synthesized definition of covariational reasoning, Thompson describes the formulation in the mind of a multiplicative object that unites the quantities simultaneously and persistently (P. W. Thompson, 2011). This conceptualization of variation and, subsequently, of covariation provides the backbone for interpreting the results of the present study.

In another work, Moore explains how crucial this understanding is to being able to create and interpret graphs that represent how two quantities are related (Moore, 2012). Furthermore, Castillo-Garsow discusses the links that researchers have identified between covariational reasoning and understandings of geometric growth, function, trigonometry, calculus, and differential equations (Castillo-Garsow, 2012). Both authors, along with their predecessors, lay the task for researchers to direct their attention to how students think about reasoning covariationally.

While the above definition remains consistent throughout literature, images of what it takes to construct an understanding of covariational reasoning do not (Castillo-Garsow, 2012). Recognizing the need for further study into the mechanics of this construction of meanings for covariational reasoning, Castillo-Garsow analyzes continuous quantitative reasoning as a building block to understanding. His study (2012) identified distinctions between two approaches to thinking about quantities varying continuously, calling them "chunky" and "smooth." Resonating with Núñez's (2006) work on the inferential organization in which humans construct their mathematics in the rich language found in everyday use, Castillo-Garsow recognized that imagining continuous change through physical-world-driven characteristics (i.e., "chunks" of fixed amounts of change and "smoothed" out accruals of change) can perhaps be more beneficial to students attempting to arrive at continuous reasoning than images based solely in discrete reasoning (2012). While Castillo-Garsow argues that images of "smooth" reasoning are more propitious for learning than their "chunky" counterparts, he further believes that images entailing a mix of *both* ways of thinking are more beneficial to students than either on its own (Castillo-Garsow, 2012). This research paves the way for further study into which paths of thinking allow students to arrive at an understanding of covariation with the fewest potential obstacles.

In reference to today's K-14 mathematics curriculum, Thompson states that textbooks lack an emphasis not only on function as covariation, but also on the concept of variation altogether (P. W. Thompson, 1994b). He explains that, with too much focus on a variable as merely a literal representation of a number, students lose opportunities to build more abstract images of variable magnitude (1994b). These images, which entail a rich understanding of variation, would be as helpful to students attempting to analytically represent covarying magnitudes as, say, images of addition for understanding concepts of infinite series (1994b). Instead of cultivating these building-block concept images for covariation, Thompson fears that today's curricula restrict students' thinking of function to a correspondence between two sets of elements (1994b). With meanings rooted in this understanding of function, one could certainly encounter difficulty in imagining the simultaneity of varying quantities, which lies at the crux of Moore and Carlson's aforementioned definition for covariational reasoning.

Recalling Thompson's discussion of variation, to understand that the value of a quantity varies is to *anticipate* that the quantity's measure holds different values at different moments in time (2011). For researchers to assess an individual's understanding of covariational reasoning, they need to look to that individual's ability to represent that anticipation in some visible way. Thompson advocates the importance of mathematical modeling in that it is composed of the same operations that allow students to "see invariant relationships among quantities in dynamic situations" (2011, p. 46). Through the methods they employ for their studies, other researchers that examine covariational reasoning in the context of instruction share this stance on the role of mathematical modeling (Castillo-Garsow, 2012; Moore, 2012; Moore et al., 2013). Hence, I will orient the methods of the present study in a similar way, emphasizing representative modeling as a lens for viewing how undergraduates currently enrolled in pre-calculus and calculus think about problems that require covariational reasoning.

Methods

In the present study, I conducted one-on-one interviews with pre-calculus and first-semester calculus students, presenting them with a task aimed at revealing their ways of thinking for covariational reasoning. The study was guided by the following research question:

Which of students' ways of thinking are propitious for developing an understanding of covariation with minimal instructional support?

Participants

This study utilized four participants, 'Coleen,' 'Brendan,' 'Kaitlyn,' and 'Dakota,' who were undergraduate students enrolled in pre-calculus or calculus courses at a large public university in the United States. These students were chosen on a volunteer basis from two separate courses, one pre-calculus and one calculus, and they were offered monetary compensation for their participation in the one-hour interview. Students were chosen from these particular classes because they provided an appropriate sample population for exploring the ways of thinking with which calculus and pre-calculus students operate. Additionally, the instructors of these courses were members of the mathematics education research team that I had worked with throughout the year.

In choosing to work with pre-calculus and calculus students, I expected that the students would have, at a minimum, a basic background in high school mathematics. With this background, these students should have at least some exposure to or familiarity with graphical representations of quantities. Since the interviews were conducted near the beginning of a spring semester (i.e., near the beginning of their present pre-calculus or calculus course), I suspected that each of the four students would be equipped with a unique and varied set of meanings for covariational reasoning. While each would have studied similar topics, their ways of thinking would be as varied and multi-dimensional as the paths that led them to enroll in their course. All four subjects were assigned to perform the same task in the same setting at separate times, to improve reliability of a cross-analysis of the interview data.

Apparatus and Materials

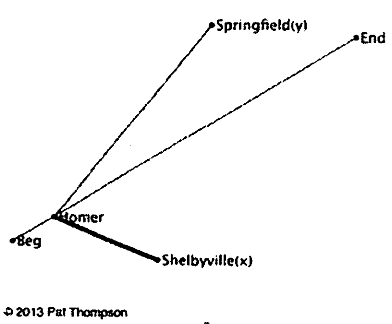
To conduct the four interviews, I implemented a covariation item that Dr. Patrick W. Thompson designed with Graphing Calculator software and provided me to use as the central task for this study. The students were able to view the animated item within a Keynote presentation that I displayed on a 13-inch MacBook Pro throughout the interview. To record the students' written work during the task, I provided them with a Livescribe pen and as many sheets of specialized Livescribe paper as they elected to use. This "smart" pen recorded the students' sketches during the interview and produced an audio file to match the written data. With the use of this technology, I sought to record exactly what they were writing and saying simultaneously.

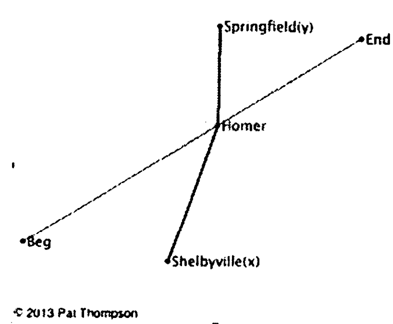
I provided coordinate axes at the top half of each sheet of paper for the students to represent their graphs, along with a snapshot of the animation item at the bottom for them to make marks with their pen if they so desired. Plenty of blank space was left available to the students should they feel the need to show any additional steps or scratch work. All of these materials were geared toward accurately recording the students' work clearly so that it could be properly interpreted for analyses.

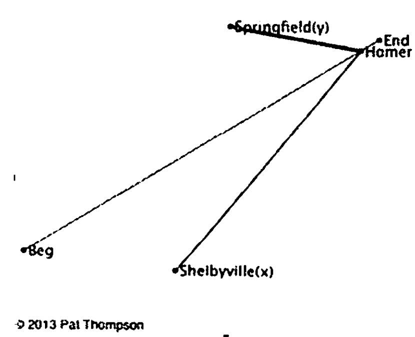
Procedure and Task

The students were given the Livescribe pen and the sheets of Livescribe paper, as described in the previous section. As a preface to the task, they were encouraged to share as much about their thought processes as they felt appropriate, whether in writing or verbal discourse.

The setting and task, which I will explain in detail, presented students with an animation that proposed an idealized representation of a person (Homer) driving along a straight road that went between two cities, Shelbyville (SV) and Springfield (SF). The following images show the animation at three instances, illustrating how Homer moves along the road:







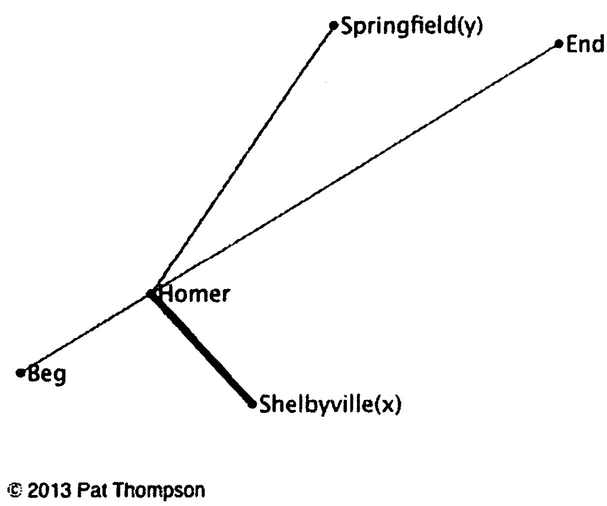
The interview process unfolded in five phases, with emphasis given to the last four. While each phase is characterized by the introduction of a new means of support, the purpose of the phases was not to directly help the students solve the task. Rather, the intent behind the progression of phases was to discern which of students' ways of thinking are more propitious for developing an understanding. According to this model, a student who demonstrates covariational reasoning only in the final interview phase would have a somewhat *less* propitious way of thinking than that of a student who did the same in an earlier phase. Hence, the phases standardized the type and degree of support that the students received in the interview, with the expectation that this method would potentially improve analysis regarding distinctions between different ways of thinking. The five phases were as follows:

0. Introduction to the animation,
1. Request to graph the distance from Homer to SF in relation to the distance from Homer to SV,
2. Introduction of dynamic visual support showing a red line on the *y*-axis representing Homer's distance to Springfield,
3. Introduction of dynamic visual support showing both the red line on the *y*-axis representing Homer's distance to Springfield *and* a blue line on the *x*-axis representing Homer's distance to Shelbyville, and
4. Introduction of dynamic visual support of a correspondence point—the point whose location represented Homer's distance to both cities simultaneously

Phase 0: Introduction to the animation

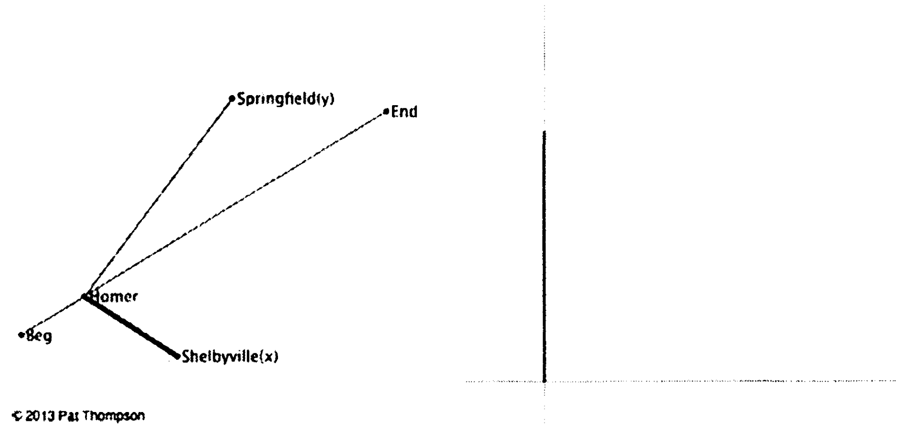
I presented the first slide of the Keynote presentation, which told the students that they were about to see an animation of a point, labeled Homer, traveling along a straight road that lies between two cities, Springfield (SF) and Shelbyville (SV). They were directed to note the distances between Homer and each of the cities. The second slide presented the animation alone. After having some time to grow familiar with the animation, I asked if they were ready to view the task and then proceeded to Phase 1. From this phase on, when I determined that the student had shown signs of exhausting their ways of thinking about the task and were effectively "stuck," I proceeded to the next phase of the interview, which provided a new means of visual support.

Phase 1: Request to graph the distance from Homer to SF in relation to the distance from Homer to SV (Slide 3)



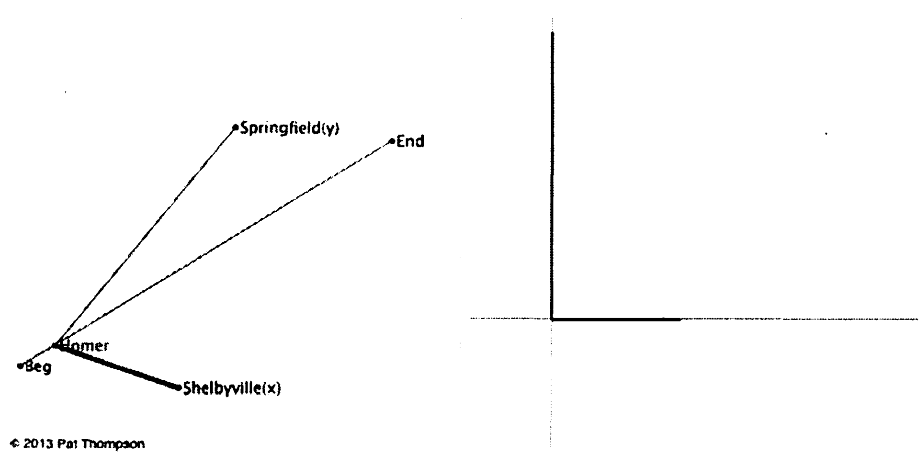
The above image shows a static representation of the animation presented in this and all remaining phases. As the students viewed the animation, the point labeled Homer moved back and forth along the road from 'Beg' to 'End' at a constant rate. His movement looped repeatedly in the animation for as long as the phase continued. Slide 3 presented just this animation and a request to sketch a graph of the distance from Homer to Springfield in relation to the distance from Homer to Shelbyville. Subjects were then left to their own approaches in solving the problem, with occasional interviewer questions for clarification of meaning.

Phase 2: Introduction of dynamic visual support showing a red line on the y-axis *representing Homer's distance to Springfield (Slide 4)*



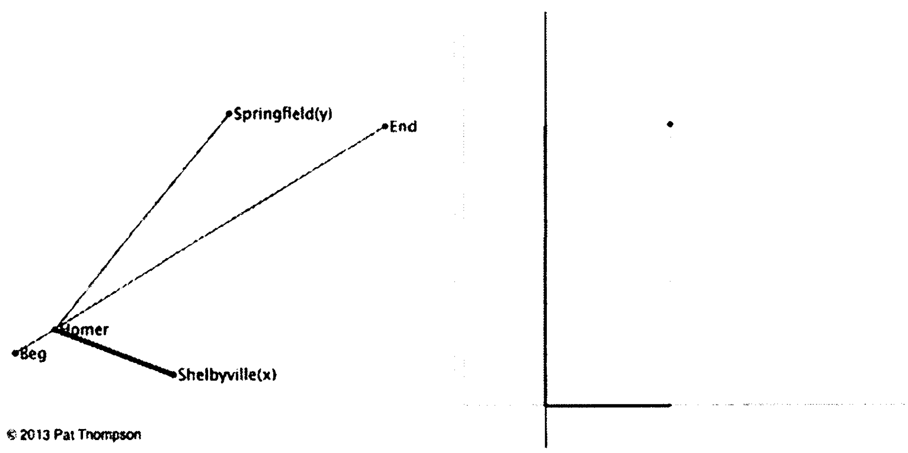
The above image shows a static representation of the animation presented in Phase 2. The red line on the vertical axis increased and decreased in magnitude in direct correspondence to the red line representing Homer's distance from Springfield in the animation.

Phase 3: Introduction of dynamic visual support showing both the red line on the y-axis *representing Homer's distance to Springfield* and *a blue line on the x*-axis *representing Homer's distance to Shelbyville (Slide 5)*



The above image shows a static representation of the animation presented in Phase 3. Additional to the support offered in Phase 2, the blue line on the horizontal axis also increased and decreased in magnitude in direct correspondence to the blue line representing Homer's distance from Shelbyville in the animation

Phase 4: Introduction of dynamic visual support of the correspondence point representing Homer's distance to both cities simultaneously (Slide 6)



The above image shows a static representation of the animation presented in Phase 4. Additional to the support offered in Phase 3, the correspondence point between the two magnitudes tracked the covariation of the two quantities.

My intention with this interview structure was to allow enough time to gather as much information as possible about how the subjects initially thought about representing the covarying distances, so that I would be able to attribute any revelations or changes in their thinking directly to the particular level of support (Phase) given. If a subject were to sketch an accurate graph and show an understanding of covariational reasoning at any point in the interview, then no further phases would be necessary, and the interview would end after a few follow-up questions targeted at how the student was able to arrive at that understanding—what, exactly, was her or his way of thinking. As an interviewer I was careful to limit my speech just to clarify what each subject meant when she wrote or said something about how she was thinking about the problem. My intention was that the student would feel free to think, write, and speak in an unrestrained manner so as to reveal as much as possible about his or her ways of thinking.

Results

Here I will present some of the predominant ways of thinking about covariation that students showed throughout the interview process. A complete listing of every different way of thinking displayed by the interviewees is both beyond the scope of and unnecessary to the goals of the present study. Instead, I focus on which ways of thinking arose most frequently in students' writing and speech and to those that eventually led students to represent the invariant relationship between the two distance quantities. While students commonly mixed ways of thinking within relatively brief periods of time, I will focus the information in each subsection on the particular way of thinking it describes. To clarify the information provided in any images pulled from interviews, green writing shows what they had drawn up until that point in the interview, while the light grey writing shows what they wrote at some later moment. This distinction is important in interpreting the order of each student's processes. I will use both excerpts and student sketches from interviews as evidence for the presence of each way of thinking.

A Disposition to Plot Discrete Points

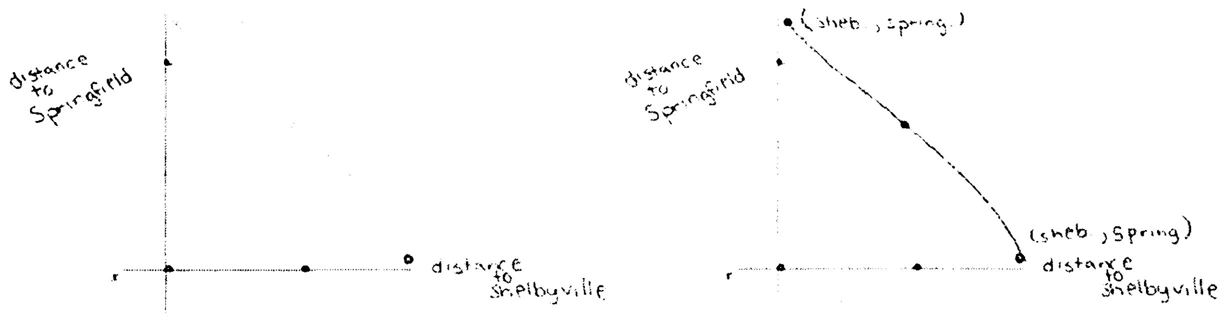
When tasked with sketching a graph on a coordinate axes, some students' thinking reflected a disposition to plot discrete points to help them understand how to construct the graph[1]. If they drew points on the coordinate plane as a first step, before attempting to draw a line, curve, or other shape, then they have shown this particular way of thinking. Additionally, students who, after sketching a graph of some sort, plotted points that held meaning for them and

[1] By "graph" I shall mean the set of points $(x(t), y(t))$ that reflect the record of simultaneous distances from Homer to Springfield and to Shelbyville.

then used this process as a step in generating a different graph also displayed this disposition. The essential feature in identifying this way of thinking lies in the students' methods of utilizing points in the coordinate plane to help drive the creation of their graph. Three of the four subjects of this study, Coleen, Brendan, and Dakota, revealed this disposition to different extents.

In Phase 1 of her interview, Coleen expressed her initial struggles with the task, saying, "I don't know, like, it's hard to explain.... Like, of course he's getting farther away from one as he gets closer to the other, but the rate at which it's changing is what I don't really know how to graph or describe" [Phase 1]. In an effort to focus her thinking, she plotted three points as shown in Figure 1—one at the origin, one on the x-axis, and one on the y-axis.

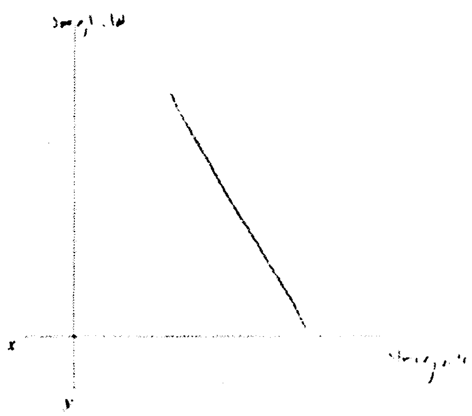
Figure 1. Coleen's plot of two points (left) and a line segment connecting them (right) [Phase 1]



The transition from her initial point-sketching to her subsequent line-sketching illustrates her disposition to plot points as a step to making a graph while simultaneously making sense of the situation. When asked about the points she drew, Coleen explained, "Okay, so, if his distance from Springfield is not that much, then his distance to Shelbyville is a lot, so the point would be right there [*she refers to the point farther along the* x-axis *at* $y=0$]. Or, like, if his distance from Springfield is, like—um—really high, then his distance to Shelbyville would be really small" [Phase 1]. Her understanding of what the axes represent arose out of this discussion of specific points on the coordinate plane. Coleen further described her reasoning for drawing the point on the middle of her line segment when she said, "I picked the middle point because it's kind of the easiest one. It's halfway between both of them, so the distance to both cities is pretty much the same because it's like when the line is straight for less than a second" [Phase 1]. When facing difficulty with the task, she opted to draw something that she knew she could: points.

Other students revealed that their thinking of graphing was linked to point plotting in a less direct way. Brendan, when initially confronted with the task, first sketched a line segment (see Figure 2).

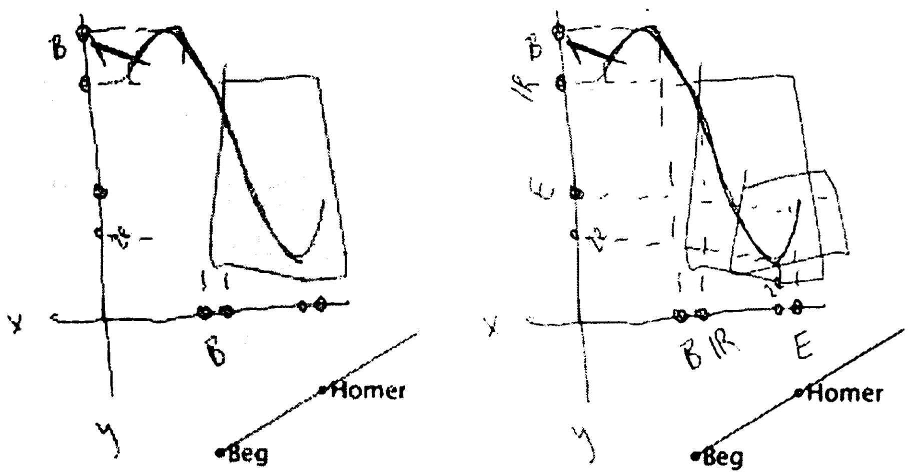
Figure 2. Brendan's initial sketch [Phase 1]



Unconfident with what he had drawn, he began trying to make sense of the situation in different ways. He said, "If they're both distances, then [what I drew] doesn't really represent anything. So, I think you probably have to make a point for Springfield on the y-axis, and a point for Shelbyville on the x-axis.... I have no idea, I think I've just confused myself more" [Phase 1]. When stumbling to make sense of how to graph the invariant relationship between these two quantities, Brendan sought to broaden his approaches, searching for other aspects of his thinking that may serve him well. By saying that "you probably have" to plot points at certain locations on the coordinate plane, he revealed this particular aspect, which entailed a disposition to plot points to make accurate graphs of lines and curves.

Another student, Dakota, displayed a disposition to plot points throughout his work in all phases. During Phase 3, he organized his thoughts by constructing a separate set of axes from the one provided, indicating that he may not have expected the work to reflect a "final answer" to the problem. The progression of his work is shown in Figure 3.

Figure 3. Dakota plots points and then guesses at behavior between them [Phase 3]



Before interpreting the thinking revealed in Dakota's approach, it should be noted that by 'B' and 'E,' he meant 'beginning' and 'end,' and by 'R,' he meant 'river,' which was a real-world feature that he repeatedly employed to refer to the two locations on the road with a minimum distance to Shelbyville and a minimum distance to Springfield (this working meaning for 'river' is what he offered when I asked about it in Phase 3). While these marks on the axes labeled 'B,' 'E,' and 'R' seem to represent points; his intent in drawing them was to plot landmarks—significant locations of correspondence of the two quantities. We can see his use of points as landmarks in his comment: "So, I'm trying to show the key marks in the road, and where both bars are at that point. And I came up with this" [Phase 3]. Rather than plot these first and then draw the curve, Dakota drew the curve first and then attempted to fit correspondence points to that curve. However, since he did this work off the main axes, as a step in his process to understanding how to accurately graph the relationship, it still stands as evidence of his disposition to plot points to help create or modify a graph. Near the end of his interview, he even stated, "So, the correspondence points kinda help me out to kinda see where I'm at—at a certain distance from both Springfield and Shelbyville" [Phase 4]. This quote exemplifies the way of thinking described in this section: he looked at specific points to understand how to construct a complete graph of the covarying distances. It is also worth noting that Dakota saw points' locations as representing two distances simultaneously.

Shape Thinking when Graphing

Some students' thinking led them to sketch a graph with a focus on matching the *shape* of the road on which Homer traveled. While multiple interviewees (Coleen, Brendan, and Dakota) initially sketched graphs of diagonal line segments, this work alone did not provide sufficient evidence that the shape of the road directly influenced the shape of their graph. To determine with more concrete data that a student actually considered the animation's shape as a model for how their graph should look, I look to the students' speech when referring to the graphs they drew. Coleen, when prompted, revealed this type of shape interpretation as a predominant way of her thinking about how to create a representative graph.

As illustrated in Figure 1, Coleen's first graph had the shape of a line segment with a negative slope. She revealed how she thought about constructing this graph in Excerpt 1.

Excerpt 1. Coleen talks about graphs as shapes [Phase 1].

1. *Ben*: So, what made you draw a straight line between the two points?
2. *Coleen*: Wow, that's a good question because I could have just added in the other points—or, like, looked at different points—but I guess I kinda was really focusing on the beginning and the end. So, this would be the beginning and that would be the end, so I just connected them, and they kind of *make* the road that he's going on, but they also describe how being close to another one is farther from the other one.
3. *Ben*: So, these points, here, the endpoints of the line you drew, those correspond to—just to confirm—the beginning and the end on the graph, or on the animation?

4. *Coleen*: Well, yeah, so this one would be Shelbyville and this one would be Springfield [*she then wrote "(sheb., spring)" with ordered pair notation by both endpoints of her line segment*]. And when you look at it—at the beginning—he's pretty close to Shelbyville, so it's closer to the beginning of the x-axis. And he's far from Springfield, so it's higher up the y-axis, and that's the same over here. So, for me, yeah, it's the beginning and end of the animation.... But I don't know why I drew a straight line....
5. *[Some omitted text]*
6. *Ben*: With the straight line, are you envisioning it partly because of the way the road looks itself, do you think?
7. *Coleen*: Yeah, definitely, because if the road were to look curved, I probably would have thought, maybe, 'Oh, definitely, the points are going to be, you know, as a constant rate.' But, because the line is straight, and the distance from Springfield to the end and beginning to Shelbyville looks like it's about the same distance, it's making me think that it's a straight line.

Her statement in Line 2 about how connecting the two points, which she associated with the beginning and the end, on her graph effectively "make[s] the road" provides strong evidence that shape thinking was central to her graphing process. To further support this claim, her response in Line 7 clarifies that she indeed sketched a line segment for her representative graph "because the line [Homer's road] is straight." Also, by suggesting that changing the shape of the road—specifically, so that it was not straight—would entail a similar change in the shape of the graph, she reveals that she thinks the graph's shape should resemble the shape of the animation. With these statements as evidence, shape thinking when graphing constitutes a significant component of Coleen's ways of thinking about representing covarying quantities.

Graphs of Functions Must Represent an Independent and a Dependent Quantity

Even though the task did not mention function explicitly, several students showed that their thinking of constructing graphical representations was wrapped up in their meanings for function. Both varying quantities that the subjects were asked to represent on the axes were dependent on a third, implicit quantity—namely, Homer's location on the road. The graph of Homer's distance to Shelbyville in relation to his distance to Springfield is of a function defined parametrically. Each coordinate is a function of a third quantity. Thus, students' underlying concepts of function as input-output or independent-dependent variables were a source of confusion for them because neither distance in the task was an independent variable. However, some students revealed that they thought that any graph they draw must represent a function with a one-to-one mapping from one quantity to another (usually from the x-quantity to the y-quantity). This way of thinking was frequently manifested in interviewees' disposition to place time as a quantity on the x-axis, despite being directed to make that axis represent Homer's distance to Shelbyville. Brendan showed this way of thinking in his approach to the task.

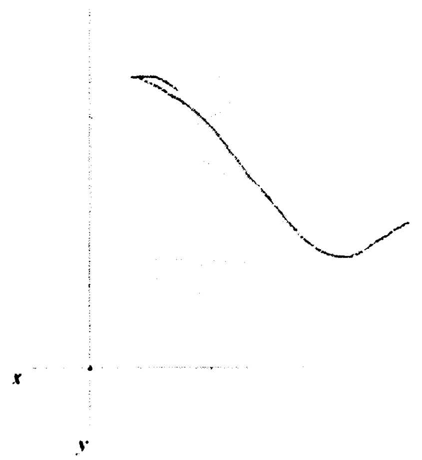
Near the beginning of Phase 3, before sketching a graph in that phase, Brendan began to reveal one potential root of his struggles in trying to represent the relationship. Excerpt 2 details his challenges with the task.

Excerpt 2. Brendan thinks that a graph should reflect y as a function of x [Phase 3].

8. *Ben*: So, how are you thinking about this added bit [*I referred to the added visual support in Phase 3*]? Is it changing anything for you?
9. *Brendan*: It's making me feel like I'm still wrong. It's that dip at the end that's on both of them. [*Pause*] On the y-axis, it's kind of easy to do.
10. *Ben*: But not on the x-axis?
11. *Brendan*: Not to me, because I just kind of think you're going back and forth with the x, along the x-axis?
12. *Ben*: Can you elaborate?
13. *Brendan*: Well, you can do the dip to go up and down with the y-axis. To do the dip on the x-axis, it would have to go kind of like left-to-right across it, which would look kinda weird—more like a scribble.
14. *Ben*: What about it do you think is weird?
15. *Brendan*: It just doesn't—it wouldn't be a function. I guess that would be the weird thing because then you'd have two spots taking up the same space, or one point be in two different spots, [*Pause*] which, I guess, kinda has to be possible in order to do that anyway. Is this supposed to be a function?

Brendan's comments in Line 8 show how he was initially thinking that his graph should somehow reflect a conventional functional relationship between the two quantities. Even though in Line 9 he noted a "dip at the end that's on both of them," seemingly displaying an understanding of the pattern of the quantities' relationship, his graph (see Figure 4) showed how he did not transfer his understanding to the graph.

Figure 4. Brandon's graph of the relationship between distance to SV and distance to SF [Phase 2]



The graph in Figure 4, coupled with the discussion in the above excerpt, illustrates how Brendan's thinking more readily allowed for two values of the y-quantity to map to a particular x-value than it did for two values of the x-quantity to map to a particular y-value. As he described in Lines 13-15, encountering such a situation is "weird" for him. He did not begin to represent this non-injective correspondence between the x- and y-quantities until he was provided with the means of support offered in Phase 3.

Conceiving of Just One Distance Quantity Varying with Time

As discussed in the previous subsection, a time quantity, in the form of Homer's motion along the road, is implicit in the covariation of both distances that the students were told to represent on their axes. Despite being instructed explicitly to represent Homer's distance to Shelbyville on the x-axis, some students still elected to plot time on that axis, and then map it to distance on the y-axis. In one case, this choice was manifested in students' direct responses to the task (i.e., they intended their graph to represent the covarying distance quantities, regardless of which quantities they actually represented). In another case, the choice to plot time on the *x*-axis marked a step in their process to create a finalized graph. Both cases are identified as a way of thinking that entails a disposition to plot time on the *x*-axis as it relates to some distance quantity, despite receiving instructions to do otherwise. In their respective interviews, Brendan and Dakota both showed this way of thinking.

In Phase 2 of Brendan's interview, he sketched the graph shown above in Figure 4 and then rhetorically asked, "Would there be more than one graph?" [Phase 2]. Looking at what he drew in the context of this question, it appears that this graph represents Homer's distance from Springfield as he moves along the road (as time passes). A few moments later, he made another remark that supports the interpretation that he graphed one distance in relation to time: "I like this graph—it works for me—because I kinda get the feeling it would do the same thing for Shelbyville. 'Cause at the end, it's far away, and then it kinda does the same dip" [Phase 2]. His statements imply that the graph in Figure 4 does not represent Homer's distance to Shelbyville; rather, it represents the distance to Springfield on the *y*-axis as it relates to some unidentified quantity. Although Brendan did not say that time was the quantity he plotted on the *x*-axis, his graph suggests that this is what he did, especially given that his words suggest that the quantity represented is *not* Homer's distance to Shelbyville. His way of thinking here exemplifies a fixation on just one distance as it varies with time.

Dakota's work in Phase 1 of his interview demonstrates a fixation on just one distance more overtly. As he worked through the problem, he drew the graph seen in Figure 5[2].

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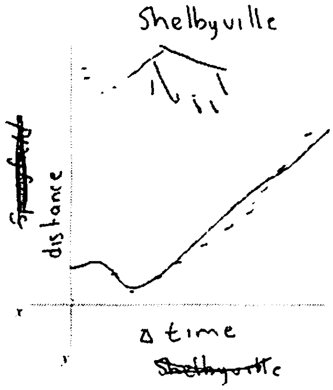
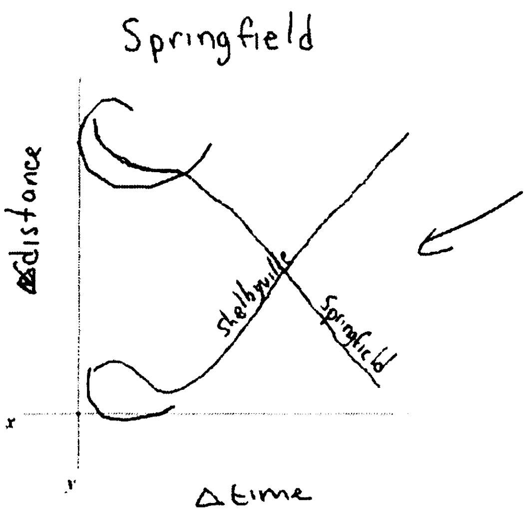
[2] Unfortunately, Figure 5 is without color.

Figure 5 [Phase 1]



After initially labeling the *x*- and *y*-axes 'Shelbyville' and 'Springfield,' respectively, Dakota crossed the names of the cities out and renamed the axes 'distance' and 'Δ time.' When asked to explain what he had graphed, he said, "I guess I was just using the *y*-axis to show increase in distance and the *x*-axis to show change in time" [Phase 1]. Dakota's label at the top of Figure 5 and the shape of the graph imply that the 'distance' that he plotted was Homer's distance to Shelbyville, specifically. Without further prompting, he said, "If I could get another graph [another sheet of Livescribe paper], then I could do that with Springfield" [Phase 1]. His next sketch is shown in Figure 6.

Figure 6 [Phase 1]



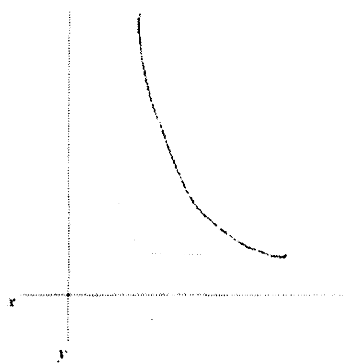
In this sketch, Dakota added a second graph, so that one graph represents Homer's distance from Springfield and the other represents Homer's distance to Shelbyville, as both quantities vary with time. He expressed recognition that these separate graphs did not show the relationship as directly requested by the task when he later said, "So, these are the individual graphs there...and to put them together...[*he doesn't finish this sentence*]" [Phase 1]. So, even with awareness that the covarying distances should be represented on a single graph relating them, Dakota's way of thinking necessitated the direct representation of time. Rather than conceiving of both distance quantities' simultaneous variation, he thinks of the variation of the quantities individually, as they relate to change in time.

Persistent Consideration of Both Quantities Simultaneously

One of the undergraduates, Kaitlyn, displayed a persistent disposition to conceive of both quantities without ever separating them in her speech or work. From the time she first encountered the task, Kaitlyn always spoke of one quantity as it related to the other. While Kaitlyn did not initially have a complete understanding of covariational reasoning, her way of thinking about the quantities entailed an ever-present marriage of the two in her work and *speech*. For anyone to display this approach, he would rarely or never describe the behavior of a single quantity without also noting the behavior of the other. He would instead persistently note both simultaneously.

Kaitlyn showed this way of thinking most frequently. As early as Phase 1, her discussion of either quantity's variation—when it increased or decreased—always made mention of the other. After she sketched the graph seen in Figure 7, I asked her why she started her graph where she did (she notably traced the relationship from right to left). She responded, "So, relative to the distance, Homer is closer to Shelbyville and farther from Springfield at the beginning, so it wouldn't be right to start it here [*she points to the* x-axis] because that's too close to Springfield. So, as he's travelling, he's getting farther...farther-ish.... It's kinda close still over here [*she refers to the animation as it moves*], but he's getting farther from Shelbyville and closer to Springfield, so the distances right here are almost the same, too" [Phase 1]. As Kaitlyn followed Homer's movement along the road, she always stated what she noticed about *both* distances at each moment. In her thinking, referring to just one quantity's variation at a time was insufficient for explaining the relationship that her graph represented.

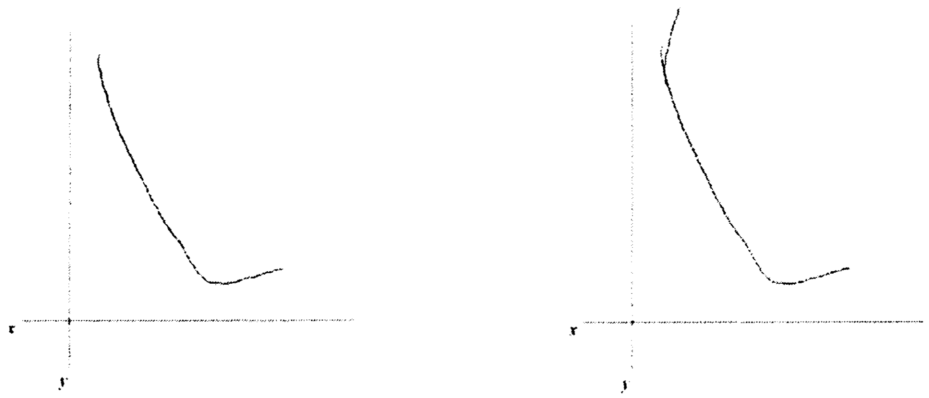
Figure 7. Kaitlyn's initial graph [Phase 1]



Given no further means of support beyond more time to think about the task, Kaitlyn opted to refine her work with the construction of a new graph (Figure 8, left graph). After I asked her what she changed with this new graph, Kaitlyn replied,

> I tried to start it at the same point, but I figure he gets closer around this area, and then he starts to get farther and closer to Shelbyville. But I see that this would probably want to be more like this...[She extended the graph, as seen in the image on the right in Figure 8] because the distance is farther and he's closer to Springfield than here, I think, than he would be at the end of his trip. [Phase 1]

Figure 8. Kaitlyn's refined graph [Phase 1]



In an effort to hear more about how Kaitlyn was thinking, I posed a few supplementary questions (see Excerpt 3)

Excerpt 3 [Phase 1]

16. *Ben*:	If the road was curved—if it waved back and forth—do you think that you would still be able to construct a graph?
17. *Kaitlyn*:	Mhm. It might take a little more time.
18. *Ben*:	How do you think you would go about it?
19. *Kaitlyn*:	Just trying to picture where Homer is relative to the two points.... Yeah, I could do it. It wouldn't be that hard, though it would be a very rough sketch.
20.	*[Some omitted text]*
21. *Ben*:	So, When you envisioned this task first, were you thinking of a function at all?
22. *Kaitlyn*:	Yeah, well, I was nervous that...if it was crossing the x—you know—if there was more than one line, umm, that it wasn't going to be right, but it doesn't really matter because—I don't know—it just doesn't. So—um—yeah, I was mostly just thinking of it as a relation between him and the points.

Kaitlyn's response in Line 19 again shows that she her thinking entailed constant attention to Homer's distance from *both* cities. She understood that, no matter how you changed the shape of the road, she could succeed in graphical representation if she thought about both quantities together. Her words in Line 22 hint that while she may have initially had other ways of thinking (i.e., that the mapping of the x-quantity to the y-quantity should be injective in her graph), those ways were trumped by a persistent focus on both distances, as they are related.

Discussion

The five ways of thinking I identified were: a disposition to plot discrete points, shape thinking when graphing, graphs of functions must represent an independent and a dependent quantity, conceiving of just one distance quantity varying with time, and a persistent consideration of both quantities simultaneously. Coleen, Brendan, Kaitlyn, and Dakota showed different levels of understanding as they revealed these ways of thinking in their interviews. I will orient my discussion of the propitiousness of these mental approaches away from a simple 'right-or-wrong' assessment, instead analyzing students' sketches and conversations to look for links from ways of thinking to an understanding of covariational reasoning.

All three students who displayed a disposition to plot discrete points did so as an organizational tactic for attempting to construct graphical representations. These students—Coleen, Brendan, and Dakota—all needed the dynamic visual support of a correspondence point that was provided in Phase 4 to eventually construct a more-or-less accurate graph of the relationship. Even though Coleen and Brendan both plotted points during Phase 1 of the interview, they still continued to struggle with the task until Phase 4, when they could clearly see the path of the correspondence point in tandem with the animation. Creating a representative graph after viewing this means of support (from Phase 4) cannot necessarily be attributed to a newly developed understanding of covariation, since it is altogether possible that a student could have merely traced the movement of the correspondence point when looking to draw their graph. Were a disposition to plot discrete points propitious for arriving at this understanding, Coleen and Brendan would have been able to sketch a graph and explain how they visualized it representing covariation after plotting points. However, since both students plotted points in Phase 1, continued to face difficulties throughout Phases 2 and 3, and only began to show signs of coming to an understanding in Phase 4, this way of thinking with plotted points was not especially propitious to developing an understanding of covariational reasoning.

In Phase 3 of Dakota's interview, he also showed a disposition to plot discrete points. However, a slight difference in his approach distinguishes his thinking from Coleen's and Brendan's. Instead of plotting points first and then connecting them to make a graph like the other two students did, he tried to plot points to fit onto the curve he had sketched [Phase 3]. This approach yielded no greater revelations in his thinking than did Coleen and Brendan's approach to point plotting. He still required the dynamic visual support in Phase 4 to draw a graph representing the distance quantities' invariant relationship. It is possible that a disposition to plot points would have been shown to be a propitious way of thinking if the three aforementioned students had placed the points more accurately on the graph. However, given that plotting points did not provide Coleen, Brendan, or Dakota with more information on how the two distance quantities varied in tandem, this disposition stood as a distraction from, rather than a mental tool for understanding, the covariation.

Perhaps the least propitious of these five ways of thinking for development of covariational reasoning was the shape thinking that Coleen displayed. As with her point-plotting disposition, she showed shape thinking when graphing in Phase 1 of her interview. The limitations of this way of thinking are evident not only in the graph she drew in Figure 1, but also in her discussion of that graph. In Excerpt 1, Line 2, Coleen said that she tried to "describe how being close to [one city] is farther from the other [city]," but with the added qualification that her line segment "kind of *make*[s] the road that he's going on." Even though she seemed aware that her graph should represent the distances from the cities, she thought that the resemblance in shape between the road and the graph was a sign of accuracy. She employed shape thinking in this way to justify what she had done, reassuring herself that she had accurately graphed the situation *because* it looked like the road. She further confirmed that she thought of the straightness of the road as a specific indicator to make her graph straight as well, when she said, "But, because the line is straight…it's making me think that it's a straight line" [Excerpt 1, Line 7]. To reason covariationally, attention must be given to the quantities and how they vary in tandem, not to shape of the model. This way of shape thinking was highly unproductive for Coleen because it distracted her from focusing on the quantities she was asked to represent.

Brendan's way of thinking that graphs of functions must represent an independent and a dependent quantity limited his capacity to graph the relationship between the two quantities, regardless of whether or not he was able to actually conceive of that relationship in his mind. His discussion in Excerpt 2 [Phase 3] of how the "dip at the end that's on both of them" suggests that, with the means of support in this phase, he began to see how he should construct his graph. Yet, even with this awareness, Brendan expressed how he thought it would be "weird" for his graph to "have two spots taking up the same space, or one point [x-value] be in two different spots [y-values]" [Excerpt 2, Line 15]. Brendan's meanings for constructing graphical representations of functions necessitated the direct representation of an independent quantity on the x-axis, which would never map to more than a single y-value. His graph in Figure 4 [Phase 2] further emphasizes how this way of thinking led him astray in his attempt at graphing the quantities. Not only was this way of thinking not propitious for coming to think of the distances covariationally, it also stood as a major obstacle to his capacity to graph the multiplicative object created by the implicit third quantity throughout Phases 1-3. The difficulties he encountered with this way of thinking are nontrivial, especially considering that he likely would not have revealed the nature of his meanings for graphing functions in classes that only presented examples of graphing an independent vs. a dependent quantity. While his way of thinking did not hinder his understanding of how the graph should look once he got to Phase 4, it prevented him from displaying that understanding earlier in the interview. A way of thinking in which graphs of functions must represent both an independent and a dependent quantity is not propitious for students attempting to develop an understanding of how two or more *dependent* quantities can vary in tandem.

This way of thinking may share some of the qualities, in terms of propitiousness for covariational reasoning, of another: conceiving of just one distance quantity varying with time. The main connection between these two ways of thinking lies in an understanding that time *is* an independent quantity, so a disposition to plot time on the *x*-axis as it relates to Homer's distance from just one city certainly ties in to a way of thinking that necessitates representation of both an independent and a dependent quantity. This way of thinking is distinguished from that which I discussed in the previous paragraph in that it does not directly feature students' meanings for function. Instead, it deals with students conceiving of the two distance quantities as separated in their mind, not covarying with each other but, rather, varying with time. Both Brendan and Dakota showed this way of thinking during their interviews. Not only does Brendan's graph in Figure 4 suggest that he was only thinking of Homer's distance to Springfield as it related to time, but also does his thought that there may be more than one graph [Phase 2]. Just as his way of thinking described in the previous section hindered his ability to conceive of both quantities varying simultaneously, so did his fixation on the variation of just *one* distance with time. He showed an understanding of how to conceive of a quantities' variation, but could not translate it to an understanding of the covariation of two or more quantities. Dakota's work in Phase 1 of his interview (see Figures 5 and 6) also illustrates how he focused on just one distance quantity varying with time. Even though he recognized that both distances should be represented on a single graph, he showed that he could only thinking of them separately as they related to time. While conceiving of how either quantity varies with time could be a step to being able to conceive of how they both vary in tandem, the fixation on time only distracted Brendan and Dakota from forming a multiplicative object of both distances' variation in their minds. They struggled to represent a relationship between the two covarying distances partly because of this way of thinking that would not allow time to go unrepresented. Conceiving of one distance at a time may be somewhat propitious for a student first developing an understanding of covariational reasoning; however, persistent attention to how that distance varies *just* with time made it difficult for the students in this study to see how a distance could vary with another distance.

Given Kaitlyn's quick and direct path to solving the task and forming a more complete understanding of covariational reasoning, her way of thinking, characterized by a persistent consideration of both quantities simultaneously, is the *most* propitious for students attempting to develop this understanding. While she did not initially possess a complete understanding of how to reason covariationally (see Figure 7), she came to develop that understanding with no means of support beyond that provided in Phase 1. I attribute her success to the way of thinking that she alone displayed repeatedly: she consistently spoke of one quantity's variation simultaneously with the other. Although her first graph was not drastically different from those drawn by the other three students, she was able to refine it with no further dynamic visual support because her way of thinking compelled her to look at both changing magnitudes together and speak of what they both did at any given moment or interval. This way of thinking may at a glance seem synonymous with merely stating what it means to have covariational reasoning, but I emphasize that it does not necessarily entail an understanding of variation at all. Instead, in a simpler sense, it can be attributed to how a student would mention what they notice in *both* quantities' behavior whenever they talk about either. With Kaitlyn's interview as evidence, the persistent consideration of both quantities simultaneously was highly propitious for developing an understanding covariational reasoning.

Synthesizing the discussion of each of these five ways of thinking, I will again delineate them—this time, in order of least to most propitious for developing an understanding of covariational reasoning, based on the findings of this study.

- Shape thinking when graphing, which led students to false understandings and even served to justify inaccurate work, is the least propitious.
- A disposition to plot discrete points is not especially propitious, since it coincided with students' inattention to how the quantities vary.
- Both conceiving of just one distance quantity varying with time and thinking that graphs of functions must represent an independent and a dependent quantity are somewhat propitious; however, each of these ways of thinking appeared to distract students from conceiving of how two distance quantities can covary in tandem.
- The most propitious way of thinking for developing an understanding of covariational reasoning entails persistently considering *both* quantities simultaneously.

I end my discussion with a call to other researchers to investigate both the veracity of these claims and to search, in greater depth, for more ways of thinking that can better assist students to develop a complete understanding of how to reason covariationally.

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