

Characterizing Teacher Change Through the
Perturbation of Pedagogical Goals

by

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ABSTRACT

A teacher's mathematical knowledge for teaching impacts the teacher's pedagogical actions and goals (Marfai & Carlson, 2012; Moore, Teuscher, & Carlson, 2011), and a teacher's instructional goals (Webb, 2011) influences the development of the teacher's content knowledge for teaching. This study aimed to characterize the reciprocal relationship between a teacher's mathematical knowledge for teaching and pedagogical goals.

Two exploratory studies produced a framework to characterize a teacher's mathematical goals for student learning. A case study was then conducted to investigate the effect of a professional developmental intervention designed to impact a teacher's mathematical goals. The guiding research questions for this study were: (a) what is the effect of a professional development intervention, designed to perturb a teacher's pedagogical goals for student learning to be more attentive to students' thinking and learning, on a teacher's views of teaching, stated goals for student learning, and overarching goals for students' success in mathematics, and (b) what role does a teacher's mathematical teaching orientation and mathematical knowledge for teaching have on a teacher's stated and overarching goals for student learning?

Analysis of the data from this investigation revealed that a conceptual curriculum supported the advancement of a teacher's thinking regarding the key ideas of mathematics of lessons, but without time to reflect and plan, the teacher made limited connections between the key mathematical ideas within and across lessons. The teacher's overarching goals for supporting student learning and views of teaching mathematics also had a significant influence on her curricular choices and pedagogical moves when

teaching. The findings further revealed that a teacher's limited meanings for proportionality contributed to the teacher struggling during teaching to support students' learning of concepts that relied on understanding proportionality. After experiencing this struggle the teacher reverted back to using skill-based lessons she had used before.

The findings suggest a need for further research on the impact of professional development of teachers, both in building meanings of key mathematical ideas of a teacher's lessons, and in professional support and time for teachers to build stronger mathematical meanings, reflect on student thinking and learning, and reconsider one's instructional goals.

DEDICATION

To my grandparents, Stephen and Maria: thank you for your encouragement at key points of my life. Your guidance and wisdom set me on this path, and your optimism lighted the way. I love you and miss you both.

To my mom, Marika: words cannot express my admiration and gratitude for the sacrifices you made throughout the years. Your North Star was your children's well-being and their success in life. Thank you for your support, love, and encouragement.

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CHAPTER 1: INTRODUCTION

Statement of Problem

Mathematics instruction in the United States has been described as procedural and disconnected, with a primary focus on developing students' calculational abilities rather than their understanding of concepts and how they are connected (Ma, 1999; Stigler & Hiebert, 1999). In recent work it has also been documented that it is common for teachers to teach in a manner in which they were instructed as students, and that making the transition to value conceptual learning and teaching is a difficult transition for teachers to make (Sowder, 2007).

Researchers have identified mathematical knowledge for teaching (MKT) as a key link between content knowledge and support of student learning (Hill, Ball, & Schilling, 2008; Silverman & Thompson, 2008). Earlier research has also shown that a teacher's image of mathematics, referred to as a mathematical teaching orientation, influences her classroom practice (A. G. Thompson, Philipp, Thompson, & Boyd, 1994).

Other research (Webb, 2011) had shown how teachers' goals (mental images of what teachers are trying to accomplish) had influenced the development of content knowledge that was pedagogically powerful. Meanwhile, other studies have shown that a teacher's mathematical knowledge for teaching also influences his/her pedagogical goals and actions (Marfai & Carlson, 2012; Moore, et al., 2011).

The relationship between a teacher's goals and his/her mathematical knowledge for teaching are reciprocal. Although researchers have made significant contributions with regards to professional development supports that promote growth of teachers' goals for attending to students' thinking of mathematics and ways to support such thinking (M.

S. Smith, Bill, & Hughes, 2008; M. S. Smith & Stein, 2011; P. W. Thompson, 2009), characterizing the stages of growth of a teacher's goals for student learning, or how these goals may change have not been researched.

In studying teacher's goals for student learning, it is important to be mindful of scripts that exist as part of a school culture regarding goals. Part of many mathematics teachers' classroom norms embodies a specific viewpoint on how goals for student learning are to be interpreted. For example, some school districts in Arizona have incorporated rubrics to assess a teacher's goals for student learning as part of the protocol used in their evaluation which uses the Marzano teacher evaluation model (Marzano, Carbaugh, Rutherford, & Toth, 2013). Teachers often write or post each day's learning goals on the wall or board so that students (and evaluators who visit their classroom) can clearly see them. One school district defines a learning goal as "a clear statement of knowledge or information as opposed to an activity or assignment" (Chandler Unified School District, 2013, p. 16). However this perspective on teachers' goals for student learning is not unique to Arizona, or the United States. In guidelines for assessment, evaluation, and reporting by the Ministry of Education in Ontario, Canada, the section regarding assessment for learning states "learning goals clearly identify what students are expected to know and be able to do, in language that students can readily understand." (Ontario Ministry of Education, 2010, p. 33). Similar to school districts in the United States, some school districts in Ontario follow similar classroom practices in support of the ministry's guidelines, such as having learning goals visible to students throughout learning. Researchers such as Courtney (2011) had noted some teachers' views of mathematics were inclined to focus on the visible products of student reasoning rather

than the reasoning process itself, and he referred to this view as an “empirical orientation”. Such a view of mathematics was shown to constrain teachers’ ability to reflect meaningfully on their practice. In light of school cultural norms that prescribe what a teacher’s goals for student learning should look like, finding teachers with an empirical orientation should not be entirely unexpected.

A teacher’s goals for student learning, and the nature of these goals, can be viewed as an unarticulated part of lesson planning. However, research into teachers’ planning (A. L. Ball, Knobloch, & Hoop, 2007; Clark & Peterson, 1986; Morine-Dersheimer, 1979) showed that teachers’ lesson plans typically had few details in written form, but the mental images teachers had of these lessons had greater detail than what was written. From this perspective, teachers’ stated goals are likely to have less detail than the mental image of the stated goal.

In addition, the literature showed that experienced teachers did not find planning at the lesson level of particular importance, while novice teachers felt they needed more planning at the daily level due to lack of knowledge or experience. Although I will discuss Robert (a high school mathematics teacher who participated in one of the exploratory studies) later in this report, it was interesting to note how his comments resonate with regards to the literature on planning. In the following excerpt of data collected during a clinical interview, the researcher asked Robert how his planning for Precalculus had changed during the three years he had used a conceptually oriented curriculum created by the Pathways project (Carlson & Oehrtman, 2012). In the excerpt, Robert is noted as “Rob” and the researcher as “Res”.

Excerpt 1. Robert's comments

- 1 Rob: The first year was a lot of reading the instructor notes, and notes in the
2 PowerPoint to make sure that I knew all of the mathematical ideas that you
3 guys thought were important for the particular worksheet or investigation. So
4 that if things did come up, I could address them, and also to know where
5 everything was going.
- 6 Res: Uh huh.
- 7 Rob: So that if there was a particular mathematical idea that was important for the
8 future, that, you know that was, that was stressed. Umm, I'm sorry, what was
9 the question?
- 10 Res: The second, the second time, and what, how you currently plan.
- 11 Rob: Umm, second time, I don't know, it was, it was still a little bit of, uh, looking
12 through the teacher notes. This year I haven't really spent any time at all
13 looking through the instructor notes. I think because I feel by now that I
14 know, that I should know what's important.

Although Robert was an experienced teacher of 13 years when the study was conducted, the changes in how he planned for a lesson using a conceptual curriculum followed the same trajectory as a teacher transitioning from novice to expert.

Gap in the Literature

While the literature suggests a reciprocal relationship between a teacher's goals and his/her mathematical knowledge for teaching, no known framework exists to characterize a teacher's mathematical goals for student learning at the lesson level. Furthermore, the effect of a professional developmental intervention designed to impact a teacher's mathematical goals, or measure how they change, has not been analyzed in the context of such a framework. The relationship between a teacher's mathematical goals for student learning and his/her overarching goals, views of teaching, and mathematical teaching orientation are not well outlined in the literature.

Statement of the Research Questions

In the pages that follow, I will discuss what I had learned from my preliminary studies to gain insight into characterizing teachers' goals for student learning. In addition,

I will share how lesson planning and teacher reflection influenced these goals in the course of the dissertation study. My primary research questions for the dissertation study were as follows.

1. What is the effect of a professional development intervention, designed to perturb a teacher's pedagogical goals for student learning to be more attentive to students' thinking and learning, on a teacher's views of teaching, stated goals for student learning, and overarching goals for students' success in mathematics?
2. What role does a teacher's mathematical teaching orientation and MKT have on a teacher's stated and overarching goals for student learning?

CHAPTER 2: THEORETICAL PERSPECTIVES

Background

In this chapter I will discuss the theoretical perspectives I used as a basis for the preliminary and dissertation studies. It is well-known that teacher's instructional practices stem from experiences as a student in their formative years (Stigler & Hiebert, 1999). This implies that a schema of what teaching entails is formed prior to a teacher's first experience in the classroom.

Consistent with Bartlett (1932), I use the word *schema* to mean an active organization of past reactions and experiences. A *schema* is a pattern of thought "in storage." Schemas exist as patterns of units that are formed in complex ways given the situations. The arrangement exists based on an individual's effort after meaning. Schema theory is explanatory in terms of mental structures that exist activated upon perceptions of similar situations, and is useful when characterizing the meanings of *remembering*, *teaching*, and *learning*.

Remembering is a reconstruction of an individual's perceived experiences through the activation of the individual's schemata. This is consistent with Bartlett's (1932) characterization of remembering, which he stated as "an imaginative reconstruction, or construction, built out of the relation of our attitude towards a whole active mass of organized past reactions or experience, and to a little outstanding detail which commonly appears in image or in language form" (p. 218).

Teaching is viewed as a form of remembering that draws upon existing schemata and is a goal-directed activity, while *learning* can be characterized as a mental process in which an assimilation to a scheme or an accommodation (reorganization) of a scheme

occurs. However, the schemata upon which a teacher draws is formed from their beliefs and their knowledge.

I adopt a cognitive constructivist perspective (Piaget, 1970) as my theory of learning when conducting this study. In the philosophy of cognitivist constructivism, learning does not occur as a result of knowledge being passed on from others, but rather it is constructed by the individual through activities, experiences, and actions by the repeated mental processes of assimilation and accommodation.

Beliefs

Beliefs are one component of affect. Debellis and Goldin (2006) view affect as a state of emotional feeling that carry meaning for the individual, and both encode and exchange information with other internal systems of information; affective pathways are local states of feeling interacting with cognitive configurations. Philipp (2007) states that affect – which he defines to be a disposition, emotion, or feeling attached to an object or idea – consists of emotions, attitudes, and beliefs. Philipp summarizes that emotions are states of feeling or consciousness, while attitudes are manners of thinking, feeling, or acting that show one’s opinion or disposition, and change slower than emotions but faster than beliefs. Beliefs are “psychologically held understandings, premises, or propositions about the world that are thought to be true.” (Philipp, 2007, p. 259). In the affective domain, beliefs are the most cognitive, the most stable, and are harder to change than attitude. A value is viewed as the worth of something, and is a type of belief that tends to be context independent (Philipp, 2007). Values follow a desirable/undesirable dichotomy, rather than a true/false dichotomy.

My perspective of beliefs and knowledge are similar to each other, in that they result from successive assimilation and accommodation. The sources of information that are reorganized are different however. Whereas knowledge is reorganized information from a purely cognitive source, the information reorganized that forms stable beliefs originally came from affective sources. Both beliefs and knowledge are held like abstract possessions in the mind (Abelson, 1986) with varying degrees of certitude. Although the information sources from which beliefs and knowledge come may be different, beliefs and knowledge do not act independently within a teacher. Beliefs and teacher knowledge impact each other. A. G. Thompson (1992) stated that looking at teachers' conceptions and beliefs about mathematics in isolation with their subject matter knowledge would lead to an incomplete model. Hence, some researchers such as Zawojewski, Chamberlin, Hjalmarson, and Lewis (2006) coined the term "interpretive system" which is a conceptual system used in a teacher's educational practice for both the teaching and learning of mathematics. This conceptual system operates from the perspective of the teacher's beliefs, values, and subject matter knowledge. Based on researchers' perspectives, it is clear that beliefs impact teacher knowledge, mathematical knowledge for teaching, and teachers' goals for student learning.

Teaching Orientation and Mathematical Knowledge for Teaching

Teacher's orientation for teaching mathematics plays a significant role in how they teach and sheds light of their image of mathematics. A teacher with a *calculational orientation* (A. G. Thompson, et al., 1994) has an image of mathematics as an application of rules and procedures for finding numerical answers to problems. A teacher with a *conceptual orientation* has an image of mathematics as a network of ideas and

relationships among these ideas, and strives for coherence among these ideas. Reform efforts have focused on interventions that help teachers transition from a calculational to a conceptual orientation in mathematics. Teaching orientation has a profound affect on a teacher's mathematical knowledge for teaching (MKT). The literature contains two primary characterizations of MKT (D. L. Ball, 1990; Silverman & Thompson, 2008).

One characterization of MKT stems from the work of Deborah Ball and her collaborators. Ball (1990) states that subject matter knowledge for teaching has two dimensions— substantive knowledge of mathematics (correct knowledge of particular concepts and procedures, understanding underlying principles and meanings, appreciating and understanding the connections between mathematical ideas) and knowledge about mathematics (understandings about mathematics as a field and the nature of mathematical knowledge). The idea of substantive knowledge of mathematics is comparable to the construct of profound understanding of mathematics, which is “deep, broad, and thorough” (Ma, 1999, p. 120). Understanding a topic with depth meant connecting with a more conceptually powerful idea, while understanding a topic with breadth meant connecting it with similar ideas of the same or weaker conceptual power. In later work, Hill, et al. (2008) map the domain of mathematical knowledge for teaching into six strands, three comprising subject matter knowledge (Common Content Knowledge, Specialized Content Knowledge, Knowledge at the Mathematical Horizon) and three strands comprising of aspects of pedagogical content knowledge (Knowledge of Content and Students, Knowledge of Content and Teaching, Knowledge of Curriculum) that built off the work of Shulman (1986).

Another characterization of MKT has been provided in the literature by Patrick Thompson and other researchers. Silverman and Thompson (2008) define mathematical knowledge for teaching as a transformation of a key developmental understanding (Simon, 2006) into knowledge that is pedagogically powerful. In other words, mathematical knowledge for teaching is defined to be mathematical understandings “that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas that does significant work in students’ reasoning” (Silverman & Thompson, 2008, p. 501). Simon (2006) states that a key developmental understanding is a type of understanding that is built upon self-reflection and multiple experiences – it requires a conceptual advance and is not acquired as a result of a demonstration or via explanations. A conceptual advance is defined to be “a change in students’ ability to think about and/or perceive particular mathematical relationships” (Simon, 2006, p. 362).

This dissertation investigation will leverage the Silverman and Thompson (2008) characterization of MKT as the basis for my theoretical framework since I am interested in the way a teacher understands ideas and connections among ideas, and how this influences his or her pedagogical decisions and actions. As part of the professional development intervention described later is this investigation, reflection on the key ideas of a lesson will be used as a way to advance a teacher’s thinking about the mathematics and his/her goals. Since reflection can lead to a conceptual advance in a teacher’s mathematical meanings, this new knowledge has the potential to be used by the teacher in robust ways to advance student learning. Silverman and Thompson’s characterization focuses on the cognitive aspect of this relationship.

Goals

In looking at goals, Skemp (1979) provides an explanatory framework for human behavior. He describes the human being as a complex system that organizes and directs actions in order to reach a desired goal state, while at the same time avoiding what he referred to as an anti-goal state. An anti-goal state is a goal state that an individual tries to either get away from or avoid altogether. Skemp's (1979) framework relating emotion and consciousness consists of following four categories: (1) pleasure/unpleasure: indicating changes either towards, or away from a goal state; (2) fear/relief: indicating changes either towards, or away from an anti-goal state; (3) confidence/frustration: signaling either ability, or inability to change towards to a goal state; (4) security/anxiety: signaling either ability, or inability to change away from an anti-goal state. Figure 1, included next, illustrates the idea.

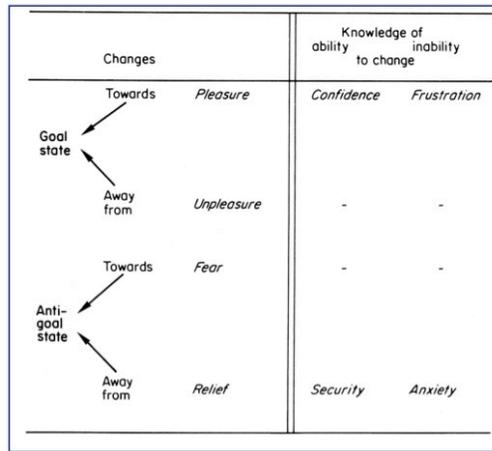


Figure 1. A model representing affective feedback (Skemp, 1979, p. 13)

Therefore an individual's affect provides feedback to himself or herself, and in the context of teaching, this can be viewed in light of actions a teacher makes to approach a particular goal state (such as having students explaining their reasoning when doing

mathematics) or avoid an anti-goal state (such having an unruly class, despite best efforts).

Researchers have categorized a goal as a mental representation of what a teacher is trying to accomplish (Locke & Latham, 2002; Norman, 2002; Schoenfeld, 1998). Other researchers have defined goals similarly, but expanded this definition to explain possible purposes or reasons why a teacher may pursue a goal (Pintrich, 2000). Some have defined a teacher's goals as learning goals for the student, such as those described in the hypothetical learning trajectory of Simon (1995). Research by Webb (2011) has shown that a teacher's goals and pedagogical powerful content knowledge interact.

Other studies have shown that a teacher's mathematical knowledge for teaching also influences her pedagogical goals and actions (Marfai & Carlson, 2012; Moore, et al., 2011). The relationship between a teacher's goals and her mathematical knowledge for teaching are reciprocal; each influences the other. A summary of the study described in Marfai and Carlson (2012) has been included in this proposal (see Appendix A) for additional detail.

Goals can be stable or ad-hoc. When teaching, an instructor may have stable goals, and goals that are instantiated during a lesson (ad-hoc goals). Regular categories differ from ad-hoc categories in that a regular category is a graded structure that is well represented in memory, whereas an ad-hoc category is not. Both stable goals and ad-hoc goals have stable category structures. According to Barsalou (1983) a graded structure has three aspects: (1) some instances are better examples of the category structure than others, (2) there is the presence of unclear cases in the categories, and (3) non-members of a category vary in how dissimilar or similar they are to the concept of the category. A

distinguishing feature of an ad-hoc category is that it violates a correlational structure and is not thought of by most people. Correlated structures share properties in common – for example, objects with wheels are much more likely to be associated with vehicles than being associated with traffic signs. Common categories are context independent, while ad-hoc categories are context dependent in that they only exhibit the graded structure in context given. For example, consider the ad-hoc category ‘questions to ask students regarding proportionality of two quantities’. It may facilitate a teacher’s goal of having students think about proportionality of two quantities in flexible ways (Carlson, Oehrtman, & Moore, 2013b). In looking at categories and ad-hoc categories, Barsalou (1983) states that ad-hoc categories are spontaneously built in order to achieve goals, but lose their ad-hoc status when they become well established in memory after frequent activation.

Based on Barsalou’s work, I infer that overarching instructional goals of teachers are stable because they are independent of the context of a lesson. With respect to lesson specific goals, as they are context dependent, these ad-hoc goals become well-established and stable in memory over repeated experiences. New connections formed through growth of a teacher’s mathematical knowledge for teaching allows the repertoire of instances within an ad-hoc category to expand and reorganize. To illustrate what I mean by expand and reorganize, I’ll refer back to the ad-hoc category ‘questions to ask students regarding proportionality of two quantities’. A teacher with an impoverished understanding of proportionality might think that questions centered around checking the cross product of two ratios are the best type of questions to get students to understand when two quantities are proportional. As her knowledge regarding proportionality grows,

she realizes there are multiple ways to think about the proportionality of two quantities (e.g., constant ratio, constant multiple, or by constant scaling factor). Expansion of the ad-hoc category occurs as she begins to think of additional questions to ask students that reveal these other ways to think about proportionality. By reorganization of the ad-hoc category, the teacher notes that some questions are better than others at having students determine the proportionality of two quantities. What seemed to be a good example of a question to ask students regarding proportionality (such as cross-ratios) is now viewed by the teacher as a poor choice for a question, since more robust ways of understanding proportionality are known. Additionally, new ad-hoc categories can emerge in support of new ad-hoc goals that are the result of professional growth. For example, the ad-hoc category ‘conceptually rich tasks that require students to engage in proportional reasoning’ could be used to support a teacher’s emerging goal of engaging students in meaning-making activities for a lesson on proportionality.

Pintrich (2000) stated that “ ‘strong’ classroom contexts or experimental manipulations (where the context defines the situation and appropriate behavior in many ways) can influence individuals to activate different goals than the ones they would normally or chronically access.” (p. 102) So it follows that teachers using a conceptual curriculum for the first time and whose network of mathematical connections is growing may begin setting different goals than ones they would have set prior to its use.

Describing Interactions Within the Theoretical Framework

I am leveraging Silverman and Thompson’s (2008) construct of MKT as a lens for examining how a teacher understands ideas and connections among ideas, and how this influences his/her pedagogical decisions and actions. The transformation of a

teacher's key developmental understandings (Simon, 2006) into MKT is developmental as a teacher's orientation shifts from calculational to conceptual. Using this theoretical lens, I plan to examine teachers' pedagogical goals for a lesson to gain insights underlying this process of growth. Figure 2, that follows, illustrates the interactions within this theoretical framework.

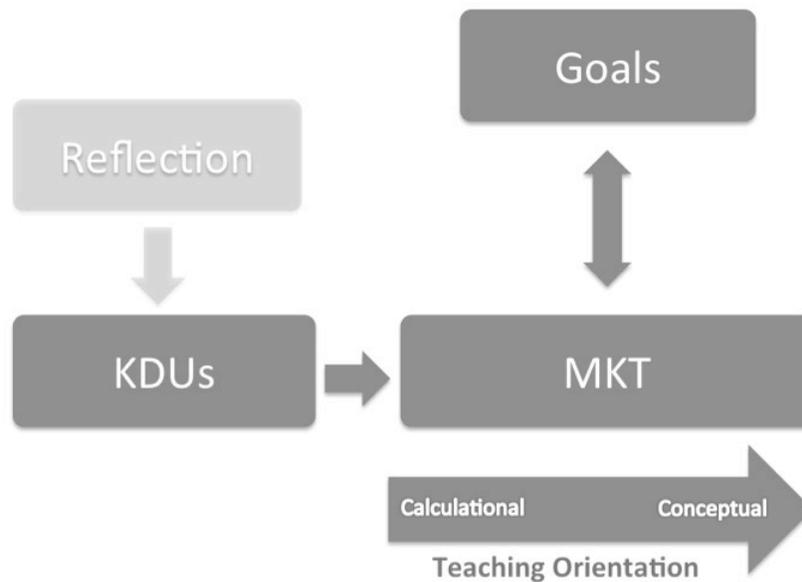


Figure 2. Interactions within the theoretical framework

As illustrated in Figure 2, I make the claim that through the process of self-reflection, a teacher's orientation may shift from calculational to conceptual, and not the other way around. A teacher's mathematical teaching orientation is influenced by his/her MKT and this knowledge impacts the goals a teacher has for his/her students' learning and his/her teaching. A teacher forms new KDUs as he/she makes more connections between key ideas of mathematics through the process of self-reflection. These conceptual advances in a teacher's understanding of mathematics are supportive of a conceptual orientation. At the start of this study, I make no claims of a direct link

between a teacher's goals and her KDUs, but this framework is open to revision if additional data supports such a linkage. Earlier findings showed the relationship between a teacher's goals and her MKT to be reciprocal and this finding suggests part of a teacher's goals is part of her MKT, and vice versa.

Classroom Practices

I characterize a teacher's classroom practice as the actions and the observable effects of these actions that are based on a teacher's thought process. Imagining teachers' thought processes as a psychological domain (Clark & Peterson, 1986), it is composed of teachers' theories and beliefs, teachers' interactive thoughts and processes, and teacher planning. Teachers' thought processes are not observable and, therefore, present methodical challenges. Clark and Peterson (1986) also describe the domain of teachers' actions and observable effects. The domain contains teachers' classroom behavior, students' classroom behavior, and student achievement, with all three components in the domain reciprocally related and directly observable.

Planning

While interactive thoughts and processes can be thought of as occurring during the interactions in the classroom, planning encompasses both thoughts and processes that occur both before and after classroom instruction. It follows that reflection is considered part of teacher planning in this construct. One way planning can be conceptualized is as a set of mental processes in which teachers imagine the future and think about resources and ways to accomplish their intended goal. Setting goals for student learning can be viewed as an unarticulated aspect of lesson planning.

Prior research has shown that teachers, regardless of experience level in teaching, have few details in the lesson plan they recorded in written form (A. L. Ball, et al., 2007; Clark & Peterson, 1986; Morine-Dershimer, 1979). Key decisions made prior to teaching the lesson were also not expressed in the lesson plan. However upon interviewing the teachers about their lesson plan, Morine-Dershimer (1979) uncovered that teachers had mental images in greater detail than what was written, and these images included other aspects of the lesson not present in the written lesson plans, and were nested within larger planning structures referred to as “lesson images.” The purposes of writing lesson plans was found to serve multiple roles for teachers: (1) to meet immediate personal needs, (2) to serve a direct function during instruction, or (3) as a means to an end (Clark & Peterson, 1986). For some teachers lesson plans were simply created to satisfy administrative requirements. The research shows that experienced teachers do not find that planning at the lesson level was a significantly important to them. Novice teachers and student teachers however expressed that having a lack of knowledge or teaching experience in a content area required them to do more daily planning (A. L. Ball, et al., 2007) focused on learning and conceptualizing the content, what they wanted to accomplish, rather than writing a formal lesson plan. The novice and student teachers in Ball et al.’s study commented that writing a detailed lesson plan was a waste of their time.

What these findings from the lesson planning literature suggest is that as part of this dissertation study, teacher planning needs to be re-conceptualized so that such an activity is both of value to the teacher, and can be used as a catalyst to promote teacher growth.

Noticing

Mason's (2002) framework of noticing will be leveraged to characterize the teacher's changing sensitivities in the processes of preparation, reflection, and in the moment of teaching. As Mason articulates, *noticing* requires making a distinction between the 'object' in consideration and its surroundings, thus implying a creation of some foreground and background by which this distinction is made. Mason describes various levels of intensity with regards to noticing: (1) ordinary noticing – perceiving that requires an external reminder to recall, (2) marking – a heightened form of noticing that can be accessed and recalled, and (3) recording – creating a note that becomes an object for later analysis). From the context of Barlett's (1932) schema theory, the stages of noticing are intentional forms of remembering; different levels of focus are given to the patterns of thought that are to be placed into "storage" and accessed later. Mason discusses sharpening sensitivities so that noticing in the moment (during the event) occurs so that researchers or teachers can act in a different way in commonly occurring situations. To do so, it requires preparation and reflection on possible actions that could be made in the future when the situation occurs again, and then knowing what to do when that moment occurs. Thus, being able to act in the moment first requires a prior reflection of how to act differently. Jacobs, Lamb, and Philipp (2010) also provide similar uses of noticing in the moment. They define professional noticing of children's thinking as a set of three interrelated skills: attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings.

Didactic Triad and the Thinking Through a Lesson Protocol

In planning lessons, several models have been described in the literature that I intend to build on. The didactic triad described by P. W. Thompson (2009) is a tool for professional development and consists of three interacting components: (1) creating a clear statement about ways of thinking the instructor intends students to learn via instruction, (2) creating tasks and materials to support student's learning through this instruction, and (3) design of instruction to support student learning while students are engaged in the task and materials.

Another model for lesson planning described by M. S. Smith, et al. (2008) is the TTLP (Thinking Through a Lesson Protocol) designed to orient teacher's thinking towards the mathematical horizon (Hill, et al., 2008). The TTLP consists of three main parts: selecting and setting up the mathematical task, supporting the students' exploration of the task, and sharing and discussing the task.

Although there are elements that overlap in the two models, coordinating and building upon their ideas so that redundancy does not occur will lead to a greater theoretical sensitivity. The artifact produced by the teacher can then also be used as a didactic object (P. W. Thompson, 2002) to further professional development at the high school level.

Thought Revealing Activities

One way in which teacher change (or resistance to change) will be documented is based on the idea of thought-revealing activities designed for teachers (Lesh, 2010; Lesh, Hoover, Hole, Kelly, & Post, 2000; Zawojewski, et al., 2006). Each of the activities is designed with the potential to go through successive iterations, and thus change is

measured through the submission of artifacts of teacher thinking during the process of an activity, and also across a series of activities. In order to maximize engagement with the activity, the design of these activities is meant to be useful for the participating teacher, with a finished product they can use in the classroom. In addition, the design of the tasks should allow students to have opportunities to make conjectures and reflect, thereby allowing for student thinking and engagement to emerge at the onset and throughout the activity.

In this study, these thought-revealing activities will be used as a research tool to both perturb and measure the evolution of teachers' goals for student learning as their mathematical knowledge for teaching grows. In asking teachers to reflect on the key ideas of a lesson and what tasks can be used to support these ideas, some guiding questions could be asked to focus teachers on the ways students may come to think of a task, helpful ways of thinking about a task, and ways of thinking that may hinder a student's successful completion of the task. By making these goals explicit as part of the task, it is intended for teachers to imagine building conversations with students around the tasks they plan to use, and supporting the goal of focusing on student thinking and reasoning, instead of solely the products of student thinking and reasoning. This aspect of planning therefore would give the teacher a better chance of being able to notice an opportunity to act in the moment by this intervention designed to sharpen sensitivity to student thinking.

CHAPTER 3: CONCEPTUAL ANALYSIS

Since the mathematical context of the dissertation study includes ideas of constant rate of change, average rate of change, and proportionality, this chapter provides an overview of what is involved in understanding these key ideas. Since my dissertation study spanned a series of lessons in which these foundational ideas would be used to build a conceptual understanding of exponential functions, I will close this chapter by discussing how the key ideas of covariation and proportionality could be leveraged to develop a well-connected meaning for n -unit factors, partial unit factors, and the continuous (or natural) growth factor e . I use the tool of conceptual analysis (Glaserfeld, 1995; P. W. Thompson, 2008) to discuss possible understandings of a particular idea. My overview includes a description of foundational reasoning abilities for learning these ideas, including a more extended discussion of what is involved in conceptualizing quantities and engaging in proportional and covariational reasoning. My discussion begins by describing these reasoning abilities and the associated constructs (e.g., quantity).

Key Ideas – Quantity, Covariation, and Proportionality

The idea of quantity in the sense of Thompson (1994b) shall be used. A quantity consists of the object, a measureable attribute of the object (or imagined to be measured), and the units of measure. Quantification is the process of assigning a number to a quantity through the process of measurement (either direct or indirect).

Two quantities are said to covary when the values they assume change together. Covariational reasoning is defined to be a cognitive activity where one coordinates how two quantities vary while also keeping account of how these quantities change with

respect to each other (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Covariational reasoning is developmental (i.e., it develops in stages). Carlson et al. (2002) developed a framework for covariational reasoning in two parts. First, mental actions of a student involved in a covariational reasoning task were categorized based on observable behaviors. Second, the framework classified the level of development of covariational reasoning based on overall images conveyed by the mental actions of a student.

The mental actions are classified into five categories, mental action 1 (MA1) being the least complex to mental action 5 (MA5) being the most complex. Mental action 1 involves relating a change in the input to a change in the output, but it is not specific. In MA1, a student would say as x changes, y also changes. Mental action 2 (MA2) involves coordinating a specific change (increase/decrease) in the input to a specific change (increase/decrease) in the output. In MA2, a student could say x increases, then y also increases. Mental action 3 (MA3) involves coordinating a specific amount of change in the input to a specific amount of change in the output. In MA3, the student may say that as x increases by 3 units, y then decreases by 5 units. Mental action 4 (MA4) involves coordinating uniform increments of change in the input values with average rates of change in the output. Observable behavior would be a student drawing contiguous secant lines along uniform intervals to model the covariation. Mental action 5 (MA5) involves coordinating a continuous change of input values with the instantaneous rate of change of the output. One would observe a student creating a graph as a result of the covariation of the input and output with the correct concavity and inflection points.

The covariational level of development also has 5 levels, based on the overall image of the mental actions performed. A ranking of a covariational level means the

participant can reason with corresponding mental action, along with being able to unpack and reason using all the lower ranking mental actions as well. For example, if a person is ranked to have a level 3 covariational reasoning ability, it means they can reason by mental action 3, and also be able to unpack and reason with mental actions 1 and 2. The developmental levels are as follows:

Level 1 – Coordination Level

Level 2 – Direction Level

Level 3 – Quantitative Coordination Level

Level 4 – Average Rate Level

Level 5 – Instantaneous Rate Level

A person having level 5 covariational reasoning ability can reason with mental actions 5 and below, but the converse is not true. A person performing activities categorized as mental action 5 might not necessarily have level 5 covariational reasoning ability; they may have simply memorized procedures to construct an acceptable response; such reasoning is referred to as “pseudo-analytical” by Carlson et al. (2002).

Upon reviewing current textbooks used in Precalculus and College Algebra courses, it was clear that key ideas of quantity and covariation are absent (Gustafson & Frisk, 2004; Stewart, Redlin, & Watson, 2002; Sullivan & Sullivan, 2008). Furthermore, in many textbooks, ideas of proportionality are discussed from a purely calculational orientation (A. G. Thompson, et al., 1994). Even in exemplary textbooks, such as *College Algebra* (Gustafson & Frisk, 2004) and *Precalculus: Mathematics for Calculus* (Stewart, et al., 2002), proportionality seems to be ill-defined in lacking of meaning. For example, Gustafson and Frisk (2004) define proportionality as “An equation indicating that two

ratios are equal is called a **proportion**.” (p. 218, emphasis in original) In discussing properties of proportions, the authors note that “In any proportion, the product of the extremes is equal to the product of the means.” (Gustafson & Frisk, 2004, p. 218) Stewart, et al. (2002) define proportionality in the presentation of direct variation as follows:

If the quantities x and y are related by an equation

$$y = kx$$

for some constant $k \neq 0$, we say that y varies directly as x , or y is **directly proportional** to x , or simply y is **proportional to** x . The constant k is called the **constant of proportionality** (p. 168).

Here the emphasis is from the original text. Note that in these textbooks, the word quantity is never explicitly defined, although the term is used when solving word problems. The interpretation of the meaning of quantity is left to the reader. In such calculationally oriented textbooks, the key developmental understanding (Simon, 2006) is a schematical association of proportionality with appropriate formulas and procedures. Let us now look at a view of proportionality that supports a conceptual orientation.

From a covariation of quantities perspective, proportionality can be viewed three different ways from the (1) constant ratio, (2) constant multiple, and (3) scaling perspectives (Carlson & Oehrtman, 2011, p. 12, Module 2):

1. **Constant Ratio**: Two quantities whose measures vary are said to be proportionally related when the **ratio** of their measures remains the same. Another way of saying this is, if the measures of A and B vary, their ratio $\frac{A}{B}$ is always

equal to some value k that does not vary. The constant ratio k is called the **constant of proportionality**.

2. **Constant Multiple:** Two quantities whose measures vary are said to be proportionally related when the measure of one quantity is always the same **multiple** k of the measure of the other quantity. This constant multiple k is the **constant of proportionality**.

3. **Scaling:** Two quantities whose measure vary are said to be proportionally related when scaling one quantity by a factor (that is, multiplying the measure by the factor) results in the other quantity scaling by the same factor (multiple).

The emphasis is in the original text. While covariation and the three perspectives of proportionality are mutually supportive, based on the covariational framework, the research suggests that students (and teachers) who have a covariational reasoning level lower than the Quantitative Coordination Level will struggle with the idea that if quantities A and B are proportional, then corresponding changes in these quantities will be also proportional. With the ideas of quantity, covariational reasoning, and proportionality summarized, the foundational understandings of constant and average rates of change can be discussed in relation to this background knowledge.

Key Ideas – Constant and Average Rates of Change

Key developmental understandings of constant rate of change and average rate of change that support a calculational orientation are based on knowledge of formulas and procedures. As defined in one widely used textbook, “slope in a linear model can be interpreted as a rate of change.” (Stewart, et al., 2002, p. 121), which is followed by two

examples and not expanded on afterwards. Slope is defined in the following way (Stewart, et al., 2002, emphasis in original):

The **slope** m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined (p. 114).

In the same page as this definition of slope, it is mentioned that slope is independent of points chosen on a line, with a nearby diagram in the text showing two similar triangles on the same line. The average rate of change in the same textbook also supportive of a calculational orientation as defined as follows (Stewart, et al., 2002, emphasis in original):

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change is the slope of the secant line between $x = a$ and $x = b$ on the graph of f , that is, the line that passes through $(a, f(a))$ and $(b, f(b))$ (p. 175).

Key developmental understandings of constant rate of change that support a conceptual orientation are grounded in proportionality (Carlson, O'Bryan, & Joyner, 2012; P. W. Thompson, 1994a) as follows:

Two quantities or variables in a functional relationship, x and y , are related by a constant rate of change if any of the following (equivalent) constraints that imply their changes are proportionally related is true:

- i) A change in the value of one quantity is always some constant c times as large as the corresponding change in the value of the other quantity ($\Delta y = c \cdot \Delta x$ where c is constant).
- ii) The ratio of corresponding changes in the two quantities is constant ($\frac{\Delta y}{\Delta x} = c$ is constant).
- iii) If the change in one quantity is scaled by some factor (Δx is changed to $k \cdot \Delta x$), then the corresponding change in the other quantity is scaled by the same factor (Δy is changed to $k \cdot \Delta y$).

A linear model can be used to represent two quantities that covary by a constant rate of change. The slope is then viewed as the constant rate of change of the output quantity with respect to the input quantity.

Key developmental understandings of the average rate of change are built on the constant rate of change. The average rate of change of two covarying quantities is the constant rate of change between the two quantities that would be necessary to accumulate the same net changes in the quantities under consideration. P. W. Thompson (2013) states the average rate of change in two parts saying that “First, two quantities, A and B, need to change simultaneously, and each has a total change. The average rate of change of Quantity A with respect to Quantity B is that constant rate of change of A with respect to B that would produce the same change in A in relation to the change in B that actually happened” (p. 72). It follows that since the constant rate of change of A with respect to B is reported in finding an average rate of change, we have information about what the

slope of the line representing the constant rate of change would have been, had it passed through the input-output pairs of A and B that represent the total change in each quantity.

Leveraging Key Ideas of Proportionality

In the study that was a focal point of this dissertation, a participant's understandings of proportionality played a role in her pedagogical moves to support student learning. In this portion of conceptual analysis, I will outline perspectives of proportionality that can be leveraged to build a conceptual understanding of functions in both the additive and multiplicative conceptual realms.

I will define an additive conceptual realm as one in which the comparison between the measured values of quantities achieved additively (for example, by the operation of subtraction). I will define a multiplicative conceptual realm as one in which the comparison between the measured values of quantities is made multiplicatively (for example, by the operation of division). I see my definition of conceptual realm as a subdomain that fits within Vergnaud's (1994) conceptual field. My description distinguishes conceptual realms from additive and multiplicative worlds (Confrey & Smith, 1994). Although some commonalities exist between Confrey and Smith's descriptions of the additive and multiplicative worlds and my conceptual realms, I do not view splitting as a primitive operation as the authors did in their research, and this may simply be due to my choice of participants (adults rather than children). Other researchers (Steffe & Olive, 2009) advocate that neither splitting nor sequencing are more primitive, but that both occur together; I will use this perspective in my analysis, since scaling covarying quantities requires attention to both the equal partitioning and iteration of

intervals of the independent and dependent quantities in order for the scaling operation to be meaningful.

For functions whose quantities are in proportion as they covary, equal changes in input correspond to equal changes in output. Researchers (Carlson, O'Bryan, & Joyner, 2013a; Carlson, et al., 2013b; A. G. Thompson & Thompson, 1996; P. W. Thompson, 1994a; P. W. Thompson & Saldanha, 2003) have used the double number line as a tool to track the covariation of two quantities while attending to how the quantities change individually. Similar to the original use of the double number line to visualize speed as a proportional relationship between the quantities of distance traveled and the amount of time to travel such a distance by Thompson and Thompson (for example, see A. G. Thompson & Thompson, 1996; P. W. Thompson, 1994a; P. W. Thompson & Thompson, 1994), I have observed that this visualization is particularly helpful to highlight the covariation of functions that are strictly increasing or decreasing (i.e., monotonic). An example of such a scenario is illustrated on the double number line that conveys a proportional relationship between the values of two quantities (Figure 3). The bottom line is used to designate the values of one measured quantity, and the top line is used to represent the associated values of another measured quantity.

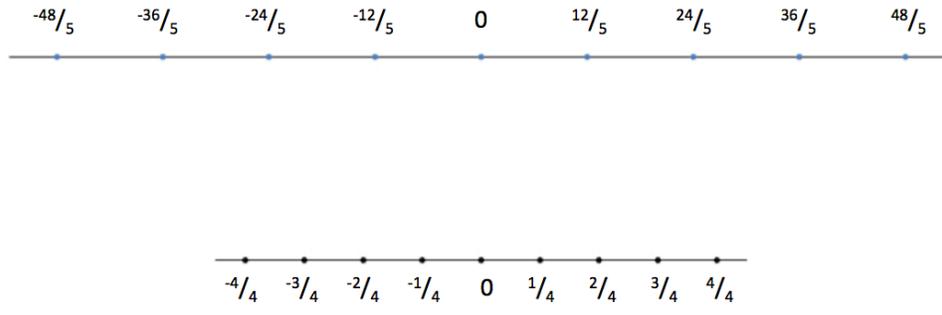


Figure 3. Covariation between proportional quantities

In Figure 3, for each increase of $\frac{1}{4}$ in the value of the input quantity, the value of the output quantity increases by $\frac{12}{5}$, and an input of 0 corresponds to an output of 0. Based on the way the increases in the input and output values of the quantities both occur in this scenario, the comparisons made within the values of each quantity occur in an additive realm while meaningful comparison between the input and output values of the quantities occur in a multiplicative realm.

Since the quantities are proportional, as one quantity is scaled by a factor of c , the corresponding quantity is scaled by the same factor of c . For example, if an input of $\frac{3}{4}$ is scaled by $\frac{4}{3}$ to become 1, then the output of $\frac{36}{5}$ is scaled by $\frac{4}{3}$ to become $\frac{48}{5}$. This implies that if the input to the function were 1, the output would be $\frac{48}{5}$, which is true because the quantities are in a proportional relationship. Furthermore, the same holds true for the changes in the quantities. Since the changes in quantities are proportional, as the change in one quantity is scaled by a factor of c , the corresponding change in the other quantity is scaled by the same factor of c . For example, if a change in output of $\frac{36}{5}$ is scaled by $\frac{1}{3}$ to become a change in output of $\frac{12}{5}$, then the corresponding change in input of $\frac{3}{4}$ is scaled by $\frac{1}{3}$ to become a change in input of $\frac{1}{4}$.

Unitization

The unit rate can be determined by finding the ratio between the covarying quantities, where the desired quantity (or change in quantity) to be unitized is the divisor of the associated ratio. The quotient can be thought of as scaling of the dividend and divisor by a factor that was the reciprocal of the divisor.

Researchers (Weber, Pierone, & Ström, 2016) have found unitization to be a common approach used to scale quantities in an effort to construct meaning between quantitative relationships. In the article by Weber, et al. (2016), part of an activity called the Shape Task required participants to express the area of 2 triangles, measured in rectangle areas, when given that the area of 4.5 triangles was equivalent to 5 rectangle areas. Teachers with a robust meaning of proportionality would know that scaling the 4.5 triangle areas and 5 rectangle areas by $\frac{2}{4.5}$ could determine the number of rectangle areas equivalent to 2 triangle areas. However, a common strategy used by teachers who responded to this task was to unitize the triangle areas, first as an intermediate step, and then for the purpose of determining the quantitative relationship between the two quantities. Participants would try to determine how many rectangle areas were equivalent to 1 triangle area, and double that result to find the number of rectangle areas equivalent to 2 triangle areas. This unitization approach can be seen in other literature. For example, P. W. Thompson and Saldanha (2003) discuss the ways of reasoning multiplicatively through a posed question called the Melissa problem: *Melissa bought 0.46 lb of wheat flour for which she paid \$0.83. How many pounds can she buy for a dollar?* The first way the authors discuss reasoning through the solution uses unitization as an intermediate step, because to scale \$0.83 to \$0.01 as the first step in the solution process means

conceiving of the penny as the unit. Both 0.46 lb. and \$0.83 are scaled by a factor of $\frac{1}{83}^{\text{rd}}$, which is then followed by scaling both measured quantities by a factor of 100 to determine how many pounds of flour can be purchased for \$1.00.

Unitization provides a benchmark between quantities from which the constant multiple perspective of proportionality can be used to build a rule for a function that relates the values of the input quantity to the values of the output quantity. For example, in Figure 3, as the input was scaled to 1, the output was scaled to $\frac{48}{5}$. It follows that the value of the output will always be $\frac{48}{5}$ times as large as the value of the input. Stated in an alternative way, $\frac{48}{5}$ copies of the value of the input to the function is the output value of the function, and this can be expressed as the rule $f(x) = \frac{48}{5}x$.

Visualizing an Analogue to Scaling

A function is exponential if for equal changes in input, the ratio between the outputs at the end and beginning of each successive interval of change is a constant. Determining if a function is exponential requires both an additive comparison of the values of the input quantity while attending to a multiplicative comparison of the corresponding values of the output quantity.

To visualize the analogue of scaling in the multiplicative realm, imagine making a graph of the function $f(x) = 3^x$ while using the double number line representation. Using an additive scale to represent both quantities does a poor job in visualizing the information when the values of the output vary by several orders of magnitude, as seen in Figure 4 that follows.

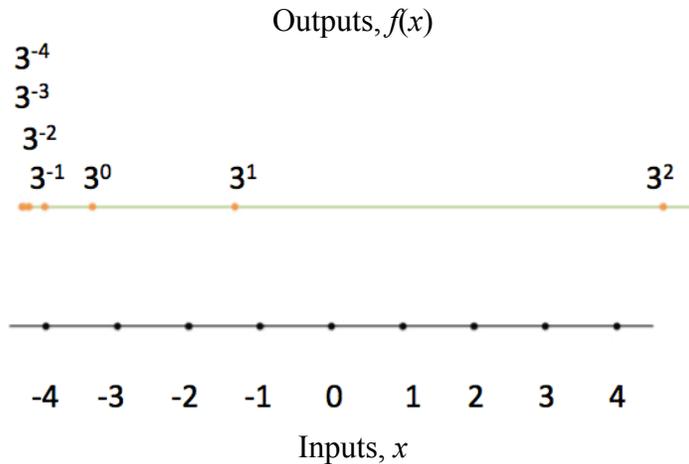


Figure 4. Graphing an exponential function using a double number line

To aid in visualizing the quantities and the covariation between them, I will represent the plot of the output on a logarithmic scale, as seen in Figure 5.

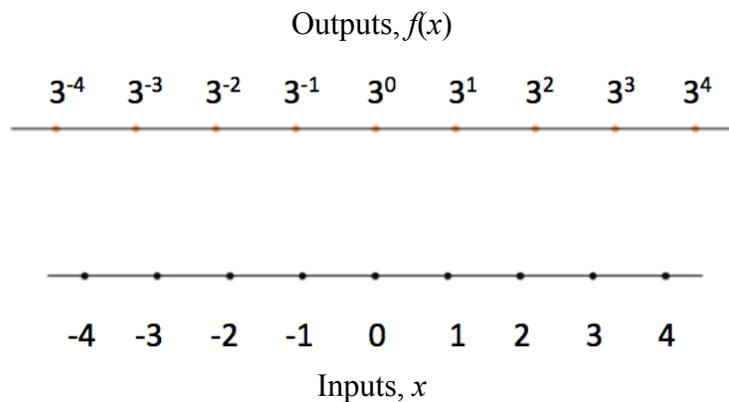


Figure 5. Representing covarying quantities using a logarithmic scale

I see this visualization as a tool that will serve multiple purposes. First, it serves the immediate need of seeing the relationship between the quantities being tracked more conveniently. Logarithmic plots are used in science to plot data that differ in orders of magnitude (Gaudet, Meacham, Bohart, Volpe, Knop, & Guhse, 2014). Researchers (E. Smith & Confrey, 1994) noted that both the construction of a number line that operated through the operation of multiplication and the idea of function as the covariation of two quantities led to historical development of logarithms. I plan to use the double number

line, covariation, and logarithms as a conceptual tool that leverages the idea of scaling in an additive realm to its analogue in a multiplicative realm. Furthermore, I will choose the base of the logarithmic scale in this representation to be e , the continuous growth factor. The rationale behind this choice of base is intentional, and will be discussed shortly.

If the output of an exponential function is plotted on a logarithmic scale, for equal changes in input, the corresponding changes in output will be equal. This is a consequence of a property of logarithms:

$$\ln\left(\frac{f(x + \Delta x)}{f(x)}\right) = \ln(f(x + \Delta x)) - \ln(f(x))$$

In particular, the constant ratio of an exponential function on an additive scale is visualized as a constant difference on a logarithmic scale. This fact was known historically; one of John Napier's key insights during his development of logarithms in the early 17th century was that "the logarithms of proportionals was 'equally differing' " (Moulton, 1915, pp. 13-14). However, what is gained through representation on a logarithmic scale is the ability to make meaningful comparisons of the outputs on a logarithmic scale additively, as this enables leveraging ideas of scaling for an additive realm into the multiplicative realm. Since the changes between the input quantities (on an additive scale) are proportional to the changes in the output quantities (on a logarithmic scale), it follows that both remain in proportion if scaled by a factor of c in their respective scales. But scaling by a factor of c on the logarithmic scale corresponds to raising the value of the constant ratio to an exponent of c for underlying output values.

This is true since:

$$c[\ln(f(x + \Delta x)) - \ln(f(x))] = \ln\left(\frac{f(x + \Delta x)}{f(x)}\right)^c$$

Following a similar line of reasoning, the relationship can be generalized for any exponential function of the form $g(x) = ab^x$, where $g(0) = a$. Scaling the change in input values by a factor of c in an additive scale results in a scaling of c of the changes between the output quantities in the corresponding logarithmic scale, which means the underlying ratio of the output values is raised to the c power. Comparing the two sides of the relationship in the given context,

$$\ln\left(\frac{g(x + \Delta x)}{g(x)}\right)^c = \ln\left(\frac{ab^{x+\Delta x}}{ab^x}\right)^c = \ln(b^{\Delta x})^c, \text{ and}$$

$$c[\ln(g(x + \Delta x)) - \ln(g(x))] = c[\ln(ab^{x+\Delta x}) - \ln(ab^x)] = c \ln(b^{\Delta x})$$

Clearly, $c \ln(b^{\Delta x}) = \ln(b^{\Delta x})^c$. However note that the change in the value of the growth or decay factor in the output quantity is scaled by a factor of c on the logarithmic scale, not the value of the quantity itself. Hence only the growth (or decay) factor is raised to the power of c when represented in the multiplicative realm. The thinking described, using mathematical notation, thus is $g(\Delta x \cdot c) = a(b^{\Delta x})^c$. When the change in input is taken from the reference value of zero, this function can be thought of as $g(x \cdot c) = a(b^x)^c$.

The n -unit growth or decay factor represents the constant ratio between the output quantities that results from any change of input over an interval $\Delta x = n$ units. The scaling perspective can be used to find n -unit factors over any interval of input. For example, given the function $f(x) = (3^{4/5})^x$, a plot of a double number line in which the output is

scaled on a logarithmic scale that illustrates this relationship can be seen in Figure 6 that follows, for changes in x over intervals of 1-unit. The 1-unit growth factor is $3^{4/5}$.

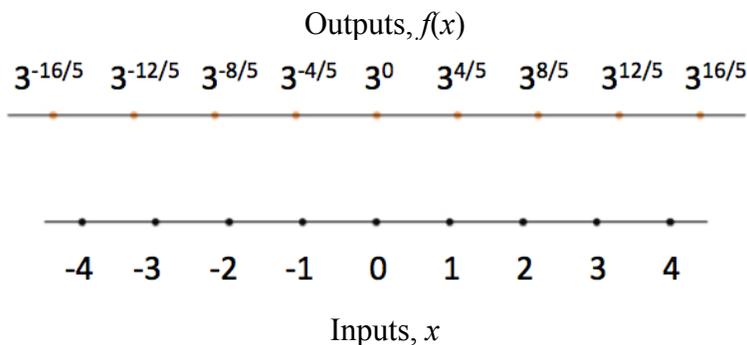


Figure 6. The function $f(x) = (3^{4/5})^x$ on a double number line

To find the 3-unit growth factor, we could scale any input change of 1-unit by 3 in the additive realm. The analogue of scaling for the corresponding ratio between the output values, $3^{4/5}$, in the multiplicative realm, is to raise this constant ratio to the third power. Thus the 3-unit growth factor is $(3^{4/5})^3 = 3^{12/5}$.

Another approach, using the ideas of proportional reasoning, would be to start with a 2-unit growth factor $(3^{8/5})$, and scale a 2-unit change input by $3/2$. Then scale the change of output (on the logarithmic scale) by $3/2$, which in the multiplicative realm corresponds to raising this growth factor to the $3/2$ power. It follows the 3-unit growth factor is $(3^{8/5})^{3/2}$, which is equivalent to $3^{12/5}$.

Flexible ideas of scaling can be leveraged when unitizing growth or decay factors. For example, in Figure 7 that follows, the $5/4$ -unit growth factor is 3. Using the idea of scaling: scaling the change of input from $5/4$ by $4/5$ would result in an input interval of 1, and that the analogue of scaling the growth factor of 3 by $4/5$ would be raising 3 to the $4/5$ power. Therefore the 1-unit growth factor is $3^{4/5}$.

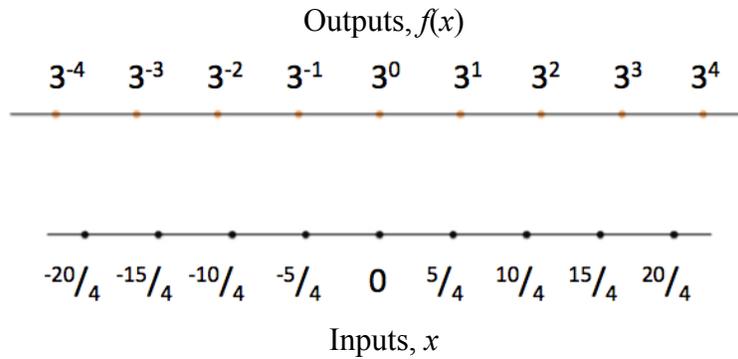


Figure 7. Using flexible ideas of scaling

Visualizing Covariation of Exponential Functions

When analyzing the covariation of quantities with regards to exponential growth, it follows that the input and output values increase together in tandem (or decrease together in tandem), depending on the direction used to read out the values on the graph (either left to right or right to left). Using equal intervals of input, an additive comparison of the corresponding output values of the measured quantities also can give insight to whether the rate of increase (or decrease) is either increasing or decreasing. For example, in Figure 7, as the input changes from $\frac{5}{4}$ to $\frac{10}{4}$, the output of the function increases from 3^1 to 3^2 , which is an additive change of 6, while as the input changes from $\frac{10}{4}$ to $\frac{15}{4}$, the output changes from 3^2 to 3^3 , which is an additive change of 18. As this relationship between successive input intervals holds (that the additive change in output in preceding interval is less than the additive change of output in the succeeding interval) regardless of the size of equal input intervals compared, it follows that exponentially growing functions increase at an increasing rate.

The visualizations thus far have focused on exponential growth, with the double number line used as a tool to highlight the covariation of the values of input and output quantities as they increase in tandem together. A similar process could be used to create a

visualization of exponential decay, for example by having the inputs on the number line displayed in decreasing order, while maintaining an increasing order for the output values. Using such visualization, as the input values decrease, the output values increase in tandem, when the double number lines are viewed from left to right. When the number lines are viewed from right to left, as the input values increase, the output values of the exponentially decaying function decrease.

Visualizing the Relationship Between the Instantaneous Rate of Change of an Exponential Function and Its Output

Earlier I had mentioned choosing the natural logarithm as the scale that can be used to build a logarithmic scale, and that such a choice was intentional. For an exponential function, when creating a graph of output values in which the intervals of input values are 1 unit apart, the length of intervals between the output values corresponding to these input values will be constant on a logarithmic scale. Furthermore, if the base used for this scale is the continuous (or natural) growth factor e , the length between output intervals corresponding to 1-unit growth factors will accurately represent the ratio between the value of the instantaneous rate of change of the function and the output value of the function. Three examples are illustrated below, along with a proof of why the length between output intervals on a logarithmic scale to base e corresponds to this ratio. Let $f(x) = 3^x$, $g(x) = 2^x$, and $h(x) = e^x$ be real valued exponential functions.

In the case of $f(x) = 3^x$, the length of interval between any 1-unit growth factor plotted on a natural logarithmic scale is $\ln 3$, or approximately 1.0986. This means the ratio between the instantaneous rate of change of the function and the output of function will always be $\ln 3$, or alternatively, the value of the instantaneous rate of change will be

In 3 times as large as the corresponding output value. When carefully plotted, this visualization of the scaling is helpful in seeing that the instantaneous rate of change of this function will be slightly more than the output of the function for a given value of input.

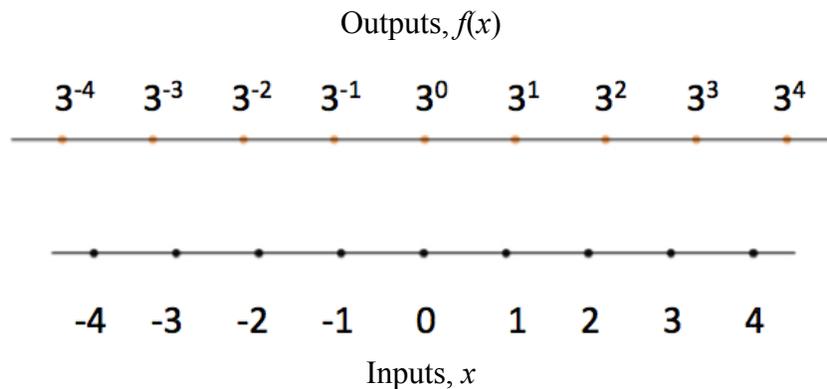


Figure 8. Depicting the output of $f(x) = 3^x$ using a natural logarithmic scale

For example, in Figure 8 the scaling between marks of the output values corresponding to one-unit growth factors are slightly more than 1 unit apart. Therefore the instantaneous rate of change of $f(x)$, when $x = 3$, can be estimated to be somewhat more than $f(3)$, the output value of the function; it follows that the instantaneous rate of change should be more than 27. In fact, the exact value of the instantaneous rate of change, when $x = 3$, is $27 \ln 3$, which approximately equals 29.6625.

In the case of $g(x) = 2^x$, which is given in Figure 9 that follows, the length of interval between any 1-unit growth factor plotted on the natural logarithmic scale is $\ln 2$, or approximately 0.6931. This means the ratio between the instantaneous rate of change of the function and the output of function will always be $\ln 2$. In other words, the value of the instantaneous rate of change of $g(x)$ will be $\ln 2$ times as large as the output value of $g(x)$.

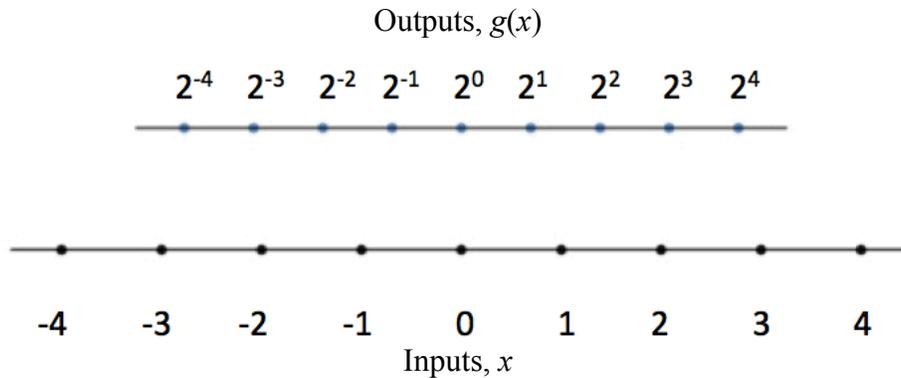


Figure 9. Depicting the output of $g(x) = 2^x$ using a natural logarithmic scale

In Figure 9, we note that the scaling between the output marks corresponding to the interval between 1-unit factors is somewhat less than 1 unit, and that roughly three marks on the output scale correspond to two marks on the input scale. So the instantaneous rate of change of $g(x)$, when $x = 3$, can be estimated to be roughly two-thirds of $g(3)$, the output value of the function. The instantaneous rate of change therefore will be estimated to be two-thirds of 8, which is $16/3$ or $5\frac{1}{3}$. The exact value of the instantaneous rate of change is numerically close to the estimated value. When $x = 3$, the instantaneous rate of change for $g(x) = 2^x$ is $8 \ln 2$, which is approximately equal to 5.5451.

In the case of $h(x) = e^x$, illustrated in Figure 10 that follows, the length of interval between any 1-unit growth factor (when plotted using a natural logarithmic scale) is 1. This means the ratio between the instantaneous rate of change of the function, and the output of function, will always be 1.

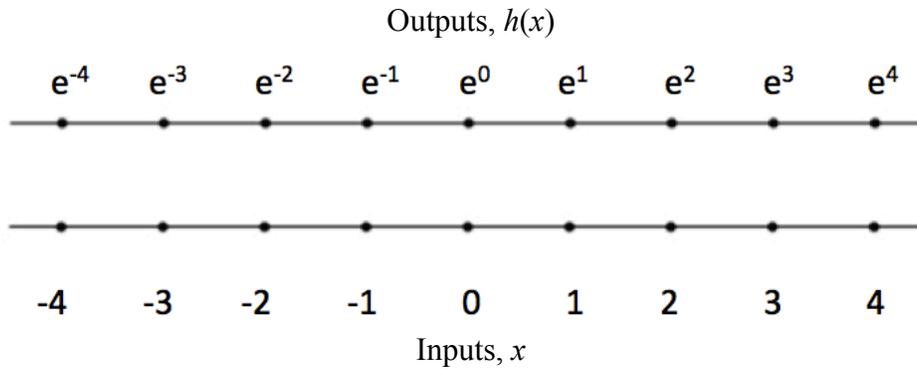


Figure 10. Depicting the output of $h(x) = e^x$ using a natural logarithmic scale

Since the scaling between marks is identical, it is visually easy to interpret that the output value of the function will be the same as the instantaneous rate of change of the function. When $x = 3$, the output of the function, $h(3)$, is e^3 (approximately 20.0855), and the instantaneous rate of change when $x = 3$ is the same value. When e is the base of an exponential function, the ratio between the instantaneous rate of change of the function for any value of input, and its output value for the same input, will always be a constant value of 1. This makes e special, as this is the only base for exponential function in which this relationship holds.

This leads to a working definition of e (adapted from Carlson, et al., 2013b):

Imagine a function in which the instantaneous rate of change of output with respect to the input is the same as the value of the output of the function as the input varies. Such a function is exponential, and the unique value of the 1-unit growth factor of this function is the constant e . e is referred to as the continuous (or natural) growth factor.

If the instantaneous rate of change of output with respect to the input of a function is proportional to the output of the function as the input varies, such a function is also exponential. The constant of proportionality k relates the values of the instantaneous rate of change of the function to its output, with the 1-unit factor of the function being e^k .

From Calculus, $\frac{dy}{dx} = ky$ defines the first situation, whose solution is $y = ae^{kx}$, where

$y = f(x)$, and a is a constant not equal to 0. In the second situation, $\frac{dy}{dx} = ky$, and the

solution to it is $y = ae^{kx}$, where $y = f(x)$, and k and a are constants not equal to 0.

To see why the ratio between the instant rate of change of an exponential function and the output of the function $f(x) = ab^x$ is always $\ln b$, consider the following.

$$f(x) = ab^x = a(e^{\ln b})^x = ae^{(\ln b)x}, \text{ and}$$

$$f'(x) = \frac{d}{dx}(ae^{(\ln b)x}) = (\ln b)ae^{(\ln b)x} = (\ln b)ab^x$$

Therefore $\frac{f'(x)}{f(x)} = \ln b$. Since e^k is the 1-unit factor of ae^{kx} and b is the 1-unit factor of

ab^x , it follows that $b = e^k$. The exponent k of e^k is equivalent to $\ln b$.

In short, this brief conceptual analysis supports the idea that leveraging the ideas of proportionality and covariation of quantities through the tool of the double number line can aid in a conceptual understanding of the constant e , and of exponential functions in general.

Describing Scaling in the Additive and Multiplicative Realms

For the purposes of being precise in communicating my later findings, when describing scaling in the additive realm and the analogue of scaling in the multiplicative realm, I will use the term “copies of” or “times as large as” as a way to describe a multiplicative comparison (scaling) in the additive realm, and “factors of” or “partial factors of” to denote an exponentiation (the analogue of scaling) in the multiplicative

realm, using language researchers (Ström, 2008; P. W. Thompson & Saldanha, 2003) have used before to support productive ways of thinking and communicating about such comparisons within these respective realms.

Summary of the Purpose of This Conceptual Analysis

This chapter provides the framework to characterize and analyze a teacher's ways of thinking about the underlying mathematics of the content they teach grounded on the ideas of quantity, covariation, proportionality, constant rate of change, average rate of change, and leveraging proportionality in the additive and multiplicative realms. To give an example of how such a framework is used, suppose a teacher had impoverished meanings of proportionality; this would limit the perspectives a teacher can take when relating two quantities that covary. For example, suppose the ratio perspective of proportionality is accessible to this teacher, but the scaling perspective is tenuous for him or her. That in turn could limit the ways accessible to the teacher when reasoning about exponential functions. He or she might be able to reason that a function is exponential between equal intervals of input when the ratio of the values between preceding and succeeding quantities remains the same as the quantities covary. However, he or she may struggle to find a way to leverage the idea of scaling when trying to find different unit growth or decay factors. He or she may resort to a procedure to find a known benchmark, such as a 1-unit growth or decay factor, in order to make sense of the problem. I intend to use the conceptual analysis outlined in this chapter as a tool for this dissertation study.

In the next chapter, I will outline pilot investigations that informed the dissertation study, along with the methods utilized for carrying out this research.

CHAPTER 4: EXPLORATORY STUDIES

The theoretical perspectives I have taken and the conceptual analysis from the prior chapters framed the planning and analysis of the studies that follow, and provide the backdrop from which the goal framework and the lesson planning protocol emerged. Three qualitative studies were conducted. The first two studies were preliminary pilot studies, while the third study was the dissertation study. All three studies were supported by the National Science Foundation Grant No. 1050721 (Pathways to Calculus: Disseminating and Scaling a Professional Development Model for Algebra Through Precalculus Teaching and Learning). In this chapter I will discuss the two exploratory studies in which a goals framework emerged that was used in the dissertation study.

First Study

Dorothy and Margaret

The first study was a qualitative study that was conducted in the Spring 2012 semester (April through May); this was my initial pilot project. Dorothy and Margaret (both pseudonyms) were high school teachers at Flat Vista High School (pseudonym) in a Southwestern state who had taught with a conceptually based Precalculus curriculum for the first time. Based on data collected, Dorothy has been teaching 15 years, while Margaret has been teaching 8 years. My first meeting with each teacher was for the purpose of building a rapport, and to discuss the teacher's goals for teaching. I used their input as a foundation for our first professional development meeting. The pilot project consisted of characterizing teacher's overarching goals, and gaining insight into lesson specific instructional goals within the context of reflecting on a specific chapter of interest to the teachers (Module 2 in the Pathways curriculum (Carlson & Oehrtman,

2012), which focused on quantities, proportionality, constant, and average rates of change). From these interactions a goal framework emerged that will be described later.

Research Question for the First Study

The purpose of the first study was to characterize a teacher's lesson specific goals, and to develop a better understanding of how a teacher's mathematical knowledge for teaching and her teaching orientation influence these goals. The primary research question follows.

How might a teacher's goals for student learning be characterized after a curricular intervention that promoted a conceptual orientation in mathematics?

Methods

Six professional development sessions, each lasting approximately one hour were conducted over a period of six weeks. All sessions were audiotaped for future analysis. The principle guiding my approach to data analysis was the based on the methodology by Clement (2000) in which clinical interviews of case studies were used for both generative purposes and for convergent purposes. Interpretive analysis of the data in a generative study leads to new observational categories, which was the intent of the first study. A follow-up study could be used for convergent purposes, to see if created categories were viable or needed additional refinement.

In attendance at the professional development meetings were Dorothy, Margaret, and myself. The first portion of each meeting a short interview focused on clarification and verification of the overarching and lesson specific goals stated in the prior meeting. The second part of the meeting focused on reflection and refinement of the lessons in the chapter of interest to Dorothy and Margaret. The purpose of characterizing overarching

goals was to understand the types of teaching goals they had independent of context in order to gain possible insight into their teaching orientation. Other aspects of the sessions with Dorothy and Margaret extended beyond the scope of this specific study. A final listing of Dorothy and Margaret's overarching goals, lesson specific goals for the chapter of interest, and interview questions can be found in Appendix B.

Lesson Specific Goals

To characterize teachers' goals for student learning after a curricular intervention that promoted a conceptual orientation in mathematics, I decided to ask the teachers to verbalize their goals for student learning in the context of each lesson. The teachers were asked to contemplate this question in the context of their refining the lesson, and I looked for emerging patterns in their goals statements. In reviewing the data, grounded theory (Strauss & Corbin, 1990) with open coding was later used to help develop the goal framework, keeping in mind teaching orientation (A. G. Thompson, et al., 1994) and Silverman and Thompson's (2008) characterization of mathematical knowledge for teaching (MKT). Grounded theory is a qualitative research method that uses an inductive method to build theory to explain the phenomenon that is being observed.

Dorothy and Margaret's overarching goals suggested a view of mathematics shifting toward a conceptual orientation that focused on ideas and connections in mathematics and that valued students' thinking and reasoning. The curriculum Dorothy and Margaret used consisted of a workbook in which students wrote. For example, one of Dorothy and Margaret's overarching goals was for students to reason through problems by having them make connections to the main idea of the worksheet [in their workbook]

and the prior learning needed to complete the lesson. Another overarching goal was on questioning strategies to guide students to the main idea of the lesson.

Collaborating evidence from an instrument designed to measure shifts in in teachers’ beliefs about mathematics teaching and learning also give additional support to the claim that Dorothy and Margaret were shifting toward a conceptual orientation in mathematics (see Appendix C, for survey details and participant data). This instrument had emerged based on prior work by Carlson, Buskirk, and Halloun (1999) and Carlson and Rasmussen (2010). Responses to specific questions related to teaching orientation (questions 23, 27, 28, and 29) suggest that Dorothy and Margaret possessed a view of mathematics consistent with a conceptual orientation.

A Goal Framework

As a result of using grounded theory to code the utterances of teachers’ goals for student learning (TGSL), a framework emerged from the data (see Appendix B) that describes a trajectory ranging from levels 0 through 6 in the table below.

Table 1.

Characterization of Levels in a Teacher’s Goals for Student Learning

Goal Coding	Description
<i>TGSL0</i>	Goals for student learning are avoided or not stated by the teacher, or he/she states that the goals of the lesson are unknown.
<i>TGSL1</i>	Goals are a list of topics that a teacher wants his/her students to learn in the lesson, each associated with an overarching action.
<i>TGSL2</i>	Goals are a list of topics that a teacher wants his/her students to learn in the lesson, each associated with a specific action.
<i>TGSL3</i>	Goals are doing methods of mathematics that a teacher wants his/her students to learn in the lesson.
<i>TGSL4</i>	Goals are getting students to think about the mathematics in the lesson, without the ways of thinking articulated.
<i>TGSL5</i>	Goals are getting students to think about the mathematics in certain ways during the lesson.

Goal Coding	Description
<i>TGSL6</i>	Goals are about developing ways of thinking about the mathematics in the lesson, with attention to how that thinking may develop.

The dimensions of this framework were arranged on a spectrum representing no focus on student thinking at level 0 to a maximal focus on student thinking at level 6. In other words, lower levels in the framework describe goals that focused on the visible products of student reasoning while higher levels of the framework describe goals that are focused on the student reasoning itself. Other researchers have characterized a teacher's attention to student thinking through the lens of teaching disposition (Courtney, 2010, 2011) or through their pedagogical actions (Carlson, Oehrtman, Moore, & Bowling, 2008; Marfai, Moore, & Teuscher, 2011; Teuscher, Moore, & Carlson, 2015). The lens I use in this study is through a teacher's goals for student learning.

Goal level TGSL4 was not observed in this study because of norms established by me stressed that Dorothy and Margaret clarify their statements regarding the ways of student thinking they were trying to promote. However, without these established norms, goal level TGSL4 could have likely emerged in some of their stated goals. Lesson specific goals were initially stated vaguely, and the literature from planning had discussed that teachers had mental images of lessons in greater detail than what was written down. Since goals for student learning are an unarticulated aspect of lesson planning, I decided to press further in such instances to see what mental images Margaret and Dorothy had structured under the initial statements.

Goal level TGSL6 was not observed in this study either, but was hypothesized to exist based on researchers' work on professional development supports that promoted

growth of teachers' goals relative to students' thinking of mathematics and ways to support such thinking (M. S. Smith, et al., 2008; M. S. Smith & Stein, 2011; P. W. Thompson, 2009).

An Example of the Goal Framework in Context: Constant Rate of Change

My prior conceptual analysis of the key ideas of the constant rate of change helped inform my model of what a teacher's goals might look in a lesson whose key ideas involved the constant rate of change. To illustrate, the following is an idealization of the goals for student learning in the context of the constant rate of change.

Table 2.

Goal Framework in the Context of Constant Rate of Change

Goal Coding	Description
<i>TGSL0</i>	Statements such as: Get through the lesson- not sure about the ideas it is conveying (or its purpose).
<i>TGSL1</i>	Statements such as: Students should understand constant rate of change (and/or linearity).
<i>TGSL2</i>	Statements such as: Students should understand constant rate of change using proportionality and the slope of a line.
<i>TGSL3</i>	Statements such as: Students should understand constant rate of change using proportionality of the changes between the output and input quantities and by calculating the slope of a line.
<i>TGSL4</i>	Statements such as: Students should be able to think about constant rate of change using proportionality of the changes between the output and input quantities in flexible ways; students should view the slope of a line as the constant rate of change.
<i>TGSL5</i>	Statements such as: Students should be able to think about constant rate of change using proportionality of the changes between the output and input quantities in flexible ways, such as the constant multiple, ratio, and scaling perspectives; students should understand that a linear model can be used to represent two quantities that covary by a constant rate of change and view slope as the constant rate of change of the output quantity with respect to the input quantity.

Goal Coding	Description
<i>TGSL6</i>	Statements such as: Support students' development of the idea of the constant rate of change through using different representations such as tables, graphs, and formulas so that they are able to think about constant rate of change using proportionality of the changes between the output and input quantities in flexible ways, such as the constant multiple, ratio, and scaling perspectives; help support the development of a linear model to represent two quantities that covary by a constant rate of change through multiple representations, with conversations about the slope being the constant rate of change of the output quantity with respect to the input quantity.

How the Goal Framework Originally Emerged

The goal framework emerged over time based on the idea for characterizing teachers' goals for student learning. For one lesson, the content did not fit into Margaret's schema of the lessons that preceded it, so her stated goal was simply to "Get through it" (TGSL0), justifying the creation of a level where the teacher's existing understandings do not support the creation or utterance of a coherent lesson goal. Although TGSL1 and TGSL2 appear similar, the distinction was given to allow identification between overarching versus specific actions when stating goals for student learning. In the same lesson in which Margaret's goal was coded as TGSL0, Dorothy's goal was to "Develop the equation of a circle." This was coded as TGSL1 because the word "develop" did not convey a specific action. In a different lesson, one of the stated goals for student learning that both Dorothy and Margaret agreed on was "State quantities precisely - don't use pronouns (want the object, the attribute of that object, units of measure)". This was coded as TGSL2 because there was clarity in the teachers' goals, representing a clear mental image of an intended pedagogical focus for the lesson.

An example of a lesson goal that was coded as TGSL3 was “Discuss three ways (Ratio, constant multiple, scaling) quantities are proportional”. It was focused on the methods of mathematics the teachers wanted their students to reproduce. Regarding a lesson introducing the average rate of change, the teachers’ appeared to set the following goals by mutually focusing on the following two ideas: (1) the meaning of average speed and how it relates to constant speed, and (2) the average speed as equivalent to the constant speed you need to travel to cover the same distance in the same amount of time. This goal was coded as TGSL5 based on the clarifying statement that was given in the stated goal, since it was oriented toward a way they wanted students to think about the idea of constant rate of change. Had this goal been stated as “Have students think about how average speed relates to constant speed and what average speed means”, it would have been coded as TGSL4.

As stated earlier, TGSL6 was not observed in the pilot data of the first study. Justification for retention of this level in the framework for charactering teachers’ goals for student learning is described in the follow-up to the pilot study.

Results and Discussion of the First Pilot Study

My first pilot study produced the first draft of a goal framework for classifying teachers’ goals for student learning. I studied two teachers’ goals for student learning. The results of this study made me realize that a follow-up study would be necessary to investigate the generalizability of the goals I had identified, and to refine my characterization of the higher goal levels in the framework. To illustrate, I will use an example when I had asked Dorothy and Margaret about their goals for student learning in a lesson regarding the constant rate of change. (See Appendix B, for their stated goals for

student learning in the lesson associated with Module 2, Worksheet 4a). In the excerpt that follows, Mar stands for Margaret, Dor for Dorothy, and Res for researcher (me). I asked Dorothy and Margaret to describe their goals for student learning, and their responses follow:

Excerpt 2. Teachers' goals for student learning for constant rate of change

- 1 Mar: Um, constant rate of change.
2 Dor: And linearity.
3 Mar: Yep.
4 Res: So the, what you, that is your goals for it, okay.
5 Mar: (laughing)
6 Res: Could you say a little more, because I'm going to come back next
7 week and say, I wrote down constant change and linearity and all, can
8 you please tell me more about that? (laughs)
9 Mar: The constant rate of change.
10 Res: So let's, I'll ask now what do you mean by both? (laughs)
11 Mar: Well, we're trying to get them to used to this idea, that something,
12 you know, that we're building off of our, um, proportional changes in
13 quantities.
14 Res: Uh huh.
15 Mar: Because that's what we did in our last one *[referring to previous*
16 *lesson's worksheet]*.
17 Res: Okay.
18 Mar: Because we're taking that now and now we're talking about that
19 constant rate of change, because in proportional, we see that in the
20 changes, the changes is going to be a constant. Does that make sense?
21 And um how that relates as far as linearity, because that's what we
22 get into at the end of (4) and (5) *[referring to problems in current*
23 *lesson's worksheet]*.
24 Dor: Uh huh.
25 Res: Uh huh.
26 Mar: And now we can use this model, um, that's what we're building up to,
27 is using the linear model.
28 Res: Using the linear model. Okay, so your goal is...
29 Mar: Based on...
30 Res: ...using the linear model based on proportionality?
31 Mar: Based on the proportionality of the changes in the quantities.
32 Res: Okay. And that's one of your goals? Are there others?
33 Mar: I want them coming out knowing that. That's what I think is the core.
34 Res: Okay, that's like one of big ones, right.
35 Dor: Yeah, absolutely, yeah.
36 Mar: Yeah, that's our, I guess our instructional goal I guess.

37 Res: Okay.

In this and other interactions in which I had posed questions to understand Dorothy and Margaret's goals for student learning I noticed that their responses were typically vague at first (lines 1, 2). After prompting them for further clarification my assessment of their goal level for student learning was changed. This is not surprising since literature (Morine-Dershimer, 1979) had reported that teachers' lesson plans had few details in written form, but that teachers often had mental images of lessons in greater detail than written. These findings suggest that goals for student learning are sometimes not articulated when completing a lesson plan. This led me to press further (lines 6-8) to uncover what mental images Margaret and Dorothy had for student learning that had not been initially stated.

Based on Margaret's statements, and Dorothy's agreement with these statements, this interaction places their goals at a TGSL3 with regards to this lesson (see lines 18-22, 26-27, 31). Summarizing the interaction, Margaret and Dorothy's goal for student learning in the interaction involved their using the linear model based on proportionality of the changes in the quantities. These statements focused more on what they wanted their students to *do*, and not on flexible ways they wanted their students to *think* about constant rate of change (lines 26-27, 31, 33, 35-36).

Each of the lesson specific goal statements in Appendix B was generated through a process in which I pressed for clarification of Dorothy and Margaret's goals for student learning. Table 3 quantifies their goals for student learning for one complete chapter/module of the *Pathways Precalculus* curriculum (Carlson & Oehrtman, 2012)

being used by both teachers. The coding that led to this summarization can be found in Appendix G.

Table 3.

Dorothy and Margaret's Goals for Student Learning

Goal Level	Coded Goals
TGSL0	1 (10.0%)
TGSL1	3 (30.0%)
TGSL2	1 (10.0%)
TGSL3	3 (30.0%)
TGSL4	0 (0.0%)
TGSL5	2 (20.0%)
All Stated Goals	10 (100.0%)

Had I not pressed for clarification with follow-up questions, several of the initial goal statements would not have been stated precisely. This may have resulted in my classifying them at a lower level in the goals framework. For example, the original goal statement in Excerpt 2 (lines 1, 2) would have been ranked at a TGSL1 level. Since goals and MKT are reciprocally related, I conjectured that a follow-up study with an experienced teacher whose MKT using the *Pathways* curriculum was profound would have goals for student learning that would rank higher in the framework. This was Dorothy and Margaret's first time teaching with the *Pathways Precalculus* curricular materials and they were in the process of making connections that contributed to their understandings of the underlying mathematics they taught; it follows that the transformation of their key developmental understandings (KDUs) into their mathematical knowledge for teaching (MKT) was also at a developmental stage.

While prior research has shown that teacher goals can influence their MKT (Webb, 2011), I conjectured that key developmental understandings need to exist before

they are transformed into pedagogically powerful moves. Supporting evidence comes from a teacher thought-revealing activity administered during a professional development workshop that I led for Algebra 1 teachers (see Appendix D). Feedback from more experienced teachers who had used Pathways before described the activity as being useful to them and engaged with questions designed for the activity. Meanwhile, teachers who were about to teach with the curriculum for the first time answered question 5, part (b) (asking about goals for student learning) at varying levels in the framework, and did not fully engage with the questions in the activity. Based on the study with Dorothy and Margaret, and the thought revealing activity from the Algebra 1 workshop, I conjectured that having goals of helping students develop ways of thinking about the mathematics of the lesson (with attention to how these ways of thinking could be developed to be more aligned with goal level TGLS6) was not accessible when a teacher's KDUs of the underlying mathematics was still forming. However, a teacher who exhibited well-connected understandings of the underlying mathematics appears to have the ability to operate at the TGSL6 level of the goals framework.

The Rationale for the Second Pilot Study

There were two major shortcomings in the first pilot project. First, the classroom data I collected when studying Dorothy and Margaret was not aligned with the lessons they chose for their reflection, so I could not characterize the relationship between their stated goals for student learning and what actually happened in class. Second, Dorothy and Margaret had only taught the course using a conceptual curriculum once, so their mathematical connections were still developing, and therefore I hypothesized that these higher levels in the goal framework could be observed (and possibly refined) in working

with an expert teacher whose mathematical knowledge for teaching was profound. As a result, I conjectured that a follow-up study would reveal the degree to which a teacher's stated goals for student learning are developmental. Would use of the conceptual curriculum naturally shift stated goals for student learning toward higher levels in the framework for teachers who had taught with the conceptual curriculum for a longer period of time?

Also, additional reading of literature (A. L. Ball, et al., 2007; Clark & Peterson, 1986; Morine-Dershimer, 1979) informed my approach to a second study with regards to recruitment of another participant. This resulted in my recruiting a teacher who had taught with the Pathways curriculum before. It also afforded an opportunity to gain insight in how the teacher's planning process changed during the course of teaching with a conceptual curriculum.

Second Study

Robert

In the second pilot study, one expert high school teacher, Robert (pseudonym) from Salt Valley High School (pseudonym) in a Southwestern state was selected for observation during two chapters in which he taught Trigonometry during the Spring 2013 semester (February through May). Robert was teaching Precalculus for the third time using the same conceptually rich curriculum provided by the Pathways project (Carlson & Oehrtman, 2012), although he had supplemented the course with materials from a traditional textbook. Robert had been teaching for 13 years, all at the same high school. The criteria for labeling Robert as an expert was that he was a teacher whose key developmental understandings of the Precalculus curriculum were well connected and

whose pedagogical actions indicated an inclination to act on student thinking. His score of 25 on the PCA (Carlson, Oehrtman, & Engelke, 2010) suggested complete mastery of the concepts he was teaching, and he was the leader of his Precalculus professional learning community (PLC) at Salt Valley High School. Prior observations by *Pathways* project team members suggested that he was a teacher whose beliefs and dispositions included valuing the development of students' mathematical thinking and reasoning. Data from the beliefs instrument (Appendix C) also provided corroborating evidence that Robert's view of mathematics was more conceptual than calculational.

The purpose of the observations was twofold. First, the study aimed to characterize the nature of overarching and lesson specific goals of an *expert* teacher in contrast to teachers who were using the curriculum for the first time (such as Dorothy and Margaret). I believed that a teacher with this profile would allow me to continue refinement of my proposed framework. In other words, was the goal framework that emerged from the study with Dorothy and Margaret a generalizable representation of a trajectory of growth in teachers' goals for student learning, and would studying Robert help characterize later parts of this trajectory of growth when a teacher develops profound understandings? Or, was the developed goal framework with its levels only specific to Dorothy and Margaret? Second, the chapters under which the observations were performed had a conceptually rich chapter from reform oriented curricular materials (Carlson & Oehrtman, 2012), followed by a skills based chapter and sections from a traditional textbook (Sullivan & Sullivan, 2008). I was curious to observe how curricular context affected an expert teacher's goals for student learning, and once characterized, if

these goals could be perturbed to form new goals that are at higher level in the framework.

Research Questions for the Second Study

The rationale for this follow-up study was to investigate whether the developed teacher's goals for student learning framework had merit, and whether the higher levels of the framework could be observed in an expert teacher. Additional research from the literature had shown to me that planning had an effect on teachers' goals, so characterizing Robert's planning was one of this study's aims. In addition, the study provided the opportunity to study the effect of curriculum on a teacher's goals and to characterize how sensitive a teacher's goals are to perturbation. The research questions for this second study were as follows.

1. How might a teacher's pedagogical goals for student learning be characterized in the context of using a curriculum promoting a conceptual orientation of mathematics, and how are they similar or different than when using a curriculum that promotes a calculational orientation?
2. How might the goal framework be used as a tool for professional growth, and how stable or fragile are a teacher's goals in the context of perturbations created by a researcher to encourage self-reflection?
3. How does the planning process change as a teacher's mathematical knowledge for teaching grows when using a conceptually rich curriculum?

Methods

Twenty-nine classroom observations were videotaped in Robert's class that focused primarily on trigonometry, in particular angle measure, trigonometric functions,

identities, and applications using trigonometric functions. At the end of each class session, I had Robert complete a short questionnaire that prompted him to explain his instructional goals, and his goals for student learning that day. He answered the questionnaire via e-mail on the same day that he taught the lesson. Prior to the series of classroom observations, an initial questionnaire was given to characterize Robert's overarching goals. When follow-up questions were asked regarding goals, they were sent via email on the same day as the responses. Responses to follow-up questions varied from arriving the same calendar day to the next calendar day. Field notes were taken during each of the observations, and additional notes were made when documenting my original intent behind the follow-up questions. The last meeting with Robert was an interview. One post study query was submitted to Robert after the study was complete, which was intended to be used as a comparison query with the participant selected in the dissertation study. For full disclosure, I was the researcher who visited the classroom, made the observations, and conducted the interview.

The transcript of questions I asked and received from Robert regarding his goals, along with the annotations made to the transcript during the course of the interview, can be viewed in Appendix E. The interview questions asked during the final interview can be found in Appendix F. Portions of the interview asked Robert to remember prior lessons he taught. I brought the transcript (given in Appendix E) of the questions I had asked during the course of the study and Robert's answers regarding his goals to the interview, in addition to sending the full transcript electronically to him after the last classroom observation, but before the interview. Robert had access to the *Pathways* workbook, the traditional textbook, and materials he gave to students on days in which

the classroom was observed. During the interview, annotations were made on the transcript: my marks are denoted in blue, while Robert's marks are shown in red.

Evolution of the Questionnaire

The questionnaire Robert was asked to complete after each class observation went through three main iterative cycles. From February 7th to February 22nd, the three core questions present for each observations were: “What were your goals of instruction with regards to student learning for the lesson you had today?”, “How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?”, and “Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?” The purpose of the first question was the same as in the study for Dorothy and Margaret, which was to study Robert's goals for student learning in the two curricular contexts and to determine the viability of the goal framework developed in the pilot study. The purpose of the second question was to record Robert's perceptions of the enacted lesson relative to his stated goals, which could then be compared to the data collected regarding that day's observation for additional analysis. The purpose of the third question was to inform me on what was Robert noticing, and designed to make Robert reflect on that day's lesson.

The questionnaire had four core questions from February 27th to April 2nd during the period of observation. The additional question that was placed as the first question was the following: “What were your teaching goals for the lesson you had today?” The original purpose of the question had been to characterize Robert's overarching goals in relation to his goals for student learning. The phrasing of this question had been suggested by Robert a couple of sessions earlier, at the end of the February 14th

questionnaire. The questionnaire had five core questions from April 11th to May 7th. The last question was a statement saying: “One or two questions will follow, based on your responses to 1 – 4.” This was the period of the study where I made moves to perturb Robert’s goals for student learning through the follow-up questions. With the methodology of the study fully described, the focal points of this study will be discussed in the next section.

Research Question 1

The first research question had asked: “How might a teacher’s pedagogical goals for student learning be characterized in the context of using a curriculum promoting a conceptual orientation of mathematics, and how are they similar or different than when using a curriculum that promotes a calculational orientation?” In this section I will describe the methods and discuss the results to the first research question.

Methods of Analysis

Analysis of the first research question specifically focused on Robert’s initial response to the question “What were your goals of instruction with regards to student learning for the lesson you had today?” Robert’s goals for student learning were then coded using the goal framework described in Table 1 (see page 47). A total of 62 goals was coded using this framework. For methodological reasons, clarifications to Robert’s stated goals that sometimes occurred through the process of follow-up questions were not coded.

To illustrate an example of how the coding was done in this study, in a lesson using the conceptual curriculum, the key idea was having students make the connection that a measure of an angle's openness is the quantification of the fraction of any circle's

circumference subtended by the angle (with tasks designed to help support students' development of meaning for angle measure in both degrees and radians). Robert's response to his goals for student learning for that lesson statement (for additional detail, please see Appendix E for his response to the 2/13/2013 questionnaire) was as follows.

One goal was for students to gain an understanding of what it means to measure an angle. Another goal was for students to gain an understanding of what it means for an angle measure to be 1 degree. I wanted to stress the importance of thinking about an angle as an object that cuts off a certain fraction of a circle's circumference whose center is the vertex of the angle.

Robert's statement of his goals for student learning that day had two goals followed by one clarification. His first stated goal of what it means to measure an angle, which included the clarifying statement, was rated at a TGSL5 level, since the desired way of student thinking was described specifically. However his second stated goal of having students "gain an understanding what it means to measure an angle of one degree" suggested a desired way of student thinking but was not articulated; therefore this goal was rated at a TGSL4 level.

Results

Table 4 quantifies Robert's goals for student learning, and is subdivided according to whether he was teaching the conceptually rich chapter, or had switched over to the skill-based chapter and sections during which the intervention was also conducted. This split was purposeful for characterizing Robert's goals in the context of the first research question.

Table 4.

Robert's Goals for Student Learning - Count (Percentage)

Goal Level	Conceptual Curriculum	Skill-based Curriculum
TGSL0	0 (0.0%)	0 (0.0%)
TGSL1	7 (17.1%)	2 (9.5%)
TGSL2	10 (24.4%)	1 (4.8%)
TGSL3	3 (7.3%)	8 (38.1%)
TGSL4	17 (41.5%)	7 (33.3%)
TGSL5	4 (9.8%)	3 (14.3%)
TGSL6	0 (0.0%)	0 (0.0%)
All Stated Goals	41 (100.0%)	21 (100.0%)

In looking at the top two categories where Robert's goals for student learning clustered, when using a conceptually rich curriculum the top two ranked categories were TGSL4 (41.5%) and TGSL2 (24.4%), while when using a skill-based curriculum the top two ranked categories were TGSL3 (38.1%) and TGSL4 (33.3%). The coding from which this summary was generated can be found in Appendix G.

After coding Robert's goals, statements of his goals for student learning ranged from levels 1 to 5. Based on field notes and classroom observations, Robert made pedagogical moves to model student thinking and he made decisions to act on his model of student thinking either at the group level or in a whole class discussion in a majority of class sessions, with varying levels of success. Robert's pedagogical moves in more successful interactions initially suggested that he was mindful of student thinking and that he had thought of ways to support student thinking during the planning process. This implied that goals rated at TGSL6 were accessible to Robert, however he did not state such goals explicitly, and therefore they were not coded. Goals rated at a TGSL6 level only emerged during the process of follow-up questions. In a few instances, Robert's

responses and his pedagogical moves in the subsequent lesson suggested reflection on how student thinking about the mathematics of the lesson could be promoted or developed. I will elaborate on this finding in greater detail when discussing the results of the second research question.

Discussion

Regardless of the curriculum type used, Robert's goal of having students think about the mathematics in the lesson (TGSL4) was prominent. When using the skill-based curriculum, articulating specific methods of mathematics Robert wanted his students to use (TGSL3) topped the types of goals that Robert stated. Goals stating specific actions (TGSL2) in support of mathematical topics were common when Robert used the conceptually rich curriculum. Pintrich (2000) found that "strong" curricular or classroom contexts influenced the types of goals teachers would normally access, so the findings of this study might not be entirely surprising. However, the results may be viewed as surprising if the goal framework is thought of as a trajectory of teacher growth representing a teacher's developing MKT. From this perspective, Robert's goals would seem to show that his mathematical knowledge for teaching was more well-connected when using a skill-based curriculum than when using a conceptually rich curriculum, since from a hierarchical perspective, goals ranked at TGSL3 are higher than TGSL2. However such an analysis would not be appropriate given that we are comparing two different curricula that are supportive of contrasting teaching orientations. A skill-based curriculum and the activities found in it are supportive of a calculational orientation, so goal statements focusing on methods of mathematics a teacher wants his or her students

to perform are representative of a different view of the mathematics than goal statements focusing on methods of mathematics viewed from a conceptual orientation.

I do suggest the goal framework can be thought of as representing stages of growth of a teacher's goals supportive of student learning. How these goals manifest themselves with teachers having a calculational orientation, versus how these goals manifest themselves with teachers having a conceptual orientation, however would need further study. Also, recalling Pintrich (2000) findings, the lesson itself can influence a teacher's goals for student learning.

Retrospective Analysis

Although it may seem that the different curricula Robert used promoted different types of goals for student learning, a retrospective analysis of Robert's overarching goals revealed that the results of this and the distributions that were determined by Robert's stated goals might have been foreshadowed by his response in the first questionnaire given on February 5th (Appendix E). In it, one of the questions I asked was "Are these goals [for student learning] affected by the type of lesson you have-for example, a conceptual versus skill based lesson? If yes, how are they affected? If not, how are not they are not affected?" Robert's response was the following.

The over-arching goal of improving student understanding remains for any lesson, regardless of the emphasis of skills vs. concepts. However, the trajectory and/or delivery method of the lesson can be affected. I visualize a concept-based lesson as having student investigation as a major portion of the activities, while a skill-based lesson is still focused on "why" certain procedures are done but there is more direct instruction of those procedures.

Reflecting on Robert's response, improving student understanding was consistent with goals that promoted students' thinking about the mathematics in the lesson (TGSL4)

and the “why”, regardless of curriculum type used. Robert’s answer to his overarching goals for student learning in a conceptual lesson was having investigations as a major part, which suggests goals promoting specific actions (TSGL2), while overarching goals of skill-based lessons focused on procedures and methods of mathematics (TGSL3). Looking back at the findings from the study, Robert’s lesson specific goals clustered around TSGL4 and TSGL2 in the conceptual curriculum, while in the skill-based curriculum the goals TGSL3 and TGSL4 were most prominent. The fact that Robert’s overarching goals were strongly predictive of his lesson specific goals is an important finding, especially in light of findings discussed in relation to the second research question (the effect of a researcher’s interventions on a teacher’s goals).

Another surprising finding was that Robert’s initially stated goals were never above TGSL5. Although the goal framework contained goals rated at TGSL6 based on other researchers’ findings, these goals did not emerge as a response to the initial question “What were your goals of instruction with regards to student learning for the lesson you had today?” Robert’s pedagogical moves made to support the development of his students’ ways of thinking about mathematics occurred during classroom interactions, indicating unstated goals rated at TGSL6. The rationale behind why the higher goals did not initially emerge as stated goals became clearer during the investigation of the second research question.

Research Question 2

The second research question had asked: “How might the goal framework be used as a tool for professional growth, and how stable or fragile are a teacher’s goals in the context of perturbations created by a researcher to encourage self-reflection?” In this

section I will describe the methods and discuss the results to the second research question.

Methods of Analysis

The purpose of the second research question was to gain insight into how the goal framework be used as a tool for professional growth, and how stable or fragile were a teacher's goals in the context of perturbations created by a researcher to encourage self-reflection. In particular, I used the follow-up questions during the time period of April 11th to May 7th as opportunities to attempt to perturb Robert's goals to levels higher in the framework that I viewed as accessible to him.

To give an example on how the framework was used in an attempt to perturb Robert's goals, I will use the lesson on identities (for additional detail, please see Appendix E for his response to the 4/11/2013 questionnaire). One of Robert's goals for student learning that day was "Students should understand that an identity is an equation that is true for any value of the input." This goal was rated at a TGSL5 level, since a desired way of thinking was described. However, I had observed that Robert's lesson on trigonometric identities focused on algebraic methods only, indicative of a lack of attention to how students might come to think about this concept, even though a desired way of student thinking about identities was stated clearly in his goal for student learning (rated TGSL5). In a move to push Robert to a TGSL6 level (a goal supporting attention to how student thinking may develop), I asked the following question.

In today's goals for students, you mention "Students should understand that an identity is an equation that is true for any value of the input". In what ways might such an understanding of identities be promoted or developed?

Robert's response was "One way would be to have students check the truth of the statement for specific values of the input. A good way to do that would be to use the graphing calculator. Either the graph or the table of values should do the trick." Robert's response indicated that he had the key developmental understandings to think about ways he could support the development of a mathematical idea in his students, but planning for ways to support or promote ways of student thinking had not been originally part of Robert's goals for student learning in the lesson for that day. However this process of reflection carried the potential to transform Robert's key developmental understandings into MKT.

Results

Robert's pedagogical moves the following class session indicated further reflection on this follow-up question, in that he began using multiple representations to support student thinking about identities. Although Robert's stated goals for student learning that day with regards to identities did not change in terms of its rated level, his unstated goals changed towards supporting student thinking, as noted from his shifts in classroom practice.

After a couple of sessions in which follow-up questions became a new norm associated with questionnaire, Robert had commented at the end of April 22nd questionnaire about uncertainty in differentiating between teaching goals and goals for student learning.

I'm having a harder and harder time differentiating between questions 1 and 2. When I'm preparing, I'm mostly focused on what I want students to learn. Then I adjust my teaching based on where they are and what they need to learn. So I feel like I don't have much in the way of teaching goals. My goal as a teacher is for students to meet their learning goals.

Question 1 had asked “What were your teaching goals for the lesson you had today?” while the second question asked “What were your goals of instruction with regards to student learning for the lesson you had today?” Robert’s comment appeared after two classroom observations in which I had made moves to perturb Robert’s stated goals to higher levels in the framework via follow-up questions. I had asked for additional explanation regarding Robert’s comment during the post-study interview (for further detail, please see Appendix F), thinking they highlighted a process of mental reorganization occurring in his goals at the time, since in the following classroom observation, Robert gave a qualitatively different response to the question “What were your teaching goals for the lesson you had today?” The exploration of the qualitatively different response (on April 23rd) will occur later in this analysis. Robert response to my request for clarification on the comment as the end of the questionnaire on April 22nd follows. In the excerpt, Rob stands for Robert, while Res represents the researcher (me).

Excerpt 3. Robert’s interpretation of questionnaire

- 1 Rob: Well obviously, the questions are connected. I mean what I was
2 thinking when I wrote that was that, my goal in general for teaching is
3 that my students learn. So I mean I could write down the goals for
4 student learning, and then my teaching goals would basically be to
5 help the students reach those goals, those learning goals.
6 Res: Right.
7 Rob: So I was having a, I guess I was having a hard time deciding what to
8 write down for teaching goals, and it seemed like it was boiling down
9 to what was the topic that I was going to investigate that day.
10 Res: So that was how you were interpreting the question, right?
11 Rob: Yeah.
12 Res: So you were interpreting what are your teaching goals as what is the
13 topic?
14 Rob: I think so, because I found that I wanted to write down exactly the
15 same things I was writing for student goals. They didn’t, they weren’t
16 different enough for me, I guess.

- 17 Res: Uh huh. Now with the goals for students, were you also writing
18 topics, or were they a little bit different? Your goals for students?
19 Rob: No, I was trying to write them in terms of what I wanted students to
20 be able to do or understand.

In restating his response (lines 1-5), it became apparent that Robert first thought about his goals for student learning, followed by crafting the teaching goals to support the student learning. The original intention in the design of the questionnaire which was to have the first question convey Robert's overarching goals of instruction in relation to the lesson, in a sense like the overarching goals Dorothy and Margaret had originally stated they had for any lesson. However it was not interpreted in the way I had intended, rather Robert interpreted the question as what topics was he planning to teach that day (lines 7-9, 11, 13) which was confirmed by me (lines 12-13). As for his interpretation of the second question, he interpreted it to mean what he wanted students to do or understand (lines 19-20). Looking at the goals framework (Table 1) such an interpretation would restrict possible stated goals between TGSL1 (goals with vaguely stated overarching action) up to TGSL5 (goals indicating specific ways a teacher wants students to think about the mathematics). His interpretation of the question did not allow for any goals stated at a higher level beyond level 5, and may be a root cause why goals levels rated at TGSL6 were not present in the coded data.

Goals rated at TGSL6 emerged through follow-up questions to stimulate reflection, or were evident based on Robert's pedagogical moves made to support the development of his students' ways of thinking about mathematics. However, the follow-up questions did not perturb Robert's responses to the initial question at any time during this study. This is an important finding that informed the methods used in the dissertation

study, since this interpretation of what goals for student learning means was not unique to Robert.

Discussion

Many school districts in Canada and the United States, specifically Arizona, have incorporated rubrics to assess a teacher's goals for student learning as part of the protocol used in teacher evaluation. For example, teachers must write or post each day's learning goals on the wall or board. It is required so that students, parents, administrators, and evaluators who visit can view them. In assessing a teacher's learning goals, the following is stated as a rubric in Canada: "learning goals clearly identify what students are expected to know and be able to do, in language that students can readily understand." (Ontario Ministry of Education, 2010, p. 33). This statement is not much different from how Robert's interpreted the first two questions regarding his goals for teaching and student learning in the questionnaire (lines 19-20). Similar statements regarding learning goals can be found in evaluation rubrics for school districts across the United States, so it should be no surprise that teachers' stated goals for student learning do not mention ways to promote or support student thinking; there are strong societal norms that push back against and undermine such an interpretation. Any future interventions designed to perturb teachers' goals for student learning need to take this finding into account.

Effect of Perturbation or Anomaly

On April 23rd, the day following Robert's comments regarding uncertainty between differentiating between the first and second questions in the questionnaire, Robert's goals for teaching had qualitatively changed, from topic-focused to ways he wanted students to think about the mathematics that day. The first stated goal for teaching

that day was “To get students to think about what would be different when solving an equation when the argument is not just theta” while the second goal for teaching was “To get students to think about how they would verify the solution to an equation using a graph”. During the interview (for further detail, see Appendix F, question 8), I mentioned to Robert that his response to goals for teaching on April 23rd were different than in the past. When Robert initially read his response, he originally commented that it sounded like reflective questions for the students. The excerpt that follows continues this thread of conversation between Robert and myself.

Excerpt 4. A teaching goal perturbed?

- 1 Rob: Well it was the second, it was at least the second day on the topic I
2 think. So we had already talked about solving equations, and then
3 we’re just going to make a little bit of a change.
4 Res: Uh huh.
5 Rob: It’s what the equation looked like. Umm. [pause] So I guess I was
6 hoping that they could [pause]. I don’t know, I was hoping that they
7 could, think about those things, in relation to what they had done the
8 previous day.
9 Res: Right.
10 Rob: Since it, you know, wasn’t so brand new, it was related to what we
11 had already done.
12 Res: So you were thinking, because your original statement was it sounded
13 more like a reflection question, so your goal for that day was related
14 to getting students to reflect? Or? Because it’s different.
15 Rob: I was hoping they could use reflection, maybe. Maybe they are not
16 reflection, I don’t know. Somebody else could tell if these are
17 reflective questions or not. But I was hoping they, could sort of use
18 that thinking to help them.
19 Res: So your focus, you’re focusing more helping students have a certain
20 way of thinking in order to do the mathematics.
21 Rob: Yeah. Or to just think that, or simply think that, oh something about
22 this situation is different.
23 Res: Okay.
24 Rob: Now that it’s different, how am I going to, how is my strategy in
25 answering the question going to change, based on how it’s different
26 now.

Robert's most specific comments regarding his goals (lines 6-8, 24-26) indicated that he wanted students to think about what they had done on the day before, reflect, and leverage this prior knowledge to solve different types of trigonometric equations than they had not seen before. The way Robert had stated his goal (lines 6-8) suggests that he wanted students to use a prior way of thinking to build a more general way of thinking about solving trigonometric questions. The excerpt suggests Robert was oriented towards having students use their prior ways of thinking about trigonometric equations in this lesson since his goals and his comments (lines 17-18) indicate that intention. Robert expressed uncertainty (lines 15-17) whether such goals are student reflection questions or not. In a way yes- students need to reflect on a prior way of thinking in order to make generalizations needed to make sense of the mathematics in this new context. What was unique about Robert's goals for teaching that day was this was the first (and only) lesson in which Robert explicitly mentioned using prior student thinking to build new knowledge. When Robert's teaching goals on April 23rd are taken together with his goals for student learning, the statements taken together would have yielded a stated goal for student learning rated at TGSL6, which is a significant finding.

However, the other aspect of this finding was the fragility of the perturbation of Robert's goals. The following day (April 24th), the rated levels of teaching goals had reverted back as the next topic to introduce. So it is possible the effects of perturbation are limited to contexts in which the lesson was part of a longer trajectory spanning more than one day, in which case perceived opportunities for student reflection opened possibilities to consider the leveraging of prior student thinking as stated goals. However, this conjecture would need further investigation.

Research Question 3

The third research question had asked: “How does the planning process change as a teacher’s mathematical knowledge for teaching grows when using a conceptually rich curriculum?” In this section I will describe the findings with regards to the third research question.

Robert’s Planning

To gain insight into the third research question, working with Robert gave an excellent opportunity to shed light into Robert’s planning process and how it had changed as he grew more experienced in teaching with a conceptual curriculum. Teacher planning lies at the heart of a teacher’s goals, and the literature review had indicated that the written artifacts of planning did not necessarily convey a full representation of a teacher’s planning; much is unstated. The excerpt that follows relates to Robert’s planning over the first, second, and third years when teaching with the conceptually rich curriculum he used; the reader may recall a short portion of this excerpt in the Introduction chapter.

Excerpt 5. Robert’s planning when using a conceptually rich curriculum

- 1 Res: How has your planning for class changed first, second, and third
2 times you’ve taught Precalculus, using the curriculum?
3 Rob: The first year was a lot of reading the instructor notes, and notes in
4 the PowerPoint to make sure that I knew all of the mathematical ideas
5 that you guys thought were important for the particular worksheet or
6 investigation. So that if things did come up, I could address them, and
7 also to know where everything was going.
8 Res: Uh huh.
9 Rob: So that if there was a particular mathematical idea that was important
10 for the future, that, you know that was, that was stressed. Umm, I’m
11 sorry, what was the question?
12 Res: The second, the second time, and what, how you currently plan.
13 Rob: Umm, second time, I don’t know, it was, it was still a little bit of, uh,
14 looking through the teacher notes. This year I haven’t really spent any
15 time at all looking through the instructor notes. I think because I feel
16 by now that I know, that I should know what’s important.

17 Res: Uh huh.
18 Rob: Already having done that before.
19 Res: So you glance at it, it comes back and you're good to go.
20 Rob: I agree. So sometimes working, working through the problems, there
21 may be a few sections here and there, where I will make sure, you
22 know, do that worksheet again. Just to make sure that as the kids go
23 through it, I know what they're going to run into.
24 Res: Can you elaborate on that?
25 Rob: Well sometimes there's a little you know, tweak in the problem,
26 where I know that students, you know once I do it, I can see that
27 students might have difficulty there.
28 Res: Okay.
29 Rob: So I want to make sure that I address that in my own mind before,
30 before the kids get there.

Robert's comments (lines 3-7) regarding the first year are typical of experiences and comments made by other teachers using the conceptual curriculum provided by the *Pathways* project (Carlson & Oehrtman, 2012) for the first time. Robert's focus in the first year was on developing key developmental understandings of the mathematics he taught (lines 3-6), being able to address student misconceptions or difficulties as they came up (lines 6-7), and to understand how mathematical ideas being developed in the current lesson connected to future mathematics that students would learn (lines 7, 10-11). To note, I took the meaning of "things" to mean 'student misconceptions or difficulties'. This meaning would be consistent with his later comments in the excerpt (lines 25-27, 29-30). In the particular curriculum Robert used, the instructor notes contained a summarization of the key mathematical ideas being developed in the lesson, possible ways students might engage the activities within an investigation, and possible misconceptions they might have.

In the second year Robert used the conceptually rich curriculum, he did not clearly recall how he planned for class. He mentioned that during the second year he still

looked at the instructor notes a “little bit” (lines 13-14). There was no mention of any other thought processes, reworking problems, or reflections on prior experiences with the lesson from the year before in planning for class; although this may have occurred, Robert did not mention it. Robert did not appear to clearly remember the second year; so I made the decision not to press this point.

In the third year, which was the year in which Robert was observed, he did not need to prepare by reading through the instructor notes (lines 14-16, 18) because he felt he knew the mathematics he was teaching, having taught it before. Classroom observations also supported that Robert’s mathematical knowledge for teaching was relatively strong. This contributed to him being confident in his teaching and him only glancing at the instructor notes (lines 19-20) prior to teaching, although he did mention that he occasionally reworked problems to remind himself where students might experience difficulties or have misconceptions (lines 20-23, 25-27, 29-30). In the case of Robert, the most significant advancement of his mathematical content knowledge relative to teaching the ideas of Precalculus appeared to happen during the first and second years of him teaching with the conceptual curriculum. His lesson planning processes was more about remembering prior ways a lesson could unfold rather than reflecting on new ways to build on the lesson, such as by attending to how student thinking about the concept could be promoted or developed.

Although this finding is not surprising based on the literature, it matters because based on the theoretical framework, new KDUs (that have the potential to transform into MKT) occur only during the process of teacher reflection. I observed that absence of teacher reflection on practice implies no or little opportunities for teacher professional

growth (in particular in MKT and shifts in teacher's goals for student learning). A compelling reason that is meaningful to a teacher who has well-connected understandings of the mathematics could create the perturbation and provide an opportunity to reflect back on prior lessons in new ways.

Opportunities for Professional Growth

Robert's stated goals for student learning, his resistance to perturbation of these stated goals, and his limited planning for class in his third year teaching from a conceptual curriculum (basically, to remember where students may encounter difficulties or have misconceptions in a lesson) suggest an inattention to specific goals on ways students may come to think of a mathematical idea during the lesson planning and post-lesson reflection stages. Reflective thinking about how students might come to learn a concept did not appear to be of Robert's habit of mind, although the possibility exists that he did reflect on ways students might come to learn a concept even though he did not mention this during the interview. His profound MKT allowed him to model and adapt a lesson based on student thinking in real time. However, not planning for how students may come to think of a concept appeared to diminish the potential for Robert to support student thinking or to help students promote connections between divergent ways of student thinking. For example, during the course of the post-study interview, Robert had mentioned one of his roles as a teacher is as the expert to guide them to a way of thinking when there is too much student disagreement. The disagreements may have resulted by not carefully attending to how the thinking may have developed in the first place.

Also, an open question is whether planning for ways students might think about the mathematics (aligning with goals ranked at TGSL6) could lead to different

pedagogical moves in which connections are found between the divergent approaches as a way to resolve disagreement rather than settling on a specific way of thinking, or misconceptions are identified earlier and leveraged into productive ways of thinking about the mathematics. In other words, disagreements might be settled by helping students make connections between their divergent approaches; helping students build capacity to reflect on their own thinking (from day one of class) through sustained effort may facilitate the process of developing a disposition to work through misconceptions. The lack of any stated goal for student learning related to having student reflect on their mathematics was not surprising based on classroom observations, but noteworthy given that the lack of student reflection in class troubled Robert. He mentioned being troubled by the lack of student reflection in class on several occasions.

I hypothesized that a consistent lack of teaching goals regarding promoting student reflection, and absence of lesson specific goals of student learning that incorporated student reflection, impeded development of a student disposition towards reflective thinking and autonomous learning in class. Robert's comments regarding student reflection merit discussion, since it helps identify both challenges and opportunities in affecting teacher change through professional development. In other professional development workshops I have either attended or facilitated, other teachers' comments about the lack of student reflection in the class have mirrored that of Robert.

The excerpt that follows is in response to a question during the post-study interview (for further detail, see Appendix F, question 10) in which I asked Robert his thoughts regarding what students are thinking about when they are not paying attention or not reflecting in class.

Excerpt 6. Robert's comments regarding student reflection

- 1 Rob: I think part of it, either they're not reflecting, or they don't know how
2 to reflect. [pause] Umm, you know you, so you can lead a horse to
3 water but you can't make them drink.
- 4 Res: Uh huh.
- 5 Rob: Alright so you can, you can have great worksheets and great
6 investigations, but if the, if at some level the student doesn't commit
7 to try to understand what was important about that investigation, then
8 it's not going to stick, no matter how good my questions are, or how
9 much time we spend talking about it. The student has got to do
10 something.
- 11 Res: Uh huh. So, umm, are most students disinclined to reflect, or?
- 12 Rob: Yeah. I don't think it's uh, I don't think it's something they're really
13 trained to do very well. Umm, hopefully we're moving towards that.
14 But if Pathways is the first time that they, that it becomes really
15 important, um, then it's, it takes a while, they're not used to that sort
16 of thing. They want to, get the answer and move on.
- 17 Res: Uh huh.
- 18 Rob: So here's the next problem, you get the answer so I can move on to
19 the next problem. It's all taking time to, think about whether or not
20 what they did makes sense, or how it's, how what they did is related
21 to other worksheets they did earlier in the week.

The comments in this excerpt, and the classroom observations that substantiated students' disinclination to reflect, lead to three insights that influenced the design of my dissertation study. Robert stated that students are not trained or do not know how to reflect (lines 1-2, 12-14), saying that the *Pathways* curriculum may have been the first time they engaged in reflection (lines 14-16). If it was the first time, he noted that it took time for students to develop a disposition towards reflection (lines 14-15). If the teacher values student reflection, then active attention towards developing this disposition to reflect means that it should become an over-arching goal. Promoting a student's capacity to reflect on their reasonableness of their own mathematical thinking and responses is supportive of autonomous learning. Since such a stated goal was not observed in this study nor in the prior study with Dorothy and Margaret, it is likely that this type of goal

for teaching that needs to be nurtured during professional development, since Robert, Dorothy, and Margaret are representative of teachers at different stages of professional growth in the *Pathways* project. Perturbing teachers' prior goal structures to value and promote student reflection comes with the realization that for teachers new to teaching conceptually, it may not yet be an accessible goal, since planning for meaningful reflection requires well-connected understandings of the mathematics that is being reflected on. In saying meaningful reflection, this goes beyond telling students to check the answer to verify a solution is correct. Examples of meaningful reflection include: ascertaining whether the solution found is reasonable or not, justifying why a solution is correct, looking for how the completed activity is both similar and different from prior mathematical concepts learned, and thinking of ways to generalize the situation to see what is different and what stays the same. In a professional development intervention designed to help develop teachers' goals to promote student reflection, it would only be meaningful to teachers who have experience teaching with a conceptual curriculum; well-connected KDUs need to be in place first.

Second, Robert stated that students do not reflect because they want to get the answer and move on (lines 16, 18-19), because it takes students time to reflect on meaning and make connections (lines 19-21). While it is true that in our current state of education in the United States, most students' prior educational experiences will have promoted a calculational orientation in their perception of what mathematics is about over a number of years, it does not mean that Robert believes such views are immutable. As Robert had noted (lines 14-16) it takes time for a disposition for reflection to develop. Allowing time for teachers to share ideas and to develop strategies to help students

develop the disposition to reflect and promote meaningful student reflection during professional development may help in this endeavor. This also would provide a meaningful opportunity to develop goals that build on student thinking (TGSL6).

Linked to student reflection is making conjectures, since prior to asking students whether the solution is reasonable or not, it might make sense for students to make conjectures about the solution, what the solution should look like, and what solutions what be deemed unreasonable (for example, values that are too high or too low). Making conjectures may also help in promoting a disposition to reflect, since part of determining reasonableness is reflecting back on the conjecture the student made in the first place; it might be intrinsically motivating for a student to see how close their initial estimate was to the solution they found, and think back to the assumptions they initially made. Giving students opportunities to conjecture, and using time in professional development to plan for such conjectures, could give teachers opportunities to develop goals that would build on initial student thinking (TGSL6).

Third, Robert expressed the belief that some students are not intrinsically motivated to make meaning or reflect (lines 2-3, 5-10) no matter what he does to promote meaning making in class. While it may be true for some students whose affective reaction to mathematics from past experiences has led to a belief that success in mathematics relies on accuracy, performance, and memorization of procedures (Goldin, 2002), Robert had not contemplated or set goals about how student conjectures and reflection may be promoted in a lesson. Extrinsic motivations such as points credited through a rubric stating what is expected during a problem solving activity may be one approach that may work for such students until their disposition towards making meaning and reflecting shift.

For example, in another project that I have been a member of, the National Science Foundation Grant No. 1103080 (Arizona Mathematics Partnership Project), one of the participating middle school teachers shared with other colleagues and the project members her problem solving rubric that she adapted through an iterative process that she used with students in conjunction with problem solving activities (Vicich, 2015). On a single sheet template that students used, there was space allocated and point values listed for the given/goals of the problem, the student's conjecture, a plan for solving (what strategies will students use), the solution (showing all work and labeling), verification (checking answer and explaining why correct), and the answer (written in a complete sentence). The rubric the teacher used established a norm of what she expected of students when solving a problem. What is interesting is that the conjecture and an aspect of student reflection (the verification) were incorporated into every problem solving activity her students are asked to do. Over time, students developed a disposition to make conjectures and reflect on their solutions relative to the rubric. Her rubric created the scaffolding needed to promote such a disposition, and if students like hers end up in classes like Robert's, they had been trained to reflect in some capacity beforehand. If classes similar to the type Robert teaches are the first time students engage in meaning making activities, making conjectures, and reflecting on solution process, an approach similar to this teacher's rubric may be a viable option to consider.

Summary of Thoughts and Conjectures

I conjectured that Robert had well-connected KDUs that were leveraged in pedagogically powerful ways, but that his MKT had limitations in relation to the conjecture (preflective) and reflective stages in problem solving activities. My

observations of other teachers suggested that Robert was not unique in having this disposition. The idea of prefection, meaning ‘to look ahead’, was a construct that comes from Mason (2002, p. 84) and was used as part of the neologism prefection-flection-reflection, with flection being thought of as noticing in the moment. The concept of prefection is interesting in context of a teacher’s sensitivities to notice students’ mathematical behaviors. I think of prefection, in the context of noticing, as sharpening sensitivities to notice what may happen in the future based on what is currently is happening. An example of prefection may be as follows: making a conjecture of how students will come to think of a concept based on the way the current interaction is unfolding, and then marking the interaction along with the way of thinking that emerged as a result of the interaction; the event is recalled during reflection to inform future conjectures. Developing a disposition to prefect may help in promoting goals for student learning that state ways in which students might come to think of a mathematical idea (TGSL6).

How This Study Informed the Dissertation Study

I hypothesized that planning for how students come to think of concept (from the teacher part) and making conjectures about a possible solution (from the student part) could describe a reciprocal relationship in which both parties (teacher and student) developed the disposition to look ahead. I believed a focus on this aspect of planning was at the core of the perturbation needed to impact goals and classroom practice. By creating an opportunity and rationale to promote goals for student learning that attended to how student thinking may develop, this intervention by a researcher might promote change in a classroom culture so that student autonomy and a disposition to reflect have an

opportunity to grow. As evidenced by Robert, a teacher having well connected understandings of the mathematics they taught had a potential to develop goals focused around student thinking. The ideal candidate would be a teacher who taught with a conceptual curriculum before and had sufficient opportunities to build their network of meanings in order to leverage these connections in pedagogically powerful ways. The ideal candidate would be open to professional growth through different means. For example, researchers (Jacobs, et al., 2010) had used the construct of noticing for teachers reflecting on videos of their own practice as a model for professional growth. Another model for professional growth would be through collaborative lesson planning, and building upon prior work of researchers (M. S. Smith, et al., 2008; Stein, Engle, Smith, & Hughes, 2008; P. W. Thompson, 2009) in lesson planning.

While classroom observations ascertained that Robert's pedagogical actions were supportive of student thinking and reasoning, one limitation of the second study was that thought revealing activities (Lesh, et al., 2000; Zawojewski, et al., 2006) that would reveal Robert's ways of thinking were not used. These could have been developed (pre-study and post-study) and used as artifacts to better characterize Robert's key developmental understandings of the mathematics he taught. The second study had also revealed that that word "goal" had specific meaning for Robert (see Excerpt 3, page 69), which depended on context (goals for teaching versus goals for student learning). His interpretation of the word "goal" limited the scope of his response and subsequent ranking in the goal framework. The findings of the second exploratory study, as well as the findings of the first exploratory study, informed the methods that were used in this

dissertation study. In chapter that follows, I will discuss the methods used for the dissertation study.

CHAPTER 5: METHODS

Research (Marfai & Carlson, 2012; Moore, et al., 2011) had shown that a teacher's mathematical knowledge for teaching impacts his or her pedagogical actions and goals. Other researchers (Webb, 2011) had found that teachers' goals had influenced the development of content knowledge that was pedagogically powerful, which was characterized in this study as a teacher's mathematical knowledge for teaching (MKT). The prior preliminary studies with Dorothy, Margaret, and Robert had shown relationships existed between a teacher's key developmental understandings, mathematical knowledge for teaching, teaching orientation, and their goals. However these studies focused on particular aspects of this relationship, and a study was needed to characterize these interactions as a whole that built on these initial findings. The second exploratory study had revealed specific interpretations of the word "goal" that constrained one teacher's development of higher ranked stated goals in the framework, even though they were accessible to the teacher upon further questioning and follow-up. However this finding is not surprising based on the literature of the meaning of goals (Locke & Latham, 2002, 2006; Norman, 2002; Pintrich, 2000) and supporting evidence of moves made by school districts and ministries (e.g., Chandler Unified School District, 2013, 2015; Ontario Ministry of Education, 2010) to adopt specific interpretations of learning goals, especially with recommended teacher actions (Marzano, et al., 2013, p. 25), such as posting their goals for student learning for a lesson in a place that is clearly visible to students (or the evaluator). Findings from the second preliminary study showed that planning for a lesson was minimal by the third time a teacher taught with a lesson, meaning that planning an individual lesson was more a form of remembering a

previously taught lesson, rather than a reflection on ways to think about and improve a previously taught lesson with an attention towards student thinking. This study sought to gain further insight into characterizing this reciprocal relationship between a teacher's mathematical knowledge for teaching and his or her pedagogical goals to better understand the effects of a professional development intervention designed to impact a teacher's goals. The study also sought to better understand how a teacher's views of mathematics and their mathematical teaching impacted his or her goals. My research began with me developing a framework to characterize teacher goals. Based on theoretical constructs contributed by other researchers (Locke & Latham, 2002; Pintrich, 2000; Silverman & Thompson, 2008; Simon, 2006; A. G. Thompson, et al., 1994), I created a model that described how these key constructs might interact with a teacher's goals. This framework of interactions was described earlier (Chapter 2, p. 15). With the framework of interactions in mind and the goals framework developed in the exploratory studies, I formulated my research questions.

Research Questions

The research questions that framed this dissertation study were as follows.

1. What is the effect of a professional development intervention, designed to perturb a teacher's pedagogical goals for student learning to be more attentive to students' thinking and learning, on a teacher's views of teaching, stated goals for student learning, and overarching goals for students' success in mathematics?
2. What role does a teacher's mathematical teaching orientation and mathematical knowledge for teaching have on a teacher's stated and overarching goals for student learning?

The sections that follow will outline the rationale used to select the participant in this study, the methods used to conduct the study, and methods used to analysis the results.

Carolyn

This dissertation focused on Carolyn (pseudonym), from Atlas High School (pseudonym) in a Southwestern state, during the Fall 2014 semester (September through November). Carolyn had previously taught Precalculus twice using the same conceptually rich curriculum provided by the project (Carlson, et al., 2013b), but in a pre-study conversation, she mentioned that she also used materials from lessons that she created. The observations occurred during the instruction of two chapters from the *Pathways* curriculum: *Module 4: Exponential and Logarithmic Functions*, and *Module 5: Polynomial and Power Functions*.

Carolyn was recruited to participate in this study because her background was similar to that of Robert from the exploratory study. Based on data from the Beliefs survey, Carolyn had been teaching high school for over 16 years, but based on personal conversations, she has been teaching at Atlas High School over 20 years. During the period of the study Carolyn taught Calculus AB, Honors Precalculus, and Integrated Math. The study occurred during her second to last academic year of teaching, as she was planning to retire at the end of the following academic year. The characteristics of Carolyn's background in terms of years teaching, the types of courses she was teaching, her years teaching with a conceptual curriculum, and use of conceptual curriculum combined with lessons from a skills based curriculum were similar to that of Robert. As a result, she was selected as a participant for this study.

What also made Carolyn an ideal candidate is that she was willing to participate in a study that was framed to her as an opportunity to grow professionally through a lesson planning collaboration. This collaboration offered the researcher an opportunity to test the theory that lesson planning in the context of questions that cause the participant to reflect on the mathematics of the lesson and how students might think about the key ideas of the lesson would shift a participant's mathematical goals for student learning toward higher level goals in the framework over the course of the study. A lesson planning protocol informed by the findings of the professional development workshop for Algebra 1 teachers (see Appendix D), the second preliminary study with Robert, and researchers' prior work (M. S. Smith, et al., 2008; P. W. Thompson, 2009) was developed to guide the lesson planning process.

Methods

The purpose of this study was to characterize a teacher's goals and mathematics in the context of my proposed interventions to shift a teacher's goals and classroom practices to be more aligned with the constructivist philosophy of learning (Glaserfeld, 1995; Piaget, 1970; P. W. Thompson, 2000). Two exploratory studies were used to create a framework to characterize a teacher's mathematical goals for student learning, and these were discussed in the prior chapter. A case study was then used to study the effect of a professional developmental intervention designed to impact a teacher's mathematical goals.

Based on the data collected from the exploratory studies, these had shown that teachers' goals for student learning mostly focused around topics to cover, actions that they wanted their students to perform, methods of mathematics they wanted students to

focus on, and/or ways they wanted students to think about mathematics. Goals for supporting or promoting ways of student thinking about mathematics were not explicitly stated in these studies, although my prior results showed instances where a teacher's MKT allowed for these goals to be accessible. However, these types of goals only emerged after following up with specific questions that would highlight these goals.

The premise of this study was that lesson planning needed to be re-conceptualized in order to provide a catalyst for teacher growth. This hypothesis was grounded in prior work by researchers on professional support tools (M. S. Smith, et al., 2008; M. S. Smith & Stein, 2011; P. W. Thompson, 2009) and from the findings of my preliminary study with Robert. Planning, as teachers had conceptualized it, was simply a form of remembering when it came to lessons previously taught, with recollection aided by looking at a prior lesson plan or notes used, rather than as a reflective activity used to continuously improve a taught lesson. I designed a professional development intervention focused on lesson planning. The intervention was designed to perturb and shift a teacher's goals toward greater attention on students' ways of thinking when planning and teaching mathematics. I conducted convergent clinical interviews (Clement, 2000) with Carolyn in both the lesson planning and debriefing sessions. While the teaching experiment methodology (Steffe & Thompson, 2000) influenced my approach for refining my model of Carolyn's mathematical understandings and her learning goals for students, the lesson planning sessions with Carolyn were lessons that she selected to teach.

Overview

The design of the dissertation study used convergent clinical interviews (Clement, 2000) in the context of a case study. The clinical interview employed clinical interview strategies in which I posed questions for the purpose of shifting her goals to have a greater focus on student thinking. I also employed a think-aloud format for both the lesson planning and lesson debrief portions aimed at modeling and perturbing Carolyn's goals and ways of thinking about the key ideas of the lesson. In these sessions, I planned to make moves to shift Carolyn's mathematical goals for student learning toward the higher goals in my goal framework. The questions used during these planning and debriefing sessions were selected from a pool of questions that appeared on the "Lesson Preflection and Reflection Protocol" and the "Post Class Observation Protocol." These will be explained in greater detail in the sections that follow. For full disclosure, I was the researcher who worked with the participant and who collected the data.

Data was collected from multiple sources: the Beliefs survey, Carolyn's lesson plans, pre-study and post-study interviews, pre-study and post-study tasks, lesson planning and debrief sessions, and through in-class observations. One post-study query was submitted to Carolyn after the study was complete; the question asked was identical to that which was used with Robert.

Written artifacts in this study were obtained through the Beliefs Survey (Appendix C), the pre-study and post-study tasks, and through copies of the lesson plans that Carolyn shared with me for the observed classes that had emerged through the lesson planning sessions. One of the debrief sessions was dedicated to gathering written feedback regarding Carolyn's views of the two protocols. Video and audio data were

collected for the in-class observations and both the pre-study and post-study interviews. Real time screen capture technology (Camtasia) was used to record lesson planning and debrief sessions that were conducted between myself and the participant via online video chat (Skype/Facetime). The software tool F5Transkript (Dresing & Pehl, 2015) was used for both video and audio transcriptions.

With an overview of the methods used in this study described, I will go into each method of data collection in greater detail.

Beliefs Survey

The beliefs survey used in the study with Robert was also administered to the selected participant for the purposes of characterizing her mathematical teaching orientation. The beliefs survey was adapted from prior work by researchers (Carlson, et al., 1999; Carlson & Rasmussen, 2010). In particular, questions 23, 27, 28, and 29 were selected as focal questions, as these questions corresponded to the items used in characterizing the mathematical teaching orientation of the teachers in the preliminary studies and are listed in Table 5.

The survey included two types of response formats. The first part of the survey used a Likert scale in which Carolyn rated her level of agreement or disagreement with a given statement. The second part of the survey used a contrasting alternative scale, in which Carolyn must choose between two alternatives that represent opposite ends of a theoretical construct. The selected items came from the second portion of the survey.

Table 5.

Selected Items from Beliefs Survey

Item number	Item text
23	Student success in my course relies on their ability to (A) solve specific types of problems (B) understand key ideas of the course
27	For my students, making sense of a problem is best accomplished by (A) knowing the sequence of steps to solve the problem (B) knowing the ideas that are the focus of the problem
28	When preparing for class, I spend more time thinking about (A) presenting the material so that students are prepared to complete the problems in the homework and tests (B) how to engage students in making sense of and using the ideas that are the focus of the lesson
29	My teaching focuses more on (A) helping students understand ideas of my courses (B) helping students learn how to work specific problems

The survey was administered in paper and pencil format. An area with space was included at the end of the part that asked Carolyn for comments or feedback regarding any of the questions from that section. Carolyn provided feedback and these responses were *marked* (Mason, 2002) as comments for future investigation in the context of the study. Unusual responses to the items, such as skipping an item or circling more than one answer choice was also used to inform future directions of inquiry. The Beliefs survey and Carolyn's responses to the items are included in Appendix I.

Pre-study Interview and Mathematical Task

The purpose of the pre-study interview was to characterize Carolyn's overarching goals, and her interpretation of goals in different contexts (e.g., personal goals, mathematical goals for student learning, teaching goals, goals for student interactions). The pre-study interview was to determine, in part, whether her goals were influenced by external criteria, such as posting them to satisfy administrative requirements. In addition,

I wanted to characterize how she planned for lessons that were new to her, and also those that she had taught before. The choice of pre-study mathematical task I created was designed to characterize Carolyn's key developmental understandings of quantity, covariation, proportionality, and constant rate of change. An excerpt of the task is included below. The task was inspired from one of questions that had been available in prior homework (Carlson, et al., 2013b, p. 68). The full task with her responses can be found in Appendix I.

Neil and Cameron live on opposite sides of town connected by a long road. They are studying for a test and agree to meet at a library located somewhere between them on the same road.

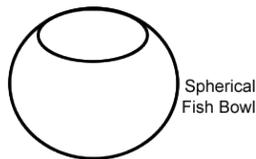
When Neil and Cameron start heading toward each other, Cameron is 5 miles away from the library, while Neil is 6 miles away from the library.

The task was based on content from the prior chapters she had recently taught from the *Pathways* conceptual curriculum (Carlson, et al., 2013b) that she was using with her Precalculus class. To maximize engagement of this thought-eliciting activity, the task was framed as a prototype of a lesson that she would find of potential use in her own classroom. I also wanted to make an initial characterization of how her understandings of the mathematics of this untaught lesson informed her goals and the questions she might ask, specifically relative to student thinking. In short, the purpose of the pre-study interview and associated task was to get an initial snapshot of Carolyn's KDUs, MKT, and goals. Out of necessity, the pre-study interview was broken into two sessions, in order to give Carolyn an opportunity to work on the mathematical task introduced during the interview. The pre-study interview protocol can be found in Appendix J.

Post-study Interview and Mathematical Task

The post-study interview which was conducted and accompanying post-study mathematical task had several purposes. The choice of mathematical task I created was designed to characterize Carolyn's KDUs of quantity, covariation, and constant rate of change in the context of the polynomials chapter she had recently completed teaching from the *Pathways* conceptual curriculum (Carlson, et al., 2013b) that she used during the study. An excerpt of the mathematical task is given below. The task was inspired from one of questions that had been available in prior homework (Carlson, et al., 2013b, p. 201). The full task with Carolyn's responses can be found in Appendix I.

It takes 20 full pitchers of water to fill an empty spherical fish bowl in the diagram below to the top. Suppose that the fish bowl is 10 inches high and can hold 10 quarts of water.



To maximize engagement of this thought-eliciting activity, this lesson was also framed as a prototype of a lesson that she would find of potential use in her own classroom. I also wanted to make a characterization of how her understandings of the mathematics of this untaught lesson informed her goals and the questions she might ask. I was curious to investigate whether participation in this study had any notable effects on Carolyn's KDUs, MKT, or goals. The post-study interview was also designed to give the researcher an opportunity to confirm and follow-up on any significant findings that were observed or noted during the study, but not documented on camera, through questions

that would help highlight these findings. The post-study interview protocol can be found in Appendix J.

Think Aloud Lesson Planning Sessions

Carolyn and I collaborated in nine lesson planning sessions. Although I intended that Carolyn take the lead at the start of a lesson planning session, I introduced perturbations during the lesson planning process using questions from a verbal protocol that I refer to as the *Teacher Preflection and Reflection Protocol*, or the *TPRP*. The list of items that I used as a potential source of questions to characterize and perturb her goals during this lesson collaboration process is given in Table 6 below. The questions listed below represent the version of the protocol that was used for the majority of the study. The final refinement of the protocol was shared with Carolyn during the post-study interview (Appendix K). To Carolyn, the TPRP was referred to as the “Lesson Preflection Questionnaire.”

Table 6.

Teacher Preflection and Reflection Protocol

Item number	Item text
1	What are the primary ideas being developed in this lesson?
2	<ul style="list-style-type: none"> • What are the key ideas of this lesson that you find important? What are 3 other questions you think will be useful to pose to your students as they complete this investigation? <ul style="list-style-type: none"> • What ways of thinking do you hoping emerges from these interactions?
3	What are your mathematical goals for student learning as you plan to teach this lesson? <ul style="list-style-type: none"> • Are there other goals you have for this lesson? • What criteria will you use to see your goals (for student learning, teaching, interacting with students) are achieved in a lesson? • How might the understandings that are suggested by your goals develop or be supported for students?

Item number	Item text
4	<p>What ways of thinking about the key ideas in this lesson do you think will be expressed by students during class?</p> <ul style="list-style-type: none"> • How might these ways of thinking be helpful for students when learning/using related concepts? • Are there possible ways of thinking that may emerge that hinders future learning?
5	<p>How do you plan to help your students make meaningful conjectures?</p> <ul style="list-style-type: none"> • How will this help your students develop their understandings and reasoning abilities associated with the key ideas of the lesson? • How did you plan to incorporate student reflection back to the conjectures they made?
6	<p>Before using this lesson with your students, think about the specific questions that you plan to ask students that will enhance their learning. What questions will you ask to:</p> <ul style="list-style-type: none"> • get your students to make meaningful conjectures related to this lesson • envision the relevant quantities in the situation • probe students in conceptualizing how the quantities in the situation are related • address possible misconceptions • hold students accountable for expressing their meanings • help them reflect on the reasonableness of their responses
7	<p>How will you have students share their solution approach with the class?</p> <ul style="list-style-type: none"> • What criteria will you use to select students? • What tools will students use (document camera, whiteboard, mini-boards, etc.) to share their solutions and thinking?
8	Summarize how you envision this lesson unfolding in your class.
9	What do you plan to do to hold <i>all</i> students in class accountable for expressing his/her thinking and constructing the understandings and reasoning abilities associated with the key ideas of the lesson?
10	<p>Your Reflection (do not fill this in until you've taught the lesson): Write down what you noticed about the lesson or what a student said in class while completing the lesson that you found noteworthy, interesting, or surprising. <i>Do this the same day as you taught the lesson, and use this information when (1) you plan and refine this lesson for the next school year, and (2) to possibly inform you for tomorrow's lesson.</i></p>

This protocol had emerged from earlier *Pathways* workshops. The design of the TPRP was based on ideas from the *Thinking Through a Lesson Protocol* (M. S. Smith, et

al., 2008; M. S. Smith & Stein, 2011) and the *Professional Development Spiral* (P. W. Thompson, 2009). It was also grounded in the theoretical framework of interactions described earlier (Figure 2, p. 15) and informed by the exploratory studies. The TPRP differed from these prior protocols by asking teachers to adapt a conceptually oriented lesson so that student learning was maximized. The second main difference was that questions were placed within the protocol to prompt teachers to consider encouraging student conjecturing and reflection. Other questions in the TPRP were designed to shift Carolyn's goals to higher levels in the framework by having Carolyn reflect on a goal that was slightly higher than her stated goal, following a method that was used to address the second research question in the exploratory study with Robert. Asking such questions during the planning phase of a lesson would do more to perturb her goals than asking the question after a lesson was taught, which was one of the limitations of the study with Robert.

Each planning session using the TPRP was intended to refine the lesson Carolyn was teaching, although the protocol was optimized for lessons taught with a conceptual curriculum (Carlson, et al., 2013b). A teacher's MKT and her goals for student learning influence each other, and the questions in the protocol were designed to have teachers reflect on the mathematics they teach and ways to support students in their thinking about mathematics. Opportunities to think about questions to ask in order to encourage student conjectures and reflection were used to promote further growth of MKT during the lesson planning stage. This thought revealing activity was designed to promote teacher reflection in order to build new KDUs, and carried the potential to transform lesson specific goals for student learning. The purpose of the TPRP was to create a structure that

supported a cycle of reflection and refinement of lessons previously taught, by prompting a teacher to focus on aspects of their students' thinking about key mathematics ideas of the lesson. The protocol was also intended to promote the development of a teacher's disposition to think ahead (to *preflect*, using verbiage borrowed from Mason (2002)) in order to notice ways students may be thinking about the mathematics of the lesson, thereby sharpening sensitivity to how the lesson may unfold. All lesson planning sessions were captured on video via computer screen recording technology (Camtasia).

Classroom Observations

I observed and video recorded eleven lessons from Carolyn's class, of which nine were products of lesson collaboration between Carolyn and myself. The observed lessons are listed in the table that follows.

Table 7.

Lessons Observed

Observed Lesson	Date	Lesson Name	Chapter in Text	Pathways Lesson
1	9/16/14	The Meaning of Exponents	4	Yes
2	9/18/14	Comparing Linear and Exponential Behavior	4	Yes
3	9/19/14	1-unit Growth and Decay Factors, Percent Change, and Initial Values	4	Yes
4	9/23/14	n-unit Growth and Decay Factors, $n \neq 1$	4	Yes
5	9/26/14	Logs	4	No
6	9/30/14	Properties of Logs	4	No
7	10/2/14	Solving Log Equations	4	No
8	10/3/14	Modeling Exponential Growth	4	No
9	10/21/14	Concavity and Average Rate of Change	5	Yes

Observed Lesson	Date	Lesson Name	Chapter in Text	Pathways Lesson
10	10/23/14	Zeros of Quadratic Functions	5	No
11	10/27/14	Polynomial Functions of Higher Degree and Multiplicity of Zeros	5	No

The designation between lessons (Pathways or not) describes whether Carolyn used the conceptual curriculum as the primary source for the lesson (hence the lesson plan consisted of instructor notes from the authors (Carlson, et al., 2013b), with Carolyn’s annotations) or Carolyn created the lesson herself (hence the lesson plan consisted of Carolyn’s own notes based on other sources with examples, with her annotations). In this study, lesson observations occurred across two chapters of the textbook, one pertaining to exponential functions, and the other focusing on polynomial functions. Five of the observed lessons were Pathways lessons, while six of the lessons were non-Pathways lessons.

Lessons 3 and 8 did not have a lesson planning collaboration. This was by design since I made the decision to focus on using the session with Carolyn strictly as a lesson debrief session of the recently taught lesson that day, rather than as a planning session for the next day. I thought reflection of the recently taught lesson taught to be a stronger catalyst for professional growth, based on the Systems of Interactions framework developed earlier (Chapter 2, p. 15).

In addition to the classroom observation video of Carolyn while teaching, field notes of significant moments were noted in which purported goals did not align with classroom practice. These were noted and used as a source of follow-up questions during the debrief session. The following is an example of how non-alignment might be noted.

A teacher may profess the goal of helping support thinking about mathematics (TSGL5) in specific ways, but this teacher may instead make moves of telling students what to do to solve the problem. This is indicative of goals about methods to use (TGSL3) to solve without moves to attend to student reasoning. Other significant moments may occur when a teacher's statements do not match statements made during lesson planning, such as missed opportunities to use conjecture or reflection to build on a student's thought process, or by not asking questions they claimed they would when the situation merited it.

Also I planned to introduce video clips to perturb Carolyn's goals and practice at some time during the study as part of the professional development intervention, for example: contrasting a student-centered lesson in which many students were making meaning of the mathematics and fully engaged, against a teacher-centered lesson in which Carolyn lectured with few students engaged. The intention of this method was to use it as a way to perturb her practice and goals towards to higher level in the framework (toward one that focused more on student thinking). Prior to the start of the study, Carolyn mentioned she used a mixture of conceptual and skill based lessons, so I thought that after a sufficient number of observations, I would have video clips highlighting both type of practice. I conjectured that the construct of noticing used for a teacher reflecting on videos of her own practice would start a conversation that could be used as a model for professional growth, based on finding of researchers (Jacobs, et al., 2010) who had used videos in this capacity with success in other studies.

Lesson Debrief

Following each lesson, a post observation debrief was conducted. Ten in-person sessions were held via online video chat (Skype/Facetime) which were recorded using Camtasia, and one written debrief was conducted via email. The purpose of the in-person sessions was to characterize and clarify observations the researcher made in class, and to learn about Carolyn's perceptions of the class session. The sessions were used to help the researcher learn what effect a particular perturbation introduced during the lesson planning process had on Carolyn's practice, goals, mathematical knowledge for teaching, and key developmental understandings. The sessions were also used as another opportunity to perturb Carolyn's goals.

The purpose of the one written debrief was to have Carolyn reflect on the two protocols introduced during the study. The list of items that I used as a potential source of questions to perturb Carolyn goals and practice during post observation debrief process is given below and referred to as the Post Classroom Observation Protocol. The questions listed in Table 8 represent the version of the protocol that was used for the majority of the study. The final refinement of this protocol was shared with Carolyn during the post-study interview (Appendix K). To Carolyn, this Post Classroom Observation Protocol was referred to as the "Lesson Debrief Questionnaire".

Table 8.

Post Class Observation Protocol

Item number	Item text
1	How you think today's lesson went?
2	Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

Item number	Item text
3	What were your goals for student interactions in the lesson you led today? <ul style="list-style-type: none"> • Are there other goals you had for today's lesson?
4	What were your mathematical goals for student learning for the lesson you led today? <ul style="list-style-type: none"> • What methods did you envision students would use? • What ways of thinking did you hope emerge?
5	How might the understandings that are suggested by your goals develop or be supported for students?
6	How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended? <ul style="list-style-type: none"> • Did you achieve your goals during this lesson? • Which goals are these? • If so, what is your evidence? • If not, why?
7	What are the key ideas of mathematics you felt were important in today's lesson?
8	What pedagogical moves did you make to hold <i>all</i> students in class accountable for expressing their thinking and constructing the understandings and reasoning abilities associated with the key ideas of the lesson?
9	Where in this lesson did you see opportunities for students to make conjectures about a problem statement or activity, a solution to a question, or the appropriate mathematics to use in a task?
10	Where in this lesson did you see opportunities for students to reflect about a question, the solution to question(s), or its relationship to prior activities/other mathematical ideas?
11	Based on your observation of how the lesson unfolded today, how have your plans for tomorrow's lesson changed (or not changed)?

In the article regarding the *Thinking Through a Lesson Protocol* (TTLP), M. S. Smith, et al. (2008) commented that the intent of their protocol was that teachers use it periodically and collaboratively to prepare lessons so that over time, a repertoire of carefully designed lessons is developed. The purpose of the TTLP was to shift how teachers thought of lesson planning without having to explicitly refer to the protocol itself (M. S. Smith, et al., 2008, p. 135). This idea, and the findings from the follow-up study with Robert, guided my development of the Post Classroom Observation Protocol. The protocol

provided another opportunity for Carolyn to reflect on a lesson on the same day that it was taught.

Views on the Two Protocols

Both the “Teacher Preflection and Reflection Protocol” and the “Post Classroom Observation Protocol” went through minor changes as I refined the phrasing of questions. My plan was to share the two protocols during the course of the study as a further move designed to shift Carolyn’s mathematical goals for student learning toward higher goals in the framework. This was done after the class observation of Lesson 8 as part of a written debrief of the lesson that day. When asking for Carolyn’s feedback, I framed it as seeking input to improve both questionnaires that could be used in future lesson collaborations with other teachers in order to maximize her reflection and engagement with both protocols. See Appendix K for Carolyn’s views on the two protocols.

Post-study Query

A post study query was conducted after completion of the dissertation study with Carolyn. The following question was asked via email, once the study was completed.

Describe what it means to you to be an effective teacher. What are characteristics of a successful lesson?

The purpose of the question was to characterize Carolyn’s beliefs about her role as a teacher of mathematics and to better understand how she viewed a successful mathematical lesson. Comparing Carolyn’s responses with Robert’s responses from the exploratory study had the potential to reveal other subtle differences that would make sense from the context of two completed studies.

Methods of Analysis

This section will give an overview of the methods used to analyze the data in this study. These include the methods used to analyze the data from the Beliefs survey, lesson planning sessions, classroom observation sessions, lesson debrief sessions, pre-study interview, post-study interview, and the mathematical tasks used in the interviews.

Beliefs Survey

In this study, Carolyn left comments in the feedback section of the survey and she also answered some items that fit my criteria of unusual. Reviewing questions in which an unusual response was given (either by skipping or selecting more than one choice) and also by looking at Carolyn's written comments, the method I used was to follow-up by looking at her other responses to items on the survey. The criteria used for item selection for follow-up was by looking at a survey items (or items) that described the construct being investigated by the written comment or unusual response to the item.

For quantitative analysis of the four contrasting alternative items characterizing mathematical teaching orientation, I assigned a numerical scale to the responses (1 to 4), and reverse coded one of responses in which the dimensions of the scale's contrasting statements was opposite to the other three items' statements. A numerical mean score was determined for the four items to inform my initial characterization of Carolyn's teaching orientation.

Lesson Planning, Classroom Observations, Lesson Debriefs

The video recordings of all episodes were reviewed and additional notes for each session was typed up that supplemented the original field notes. The additional notes and field notes were reviewed and open coding from grounded theory (Strauss & Corbin,

1990) was used, keeping the key constructs framing the study and the research questions in the background. As categories emerged, I identified key episodes from the data for additional analysis and entered them in an Excel spreadsheet. Portions of these key episodes were then transcribed for future use. Carolyn's stated mathematical goals for student learning were rated using the goals framework that had emerged during the exploratory studies.

Pre-study Interview, Post-study Interview, and Mathematical Tasks

The video recordings of the pre-study and post-study interviews were completely transcribed. Using grounded theory, open coding was used, while keeping the key constructs framing the study and the research questions in mind. As categories emerged, I identified the portions of the videos that addressed the construct or research question for transcription and further analysis. Results from the analysis of either mathematical task that were marked as interesting but not related to the focal research questions of this study were placed in an Appendix (see Appendix O).

I did a conceptual analysis of well-connected meanings (Chapter 3, pp. 21-28) of quantities, covariation, proportionality, and constant rate of change on each task prior to administering the tasks to Carolyn. This informed my approach when analyzing her meanings of the key ideas of mathematics of the task. Furthermore, I analyzed Carolyn's responses to her stated mathematical goals for teaching the lesson described by the task using the goals framework.

Summary

In this chapter both the methods used to conduct the study and the methods used to analyze the study were discussed. Also, the rationale behind the selection of Carolyn was stated. In the next chapter, I will discuss the results of this dissertation study.

CHAPTER 6: RESULTS AND ANALYSIS

The research questions of interest that shaped this dissertation study were as follows.

1. What is the effect of a professional development intervention, designed to perturb a teacher's pedagogical goals for student learning to be more attentive to students' thinking and learning, on a teacher's views of teaching, stated goals for student learning, and overarching goals for students' success in mathematics?
2. What role does a teacher's mathematical teaching orientation and MKT have on a teacher's stated and overarching goals for student learning?

In order to answer these questions, findings from this study that influenced my interpretation of the results will be discussed first. In the first part of this chapter, I will discuss and analyze these findings, and in particular, I will discuss what I learned about Carolyn's beliefs, her overarching goals, and some early insights into her key developmental understandings of the mathematics she taught.

In the excerpts that follow in this chapter, Carolyn is noted as "Car", while the researcher is "Res." I will identify individual students speaking using "Stu" followed by a number. If many students are speaking in unison "StuS" will be used at a class level exchange and "Grp" will be used at the group level of interaction. When used in an excerpt, words in parenthesis will be used to describe an action, while words in brackets will represent clarification of a speaker's utterances. For full disclosure, the researcher of this study and the author of this manuscript are the same person.

External Influences

To better understand the qualitative results of this study, the perspective I had taken was that the teaching that occurs in a school classroom is not isolated from external influences in the system in which the teaching is embedded. For example, a teacher's outside commitments in support of school functions or administrative duties could impact the time spent lesson planning and considering the key ideas of a lesson. In addition, availability of professional support resources influences a teacher's professional growth. This portion of the analysis reviews findings that revealed how external influences impacted one teacher.

Carolyn Smith (pseudonym) is a high school teacher at Atlas High School (pseudonym) located in the southwestern United States. Based on her responses to the Beliefs survey (see Appendix I), Carolyn had been teaching for more than 16 years. In conversation with her she further revealed that she had been teaching at Atlas High School more than 20 years. During the period of the study Carolyn taught Calculus AB, Honors Precalculus, and Integrated Math. At the time of the data collection Carolyn was in her third year of teaching Precalculus with a conceptual curriculum (Carlson, et al., 2013b). In the first year, in addition to professional development and resources given by the Pathways project, she worked one-on-one with a member of the Pathways team to discuss the key mathematical ideas of the lessons she taught. In describing her experiences with the curriculum, Carolyn mentioned she was the only one using the conceptual curriculum (Pathways) for honor Precalculus at her high school. She did not work with other teachers in her school because the other Precalculus teachers at her school had chosen to continue using a formerly adopted traditional textbook when

teaching Precalculus. After the first year of teaching using the curriculum with professional supports from project members, Carolyn worked in isolation when it came to teaching the honors Precalculus course. Carolyn was also chair of the math department and mentor/advisor to the high school volleyball team during the period of this study, which also meant additional time commitments due to school and administrative functions. An example of one of her time commitments is illustrated in Excerpt 7 below, as the researcher and Carolyn were arranging a time to meet for the next lesson planning session.

Excerpt 7. Carolyn's additional time commitments (Lesson 3 Debrief)

- 1 Car: Well here's the deal on Monday. I score for Volleyball.
- 2 Res: How fun.
- 3 Car: And we have Volleyball on Monday, which starts at 3:45 and probably
- 4 goes to 7:30 or 8:00.
- 5 Res: Okay.
- 6 Car: And then I have to eat (*laughs*).

Carolyn's additional time commitments created a need for an alternate way to debrief and plan (Skype/Facetime) for this study as was described in the methods section of the previous chapter, since the school games lasted approximately 4 hours (lines 3-4) and spanned into the evening. Also, Carolyn oftentimes tutored after school. More importantly, these external time commitments contributed to the research findings because the duration of the Volleyball season lasted through most of the study, as documented in Excerpt 8 below.

Excerpt 8. One additional time commitment ends (Lesson 11 Debrief)

- 1 Res: So volleyball is done, or do you have more games?
- 2 Car: Oh we're done.
- 3 Res: You're happy and relieved I think.

- 4 Car: I, yes. Yeah, I was ready for the season to be over. It's a lot of fun, but
5 they back, back to back here, and their long match, you know both, both
6 last night and tonight was four hours plus.
7 Res: It's long, long days.
8 Car: Yeah. That's long. And I had after school tutoring before that, and so,
9 yeah, it was a long day.

Although the Lesson 11 Debrief was the last lesson debriefing of the study, Carolyn had tutoring and volleyball that day (lines 5-6, 8-9). Many teachers, including Carolyn, take on external time commitments in support of the school community as educators at the K-12 level. However, such time commitments can influence the amount of time spent on lesson planning and reflection. In addition, the absence of colleagues teaching with the Pathways materials at her school appeared to limit her opportunities for reflection and professional growth.

Beliefs

While external factors can influence the time a teacher has to spend on thinking about key ideas of a lesson, a teacher's mathematical teaching orientation influences a teacher's view of mathematics and how he/she teaches mathematics. In this section, I will discuss results that helped characterize Carolyn's mathematical teaching orientation and that gave insight into Carolyn's views of mathematics teaching and learning.

Carolyn's responses to the Beliefs Survey (Appendix I) to Questions 23, 27, 28, and 29 were identified as items that would be used to make an initial characterization of Carolyn's teaching orientation. To review, the four questions identified as interesting were the following.

Table 9.

Selected Items from Beliefs Survey (Restated)

Item number	Item text
23	Student success in my course relies on their ability to (A) solve specific types of problems (B) understand key ideas of the course
27	For my students, making sense of a problem is best accomplished by (A) knowing the sequence of steps to solve the problem (B) knowing the ideas that are the focus of the problem
28	When preparing for class, I spend more time thinking about (A) presenting the material so that students are prepared to complete the problems in the homework and tests (B) how to engage students in making sense of and using the ideas that are the focus of the lesson
29	My teaching focuses more on (A) helping students understand ideas of my courses (B) helping students learn how to work specific problems

Carolyn’s response of “4” on Questions 23, 27, and 28 represented a view of mathematics consistent with a conceptual orientation, while a response of “1” was indicative of a calculational orientation. In Question 29, the ordering of the scale was reversed, with a “1” representing a view suggesting a conceptual orientation, while a “4” being a view indicative of a calculational orientation.

For context, I listed the other participants’ responses from the preliminary studies as a benchmark in making an initial characterization of Carolyn’s teaching orientation.

Table 10.

Beliefs Survey Results (All Studies)

Participant	Question 23	Question 27	Question 28	Question 29
Carolyn	3	3	Both	both
Robert	3	3	3	2
Dorothy	4	4	4	1
Margaret	4	4	4	1

When responding to questions 28 and 29, Carolyn did not circle any of the four choices. Instead, she wrote the word “both” between the 2 and the 3, suggesting that she valued both views that were expressed. For the purposes of quantitative analysis, I treated annotation between two numbers as the average of those values. Based on the mean score of these four items (and reverse coding Question 29), Carolyn’s mean score was 2.75, Robert’s mean score was 3, and Dorothy and Margaret’s mean score were both 4. From an ordinal perspective, the data suggests that Carolyn was more calculationally orientated than the other teachers who had participated in prior studies.

One might argue such a comparison is potentially flawed, and if the scale used to find this score were taken from a Likert scale, such an interpretation of intensity is subjective to the participant, and this objection would have merit. While a within-participant comparison might be suitable for measuring shifts in beliefs through a course of professional growth, a between-participant comparison might be valid under conditions where all participants taking the survey held a shared meaning of the intensity on a scale. This potential concern was not viewed as an issue for the items analyzed, as these were contrasting alternative items based on existing beliefs surveys (Carlson, et al., 1999; Carlson & Rasmussen, 2010) that also used a contrasting alternative scale.

The second part of my investigation with regards to the Beliefs survey was to look for survey items in which Carolyn gave additional feedback. I conjectured that such responses could suggest areas to *mark* (in the sense of Mason (2002)) for further study with regards to her views of mathematics teaching and learning. In addition to writing “both” as a response to items 28 and 29, Carolyn also put a question mark by item 3, and further elaborated on her response in the additional comments section. Item 3 asked

whether a teacher strongly disagrees or strongly agrees with the statement “I try to make learning easy for my students”. Her elaboration on the survey item was as follows.

#3 is difficult for me to answer. I like to challenge my students, but I also then try to break down problems in chunks so that problems are easier to grasp.

Carolyn’s response in Excerpt 9 reveals reiterating this view. In a lesson that focused on methods of solving logarithmic equations, she first showed examples of problems that require a method she called *Hint #2* that she described as solving a logarithmic equation by writing the corresponding exponential equation. After demonstrating three examples for *Hint #2*: $\log_2 x = 4$, $\log_5 2x = 3$, and $\ln x = 5$, she had students working in groups on another question. She then wrote the problem $\log_2(x+4) - \log_x(x-3) = 3$ on the board, citing it as another example for students to practice.

Excerpt 9. Making a problem easier (Lesson 7 observation)

- 1 Car: I’ll give you some practice. Alright let’s try this one (*erases board*). One
- 2 I like.
- 3 Car: (*Writing question*) Log base 2 of x plus 4, minus log base 2 of x minus 3,
- 4 equals 3.
- 5 Car: Unlike one of those first problems that we did, we don’t have any
- 6 logarithm equals a logarithm. So can’t use that first hint. But could we
- 7 write this as a logarithm equaling something?
- 8 Stu1: No.
- 9 Car: Would we, we got two logarithms. What do we have to do?
- 10 Stu2: Divide.
- 11 Car: Mmm, well okay. I see what, I know what you’re saying when you say
- 12 ‘divide’. But do you mean by that: condense the logarithm?
- 13 Stu3: Yes.
- 14 Stu2: Yes.
- 15 Car: Yeah let’s do that (*writing on board*).
- 16 Car: I can condense the logarithm like that, because it’s a difference. We can
- 17 make it a logarithm of a quotient. Yes?
- 18 Car: Now, does that hint number 2 work? Can we write this as an exponential
- 19 equation, what was the base?

- 20 StuS: Two.
 21 Car: Two. 2 to the what power?
 22 StuS: Third.
 23 Car: *(Writing on board)* 2 to the third power equals x plus 4 over x minus 3.
 24 Car: What is 2 to the third power?
 25 Stu2: Eight.
 26 Car: *(Writing on board)* 8, equals x plus 4 over x minus 3. So here we go,
 27 we're back to Algebra 1. How do we solve for x ?
 28 Stu2: Multiply by x minus 3.
 29 Car: Multiply both sides by....
 30 StuS: x minus 3.
 31 Car: x minus 3. Thank you.
 32 *(Carolyn stops writing on the board. She circulates to different groups as*
 33 *students continue solving the problem.)*

An image of Carolyn's board work, as she pivoted from direct teaching to work with students at the group level (lines 32-33) is given in Figure 11 that follows.

$$\log_2(x+4) - \log_2(x-3) = 3$$

$$\log_2 \frac{x+4}{x-3} = 3$$

$$2^3 = \frac{x+4}{x-3}$$

$$8 = \frac{x+4}{x-3}$$

Figure 11. Making a problem easier (Lesson 7 Observation)

The first hint (*Hint #1*, lines 5-6) refers to a method Carolyn discussed earlier in the lesson, in which she wrote down “If $\log_b x = \log_b y$, then $x = y$ ”; she followed *Hint #1* with examples that showed her applying *Hint #1*. After asking the question to her class whether the given equation can be written as a “logarithm equaling something” (line 7), one student replied “no” (line 8). Carolyn asked a leading question in order to reduce the complexity of the problem (line 9) and clarified one student's response to her question (lines 11-12) as a cue to proceed with and “condense the logarithm” by rewriting the difference of logarithms as a logarithm of a quotient (lines 16-17). Asking another

leading question “Now, does hint number 2 work?” (line 18) followed rapidly by “Can we write this as an exponential equation, what was the base?” (lines 18-19) were moves to further break down the problem and make the question easier to solve through the process of direct teaching.

Carolyn continued to make moves to reduce the complexity of this problem for her students. She rewrote the logarithmic equation as an exponential equation (line 23), then asked students to state what the value of 2^3 was (line 24), and rewrote the equation with 8 in lieu of 2^3 (lines 26-27). Carolyn proceeded to ask the class how would they solve the resulting equation (line 27). After enough students stated a method (line 30) that she agreed with (line 31), she gave her students an opportunity to solve the remaining part of the problem in their groups. All that remained of the original problem was an algebraic procedure and a calculation to make.

This excerpt was also an example of one of several exchanges observed during the study in which Carolyn made moves to make a more complex problem simple. In the context of procedure or skill based lessons, her pedagogical moves, as highlighted in Excerpt 9, were representative of her practice. Carolyn’s calculational orientation was visible through her pedagogical moves via her questioning, and centered on next steps to take (lines 9, 18-19, 21, 28, 30, 32) and intermediate calculations (line 25). Procedural and skill based lessons, such as the one from which Excerpt 9 was taken, had a significant portion of the lessons taught via direct instruction; her practice in these lessons could be characterized as teacher-centered. With respect to lessons using a conceptual curriculum, the results were more nuanced, and will be discussed in greater detail in the context of the second research question.

Reflecting on Carolyn's response to "I try to make learning easy for my students" (survey item 3) in her additional comments, by her statement of breaking things down into smaller chunks to make math easy for students, I conjectured that her statement of breaking things down also represented a schema of how she thought of solving any particular problem herself. Therefore it should impact how she modeled student thinking (or student misconceptions) since it followed that her attention would focus on matching her own problem solving process to her students' processes. Based on this conjecture, I thought it likely that Carolyn would attend to a student's described steps in solving a problem, rather than attending to a student's description of her or his thinking process or rationale behind their work. To see whether this conjecture had supporting evidence, item 30 on the Beliefs survey best matched my follow up question. In item 30, Carolyn was required to choose between the two contrasting alternatives: "When a student gives an answer that is perplexing to me, I find it more helpful to ask questions focused on (A) The sequence of steps leading to an answer, or (B) The thinking used to understand and respond to the problem". Carolyn circled "2", which meant she agreed with statement (A) more than (B).

Furthermore, classroom observations of Carolyn while she was helping students in small groups corroborated her statement. One such example is given in Excerpt 10 where Carolyn asked her students to find out the time when the enrollment at a particular high school had reached 1600 students. It was part of a longer task in which students had already modeled the projected student enrollment at this high school as a function of time using an exponential function; time was measured from the reference year the enrollment projection was made. After giving the task to the class, Carolyn circulated around the

room, observing students as they worked in small groups. She gave her feedback to the individual groups at a speaking voice level intended for all the students in class to hear as she kept walking. In the following excerpt, students were engaged in the task while Carolyn was walking from group to group.

Excerpt 10. Carolyn's focus on certain steps (Lesson 2 Observation)

- 1 Car: *(With group in back of room)* How are you going to do that?
- 2 Car: *(Walking to next group in front of room)* How are you going to find out
- 3 when it's 1600?
- 4 Car: *(Walking to next group in front of room)* How are you going to find that
- 5 when it's 1600, I'm not seeing that. Think about what you can do.
- 6 Car: The table is not going to give it to you. *(Walking to next group in middle*
- 7 *of room)*
- 8 Stu1: *(To another student)* How do you find it?
- 9 Car: The table is not going to give it to you. *(Walking to next group in back of*
- 10 *room)*
- 11 Stu2: I found it.
- 12 Car: What did you all get?
- 13 Stu2: The graph.
- 14 Car: Yes. Yes. Oh no! What did you do? You don't have it.
- 15 Grp: *(laughter)*
- 16 Stu3: *(From front of room)* Ms. Smith, we got it. We got it.
- 17 Car: *(To Stu3)* What did do you? *(Walking to next group in back of room)*
- 18 Stu4: *(To group members)* 6.996.
- 19 Stu4: Ms. Smith I found it.
- 20 Stu3: We set a line at y equals 1600.
- 21 Car: *(To Stu4)* Exactly. *(To Stu5)* Do you have it?
- 22 Stu5: He *[Stu4]* beat me. *(To Stu6)* You do Calc, Intersect.
- 23 Car: *(Carolyn walks to group in front of room)* What did you do? *(To Stu3)*
- 24 Yes.

While visiting the groups, Carolyn had specific steps in mind (lines 5, 6) and she did not observe students doing those steps. Rather, as she walked around, she noted various groups using the table method (lines 6, 9) on a TI graphing calculator (putting the exponential function into $y1$, putting 1600 as $y2$, and looking at a table of values to see for which input value are both output values 1600). Carolyn made moves to discourage students from using the table approach (lines 6, 9). The method she had in mind as the

best approach to solving was the graphing by intersection method (putting the exponential function into $y1$, putting 1600 as $y2$, finding where the two graphs intersect using an appropriate window size). An algebraic approach was not considered (nor used) in this lesson since logarithms had not been introduced yet. Carolyn affirmed her view of the best approach to solving this problem (via graphing) for all in class to hear (lines 14, 21, 24). Analysis of the interaction suggested Carolyn's mathematical teaching orientation was calculational. In addition to the focus on specific steps, the teaching episode had focused on the answers students were finding (lines 14, 21), rather than the thinking process used to find such answers. Although Carolyn had a specific method in mind, she did not attend to students' reasoning used when using the method; a chosen method was used as a means to find an answer.

While one possible interpretation of this interaction can be taken from the lens of her responses to the Beliefs survey items 3 and 30 – making a problem easier and focusing on steps students take – the researcher brought up this teaching episode during the lesson debrief, to further model the rationale behind her pedagogical moves. In Excerpt 11 that follows, the researcher framed the discussion by asking Carolyn if she thought the prior activity in which the function modeling two high schools' enrollments had discrete input values, may have influenced students' thought process during this teaching episode.

Excerpt 11. Discussion of Carolyn's pedagogical moves (Lesson 2 Debrief)

- 1 Res: I remember what I think it was a girl who was sitting near the back,
- 2 which is near the camera, who actually was using a table method for
- 3 intersection. Do you think that way of thinking may have emerged
- 4 because the student thought that the discrete representation was the only
- 5 way that could of, that was the appropriate model? Hence looking at
- 6 intersection of a graph may not be the best choice.

- 7 Car: No, I think that just the way she would do it.
8 Res: Okay.
9 Car: I don't think that the discrete part had anything to do with that.
10 Res: So it was a preference of using the table method versus the graph
11 method?
12 Car: They find it, effective, but I'll try to talk her out of that (*laughs*).
13 Res: I guess what limitations do you see on the table method?
14 Car: I don't think you can get the accuracy, can you?
15 Res: Right, so you lose some of the accuracy?
16 Car: I think so. Unless you're willing to have a delta table that is 0.001. And
17 then you might be hunting around for a long time.
18 Res: Right.
19 Car: I don't know. That's never a way I choose to do it.

After the researcher framed the question (lines 1-6) posing a possible reason why a student might choose the table method over the graphing method, Carolyn's response indicated that she had in mind one way to approach the problem and that she hadn't considered the pros and cons of determining the solution graphically versus using a table; nor had she thought of connecting the graphing and table approach and probing students see how the two methods, although different, both represent an attempt to find the x value for which the y -values are the same (lines 7, 9). Carolyn's response (line 12) highlights that she thought students find the table method effective, but they can be convinced to use a different method. This is further support that she was focused on her students getting an answer rather than helping them understand why both methods work and how they were connected.

In trying to ascertain Carolyn's thought process behind her moves to have all students use a particular method (in this case the intersecting of graphs) to solve the problem, the researcher probed further (lines 13, 15) on Carolyn's perceptions of the limitations of the table method. She mentioned the method was neither accurate (line 14, 16) nor efficient (line 17), and included a comment about the method only being accurate

when the Δx (step size between successive input values) of the table was 0.001 (line 16). At the time when Carolyn made the comment of a specific step size, I did not note its significance until I viewed it in context of her overarching goals, which will be discussed in this chapter. Carolyn's statement "that's not the way I'd do it" as this discussion finished was noteworthy (line 19). It added supporting evidence to earlier findings during the class observation that Carolyn's image of the way to solving this problem (intersection of graphs method) was the "best" way to solve the problem, and that alternative methods deemed not as efficient by her were to be discouraged (line 12).

In summary, Carolyn's responses to items 3 and 30 on the Beliefs survey, combined with the qualitative data from classroom observations, provides insights into the beliefs she held and how they may have influenced her pedagogical actions throughout the study. Carolyn expressed that she believes in breaking down problems to make them easier for her students. The pedagogical moves she made in breaking down these problems were characteristic of a teacher having a calculational orientation. Secondary findings included early evidence suggesting that the way she viewed a problem in her mind was predictive of what she valued as a preferred solution approach for her students. Solution methods not matching her preferred approach were discouraged as either inaccurate or inefficient and appeared to result in pedagogical moves aimed at steering students away from these approaches. Lessons in which procedures or skills were the focus were teacher-centered.

Other External Factors Impacting Beliefs

Earlier I had discussed external influences at the school level that impacted Carolyn's time to reflect on the mathematics and planning lessons, along with an absence

of colleagues to work with whom she could have collaborated in a Precalculus professional learning community. In addition to influences at the campus level, outside influences at the school district level can support or thwart changes to a teacher's beliefs and goals regarding the teaching and learning of mathematics, particularly when incentives are associated with students' scores from external assessments. This section will discuss findings from this study.

During the study the researcher noted that Carolyn had won an award from the school district because of her students' high scores on the Calculus AP exam. Carolyn had posted the certificate she received that acknowledged her award on the wall. The certificate read "Outstanding achievement by your students on an advanced placement test". Based on the researcher's observations and notes during the course of the study, the subsequent Calculus course had a contributing impact on her beliefs about mathematics teaching and her overarching goals. A discussion about her award occurred during the first part of the post-study interview, and Excerpt 12 documents the beginning of this conversation.

Excerpt 12. Carolyn's AB Calculus award

- 1 Res: First of all, congratulations.
- 2 Car: Well thank you. I believe they did it the year before, but I hadn't had
- 3 such wonderful, I didn't meet the 60 percent passing. I had 58 or
- 4 something like that. So, it was the first time I won the award, but I think
- 5 they've only done it twice.

When sixty percent of the students taking the exam passed the AP Calculus exam (line 3) the teacher received recognition from the district. Although it was a relatively new award having being done twice so far (line 5), she came close to achieving this award the first year (line 3), missing the cutoff by 2 percent. Although Carolyn dropped a

word midsentence, I believe she meant to say “such wonderful students” (lines 2-3) or something analogous when referring to the class that took the AP exam. She attributed the high pass rate on the exam (thus her award) to the students who took the exam. Based on this interaction, the researcher asked further questions to see Carolyn’s perceptions about what other factors she might attribute her successes to. The following excerpt gives insight into Carolyn’s beliefs about the curriculum used.

Excerpt 13. Carolyn’s AB Calculus award and her perceptions

- 1 Res: What percentage of those [*who took the test*] were your students from
- 2 before? The ones who did it? Who took Precalculus with you who took
- 3 Calculus with you?
- 4 Car: Maybe half.
- 5 Res: Half, okay.
- 6 Car: I’m not sure that I think the kids who come from *Pathways* are in fact
- 7 better prepared, than kids who come from another route.
- 8 Res: Uh huh. So I guess my question is what percentage of your Precalc move
- 9 on to Calc, based on your experience?
- 10 Car: Most.

Carolyn did not attribute the conceptual curriculum she used, *Pathways*, as a source of success with regards to the advanced placement exam (lines 6-7), although most of her Honors Precalculus students (line 10) went on to take Calculus AB. This finding helped in understanding the background in which analysis of other results were made during the course of this study. The award from the school district supported her beliefs that student success in mathematics, as measured by passing scores on the AP exam, was supported by either the conceptual curriculum or skills-based curriculum.

Knowledge of the Curriculum

While Carolyn’s beliefs and external influences provided a backdrop in which this study was conducted, Carolyn’s knowledge of mathematics curriculum also influenced her goals. In referring to Carolyn’s knowledge of the curriculum, I am talking about her

network of connections between the key ideas of mathematics she is teaching; aspects of these connections can be thought of between concepts, between procedures, between concepts and procedures, between representations, and within representations. Different curricula emphasize different aspects of key ideas of mathematics. In particular, my findings suggest there were three sources of curricula that played a prominent role with regard to Carolyn's goals, mathematical knowledge for teaching (MKT), and classroom practices through the study: (1) the AP curriculum, (2) prior Precalculus course materials from traditional textbooks, and (3) the conceptual curriculum she used as her primary textbook (Carlson, et al., 2013b) which was referred to as the *Pathways* curriculum. Prior researchers (Pintrich, 2000) mentioned that strong curricular contexts have an influence on a teacher's goals. While the findings of this study confirmed earlier research, the extent of the curricular context went beyond the immediate curricular materials used in the Precalculus course in which this study had been framed. The broader curricular context in which Carolyn viewed Precalculus had a significant role with regard to the findings of this study; this will be discussed in greater detail with Carolyn's overarching goals. I refer to the context in which the three sources of curricula were used as the *curricular context* of this study.

Overarching Goals

In the exploratory study with Robert discussed earlier, the results had suggested that the overarching goals of a teacher had a major impact on a teacher's mathematical goals for student learning at the lesson level. In this section, results of this study were analyzed to better understand Carolyn's overarching goals.

Carolyn's overarching goal in Calculus was to have students be successful on the AP exam, and this translated to decisions she made in Precalculus to prepare students for Calculus. The following excerpt from the post-study interview was in response to how she viewed her role as a teacher of mathematics.

Excerpt 14. Carolyn's role as a teacher

1 Well, in a senior level class, I mean I see my role as, getting those, having
2 students who leave my class prepared to move on to whatever institution of
3 higher learning that they're going to. That they have a good foundation in the
4 class that I'm supposed to be teaching them. I know that some of my Calculus
5 kids may never ever take another math class. If they do well on the AP test, and
6 their major doesn't require math, this is their ultimate class. And so, I want to
7 make sure that all of the topics that are in the AP curriculum, I give them a good
8 foundation in those topics. That I'm very, I make sure that I don't leave anything
9 out (*gesturing in a left to right motion, partitioning the space with her hand in a*
10 *sequential order on the desk*). I teach that curriculum (*sweeping hand left to*
11 *right along the desk*) to the best of my ability.

Carolyn's goals to have students succeed in the AP exam in Calculus had direct bearing on her teaching of Precalculus (lines 7-9) since she wanted to give her students a good foundation to succeed. She did not want to leave anything out (line 9) and doing well on the AP test meant saving students some time in their coursework at the college or university level (lines 5-6). The researcher followed up by asking how Precalculus and Calculus relate to each other. Her response to this query is given in Excerpt 15.

Excerpt 15. How Precalculus and Calculus relate

1 Well, quite a bit. The trig. There are some things that really just move right into
2 Calculus. It's what we're doing now, finding zeros of functions, and having a
3 feel for exponential functions and growth, and knowing what to expect when
4 you see a function. And I think that Precalculus is really important for Calculus,
5 I really don't know we have all that many topics that they've never ever seen
6 before. There are certainly some. They haven't spent- they have never seen
7 parametric equations. They have never seen polar equations. But there are a lot
8 of things that we do that they have at least had a sampling of before. And so it's
9 a year to cement.

In analyzing Carolyn's response, it suggests her view of Precalculus seems to be as a preparatory course for Calculus (line 9), with some additional topics to learn (lines 7-8). Carolyn's choice of lessons and topics in Precalculus were informed by her experiences in teaching Calculus and her image of what students needed to be prepared to be successful in Calculus.

Furthermore, the AP curriculum impacted her views of what was seen as an accurate response to a mathematical question in the Precalculus course. The researcher noted that on occasion Carolyn emphasized an in-class rule (norm) that all answers should be expressed to three decimal places whenever appropriate. It seemed to permeate into how she thought of methods that she deemed mathematically accurate. Her prior comment about the table method being accurate only for a "delta table that is 0.001" during a lesson debrief (Excerpt 11, line 16, page 119) had significance in the context of the findings regarding Carolyn's overarching goals. The rationale behind this class rule aligned with her overarching goals of preparing her students for the AP exam, but it was not immediately apparent until she stated this goal explicitly during one of the lesson planning sessions. In Excerpt 16 that follows, Carolyn talked about a quiz she recently finished grading and spontaneously brought up the issue of decimal places.

Excerpt 16. Rationale of three decimal places (Lesson 6 Planning)

- 1 Car: They did pretty well on it. I just finished grading them, just a few
- 2 minutes ago. So, yeah.
- 3 Res: Any surprises?
- 4 Car: No.
- 5 Res: No.
- 6 Car: You know, I will stress as we get into the solving equations later this
- 7 week, how they have to not round in the middle of a problem.
- 8 Res: Ah.
- 9 Car: That store their values [*in the calculator*], because they're really not
- 10 doing that and I've tried to emphasize to them that AP rules are that you

- 11 are always going to go out three decimal places. And so that's the rule of
12 this class, even if it doesn't say that on the instructions. That's the rule,
13 and I still have a lot of people who aren't doing that.
14 Res: Mmm. Interesting.
15 Car: You know, it's a habit they don't go out three decimal places or more.

After talking about the quiz results (lines 1-2) and stating her plan to tell students not to round in the middle of a problem (lines 6-7), Carolyn shared the rationale behind her attention to numerical results being rounded to three decimal places (lines 9-13). Her in-class rule (lines 11-13) was based on requirements of the AP Calculus test, in which students' numerical answers should be rounded to three decimal places of accuracy (lines 10-11). In short, Carolyn was training her students for next year's AP test and one of her goals was to have students adopt this practice when expressing their solutions. This finding is consistent with earlier research that mentioned curriculum influenced goals.

Goals Influencing Curricular Choices

In the prior section, the results from this study had supported other researchers' findings (Pintrich, 2000) that curricular context had impacted a teacher's goals. A follow-up question was whether a teacher's goals influenced her curricular choices. The findings from this study suggest that a teacher's goals have an impact on the curriculum used and the value placed on it.

Carolyn's lesson selections, and the value she placed on a lesson, were influenced by the topics that were on the AP Calculus exam. For example, in Excerpt 17 that follows, Carolyn questioned the value of a particular lesson in Precalculus that she taught with the conceptual curriculum, when seen from the perspective of the course that follows (Calculus). The researcher, noting Carolyn's comments after the in-class observation of Lesson 4 regarding her dislike of the particular lesson she had just taught,

chose to revisit the lesson during the post-study interview. The excerpt that follows recounts her reaction to a lesson having to do with exponential growth and partial growth factors (Lesson 4: Module 4, Investigation 5, n -unit Growth and Decay Factors, $n \neq 1$) in which the researcher asked Carolyn to elaborate on what she said earlier about disliking the lesson. During the interview, Carolyn's lesson plan for Lesson 4 using the conceptual curriculum was handed to her to aid in her recollection of the taught lesson; it included printed instructor notes, along with her annotations on them.

Excerpt 17. Carolyn's comments on lesson using partial growth factors

- 1 Car: Maybe, maybe my math is lacking. I'm not sure I see the point in
2 spending that much effort on partial growth factors.
3 Res: Okay.
4 Car: In Calculus, I never see that when we do exponential growth. It's always
5 with the base of 'e'. Always. I shouldn't say always, but I can't think of
6 an example that we use in our Calculus book that has an exponential
7 problem where the base is something other than 'e'.
8 Res: Right. So you don't see this (*indicating Pathways Lesson 4*) as
9 particularly useful for your students?
10 Car: (*shaking head no*)

Carolyn's reaction to the lesson on partial growth factors stemmed in part from her tenuous connections of the mathematics of this lesson, something she pondered herself (line 1). Her troubles with this lesson were noted during the in-class observation of Lesson 4, and will be discussed in greater detail in the context of the findings of the second research question. However, I suggest that a lesson specific goal of building the mathematical connections to effectively teach the lesson was secondary to her overarching goal of preparing students for Calculus. In particular she cited never seeing exponential functions other than those to base 'e' in Calculus (lines 4-7). Therefore she questioned the value of the lesson (lines 1-2, 10). In addition to data acquired through the classroom observation, an analysis of Carolyn's mathematical connections during the

pre-study task also shed light as to other possible reasons why she did not find value in this particular lesson for her students; her ways of thinking about covarying quantities was limited and she had impoverished ways of thinking about proportionality. Findings from the pre-study task that analyzed Carolyn's key understandings of quantities, covariation, and proportionality will be discussed later in this chapter.

Decisions about where to place content that she perceived as needed for Calculus were based on the connections she saw in a lesson to Calculus. Carolyn incorporated curriculum from prior Precalculus materials when planning to augment Pathways lessons. The following excerpt comes from a lesson planning session (Lesson 2: Module 4, Investigation 2, continued, Comparing Linear and Exponential Behavior). During the session, Carolyn discussed one of her mathematical goals for student learning in the upcoming lesson.

Excerpt 18. Adding content for future needs (Lesson 2 Planning)

- 1 Car: Then what I would like to do is go back to the comparing and have them
2 get out the big boards and compare: Make a table of values for $y=2x$,
3 $y=x^2$, and $y=2^x$. And have them input values, maybe from 0 to 10. And
4 look at the changes in those three graphs as we look at how each one of
5 them grows and where's the growth is more rapid. And then maybe
6 throw in, not actually put it on the table, put in things like x^3 : Where
7 would that fit: between the $2x$ or between the x^2 ? Or 3^x , where would that
8 fit, if we're looking at how the growth patterns go?
- 9 Res: And your focus, you're trying to focus the students the way the rate of
10 change is increasing, or what is your objective behind the comparison?
- 11 Car: Yes, I find in Calculus when I'm looking at functions that: if you have a
12 polynomial, well not a polynomial. But if you have a function that's a
13 sum of functions, which function is going, which part is going to
14 dominate? Sometimes they're not sure. Is x^4 going to dominate or is 4^x
15 going to dominate? And so I would like to maybe look at some examples
16 of that. Which is maybe not what the objective of the *Pathways* lesson is,
17 but I think this might be a good time to compare those.

The rationale behind adding this topic to the existing lesson was based on the need of a future course (lines 11-14). Whether it was a good fit with the current lesson was less of a concern (lines 16-17) to Carolyn. One of the intentions of this *Pathways* lesson was to compare linear and exponential growth from the perspective that a function was linear if: as the input increases, the output either increases or decreases at a constant rate. The function was exponential if: the output increases or decreases by a constant factor for equal changes in input. Placing the topic of comparing linear, polynomial, and exponential growth in the same lesson as a means to compare which terms “dominates” seemed to be incongruous with the intention of the lesson, but congruous in terms of Carolyn’s goal of coverage of a Precalculus topic in service to the Calculus course.

Carolyn also used materials from prior Precalculus courses she had taught to augment lessons in which she felt the conceptual curriculum did not address. At times, she substituted multiple lessons from a trajectory in the conceptual curriculum and replaced them with a trajectory from prior sources in which she felt the content area received a more thorough treatment. To illustrate, Excerpt 19 that follows is from a conversation in which Carolyn chose to use her own source material in lieu of the *Pathways* provided materials during the course of the study. Carolyn’s initial response was in regards to the researcher’s question of whether the lesson she was planning was based on a *Pathways* lesson, or one of her own lessons. In the lesson, Carolyn was planning to introduce the logarithm as the inverse of the exponential function.

Excerpt 19. Supplementing course materials (Lesson 5 Planning)

- 1 Car: I guess I’m really sort of doing my own [*lesson*].
- 2 Res: That’s my sense of it.
- 3 Car: The graphs, I don’t think that the graphical approach is in the *Pathways*
- 4 lesson.

5 Res: Correct.
6 Car: And I like that [*the graphical approach*]. And I'm also going to talk just
7 a little about shifting that logarithmic function left and right, up and
8 down, and how that changes the look of the function itself. Which is one
9 thing that my Precalculus book spent quite a bit of time on every type of
10 function doing that, and the *Pathways* book doesn't do that at all.

This was Carolyn's own lesson (line 1). However she wanted to highlight the inverse relationship between the exponential function and the inverse through a graphical representation (lines 3-4, 6) which she liked to do, by characterizing the inverse as the reflection across the line $y = x$; this was not an approach the conceptual curriculum either advocated or used, since such a procedure was not grounded in the meaning of the inverse relationship. In addition, she felt it necessary to teach topics (lines 6-8) that were not in the *Pathways* curriculum but she that had taught before (lines 8-10) from prior course materials.

In summary, Carolyn's choices for curriculum came from three sources: the AP curriculum (The College Board, 2012), prior Precalculus course materials, and the conceptual curriculum (*Pathways*). The AP curriculum was the major driver of Carolyn's decision to choose a *Pathways* lesson or use other materials, in so far as Carolyn thinking about what would serve students' needs in Calculus (and the exam that they would take). Based on my observations, I surmised that if a *Pathways* lesson fit into a topic that was in Calculus, which Carolyn felt would help students succeed in the AP exam, it was used. Otherwise, Carolyn's own notes that had been developed when preparing to use traditional curricular sources were used. The choice to use *Pathways* was not made because Carolyn viewed the conceptual curriculum as offering students superior opportunities in building richer connections of the underlying mathematics. Rather, the

Pathways curriculum gave her students more opportunities to be engaged with the mathematics, and Carolyn liked when students were engaged. In Carolyn's response to the post-study query (Excerpt 20, that follows), student engagement was one of the hallmarks of a successful lesson. In essence, the results suggest Carolyn's curricular choices were influenced by one overarching goal on the horizon: preparing students to pass and succeed in the AP Calculus exam. This finding suggests that Carolyn's instructional goals were influenced by her perception of abilities students would need in their Calculus course the following year, which influenced the curricular choices she made to support student learning. The findings of this study demonstrate that a teacher's goals influence curriculum used. Taken with prior research (Pintrich, 2000) and supporting findings that curricular choices also influences a teacher's goals, it follows that the relationship between a teacher's goals and the curriculum are reciprocally related; they influence each other.

Views on Learning Mathematics

Another finding emerged while analyzing the relationship between Carolyn's goals and curriculum. It merits discussion because of how Carolyn's view of mathematics impacted her curricular choices during the study. To keep my method consistent with the one used in the exploratory study, the following query was submitted to Carolyn after completion of the dissertation study.

Describe what it means to you to be an effective teacher. What are characteristics of a successful lesson?

Her response to this query is given in Excerpt 20.

Excerpt 20. Being an effective teacher (Carolyn)

- 1 An effective teacher is enthusiastic and loves to teach. She has high expectations
- 2 for herself and for her students. She can manage a classroom and design lessons
- 3 that are appropriate to grade level of students. She has a rapport with students
- 4 such that the students respect and trust the teacher. She has a solid knowledge of
- 5 the subject matter.
- 6 In a successful lesson, students are engaged. A successful lesson is flexible – it
- 7 can change gears when necessary, either to remediate or enrich. In a successful
- 8 lesson, students are called upon to think, not just regurgitate information.

In analyzing Carolyn's response to the characteristics of a successful lesson and what it means to be an effective teacher, she expressed that managing a class and lesson design was important, along with solid knowledge of the subject matter (lines 2-3, 4-5). She did not mention how a successful lesson might be flexible, although she expressed that the teacher should be prepared to adapt a lesson if needed (lines 6-7). According to her statement, the amount of student engagement in a lesson was used as a measure of a lesson's success (line 6). Carolyn's classroom actions did not align with her view of what a successful lesson entailed (lines 7-8); her pedagogical moves often suggested the opposite was true, with efficient methods and correct answers being valued more than student thinking.

Carolyn's view of mathematics had some key differences to that of Robert, who participated in the second exploratory study. When Robert was asked the same question after completing the exploratory study, his response to the query is given in Excerpt 21 below.

Excerpt 21. Being an effective teacher (Robert)

- 1 I think that an effective teacher is one who can structure classroom experiences
- 2 so that students have the most potential to making meaning and form
- 3 connections between ideas. The effective teachers also uses questioning
- 4 techniques that serve to clarify student thinking and in some cases challenge
- 5 student thinking (either to solidify what's been said or to force the student to

6 re-think an incorrect thinking).
7 A successful lesson has the same components. The final outcome should be that
8 student learning is maximized, both in terms of how many connections students
9 are able to make and in terms of how many students walk out of the room with
10 more understanding than what they entered with.

From earlier, one of the findings from the exploratory study was that Robert's view of mathematics learning was about forming meaningful connections between key ideas. Such a view was highlighted in his response to my query (lines 3, 8-9). In contrast, this view of mathematics was notably absent in Carolyn's responses. Her view of learning mathematics did not appear to include making connections between ideas. In fact, her view of mathematics learning suggested that coverage of topics was a priority (Excerpt 14, lines 8-11; Excerpt 15, lines 4-9). I do not think Carolyn's gestures (Excerpt 14) during the post-study interview were coincidental, such as going from left to right while partitioning the space along the desk with her hand in a sequential order when describing thorough coverage of topics. She appears to be visualizing mathematics as a sequential order of topics to be covered. This ties back to Carolyn's overarching goal of preparing students for the AP exam. The course description for the Calculus AP exam (The College Board, 2012) gives an outline of topics and sample problems that will be on the exam, and I think this interpretation of mathematics, (as a list of topics and methods of mathematics to learn in order to solve problems) when combined with other methods and skills based curricular sources that are supportive of this view, contributed to Carolyn's choices in lesson planning. So as viewed through the perspective of a teacher's interpretive system (Zawojewski, et al., 2006), the context in which the curricular

materials were used influenced the connections Carolyn made among the mathematical ideas she was teaching (her KDUs).

Although this finding when viewed in isolation appears as a non-finding, I see it from the perspective of factors that thwart substantial shifts in a teacher's mathematical practice. When one of Carolyn's overarching goals was to prepare students for an important exam, viewing mathematics as connections between ideas was not a viewpoint that could easily emerge, and so pedagogical goals and actions to support students making those connections are unlikely to be observed. Furthermore, during the analysis of the second research question in this study, her weak MKT emerged in greater detail and was found to be a contributing factor of her inability to see connections between the *Pathways* curriculum and AP Calculus.

A Revised Framework of Interactions

One of the results from the findings of this dissertation study was that my original model of interactions (Chapter 2, p. 15) was incomplete. While all models are imperfect, they are designed to capture the essential elements relevant to the situation being studied. My finding was that leaving out curricular context in the framework of interactions missed an important element that, if not accounted for, would have led to a superficial characterization of the results from this study. My findings revealed that curriculum had major role in both Carolyn's goals and her KDUs. The revised framework that emerged as a result of this dissertation study is in Figure 12.

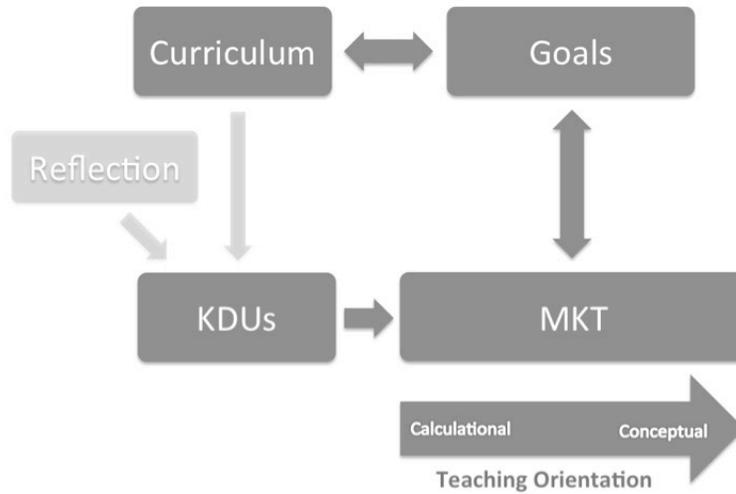


Figure 12. A revised Framework of Interactions

Results from this study support that a teacher’s goals and the curriculum she uses mutually influence each other, and that curriculum she uses also influences a teacher’s key developmental understandings (KDUs) of the mathematics she teaches.

Personal Goals

Although the effect of Carolyn’s overarching goals on her curricular choices and pedagogical moves has been analyzed, Carolyn’s personal goals provide additional information about background events that may help explain the results of this study. This section characterizes Carolyn’s personal goals in the context of her teaching mathematics.

Carolyn’s personal goals influenced her interactions with her students in class. The researcher noted that during the course of the study, Carolyn was mindful of the time that would be potentially spent on particular tasks in class during lesson planning. This disposition made sense in context of her pedagogical moves to have students use what she viewed as efficient methods of solving math problems, but it also spoke to Carolyn’s use of time with her students. During the post-study interview, the researcher asked

Carolyn about her personal goals for a lesson she teaches, and her response in the excerpt that follows highlights this view.

Excerpt 22. Carolyn's personal goals

- 1 Car: I hope that at the end of the lesson, students don't walk out of room
2 saying "What, what was that all about?" Um, you know that they, that
3 they know what it was they were supposed to be learning that day. And
4 they may be unclear about some of it, but they may not 100 percent get
5 what we were talking about. They may have to come in after school with
6 some more questions or whatever. But that they know where we were
7 going. That they - things are not disjointed enough that they say, "Boy.
8 That was a waste of time."
9 Res: Uh huh.
10 Car: I mean, on a daily basis that's, I hope that they walk out knowing
11 something that they didn't know before. Or have, understand more
12 clearly what they didn't understand before.
13 Res: Uh huh.
14 Car: I don't want to waste their time. I don't want to waste my time.
15 Res: Make it efficient.
16 Car: That's right.

Carolyn did not want to waste her students' time (lines 8, 14) and she did not want to waste her time when teaching a lesson (line 14). She agreed with the researcher's statement about making the lesson efficient (line 16); the researcher had made this statement to test whether she agreed or disagreed with the assertion made. In isolation, this exchange may not mean much, but in the context of the study, this disposition influenced both her interactions with students, and the amount of time she allowed for students to construct their own meaning of the mathematics being taught. At the end of each lesson debrief (with the exception of one lesson) Carolyn's attention to staying on a pre-planned lesson schedule did not change. Her response to the question "Based on your observation of how the lesson unfolded today, how have your plans for tomorrow's lesson changed or not changed?" was "Not changed". In the one lesson debrief where she

struggled with the key ideas of the taught lesson (Lesson 4), her response was to use direct teaching the next day in order to catch up with already planned lesson for the next day (the analysis of Carolyn's goals and MKT starting on page 164 will discuss this exchange in more detail). Carolyn's desire for efficient use of time also influenced the efficacy of the researcher's moves to perturb Carolyn's goals during the sessions they collaborated, and this will be discussed in greater detail on when analyzing Carolyn's goals in the context of reflecting on her teaching practice (page 169).

Carolyn's Key Developmental Understandings

In order to characterize the relationship between a teacher's mathematical knowledge for teaching and her mathematical goals for student learning, the key developmental underlying of a teacher's mathematics was analyzed. Because the conceptual curriculum Carolyn used was built on well-connected meanings of quantities, covariation, proportionality, and constant rate of change, this study investigated her key ideas of these mathematical concepts. This section will discuss results of analyzing Carolyn's mathematical connections.

As part of the pre-study interview, Carolyn was given a mathematical task to help the researcher characterize her KDUs with regards to quantities, covariation, proportionality, and constant rate of change. The pre-study task, with Carolyn's responses, can be found in Appendix I). This task was inspired by one of the homework problems (two people meeting at a park bench) in a previous chapter Carolyn had taught with using the course materials from the conceptual curriculum (Carlson, et al., 2013b, p. 68).

Quantities

In the problem statement of the pre-study task, two students, Cameron and Neil, live on opposite sides of town connected by a long road. They are studying for a test and agree to meet at a library located somewhere between them on the same road. When Neil and Cameron start heading toward each other, Cameron was 5 miles away from the library, and Neil was 6 miles from the library.

After a request to draw a diagram, part of the task asked Carolyn to represent the quantities in a diagram and describe the quantities that could be relevant to the situation. A second part of the task aimed at assessing Carolyn's attention to quantities prompted her to draw a plot of the relationship between: distance Cameron is from library and time elapsed, for the duration of time that both Neil and Cameron are travelling, given the constant speeds for Neil and Cameron, respectively. The third part of the task was asked to characterize her attention to quantities when plotting the relationship between: distance Cameron and Neil are from each other and time elapsed, for the duration of time until Neil and Cameron meet each other at the library; their respective constant speeds were changed for the third part of the task.

In order to do the task, Carolyn stated she had to assume that Cameron and Neil started at the same time, and she said the task was missing this essential piece of information. She had "trouble answering some of those questions" without making this assumption first. The researcher agreed that she could make that assumption in order to not limit her reasoning through the remaining parts of the task, but this was a noteworthy finding. Carolyn needed a fixed reference point of time (a common starting time from their homes) in order to address the task. In the task itself, the problem statement "When

Neil and Cameron start heading toward each other, Cameron is 5 miles away from the library, while Neil is 6 miles away from the library” was not sufficient information for Carolyn to reason that at the same moment in time, Neil and Cameron were their respective distances from the library, regardless of who actually left home first. It suggested that Carolyn needed a specific view of the reference time in order to address the task. For example the quantity “time since Cameron and Neil starting walking toward each other” had to mean “time since Cameron and Neil left their homes” in order for her to make sense of other parts of the task.

In addressing the second part of the task, additional information about two students’ constant rates of travel was given (Cameron traveled at a constant rate of 20 miles per hour, and Neil traveled at a constant rate of 25 miles per hour).

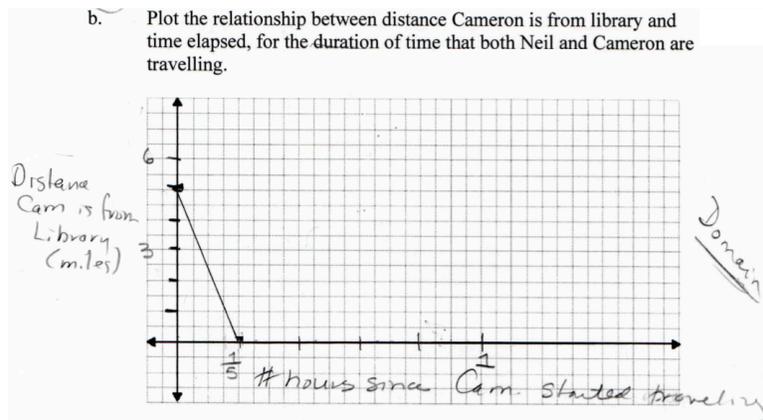


Figure 13. Carolyn’s initial response to question 2, part b

The problem statement with Carolyn’s initial response is given in Figure 13; a slight photo edit of her final response was used to reconstruct her original response. In addressing this question, Carolyn did not attend to the input quantity in the problem statement, which was the duration of time both Neil and Cameron were travelling. She did attend to quantities when making the plot as indicated by the labels “# hours since

Cam started traveling” for the input quantity and “Distance Cam is from Library (miles)” for the output quantity, indicating attendance to the attribute and units of the quantity. However she attended to the time Cameron travelled, not when both students travelled. The researcher followed up to learn whether Carolyn meant to do this, or whether it was an oversight, by making moves to attend to the graph she drew. At first the researcher’s moves were not overt.

Excerpt 23. Carolyn’s attention to quantities

- 1 Res: So where does your graph stop? Just curious.
- 2 Car: That’s when he started (*pointing to graph at $(0,5)$*) and that’s when he
- 3 got there (*pointing to graph at $(\frac{1}{5},0)$*).
- 4 Res: And that’s it, right?
- 5 Car: Huh?
- 6 Res: And that’s it, right? So you didn’t really need all that extra axis?
- 7 Car: No I didn’t need any there (*laughing*).
- 8 Res: That’s what I was trying to see.
- 9 Car: Maybe I put that on there first, I don’t know, then I realized all that over
- 10 there. He only went for 12 minutes so I don’t really need all that room.
- 11 Res: Okay. So officially I should only be looking at just up to there (*pointing*
- 12 *to $(\frac{1}{5},0)$*), and nothing beyond that.
- 13 Car: Yeah.

The researcher’s questions (line 1, 4) and Carolyn’s response (lines 2-3) documented her initial response. The researcher’s queries (lines 6, 11-12) clarified that Carolyn really meant to graph the quantities stated. At no time during this initial conversation did she reread the problem. However as the interview went on, the researcher noted Carolyn’s comments about the two graphs in the task as she discussed her thoughts on the key ideas of the lesson, which is given in the following excerpt.

Excerpt 24. Carolyn’s comments about the graphs from pre-study task

- 1 Car: Well, I think one of the key objects is we’re doing two graphs when
- 2 you’re graphing different things. And kids are not always paying
- 3 attention to what it is they are graphing. Because this (*pointing to*
- 4 *problem 2*) is just the distance Cameron is, and this (*pointing to problem*

5 3) is the distance between two of them. And, lots of times, they're not
6 going to make, they are going to do them both the same. They are not
7 going to make that distinction, because they don't read. Closely.
8 Res: Okay.
9 Car: In fact I said (*turning page to problem 3*), is that really what they wanted
10 me to graph? (*laughs*)
11 Res: Uh huh. So paying attention to the ideas.
12 Car: So, (*turning page to problem 2*) and this one was so much easier to do.
13 Res: Oh yeah.
14 Car: Than this one (*turning page to problem 3*). Yeah.
15 Res: So this one (*pointing to problem 2*) is not piecewise, the first graph?
16 Car: No, I don't think so. Not the way I did it.

The researcher noted Carolyn's comments about her students not paying attention to the quantities they are graphing (lines 2-3) because they do not read a problem statement closely (lines 6-7). Carolyn wondered about complexity of the third problem (line 9-10) whose plot was a piecewise function, when compared to second problem that she viewed as simpler (line 12). Based on the way she constructed the graph of the second problem (line 16), the researcher confirmed that Carolyn did not view the graph as a piecewise function (line 15). Based on Carolyn's comments about her students not reading closely and the previous exchange (Excerpt 23) in which Carolyn did not reread the problem statement when she discussed her graph, the researcher conjectured Carolyn simply misread the problem. The question remained to the researcher whether she could attend to the covarying quantities in the situation and revise the plot to represent the quantities being tracked in the problem statement. In the following excerpt, the researcher asked Carolyn to look back at problem 2 and made overt moves to have Carolyn attend to the quantities in the situation.

Excerpt 25. Reviewing Carolyn's attention to quantities

1 Res: Let's go back to the other one, I'm just curious. So in that one [*problem*
2 2*b*], what is it that they're asking us to do?

- 3 Car: The distance Cameron is from the library (*pointing to vertical axis*) as a
4 function of the time elapsed (*pointing to horizontal axis*).
- 5 Res: Of what?
- 6 Car: Since Cameron has started walking (*tracing horizontal axis*).
- 7 Res: Okay.
- 8 Car: (*tracing horizontal axis*) Number of hours since Cameron has started
9 walking.
- 10 Res: Right. And that was the only quantity that was being attended to, right?
- 11 Car: Yes. Or the duration, oh actually, I guess didn't read that correctly. For
12 the duration of the time that both of them are walking.
- 13 Res: Would that change how...
- 14 Car: Oh. Well then it just goes, (*drawing horizontal line segment on axis in*
15 *problem 2b*) yeah I guess because now Cameron is already there.
- 16 Res: So there then it would be also a second piecewise function?
- 17 Car: (*laughs*) Yeah, you got me on that one.

The researcher persisted (lines 1-2, 5) in finding out if Carolyn could describe the quantities in the situation precisely (lines 3-4, 6, 8-9). After asking Carolyn to reflect on the quantities being modeled in the situation (line 10), she realized her error of misreading the question (lines 11-12) and made corrections to her graph (lines 14-15). The researcher asked if this problem was a piecewise function, like the other problem (question 3 in the task) that she had completed in addressing this task (line 16), which she confirmed (line 17). She later underlined the part of the problem statement she misread. Carolyn's revised graph with her annotations are given in Figure 14. The length of the horizontal line segment was intentional, since the horizontal axis was marked off in increments of $\frac{1}{5}$.

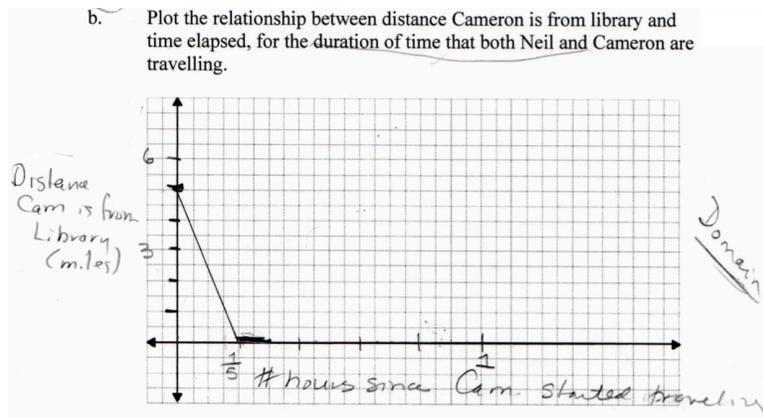


Figure 14. Carolyn’s revised response to question 2, part b

After mentioning that the duration of time when Cameron and Neil are both walking goes on for another tenth of an hour beyond the time that only Cameron was walking, the researcher asked Carolyn to explain her revision to the plot. The discussion is given in the excerpt below.

Excerpt 26. Carolyn explains her revised plot

- 1 Car: Something like that (*pointing to horizontal segment on graph*) because
- 2 he’s at the library so his distance isn’t changing. But if I wanted to go for
- 3 the duration of the time they are both walking, Neil is still walking so
- 4 that would have to go like more.
- 5 Res: So, I’m looking at it and it looks very flat. Why would it, why...
- 6 Car: Because he’s [*referring to Cameron*] already at the library.
- 7 Res: So the distance is not changing from the library.
- 8 Car: Yeah, he’s already there. But if I have to go for the duration of the time
- 9 that they are both traveling, then I guess I would really do have to have
- 10 that (*pointing to horizontal segment on graph*), that piece.

Carolyn’s description of her rationale behind her plot of the horizontal line segment (lines 1-4, 6, 8-10) is consistent with that of a person *making meaning* of the quantities and being able to represent and interpret the quantities in graphical form. I concluded that Carolyn’s mathematical conception of the quantities were sufficient to respond to the questions posed in the task. She could make meaning of quantities

representing the situation and she could attend to the quantities in graphical form, but that she had the potential to make errors due to inattentiveness.

Covariation of Quantities and Constant Rate of Change

As Carolyn described her reasoning of parts of the task from both the lens of a student and then a teacher, the researcher found a notable lack of attention to how the quantities covaried. She described the calculations she made to find the associated times for the situation using completed distances and the fact that each of Neil's and Cameron's speeds were constant. Her utterances described a process where line segments were drawn in to represent the situation, based on her knowledge of the initial and final states of the two quantities being tracked. When Carolyn mentioned that the constant rate of change was a key idea of the lesson, the researcher saw this as an opportunity to direct her attention to covarying quantities. The researcher asked her to clarify her meaning of constant rate of change (Excerpt 27).

Excerpt 27. Carolyn discusses constant rate of change

- 1 Res: So, can you say more about what you mean by: the key idea is the
- 2 constant rate of change?
- 3 Car: Well that you can figure out the time each one of them travels, because
- 4 they are going at a constant rate of change. And then that would be a way
- 5 to compare how long it takes them. We know, we know their constant
- 6 rate of change so the graph is going to be linear. For each one of them,
- 7 how fast they are walking, we talk about that.
- 8 Res: So why would students know it's linear, I guess is the question?
- 9 Car: Well I guess that depends on where this comes in the Module, because
- 10 we talked a lot about that.

Carolyn used the justification that Cameron and Neil traveled at a constant speed to justify the linear graphs (lines 5-6). However she calculated the completed time first (line 3-4) in order to make the graph and to determine which student arrived at the library first. The researcher's attempt to understand Carolyn's meaning of linearity (line 8) was

unsuccessful (lines 9-10), although her response did suggest that she thought a student's answer would be different depending on its placement in the trajectory of lessons.

The researcher mentioned that students would see this lesson at the end of Module 2 in which the concepts of quantities, covariation, and constant rate had been already discussed. This move was designed to free Carolyn of any perceived constraints, with the intention of uncovering her conception of constant rate of change and covariation. The researcher reiterated his prior question, framed as: how students might think about linearity.

Excerpt 28. What does linearity mean to Carolyn

- 1 Res: How would they [*students*] know that: I can draw a straight line- I guess
- 2 is the question. Why do you think students would know that, rather than
- 3 just drawing something squiggly?
- 4 Car: They would know for equal changes in input you have correspondingly
- 5 equal changes in output. They would know then that was linear.
- 6 Res: Okay.

In rating Carolyn's covariational level of reasoning (Carlson, et al., 2002), this suggests Carolyn's thinking about covariation could potentially be at the Quantitative Coordination Level, which is the third level of the framework. This meant her KDUs supported coordinating a specific amount of change in the input to a specific amount of change in the output, where the specific amounts of change were "equal". Since Carolyn was not specifically attending to coordinating uniform change in the input values with rates of change in the output, I did not classify her at the Average rate of change level, the fourth level in the framework.

Carolyn could think of constant rate of change from the perspective of covarying quantities; it was accessible to her, but only after questioning. However, Carolyn's use of

covariational reasoning was not part of her regularly accessed ways of problem solving and meaning making, which was an early finding that had implications later in this study. For example, Carolyn's limitation in covariational reasoning in the context of constant rate of change was predictive of possible struggles with covariational reasoning in the context of exponential functions and polynomials, which were to be the lessons to be observed during the study.

Attention to Proportionality

From a covariation of quantities perspective, proportionality can be viewed from the ratio, constant multiple, and scaling perspectives. These ways of reasoning were supported in the conceptual curriculum Carolyn used. Problems 2c, 3b, 3c, and 3f in the interview pre-study task were originally designed to characterize a participant's KDUs with respect to proportionality. For example in problem 3b of the task, a participant could use the scaling perspective of proportionality, given that Neil was 6 miles from the library at the same time Cameron was 5 miles from the library, as follows.

Since Neil was travelling at a constant speed of 25 miles per hour, he travelled 25 miles in one hour. It follows that he travelled 1 mile in $\frac{1}{25}$ hours, and therefore Neil travelled 6 miles in $\frac{6}{25}$ hours. Cameron travelled at a constant speed of 21 miles per hour. Following a similar way of reasoning using scaling, Cameron traveled 5 miles in $\frac{5}{21}$ hours. Therefore Cameron arrived at the library before Neil, since $\frac{5}{21}$ is less than $\frac{6}{25}$. For the first $\frac{5}{21}$ hours they were both travelling to the library. For remaining $(\frac{6}{25} - \frac{5}{21})$, or equivalently $\frac{1}{525}$, of an hour Neil continued travelling to the library until he arrived.

With regard to problem 3c, to draw an accurate graph, a participant could determine the exact time and distance Cameron and Neil were from each other until

Cameron arrived at the library using the scaling perspective of proportionality as follows. When both were travelling to the library, the distance between them was decreasing at a constant speed of 46 miles per hour. This means that in $\frac{5}{21}$ of one hour, the distance between them decreased by $\frac{5}{21} \cdot 46$ miles. Since Cameron and Neil were 11 miles apart at the start of the situation, it means they were $(11 - \frac{5}{21} \cdot 46)$, or equivalently $\frac{1}{21}$, of a mile apart when Cameron arrived at the library. For the remaining $\frac{1}{21}$ mile, Neil travelled to the library at a constant speed of 25 miles per hour, and this fact could be used to verify that the interval of time associated with the remaining distance and constant speed was $\frac{1}{525}$ of one hour (or equivalently, $6\frac{6}{7}$ seconds).

Other ways a participant could engage in the proportion task would be purely calculational without leveraging ideas of covariation, either by the procedure of solving for t in the $d = rt$ equation (this is called the “distance formula” in American classrooms—it is a statement that distance travelled, d , is the constant speed, r , times the amount of time elapsed while travelling, t), or by using a procedure where a proportion statement is created by setting two equivalent ratios equal, where the time it takes to arrive at the library is the unknown value.

Of the possible ways of thinking about proportionality described, Carolyn used the distance formula for problems 2c, 3b, and 3f. A representative sample of her work in problem 3, part b is illustrated in Figure 15.

3. Let's change the situation. Let's suppose that that Neil traveled at a constant speed of 25 miles per hour and Cameron traveled at a constant speed of 21 miles per hour.

a. Whom do you conjecture will arrive at the library first, or do you think will they arrive at the same time? Explain why you believe this is true.

b. Determine who arrives at the library first (or if they tie) using mathematics. Show your work. Explain why your process is correct.

Plot the relationship between distance and time for each person.

Handwritten notes in the right margin: $D = RT$

Handwritten calculations for part (b):

Neil	60m	25 mph	.24 hrs
Cameron	5m	21 mph	.238

Figure 15. Sample of Carolyn's work for proportional tasks

While I could discuss Carolyn's cursory responses in addressing the questions in the task from the written artifact, I will focus on the mathematics. In the right margin, Carolyn had written the distance formula, and algebraically solved it for t . She then used that approach to find the amount of time it took Neil to arrive at the library, and the amount of time Cameron travelled before arriving at the library. She wrote out a decimal representation of time elapsed for each traveller, to three places of accuracy, putting ".24 hrs." for Neil's time and ".238" for Cameron's time. In the case of problem 3b, under magnification and a bright light, the researcher noted that Carolyn had originally written $\frac{5}{20}$ for Cameron's time, which she simplified to $\frac{1}{4}$ and represented as ".25 hr." as the time Cameron travelled; however she erased these steps and redid her calculations for the correct constant rate at which Cameron now travelled. She erased the circle around Neil and placed a box around Cameron to signify that Cameron arrived at the library first. It appeared however that the edits she made to correct the error in problem 3b occurred after completing problem 3c, since based on the numbering of the scale, the plot was still representative of the original situation she had modeled (Figure 16, page 152).

After Carolyn finished discussing her approaches to problems 2c, 3b, and 3f, the interviewer noted that she used the same method to address each of the tasks. This conversation is highlighted in the excerpt below.

Excerpt 29. Carolyn's approach to the proportion tasks

- 1 Res: Uh huh. So the process you used in any of the three examples where Neil
2 and Cameron were coming together was utilization of the distance equals
3 the rate times time formula.
4 Car: I did. I'm old, I've done it for so many years, that's the way I going to
5 approach it.

The researcher stated Carolyn's consistent use of the distance formula in the problems she addressed (lines 1-3) which she affirmed (line 4). It was a stable approach she used to address this type of question, and she had been using this method for a long time (lines 4-5). The question remained was whether the method she described was the only method she knew to solve the proportionality tasks, or did she have other ways of thinking in mind. In talking about the key ideas of the lesson in the excerpt that follows, Carolyn recognized students might utilize different methods when addressing the tasks in problems 2 and 3.

Excerpt 30. Carolyn anticipates different solution approaches

- 1 I think another idea is that although we have the same situation, we have two
2 totally different graphs because we are comparing different things, and I think
3 again that that's something my kids might tend to gloss over. And I guess talk
4 about, if I calculated this in one fashion, I think we need to talk about how
5 somebody else did it, because I'm sure we wouldn't all do it the same way.

Carolyn was aware that different students might do the same task differently (lines 4-5), and this provided the researcher the opportunity to characterize other ways she was thinking about the task. The follow-up question was couched as what are other ways her students might think about a problem like the one in problem 2, given that she had already used the distance formula approach to find the associated times.

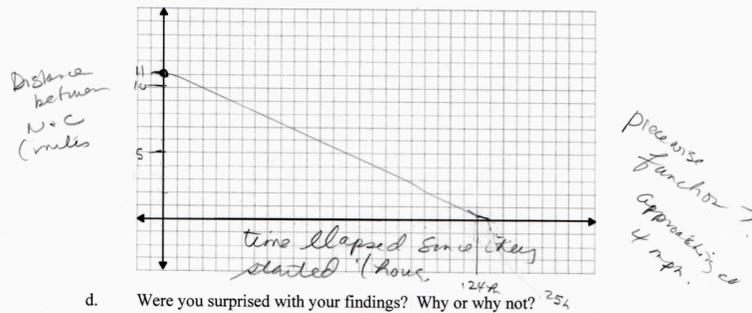
Excerpt 31. Carolyn describes an alternate approach

- 1 Res: I'm just thinking, what other ways would you imagine students doing it?
2 (*pointing to Carolyn's work in problem 2*). So this is probably the most
3 popular way a student would do it. Would there be other ways?
4 Car: Oh I imagine there would. Um (*pauses*). If you will we could do if, let's
5 say that Neil was going 20 miles in 1 hour, he's going to go 6 miles in
6 how many hours? (*writes $\frac{20 \text{ mile}}{1 \text{ hr}} = \frac{6 \text{ m}}{x \text{ hrs}}$ in left margin near problem 2, part*
7 *c*)
8 Res: Uh huh. And try to figure it out. Okay.
9 Car: I would say there are probably some kids who would do that.
10 Res: That would definitely be another way that they would, another approach
11 or way of thinking that they would approach it. So using a ratio?
12 Car: Uh huh.

After the researcher posed the question (lines 1-3), Carolyn proceeded to set up and describe a proportion in which two ratios were set equal (lines 4-6), stating that some kids would use this type of approach (lines 9, 12). At no time during the interview did Carolyn mention any other approaches that would indicate a view of proportionality from a covariation of quantities perspective. It is noteworthy that she had experienced the task from a covariation of quantities perspective on several occasions during workshops aimed at supporting teachers' implementation of the Precalculus materials.

What remained to be seen was how Carolyn's views of proportionality influenced her construction of a graph representing how the two quantities change together, the original intention of problem 3c. Figure 16 shows Carolyn's work in problem 3, part c.

- c. Plot the relationship between distance Cameron and Neil are from each other and time elapsed, for the duration of time until Neil and Cameron meet each other at the library.



- d. Were you surprised with your findings? Why or why not?

Figure 16. Carolyn's plot for a piecewise function

There were a couple of noteworthy features with regards to the graph. Based on the scale of the graph, it appeared to have been drawn for Carolyn's original characterization of the situation before she fixed her error in problem 3, part b. Hence the plot on this graph has a starting point at $(0, 11)$ and an ending point at $(0.25, 0)$, with a small bend in the graph at the approximate location $(0.24, \frac{1}{4})$. With regards to the plot in problem 3, part c, Carolyn did not use proportional reasoning to make the graph, as was revealed in the excerpt that follows.

Excerpt 32. Carolyn describes how she made the plot

- 1 Res: And, how did you plot the relationship that you have?
- 2 Car: Boy that was the hardest thing in this whole thing.
- 3 Res: Oh.
- 4 Car: Very much hard. Because it really has to be a piecewise function.
- 5 Res: Uh huh. Tell me more.
- 6 Car: Well because they don't arrive at the same time. And if this is the rate
- 7 between them (*pointing at paper*), then the rate of change changes when
- 8 only one of them is walking.
- 9 Res: Uh huh. So what is the effective rate of change that you had determined
- 10 between them?
- 11 Car: I didn't. I just drew. (*laughs*)
- 12 Res: Okay.
- 13 Car: I didn't. I just knew they started 11 miles apart and after .24 hours they
- 14 were almost there. And then the rate of change slowed for that last little
- 15 bit. So, I didn't actually calculate that.
- 16 Res: You used a position as a way to determine (*pauses*)
- 17 Car: Yeah.

- 18 Res: the graphs. So the part which is right here (*pointing at bend in graph*)
19 which is I guess where the function's definition changes, that is a
20 conjecture of the approximate location?
21 Car: Yes.
22 Res: As opposed to an exact calculation?
23 Car: Yes.
24 Res: Okay. And then you made the last leg of the graph, right?
25 Car: With a slope that was a little bit less, because only one of them was
26 walking.

Carolyn found this aspect of the interview task the most challenging part (line 2, 4) and she realized the situation being modeled was a piecewise function. She knew that the rates of change changed at the moment in time when the situation being modeled switched from two people walking to one person walking (lines 6-8). The researcher incorrectly modeled Carolyn's approach to the problem (lines 9-10) at first, thinking she used the effective rate of change between Cameron and Neil as a way to start the problem.

Carolyn did not use the rates of change to make the graph (line 11). She used the information that Cameron and Neil started 11 miles apart, and assumed that after 0.24 hours they were almost at the library (lines 13-14). She sketched the remaining segment representing the remaining traveller with a slope that was less steep (lines 25-26) because the rate of change was less (lines 14-15). Carolyn stated the location of the bend in the graph was an approximate location (line 15), not an exact location (lines 18-23). Because her graph of the relationship of the distance between Cameron and Neil as a function of the time since they both started travelling was constructed based on estimations of key landmarks on the graph, I do not think her later revision to problem 3, part b, was put into consideration. Had she looked back and thought about the quantities in the situation, 0.24 hours represented the later of the two times, meaning that both students were at the

library. Cameron, not Neil, arrived at the library first, and the 0.24 hours associated with Neil's time to travel to the library would represent the end of the situation being observed, instead of being associated with the bend in the graph.

The results from her plotting portions of the graph revealed more about Carolyn's approach to graphing. She did not conceptualize the quantities and how they changed together or use ideas of proportionality to plot her graph. Rather, she used her understanding of the relative distances and how the quantities related to each other at specific points as a way to draw essential characteristics of the graph.

Interpretation of Task

Results reported earlier from this study had shown that Carolyn's overarching goals were preparing kids for the AP exam, and as part of doing so, she had created a classroom norm that all answers should be rounded to three decimal places (Excerpt 16, page 126). What was interesting is that this seemed to effect how Carolyn perceived to the intention of the task on problem 3, part b, as shown in the excerpt that follows.

Excerpt 33. How the task relates to the AP exam

- 1 My thought, as I got thinking, why did they [*the designers of the task*] do this?
- 2 Did they think that we were going to round both of these and say they got there
- 3 at the same time? Was that your conjecture? Was that your assumption what
- 4 would happen, or is this what you really wanted to happen, because I wouldn't
- 5 round that to .24, because I'm trying to tell, I'm trying to get these kids in the
- 6 habit that for advanced placement, if it's .238 and you wrote .24, you are not
- 7 going to get credit on the advanced placement exam. So we have to go to three
- 8 decimal places always.

At the time of the interview, the relevance of Carolyn's statement was missed, but in retrospective analysis this comment took on a greater significance. To Carolyn, this activity was designed by the *Pathways* team to get feedback on a prototype lesson. Her perception of the possible intention of the question was to determine whether students

would follow the AP rule (rounding to three places) or not (lines 1-3), noting that students in her class would get the problem wrong if they did not follow the AP rule (line 2-3). She wondered if this was or was not an intentional part of the question's design (lines 3-4). Carolyn further mentioned that in her class she was trying to get kids to write decimal answers to three places (lines 5-6, 8), since in the AP exam students would not get credit if they wrote answers to two decimal places (lines 7-8).

Possible Results to Research Questions Foreshadowed

Analysis of the pre-interview task with regard to Carolyn's key developmental understandings of quantities, covariation, proportionality, and constant rate of change had several implications that foreshadowed the results of this study with respect to the research questions. First, Carolyn could attend to quantities and make meaning of the quantities plotted on a graph. With regards to covariation, Carolyn's covariational reasoning seemed mostly absent, but was accessible in the context of direct questioning. She was able to interpret a constant rate of change from the third level of covariational reasoning, which was the Quantitative Coordination Level, but she did not appear to value engaging her students in this way of thinking. Carolyn's approach to proportions was calculational, and there was no evidence to suggest she viewed proportionality from a covariation of quantities perspective, even though the *Pathways* curriculum supports were in place that fostered this perspective. However, due to the finding that Carolyn's covariational reasoning was limited, I conjectured that this would influence her conception of proportionality. I did not find evidence that Carolyn had a scaling view of proportionality from the pre-study interview; it would be reasonable to predict that Carolyn would struggle in contexts where well-connected meanings of proportionality

would be required to make sense of a problem and/or construct a solution. Because Carolyn did not have a flexible view of proportionality, it was unlikely that she would leverage a scaling perspective as a way as described in the analysis in Chapter 3 (p. 32) to build a conceptual understanding of exponential functions. Since Carolyn's responses to a pre-study interview task revealed that she had a limited view of a reference point from which a quantity is being measured, contexts in which a flexible view of the referent quantity was needed presented challenges for her.

With the relevant background findings discussed, I will address the two primary research questions posed in this study.

Research Questions

The guiding research questions of this study were as follows.

1. What is the effect of a professional development intervention, designed to perturb a teacher's pedagogical goals for student learning to be more attentive to students' thinking and learning, on a teacher's views of teaching, stated goals for student learning, and overarching goals for students' success in mathematics?
2. What role does a teacher's mathematical teaching orientation and MKT have on a teacher's stated and overarching goals for student learning?

In short, the background findings discussed earlier had a significant influence on the findings to these research questions. Many of the professional development interventions designed and informed from earlier research had very little effect on Carolyn's goals for student learning, and are worthy of further discussion. Other findings revealed insights into potential obstacles for improving Precalculus teaching, and suggest areas of focus for both professional development leaders and researchers.

To address these research questions, the first part of analysis entailed my characterizing Carolyn’s stated goals. Carolyn’s mathematical goals for student learning were analyzed using the same framework as in prior studies (Marfai, 2014). For convenience, the goal framework is restated below.

Table 11.

Teacher’s Goals for Student Learning Framework (Dissertation Study)

Goal Coding	Description
<i>TGSL0</i>	Goals for student learning are avoided or not stated by the teacher, or he/she states that the goals of the lesson are unknown.
<i>TGSL1</i>	Goals are a list of topics that a teacher wants his/her students to learn in the lesson, each associated with an overarching action.
<i>TGSL2</i>	Goals are a list of topics that a teacher wants his/her students to learn in the lesson, each associated with a specific action.
<i>TGSL3</i>	Goals are doing methods of mathematics that a teacher wants his/her students to learn in the lesson.
<i>TGSL4</i>	Goals are getting students to think about the mathematics in the lesson, without the ways of thinking articulated.
<i>TGSL5</i>	Goals are getting students to think about the mathematics in certain ways during the lesson.
<i>TGSL6</i>	Goals are about developing ways of thinking about the mathematics in the lesson, with attention to how that thinking may develop.

Eleven lessons were observed in class. These were summarized in the previous chapter in the table with the lessons observed in the Methods chapter on page 99.

Carolyn’s mathematical goals for student learning were rated using the above framework. Carolyn was asked the following question: What are your mathematical goals for student learning in the lesson you plan to teach today? Her response to this question was used to then characterize her goals.

Goals from the Perspective of Curriculum

In the table that follows, the designation between lessons (*Pathways* or not) described whether Carolyn used the conceptual curriculum (Carlson, et al., 2013b) as the primary source for the lesson, or Carolyn created the lesson herself based on her notes from teaching the idea before. The lesson observations occurred across two chapters of the textbook, one focused on exponential and logarithmic functions (Module 4), and the other on polynomial functions (Module 5). Five of the observed lessons were Pathways lessons, while six of the lessons were non-Pathways lessons. The characterization of Carolyn's mathematical goals for student learning is summarized in the table that follows.

Table 12.

Carolyn's Mathematical Goals for Student Learning

Observed Lesson	Date	Lesson Name	Chapter in Text	Pathways Lesson	Ranking of Stated Goals
1	9/16/14	The Meaning of Exponents	4	Yes	TGSL1, TGSL2
2	9/18/14	Comparing Linear and Exponential Behavior	4	Yes	TGSL1, TGSL1, TGSL1, TGSL2
3*	9/19/14	1-unit Growth and Decay Factors, Percent Change, and Initial Values	4	Yes	TGSL2
4	9/23/14	n-unit Growth and Decay Factors, $n \neq 1$	4	Yes	TGSL3
5	9/26/14	Logs	4	No	TGSL2, TGSL4
6	9/30/14	Properties of Logs	4	No	TGSL2, TGSL2
7	10/2/14	Solving Log Equations	4	No	TGSL1
8*	10/3/14	Modeling Exponential Growth	4	No	TGSL3, TGSL1
9	10/21/14	Concavity and Average Rate of Change	5	Yes	TGSL4
10	10/23/14	Zeros of Quadratic Functions	5	No	TGSL4
11	10/27/14	Polynomial Functions of Higher Degree and Multiplicity of Zeros	5	No	TGSL4, TGSL5, TGSL3

* = Lessons in which lesson planning collaboration did not occur

In all, 20 mathematical goals were stated during the lesson planning process, or during the lesson debrief. Of the ranked goals using the framework, 6 (30%) of them were rated at TGSL1, 6 (30%) of them were rated at TGSL2, 3 of them (15%) were rated at TGSL3, 4 of them (20%) were rated at TGSL4, and 1 goal (5%) was rated as TGSL5. Goals rated at TGSL6 were not observed in this study, which was a finding that will be discussed on page 162. The perspective of these results is overarching since it spans all observed lessons, but initial analysis conveys the fact that 75% of Carolyn’s initially stated goals were TGSL3 or lower (overarching topics, topics with specific actions, and methods of mathematics). In light of the background findings of Carolyn’s overarching goals to provide thorough coverage of topics to prepare students for the Calculus course and the AP Calculus exam, this finding was not entirely surprising. Based on the way the spectrum of this framework was arranged, it suggests that the majority of her goals reflected that she did not consider student thinking or design her lessons for the purpose of affecting student thinking. Although the AP Curriculum had a major influence on her goals, an immediate follow-up question related to the curricular sources Carolyn used in class and how they impacted her goals for student learning. A summary of her goals and their ranking (using my Goal Framework) in the context of the primary curricular source used in that lesson is revealing.

Table 13.

Comparing Carolyn’s lessons: Curriculum Source

Stated Goal	Pathways Lesson	Non-Pathways Lesson
TGSL1	4	2
TGSL2	3	3

Stated Goal	Pathways Lesson	Non-Pathways Lesson
TGLS3	1	2
TGSL4	1	3
TGSL5	0	1
Total	9	11

If only the curricular source was the major driver of Carolyn's goals, Carolyn's goals would likely be rated higher in a conceptual curriculum. But in fact, the opposite seemed to be true. While it was true that a majority of Carolyn's goals were ranked at TGSL3 or below regardless of curricular context, her goals alone suggest that more attention to student thinking about the mathematics in lessons occurred when using lessons created from her own notes. At first the findings with Carolyn seemed to contradict the findings from the study with Robert, in which the type of curriculum (skill-based or conceptual) influenced the clustering of his lesson specific goals.

The majority of Carolyn's goals being TGSL3 or lower regardless of curricular source can be explained by different causes, or a combination of causes. The method used in ranking the goals had inherent limitations, as it captured the first utterances of a teacher's stated goal. Possible clarifications made later on in conversation about a lesson during planning that would possibly elevate the level of the goal were not used. Therefore the method used tended to underrate the teacher's actual mathematical goals for student learning.

Other Constraints on Carolyn's Goals

As was revealed in the earlier study with Robert, another contributing factor for her lower rated goals might be her interpretation of the word 'goals', as this could constrain the response that was given. In the pre-study interview, I asked Carolyn to

interpret the statement, what does it mean if someone asks “What are your goals for student learning” with regard to a lesson? In the conversation that followed, Carolyn stated she had two interpretations for the question, one being her objectives of the lesson, and the other being her goals for student understanding. Asking a follow-up question added insight to Carolyn’s interpretation of “objective”.

Excerpt 34. Carolyn’s interpretation of the word ‘goal’

- 1 Res: When you say objectives what does this word, what does “objective”
- 2 mean to you?
- 3 Car: A goal.
- 4 Res: Okay. What does a goal mean to you?
- 5 Car: *(laughs)* What I hope will, what I hope will happen.

While this meaning of goal (what will happen) is consistent with how goals are defined in the literature (Locke & Latham, 2002) Carolyn’s goals tended to focus on the visible, such as actions and methods. Although this view of goals was dominant in the study, her second interpretation (goals for student understanding) at times allowed other goals (TGSL4 and TGSL5) to emerge.

With regard to the absence of goals ranked at TGSL6 (goals are about developing ways of thinking about the mathematics in the lesson, with attention to how that thinking may develop), this goal did not appear to be a part of her thinking. TGSL6 represented goals in which attention to student thinking was maximal. In the post-study interview, the researcher had asked Carolyn to review and provide feedback on the list of questions on both protocols that were shared with her. With regards to the *Post Classroom Observation Protocol*, she had mentioned not understanding the item 5, which read “How might the understandings that are suggested by your goals developed or supported for students?” The excerpt of this conversation is below.

Excerpt 35. Inaccessible goals for student learning

- 1 Res: You mentioned not understanding 5. Can you clarify? This is the one
2 about what...
- 3 Car: (*Reading question 5*) How might be the understandings that are
4 suggested by your goals develop or be supported for students? I guess I
5 really don't understand how would I answer that question either.
- 6 Res: Okay, so what does that question mean to you? I'm just trying to
7 understand what I would need to fix with it, or is it ...
- 8 Car: What do you mean by "supported for" goal? I don't know the "supported
9 for students", I am not sure.

Although it is possible that the wording of the question stumped Carolyn without context, when Robert was asked this question, he was able to infer its meaning because his views of mathematics as connections (and student making connections between ideas) influenced his goals for student learning. Carolyn's comments (lines 3-5) and question (lines 8-9) indicated that her goals were not centered on student thinking; therefore the question did not make sense to her. When the researcher referred to a prior lesson to provide a context to the question, her feedback for revising the question was to rewrite it as "What specifically will you do to help your students understand?" which suggested a teacher centered view of a lesson focused on concrete actions to take rather than attending to the ways student thinking might develop and be supported by her (e.g., thinking of questions to support student thinking, misconceptions that might occur, or analyzing the task itself) during the course of a lesson.

Her comment echoes an earlier finding during the study in which the researcher made moves to have Carolyn attend to student thinking. After Lesson 4 observation, in which Carolyn had struggled when teaching, the researcher tried to use the debrief session to get her to reflect on how key ideas might develop in her students. Her mathematical goals for student learning expressed during the lesson planning session had

originally been “That they [*the students*] can express those growth factors. And that they can express a function to describe the situation they are reading about, or looking about. You know, some are from information, some are from tables, some are from graphs. So I’d like them to get a feel for approaching those problems three different ways.” This goal had been rated as TGSL3 in the framework. Based on her stated goal of having students express growth (or decay) factors via multiple representations seemed to be an appropriate question to inquire about her attention to student thinking.

Excerpt 36. A question with scaffolds (Lesson 4 Debrief)

- 1 Res: The goals you had were with regard to finding the growth and decay
2 factor. How might the understandings that are suggested by your goals
3 develop? And I think as you taught the lesson, you have, I think
4 especially after the second lesson, you have I think even stronger insights
5 into it. So can tell me a little bit about what you noticed?
- 6 Car: Well I noticed that I needed to rewrite the function, pulling these
7 exponents apart, so that I could write it as: something, raised to the
8 variable. Something raised to the x power, which then I think they could
9 see it that was the 1-unit growth factor. And then work from there. And it
10 appears to me, and maybe this is not true, but it appears to me that it’s
11 almost, almost always easier to find the 1-unit growth factor first. And
12 work from there. Now on those with the tables of course you didn’t. You
13 found the 2 or 3 unit first but then I would go to the 1-unit and then I
14 would work from there.

The intention of the question (lines 2-3) was to push Carolyn to think about how the key ideas of the lesson could develop for her students after teaching the lesson twice that day (a move to toward goals TGSL4, TGSL5, or TGSL6), since she had mentioned earlier in the debrief that she used her challenges with her first class (the one observed) to inform the second time she taught the same lesson (lines 3-5). The researcher asked what her to explain what she noticed (line 5), which had the potential for a student-centric or a teacher-centric response. It appeared that Carolyn was focused on the connections she was making (lines 6-9), rather than noticing how her students were learning in response

to her pedagogical moves (lines 10-14), so her response was teacher-centric. However, the goal of thinking about the ways student thinking could develop was not attended to. When the researcher asked a similar question in a written lesson debrief for Lesson 8, she gave the following response.

Excerpt 37. Goals not supportive of student thinking (Lesson 8 Debrief)

- 1 Res: What ways of thinking did you hope emerge during the lesson? How
2 might these ways of thinking develop or be supported for students?
3 Car: I wanted them to confirm that to solve an equation where the variable is
4 the exponent that they needed to use a logarithm. I hoped that they would
5 see that using a logarithm is more expeditious than graphing to find a
6 solution.

In analyzing Carolyn's response she focused on a specific method (supportive of TGSL3, lines 3-4) for solving for a variable exponent, that of using the logarithmic function (lines 4-6); this goal was consistent with one of her personal goals (being efficient and not wasting anyone's time, see Excerpt 22, page 137). The fact that she did not address the question suggests that she did not attend to student thinking even when the question was stated in written form. In light of earlier findings in which Carolyn's overarching goals focused on coverage of topics, her ways of thinking about what is involved in learning mathematics content did not include affecting student thinking.

Goals from the Perspective of MKT

Her MKT provided yet another perspective for her broader range of goals for a non-Pathways lesson. I analyzed her stated mathematical goals for student learning, the first being from the perspective of *Pathways* lessons versus non-Pathways lessons.

Looking at Table 13 on page 159, four (36 %) of Carolyn's goals ranked at TGSL4 or higher. In *Pathways* lessons, only one goal (11%) ranked at TGSL4. One explanation may be that because she taught her own lessons so many times, she had goals about the

ways she wanted students to think about the mathematics as she was well-versed in the key ideas she wanted to convey. This would make sense because when using *Pathways*, she was relatively new to teaching with the curriculum (teaching it the third time, as opposed to teaching with another curriculum numerous times).

Using Carolyn’s MKT as a lens for examining her goals relative to student learning the content provided yet another perspective for examining and explaining her goals. The table that follows gives a summary of her goals with respect to the curriculum content.

Table 14.

Comparing Carolyn’s Lessons: Curriculum Content

Stated Goal	Chapter 4 (Exponential Functions)	Chapter 5 (Polynomial Functions)
TGSL1	6	0
TGSL2	6	0
TGSL3	2	1
TGSL4	1	3
TGSL5	0	1
Total	15	5

Although there are few data points, the contrast is apparent. Carolyn’s comfort level with polynomials functions was palpable during the study, in contrast to her comfort level with some of key ideas of exponential functions in which her content weakness was sometimes revealed. The key ideas of polynomial functions were reinforced in the AP Calculus curriculum, whereas Carolyn did not see how the key ideas of exponential functions (as built in the conceptual curriculum) had utility in Calculus. In Excerpt 17 on page 128, Carolyn had questioned the value of the partial growth factor lesson in the exponential lesson trajectory, and the reader might notice that the partial growth factor

lesson was the last conceptual lesson used for the observed lessons in that chapter (see Table 12, page 158). The exposure of her weak connections about partial growth factors during teaching led to her changing her lesson plan for the next day.

Excerpt 38. Carolyn's future lesson plan changes (Lesson 4 Debrief)

- 1 Res: Based on your observations on how the lesson unfolded today, have your
2 plans for tomorrow's lesson changed or not changed?
3 Car: Well, yeah. Yes in fact they have. I'm going to jump right in with just
4 being straightforward about how we do this, and not try to get off task,
5 and hope that I can convince them in number 2 - what's the 1-unit and
6 what's the $\frac{1}{4}$ unit decay factors and then get into the problems with the
7 table and the problems with the graphs, which I don't think will be as
8 difficult for them. I think having those visuals is actually going to help.
9 Because I found that in numbers 3 and 4, when they could see exactly
10 what a 2-unit growth factor was, that helped.

Carolyn valued efficiency and not wasting her or her students' time and her personal goals reflected that (see Excerpt 22, page 137). Being straightforward (lines 3-4) meant direct teaching to make up for lost time in Lesson 4. Her experiences with the second time she taught the lesson that day had informed her of the methods she could use to help convince students (line 5-6) of the preferred way to reason with regards to partial growth factors. She saw the multiple representations in the lesson as a means to help teach students the concept (lines 6-10), commenting that the visuals helped. But it is possible that her struggles, along with her desire to have learning math be efficient for her students, influenced her goals with regards to her choice for her lessons on exponential functions going forward.

During the post-study interview, when discussing how her planning had changed since the first time teaching Precalculus with the *Pathways* materials, she expressed discomfort with the partial growth factor lesson. This was unprompted, but noteworthy. She described first printing out the instructor notes version of the lesson, and then

annotating the instructor notes with her own notes, and using this as her lesson plan when teaching a lesson from the conceptual curriculum. Among marks, circles, and underlining of statements that Carolyn found relevant, her annotations to the instructor notes typically included worked solution steps to the answers, and questions she planned to ask students during the lesson.

Excerpt 39. Carolyn's lesson plan for partial growth factors

- 1 Car: I wrote right on those [*the instructor notes*].
- 2 Res: Yeah. So you wrote on those?
- 3 Car: Yes. I wrote on, I, when I was planning a lesson in *Pathways* I printed
- 4 out the instructor notes, and I made...
- 5 Res: And you did the annotations that I see on the notes that you are referring
- 6 to.
- 7 Car: Yes. On the *Pathways* notes. What's important. How did they get this
- 8 answer- you know I had to do sometimes I had to do my own
- 9 calculations because I'm not really sure if they approach something a
- 10 little bit differently. Especially in the exponential growth chapter because
- 11 that was done, I had never done partial growth factors. I had never, I
- 12 found that- still now I'm not 100 percent comfortable with that approach.

Carolyn mentioned annotating what was important to her (line 7) and her worked out solutions (lines 8-10). She mentioned a particular focus on annotations in the exponential chapter (line 10) because she had not done partial growth factors (line 11). Even though this was the third time she had taught with the lesson, at the time the study was completed, she was still not comfortable with the partial growth factor lesson (lines 11-12). As was predicted with the analysis of Carolyn's pre-study task (see page 155) and using the lens of conceptual analysis, a limitation in her accessible levels of covariational reasoning along with inattention to quantitative reasoning, contributed to her difficulties with this lesson.

Based on Carolyn's preference for the natural base of 'e' to discuss exponential functions in an earlier part of the post-study interview (see Excerpt 17, page 128), and her

self-reported discomfort with partial growth factors lesson, the researcher attempted to characterize Carolyn's conception of 'e'. The excerpt below shares these findings.

Excerpt 40. Carolyn's personal meaning of 'e'

- 1 Res: What is the main idea you want students to take from 'e'?
- 2 Car: How an exponential function works.
- 3 Res: Uh huh. But the meaning of 'e' itself I guess is what I'm asking.
- 4 Car: Well let's ask Dr. Euler here. (*laughs- points to a book about Leonhard Euler off camera*)
- 5
- 6 Res: (*laughs*) I guess my question is you know there is many interpretations, there's so many ways have a take on 'e'. What is it the way you want your students to have in Precalculus, going into Calculus?
- 7
- 8
- 9 Car: I'm not sure I do an outstandingly good job at that. There have been times when I have started with, a compound interest, and changed that to a base of the 1 plus $1/x$, kind of thing, but I still have a lot of 'I don't kind of get it' looks. I don't know a good way to describe except it's this magic number (*laughs*).
- 10
- 11
- 12
- 13
- 14 Res: Okay.
- 15 Car: It's a magic function, I think, just because its derivative is itself. They come to me having used natural logs.
- 16
- 17 Res: Okay. From their science courses or from earlier math courses?
- 18 Car: From earlier math courses.

Carolyn's meaning of 'e' (lines 13, 15) was this magic number, 2.71828... after the researcher's attempts at clarifying her meaning of 'e' (lines 1, 3, 6-8). Further inquiry may have revealed what Carolyn meant by 'e' as representing how an exponential function works (line 2), although based on classroom observations (Lesson 5), her introduction to 'e' was the exponential value of the base "that was used in exponential growth and decay frequently". Interestingly students in her class had been introduced to the natural logarithm in prior math courses (lines 15-16, 18). While not the scope of this study, this line of inquiry (what does 'ln' mean) may be a valuable line of inquiry for future research.

Similar to earlier analysis, Carolyn's lack of attention to quantitative reasoning and covariation predicted a weak meaning associated with 'e'. It simply is a magic

function whose derivative is itself, but important to know in Calculus and the AP exam. She admitted when she had taught the lesson regarding 'e' using a conceptual curriculum, she felt she did not do a good job with it (lines 9-10) and that the lesson had confused students (lines 9-12). It appears that in this study, Carolyn retreated to old ways of teaching exponential and logarithmic functions after the lesson with partial growth factors, in which she started to experience pedagogical discomfort. Carolyn did not use the conceptual lesson introducing 'e', such as the one with compound interest (line 10) in this study in her introduction to 'e'. Rather, she used a population of bacteria as the context to introduce 'e' as this constant approximately equal to 2.71828... that is typically used when describing exponential functions.

Carolyn's weak meaning for partial growth factors and 'e' did not get resolved. As a result her lesson goals remained at a low level. Had she been in a context in which she was supported in understanding how these ideas might develop meaningfully, it remains a question as to whether her goals for student learning may have changed. Rather, it was how well she understood the key ideas of the lesson in relation to her overarching goal of preparing her students for Calculus that made the difference in her goals of which lesson she would teach as part of her overall lesson trajectory.

Goals of Efficiency Trumps Reflection on Practice

In this section I discuss findings related to Carolyn's personal goals and how they impacted her response to the professional development intervention and her mathematical goals for student learning. In findings discussed earlier (page 137), Carolyn's goals for efficiency and not wasting student's time influenced the interactions she had with students in class. This perspective also suggests that she did not appreciate the process,

from the students' perspective, of what is involved in learning and understanding a new and complex idea. However, it appeared that these findings on not wanting to waste time were also applicable to interactions with the researcher, and this will be discussed next.

Earlier work by researchers (Jacobs, et al., 2010) used the construct of noticing for teachers, in which they reflected on videos of their own practice as a model for professional growth. However the underlying assumption was that a teacher was open to professional growth through different means. As part of the methods used during classroom observations described in an earlier chapter, I had planned to introduce video clips of Carolyn's practice at some time during the study as part of the professional development intervention: for example by contrasting a potentially student-centered lesson in which many students were making meaning of the mathematics and were fully engaged, against a teacher-centered lesson in which Carolyn lectured with few students engaged. The intention of this intervention was to use it as a way to perturb her practice and goals towards to higher level in the framework (toward one that focused more on student thinking). The opportunity to share clips occurred during Lesson 5 debrief. The clip selected was a conceptual lesson that she facilitated and felt went well (Lesson 2) due to maximum student engagement, as compared to a skill-based lesson she taught (Lesson 5) that she also felt went well in which she used direct teaching with little student engagement. At the start of the lesson debrief, the researcher indicated that he wanted to show some video clips to her of class, starting with a lesson she felt good about (Lesson 2), in hopes of eventually highlighting the contrast between the two. The excerpt follows.

Excerpt 41. Carolyn seems pressed for time when shown video clips

- 1 Res: Before we start the lesson, I just wanted to highlight that fun lesson that I
- 2 think you felt pretty good about. So I just wanted to show you how that
- 3 played out, if you want to see it. Are you ready?
- 4 Car: For a little bit, okay.
- 5 Res: Sure. It's three minutes. (*Video clip starts to play*)

Although Carolyn's comment playing a video clip for a short time would be okay (line 4) caught the researcher off guard at first, he played the video to see what she noticed. However, the comment indicated to the researcher that she was not interested in this form of professional development, even when reviewing a lesson she liked. She seemed pressed for time. After viewing the video, her comment was that "it was really fun to see that they [*the students*] were so excited". The researcher thought student engagement could be used as the source of perturbation to get her to contrast this clip with the video clips from Lesson 5 (to be shown next). When the researcher mentioned showing her clips of the prior lesson she taught, her first question appears in the dialogue as follows.

Excerpt 42. Carolyn is not interested in video clips

- 1 Car: Are they short clips?
- 2 Res: Sure. They're one minute clips each.
- 3 Car: Sure, we could do that.
- 4 Res: Let's do it. (*Video clips of prior lesson clips played*)

Carolyn simply was not interested in looking at her practice (line 1). Her personal goals (Excerpt 22, page 137) valued efficiency and not wasting time, and the researcher's sense of the interaction unfolding with Carolyn is that she perceived viewing video clips as a waste of her time. When asked about what she noticed about the clips shown to her, she did not note the absence of student interactions during the clips. Rather she focused on the part of a clip where she made an error: "And I really said that 27 is not a multiple,

9 is not, what did I say? 27 is not a multiple of 9, did I really say that? (*laughs*) Oops!”

While what she noticed might be interesting data for a different research lens, the finding relevant to this focal point of the study was that using video clips to foster her professional growth was ineffective.

Successful Perturbations Not in Conflict with Goals

While my attempt to get Carolyn to reflect on her practice through videos was met with resistance, the same was not true in reflecting on the mathematics she was planning to teach; she valued learning new methods of doing mathematics. The analysis of this section discusses successful professional interventions occurred when the researcher focused on aspects of the lesson she valued did not conflict with her personal goals of being efficient. In the first vignette, I discuss a successful perturbation to influence Carolyn’s practice, while in the second vignette discusses how the goal framework was used as a way to inform the perturbation I introduced during lesson planning. In the third vignette, I discuss a successful intervention based on an aspect of lesson planning where Carolyn had specifically asked for the researcher’s input.

First Vignette: Lesson 7 Planning

Moves to increase Carolyn’s connections of the key ideas of mathematics that she was teaching (and in doing so, potentially led to her students’ connections) in a lesson were received positively. While planning Lesson 7 (Solving Logarithmic Equations), Carolyn had discussed her plans to have students solve logarithms of the form $\log_b x = \log_b y$ (under the heading *Hint #1* in her lesson notes, with a prescribed solution method and examples) and of the form $\log_b x = y$ (under the heading *Hint #2* in her lesson notes, with a prescribed solution method and examples). The methods Carolyn

emphasized in the lesson would rely on a combination of algebra (e.g., combining like terms, factoring, solving linear and quadratic equations) and known properties of logarithms. Seeing an opportunity for professional growth, the researcher asked her if her students knew how to solve a logarithmic equation in the form $\log_a x = \log_b y$, where the bases of the logarithms were not the same. After Carolyn wondered about and confirmed that the graphing method would be the most direct approach to solve this type of equation, the researcher worked with Carolyn to create a problem that her students could solve using the graphing method. While the researcher used the graphical approach on *Desmos.com* (an online graphing calculator), Carolyn worked on her graphing calculator to solve the equation $\log(2x) = \log_3(2-x)$. Both the researcher and Carolyn agreed on an approximate solution to the equation, and Carolyn incorporated this question into her notes for the lesson, as seen in the figure below, under the heading “Hint #3”.

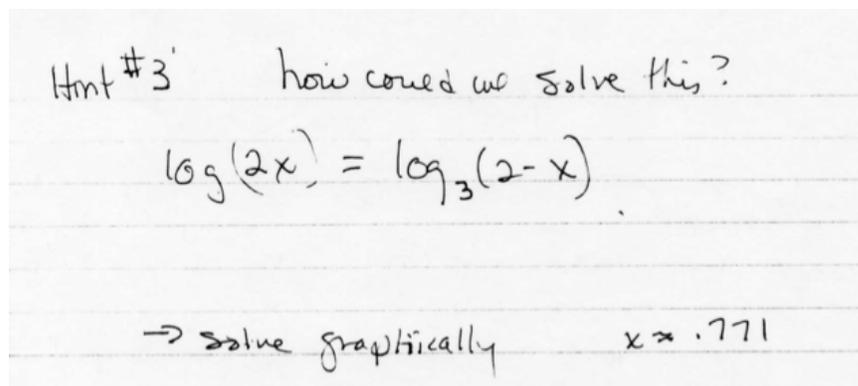


Figure 17. Proposed problem using graphical method (Lesson 7 Planning)

She mentioned liking the problem, and stated that it showed students another way to solve logarithms. While teaching the lesson, she introduced this method as the third way to solve a logarithmic equation. During the lesson debrief, Carolyn’s comments suggested a positive experience from this collaboration.

Excerpt 43. Carolyn values suggestion (Lesson 7 Debrief)

- 1 Res: I guess my question was you know, tell me about your experiences with
2 the activity. Because at this, students were engaged from that last part.
3 Car: Yes, and I hadn't done that before, and I appreciate your input on doing
4 that. I liked the fact that we had one that they couldn't solve any other
5 way. To see that it, actually that that does work, that you could do it that
6 way. And I hope that I get to that tomorrow when we do some of the
7 others that I, that whether we actually have time to do it or not, but talk
8 about the fact that you could do this with the graph.
9 Res: Uh huh.
10 Car: And do you want to? You know, and I will tell them I think it's going to
11 take you longer because you're going to have to enter everything you're
12 going to have the get the parenthesis right, you're going to have to set an
13 appropriate window and when you're taking an exam, you've only got a
14 limited amount of time, and you have to use your time wisely and is that
15 the best use of your time, to be doing that on your calculator?

One of the findings was that Carolyn incorporated suggestions that did not contradict her goals or what she valued. She liked learning new methods to solve mathematics problems that could not be approached in any other way (lines 3-5) or that provided students a different method to solve a class of problems (lines 5-8). She liked the opportunity to discuss with students the advantages and drawbacks of different methods, such as when comparing the graphical and algebraic approaches to solving logarithms (lines 10-15). In particular, since Carolyn's overarching goals were to prepare her students to take timed exams (lines 13-14), she planned to discuss which solution approach was more efficient, which was an aspect of doing mathematics that she valued. In retrospect, moves to build on Carolyn's key developmental understandings of the concepts she taught through alternative solution methods were received positively through the collaborative process.

Second Vignette: Lesson 10 Planning

Researcher attempts to help Carolyn make connections between methods in a procedural lesson were received positively also. In the planning session of Lesson 10, her stated goal of the lesson was “For them [the students] to be able to find zeros of *any* quadratic function – the real zeros”. The ‘any’ part of the statement and the ‘real zeros’ part of statement suggested to me she had a way she wanted her students to think about the key ideas of this lesson without it being articulated. She contrasted this statement immediately with a goal of her students finding imaginary zeros for polynomial functions for a lesson in the next week. Carolyn’s mathematical goals for student learning were ranked at TGSL4, as her statement implied getting students to think about ways to solving a quadratic equation, without the ways of thinking articulated. During the planning session she later clarified this statement to mean students would choose an appropriate method to find the real zeros of a quadratic function; she listed all possible available methods she had in mind (factoring, graphing, extracting square roots, completing the square, and quadratic formula), which would have refined the goal to TGSL3. I had used my initial characterization of TGSL4 (ways she wanted students to think about the key ideas of the lesson, without the way articulated) to scaffold my questioning. I had modeled Carolyn’s goals to be at a level in which a question asking her about ways students might think about the task in the lesson to be accessible. Since the lesson was skills-based, I asked Carolyn to consider how students might better make the connections between the methods available to them in solving a quadratic equation.

Excerpt 44. Researcher suggests making connections (Lesson 10 Planning)

- 1 Res: Okay, so you have your goals of you want them [*students*] to be able to
- 2 do these [*find solutions to quadratic equations*] four different ways.

- 3 Car: Yeah.
- 4 Res: How do your students get to that point? Or how your, those
5 understanding that your goals suggest, are developed or supported for by
6 students?
- 7 Car: Well some of the things, they've pretty much done all of this before.
8 Maybe not all tied together in one lesson, so I'm hoping that they will
9 see how the different methods already produce the same result.
- 10 Res: Uh huh.
- 11 Car: Although I'm not going to use the same, I'm not going to use the same
12 function for each of the different methods. But they're similar enough
13 that I think it should be clear that we're doing the same thing.
- 14 Res: Do you think it would be more powerful...
- 15 Car: That we're getting the, we're getting the same result.
- 16 Res: Do you think it would be powerful if you use the same example over and
17 over again? Or not really?
- 18 Car: I don't know.
- 19 Res: Meaning like the four different methods: here's this, notice you get the
20 same answer, and then here you do this, you get the same answer. And
21 then having a secondary example where obviously there's certain ones
22 that you can only use particular methods for because the solutions are
23 radicals, therefore factoring is hopeless.
- 24 Car: Well I suppose we could do that. I guess I hadn't really thought about
25 that, but I suppose that that would certainly be something we could do.
26 Yeah.

Although the researcher misspoke about the number of ways Carolyn had envisioned students' solving quadratic equations (she had mentioned five different ways) (lines 1-2), the researcher made specific moves to have her attend to how students might develop the understandings to tie the methods together (line 4-6). While Carolyn hoped that students would see that the different methods would yield the same result (lines 9, 13, 15), she did not plan on using the same function to support this insight (lines 11-12). The researcher asked Carolyn if she thought using the same function would better support students making connections between the methods (lines 16-17), and after saying she had not considered doing that before (lines 24-25), she decided to try it in the lesson (lines

25-26). During the lesson debrief, Carolyn thanked the researcher for the suggestion, as noted in the excerpt below.

Excerpt 45. Carolyn valued suggestion (Lesson 10 Debrief)

- 1 Car: I very much, thank you for suggesting that I do all those problems with
2 the same function. I do think that maybe I should choose a function that
3 I'm not dealing with fractions to begin with. Although I guess on the
4 other hand it's good for them to see that it's nothing scary about, when
5 you add a number that's not an integer. Because they do shy away from
6 fractions every chance they get.
- 7 Res: Thank you for the thank you. So what did you notice that went I guess
8 more smoothly because of the using the, exploiting the one example in
9 multiple ways?
- 10 Car: Well I don't know necessarily that it went more smoothly, but I thought
11 the good news was, it was justification that they were getting, that they
12 were doing things correctly, because we got the same answer three
13 different ways. We didn't do it, we didn't do the quadratic formula
14 because we ran out of time, but when you graph it, you know what
15 you're looking for, and make sure that that's what you find. And so
16 that's way to see that, that they're doing it correctly if they know they're
17 getting the right answer and then they get it. They know what answer
18 they're looking for ahead of time I guess.

The thanks by Carolyn (lines 1-2) meant that she had valued the researcher's suggestion when reflecting of the taught lesson; she liked the idea of using the same function. She had felt that finding the zeros of the same function in multiple ways provided a source of justification for students (lines 11-13), as they were able to verify that the different methods they used produced the same answer (lines 13-18).

Third Vignette: Lesson 9 Planning

Other attempts to perturb Carolyn's instructional goals were met with some success during the study. The researcher attempted to raise Carolyn's awareness of the value of having her students share their thinking and knowledge.. These moves were

motivated by the researcher's goal to have her confront her inclination to to explain everything to her students; and to give her an opportunity to listen to her students.

During planning for Lesson 9, Carolyn was considering how she might ask students to create a graph that related the distance Karen is from home, to the time elapsed since she started walking. This particular task necessitated a scale break (an intentional break, often with a zig-zag symbol, that is drawn on an axis to denote an omission of an interval of values on the drawn scale; a graph with a scale break is called a broken axis graph) on for the graph to be meaningful, and it was part of a longer task in which data from a table related the values of two quantities and was used to plot the graph. The instructor notes and these notes are visible in Figure 18 that follows.

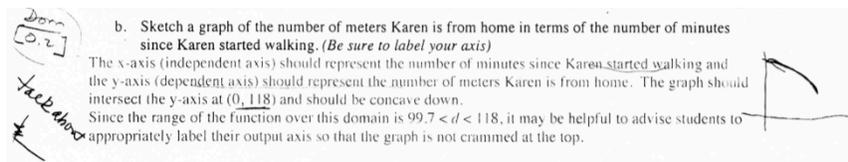


Figure 18. Thoughts on sketching a graph (Lesson 9 Planning)

There are two annotations in this image from Carolyn's lesson plan that are noteworthy. In the left margin there is an annotation that says "talk about" and below it a zig-zag symbol (denoting the scale break). In the right margin, a sketch of a concave down graph that Carolyn drew in response to the question is shown. An excerpt of a discussion Carolyn and the researcher had around this aspect of lesson planning follows.

Excerpt 46. Carolyn asks for a suggestion (Lesson 9 Planning)

- 1 Car: Tell me how do I tell them, how do I suggest, I don't know if you're
- 2 looking at the lesson right now.
- 3 Res: I am.
- 4 Car: In part (b) where they have to sketch, draw a sketch, how do I get them,
- 5 how do I suggest that they make their y-scale such that you can actually
- 6 see something?
- 7 Res: Uh huh.

- 8 Car: Because those y-values are going to range from what? 99 to 118.
9 Res: Exactly.
10 Car: And I know what they're going to do. You know: 1, 2, 3, 4, oh no, let's
11 do 10, 20, 30, but still all the y-values are going to in that little tiny area.
12 What would you suggest that I suggest, so that they get a graph that they
13 can actually learn something from?

In planning this portion of the lesson, Carolyn was concerned that students would chose an inappropriate scaling of the graph that would obscure the relationships between quantities as they change together (lines 4-6, 8). She predicted that students might first label the output scale by increase by ones (line 10) and then correct it to go by tens (line 11), but expressed that neither of these choices would be optimal for revealing how the output values vary (line 8). She indicated that she wanted the graph to be something students could learn from (line 13), although she failed to say what she hoped that they learned.

The researcher then suggested discussing the idea of the zig-zag (scale break) with her students. Carolyn was unsure whether students had ever seen a scale break before, and the researcher suggested she leverage her students' prior knowledge in the excerpt that follows.

Excerpt 47. Move to incorporate student contributions (Lesson 9 Planning)

- 1 Car: I'm not sure they've ever done that.
2 Res: Well this would be a good reason to try it.
3 Car: So you should you suggest that perhaps before I ask them to graph, that I
4 show them that that's *[drawing a scale break]* the way you could do
5 that, to skip a bunch of the y-values?
6 Res: And you would only do that at the very beginning *[of the graph]*, and
7 part of it is you'd add- or perhaps show them some data values and say
8 "If I wanted to graph this", before even I start with Karen, say: "What
9 would be a good, what would you guys suggest would be a good way to
10 graph it?"

At first, Carolyn was unsure (line 1) if her students had used a scale break before, but the researcher encouraged her to proceed (line 2), since this was a situation in which drawing a scale break had merit. By her query to the researcher (lines 3-5), it appeared that Carolyn was thinking about demonstrating how to create a scale break to her students. The researcher suggested to Carolyn to ask her students how they would graph such a situation (lines 9-10), thinking that it would be good opportunity for students to take charge of their learning, and for Carolyn to grow from the experience of allowing them to do the thinking. Carolyn's need for the scale break was based on her mathematical goal of students being able to characterize the concavity of the graph, and a poor choice of scaling would hinder such analysis. The researcher and Carolyn discussed this aspect of the lesson during the debrief, highlighted in the excerpt that follows.

Excerpt 48. Thoughts regarding lesson (Lesson 9 Debrief)

- 1 Car: I liked the fact that they knew how to put the thing, and in fact in fifth
2 hour they called it a break, just a break. Some sort of a break. Remember
3 you said you were wondering what to call that little zig zag?
4 Res: Right.
5 Car: They just said, let's put in a graph break.
6 Res: Okay.
7 Car: And I said: Oh. Well good, so they'd, and they knew how to do that and
8 that was a surprise to me because I don't guess I've really ever asked
9 kids to do that before.
10 Res: Uh huh.
11 Car: So that, they must do that in the science classroom, or in Econ, or
12 someplace, I don't know.

Carolyn liked the fact that her students came up with the idea of putting in a scale break (lines 1-2). She found the students knowing how to do a scale break pleasantly surprising (lines 7-8), but at the same time she acknowledged she had not asked kids to do that in the past (line 9). I inferred Carolyn meant "that" to mean "scale break" based

on her comments that followed (lines 11-12), in which she assumed students learned how to create scale breaks from a different discipline, such as science or economics.

Summary

To summarize, successful attempts of the researcher to perturb Carolyn's practice so that she made adjustment to her lesson can be collapsed into three categories. The first involved explicit moves to expand Carolyn's repertoire of methods to solve problems that built on her existing understandings of the idea she was teaching, as illustrated with the graphical approach to solving a logarithmic equation. The second involved attempts to make connections between representations or to make connections between mathematical methods more explicit, as was discussed in the example with connecting methods of solving a quadratic equation. The third involved researcher attempts to have Carolyn leverage student thinking and knowledge to engage students in confronting the challenging ideas in the lesson, as discussed in the example with the scale break.

Reflecting on this third category, the degree of success was limited by Carolyn's use of student contributions. She used student contribution more as a tool for engagement with the lesson or as a way to help move the trajectory of the lesson along. When using student contributions, Carolyn was sometimes surprised by what her students knew, but by the end of the professional development intervention her acknowledgement that students were capable of making sense on their own did not translate to her lessons to evoke students' ways of thinking; it was more about using student contributions as a segue into specific components of a larger lesson trajectory that she had in mind. Carolyn's use of student contributions during a class discussion was consistent with how she used individual student and group work, which I shall discuss next.

Carolyn's Views of Student Work Shaped by Other Factors

In this section, I will share findings regarding Carolyn's beliefs about the use of student work for learning. The analysis will discuss how her views influenced the use of student work as part of the classroom discourse.

One finding was although Carolyn liked when students worked with each other and helped each other during class, she did not effectively leverage student work as a tool for learning to promote learning for all, for example, when students participated by being selected for board work. On the few occasions during the course of observations when students were called to the board to write down their work, they would go up, write down their work, and sit back down. Rather than having the students explain their work and their thinking about the mathematics to peers in class, Carolyn proceeded to explain and annotate the students' work to the class. Likewise, students were not given opportunities to ask their peers to clarify their steps, since Carolyn did all the explaining, annotations, and clarifications.

Carolyn's selection of students to share solutions with the class was not random; it was intentional. She selected students whom she felt had a correct (or nearly correct) answer. Carolyn expressed early in the study she did not want her students to feel stupid (Excerpt 49, lines 1-2). She wanted them feel confident and successful. The excerpt below shares Carolyn's view.

Excerpt 49. Views about Honors students (Lesson 1 Planning)

- 1 They're skittish about asking a question if it makes them look stupid. If they ask
- 2 a question that makes them sound stupid, then they think they're stupid. They're
- 3 not, but some are shy about volunteering. They're not definitely sure of the
- 4 answer.

It followed from this rationale that student work was selected so that students could feel successful. Keeping Carolyn's view of how students might feel about sharing their work with their peers in mind, the researcher suggested having a pair of students from a group come up; that way, the presented work was the result of the group's contribution, and not any one individual's effort. At the researcher's suggestion of using paired board work during planning of Lesson 6, Carolyn gave another rationale behind her reluctance to have students come to the front of class to do board work, as illustrated in the excerpt below.

Excerpt 50. Using paired student board work (Lesson 6 Planning)

- 1 Res: Have you ever tried having them come up in pairs? Or the people who
2 are working together, next to each other, or something like that.
- 3 Car: Well, you mean put their solutions after they've done it, or just work on
4 the board, have a group's work on the board.
- 5 Res: What do you think?
- 6 Car Well, I do that sometimes, go to the board, but I really don't do that very
7 often and the reason is I like them to have it in their notes. And if they go
8 and do it on the board, they're not going to have it there.
- 9 Res: What if they have it in their notes....
- 10 Car: I guess I feel like in a traditional textbook, there are going to have lots of
11 examples that are, you know, step by step are there in your section. And
12 they don't have it with the Pathways book. They don't have that.
- 13 Res: Uh huh.
- 14 Car: They don't have any examples that are worked out. So I like to make
15 sure they get that down so that they have something to refer to.
- 16 Res: Right. What I remember you did last time is you had them work it on
17 their notes and they shared some of their solutions on the board, right?
- 18 Car Yes.
- 19 Res: Have you considered having them, let's say they worked in their notes,
20 they compare their notes and then go up to the board together, like in
21 pairs or threes.
- 22 Car: Oh. Well I could do that. You think people are less intimidated if they go
23 up with a friend?

Carolyn's reasons for her avoidance of student board work were more complex than my initial characterization. After suggesting paired board work (lines 1-2, 20-21) to

address her concerns that presenting work might make students feel stupid or intimidated (line 22-23), another concern she expressed was that having them produce their solution on the board would prevent them from writing their solutions in their notes (lines 6-8). Using the conceptual curriculum for her was a further disincentive to use board work, since she wanted students to have examples to reference (lines 10-12, 14-15), as is typical in a traditional textbook. After the researcher recalled that Carolyn had used student board work in the prior contexts where they had an opportunity to work on a problem in their notes first before sharing with peers on the board (lines 16-17), Carolyn agreed to try the paired board work. After further discussion, she thought of some questions where she anticipated different approaches by students and said, “I’ve got several examples, and some of them are pretty complicated. So yeah, I could have actually two or three partners come up and do the same problem.”

During the observation of Lesson 6, Carolyn volunteered two students from two different groups to present their work in expanding a logarithmic expression using properties of logarithms, as shared in Excerpt 51. The expression to be expanded was

$$\log_b \sqrt{xy} .$$

Excerpt 51. Carolyn volunteers students (Lesson 6 Observation)

- 1 Car: *(To Stu1 and Stu2)* Would you put your work over to the left (points to
- 2 board at front of room). The two of you go up there and write to me
- 3 these *(Stu1 and Stu2 go up to board, Carolyn walks to another group)*.
- 4 Car: What do we have over here? *(looking at work of Stu3 and Stu4)* How
- 5 about the two of you put this up onto the, they’re putting their problem
- 6 up, you put it also on that side on that board.
- 7 Car: Uh Stu2, move over a little bit to the left. Okay? So let’s see if your
- 8 method compares with their reasoning.
- 9 Stu3: Should we show our work?
- 10 Car: Yes! *(Stu3 and Stu4 prepare to go to board.)*

As students were working on the problems in their small groups, Carolyn walked around while answering questions, looking at student’s work and correcting misconceptions. She volunteered two students whom had completed the expansion correctly from two different groups (lines 1-3, 4-6) to share their solution process on the board. Based on the planning session, her criterion for selection was to compare different approaches to expanding the logarithmic expression. It is interesting to note that sharing the solution process did not appear to be a norm well established in class, with one of the students asking Carolyn if they should show their work (line 9). This is not a surprise, based on previously reported findings that Carolyn’s skill-based lessons promoted a calculational orientation of mathematics; hence higher importance was attached to the “answer” itself, not the “work” (the solution process) or reasoning leading to the “answer”. In Figure 19 below, the problem statement is given in the top center; student #1 and #2’s solution is on the left side, while student #3 and #4’s solution is on the right side.

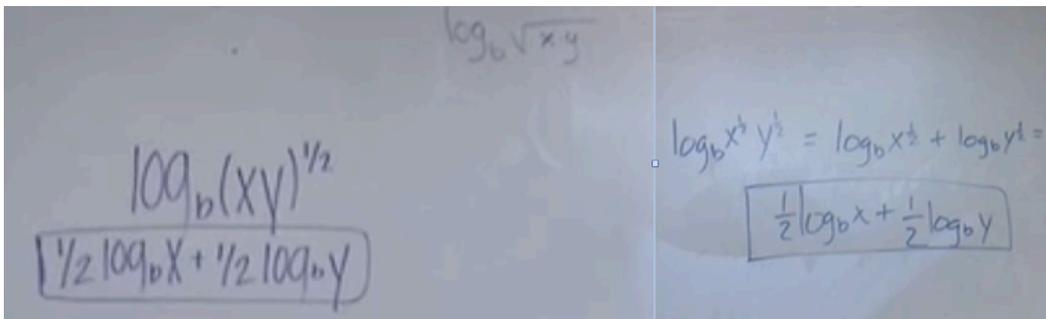


Figure 19. Students’ paired board work (Lesson 6 Observation)

Similar to observations when individual students would write their solutions on the board, she never asked any of the selected pairs to explain their approach, or the thinking behind their approach. Carolyn connected the dots and explained the methods for the class, rather than asking questions that the pair that worked on the question. First,

Carolyn stated to the class that the first group rewrote the square root of x times y as x times y to the one-half power. While stating this, she circled $\log_b(xy)^{1/2}$ and asked the class as whole if the step was “legal”, and can be seen in Figure 20. After enough students said yes, she then drew an arrow from the expression $\log_b(xy)^{1/2}$ to the second group’s expression $\log_b x^{1/2}y^{1/2}$ and asked the class if $(xy)^{1/2}$ is the same as $x^{1/2}y^{1/2}$.

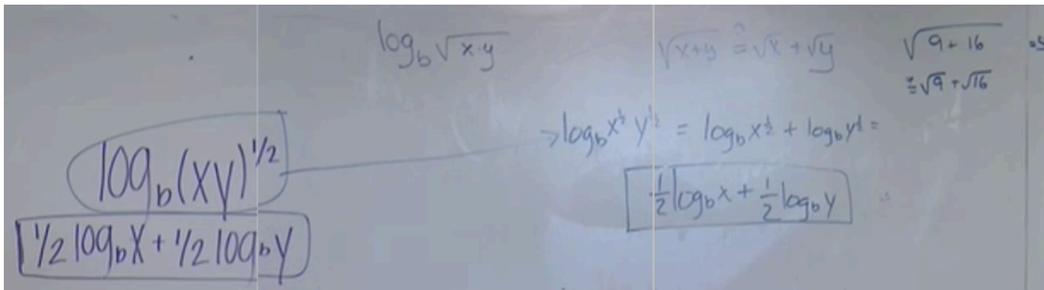


Figure 20. Carolyn’s annotations to paired board work (Lesson 6 Observation)

Then she asked students in class whether the square root of x plus y was the same as the square root of x plus the square root of y , and wrote a statement to this effect on the board (see Figure 20, upper right). To show should a claim is false, she used the numbers 9 and 16 for x and y to demonstrate that $\sqrt{9+16} \neq \sqrt{9} + \sqrt{16}$. While this seemed unrelated to the students’ work on the board, I believe her remark was the residue of a discussion Carolyn had with one group who had engaged with the task earlier; she had observed and corrected a student who had stated a misconception that the square root of x times y was the same as the square root of x plus the square root of y .

Carolyn explained that the square root of the sum of two values is not the same as the sum of the square root of those values; however, the claim that the exponent of product was equivalent to product of exponents was legal. Therefore both approaches were correct. She then referred to the theorems for logarithms used to change the logarithm of a product into a sum of logarithms (to justify

$\log_b x^{1/2} y^{1/2} = \log_b x^{1/2} + \log_b y^{1/2}$), and the theorem in which a logarithmic expression having an argument raised to a power can be rewritten with the power expressed in front (to justify $\log_b x^{1/2} + \log_b y^{1/2} = \frac{1}{2} \log_b x + \frac{1}{2} \log_b y$).

Carolyn's approach to paired board work followed similar pedagogical moves as with individual board work. Perhaps her beliefs about not wanting students to be embarrassed by being asked to account for or explain their solution process to their peers extended to pairs/groups. Another explanation is that Carolyn had no image of how to leverage student work in ways that involved students justifying their reasoning to others. After the lesson, the researcher asked Carolyn to reflect on this activity, as given in Excerpt 52, below.

Excerpt 52. Reflection on paired board work (Lesson 6 Debrief)

- 1 Res: What did think about the pair, group work, or the paired board thing?
- 2 Car: I thought, I thought that was okay. You know the one, with the group on
- 3 the left still didn't show all their work, and I thought maybe they would
- 4 when there were two of them together, but yes, I thought that was fine.
- 5 Res: Did you, do you think that would be valuable for future lessons, or you
- 6 think that was sort of like, it was okay, but not something to do in
- 7 another lesson?
- 8 Car: If I have enough time, I would do that again.
- 9 Res: Okay.
- 10 Car: Time is always an issue.

In reviewing the activity, Carolyn thought the paired board work was "okay" (line 2), but commented that she thought the group on the left would have showed all their steps if they showing their work together (lines 2-3). She felt that if she had enough time in class, she might use the paired board work approach again (lines 8, 10). Since paired board work was used for purposes of student engagement in tasks, rather than as a tool for learning, it could be sacrificed if time was lacking. Furthermore, Carolyn's use of

student work in lessons highlighting skills based tasks did not deviate greatly from direct instruction, other than students contributing the work from which direct instruction was then based.

Use of Student Work in the Context of Curricular Supports

In this section, I will discuss findings that support researchers' earlier work (Moore, et al., 2011) that curricular supports promoted shifts in a teacher's understanding of their key ideas of mathematics and her pedagogical practices. The findings add to this body of knowledge by describing the perspective from which a teacher views the student interactions in the lesson as a result of her pedagogical moves. Although students made meaning of mathematics through engagement with conceptually rich tasks, the findings suggest that Carolyn viewed these interactions primarily from the lens of student engagement; student group work was not viewed as a tool for learning mathematics.

In Lesson 9, the first lesson observed in the polynomials chapter, Carolyn had decided to use the conceptual curriculum in support her mathematical goals for student learning, which were for students "To recognize a function that's decreasing at a decreasing rate and a function that decreases at an increasing rate." Her task selection (in Carolyn's annotated instructor notes below) reflected that goal, in which students were asked to attend to the covarying quantities of: (1) Karen's distance from home and time elapsed since she started walking, (2) how these quantities changed, and (3) the average rate of change.

1. At 5:00 pm Karen started walking from the grocery store back to her house.
- a. Fill in the below table by determining the number of feet Karen is from home, d . Then use the information in the table to determine the average rate of change of the number of feet Karen is from home with respect to time (measured in minutes since Karen started walking) on the specified intervals.

Change in the number of minutes since Karen started walking = Δt	Number of minutes since Karen started walking = t	Number of feet Karen is from home = d	Change in the number of feet Karen is from home = Δd	Average rate of change of Karen's distance with respect to time
	0	118		
0.5	0.5	116.5	-1.5	-3
0.5	1	113.3	-3.2	-6.4
0.5	1.5	106.8	-6.5	-13
0.5	2	99.7	-7.1	-14.2

- b. Sketch a graph of the number of meters Karen is from home in terms of the number of minutes since Karen started walking. (Be sure to label your axis)
- The x-axis (independent axis) should represent the number of minutes since Karen started walking and the y-axis (dependent axis) should represent the number of meters Karen is from home. The graph should intersect the y-axis at $(0, 118)$ and should be concave down.
- Since the range of the function over this domain is $99.7 < d < 118$, it may be helpful to advise students to appropriately label their output axis so that the graph is not crammed at the top.

Figure 21. Carolyn's annotations (Lesson 9 Planning)

In part (a) of the task, students were asked to complete the table describing the changes in the quantities and average rate of change between the quantities, while part (b) of the task asked students to create a graph that related the distance Karen is from home, to the time elapsed since she started walking from home. During the process of lesson planning, Carolyn made a decision to have students do group work with larger white boards at their table. This exchange is shared in Excerpt 53, below.

Excerpt 53. Planning for student sharing (Lesson 9 Planning)

- 1 Car: I believe that I will get the big boards out tomorrow. And ask them to
 2 sketch these graphs together. I don't know, I guess maybe I'd like to get
 3 the values from the, that they have to fill in, up on the, I don't know if I
 4 should have them do those as a group or not too. I guess maybe I could,
 5 so have them on the board.
- 6 Res: So is this part of Investigation 1 you are continuing, or are you going
 7 into Investigation 2?
- 8 Car: Investigation 2. Where at first we have Karen walking away, walking
 9 towards her house.
- 10 Res: Uh huh.
- 11 Car: Looking at the distance she is from her house. And may I have, need to
 12 have the kids do this as a group too on their boards. So that we make
 13 sure that I can really make sure that everybody has the difference
 14 between d and delta d , and t and delta t , and the average rate of change,

15 so that I can make sure that we've got all those values, and everybody
16 has correct values to look at, and to calculate.

Carolyn planned for students to use the big white boards in small groups at their tables to sketch their graphs (lines 1-2) and to record the results from the table that tracked Karen's distance from home and the time elapsed (lines 11-12, 14), the changes between these quantities (line 14) and average rate of change (lines 14-15). She wanted to use the boards as a way to verify that all her students (line 16) were using the same correct values.

During class, as students were engaged in creating their solutions to this task, Carolyn asked representatives from two groups to take their white board work up to the front of class. Images of the two boards follow, in Figure 22.



Figure 22. Student work discussed (Lesson 9 Observation)

The notable difference between the two groups' work had to do with how the data points on the graph were connected. In the graph on the left, line segments connected the points. For the graph on the right side, the graph was concave down, which more closely matched Carolyn's annotations in her lesson plan (Figure 21, page 189). Carolyn started the discussion about the two graphs by asking students to look at the two graphs in front

of class. After consensus was reached that the two graphs did not “represent the same thing”, Carolyn probed her students further, as shared in Excerpt 54.

Excerpt 54. Discussion of a conceptually rich task (Lesson 9 Observation)

- 1 Car: What’s the difference in those two graphs?
- 2 Stu1: One line is curved. And one line is like shape connections.
- 3 Car: Okay. Let’s just, let me just say to you. If it’s curved, it’s not a line.
- 4 Stu1: The one....
- 5 StuS: Ooo!
- 6 Stu1: One of them is concave...
- 7 Car: One graph is curved.
- 8 Stu1: Curved.
- 9 Car: But this one is just a series of line segments, isn’t it? (*pauses*) Which do
- 10 you think is really happening?
- 11 StuS: (*talking over each other*)
- 12 Car: You think what, Stu2?
- 13 Stu2: We got the graph part.
- 14 Car: You think your graph is better. How come?
- 15 Stu2: (*incomprehensible*)
- 16 Car: It’s precise.
- 17 StuS: (*talking over each other*)
- 18 Car: Is yours correct? What would this mean is happening? This one
- 19 (*indicating the graph with the line segments*). I don’t know. Explain to
- 20 me. (*Selecting Stu3*) Okay yes.
- 21 Stu3: I think the first graph might be slightly less realistic, because the person
- 22 who is slowing down when they’re walking probably wouldn’t walk a
- 23 constant speed, then slow to a slower constant speed, and then slow to a
- 24 slower constant speed. It would probably be gradual.
- 25 Stu4: I don’t think she’s slowing down, I think she’s getting faster.
- 26 StuS: What?
- 27 Stu4: Isn’t she getting faster, because the distance from her house, is getting
- 28 smaller. That would mean she’s getting faster?
- 29 StuS: (*talking over each other*)

After posing the question asking students to explain the difference in the two graphs (line 1), the first student who was selected described the graphs in the shapes of the two plots using imprecise language (line 2), which then Carolyn corrected (lines 3). Carolyn then focused the class’s attention on the graph on the left and commented that it was a series of line segments (line 9). She asked students explain what it meant in the

context of the situation (lines 10, 18-19), and after the second student (line 3) claimed that his group's answer was better (because they had created the graph in question), Carolyn selected a third student who tried to make meaning of the situation that the graph represented (lines 21-24). This student claimed that the left graph would mean that Karen walked at a constant speed, then at a slower constant speed, and then at a slower constant speed, which she thought to be less realistic than the graph on the right that represented a gradual change in speed. Student 4 (lines 25, 27-28) remarked that he thought Karen's rate of speed was getting faster not slower, since the distance from home was getting smaller. Although Student 4 was correct that Karen's speed was getting faster, his rationale was flawed. Karen's average rate of change of distance with respect to time was decreasing and her distance from home was getting smaller; this is why Karen's rate of speed was getting "faster". After Student 4's comment, students began to talk over each other, and this topic of conservation was lost during the remainder of the discussion. Attending to the idea of whether Karen's speed was increasing or decreasing as time elapsed was challenging to Carolyn's students (see Excerpt 55), and it is noteworthy that Carolyn did not intervene to help students compare the different solutions, or consider what the graphs were conveying about the quantities in the situation as they changed together.

Excerpt 55. Students' misconceptions of rates of change (Lesson 9 Debrief)

- 1 Car: So I thought there was lots of good discussion, not necessarily about
2 what I thought it would be about. I did find this afternoon that in the first
3 one, where the graph was, at the rate of change was decreasing, those
4 numbers went from negative three to negative six to negative thirteen to
5 negative 14, there were lots of kids who raised their hand and said: that's
6 an increasing rate of change.
7 Res: Did that surprise you?

- 8 Car: Well it surprised me that I heard so many kids say that this afternoon, but
9 I didn't hear people say that this morning.
10 Res: Mmm, interesting.
11 Car: That is such a, that is such a good way to illustrate a decreasing rate of
12 change, looking at those, by looking at those values. Because I think
13 that's a hard concept for kids.

Although Carolyn liked the discussion, it had not gone as she anticipated (line 2). She found that her afternoon students stated that the rate of change of the numbers in the table (in the sequence -3, -6, -13, -14) represented an increasing rate of change, which surprised her (lines 8-9) since she did not observe the same utterances made by the students in the observed class. She stated this activity was really good for illustrating the concept of decreasing rate of change (lines 11-13).

Carolyn found the differing approaches to constructing the graph interesting, and that the way of drawing a graph by connecting points with line segments had emerged in both classes. She commented "I thought that it was interesting that there would be a group that would graph it that way and then we would have that, that we could talk about." She found the discussion about the graph interesting, but I contend that although engagement was considered a hallmark of a successful lesson (Excerpt 20, line 6, page 133), facilitating a conceptually rich discussion in which students were engaged in trying to make meaning of the situation as the tool for learning was not her focal point. The discussion emerged because Carolyn wanted students to explain their graphs. When the researcher asked her directly what group work did in terms of moving student thinking forward, her response was as follows.

Excerpt 56. What is the purpose of group work? (Lesson 9 Debrief)

- 1 Res: What do you think the group work did in terms of moving thinking
2 forward, student thinking forward?

- 3 Car: I saw them correcting each other sometimes when they were sketching
4 the curves, I don't know. Tell me, what was that question again?
5 Res: Yeah, so what did the group work do in terms of moving student
6 thinking forward?
7 Car: I don't know. I think maybe it made them just actually think, actually do
8 something rather than do nothing. It didn't, that didn't work in every
9 group, but it, you know. Hmm. Made them talk about how the graph
10 should look I guess, rather than just doing on their own.

When the researcher first posed the question of how she thought the group work did in terms of moving student thinking forward (lines 1-2), Carolyn's first inclination was to state what she saw students doing (lines 3-4). However, she hesitated (line 4) and asked the researcher to state the question again. After the researcher reiterated the question (lines 5-6), her immediate answer was "I don't know" at first, and commented that the activity made students think and do something rather than nothing (lines 7-8). She stated the task made students talk about the graph to each other rather than working on their own (lines 9-10). While the task itself may have been viewed as a tool for learning (see Excerpt 55, lines 11-13), student constructing meaning of mathematics and justifying their solution process to their peers as a result of working in a group was not perceived as a tool for learning; this simply was not part of her view of how students learn mathematics. Carolyn's comments in Excerpt 56 suggest a view of group work as a tool for engagement in a lesson. Student engagement and discussion were viewed positively; she liked the discussion that resulted as part of this activity, as is shared in Excerpt 57 that follows. It is noteworthy that she did not speak about students' thinking; nor did she pose questions to help students confront their misconceptions.

Excerpt 57. Carolyn likes student discussion (Lesson 9 Debrief)

- 1 Car: I liked the discussion about the line segments as opposed to the smooth
2 curve. Clearly everybody is not involved in those discussions but I think

3 that a lot of people were engaged in thinking about what the discussion
4 was. Rather than just, when multiple people were discussing I think that
5 that helps bring more people in even if they're not actively participating.
6 Res: Uh huh. So what do you think about the class discussion that resulted
7 from the engagement?
8 Car: I liked it. I liked it. I liked the fact that they didn't all agree, and maybe
9 at the end did agree, but that they were willing to express their opinion. I
10 guess I think that says it's a non-threatening atmosphere.
11 Res: Uh huh.

Carolyn felt the discussion got more students engaged with the lesson. Even if they were not actively participating (lines 4-5), her students were engaged in thinking about what was being discussed (lines 3-4). She liked that not all students agreed but because it was a non-threatening atmosphere students felt comfortable to discuss their opinions (lines 8-10).

Carolyn liked student engagement and discussion. Unlike in skill-based lessons, where Carolyn annotated and corrected student work, in conceptual lessons, her pedagogical actions suggested a curiosity of what her student were thinking. I conjecture her openness to facilitate a discussion that engaged students during some conceptual lessons occurred because she was in the process of making connections herself with regards to the mathematics of these lessons. She was learning a way of thinking about the mathematics along with her students, but she did not facilitate student discussions for the purpose of advancing student learning.

I suggest that the reason is two-fold. First, the discussion was viewed as a tool for engagement. Second, she did not yet have the conceptual tools herself to successfully facilitate a student discussion around the key ideas of mathematics: this would have required a disposition to attend to and think about ways her students might reason, and these were not reflected in her mathematical goals for student learning during the course

of the study. Her stated goals for this lesson were rated at TGSL4, which indicated a goal of getting students to think about the mathematics in the lesson, without the ways of thinking articulated. To attend to how thinking might develop required goals at level TGSL6, as these goals focus on attending to how student thinking might develop.

Characterizing Carolyn's MKT in the Context of Supporting Student Conjectures

An earlier hypothesis based on the results from the exploratory study with Robert was that lesson planning had to be re-conceptualized to promote higher level goals that attended more to students' thinking about mathematics. In my proposal for this dissertation study, I had hypothesized that lesson planning in which a teacher incorporated a question to promote student conjecture would require reflection on the key mathematics ideas. The purpose of using the tool of conjecture was intended for teachers to actively plan and set goals to promote and support student thinking (TGSL6). For student conjecture and reflection to be used effectively as a means to shift teacher's goals, I hypothesized that this meant a teacher would need 1) well connecting meanings of the underlying mathematics of the lesson, 2) the ability to leverage the conjecture effectively in the context of the key ideas of the lesson, and 3) an ability to link the ideas of the lesson back to the initial conjecture. Some examples of linking the ideas of the lesson back to the initial conjecture might include questions focused around asking students: did the mathematics you learned in the lesson revise your earlier conjecture (and why); did the mathematics you learned today support your initial guess (and why); did the mathematics you just learned help you think of a way to verify your initial conjecture (and why); how did you know if your conjecture was a good estimate or not, based on what you learned in the lesson today? This section discusses the findings

learned from this type of professional intervention. The first vignette shares what was learned from a question that Carolyn posed that related to the key ideas of a lesson. The second vignette shares what was learned from a question that Carolyn posed that did not relate to the key ideas of a lesson. While the first two vignettes discuss her use of student conjecture with conceptual lessons, the third vignette discusses Carolyn's views of using conjectures when teaching skills-based lessons.

First Vignette: Lesson 1

At the beginning of this study, a component of the researcher's planning included a request for Carolyn to write a statement to encourage her students to make a conjecture that would result in their making meaning of the situation. My thinking was that over the course of the study, this would shift Carolyn's mathematical goals toward the higher goals in the framework and that it would prompt her to consider the mathematical goals of the lesson.

In planning the first lesson together, the purpose of the lesson was to introduce students to the meaning of exponential functions, their notation, and the ways to represent their multiplicative growth pattern. The scenario of the investigation started with three people who know a secret nobody else knows. On the first day each person tells the secret to two other people, and then they do not tell anyone else. On the second day each person who had learned the secret tell two new people, and then they do not tell anyone else. This pattern of secret sharing continues for 15 days. Students are provided with a table to fill out during the course of the investigation to determine the covarying values of the day number and the number of people who learned the secret (using decimal, product, and exponential notations). Figure 23 includes the heading row of the table students were

expected to fill out during the activity, and Carolyn’s annotations with regards to conjecture she thought of to engage students.

day 2 [person 10][person 11][person 12][person 13][person 14][person 15][person 16][person 17][person 18][person 19][person 20][person 21]

1. Determine how many people learn the secret on day 2. Explain why this makes sense. Students should see from the diagram that 12 people learn the secret on day 2. This makes sense because twice as many people learn the secret on day 1 than day 0 (so 6 people) and then twice that many people learn the secret on day 2 than day 1 (so 12 people). Help students see that for each subsequent the number of people who learn the secret will be two times as large as the number of people who learn the secret the previous day.

From the diagram above, we see that 12 people learned the secret on day 2. We will represent the number of people who learned the secret on a given day in three different formats that will help us generalize the mathematics.

Decimal Notation	Product Notation	Exponential Notation
12 people	$3 \cdot 2 \cdot 2$ people <small>2 factors</small>	$3 \cdot 2^2$ people

*2. Let x = the day number and y = the number of people who learned the secret on day x . Fill in the following table:

y = Number of People Who Learned the Secret on Day x			
x = Day Number	y (Decimal Notation)	y (Product Notation)	y (Exponential Notation)

Handwritten annotations:
 - Left side: "How many days will it take for everyone to know secret?"
 - Right side: "Is this the same as how many people know the secret?"

Figure 23. Annotations from planning session (Lesson 1 Planning)

In planning the lesson together, I asked Carolyn to create a scenario that would involve her students in making a conjecture to engage them in the task, anticipating that Carolyn would be prompted to consider a meaningful task and it could be leveraged to shift the teacher’s goals to be more attentive to student thinking. Excerpt 58 describes one such conjecture used to support the first conceptual lesson that introduced exponential functions.

Excerpt 58. Creating a conjecture (Lesson 1 Planning)

- 1 Res: Before they even see that table [in question 2], is there something
- 2 that we could design which would produce something that would get
- 3 students to start conjecturing about this situation, so they are really
- 4 engaged with the activity?
- 5 Car: Like the doubling the penny on the checkerboard kind of problem?
- 6 Res: If you- yeah, that could work.
- 7 Car: Is that what you had, is that what you were meaning, maybe?
- 8 Res: Uh huh - or, just making a prediction about, if you were to imagine
- 9 this process continue with the secrets before they even saw the values
- 10 in the table, what do you predict after 15 steps? What do you have
- 11 them conjecture?
- 12 Car: How about this? How many, how many days would it take before
- 13 everybody in the United States would know the secret?
- 14 Res: Oh that would be fun! That’s exciting, because that gets them really
- 15 engaged.

- 16 Car: I wonder what the population of the United States is. I guess I would
17 have to look that up, huh? (*laughs*)
18 Res: Or have the students - are the students allowed phones in the room?
19 Car: Oh they've all got them.
20 Res: Have them look it up.
21 Car: (*writing*) Okay.
22 Res: If this model was, if this model was a fair model, right? Because that
23 also leads to a reflection question later. You can have them reflect: is
24 this reasonable?
25 Car: (*writing*) Uh huh. Yeah, and I still got that there. Okay, that might be
26 a good way, a good thing to do at the beginning, huh?
27 Res: Yeah.

At the beginning of this conversation, the researcher asked Carolyn what conjecture she might pose to maximize student engagement in the context of the lesson (lines 1-3). After Carolyn sought clarification to the researcher's question by associating the question to a related task (line 5), the researcher narrowed the scope of the conjecture around making a prediction about the number of people learning the secret on day 15, assuming the pattern of growth continued (lines 9-11), in order to force the conjecture to tie more directly to the activity. Carolyn proposed a related idea, which was to ask students to use the model to determine when the population of the United States would know the secret (lines 12-13). At first Carolyn stated that she would need to look up the population of the United States first (lines 16-17), but the researcher suggested that she have her students look up that information in class using their smartphones (line 20). After annotating her lesson plan to include the question to support student conjecture (see Figure 23 and Excerpt 58, line 21), the researcher commented that assuming the model used was appropriate (exponential), whether or not students would find their initial conjecture was reasonable, in the context of their findings as they completed the task (line 22-23). However, the researcher had not been careful in expressing his meaning to

Carolyn, which led to Carolyn interpreting the comment “is this reasonable” differently (lines 23-25). She commented, “I still got that there” (line 25) to refer to the annotation in her lesson plan “Is this a reasonable model?” as a question to ask her students.

Two points merit discussion in analyzing Carolyn’s intended classroom activity. The first was the opportunity to see how Carolyn would manage the discussion around the difference between knowing a secret and learning a secret, a question she had included in her lesson plan (Figure 23). Although the number of people who learned the secret on day d could be modeled by the function $f(d) = 3 \cdot 2^d$, the function describing the number of people who know the secret was more complex. The number of people who know the secret on day d was the sum of the original three persons who knew the secret, and all those who learned the secret thereafter. Hence a model of the function describing the number of people knowing the secret was $h(d) = \sum_{n=0}^d 3 \cdot 2^n$. I will go into more specifics of the reasoning Carolyn and her students used during the analysis of the class discussion that emerged.

The second point was the intention of the use of conjecture with regards to this professional development intervention. How would Carolyn use it to further student learning? Since this was the first collaboration, what emerged from the classroom interactions could inform possibilities and limitations of the teacher beginning a lesson by asking her students to make conjectures. How would Carolyn address the different ways of reasoning that students expressed when making their conjectures? The researcher anticipated possible ways students could reason; these included: using the idea of constant rate of change from prior lessons, or using the idea that the rate of change was

increasing on an intuitive level, and that the specific model was exponential growth. It was also possible that the conjecture was no more than an arbitrary guess informed by what other peers said. Alternatively, the conjecture could be guided by shape thinking: Carolyn’s classroom had posters of graphs of fundamental functions on the back wall of the room that were large enough that for students to easily see from where they were sitting. Some examples of these fundamental functions on the back wall included:

$$y = x, y = x^2, y = x^3, y = |x|, y = \frac{1}{x}, y = \sqrt{x}, \text{ and } y = b^x .$$

During the lesson observation, Carolyn posed the problem statement to students as follows: “Three people know the secret. And the next day each one of them tells two people. Then they keep their mouths shut. And they don’t tell anybody else. Now is this realistic?” After students remarked no, she moved on. To note, this question (and the students’ response) represented Carolyn’s move to ask whether the exponential model proposed was reasonable, which was different than the researcher’s intent to have students to reflect on whether their conjecture was reasonable or not. Carolyn read the rest of the problem statement, and clarified that the next day after each person had learned the secret, they each told two people, and never again told anyone else again. Then Carolyn made the move to introduce the conjecture she proposed during planning.

Excerpt 59. Staging of a conjecture (Lesson 1 Observation)

- 1 Car: Here’s what I’d like you to do. Without doing any calculations,
- 2 without pushing any numbers in your calculator, I would like you to
- 3 come up with a consensus. You are going to have to get out your
- 4 phone probably, because I don’t even know the number. How many
- 5 days is it going to take if this pattern would hold for everyone in the
- 6 United States to learn the secret?
- 7 StuS: *(students talking over each other)* Woah - how many people in the
- 8 United States - yeah.

- 9 Car: That's what you have to look up. I don't know.
10 StuS: *(students talking over each other)*
11 Car: Let's decide on a number though. But then I don't want you to do
12 any calculations. I just want you to talk about, how many days do
13 you think it will take.

After relaying the instructions for students to talk about the number of days they think it would take for everyone in the United States to learn the secret (lines 4-6, lines 11-13) and to look up and come up with a consensus for the population of the United States (lines 2-4, 11), a consensus of 314 million emerged. Students worked in small groups of 3-4 students, and Carolyn asked students to write their group's conjecture on the board. As representatives from each group went up to the board, the conjectures listed on the board were: 21 days, 27 days, 35 days, 87 days, 40 days, and 33 days. After Carolyn confirmed that all groups had weighed in, she begins the discussion by stating her assumption of her students' responses.

Excerpt 60. Class discussion of conjecture (Lesson 1 Observation)

- 1 Car: I'm assuming then, am I under the right assumption that this is not
2 going to be a linear growth pattern?
3 StuS: Yeah.
4 Stu1: It will be an exponential growth pattern.
5 Car: What?
6 Stu1: An exponential growth pattern.
7 Car: What does that mean?
8 Stu1: Good question. It's like this *(off camera- gesturing in the air)*. Let's
9 figure it out.
10 Stu2: Just look up there. *(off camera- pointing to back wall)*
11 Car: Just look up there! Stu1, what did you say?
12 Stu1: Oh. *(laughs)* It's going to get bigger! By a lot.
13 Car: It's going to get bigger by a lot.
14 Stu1: Yes.
15 Stu2: How much?
16 Stu3: *(incomprehensible)*
17 Car: Oh! Stu3 said what? The change in the growth is increasing.
18 StuS: *(students talking over each other)*
19 Car: Think about what that might mean.

- 20 Stu4: It's accelerating.
21 Car: It's what?
22 Stu4: Accelerating.
23 Car: Accelerating? That sounds like a physics term.
24 Stu4: Yes. Physics taught me all this. Yay Physics!
25 Car: So what does that mean, if it's accelerating?
26 Stu4: That means the rate of change is increasing with each, with each
27 passing day.
28 Car: Oh that's interesting. The rate of change is increasing with each
29 passing day. If this were linear what would be happening to the rate
30 of change?
31 Stu4: It would be constant change.

A couple of issues emerged in this exchange that merit discussion. Carolyn stated at the beginning of the exchange that her assumption that students did not think the function was linear (lines 1-2). This move may have been due to her observations of her students' discussions as Carolyn walked around while students were talking about their conjectures, or it is possible that she made this pedagogical move for the sake of efficiency. The discussion that mostly consisted of her more vocal students in class had a propensity toward shape thinking (lines 4, 6, 8-10), as demonstrated through the use of gesture and pointing to the graph of the exponential function on the back wall of the classroom. Whether these students understood exponential functions beyond the shapes they described was unclear, other than a vague meaning of exponential growth pattern is one in which the output was going to get "bigger by a lot" (line 13). As the discussion continued, Carolyn re-voiced one student's comment that an exponential growth pattern was one in which the change in growth of the function was increasing (line 17), and asked the class what such a statement might mean (line 19). A fourth student in the discussion suggested this meant acceleration (line 20). As Carolyn asked for clarification of what this student meant by acceleration (line 25), he stated that this meant the rate of

change was increasing with each passing day (lines 26-27). Carolyn asked this student to contrast this to the rate of change of a linear function (lines 29-30), to which the student replied a linear function would have a constant rate of change (line 31).

While this characterization of exponential growth is imprecise (there are numerous functions whose rate of change are increasing, but are not exponential) this was her students' first formal introduction to exponential functions in Precalculus. It was likely that students had seen exponential functions before in an earlier course, such as Intermediate Algebra. However, this exchange did suggest a lack of awareness of possible misconceptions that could result from not anticipating student utterances or by not following up for further clarification. Carolyn could have asked students to think about whether all functions whose rate of change is increasing is always exponential, or to address that possible misconception later in the lesson, however she did not do so during this lesson. Perhaps for Carolyn, such a characterization was sufficient for the first foray into exponential functions. It is also possible that she was not listening carefully to her students, or she had not thought carefully about what her students were saying until after the fact.

Notably absent both from this exchange, and from the lesson itself, was attention to quantities or by extension, the covarying quantities in the context of this activity. In the lesson debrief, Carolyn's response to the researcher's question of how she thought the lesson went is given in the excerpt that follows.

Excerpt 61. Carolyn's observations (Lesson 1 Debrief)

- 1 Car: I thought that my objectives were met. They recognized that it's
- 2 exponential growth. They were able to verbalize what that meant to
- 3 them and they did a good job of being able to write a function to
- 4 describe what they saw, and what they didn't see, even abstractly.

- 5 Res: Uh huh. What did you, what was the evidence you used to ascertain
6 that? I guess what was the evidence you used to say okay, I feel good
7 that they understood that?
- 8 Car: As I walked around, I could see them writing the functions. And,
9 listening to their discussion, I heard words like, growing at a faster
10 rate, and not linear, and I saw them, doing this [*tracing a curving up*
11 *shape with her arm*] with their hands so that I could see that they
12 weren't seeing it as linear growth.
- 13 Res: So you think they were imagining, because I remember in our
14 class...
- 15 Car: Somebody was pointing to the back.
- 16 Res: ...was pointing to exponential growth. So you think of them, in a lot
17 of their minds, that's what they have is the graph as the image of
18 exponential growth. In other words, that's the representation they're
19 using...
- 20 Car: Uh huh.
- 21 Res: ...as their way to make sense of it.
- 22 Car: I think so.

In stating that her objectives were met (lines 1-4), Carolyn was referring to her mathematical goals for student learning, in which she had stated that “students would recognize exponential growth, and be able to represent the exponential growth by writing a function.” When the researcher asked for the evidence she used to ascertain that her goals were met (lines 5-7), she used her observations of what students were writing (line 8), what students were saying (lines 9-10), and their gestures in the shape of an exponential graph in the air (lines 10-12), rather than contrasting their responses with the meaning of exponential growth. Carolyn listened for key words, such as “growing at a faster rate” and “not linear” (lines 9-10) as a way to gauge her students’ learning of this lesson. Carolyn used students’ gestures as way to assess her students’ learning, corroborating the researcher’s observation that some students used the poster of the exponential graph in the back of the room as a way to describe their thinking (lines 13-22). The gestures provided evidence to Carolyn that her students recognized the function

being studied as non-linear (lines 11-12). Based on this exchange, I found early evidence to suggest that Carolyn's MKT about exponential functions had gaps. She did not express that exponential growth described how two quantities changed together, a point of emphasis in earlier lessons. Her goal for this lesson appeared to be rooted in the notion of "getting bigger and bigger" and "increasing at an increasing rate". She responded in ways that suggest that a response that a function is "growing at a faster rate" is an acceptable explanation from students for claiming that a function is exponential. Her acceptance of students' gestures without further probing about the meaning about the shape drawn in the air also suggested either Carolyn's inattention to quantities or the covarying quantities that generated the shape, or to her impoverished conception of exponential growth.

While the conjecture about when everyone in America would know the secret (assuming the pattern held) had been used successfully as an entry point for student engagement in the lesson, Carolyn did not use as an opportunity to tease out student's possible reasoning behind their conjectures.

Students agreed that the relationship they were looking at was non-linear. Perhaps students simply guessed, or they were trying to apply novel thinking about the situation, meaning that they used an intuition about a function that grew at an increasing rate to make a conjecture about the days it would take to know this secret. It is possible that students looked at the shape of the exponential graph $y = b^x$ on the wall and used the image as a guide in making a prediction about the number of days it would take for the population of America to know the secret. From prior coursework from Algebra 2, they may have had an intuition about quadratic and exponential functions. Based on the student utterances from the classroom observation and the debrief with Carolyn, these

seemed to be potential ways students were thinking about exponential functions at the start of the lesson. However, these exchanges suggested that Carolyn did not have the inclination to question her students on their reasoning behind their initial conjecture, suggesting that using student conjecture as a way to both engage students, and build on their understanding of the mathematics, was not on her radar. At this stage of the study, Carolyn was disinclined to attend to students' thinking or the reasoning behind their utterances. The idea of having students make a conjecture to promote students' disposition to reflect on the key idea of the lesson was not effective at this stage of the study. Even though her prompt to her students to make a conjecture resulted in student discussion, her inattention to student thinking and what meanings they held for exponential growth resulted in little or no advancement in their understanding of exponential growth. These results led me to refine my approach to framing the task for the teacher to prompt students to make a conjecture about some aspects of an idea of a lesson.

The findings from this episode suggested that in order to use student conjectures in support of the key ideas of a lesson, a teacher also needed to view effective instruction as supporting student thinking and reasoning. I had hypothesized that well-connected meanings of the underlying ideas of the lesson and getting the teacher to pose questions to promote student conjecture would cause the teacher to shift to higher-level goals for student learning. My model was incomplete and therefore needed refinement. The question Carolyn asked to promote student conjecture had supported the key ideas of the lesson, but the findings suggested that well-connected meanings of the mathematics was insufficient in shifting Carolyn's goals for student learning. An orientation that valued

student thinking, and a clear image of the desired thinking and how it might develop, appeared to be obstacles to her adopting higher level goals for student learning.

Second Vignette: Lesson 4

This vignette discusses findings that resulted from my work with Carolyn in which she posed a question to promote student conjectures that introduced concepts unresponsive of the key ideas of a lesson. In particular, a finding of this study was that fragile understandings of the foundational ideas of the mathematics of a lesson boded poorly for leveraging conjectures in a way to support student learning of the key ideas of a lesson. Recalling the revised framework of interactions (Figure 12, page 136), poorly connected meanings of the underlying mathematics (KDU) inferred limitations on a teacher's ways of leveraging the desired mathematical content in pedagogically powerful ways that could support student reasoning (MKT). A failure to ascertain whether a question posed to encourage student conjecture was suitable in the context of the key ideas of the lesson led to additional challenges to the enacted lesson itself during the fourth class observation.

A discussion resulting from the question that encourages student conjecture can either support or distract from the key ideas of the lesson. The question might engage students, but the key ideas of the question might not tie to key ideas of the lesson. The discussion may even result in an exchange in which students are engaged, and this was noted during the fourth class observation. The mathematics idea around which Carolyn had her students make a prediction was self-contained; she did not leverage the discussion in a way or think about how the conjecture might have tied to main ideas of the lesson, and the resulting exchange moved the trajectory of the lesson no closer to her

goals. Carolyn had described her mathematical goals for student learning for the fourth observed lesson as follows.

That they [*the students*] can express those growth factors. And that they can express a function to describe the situation they are reading about, or looking about. You know, some are from information, some are from tables, some are from graphs. So I'd like them to get a feel for approaching those problems three different ways.

Carolyn's stated goals were to use multiple representations to highlight the idea of exponential growth as the focal point of the lesson. Carolyn's choice for the conjecture at the lesson opening was to confront possible misconceptions about exponential growth with regards to n -unit growth factors, before delving into the main lesson that explored partial unit and multiple unit growth factors through multiple representations. In Question 1 of Lesson 4, students were given the function $f(x) = 9.5(1.24)^x$, and asked to determine the $\frac{1}{2}$ -unit growth factor, the 2-unit growth factor, the 2-unit percent change, and the initial value. While discussing possible student misconceptions during lesson planning, Carolyn planned to address the student misconception using a linear growth pattern to predict outputs of non-unit growth factors. She planned to use their conjecture at the start of the lesson as an extension of Question 1. She framed the conjecture that she would pose to students to confront the misconception as follows.

I'm sure, if the growth factor is given as 1.24 and I say what, let's guess what we think that half unit growth factor would be, or the 2-unit growth factor, I don't bet anybody would be even close.

At the beginning of the lesson, each group of three to four students had a small board and markers. Carolyn asked her students to predict the value of $f(2)$ given the function $f(x) = 9.5(1.24)^x$, after students had engaged in finding the initial value, one-

unit growth factor, and one-unit percent change. Carolyn asked students to make a prediction without doing any calculations.

Excerpt 62. A question to encourage conjecture (Lesson 4 Observation)

- 1 Car: *(walking around class to individual groups)* I don't want you, I just
- 2 want you, I don't want to do calculations, I just want you to do some
- 3 thinking, what might that be? $f(2)$.
- 4 Stu1: It's not times 2. It's some power.
- 5 StuS: *(Talking in groups and working)*
- 6 Stu1: So it has to be less than 2.
- 7 Car: *(walking to a group)* I'm looking for a prediction. What do you
- 8 have? *(laughs)* I love it!
- 9 Car: *(walking to another group)* Stu2, do you have a prediction?
- 10 Stu2: I had but I just erased it.
- 11 Car: What did you have?
- 12 Stu2: Eight point five.
- 13 Stu3: I had 0.3.
- 14 Car: *(walking to next group)* What do you guys have?

Carolyn posed the question, and her students were engaged in the activity. While some students used this activity to make meaning of exponentiation (lines 4, 6) others appeared to resort to haphazard guessing (lines 12, 13). However, the students' ways of thinking were opaque, and Carolyn did not make pedagogical moves to help students draw upon their prior knowledge to make an educated guess. An example of a supporting question to move students away from haphazard guessing might have included: what number would be too small, or what number would too large (and why). The conjecture engaged students (lines 7-8), which is something Carolyn valued. As the different students engaged in this activity, Carolyn summarized some the values different groups obtained for their prediction of $f(2)$.

Excerpt 63. Isolated conjecture (Lesson 4 Observation)

- 1 Car: *(walking toward front to room and reading out other students'*
- 2 *responses)* 11.8, 14, 20, 30.

3 Car: (*At front of room, speaking to whole class*) How, what are you
4 thinking are you using to come with that prediction?
5 Stu4: We just square it.
6 Car: Square?
7 Stu4: We square 1.24, it's just 1.24 times that (*gesturing to group white*
8 *board, off camera*).
9 Car: And what is 1.24 squared? Do you have any idea?
10 Stu4: It's not greater than 2, it's probably still like 1, but we say that it was
11 bigger. But we're saying the number is bigger than 11.
12 Car: Did you hear what he said? 1.24 squared is not going to be as big as
13 2, he said. Do you agree?
14 Stu2: I guess.
15 Car: So if 1.24 squared is less than 2, then 9.5 times a number less than 2
16 is going to be less than (*pauses*). Ooo...did you think about that
17 when you made your prediction of 25 or 30 or 32?

In the class discussion, Carolyn asked her students what thinking they used to determine their predictions (lines 3-4). One student commented “we just square it” (line 5), getting Carolyn’s attention (line 6). The student commented squaring meant 1.24 times “that”, pointing to the writing on his group’s board (lines 7-8). After Carolyn pressed for his interpretation of squaring 1.24, the student commented that the product of the factor squared is not greater than 2 (line 10), so his group estimated the product of the factor squared to be bigger than 1 (line 10). Therefore group conjectured the output of the function to greater than 11 (line 11). Carolyn re-voiced the student’s comment that 1.24 squared was not greater than 2, asking the class if they agreed with this statement (lines 12-13). After one student tentatively agreed (line 14), Carolyn used this as an opportunity to rhetorically ask whether they had thought about using the fact that 1.24 squared is less than 2 when making predictions about the value of $f(2)$, citing some students’ predictions where this constraint had not been used (lines 16-17).

Carolyn did elicit student thinking in this context. However, in thinking about Carolyn’s comments that encouraged her students to reflect on the reasonableness of their

predictions (lines 15-17), this seemed to be a pedagogical move analogous to direct teaching. In effect Carolyn stated why certain predictions were wrong, rather than encouraging students to reflect on which predictions could be ruled out through the course of the activity. Carolyn moved on from this activity and continued to the remaining part of the lesson without attempting to tie together the ideas discussed.. It is possible that Carolyn did not consider how the conjecturing activity could be connected to the core component of the lesson, either because this activity itself did not relate to the key ideas of the lesson, or she did not see how to tie this activity to the key ideas of the lesson. In either case, Carolyn's understandings of the lesson's main ideas were not robust enough to make this judgment, and Carolyn's stated goals for the enacted lesson were not supported by this activity. The findings suggest that the activity in support of a conjecture must be meaningful to Carolyn in the context of her lesson; otherwise she appeared to not view the activity as supporting the development of student understanding of exponential growth. During the lesson debrief the researcher asked Carolyn about the use of conjecture during the lesson.

Excerpt 64. Conjectures should be meaningful (Lesson 4 Debrief)

- 1 Car: Well I tried to get them to do some conjectures that in first hour and
- 2 that was interesting, but I don't think that it helped them. I don't
- 3 think they made connections with what we were doing, to how to
- 4 figure those growth factors out.
- 5 Res: Uh huh.
- 6 Car: I mean it was interesting, but I don't think having them do that at the
- 7 beginning was worth the time that it took.

This exchange about the role of this conjecture in the lesson was noteworthy, since Carolyn herself stated the lack of connections between the conjecture and main lesson (lines 1-3). She felt that the time spent in this activity, although interesting, did not

help her students learn the key ideas of the lesson (lines 3-4). She also felt the activity was not worth the time it took (lines 6-7), which from earlier findings suggests the poor use of time conflicted with her values of efficiency with regards to use of class time.

To see whether Carolyn might have thought of a conjecture that would be supportive of the key ideas of the lesson, the researcher moved to see whether she had considered alternatives to the conjecture she used.

Excerpt 65. Were other conjectures considered? (Lesson 4 Debrief)

- 1 Res: So maybe a different conjecture would have served them better.
- 2 Car: Possibly.
- 3 Res: Do you have any thoughts on what conjecture would have been a
- 4 more useful, something to ponder, in relation to the lesson?
- 5 Car: I don't. Have you?

To summarize this exchange, the answer was no: Carolyn could not think of an alternate question to pose to encourage student conjecture that would tie more directly to the main ideas of the lesson (line 5). From the observation of the enacted lesson, Carolyn had struggled in supporting students' understanding of the key ideas. This supported my findings that her underlying understandings of the mathematics of the lesson itself were not well developed.

After watching the enacted lesson and reflecting on the disconnectedness of the conjecture from the main thrust of the lesson, the researcher had thought of an activity using the value of two bank accounts that grew at different rates in different time intervals as a context that could have tied more directly to the lesson. It was a moot point after the fact, although Carolyn was curious for the researcher's opinion on a more relevant conjecture. At the time, the researcher decided to steer the discussion of the lesson to other parts of the enacted lesson that were rich with data of Carolyn's tenuous

MKT, since the amount of time to discuss the lesson with Carolyn was limited. A more relevant opening question the researcher had created to promote student conjecture is given in Appendix M. The conjecture Carolyn chose for the fourth observed lesson did not support the key ideas of the lesson. Due to struggles with her own understanding of the idea of exponential growth, Carolyn reverted to more traditional practices after the fourth lesson observation.

Third Vignette: Lesson 5

This third vignette shares results learned from asking Carolyn to consider a question to pose that would promote student conjecture in a skills-based lesson. Carolyn began using skills-based lessons from her prior notes for the remainder of the exponential chapter starting with the fifth lesson observation. As a result of using skills-based lessons, Carolyn did not see the need for using student conjectures at the onset of a lesson. Her views of using conjectures for skills-based lessons are documented in the excerpt that follows.

Excerpt 66. Conjectures and skill-based lessons (Lesson 5 Debrief)

- 1 Res: Any conjectures are you planning to ask [*in the next lesson*]? This is
- 2 one more about them expanding/condensing logarithms. Are there
- 3 are any conjectures that you want to incorporate that you think would
- 4 be meaningful for their learning?
- 5 Car: Oh no, because I don't think they'll see the point of expanding and
- 6 condensing a logarithm for a while yet.
- 7 Res: Okay.
- 8 Car: But sometimes, as you know sometimes you have to learn the
- 9 logistics before you can apply it to something.

Carolyn's perspective of teaching skill-based lessons was traditional. She believed that students had to learn a skill prior to applying it (line 8-9). Furthermore, she did not think her students would find making a conjecture meaningful in the context of a skills-

based lesson (line 5-6). I suggest however that her beliefs were sustained due to her own tenuous connections with regards to exponential functions and logarithms. For example, Carolyn could have asked students to think about comparing a couple of pairs of logarithmic expressions without calculating, and to state which expressions they think represents a larger value of two (and why). Or if students thought the two expressions were equal, state why. For example, she could have asked students to compare expressions such as: $2 \log 3$ and $\log 3 + \log 3$; $\log 3 - \log 2$ and 1 ; $\log 2 + \log 3$ and $\log 6$; $2 \log 3$ and $\log 9$. Such a conjecture could have been used to confront misconceptions during course of the lesson, along with a rationale of the end of lesson that gave students an opportunity to take a second look at their initial conjectures, refine them, and justify their reasoning in light of new knowledge. However, after her recent struggles with the fourth observed lesson, the researcher did not think it would be beneficial to press Carolyn on using conjectures.

Summary

The findings of this section had suggested that robust connections of key ideas of mathematics of the lesson, and a disposition to attend to students' reasoning were necessary to support the use of student conjectures during a lesson that builds on the mathematics of the lesson. The findings also suggested that from the perspective of Carolyn, conjectures could be used as a tool for student engagement in conceptual lessons. Such engagement in an activity promoting conjecture was valued only when it did not hinder the flow of the lesson trajectory and goals. The findings suggested that student conjectures were not necessarily viewed as a tool for learning mathematics, although not surprising in the context of other findings. For example, prior results

showed that Carolyn viewed student group work also as a tool for engagement in a lesson rather than as a tool for her students' mathematical learning.

Characterizing Carolyn's MKT in Terms of Gaps That Supported Learning

In this section I share results that reveal Carolyn making conceptual connections while interacting with her students as they attempted to respond to a question in the conceptual curriculum. Her genuine engagement with the task appeared to be productive in advancing both her and her students' solution approach. Her gaps in knowledge were small enough where she was able to support students and her own learning simultaneously, and at the time she had not formed an opinion of the most efficient way to proceed with particular mathematical tasks. The episode that follows revisits the exponential growth problem from the first lesson, in the context of Carolyn's support of multiplicative reasoning as a way to make sense of the mathematics of the task. This conceptually rich task promoted meaning making and growth of mathematical knowledge for both Carolyn and her students.

Carolyn's gaps and weak connections afforded instances in which students had opportunities to make meaning of the question and leverage their knowledge to advance their solution. An aspect of her success in this situation may have been that she did not have a more "efficient" method in mind to promote in the course of the lesson; this led to honest attempts to engage in meaning making and "less efficient" ways of thinking being exhibited as a natural process of moving to a solution, rather than immediately being squelched because it is less efficient. After the fourth observed conceptual lesson with exponential functions, Carolyn's knowledge gaps lead to a retreat in her practice to skills based lessons for the remainder of the chapter pertaining to exponential functions and

logarithms. However, prior to that time, students' contributions at times were more highly incorporated in the course of the class discussions, simply because Carolyn did not "know" any better. This led to some findings in this study that merit discussion.

I will begin by discussing Carolyn's MKT with regard to Lesson 1. The question posed to students was to predict the time it took the population of the United States to know the secret, assuming the exponential pattern held (see Excerpt 58, starting page 198, lines 12-13). I was curious to see how Carolyn would facilitate the class discussion, once students had noted the pattern of exponential growth (doubling) from the initial three people who had learned the secret. The people knowing a secret versus those learning a secret were two distinct questions with different models. The table students filled out in the workbooks had been projected onto the whiteboard, and it tracked the day number and number of people who learned the secret (in decimal, product, and exponential notations). Returning to the opening question, the discussion that emerged was as follows.

Excerpt 67. Moves to support student thinking (Lesson 1 Observation)

- 1 Car: *(to class)* So do you recognize that I didn't ask you the right question
- 2 over there? But perhaps, let me think about this. When 314 million
- 3 people know the secret, how many people learned the secret that
- 4 day?
- 5 Stu1: How would we know whom to tell? What if someone already knew?
- 6 Car: *(to Stu1)* Don't be practical on me, that's a terrible habit *(laughs)*.
- 7 StuS: *(laughter)*
- 8 Car: Theoretically. Theoretically, so let's think about this please. If on this
- 9 day- 314, if everybody in the U.S. knows the secret, how many were
- 10 in the last group, who learned the secret?
- 11 StuS: *(talking over each other)*
- 12 Stu2: Half. I know, I'm like half.
- 13 Car: *(to class)* Half?
- 14 StuS: *(some say no, some say yes)*
- 15 Car: *(to Stu2)* Tell us why you think half.

- 16 Stu2: Well, because, on day 0, 3 people knew. Then on day 1, 6 people
17 knew. Day 2, 12 people knew. So it doubles every time.
18 Car: Okay, so on the day when this many people knew it, how many
19 people learned it? (*pauses*) Half. So what's half of 314 million?
20 Stu3: 157.
21 Car: 157 million. So if this number, right here (*writing on 157,000,000 on*
22 *board at bottom of second column of table*), is 157 million, then that
23 must be 3 times 2 to what power? And if we can find the what power,
24 does that tell us how many days?
25 Stu4: Yeah.
26 Car: Can you figure that out? (*pause*) Now you can use your calculators.

Carolyn was thinking in the moment (lines 1-4). After joking with a student to not ask a practical question (line 6) about the situation that had been given, she re-framed the question, with additional scaffolds to support students (lines 8-10). Carolyn had reasoned that all who knew the secret had learned the secret from the last group, and her thinking was revealed in a way she had asked the question a second time. One of her students reasoned that the last group who had just learned the secret represented half who knew the secret (line 12). When Carolyn asked the student to explain her reasoning, she described the exponential pattern but described the relationship in terms of the number of days elapsed and the number of people who knew the secret, summarizing that the number of people who knew the secret “doubles each time” (lines 16-17). Satisfied with this student’s explanation, Carolyn leveraged the student’s explanation to ask a question that she then immediately answered. If 314 million people knew the secret, Carolyn and some of her students reasoned that half of 314 million people had learned the secret (lines 18-19). After asking for what the value of half of 314 million is and getting a student response (line 20), Carolyn wrote 157,000,000 on the board and asked students to use their calculator to find the value of the power for which 3 times 2 to some power was 157 million (lines 21-24, 26). The power represented the number of days (line 24).

The class worked on this task, and students shared their solution approaches. Most students used guess and check methods and one student used a graphical method to come to a consensus that the unknown power was between twenty-five and twenty-six. This meant that sometime between 25 and 26 days since the secret was first told, half the population of the United States had learned the secret.

Applying the logic of the reasoning shared by Carolyn and her class in this exchange was that if for example, 500 people know a secret, then half of people, or 250 people, in that group just learned the secret that day, because the pattern of learning the secret was growth by doubling. In the case of 314 million people knowing the secret on a particular day, that means 157 million people just learned the secret that day.

Assuming that the number of people that learned the secret on day d can be modeled by the function $f(d) = 3 \cdot 2^d$, then the logic of the students' and Carolyn's reasoning was that double this amount, $2f(d)$, represented the number of people who actually knew the secret. Let $g(d)$ represent the number of people who (based on Carolyn's and the students' reasoning) knew the secret.

Based on this thinking, a function describing the relationship between the number of days elapsed and the number of people who knew the secret could be modeled by $g(d) = 3 \cdot 2^d \cdot 2$, or more simply, $g(d) = 3 \cdot 2^{d+1}$. Finding half of $g(d)$ then gave the number of people who learned the secret that day.

However the underlying assumption of this way of thinking was that all people who know a secret learned the secret from someone else, which is almost true. The assumption overcounts the original three individuals, who did not learn the secret from

someone else. So the expression $g(d) - 3$ represents the correct number of people who know the secret, or alternatively stated, $3 \cdot 2^{d+1} - 3$.

Let $h(d)$ represent the correct number the people who know the secret on day d .

Since $h(d) = g(d) - 3$, it follows that $h(d) = 3(2^{d+1} - 1) = \sum_{n=0}^d 3 \cdot 2^n$, which sums the number of people who have learned the secret up to day d .

It follows that the discussion in which Carolyn and her students were engaged in was technically wrong, although pedagogically useful in terms of making meaning of an exponential growth pattern. The number of people who know the secret, $h(d)$, was approximately 2 times as large as the number of people who learned the secret, $f(d)$, when values of d were greater than or equal to 6. Would it have been useful to bring up the issue that her model assumed everyone who knew the secret learned it from someone else was a faulty model? Within this extension to the original lesson existed a conundrum: the error in the model created a necessary mistake that advanced student learning. For higher powers of the exponential function, the relative error introduced into the model by this flawed assumption became negligible, since the double counting of the initial value of the function was the source of the error. A second activity that could have highlighted the inherent flaw in the model (for smaller values of the exponent) may have been useful as a later lesson, but in my opinion only after students developed robust understandings of exponential functions. Focusing on the flaw in the model would have been counterproductive to support the key ideas of this lesson. This finding reveals that a flawed mathematical model can provide useful “necessary mistakes” that advance initial student learning of a proximal key idea.

In the lesson debrief of Lesson 1, Carolyn related her experiences in using this question that promoted student conjecture with her students.

Excerpt 68. Thoughts on opening activity (Lesson 1 Debrief)

- 1 Car: It was interesting, how I screwed that that one problem up, by asking
2 a question that wasn't related to what we were doing. I really liked
3 the way they figured that out.
4 Res: Uh huh. And maybe that was good.
5 Car: Maybe.
6 Res: Did you do that in both classes?
7 Car: I did, and they had more trouble figuring out this afternoon, than they
8 did this morning. But yes, I did the same thing.
9 Res: Interesting.
10 Car: I didn't adjust it, just to see how they do would do. And they had
11 more trouble. They didn't see that having to have half as many
12 people, half of those were people who would learn it on the last day.
13 They also more trouble figuring out on the calculator, what the day
14 was.

In relating her experience with this opening task to promote student conjecture, Carolyn commented that she had asked a question that she thought was not related to the task at hand (line 1-2), since her question had asked how many people knew the secret, while the task in the lesson concentrated on how many people learned the secret. She had commented during the lesson (Excerpt 67, lines 1-2) that she had asked the wrong question. However, Carolyn was pleased with how her students figured out a way to address the question (lines 2-3). She mentioned doing this same activity both in her morning and afternoon classes, simply to see what happened (lines 7-8, 10). She noted her second class had more trouble using the relationship between those who knew the secret in relation to those who had learned the secret the last day (lines 11-12). After discussing this activity, the researcher queried if she taught this lesson again, would she use that question again, to which she replied yes. The finding from this exchange and the lesson observed was that Carolyn did not note the flaw in the model she used which

related the number of people who knew the secret to those who learned the secret. She was learning the mathematics of exponential functions conceptually along with her students; it is possible this error was benefitting her learning, as well as her students' learning.

Characterizing Carolyn's MKT in Terms of Emerging Connections

The previous section discussed how Carolyn was learning mathematics alongside her students, and that while her knowledge gaps were small, she was able to support both her learning and her students' learning. A related finding was that new ways to look at previous mathematics were beginning to emerge for Carolyn. In this episode, I discussed that Carolyn considered multiple approaches to answering the same question. The interactions from this task advanced both Carolyn and her students' thinking about exponential functions. This episode examined a lesson in which Carolyn supported a flexible view of the reference point of a function. In isolation it may seem to be a minor finding, but significant in the context of the pre-study interview. One of the findings from the analysis of the pre-study task was that in attending to quantities, Carolyn needed a specific view of the reference time needed in order to address the task itself (see pages 139-145), and in particular she needed a fixed reference point of time (a common starting time) in order to answer the questions. Knowing the positions of the two students at a moment in time after the two started heading toward each other was insufficient information for her to make progress. This implied possible limitations in Carolyn's flexibility in attending to points of reference in a function, as emerged during the third

lesson observation. For example $f(x) = ab^x$ and $g(x) = cb^{x-1}$ might represent two equivalent exponential functions, if $a = \frac{c}{b}$.

The third observed lesson had to do with 1-unit growth and decay factors, percent change, and initial values of a function. As the lesson progressed, students were working in small groups on Question 4 of the lesson. Students were given a table, as shown in Figure 24.

x	1	2	3	8
$g(x)$	260	299	343.85	691.605

Figure 24. Modeling an exponential function (Lesson 3 Observation)

As part of the instructions for this task, students were asked to find the 1-unit growth factor, the 1-unit percent change, the initial value, and a function that modeled the exponential pattern of growth in the table. As students worked in groups, Carolyn walked around, observing her students' work as they made progress on the question. After students had engaged in the task, Carolyn held a class discussion with regard to this question. She had projected the table from Question 4 up on the white board, and proceeded to annotate on the image that was projected on the board. The discussion began as follows.

Excerpt 69. Carolyn's developing MKT (Lesson 3 Observation)

- 1 Car: Alright. Do you get, at any rate I think I saw that every place.
- 2 *(Writing on board)* Did I? Did anyone have anything other than 1.15?
- 3 And so the percent change is 15 percent *(writing on board)*, and I
- 4 saw that with everybody. Now what I did see, was differing initial
- 5 values. I saw some of you having an initial value of 260 *(writing on*
- 6 *board)*, and some of you had an initial value of was it: 226.087?
- 7 *(writing on board)*
- 8 StuS: Yeah.

9 Car: Now, those of you who had this initial value (*pointing to 226.087*)
10 had this function: $f(t)=226.087(1.15)^t$ (*writing on board*). Am I
11 correct? A lot of you, most of you, have that. But I also saw this, and
12 I think this bears some discussion: $f(t)=260(1.15)^{t-1}$ (*writing on*
13 *board*). Now I heard Stu1 say that will do it. Will in fact an input of 1
14 give you? Will in fact an input of 1 here, give you? I'm sorry, an
15 input of (writing 0 in table over x on board), let's see, that this would
16 be 226.087 (writing 226.087 in table over g(x) on board). Do these
17 give us the same input? If I put in a 1 here, do I get 260?
18 Stu2: Yes.
19 Car: (*repeats question*) If I put in a 1 here, do I get 260?
20 Stu2: Yes.
21 Car: If I put in a 2 here, do I get 299?
22 Stu2: Yes.
23 Car: Think so on 3?
24 Stu2: Yes.
25 Car: So in fact, (puts a box around each function on board) does it appear
26 that either one of those functions will describe what happens?

At first Carolyn confirmed with the class that based on her observations, students had agreed on the growth factor (1.15) and percent change of the exponential function (15%) (lines 1-3). Carolyn noted, however, that the initial values students had listed was different (lines 4-7), with some students writing 260 as the initial value of the function, while other students indicating that 226.087 was the initial value. The students in class agreed with Carolyn's characterization (line 8) of the two initial values. Then Carolyn proceeded to write down both functions she saw her students use (lines 9-13).

The function associated with the students' initial value of 260 was written as

$f(t) = 260(1.15)^{t-1}$, while the function associated with the students' initial value of

226.087 was $f(t) = 226.087(1.15)^t$. Figure 25 below shows Carolyn's annotations on the board.

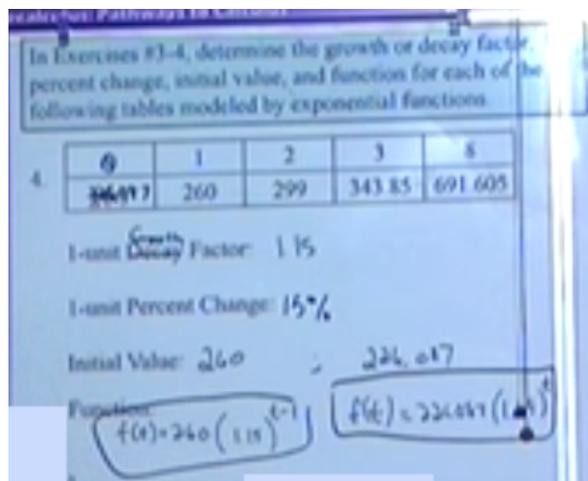


Figure 25. Carolyn's board work for Question 4 (Lesson 3 Observation)

Carolyn asked students whether the outputs of both functions agreed with the table for the differing input values, such as 1, 2, and 3 (lines 13-17, 19, 21, 23). As Carolyn asked students in class to test the two functions for each input value, one student confirmed the outputs matched those in the table (lines 18, 20, 22, 24). Carolyn then asked her class whether either of the two functions described what was happening in the situation (lines 25-26). Carolyn was doubtful that two functions modeled the same behavior, yet appeared to have two differing initial values. Excerpt 70 continues with the second half of this class discussion.

Excerpt 70. Carolyn's ponders initial value (Lesson 3 Observation)

- 1 Car: So, I guess my question is this: if you saw this function (*pointing to left side*). If you saw this function (*pointing to right side*), you would
 2 say the initial value is 226.087. Then what are we going to say is the
 3 initial value here? (*pointing to left side*)
 4
 5 Stu3: 226.087.
 6 Car: Is the initial value going to be when you plug in a 1 here? Or when
 7 plug in a - I don't know. I guess it's still if you plug 1, this is still
 8 going to give 260 for the initial value, isn't it? But 260 isn't the
 9 initial value that we found. Is the initial value what happens when t is
 10 0? I guess that depends how we define the initial value. Have we
 11 really ever defined the initial value? (*pause*)
 12 Car: I would tell you this. I may be wrong, but I would define the initial
 13 value as the value of the function when t is equal to 0. For this

14 function (*pointing to right side*), I would say the initial value is
 15 226.087.
 16 Car: (*pointing to left side*) I don't know, what is this, would somebody do
 17 this? What is one fifteen, maybe this works. What is one fifteen to
 18 the negative 1 power, times 260?
 19 Stu4: 226.
 20 Car: Never mind! Oh, what a, you know, I just had an 'aha' moment right
 21 there.
 22 Stu4: Go Ms. Smith!
 23 Car: So it doesn't really matter, does it? It's still going to give me, plug in
 24 a 0 for t , the same value. This makes it harder to find the initial value,
 25 but it's perfectly okay. Alright. Oh I love it, thank you. I just learned
 26 something pretty neat. Something I had never thought of before.
 27 (*erases board*)
 28 Car: Do you see what I just taught you right now? That you can be a
 29 lifelong learner.

While Carolyn felt confident about the initial value being 226.087 for the function

$f(t) = 226.087(1.15)^t$ (lines 1-3), she doubted 260 was the initial value for the

$f(t) = 260(1.15)^{t-1}$. She asked students what they thought the initial value of that function was (lines 3-4). Although one student answered her question (line 5), Carolyn focused on reasoning through the problem herself (lines 6-10), wondering out loud if the initial value occurred when the value of the input t was 0 (line 9-10). The researcher conjectured that Carolyn's struggles stemmed from conflating the coefficient a with the initial value of the function $f(0)$ in the canonical form of the exponential function $f(t) = ab^t$; while it is true that in the canonical form $f(0)=a$, this relationship was not applicable in a non-canonical form, such as $g(t) = ab^{t-c}$, if $c \neq 0$.

Carolyn pondered if she and her class had ever defined the meaning of an initial value (lines 10-11), and then she paused. Gaining confidence, she stated that an initial value was the value of a function when t (the input value) was 0 (lines 12-13). She then referred to $f(t) = 226.087(1.15)^t$ as the example to illustrate that the initial value of

226.087 (lines 13-15). Then Carolyn shifted her attention to the function

$f(t) = 260(1.15)^{t-1}$, and asked students to find the value of $260(1.15)^{-1}$, stating that the evaluation of the initial value might work (lines 16-18). One student replied that the value was 226 (line 19) and Carolyn immediately made the connection. In her own words, she had an 'aha' moment (lines 20-21), and she was enthusiastic about what she had just learned (lines 23-27). She thanked her students and commented that this question had made her think about something she had not thought before; she shared with her students that they too can be lifelong learners (lines 25-29).

During the lesson debrief, the researcher asked Carolyn to elaborate on her comment about being a lifelong learner. Her response to this query is given in Excerpt 71 below.

Excerpt 71. Carolyn's reflections on 'aha' moment (Lesson 3 Debrief)

- 1 Car: Well you know sometimes you just get an 'aha' moment when you
2 think, I never looked at a problem that way before.
- 3 Res: Uh huh.
- 4 Car: And um, you know, well they can't write 260 as the initial value. So,
5 then when I put the two functions side by side I realized, oh my gosh,
6 you could write it that way, and it still gives the correct initial value
7 when you put in 0 for the input. So I mean it was just like one of
8 those you could have knocked me over, I thought, I have doing this
9 how long and I never noticed that before? (*laughs*)
- 10 Res: So how did this, I guess how did this 'aha' moment emerge?
- 11 Car: Not until I wrote the two functions side by side. And plugged: one
12 had an exponent of t and the other one had $t - 1$. No, was that right?
13 Yes. Yes, and so I said, you know, an initial value happens when you
14 plug in a 0 for the input. And I said what happens over here when
15 you plug 0 in for the input and I then realized exactly the same thing
16 happens.
- 17 Res: Did that idea come from you or it came from your students, or how
18 did that that way of thinking emerge? I guess that's what I'm trying
19 to figure out.
- 20 Car: I saw a student write the function with the exponent $n - 1$. And he,
21 you know, of course that doesn't have the same- if it's ab^x , it doesn't

22 have the same ‘ a ’ as the other function has, and so I said: I don’t
 23 think, I don’t think that’s going to work. And then I thought to
 24 myself: I don’t think that’s going to work? Let’s take a look, so that
 25 was when I wrote them on the board, and wasn’t even until after I
 26 wrote them on the board and started plugging in those values that I
 27 realized that it did work.
 28 Res: Was that the one student, or many students, that were thinking that
 29 way?
 30 Car: I think I saw, I saw more than one student with the wrong ‘ a ’ value,
 31 but this was the only student I saw with the exponent $x - 1$ or $t - 1$, or
 32 whatever it was. So good for him!

In recounting her ‘aha’ moment, Carolyn mentioned looking at a problem in a new way (lines 1-2). She knew that 260 was not the initial value of the function (line 4), and yet saw a model in which students could write it “that way”, which the 260 was the constant in front of the exponential function that was correct. When she wrote the two functions $f(t) = 260(1.15)^{t-1}$ and $f(t) = 226.087(1.15)^t$ side by side and saw both gave the correct initial value to the function when input was 0, she had her ‘aha’ moment (lines 5-7). She was surprised she had not made that connection before (lines 7-9). In analyzing what Carolyn said, she said earlier that “they can’t write 260 as an initial value” (line 4) followed by “I realized, oh my gosh, you could write it that way, and it still gives the correct initial value” (lines 5-6). These utterances added support by my earlier conjecture during the observation, in which I suggested that Carolyn conflated the coefficient of the exponential function with initial value. Her use of words suggested that she used the ‘initial value’ to indicate either. Perhaps to Carolyn, the initial value of the exponential function and the coefficient of the exponent function had to the same number, and the revelation that they did not necessarily have to be the same was the source of her ‘aha’ moment.

Carolyn made this realization when she placed the exponential functions side-by-side and substituted 0 for the input. In her words, an initial value happens when you plug in a 0 for the input” (lines 13-14). While she could readily tell the initial value of the function from the canonical form, it was only after doing the calculation with 0 as the input did Carolyn realize that “exactly the same thing happens” (lines 15-16) for the function in non-canonical form.

The researcher asked a follow-up question to highlight the source of the idea of the two ways to write the exponential function (lines 17-19). This question was intentional, since the researcher had noted these different characterizations of the exponential model were due to students’ contributions to the lesson. These contributions had impacted both the learning of Carolyn and her class.

Carolyn noticed that the model of the exponential function by one student that was not in the form ab^t , but instead an exponent of $t - 1$ with a different ‘ a ’ (lines 20-22). At first she was uncertain whether or not the student’s model would work (lines 22-24). She was curious though, and wanted to “take a look” (line 24) at the student’s model. She wrote the student’s model and the canonical model side-by-side on the board (line 25), but it was only after substituting values into both models that she noticed the student’s model “did work” (lines 25-27). While she had noted other students had the “wrong” ‘ a ’ value (line 30, in referring the ‘ a ’ value of 260), this was the only student who wrote the function with exponent of $t - 1$ (line 31).

What is noteworthy about this exchange is an insight into Carolyn’s openness to student contributions. Unlike in earlier findings in which Carolyn made pedagogical moves to discourage student methods that she judged less efficient, her weak conceptions

of the mathematics she was teaching and her curiosity about the correctness of student's thinking using the "wrong" coefficient contributed to her willingness to explore her student's contribution as part of everyone's learning. She had attended to student thinking, and she persisted in make sense of the student's thinking. She was eventually successful, and as a result she learned an alternate way of looking at the initial value of exponential functions.

Not 'knowing' an answer or having the 'most efficient' method in mind appeared to contribute to her openness to explore alternate methods. Although the findings suggest Carolyn was disinclined to prompt students to provide a rationale for their responses, in instances in which, she became curious about a mathematical idea she exhibited what has been classified as "intellectual integrity" (Carlson & Bloom, 2005); she did not pretend to understand, but instead revealed her uncertainty and attempted to make sense of the question at hand. She was also open to exploring student suggestions or consider a method she had not considered but thought to be viable.

The Influence of Carolyn's MKT in Terms of Barriers to Supporting Student Learning

In this section, I will discuss a turning point in this teaching intervention that had implications for the findings reported from this study. During the fourth observed lesson, Carolyn's emerging mathematical connections were insufficient to support her students' learning during the lesson, to the point where she was unable to accomplish her stated mathematical goals for that lesson. Her struggles with the partial growth factor lesson represent the one and only time she did not meet her mathematical goals for student

learning. This section will analyze Carolyn's meanings of exponential functions with regards to growth and decay factors.

Although Carolyn exhibited weakness in her mathematical conceptions of the central ideas of the lesson, she was able to accomplish her mathematical goals for student learning in a lesson for the first three observed lessons using a conceptual curriculum. This did not hold true in the fourth observed lesson. Her weak connections contributed to her choosing a question that promoted student conjecture but did not tie to key ideas of the lesson, and these results had been analyzed earlier (pages 208-214). Furthermore, Carolyn struggled in teaching the key ideas of the lesson. In this teaching episode, I will analyze not only her pedagogical moves, but I refer back to the pre-study interview that may have been the source of her difficulties with this lesson. Furthermore, I will talk about how *well connected meanings* of proportionality and scaling (which Carolyn did not have) might have been leveraged in such a context. Although Carolyn completed this teaching episode with a more conceptual approach to teaching partial and multiple unit factors, she sometimes failed to see critical connections between a 1-unit and n -unit growth factor. Furthermore, her mathematical teaching goals for the lesson were not met for the first time in the study, and this appeared to contribute to her reverting back to a more traditional (direct) teaching approach for the remainder of the chapter. Her original mathematical goal had been: "That they [*the students*] can express those growth factors. And that they can express a function to describe the situation they are reading about, or looking about. You know, some are from information, some are from tables, some are from graphs. So I'd like them to get a feel for approaching those problems three different

ways.” Carolyn’s statement in Excerpt 38 on page 165 described her plan of going to direct teaching in the subsequent lesson to make up for time spent on this lesson.

A primary goal of Carolyn’s fourth lesson was to support her students in distinguishing between a 1-unit and n -unit growth factors. (To review my conceptual analysis of this idea, see Chapter 3, pp. 28-42.) This description of what is involved in understanding 1-unit and n -unit growth provided the primary lens for analyzing Carolyn’s discussions of an exponential growth task with her students. This analysis (see Chapter 3, pp. 28-42) outlined a possible way to leverage the key ideas of proportionality and scaling from linear functions in the context of conceptualizing exponential functions. Prior to presenting data I will remind the reader of what I consider to be a conceptual approach to a task that Carolyn used to introduce partial growth factors to her class. The task involved explicating the growth of the function f defined by $f(x) = 9.5(1.24)^x$. To understand this growth model a student would understand the 1-unit growth factor for f is 1.24 and see that as the independent quantity increases by 1, every new output value of f can be determined by multiplying 1.24 by that value. They would also see that, as input and output quantities change together, the ratio of $\frac{f(x+1)}{f(x)}$ is 1.24. Likewise the 2-unit growth factor is 1.24^2 (decimal equivalent is 1.5376) because for any change in input of two units, the ratio of $\frac{f(x+2)}{f(x)}$ is 1.24^2 . Leveraging the idea of scaling from the conceptual analysis (p. 32), it follows that when a change in input is scaled from a change in 2 units to change in 5 units, the change of input by 2 units is scaled by $5/2$. In the multiplicative realm, it follows that the factor of 1.24^2 associated with the change in

output of the function (measured as a ratio) is raised to the $5/2$ power. Therefore the 5-unit growth factor is $(1.24^2)^{5/2}$, or 1.24^5 (approximately 2.9316), which is consistent with the prior way of conceptualizing a 5-unit growth factor as the ratio of $\frac{f(x+5)}{f(x)}$.

During the fourth lesson observation, Carolyn tried to describe and use different unit-growth and decay factors, but she did not frame these growth factors in a concrete context as part of meaning making during the lesson. The details of how Carolyn described and used different unit factors will be discussed in this section. A concrete context that could have been used as a possible introductory activity can be found in Appendix N.

I will now discuss the function f where $f(x) = 9.5(1.24)^x$. For this function, a context was not provided in the lesson Carolyn taught. Attending to the quantity the variable x is tracking without a context, x can be thought to represent the number of 1-unit jumps from the reference point of 0. In Figure 26 that follows, a double number line is used below to track the relationship between the factor and exponent of the growth factor of the function (on a logarithmic scale) and the input (on a linear scale) in 1-unit jumps from 0; for the function $f(x) = 9.5(1.24)^x$, 1.24 is the 1-unit growth factor. After a sequence of five 1-unit jumps, the input, factor, and exponent are annotated on the number line.

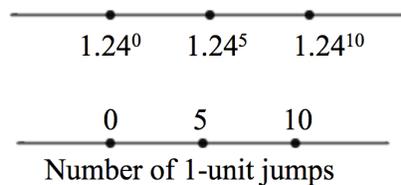


Figure 26. Tracking an exponential function (1-unit jumps)

The input can be measured in n -sized jumps rather than 1-unit jumps. Carolyn tried to use this idea during the lesson, although she did not embed the idea in a context. For example, one 5-unit jump is equivalent to five 1-unit jumps. Figure 27 illustrates this concept on the double number line.

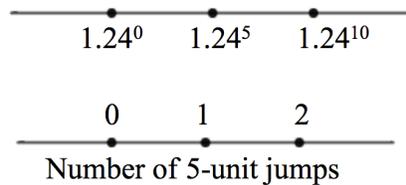


Figure 27. Tracking an exponential function (5-unit jumps)

Let v represent the number of 5-unit jumps from 0. Then $f(v) = 9.5(1.24^5)^v$ represents an equivalent model of the function, with input to the function measured in 5-unit jumps from 0; 1.24^5 is the 5-unit growth factor. In relating the two models of the function to each other, since five 1-unit jumps is equivalent to one 5-unit jump, an equation that relates the two quantities is $x=5v$, or alternatively, $v = x/5$.

Thus, another equivalent model to the original function $f(x) = 9.5(1.24)^x$ using the one-unit growth factor with an input measured in units of 5-unit jumps is $f(v) = 9.5(1.24)^{5v}$. Similarly, an equivalent model to the original function $f(v) = 9.5(1.24^5)^v$ using the five-unit growth factor with an input measured in units of 1-unit jumps is $f(x) = 9.5(1.24^5)^{x/5}$. With this discussion about different representations that produce conceptually equivalent models of the same function, I will proceed to the analysis of the lesson in which Carolyn taught exponential functions using multiple unit and partial unit factors.

In Lesson 4, Carolyn had asked a question to encourage student conjecture (see Excerpt 62, page 209) that had remained isolated from the key idea of the lesson or her

mathematical goals for student learning. After wrapping up the discussion about students' prediction of the value of $f(2)$ for $f(x) = 9.5(1.24)^x$, Carolyn pivoted to summarizing the various growth factors on the whiteboard for the function, both in exponential and decimal form. These included the $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2-unit growth factors. She asked students to report share the values they calculated in the third column, as seen in Figure 28.

x	$(1.24)^x$	
$\frac{1}{2}$	$1.24^{\frac{1}{2}}$	1.114
1	1.24	1.24
$\frac{3}{2}$	$1.24^{\frac{3}{2}}$	1.311
2	1.24^2	1.518

Figure 28. List of growth factors

Following the norms Carolyn had advocated earlier with regard to the number of decimal places to retain due to the AP Calculus exam, the decimal expansion for the various growth factors was rounded to three decimal places. She first called attention to and validated a student's earlier conjecture that the two-unit growth factor was less than two (see Excerpt 63, page 210). Carolyn then focused on the fact that the 1.24 in the table represented the 1-unit growth factor. Later she noted that $f(x) = 9.5(1.24)^x$ was the function using the 1-unit growth factor, but she did not make clear that the variable x represented the number of one-unit jumps (or chunks, using Carolyn's verbiage). She then made the pedagogical move to rewrite the function using a two-unit growth factor using another variable, as highlighted in the excerpt that follows.

Excerpt 72. Carolyn's struggles with non-unit factors (Lesson 4 Observation)

- 1 Car: If I want to use another variable, let's use 'n', and let that represent
- 2 the number of 2-unit chunks. (*writing $f(n) = 9.5$ on board*) Then that

3 function in terms of 'n' is going to be 9.5 times 1.24- uh uh uh (*stops*
4 *writing, erases 1.24*). What is the two-unit growth factor?
5 StuS: 1.538.
6 Car: 1.538. (*writing (1.538) on board*) To what power?
7 Stu1: 'n'.
8 Car: To the 'n' power. (*writing the exponent 'n' on board*) So that if I said
9 how many, what's the value when we have one 2-unit chunk.
10 (*writing $f(1)=9.5(1.538)^1$*) Is that going to be the same as what we
11 get when we have two 1-unit chunks? (*pointing to the chart on the*
12 *other side of board that displayed earlier work, in which 1.24^2*
13 *evaluated to 1.538*)
14 StuS: Yes.
15 Car: So if I wanted to do this, let's group these: what is the 5-unit growth
16 factor? Let's let w represent the number of 5-unit growth factors.
17 (*writing $f(w)=9.5()$ on board*) What would this function look like?

Carolyn defined the variable n to represent the number of two-unit chunks (lines 1-2). She then wrote the function with the decimal representation of the two-unit growth factor and variable using students' feedback (lines 5,7). The rewritten function (line 8) was $f(n) = 9.5(1.538)^n$, using the two-unit growth factor of 1.538 with the input variable n tracking the number of two-unit jumps. Carolyn then asked her students whether the value of the function $f(1) = 9.5(1.538)^1$ using one two-unit chunk was the same as the value of the function when two one-unit chunks were used (lines 8-12), pointing to the 1.538 in the table illustrated earlier in Figure 28. After hearing her students confirm yes (line 13), she gave another example of a non-unit growth factor by defining the variable w to represent the number of 5-unit growth factors (lines 14-15). She wrote the template of the function (see Figure 29, below) and asked her students what the function looks like (line 16). She awaited students' response.

Figure 29. Illustrating n -unit growth factors (Lesson 4 Observation)

After some pause, Carolyn again asked her students to identify the 5-unit growth factor. One student volunteered a response and Carolyn repeated this student's utterances for the whole class to hear, as continued in the Excerpt 73 that follows.

Excerpt 73. Fragile connections (Lesson 4 Observation)

- 1 Car: 2.932? (*writes 2.932 in the parenthesis, then writes an exponent of n*)
- 2 Now is 2.932 is the same as 1.24 to the 5th power? (*writing*
- 3 $2.932 = (1.24)^5$ *at the side of the board*)
- 4 Stu1: Yes.
- 5 Car: Could I write this as. (*pauses*) How can I write this? (*writing*
- 6 $= 9.5()$) Instead of a 2.932, with the 1.24? (*writing 1.24, pauses*)
- 7 Car: Is this 1.24 to the fifth power? (*writing the exponent 5 inside*
- 8 *parenthesis*) Raised to the n^{th} power? (*writing the exponent 'n'*
- 9 *outside parenthesis*) So how about if I write 9.5 times 1.24 to the 5n
- 10 power (*writing $= 9.5(1.24)^{5n}$*). I don't want to do that. (*erases n*) I
- 11 want to write this in terms of my original x . (*writes $f(x)$ in front of*
- 12 *equal sign by replacing n with and x*) 5x power. The growth factor,
- 13 did I do that right?
- 14 Car: (*pauses*) 5 times as much. 5 times as much. Did I confuse you more
- 15 or are we okay? We have it- oh yes, Stu2?
- 16 Stu2: Is it the same thing, as the same?
- 17 Car: Well let's think about this, if I put a 1 in here (*pointing to n in*
- 18 $f(w) = 9.5(2.932)^n$), or a 1 in there (*pointing to x in $f(x) = 9.5(1.24)^{5x}$*)...
- 19 Stu2: Oh yeah.
- 20 Car: ...do I get the same thing? One 5-unit factor (*pointing to the*
- 21 *location between 2.932 and the exponent n*), and this would mean
- 22 five 1-unit (*pointing to the exponent 5x*).
- 23 Car: (*pauses*) Yeah? Okay? Let's, let's take a look at this problem in our
- 24 workbook.

After repeating the approximate growth factor of 2.932 that the student said (line 1), Carolyn filled in the blank had she left for the growth while had waited for a student

response (Figure 29). The written function became $f(w) = 9.5(2.932)^n$ (lines 1-2). She then wrote the statement $2.932 = (1.24)^5$ to the side, and asked her students whether 2.932 and $(1.24)^5$ were equivalent (lines 2-3). The student who had originally stated the growth factor replied yes (line 4), and Carolyn took this response as a cue to proceed. Carolyn did not attend to the meaning of the exponent, or address the mismatch between the input of the function, w , and the exponent used, n . Her next step, while soliciting her students' feedback, was to rewrite the growth factor using 1.24^5 in place of 2.932 (lines 5-8). She then took an additional step and rewrote the expression $9.5(1.24^5)^n$ using the power of a power property of exponents as $9.5(1.24)^{5n}$ (lines 9-10), as can be seen in Figure 30 that follows.

Handwritten mathematical work on a board. At the top left, "w →" and "Sumit" are written. The main equation is $f(w) = 9.5(2.932)^n$. To the right, $2.932 = (1.24)^5$ is written. Below the main equation, it is simplified to $= 9.5(1.24^5)^n$, and then further to $= 9.5(1.24)^{5n}$.

Figure 30. Carolyn ponders her meanings (Lesson 4 Observation)

She soon corrected herself, saying she did not want do that (line 10). I inferred it to mean using n as the exponent, since her next action was to erase the n . She wanted to write the function in terms of the original x (lines 10-11), so she wrote an $f(x)$ in front of the equal sign, and wrote an x in place of the n (lines 11-12). Carolyn's revisions to her board work are illustrated in Figure 31.

w →
5 unit

$$f(w) = 9.5(2.932)^n$$

$$2.932 = (1.24)^5$$

$$= 9.5(1.24)^{5n}$$

$$p(x) = 9.5(1.24)^{5x}$$

Figure 31. Fragile connections emerge (Lesson 4 Observation)

Carolyn appeared to make an important connection that resulted in her explaining her approach. She wondered openly if she had determined the growth factor correctly (lines 12-13). Her utterances of ‘5 times as much’ (lines 14) were her attempts to make meaning of the rewritten growth factor. She wondered openly if she had just confused her students, or if they were okay with her explanation (lines 14-15). One student asked her whether or not this function was the same thing (line 16). Although this question was vaguely stated, Carolyn interpreted the student’s question to mean whether or not the first ‘function’ $f(w) = 9.5(2.932)^n$ and the second function $f(x) = 9.5(1.24)^{5x}$ were equivalent. Her response was that if she replaced the exponent n in $f(w) = 9.5(2.932)^n$ with 1, and the exponent x in $f(x) = 9.5(1.24)^{5x}$ with 1, then she would ‘get the same thing’ (lines 17-18, 20). The ‘thing’ to which Carolyn referred was the calculated value of the function. The student appeared to agree with Carolyn’s explanation (line 19). Carolyn continued her explanation and made a pedagogical move to tie the equal values of the two functions back to a conceptual meaning of different unit growth factors. She pointed to the space between the exponent n and 2.932 in $f(w) = 9.5(2.932)^n$ and referred to it as one 5-unit factor (lines 20-21). She then pointed to the exponent $5x$ in $f(x) = 9.5(1.24)^{5x}$

and referred to it as the ‘five 1-unit’ factor (lines 21-22). In this exchange it was unclear as to what quantity she was attending to. It seems to me that by pointing to the space she was inferring that the absence of a number implied that the exponent was $1n$, although she did express some uncertainty. I surmised Carolyn was unsure, as she did not appear confident with what she had just said. She then segued away from this exchange and referred her students to a problem in their workbooks (lines 23-24).

Carolyn was in the process of making meaning of multiple unit growth factors; her connections were emerging. She did not have robust connections, and because her connections were tenuous, she struggled to support student learning. Her pedagogical moves to justify why writing the growth or decay factors in different ways were equivalent were supported by her calculations. She had tried to build an equivalent function to the original function $f(x) = 9.5(1.24)^x$ using a 5-unit growth factor and the input w to represent the number of 5-unit ‘chunks’ (the number of 5-unit jumps of input from 0). Carolyn struggled to leverage the meanings she had used from the prior example she used with two-unit ‘chunks’ (Excerpt 72). Following her ways of thinking with the 2-unit factor example she used, she might have written $f(w) = 9.5(2.932)^w$ and possibly had a conceptual meaning for this function, beyond the algebraic rule stating the variable used in the expression must match the function’s definition. I conjecture that Carolyn’s struggle occurred when she was trying to reconcile the procedural definition of the function with her emerging conceptual meanings for n -unit factors. She wrote $f(w) = 9.5(2.932)^n$. The 2.932 held meaning to her, as it represented the 5-unit growth factor. However she did not attend to the referent quantity (the number of 5-unit jumps which would be represented by w , not n) as evidenced by the fact that she wrote n as the

exponent instead of w . Based on her utterances in this exchange, n held meaning to her as the number of 5-unit factors, however she appeared confused because of the mismatch between the n in the exponent and the w used to represent the input quantity; she knew that the variables used in a function had to match the defined input variable. She then expressed the function as $f(x) = 9.5(1.24)^{5x}$ because she wanted to rewrite the function in terms of the original variable x . She then used calculations to justify why the two forms of the function were equivalent. She used the example when the “input” is 1. According to Carolyn’s reasoning, these two functions $f(1) = 9.5(2.932)^1$ and $f(1) = 9.5(1.24)^5$ were equivalent because they “evaluate to (approximately) the same output value in two different ways”. If a strictly calculational perspective is taken, her reasoning would be correct, meaning that the variable in a function is merely a placeholder for values that will be used as inputs.

However from a conceptual perspective, variables are thought to be all the possible values an underlying measured quantity can assume. In the problem statement in question 1 of this lesson to which students were referred to at the end of this exchange, the original function was $f(x) = 9.5(1.24)^x$, where 1.24 represented the 1-unit growth factor, and x represented the number of 1-unit jumps from the input value 0.

$f(w) = 9.5(2.932)^w$ (not what Carolyn wrote) would represent an (approximately) equivalent function, where 2.932 represents the 5-unit growth factor and w represents the number of 5-unit jumps from the input value 0. Since five 1-unit jumps is equivalent to one 5-unit jumps, it follows that $x = 5w$. Using that relationship, the (conceptually) equivalent functions to the original two functions would be: (a) $f(w) = 9.5(1.24)^{5w}$ for

$f(x) = 9.5(1.24)^x$, and (b) $f(x) = 9.5(2.932)^{x/5}$ for $f(w) = 9.5(2.932)^w$. It would follow that the 5-unit growth factor in (a) is $(1.24)^5$ and w could be thought of as counting the number of 5-unit factors, whereas in (b) the 1-unit growth factor would be $(2.932)^{1/5}$ and x could be thought of as counting the number of 1-unit factors. From this perspective, both w and x have specific meaning as the value of the measured quantity, and $f(1)$ can mean two different things, depending on whether the input is measured in jumps of 5-units or in 1-units. It appeared that Carolyn failed to keep track of the referent quantities. From a conceptual perspective, $f(x) = 9.5(1.24)^{5x}$ is not an equivalent function to $f(w) = 9.5(1.24)^{5w}$ since $f(1)$ with respect to $f(x)$ means the output of the function whose input is 1, measured in 1-unit jumps; it would imply that $(1.24)^5$ is the 1-unit factor, which is false in the context of the original question.

However, Carolyn's view of exponential functions was not purely calculational. Her connections were emerging for a conceptual understanding, but they were insufficient to support her students in resolving subtle but important connections. In this exchange and the previous exchange (Excerpt 73, Excerpt 72) she made moves to connect the ideas of the mathematics together, although her connections with regard to n -unit growth/decay factors were tenuous. Her attempts to use conceptual ideas of exponential functions, although influenced by a calculational view, were observed at different times (Excerpt 72, lines 8-11; Excerpt 73, lines 20-22). In her attempt to summarize her meaning of the connection between the functions, she said: "...if I put a 1 in here (*pointing to n in $f(w) = 9.5(2.932)^n$*), or a 1 in there (*pointing to x in $f(x) = 9.5(1.24)^{5x}$*) do I get the same thing? One 5-unit factor (*pointing to the location*

between 2.932 and the exponent n), and this would mean five 1-unit (pointing to the exponent $5x$). Yeah? Okay?” Carolyn was trying to connect the idea of the equivalence between one five-unit growth factor and five 1-unit growth factors. This showed an emerging conceptual view of this idea of mathematics that was in conflict with a calculational view. It may also explain why Carolyn struggled with aspects of this conceptual lesson.

As Carolyn continued teaching the lesson, she continued making meaning of the n -unit and partial unit factors. In the question that followed $f(x) = 9.5(1.24)^x$ from the lesson, Carolyn asked her student to find the 1-unit and 5-unit decay factors for the function $g(x) = 0.46(0.874)^{4x}$. Carolyn was making connections, but some of her students held the misconception that the base of an exponential function was the 1-unit factor, while other students had become disengaged with the lesson. Carolyn made the pedagogical move to overcome students’ misconceptions by rewriting the $g(x) = 0.46(0.874)^{4x}$ as $g(x) = 0.46((0.874)^4)^x$. She annotated part of the first expression with a box and second expression with an underline to call attention to their equivalence. Carolyn stated that $(0.874)^4$ was the 1-unit growth factor, and then asked students to calculate the decimal representation of $(0.874)^4$. Students found the decimal approximation to be 0.584. She rewrote the expression the function was equal to as $0.46(0.584)^x$, and drew a line from $g(x)$ to this expression. Carolyn’s board work is illustrated in Figure 32.

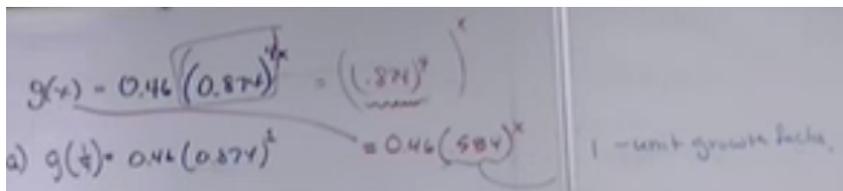


Figure 32. Carolyn tries to make connections (Lesson 4 Observation)

On one part of the board Carolyn had $g(x) = 0.46(0.874)^{4x}$, and on the other part of the board she had $g(x) = 0.46(0.584)^x$. She underlined and drew an arrow from 0.584, labeling it as the “1-unit growth factor”. It is possible that Carolyn simply made an error in labeling, or that she did not differentiate between the two types of factors (growth versus decay); class observations and field notes suggested lack of differentiation was the more likely reason. The explanation for the 1-unit factor that Carolyn gave was that if she were to evaluate the function $g(x)$ at 1: whether she puts 1 as an input when the function is in the form $g(1) = 0.46(0.874)^4$, or she puts 1 when the function is in the form $g(1) = 0.46(0.584)^1$, she would get the same value. As for which value Carolyn was attending to, it is unclear whether she was referring to the value of the output, or the value of the exponential expression. Her explanation therefore denoted an impoverished and inaccurate meaning for describing a 1-unit decay factor. The ratio of output values on an input interval that is n units apart is what determines an n -unit growth or decay factor, so for any function $h(x) = ab^{mx}$ it follows that $\frac{h(x+n)}{h(x)} = b^{mn}$ describes the n -unit factor from a ratio of output values perspective.

According to my model of Carolyn’s reasoning, an input value of 1 determined the 1-unit factor; the input value of the function determined the number of times the particular growth or decay factor was repeated. Thus for Carolyn, evaluating the growth

or decay factor at the particular input n would determine the n -unit growth or decay factor. Although her procedure found the correct values, why the process worked was not part of Carolyn's ways of thinking about exponential functions. Recalling the earlier analysis of the pre-study task, Carolyn's understanding of the relationship between factors and roots tenuous. Therefore her ways of thinking with regards to n -unit factors or partial unit factors were limited. She was unable to leverage a covariational perspective to flexibly coordinate linear growth in the independent variable with multiplicative growth in the dependent variable, and by extension, a scaling perspective of comparing quantities in the additive and multiplicative realms.

After her explanation given for the 1-unit factor, Carolyn then made the pedagogical move to connect the two representations of the same function, $g(x) = 0.46(0.874)^{4x}$ and $g(x) = 0.46(0.584)^x$, by asking students to find the 5-unit growth factor. On the left side of the board, Carolyn wrote $g(5) = 0.46(0.874)^{20}$, and proceeded to put a box around $(0.874)^{20}$, identifying it as the 5-unit growth factor. She called attention to the function on the right side of the board and wrote $g(5) = 0.46(0.584)^5$. She then circled the $(0.584)^5$. Figure 33 shows more of Carolyn's board work in relation to this question.

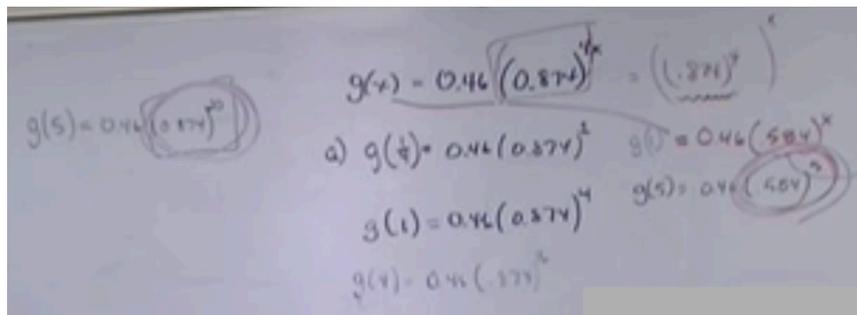


Figure 33. Carolyn's pedagogical moves (Lesson 4 Observation)

Carolyn asked the students in class to find the decimal values of $(0.585)^5$ and $(0.874)^{20}$ (to do a calculation), and students confirmed the equivalence of the two factors (discounting the error due to rounding). However, during the process of her own sense making of partial growth factors, some of Carolyn's students became further disengaged with the lesson. She summarized the class's findings by saying that "If I said to you what is the five year growth factor, you could say, it's 0.874 to the twentieth power, or you could say it's 0.584 to the fifth power. Or take the 1-unit factor (*pointing to 0.584*), and raise it to the fifth power, to get the 5-unit. Or take the $\frac{1}{4}$ -unit growth factor (*pointing to 0.874*), and raise it to the 20th power. It's a little confusing, isn't it?" The last statement acknowledges her own struggles and her students' struggles in making sense of this mathematical idea; in portions of the discussion some of her students had disengaged with the lesson and the sense making process.

In retrospect, Carolyn had struggled with this lesson, and in particular multiple unit and partial unit factors. Her views of a conceptual meaning of exponential functions were emerging, but limited in its potential for growth, based on earlier analysis of her impoverished meanings of proportionality, and in particular the scaling perspective. Her strategy rested in unitization of the growth or decay factor, based on a procedure of rewriting the exponential expression so that one-unit factor could be identified as the base of the exponential expression. For example given a function $h(x) = ab^{mx}$, rewriting it as $h(x) = a(b^m)^x$ described Carolyn's approach to finding the one-unit factor. If d represented the decimal representation of b^m , then according to Carolyn's model, either d or b^m represented the 1-unit factor. Then d^n or $(b^m)^n$ represented the n -unit factor, which she then oftentimes represented as a decimal approximation using the norm of 'AP rules'

(three places of accuracy). Through her unitization approach, Carolyn was able to leverage her emerging conceptual meanings for multiple unit growth or decay factors where n represented the number of factors.

Carolyn shared her struggles with the activities in the task during the lesson debrief. To recap the analyzed portion of the trajectory of the lesson, at first Carolyn asked students to make a conjecture about the value of the function $f(x) = 9.5(1.24)^x$ for the input value of 2 (see pages 208-214). She pivoted to asking students to determine various unit growth factors for this same function, extending on the question given in the workbook, which had originally asked students to find the $\frac{1}{2}$ -unit and 2-unit growth factors, along with the 2-unit percent change and initial value. Her extension of this question, which had included finding the $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, and 5-unit growth factors had been discussed earlier on pages 235-244. For the second question, she also extended the original question about finding the $\frac{1}{4}$ -unit and 1-unit decay factor, along with the 5-unit percent change and initial value; her extension to this question was discussed on pages 244-245. Excerpt 74 highlights Carolyn's experiences with regards to this enacted lesson.

Excerpt 74. Carolyn shares her struggles with lesson (Lesson 4 Debrief)

- 1 Car: I don't think, I think I spent way too much time talking about that
- 2 first problem for little value. I didn't think that was all that beneficial.
- 3 I don't think it cleared up any anything for them. They still had
- 4 trouble with those partial growth factors when we got into the gist of
- 5 the next problem.
- 6 Res: Uh huh.
- 7 Car: So I don't, I was just disappointed, I just didn't think that it went the
- 8 way I wished it would.
- 9 Res: That problem did not do the work you wished it would have done for
- 10 you.
- 11 Car: That is correct. And so, I adjusted for the second time I taught it.
- 12 Res: Tell me about the second time.
- 13 Car: Well I started by raising the exponent to the exponent.

14 Res: To begin with?
 15 Car: And I think that that was much clearer to them what I was doing. I
 16 tried to do that at the end of first hour, but I think I kind of lost them
 17 by then. And I don't think that they were hearing what I said. So, I
 18 think that they, that first problem, which was reasonably
 19 straightforward, we got through that in fifth hour without much
 20 problem. And then when we got to the second one, which first hour
 21 couldn't get their hands around at all, it was much better when I
 22 wrote the $4x$ as to the 4^{th} power, to the x power. And we talked about
 23 just that to the 4^{th} power was actually, you know, the 1-unit factor.
 24 Res: Uh huh.
 25 Car: And I think they're were getting, by the time we done about the third
 26 one of those, I think they were getting the hang of that, pretty well.
 27 So I felt much better about 5th hour, although I will have to tell you,
 28 I told you the other day, that this was not my favorite lesson, my
 29 favorite topic. And I will have to say that that there was one thing
 30 that I did fifth hour I just had to stand back and look and say: now,
 31 did I do that right? (*laughs*)
 32 Res: Uh huh.
 33 Car: That's embarrassing. But I can see why they struggle, because I think
 34 it's different from anything I've ever done before, so it's in essence,
 35 it's reasonably new to me also. And I understand the problems that
 36 they're having.

In referring to first problem (lines 1-5), Carolyn was referring to her pedagogical moves to discuss the partial unit and multiple unit growth factors of $f(x) = 9.5(1.24)^x$ with her students. However she felt this discussion had not helped her students in the problem that came next, which was $g(x) = 0.46(0.874)^{4x}$. She was disappointed in the outcome of that activity and mentioned not going the way it wished (lines 7-8) the first time had taught it during the lesson observation in her morning class. However, she made an adjustment the second time she taught the lesson in the afternoon (line 11). She described her move to rewrite the exponential expression as raising the exponent to an exponent (line 13). Based on the observation of the first lesson, such as the board work in Figure 32, Carolyn was describing the idea of rewriting $(0.874)^{4x}$ as $\left((0.874)^4\right)^x$, which

was confirmed by her later comments (lines 20-24). She had remarked that she made this pedagogical move at the end of the first class (lines 15-17) but by at that point in the lesson she “kind of lost them by then”. From earlier analysis, Carolyn made sense of the mathematics in the course of teaching the first lesson; by the end of the first observed lesson, she began to use the strategy of rewriting the exponential function using a unitization approach. She would write an exponential function of the form $h(x) = ab^{mx}$ as $h(x) = a(b^m)^x$. Thus the base b^m of the rewritten function was the 1-unit factor. In analyzing why the first question, $f(x) = 9.5(1.24)^x$ did not advance the learning for Carolyn’s students as she had hoped, she did not anticipate some students would have the misconception that the 1-unit factor and base b meant the same thing (this is false), as it did not visibly surface during the classroom activity. If the input x represented the number of 1-unit jumps, then ‘ b ’ being the 1-unit factor was true only for functions of the form $h(x) = ab^x$. However this was not true for functions of the form $h(x) = ab^{mx}$. In addressing students’ misconception, Carolyn began using the pedagogical move to rewrite the exponential function in the form $h(x) = a(b^m)^x$. Based on her comments during the debrief, it suggests she found this strategy effective and used it in the subsequent class. She stated that by the third example she used in the afternoon class, her student were getting the “hang of that”, and that she felt better about the afternoon lesson (lines 26-28). Carolyn commented that this was not her favorite lesson or topic (lines 29-30). A large contribution to her distaste for this lesson was her uncertainty about the mathematics (lines 30-32) she was teaching. She found this embarrassing (line 34) and could relate to her students struggling with ideas of the lesson, as the mathematics was

different than anything she had taught in the past; it was relatively new to her too (lines 34-37).

The researcher asked Carolyn to elaborate on what she meant about the math being different (lines 34-36). Excerpt 75 focuses on this exchange.

Excerpt 75. Carolyn discusses her emerging connections (Lesson 4 Debrief)

- 1 Car: Well there was one this morning, that you know, we had the same,
2 representing the same situation with two different functions, one with
3 the what? Two unit growth factor, and one with a three- I don't
4 remember.
5 Res: Right, and then switch over. Uh huh.
6 Car: And then I said, if we looked at those they would be same. And I said
7 evaluate those, and I said, oh gosh, I hope they are the same.
8 Because, you know, then I was second guessing myself again. And
9 they were, but I am generally very confident of the math that I do.
10 And I would say here I'm not as confident as I wish I were.
11 Res: What is difficult, I guess what do you identify as difficult about the
12 idea, or about this key idea, that makes this problematic to students?
13 Car: Well, I think when you see an exponent of x over 2, I mean I have
14 stop back, I have to sit back and think, now: is that a $\frac{1}{2}$ or a 2 unit
15 growth factor. I have to think about that. And now know, the fact of
16 the matter is that I've been teaching this for a long time. And it's not
17 really a topic that I've ever encountered in any of the myriad of
18 textbooks that I have taught from.
19 Res: Uh huh.
20 Car: So, I wonder why is that.

While Carolyn did not precisely remember the example where she second guessed herself (lines 1-4), based on the analysis of the observed lesson, the start of her difficulties began when trying to rewrite the function $f(x) = 9.5(1.24)^x$ using a 5-unit growth factor, after giving an example using a 2-unit growth factor. However Carolyn experienced further troubles as student misconceptions surfaced during the discussion of the function $g(x) = 0.46(0.874)^{4x}$. Some students had become disengaged as she made sense of the mathematics, while simultaneously trying to help students come to her way

of thinking. During the portion of the lesson when she asked students to find the value of $g(5)$, Carolyn placed a box around $(0.874)^{20}$ in $g(5) = 0.46(0.874)^{20}$ and a circle around $(0.584)^5$ in $g(5) = 0.46(0.874)^5$, with the boxed and circled portions representing the 5-unit factor. She asked students to find the values of $(0.874)^{20}$ and $(0.584)^5$ and said to the class “I sure hope those are the same”. This is consistent with her recollection of the lesson during the debrief (lines 6-7). She had stated that she had second guessed herself, and that normally she was confident about the math she taught. In this lesson she did not feel as confident as she wished to be (lines 9-10).

The researcher asked Carolyn what she thought made the key ideas difficult or problematic for students (lines 11-12). This was an intentional move by the researcher to reveal Carolyn’s thinking about the mathematics. Her response was in the context of an example: when she saw an exponent of $x/2$ with an exponent, she had to sit back and think about whether that was a $1/2$ -unit or 2-unit factor (lines 13-15). Unfortunately Carolyn’s phrasing of the word “that” was ambiguous, and the researcher had taken it to mean the exponent of the function during the conversation. However such phrasing was not consistent with the meaning of growth or decay factor in the context of Carolyn’s pedagogical moves in the course of the lesson, although she attended to the value of the exponent when finding the n -unit growth or decay factor. So it is possible she was still in the process of making sense of the appropriate terminology.

It is also possible that the word “that” to Carolyn meant determining that $b^{x/2}$ represents (a $1/2$ -unit or 2-unit factor) given a function of the form $j(x) = ab^{x/2}$? Depending on the choice of x , $b^{x/2}$ could represent any unit factor, but I think this was closer to Carolyn’s meaning. I think what Carolyn was trying to describe was a conflict in

trying to find the number of 1-unit factors that were required to obtain b . The coefficient of x in the exponent was $\frac{1}{2}$, leading to her first way of thinking that the given factor was the $\frac{1}{2}$ -unit factor. But the value of the input x that required her to make the exponential expression equivalent to b was 2, since $b^{2/2} = b$, suggesting that the given factor was a 2-unit factor. Making her pedagogical moves to unitize the representation of the exponential function (such as by rewriting $j(x)$ as $j(x) = a(b^{1/2})^x$) made it easier for Carolyn to see the 1-unit factor within the function. Although the following is not mathematically correct: from Carolyn's perspective of exponential functions, $b^{x/2}$ represented a 2-unit factor since $(b^{1/2})^2 = b^1 = b$. It is possible her strategy was couched in finding the value of x that resulted in the exponential expression having an exponent of 1, but her emerging connections were deeper than an algorithm suggests.

Carolyn had more to learn about exponential functions, but the end of this exchange foreshadowed a retreat to skill-based lessons. She mentioned having taught a long time and never encountering this topic in the various textbooks she had used in the past, and she wondered openly why was this was the case (lines 15-18, 20).

As had been described in earlier analysis of this chapter, this one lesson represented the only time Carolyn changed her upcoming lesson as a result of the taught lesson (Excerpt 38, page 165), due to her struggles with mathematics. She had not met her mathematical goals for student learning in this lesson, in which she had wanted students to explore exponential functions with non-unit factors using multiple representations. Her personal goals (see Excerpt 22, page 137) of being efficient and not wasting students' time were not met in this lesson.

Carolyn's Retreat in Practice Influenced by Her Goals and MKT

This section discusses the findings of this study related to Carolyn's decision to return to using a skills-based curriculum for the remainder of the chapter after her struggles with the prior conceptual lesson. The results suggest that in addition to her tenuous mathematical connections and struggles supporting student learning in the prior observed conceptual lesson, additional factors such as her overarching goals to prepare students for AP Calculus, and weak meanings for the continuous growth factor 'e' contributed to her decision making process reverting to a traditional skill-based practice.

In the following planning session, Carolyn had made the decision to introduce logarithmic functions. In describing her plans for the lesson, Carolyn wanted to use a graphical approach when discussing the logarithm as the inverse function of the exponential function, because she liked doing that (see Excerpt 19, page 130) and the conceptual curriculum did not include this approach. In response to the researcher's query for clarification on whether she was planning to use a graphical approach in lieu of looking at the relationship between input and output as a way to introduce logarithmic functions, her response was as follows.

Well yes, I'm going to do that, but I'm also, yes. Exactly. That we're doing that, and that is a reflection across the line $y=x$. So then I will make a table of values where the input becomes the output and graph that function, and that is defined to be the inverse of the exponential function, and so that will be the logarithmic function. And that the domains and ranges switch, just like they do for any other inverse function.

She discussed graphing $y = 2^x$ as a way to introduce $y = \log_2 x$ as the inverse function. She planned to graph $y = 10^x$ and then look at its inverse function as a way to introduce the common logarithm. Carolyn's comments about what she planned to do after

introducing the common logarithm were relevant in the context of the prior lesson in which she struggled in the mathematics and her overarching goals. This exchange between the researcher and Carolyn is included in Excerpt 76.

Excerpt 76. A pivot back to old ways (Lesson 5 Planning)

- 1 Car: And then I will talk a little bit about ‘ e ’ and where ‘ e ’ comes from.
2 And how we use, and that that’s often the base of exponential
3 functions. And let me stop right here and say: the more I look back at
4 the 4 point 5 [*referring to Module 4, Investigation 5*], whatever,
5 Investigation 5, I just wonder. I just wonder. I don’t know, the
6 benefit of doing all of those 2, 3, 4, $\frac{1}{4}$ unit growth factors, when we
7 could just define all of those function in terms of base ‘ e ’.
- 8 Res: Hmm. Interesting.
- 9 Car: Which is before Pathways, how I always did exponential functions.
10 Just with base of ‘ e ’ rather than, I mean most often with the base of
11 ‘ e ’. Um, and in fact some of the problems, most of the application
12 problems I would have in mind for next week will have a base of ‘ e ’.
- 13 Res: Uh huh.
- 14 Car: And something else, in fact I might even go back to some of the
15 problems we did in lesson 5, and show how those could be written
16 with the base of ‘ e ’. Working with your calculator, it’s a lot easier to
17 use a base of ‘ e ’ if you’re going to look at the logarithms. I don’t
18 know that, I guess that would be...
- 19 Res: That’s the connection to your thinking.
- 20 Car: ...my questioning of the “why do we do it”. You know, we spent
21 basically three days talking about one-unit, two-unit, three-unit, four-
22 unit, whatever growth factors. And I know what they see when they
23 get to Calculus. And they see base ‘ e ’.
- 24 Res: Uh huh.
- 25 Car: They don’t see this other stuff. At least in the Calculus, in the AP
26 Calculus curriculum that I teach, we don’t deal with exponentials
27 with any base other than ‘ e ’, hardly ever.

As part of her lesson trajectory, Carolyn wanted to discuss the constant ‘ e ’ next, since she stated that it was often used as the base of exponential functions (lines 1-3). She questioned the benefit of spending time in the prior lessons discussing the 1-unit, multiple-unit, and partial-unit factor, when they all could have been defined in terms of base ‘ e ’ (lines 4-7). Her prior experiences with other curricula before Pathways (the

conceptual curriculum in this study) had most exponential functions defined in terms of base ' e ' (lines 9-11), and she questioned the value of spending three days talking about the various unit growth factors (lines 18, 20-22). She commented that many of the application problems she was planning to do for lessons next week had base ' e ' (lines 11-12), and when students get to Calculus they see exponential functions with base ' e ' (lines 22-23). She entertained the thought of going back to the previous lesson with partial unit and multiple unit factors and showing students how the functions could have been represented with base ' e ' (lines 14-16). She remarked that working with base ' e ' was easier overall because when using ' e ' with a calculator, this made it easier to work with logarithms (lines 16-17). I surmised that Carolyn's remark was based on the ease of calculating the value of the natural logarithm when using a scientific or graphing calculator (line 19).

From earlier analysis, Carolyn's meaning for ' e ' were impoverished; the constant ' e ' was a magic number used as a base for many exponential functions (see Excerpt 40, page 168) for which the value of the inverse function could be readily determined using a calculator. A more robust meaning for ' e ' discussed in the conceptual analysis (Chapter 3, starting page 38) was that ' e ' referred to the continuous (or natural) growth factor. It was the unique 1-unit growth factor for an exponential function in which the instantaneous rate of change of output with respect to the input was the same as the value of the output of the function as the input varies. The constant ' e ' represented the unique value of the 1-unit growth factor of the exponential function for which this relationship always remained true.

In summary, Carolyn rejected a large portion of the previous lessons using the conceptual curriculum she used when teaching exponential functions, questioning its value and the time spent on it. According to her, she almost never saw any other base used for exponential functions besides base ‘ e ’ in the Calculus (lines 25-27). Teaching exponential functions conceptually by leveraging ideas of factors and roots and how they are related as an input quantity changes by incremental amounts and the outputs grow multiplicatively to build meanings of 1-unit, multiple unit, and partial unit factors did not align with Carolyn’s overarching goals of preparing students for the AP Calculus exam. Her tenuous understandings of the underlying ideas led to a less conceptual lesson for students since these lessons required richer connections to support student understanding. The turning point marking the retreat to a more traditional practice occurred when Carolyn could not meet her mathematical goals for student learning due in the prior observed lesson. When coupled with lessons not aligned with her overarching goals of being efficient, not wasting students’ time, and preparing students for the AP Calculus exam, these elements contributed to a perfect storm that fostered the conditions for Carolyn’s retreat to a more traditional teaching practice for the remainder of the chapter during this professional development intervention.

Characterizing Carolyn’s Retreat in Teaching Practice

While the episode described in the previous section contained significant findings for this study, a question the reader might be asking is how might one characterize Carolyn’s practice that is “traditional”? This section will describe the findings that characterize Carolyn’s traditional practice.

While much of Carolyn’s traditional practice was direct instruction with answers to her questions given by more vocal students setting the pace of the lesson, there was evidence of Carolyn’s professional learning at times during these lessons. The findings of this study have analyzed how Carolyn’s goals, her mathematical knowledge for teaching, and her views of teaching mathematics influenced her pedagogical decisions with regard to teaching with a conceptual curriculum. However, when she reverted to a using traditional skills-based curriculum, there were times when her pedagogical moves suggested the impact of prior professional learning. I will share one such example in the vignette that follows, which occurred during the observation of the fifth lesson.

During the planning session (see page 253) Carolyn had discussed introducing logarithmic functions. During the lesson, first she created a table of values, drew a graph, and discussed the domain and range of the function $f(x) = 2^x$. An image of Carolyn’s board work is given below in Figure 34.

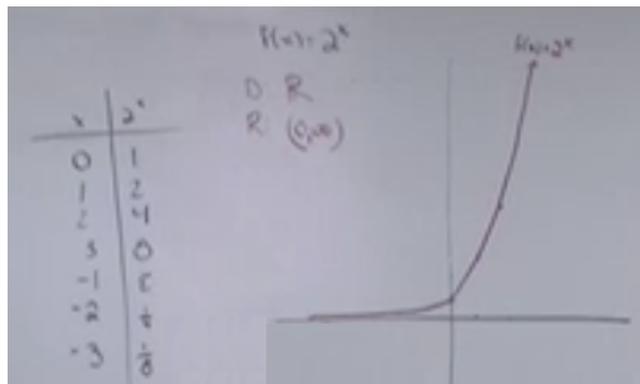


Figure 34. Graph of original function (Lesson 5 Observation)

She then volunteered a student to read the definition of a logarithmic function. The student responded by saying “It’s the inverse of the exponential function”. In the exchange that follows, it is important to note that posters were placed on the back wall of the classroom that had pictures of the fundamental graphs of the most commonly used

algebraic functions such as: $y = x$, $y = x^2$, $y = x^3$, $y = |x|$, $y = \frac{1}{x}$, $y = \sqrt{x}$, and $y = b^x$. These served a larger purpose than simply the décor of a mathematics classroom. These graphs had been used as a tool for student discussion in previous class observations (see Excerpt 60, page 202), and Carolyn referred to the graphs in this excerpt. The next pedagogical move she made after asking for the definition of the logarithm was to have students describe what the inverse of the exponential function might look like. In the excerpt that follows, Carolyn started by asking the same question in multiple ways with little wait time, reframing the question while waiting for students to respond.

Excerpt 77. Encouraging shape thinking (Lesson 5 Observation)

- 1 Car: What's the inverse of this function going to look like? Do you, can
- 2 you envision what that's going to look like? What do we know about
- 3 inverses? Do we even have to know the equation- does this function
- 4 have an inverse? First, does it have an inverse? How do we know
- 5 that: yes it does have an inverse. By looking at the graph can you tell
- 6 if a function has an inverse that's a function? How do you know?
- 7 Stu1: It's um...
- 8 Stu2: The input doesn't repeat.
- 9 Car: For every input there's one output. How else, that's exactly right.
- 10 What other words do we use to describe that?
- 11 Stu3: For each unique...
- 12 Car: Stu3, what?
- 13 Stu3: For each unique output, there's a unique input.
- 14 Car: Oh. That's right! For each input, there's one output. For each output,
- 15 there's one input? Sometimes we use the words one-to-one to
- 16 describe that. For each input there's one output. For each output
- 17 there's one input. And just by looking at the graph, what are we
- 18 going to see when we see a function like this that has an inverse?
- 19 *(pauses)* Remember, that when we looked at these functions back
- 20 here? *(walking to back of room with graphs of fundamental*
- 21 *functions)*
- 22 Car: This one doesn't have an inverse *(pointing to one of the graphs of*
- 23 *fundamental functions on back of room)*. Why not? This one doesn't
- 24 have an inverse *(pointing to another graph)*. Why not?
- 25 Stu4: It doesn't pass the vertical line test.
- 26 Car: Well they all pass the vertical line test.

- 27 Stu4: Horizontal I mean.
28 Car: Huh?
29 Stu4: Horizontal line test.
30 Car: A horizontal line test. (*To class*) Do you remember that? (*walking to front of room*) If it passes the horizontal line test, no horizontal line
31 will touch the graph more than once. Then it has an inverse. And this
32 does that, it's true. (*pointing to graph of $f(x)=2^x$*)
33

After asking the question in multiple ways that focused on how one might determine whether a function has an inverse by looking at the graph (lines 1-6), one student said it means the input did not repeat (line 8). Carolyn revoiced this student's response by acknowledged the response as correct: that meant for every input there was one output (line 9). She moved on without analyzing the meaning of the student's contribution or her own utterances (this described the conditions for a function, not an inverse function). Carolyn was seeking a specific answer, so she asked students what other ways could they use to describe inverse function (lines 10). Another student described an inverse function meant "for each unique output, there's a unique input" (line 13), which I had thought during the observation was the mathematically correct answer that Carolyn was looking for, but my model of her trajectory was wrong. She acknowledged this student's response as correct (line 14), and summarized this student and the prior student's contribution (lines 14-17).

The 'right answer' that Carolyn was seeking from her students was using the image of the plot on the graph to determine whether function had an inverse or not. Turning towards the graph on the front board, Carolyn asked the class, "And just by looking at the graph, what are we going to see when we see a function like this that has an inverse?" (lines 17-18). After some pause without a student response, she made a

pedagogical move to refer to the posters of the graphs of the fundamental functions in the back of the classroom and proceeded to walk towards them (lines 19-21).

Then Carolyn pointed to one of the graphs that was not a one-to-one function and asked, “This one doesn’t have an inverse. Why not?” She pointed to another graph that was not a one-to-one function, and repeated her question (lines 22-24). This move elicited the response Carolyn was seeking. After a student initial response (passing the vertical line test) and Carolyn’s correcting the student’s response (all these graphs pass the vertical line test), the student gave the answer she was looking for (lines 25-30): a function that has an inverse passes the horizontal line test. Carolyn then summarized what the horizontal line test entailed (lines 31-32) by saying, “If it passes the horizontal line test, no horizontal line will touch the graph more than once. Then it has an inverse.” Walking back to the front of the room, she then pointed to the graph of $f(x) = 2^x$ and stated these conditions were met for the graph (lines 32-33).

What was consistent in Carolyn’s pedagogical practice, regardless of type of lesson (conceptual or skills-based) was her insistence on particular methods. In the planning session Carolyn wanted to use a graphical approach in introducing logarithmic functions as the inverse of exponential functions. During the lesson, she wanted students to use the method of the horizontal line test as a way to determine whether a function had an inverse or not, rather than focusing on the meaning of the inverse function. She missed an opportunity to link the meaning of an inverse function, which Student 3 had correctly described, as to why the horizontal line test worked. However from earlier analysis, for Carolyn, student contributions were for the purpose of engagement in the lesson and for pacing; student contributions were not viewed as a tool for learning of the class.

Carolyn's use of the shapes of graphs was promoted as a way for students to think about mathematics, and this became more prominent when she reverted back to her prior ways of teaching, since quantities or the covariation of two quantities were not attended to in a skills-based curriculum. Her use of the graphs of fundamental functions as part of this class discussion is one such example. However, a second example that illustrated an affinity of using shapes of graphs to promote her students' learning was observed immediately after this initial exchange. It merits discussion since it also relates to Carolyn's professional learning. In the excerpt that follows, Carolyn started this teaching episode by referring back to the graph of $f(x) = 2^x$ and then asked students to sketch a graph of the inverse function in the air.

Excerpt 78. Drawing through the air (Lesson 5 Observation)

- 1 Car: Alright. So if it has an inverse, what's it look like? Can you do this?
- 2 You take your finger and sketch for me (*gestures in air*) what that
- 3 you think that inverse is going to look like? (*walks to back center of*
- 4 *room*) You think, what do you think? We're going left to right.
- 5 Going left to right, how is that graph going to look?
- 6 StuS: (*incomprehensible*)
- 7 Car: I don't see everybody off the fence here. Make a. (*pauses, looks*
- 8 *around class to observe students' sketches of graph through air*)
- 9 Stu1: Left to right.
- 10 Car: Left to right.
- 11 Stu1: Left to right.
- 12 Car: Well let's see. (*walking to front of room towards board*)
- 13 Car: Let's see if you were right. So if I'm now going to look at, if I'm
- 14 going to look at f inverse. (*Creates a two column table on the board,*
- 15 *with x and $f^{-1}(x)$ as columns*) And I'm just going to write it like that
- 16 since we don't know what it is yet. Remember what happens, the
- 17 input becomes the output. So when x is 1, what's the inverse going to
- 18 be equal to?
- 19 StuS: Zero.
- 20 Car: Zero. And when x is 2?

Carolyn's instructions to her students were to sketch a graph in air, from left to right, of what they thought the inverse function looked like (lines 1-5). She observed some of her students tracing a plot in air, and then she encouraged more students to participate (lines 7-11). Her next move was to use the definition of the inverse function to investigate whether their initial guesses were correct or not (lines 12-14). She created a two column table on the board near the graph, and labeled the left column x and right column $f^{-1}(x)$. She then used the values of $f(x)$ from the original table of values generated by $f(x) = 2^x$ (see Figure 34, page 257) as the input value x for $f^{-1}(x)$, starting with the value 1 (lines 14-18). After students responded 0 for the output value of the inverse function, she continued to the next input value (lines 19-20). This process continued until the table was completed. Then Carolyn drew a graph of the inverse function on the same coordinate axes. This process created points on a graph that was a reflection of the original graph across the line $y=x$. She then drew a curve through those points, and asked students if what was plotted on the graph matched what they had earlier sketched in the air.

Whether Carolyn was aware of it or not, it is possible that earlier sessions of collaborative lesson planning had a minor impact on her willingness to ask a question to promote a student prediction (lines 1-4, 7); however, it is also possible she might have asked this question in absence of this professional development intervention. In addition to illustrating this teaching episode as an example of Carolyn's affinity to using the shapes of graphs as a tool for student learning, the purported source of her idea to have students sketch graphs in the air was also a finding that emerged during the lesson debrief. She attributed it to her professional development.

Carolyn's professional development on the Pathways project had focused on promoting robust meanings for quantities, the covariation of quantities, and proportionality between quantities from a covariational perspective. Robust meanings could then be leveraged to support student learning when teaching with a conceptual curriculum. As part of this professional development on the Pathways project, one of the activities teachers were asked to engage in used the *Coordinating Quantities Tool* (Lima, McClain, Castillo-Garsow, & Thompson, 2009; P. W. Thompson, 2002, 2009). In the facilitator's guide (P. W. Thompson, 2009) that described the *Coordinating Quantities Tool*, after selecting an activity in which two covarying quantities were to be tracked, the professional development facilitator asked participants to: (1) track the independent variable with the right index finger, (2) track the dependent variable with the left index finger, (3) track both variables simultaneously, but without having the finger tracking the dependent variable be on top of the finger tracking the independent variable, and (4) track both variables simultaneously, but now having the finger tracking the dependent variable be on top of the finger tracking the independent variable. These four parts were done in sequential order during professional development, with attention to participants' mastery of each individual part before proceeding to the next part. Carolyn had participated in this professional development in which the *Coordinating Quantities Tool* was used.

During the lesson debrief, Carolyn discussed highlights of the lesson she had taught. Excerpt 79 focuses on her recollection of the teaching episode in which students sketched graphs in the air.

Excerpt 79. Carolyn's remembering (Lesson 5 Debrief)

- 1 Car: I think I liked doing the fact of asking them to use their fingers
- 2 (gesturing through air) to sketch what they thought the graphs would

3 look like. Not very many of them had it right on the first one, but
4 most everybody had it right on the second time that I asked them to
5 do that. I kind of like doing that, that's something that I learned in a
6 Pathways training, is having kids do that, and I like doing that.
7 Res: When you're talking about doing 'that', you're talking about the
8 finger through the air?
9 Car: Yeah. What do you think that graph would look like, I think.
10 Res: Oh okay. Did they...
11 Car: That was before we graphed log.
12 Res: Yeah, in the Pathways workshop, were you doing it with both hands
13 or one?
14 Car: I know we did it with two. But I think we did with one also.
15 Res: Okay.
16 Car: I think. You know I don't remember. I know we did with two. How
17 things were varying together.
18 Res: Right, so you were making one hand one quantity....
19 Car: Yeah.
20 Res: ...representing the, you were tracking one quantity with one hand,
21 and you were tracking the other quantity with another hand.
22 Car: Yeah. But I've done this before just with a graph, and I think that
23 that's a good way to, them to visualize what they're seeing.
24 Res: What they see with a single hand you mean?
25 Car: Yeah.

Carolyn liked that she had asked her students to sketch what they thought the graph would look like in the air (lines 1-6), and she felt that more students had sketched the graph correctly the second time they engaged in the activity (line 3-5). She attributed the idea of this activity to a Pathways training (lines 5-6) which is a relevant finding since the author of this study had been a member of the Pathways project team. Sketching graphs of functions in the air without careful attention to the covarying quantities was not part of the professional development trajectory, so the researcher was curious as to the origin of her remark. After the researcher's moves to verify that this conversation was regarding the same teaching episode observed and analyzed in Excerpt 78 (see page 261), the researcher sought clarification on how Carolyn learned this sketching in the air

method from a Pathways workshop. Did she remember the activity from the workshop, and if so, the intention of the activity?

At first the researcher asked if she recalled the workshop facilitator sketching the graph in the air using two hands or one hand (lines 12-13). Carolyn remembered the activity being done with two hands, but she thought the activity had also been done with one hand also (line 14). Technically this was true.

Carolyn she did not remember the events exactly, but she knew that in the workshop they used both hands to show “how things were varying together” (lines 16-17). This is also true, although the ‘things’ to which Carolyn referred were the covarying quantities. The researcher began to describe the process used for the *Covarying Quantities Tool* in the workshop (lines 18, 20-21), but Carolyn was not interested in listening. She liked her own method, and described that she had “done this before just with a graph” (line 22). Carolyn stated that sketching a graph in the air with one hand was helpful in getting students to visualize what they were seeing (lines 22-25). This finding suggests the professional development Carolyn had with regards to the *Covarying Quantities Tool* had been subsumed into her way of thinking about sketching a plot of a graph in the air. While Carolyn recollected the actions used with the *Covarying Quantities Tool*, the intention of the tool did not appear to part of her ways of thinking. Earlier findings had shown that Carolyn could attend to quantities if requested, but she did not attend to quantities as a natural part of her teaching. Also, prior results reported had previously revealed Carolyn’s impoverished tendency to engage in covariational reasoning or prompt her students to do so. Carolyn’s remembering of the activity were a reconstruction based on her own tenuous connections of the mathematics: her

interpretation of the *Covarying Quantities Tool* was a result of her assimilation of the activity into the context of tracing shapes of graphs in the air, without attending to the quantities or how they covaried.

In summary, Carolyn was happy with how the lesson went. She said, “I thought that it went, that it went quite well”. She mentioned a few students had questions for her after school, but that the questions were easy to answer; the students whom she had helped “were doing just fine” after the session. By retreating to her old ways, Carolyn was once again at ease with her knowledge of the mathematics and at ease with teaching the subsequent lessons.

Summary

This chapter portrays a complex relationship between Carolyn’s views about mathematics, her goals for teaching and student learning, her mathematical knowledge for teaching, and her pedagogical practices in the context of the curriculum she used. Her ways of understanding mathematics had a significant impact on her pedagogical moves, her curricular choices, and her approaches to supporting student thinking and learning. In the next chapter, I provide a summary of what has been learned from this study and highlight directions for further research.

CHAPTER 7: CONCLUSION

The intention of this study was to characterize the relationship between a teacher's mathematical knowledge for teaching and his/her pedagogical goals. In this chapter, key results relative to the following research questions will be shared.

1. What is the effect of a professional development intervention, designed to perturb a teacher's pedagogical goals for student learning to be more attentive to students' thinking and learning, on a teacher's views of teaching, stated goals for student learning, and overarching goals for students' success in mathematics?
2. What role does a teacher's mathematical teaching orientation and mathematical knowledge for teaching have on a teacher's stated and overarching goals for student learning?

Summary of Emerging Frameworks

My research began with me developing a framework to characterize teacher goals. I based this framework on theoretical constructs contributed by other researchers. The framework included constructs of: (1) key developmental understandings (KDUs) (Simon, 2006), (2) mathematical knowledge for teaching (MKT) (Silverman & Thompson, 2008), (3) teaching orientation (A. G. Thompson, et al., 1994), and (4) goals (Locke & Latham, 2002; Pintrich, 2000). This framework was used to characterize the interactions between these key constructs. I conceptualized the system of interactions as seen in the figure that follows, and this guided how I framed the research questions and analyzed the research data collected in this dissertation study.

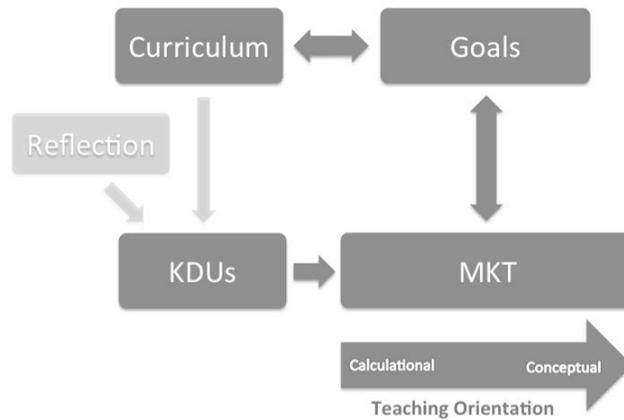


Figure 35. A revised Framework of Interactions among key constructs

To review, the arrow above teaching orientation in the framework describes a shift in teaching orientation from calculational to conceptual as a teacher’s mathematical knowledge for teaching advances. The findings of this study resulted in a revision to the *Framework of Interactions* among key constructs to include the orientation of the curriculum a teacher chooses to inform his or her lesson planning (labeled as ‘curriculum’ in the framework). The original model of interactions (see Chapter 2, p. 15) did not include the curriculum orientation used for planning a lesson.

A framework to characterize a teacher’s mathematical goals for student learning was developed. To characterize these goals, two exploratory studies were conducted with participants who taught using a conceptual curriculum (Carlson, et al., 2013b). Grounded theory (Strauss & Corbin, 1990) was used to create the goals framework, using Silverman and Thompson’s (2008) characterization of MKT as the theoretical lens from which open coding was performed. From these studies, a goals framework emerged and was subsequently used to characterize a teacher’s mathematical goals for student learning. This dissertation investigation used the same goals framework in a case study with one high school teacher who was teaching with the conceptual curriculum. The goals in the

framework ranged on a spectrum representing: no focus on student thinking at level 0 to a maximal focus on student thinking at level 6. The framework is given below.

Table 15.

Characterization of Levels in a Teacher’s Mathematical Goals for Student Learning

Goal Coding	Description
<i>TGSL0</i>	Goals for student learning are avoided or not stated by the teacher, or he/she states that the goals of the lesson are unknown.
<i>TGSL1</i>	Goals are a list of topics that a teacher wants his/her students to learn in the lesson, each associated with an overarching action.
<i>TGSL2</i>	Goals are a list of topics that a teacher wants his/her students to learn in the lesson, each associated with a specific action.
<i>TGSL3</i>	Goals are doing methods of mathematics that a teacher wants his/her students to learn in the lesson.
<i>TGSL4</i>	Goals are getting students to think about the mathematics in the lesson, without the ways of thinking articulated.
<i>TGSL5</i>	Goals are getting students to think about the mathematics in certain ways during the lesson.
<i>TGSL6</i>	Goals are about developing ways of thinking about the mathematics in the lesson, with attention to how that thinking may develop.

To study a participant’s mathematical knowledge for teaching (MKT), a conceptual analysis was performed on the key ideas of quantities (P. W. Thompson, 1994b), covariation (Carlson, et al., 2002), proportionality (Carlson, et al., 2013b), constant rate of change (Carlson, et al., 2012; P. W. Thompson, 1994a), and average rate of change (P. W. Thompson, 2013). Additional analyses were conducted to characterize how leveraging quantitative reasoning impacted a teacher’s meaning for exponential functions.

With the key frameworks and key elements of the conceptual analysis from this study summarized, I now provide a summary of the key results of this dissertation study.

Summary of Key Results

This section summarizes findings related to Carolyn's goals and how they interacted with her mathematical knowledge for teaching, teaching orientation, and the orientation of the curriculum she used. Following this, I summarize the effects of the professional intervention and other external factors that impacted Carolyn's instructional choices and teaching.

Relationship Between Goals and Curricular Source

Carolyn's stated mathematical goals for student learning for 11 observed lessons were analyzed using the goals framework. Twenty of Carolyn's initially stated goals were rated and separated into Pathways (Carlson, et al., 2013b) and non-Pathways lessons. The non-Pathways lessons used were skill-based lessons she had used in prior years when teaching. Of the lessons analyzed, five were Pathways lessons, while six of the lessons were non-Pathways lessons. The results are summarized below.

Table 16.

Comparing Carolyn's Lessons: Curriculum Source

Stated Goal	Pathways Lesson	Non-Pathways Lesson
TGSL1	4	2
TGSL2	3	3
TGSL3	1	2
TGSL4	1	3
TGSL5	0	1
Total	9	11

Unlike Robert, whose goals were strongly aligned with the conceptually oriented curriculum used (skills-based versus conceptual), Carolyn's goals for student learning did not have the same strong associations. Carolyn's goals were rated primarily at TGSL3 or

lower (overarching topics, topics with specific actions, and methods of mathematics).

While it seems to be a contradictory finding that one teacher's goals aligned with the conceptually oriented curriculum, while another teacher's didn't, these differences can be explained from the perspective of Carolyn's overarching goals for teaching.

Carolyn's Overarching Goal

With Robert, the type of curriculum (skill-based or conceptual) used aligned with the clustering of his lesson specific goals. One of Robert's overarching goals for teaching was to improve student understanding. During the exploratory study Robert also described how he envisioned the trajectory of skills-based lessons and conceptual lessons unfolding. His overarching goals were predictive of his lesson specific goals.

One of Carolyn's overarching goals for teaching was to prepare her Precalculus students for success in AP Calculus (The College Board, 2012). Her decisions about what ideas to address in a lesson, and her lesson specific goals were influenced by her desire to prepare her students to perform well on this exam. From this perspective, the finding that a majority of Carolyn's goals were classified at TGSL3 or lower (overarching topics, topics with specific actions, and methods of mathematics) was not surprising. Her overarching goal to prepare students for AP calculus exams was consistent with the values of the system in which she taught. Awards of recognition were given to teachers by the school district if 60 percent or more students took the AP exam passed it.

Relationship Between Goals and Mathematical Knowledge for Teaching

A second perspective I used to characterize Carolyn's goals was to examine the relationship between her mathematical goals and the mathematical content she choose to teach. The analyses further compared her goals for two mathematical topics, one in which

she had strong MKT (polynomial functions) and the other for which her MKT was more impoverished (exponential functions). The results are given in the table below.

Table 17.

Comparing Carolyn’s Lessons: Curriculum Content

Stated Goal	Chapter 4 (Exponential Functions)	Chapter 5 (Polynomial Functions)
TGSL1	6	0
TGSL2	6	0
TGSL3	2	1
TGSL4	1	3
TGSL5	0	1
Total	15	5

Although the data points were few, the contrast between Carolyn’s goals in the polynomial chapter and the exponential chapter were markedly different. Based on classroom observations and Carolyn’s responses to the pre-study task, the findings suggested that she did not have robust meanings for partial growth factors and therefore she could not leverage this key idea in flexible ways. Robust meanings of a partial growth factor include: (1) the number that if multiplied by itself n times results in the 1-unit growth factor, and (2) if the 1-unit growth factor is known, the partial growth factor is the n^{th} root of the 1-unit growth factor. Although her mathematical connections advanced with regard to exponential functions she was unable to support student reasoning in the context of a lesson that introduced the concept of n -unit and partial unit growth factors. The findings from this study revealed that her instruction and instructional goals were influenced by (1) her weak meaning for exponential functions, and the n -unit and partial unit factors lesson in particular, (2) her inability to meet her stated goals for student learning in the partial unit growth factor lesson, (3) her overarching goals of preparing

students for Calculus, and (4) her personal goals of being efficient and not wasting students' time. These factors were identified as influencing her decision to returning to traditional skills-based practice lessons for the remainder of the chapter. The following statement by Carolyn summarized her thoughts regarding the multiple and partial growth factor lesson.

You know, we spent basically three days talking about one-unit, two-unit, three-unit, four-unit, whatever growth factors. And I know what they see when they get to Calculus. And they see base 'e'. They don't see this other stuff. At least in the Calculus, in the AP Calculus curriculum that I teach, we don't deal with exponentials with any base other than 'e', hardly ever.

Since Carolyn attached high importance to exponential functions with the base 'e', the findings further suggested that her meanings for 'e' were superficial. When asked what was the main idea of 'e' she wanted to convey to her students, Carolyn's response was as follows: "I don't know a good way to describe except it's this magic number. It's a magic function, I think, just because its derivative is itself."

Characterizing Carolyn's MKT

Carolyn did make substantial shifts in her MKT related to exponential growth. As compared to the subject in April Ström's study who believed that any functions with exponents, such as the function $f(x) = x^2$ grew at an exponential rate (Ström, 2008, pp. 48-49), Carolyn advanced significantly in her understanding of exponential growth. In this next paragraph, I describe some of Carolyn's key meanings of exponential functions.

Let b represent the 1-unit growth factor, d be a non-unit growth factor, c be a positive real number, m be a nonzero real number, and f be the name of a function; the mathematical statements given next to verbal descriptions in brackets will be used to convey Carolyn's thinking process symbolically, along with the observed lesson in which

these findings were situated. Based on observations of the first four conceptual lessons with exponential functions, and her statements made during the lesson planning and debriefing sessions, the findings of this study suggested that her key meanings of exponential functions included the following mental images:

- (1) how to leverage the multiplicative relationship of a 1-unit growth factor and the current output of a function, to find the preceding and succeeding outputs

$$\left[\frac{f(x)}{b} = f(x-1), \text{ and } b \cdot f(x) = f(x+1) \text{ (Lesson 1)}, \right]$$

- (2) the initial value of an exponential function occurred when the input value was 0, $[f(0)$ is the initial value (Lesson 3)],

- (3) the 1-unit growth factor can be determined by finding the ratio of the succeeding and current outputs for a change in input of 1 unit

$$\left[\frac{f(x+1)}{f(x)} = b \text{ (Lesson 2)}, \right]$$

- (4) a percent change of some amount over some time period can be computed by raising the growth factor to the amount of time period elapsed, subtracted from 1, and changing to a percentage $[(d^t - 1) \cdot 100\%$ represented the percent change for some time period t (Lesson 3, Lesson 4)],

- (5) how a strategic rewriting of an exponential function with a parenthesis can help reveal the 1-unit growth factor [for $f(x) = cd^{mx} = c(d^m)^x$, this implies d^m was the one-unit growth factor (Lesson 4)], and

(6) a unitization approach (determining the 1-unit growth factor first) could be leveraged to find any unit growth factor [for $d^m = b$, the n -unit growth factor is $(d^m)^n$ or b^n (Lesson 4)].

The results of the study suggest that the ways of thinking described as items (5) and (6) emerged during the course of the lesson as it was taught. Her unitization of the growth factor helped her make meaning of multiple unit growth factors. By thinking of repeated multiplication of 1-unit factors n times, Carolyn reasoned that the result of this operation was equivalent to one n -unit factor. In Lesson 4 observation, the results had highlighted this finding as she made sense of an exponential function whose 1-unit growth factor was 1.24. As a class she and her students determined that the two-unit growth factor was approximately 1.538. She asked her class: “So that if I said how many, what’s the value when we have one 2-unit chunk. Is that going to be the same as what we get when we have two 1-unit chunks?” Carolyn did not use precise language in this exchange, but the findings suggested the ‘chunks’ to her were ‘factors’.

To make meaning of n -unit and multiple unit growth factors, her most common approach was to use a procedure to unitize the growth factor first. Given any exponential function in a canonical form, she would rewrite the function and then identify the one-unit growth factor. Then to find the n -unit growth factor, she would raise the 1-unit growth factor to the n^{th} power, reasoning this meant the number of repeated factors. Her way of thinking multiplicatively with exponential functions was limited to the mental images described in items (1) and (3).

Teaching Orientation

Although she used a conceptually oriented curriculum for approximately half the observed lessons, the findings from this study suggested that Carolyn's teaching orientation was transitioning from procedural to more conceptual (A. G. Thompson, et al., 1994). While many of her pedagogical moves (such as focusing on her students' computations or promoting specific methods or steps to answer a problem 'efficiently') suggested a calculational orientation, there were lessons observed during the study that suggested small shifts towards a conceptual orientation were underway during some teaching episodes when using the conceptual curriculum. For example, in Lesson 9 Observation, Carolyn gave students an opportunity to explain their reasoning on a task where different graphs were created when plotting the relationship between the time elapsed since Karen left the grocery store and her distance from her home (Excerpt 54, p. 191). After small groups engaged in the activity and plotted the graph of the relationship on a mini-whiteboard, she volunteered two groups to place their work in front of the class to see. She was more interested in the mathematical ideas being used by the group to create those graphs, and facilitated a class discussion around the meaning of the graphs that had been drawn. Curricular supports appeared to contribute to these shifts.

The conceptually oriented curriculum that Carolyn was using aided her in making these shifts as she planned her daily lesson she worked to understand the ideas that were being promoted in the investigations. However, without a professional learning community (PLC), a mentor, or time to reflect on her lessons she sometimes failed to make subtle and key connections needed to respond to a question or provide a coherent explanation when teaching.

Minor Changes in Carolyn's Instructional Goals

The professional development intervention affected Carolyn's practice in minor ways, and it is possible that her involvement in the research study led to improvements in her key developmental understandings of the mathematics when she was probed to reflect on her practice. The intervention that included lesson collaboration that focused on: (1) the mathematics she was going to teach, (2) her mathematical goals for student learning, (3) other goals— such as goals for student interactions, (4) questions to support students' mathematical reasoning, and (5) ways students might reason. These meetings appeared to result in only minor changes in Carolyn's instructional goals. However, my suggestions for her to try new pedagogical moves appeared to sometimes evoke novel thinking about her goals for teaching. The planning and debriefing sessions required her to reflect on her practice in ways she did not have available at her campus, as there was no active professional learning community operating for Precalculus at Atlas High School. This observation was supported in Carolyn's comments when I asked her to relate her experiences working with a colleague in collaborative lesson planning.

Excerpt 80. Experiences working with a colleague

Car: I enjoyed having a different perspective, suggestions for different ways to do things. Didn't always do what you suggested, but when I, sometimes when I did, I thought it, there were great ideas. It does help to have another perspective. When it's somebody who has a lot of experience in what you're doing. I don't get an opportunity, although we do do collaboration, I'm teaching classes nobody else is teaching so I really don't get that opportunity to do collaboration and planning. Like perhaps like Algebra 1 teachers do, where there are 6 of them. Or the Algebra 2 teachers.

Res: There's fewer resources to lean on.

Car: Yeah.

Unfortunately for Carolyn, her other colleagues did not teach Precalculus using the Pathways materials. To contrast, Robert from the second exploratory study had a

robust PLC environment in which he was the lead teacher of the Precalculus PLC. Had Carolyn also had an active PLC this may have provided more opportunities for her to discuss student thinking, and reflect on the mathematical understandings she desired for her students.

Perceived Impact of Lesson Collaboration

In reflecting on her experience in lesson collaboration, I asked Carolyn to identify the items from the *Lesson Reflection Questionnaire* she considered using in the future. Her response indicates that she believed the lesson collaboration had a positive impact on her thinking about specific questions to pose to her students.

Well, I think it has helped me to think about specific questions that would be helpful that may be a different way of looking that I hadn't done before. What other, what questions can I use to help them conceptualize. Maybe I haven't always done a good job with that. I had never, I guess I had never had a goal for student interaction. I mean I had never looked at that in that regard - in that term, in those terms.

In particular she found thinking about specific questions to ask, and the question about her goals for student interactions helpful; these were components of questions 2 and 4 from the *Lesson Reflection Questionnaire* (version 3) (see Appendix K) during the Post-study interview. As for why she picked those questions, based on the results of this study and working with Carolyn, I hypothesize that having specific questions to ask provided guidance for improving her questioning and student engagement when teaching. She made earlier comments on several occasions during the study that conveyed that student engagement was one of the hallmarks of a successful lesson. Her high value on student engagement may explain why she valued thinking about student interactions and their impact on student engagement during a lesson.

Limitations of Intervention Influenced by Goals

Although other researchers had used video reflection as way to perturb practice, it was not an effective tool because Carolyn was not interested in looking at videos of her practice; she expressed concern about the amount of time it would take to view her video taped lessons. She also appeared to be distracted and impatient during the few instances when we viewed a video from her class. This response appeared to result from the pressure she was feeling about the many other school related time commitments she had at Atlas High School. The planning sessions and debriefs for this research study were in the late/afternoons and evenings due to her after school and administrative commitments. She expressed that watching videos were not a good use of her time after a long day when she still had to plan her lesson for the following day. She had expressed the same concern when describing her personal goals earlier in the study (Excerpt 22, p. 137).

Insights Gained from This Study

This section will summarize insights gained from this dissertation study with respect to the two stated research questions. Key results from this study that informed these insights are highlighted.

First Research Question

The first research question: What is the effect of a professional development intervention, designed to perturb a teacher's pedagogical goals for student learning to be more attentive to students' thinking and learning, on a teacher's views of teaching, stated goals for student learning, and overarching goals for students' success in mathematics? The three aspects of Carolyn's profile that were investigated are (1) her views of

teaching, (2) her stated goals for student learning, and (3) her overarching goals for students' success in mathematics.

The findings suggest that Carolyn's view of teaching mathematics was influenced by the Calculus AP exam (Excerpts 16, 18, 33). Participating in the professional developmental intervention in this study did not appear to impact this view. Success in Calculus meant having a good foundation for the topics in the AP curriculum (Excerpt 14), and to her this meant not leaving any of the topics out from the AP Calculus syllabus that might be assessed. She viewed Precalculus as a preparatory course for Calculus (Excerpt 15). Carolyn viewed student engagement in a lesson as a hallmark of success (Excerpt 20) and she liked student discussion in the context of mathematical tasks (Excerpt 57). However, attending to students' construction of meaning during engagement in the Pathways mathematical tasks did not occur. As a result it appears that her view of teaching mathematics (Excerpts 56, 57) did not include a focus on student thinking.

In looking at the first research question with respect to the effect of the intervention on Carolyn's stated goals for student learning, I found efforts to perturb her instructional goals minimally effective. She found thinking about questions she could ask her students helpful to her, but she stated that she did not always follow through with suggestions I had made (Excerpt 80). Her views of teaching mathematics as a series of topics for students to learn were reflected mostly in her stated goals for student learning in the exponential functions chapter (Table 17), where 12 of 15 stated goals were rated TGSL1 or TGSL2. Attempts to have Carolyn reflect on her practice by viewing videos of her teaching were met with resistance (Excerpts 41, 42). The findings suggested that my

intervention attempts were influenced by external factors that reduced her available time to reflect on her practice (Excerpts 7, 8). Her personal goals of being efficient and not wasting the students' or her time (Excerpt 22) also influenced her pedagogical goals and her moves to support students in class. She seemed to favor specific solution methods that she viewed as more efficient and made moves to support these while discouraging "less efficient" methods (Excerpts 10, 11). The teaching episodes when Carolyn appeared more open to student contributions occurred when she was learning the key ideas of mathematics alongside her students (Excerpts 70, 71).

In looking at the first research question with respect to the effect of the intervention on Carolyn's overarching goals, the findings from this study suggest (1) they were resistant to change, and (2) successful perturbation of a teacher's classroom practice was possible when not in conflict with her overarching or personal goals. Carolyn's overarching goal for Precalculus teaching to prepare students to answer questions on the AP Calculus Exam did not change as a result of this professional development intervention (Excerpts 14, 15, 76); nor did her personal goal to be efficient and not waste time (Excerpts 10, 11, 22, 76). However, successful attempts to perturb her classroom practice occurred when they were not in conflict with her overarching or personal goals. Although Carolyn was not mindful of attending to her students' process of meaning making (Excerpt 56), she valued learning new ways to think about or do mathematics (Excerpts 43, 44, 45). Expanding her own knowledge of mathematics was not in conflict with her goals or preparing her students for AP Calculus, because she could leverage her learning to teach her students other approaches to complete problems (Figure 17, Excerpt 43).

Second Research Question

The second research question: What role does a teacher's mathematical teaching orientation and mathematical knowledge for teaching have on a teacher's stated and overarching goals for student learning? The findings of this study suggested that Carolyn's mathematical teaching orientation was predominantly calculational. While many of Carolyn's pedagogical moves were calculational (Excerpts 9, 10), some teaching episodes suggested that Carolyn was making shifts from a procedural to a conceptual orientation (Excerpt 54). With regard to the relationship of teaching orientation and a teacher's stated mathematical goals for student learning, the findings were inconclusive. While the teaching episode (Excerpt 54) described in Lesson 9 Observation, in which she used a conceptual curriculum, revealed minor shifts toward a conceptual orientation that was associated with a higher stated goal for student learning (TGSL4), Carolyn had also taught skill-based (non-Pathways) lessons in which her stated mathematical goals were also rated at TGSL4 (Table 12). Her pedagogical moves in these skill-based lessons were characterized as procedural and focused on getting answers, suggesting that her higher stated goals for developing student thinking during a lesson were not enacted when teaching. The findings of this study revealed that Carolyn's teaching orientation was primarily aligned with her overarching goals to: (1) prepare students to complete problems on the AP Calculus exam, (2) have her students engaged during class, and (3) to be efficient with both her and her students time. Carolyn's overarching goals were to prepare students for the AP exam (Excerpt 14). This overarching goal supported curricular choices (Excerpts 17, 18, 19) that emphasized procedures over concepts and

pedagogical moves (Excerpts 10, 11) that focused on getting answers as efficiently as possible.

With regard to the relationship between a teacher's mathematical knowledge for teaching and her goals, Carolyn's key meaning of exponential functions (p. 273-275) were robust enough in the first three conceptual lessons observed in this study to accomplish her stated mathematical goals for student learning (p. 230-231). In the fourth lesson observed that she taught pertaining to n -unit and partial unit growth factors, she struggled with the mathematical ideas of the lesson and was not able to support student learning (Excerpt 39). She could not accomplish her stated mathematical goals in that lesson (Excerpt 38). She questioned the value of spending time teaching partial and multiple unit growth factors, when exponential functions in AP Calculus typically had base ' e ' (Excerpt 76). Carolyn expressed that her experience teaching the partial growth factor lesson was embarrassing at times (Excerpt 74). Her view of ' e ' was not robust, and her personal meaning for ' e ' was a "magic" number frequently used as a base for exponential functions that grow or decay (Excerpt 40). After the observation of the partial and multiple unit growth factors lesson, Carolyn retreated to skills-based practice lessons when teaching the remaining lessons in this chapter (p. 266). Carolyn did not have a PLC at her school that supported her in teaching Precalculus with a conceptual curriculum (Excerpt 80). Since she was the only teacher at Atlas High School using the conceptually oriented curriculum she did not have regular opportunities to discuss issues related to understanding, learning, or teaching key ideas in the conceptually oriented curriculum with colleagues.

Summary of Results and Some Thoughts

The results of the professional development intervention showed potential growth in ways of thinking about the mathematics that did not yet manifest itself in practice. Since lesson collaboration was an experience designed to promote reflection on the mathematics of the lesson and shifts in teacher's mathematical goals for student learning, it is possible that more shifts in Carolyn's practice would have been observed had she invested time in consider learning goals that were more focused on productive ways of thinking for her students.

A main finding of the study is that Carolyn's overarching goals appeared to take precedence over lesson specific goals. An overarching goal is a stable goal, independent of context. This is consistent with other research that has reported that stable goals are context independent (Barsalou, 1983). Overarching goals, like beliefs, are resistant to change (Philipp, 2007; A. G. Thompson, 1992). The results from this study suggested that external influences such as: (1) outside assessments, (2) additional time commitments in support of non-academic school functions, and (3) district initiatives that incentivize/de-incentivize results of students' test scores have an effect on a teacher's goals. In the case study of Carolyn, these external influences included: (1) preparing students for success in the AP Calculus Exam, (2) scoring Volleyball after school, and (3) district recognition for high participation (at least sixty percent) of her students taking and passing the advanced placement Calculus exam.

This is not an indictment of Carolyn. Rather, it is an indictment of the cultural values of the education system in which Carolyn taught. Carolyn is a successful teacher and her students are successful as a result of her being in equilibrium with the education

system in which she taught. Her decision to return to traditional content focus for teaching exponential and logarithmic functions aligned with her personal goals, lesson specific goals, overarching goals, and her views of teaching. Returning to skills-based teaching represented returning to an equilibrium (Stigler & Hiebert, 1999); a place in which she felt comfortable—free of perturbation. Her views of teaching Calculus and by extension, Precalculus, were influenced by the list of topics on the syllabus of the AP Calculus Exam and its requirements for mastery. Success in mathematics was measured in the context of high stakes tests. As was observed in this study both students who performed well, and their teachers, were rewarded for such efforts.

Humanity has survived for millennia due to their ability to adapt to the environment in which they live. Simply stated, Carolyn and her students thrived in the environment to which she had adapted without an awareness that such an adaptation occurred. As was shown in prior research (Stigler & Hiebert, 1999) the system self-corrects ripples that are introduced. I conjecture that any shifts (albeit minor) that may have occurred in Carolyn's goals and pedagogical actions as a result of participation in this intervention will diminish over time.

Future Directions

Regarding the future directions for research, the results were inconclusive as to the sources of her lower rated goals for teaching exponential functions. The data suggests that weaknesses in her MKT may have contributed and her image of what students needed to know for calculus. Future research would be useful to see if fostering her in developing more robust connections would result in her valuing the instruction supported by the conceptual curriculum. In contrast Carolyn's meaning for teaching the polynomial

chapter appeared to be sufficient for aligning her instruction with the goals of the conceptual goals of the curriculum. They also appeared to be aligned with what she thought was important for preparing students for Calculus (Excerpt 55, lines 11-13, p. 192). Another explanation might be that her goals helped motivate her to make deeper connections when teaching content that she valued for their future learning. This explanation is consistent with Webb's (2011) findings. Future studies could inform the impact of impoverished meanings for multiplication, factors, roots, proportionality, and quantitative reasoning in the development of well-connected meanings for exponential functions. Future research may also study the impact of interventions designed to develop these foundational meanings prior to introducing exponential growth and exponential functions.

There were limitations in how the goals framework was implemented during data analysis. The researcher rated the teachers' first utterances of their stated goals, which tended to underrate possible unarticulated goals. In addition, any clarifying comments later in the conversation were not used to modify the goal classification of Carolyn. The framework that emerged measured one dimension of a teacher's goals organized around the level of attention to student thinking, and other dimensions of the framework could be developed to provide a more comprehensive characterization of a teacher's goals. For example, it may be useful to explore the relationship between a teacher's mathematical goals for student learning and his or her (1) goals for student interactions, or (2) goals for procedural fluency. A multidimensional framework that coordinates these differing dimensions could offer a more nuanced insight into a teacher's goals. Another direction

to investigate would be the usefulness of this framework, and how it might be extended when used in the context of a teacher using a skills-based curriculum.

Some Final Thoughts

In terms of contributions to the field, this study leveraged key constructs based on prior research to create a framework that models the major influences that impact a teacher's goals. A goals framework emerged that characterized a teacher's mathematical goals for student learning. The data provided further insights regarding the influence of a teacher's MKT and external school factors (e.g., the content focus of AP exams, limited time for planning, and absence of an intellectual colleague or mentor when preparing lessons) on her goals, choice of curriculum to use when teaching, and implementation of a conceptually oriented curriculum.

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APPENDIX A
TEACHING ORIENTATION AND GOALS STUDY

Introduction

The following is an excerpt of an unpublished report that studied the relationship between teaching orientation and a teacher's goals for teaching. An earlier form of this work can be found as a poster presentation in Marfai and Carlson (2012). The key components of the theoretical framework from the dissertation study that were used in this report were the constructs of mathematical knowledge for teaching, key developmental understandings, teaching orientation, and goals.

Research Question

The following problem was posed in Precalculus curriculum developed by Carlson and Oehrtman (2012).

In a diving competition, a diver received the following scores from 4 judges after making a dive.

Judge 1	Judge 2	Judge 3	Judge 4
8.7	9.3	8.0	8.2

What is the meaning of average in the context of computing a diver's average score for a dive? How does the meaning of the word *average* when computing the average score compare with the meaning of the word *average* when computing a diver's average speed?

Figure 36. The Diving Task

The research question was as follows. How does a teacher's orientation impact her MKT and goals for teaching in the context of a conceptually oriented task?

I used the tool of conceptual analysis (Glaserfeld, 1995; P. W. Thompson, 2008) to discuss possible understandings of an idea that is held by a teacher and how those understandings inform his or her teaching practice.

Conceptual Analysis

In the article by Moore, et al. (2011), the Diving Task (Figure 36) was posed as a follow-up question to a task designed to help students confront the difference between an arithmetic average and the idea of the average speed of a diver who had jumped into a pool from a specific height above the water level. The intent of the diver task was to help students conceptualize the average speed of the diver as the constant speed a second diver jumping from the same height would need to travel at in order to drop the same distance in the same time. In earlier lessons, constant speed had been developed as a proportional correspondence between the changes in two quantities as the quantities' values change in tandem Carlson and Oehrtman (2012).

For a teacher having a calculational orientation, the Diving Task was designed to confront students' conceptions of arithmetic average and average rate of change and how these two ideas are related. This question as posed is incompatible with the rules taught from the previous task on average speed. This would fit into the teacher's schema seeing this question as an opportunity to clarify the differences between the two types of average for students. The arithmetic average is viewed as a procedure, in which the values of the items in the list under consideration are summed up and divided by the number of items in the list. A teacher's MKT under this scenario is having knowledge of various procedures one can use to calculate an arithmetic average, and juxtaposing it to the procedures for finding average rate of change. In this analysis, I will refer to the teacher's MKT with a calculational orientation as MKT #1.

A teacher whose MKT is conceptually oriented views the Diving task as a discrete instantiation of the average rate of change. When the total accumulation of scores

are considered, then the average score would be the constant score needed to be earned by each judge, from a set of fictitious judges, in order to earn a total of 34.2 points from the 4 judges; in other words, a constant score of 8.55 points per judge would be needed to earn a total of 34.2 points from these 4 judges. This would fit into the teacher's schema that this question could be used as a way to build on the idea of the average rate of change from the previous task, but in a different context. I will refer to the teacher's MKT of the Diving task with a conceptual orientation as MKT #2.

The teacher may carry MKT #1, MKT #2, or both, although it is highly unlikely that a teacher thinks of MKT #2 by itself – a teacher's prior education ensures some form of MKT #1 exists in a teacher's knowledge structure. Depending on how the teacher interprets the intention of the Diving task, the teacher could have overarching goals such as 1) supporting each student's way of thinking to develop an "appropriate" image of average rate of change, and 2) providing each student an opportunity to meaningfully contribute to the understanding of average rate of change. The reason I put "appropriate" in quotes for goal 1) is because if the teacher lacks one of the strands of MKT, he or she might not be able to follow student thinking in that strand. For example, a teacher lacking MKT #2 would see students pursuing a line of thinking that is consistent with MKT #2 as conflating arithmetic average and average rate of change, and therefore makes pedagogical moves to steer students away from this "unproductive" way of thinking. Such a scenario, for example, was seen in research by Marfai, et al. (2011) when the teacher's view of "productive" ways of thinking used proportionality as a key idea of modeling the situation, while the students she was working with found the idea of using a step function as a productive model for the situation.

Methods

Claudia (pseudonym) is a secondary mathematics teacher teaching Precalculus at Rover High School (pseudonym) using the research-based curriculum (Carlson & Oehrtman, 2012) from the Pathways project. Project Pathways is an initiative that focuses on professional development to improve teacher's key developmental understandings (Simon, 2006) of the mathematics they teach in order to improve content knowledge through fostering a rich connection of mathematical ideas and relationships. Part of this initiative led to project leaders developing a research based conceptually oriented curriculum that teachers involved on the project would use in their classroom.

I used the conceptual analysis of a teaching orientation's interaction with a teacher's MKT described earlier with regard to the Diving Task (Figure 36), and then performed further analysis on the full video associated with the Diving Task reported in Moore, et al. (2011) with my research question in mind. The time spent in class with regards to this particular task was approximately 15 minutes. I did a qualitative analysis of the classroom interactions.

Results

In the two excerpts that follow, ... at the end of a speaker's utterances was used to mean one speaker cutting off another speaker, and words in parenthesis will be used to convey an action by the speaker. Stu was used for student (followed by a speaker number), Cla was for Claudia, and Grp refers to the current group of students Claudia was conversing with if they were speaking in unison. Students had been working on the Diving Task (Figure 36) in groups of 3 to 4 students, and Claudia initiated a class

discussion by asking groups to report their responses to the first question stated in the task.

Excerpt 81. Claudia's teaching orientation and her goals for teaching

- 1 Cla: Alright, so (b), what's the meaning of average score in the context of
2 computing the diver's average score? Okay, we're going to start over
3 here. (selecting a group) Read to me what you wrote.
4 Stu1: If all the judges had the same score, it would be 8.55, you want (b),
5 right?
6 Cla: Uh huh.
7 Stu1: It would be 8.55.
8 Cla: Okay, alright, what did you have? (going to next group of students)
9 Stu2: The constant score you need to receive the same score from all four
10 judges.
11 Stu3: Same total score.
12 Stu4: Total score.
13 Cla: Okay, total score?
14 Grp: Yes.
15 Cla: Okay, could you really say that's a constant score?
16 Stu2: Yeah.
17 Cla: Can you call it a constant score?
18 Stu4: No.
19 Stu3: It's the same score you...
20 Cla: Because they're all giving her a score at the same time, right?
21 Stu2: Yeah, but it needs to be constant score...it doesn't need to be.
22 Stu3: Yeah.
23 Stu4: The...
24 Stu2: The score that gives you the same total score.
25 Cla: Okay, okay, alright, different, did anyone else have a different
26 definition? (moving on to next group of students)

Excerpt 81 suggests that Claudia's teaching orientation was calculational (lines 7, 8) based on her acceptance of the response "it would be 8.55" as the one having "meaning", even though Claudia's interpretation of the question appears to be consistent to one of asking for the definition of the average score (line 25-26). There is no evidence to suggest that Claudia listened to the first iteration of the group's response (lines 4-5), as there was no follow up. In the group she called on after, Claudia attempted to make pedagogical moves to steer students away from what she perceived as "unproductive"

ways of thinking about the average rate of change (lines 15, 17, 20). This group's perspective on average rate of change was consistent with a way of thinking that would be supported by a teacher having a conceptual orientation (lines 9-12). However, Claudia's way of thinking was consistent with MKT #1 and she did not have a way of thinking that could be supportive of MKT #2.

Let us look at the two overarching goals described in the conceptual analysis to see how her MKT influenced her goals. In looking at goal 1), Claudia's image of average rate of change was not conceptual, and so did not support student thinking that might be conceptual (line 4-5, lines 9-12). Her MKT was supportive of the products of student thinking, but only if they were couched in calculational terms (lines 4-8). It appears Claudia was at a loss in how to respond to the second group based on her final utterances in the transcript (lines 25-26). It appears that goal 2) was valued by Claudia since she asked multiple groups to give their perspectives on the question, regardless of how the interactions unfolded. The dialogue between Claudia and the group she called on after the interactions described in Excerpt 81 is given in Excerpt 82.

Excerpt 82. Further evidence of Claudia's teaching orientation

- 1 Cla: Okay, go ahead.
- 2 Stu5: We said the overall score given by the four judges for the one dive.
- 3 Cla: So, determine, what is the meaning of average in the context of
- 4 computing a diver's average score for a dive? Read it one more time.
- 5 Stu5: The overall score given by the four judges for the one dive.
- 6 Cla: That's pretty good.
- 7 Grp: (laughing)
- 8 Cla: (laughing) Okay.
- 9 Stu5: Nice.
- 10 Cla: As long as you understand that overall score was a mean of all the
- 11 four scores, yeah.

Excerpt 82 is consistent with the hypothesis that Claudia's MKT was calculational oriented, especially in light of her last utterances in the transcript (lines 10-11), where her pedagogical move was to reinforce the idea that the students' use of the term "overall score" should be thought of as the arithmetic mean. In addition, her utterance in lines 10-11 indicates that an "appropriate" image consistent with her interpretation of average rate of change was one that was supportive of only a calculational orientation. In looking at Claudia's overarching goals, her response in lines 10-11 is consistent with the goal of helping students develop an "appropriate" image of the average rate of change, in offering a "clarifying" statement. Based on her teaching orientation, she had provided the students in the group an opportunity to contribute to the understanding of the rate of change.

Discussion and Conclusion

In Moore, et al. (2011) the authors noted that after the class session, Claudia had expressed an uncertainty with regards to the ideas of the Diving task, and how these ideas related to the previous task with a project leader. The focus in this report was on Claudia's teaching orientation in the initial way that she saw mathematics, which was at the time incompatible with intentions of the curriculum. Over time, as Claudia's mathematical content knowledge grew through intense professional development supportive of a conceptual orientation in mathematics, she was able to transform her key developmental understandings into mathematical knowledge that informed her practice. From the earlier report by Moore, et al. (2011), I see evidence that Claudia's view of mathematics transitioned from a calculational orientation to a conceptual orientation through the course of a semester that she was observed, and Claudia's goals for teaching

were a key component in making this transition. This is consistent with prior research (Webb, 2011); Claudia's MKT advanced in response to her emerging goals for her students.

I see the potential for future research in analyzing how teaching goals transition as a teacher's MKT evolves from a calculational to a conceptual orientation. Pintrich (2000, p. 102) stated that “ ‘strong’ classroom contexts or experimental manipulations (where the context defines the situation and appropriate behavior in many ways) can influence individuals to activate different goals than the ones they would normally or chronically access.” In interpreting Pintrich's statement, the goals a conceptual curriculum and professional development supports, such as meaning making, ideas of mathematics and the relationships between them, conceptual coherence, and attention to student thinking may be reciprocally linked to a teacher's orientation in mathematics. These findings suggest that goals for student learning, that are not accessible by teachers whose MKT is calculationally oriented, can become accessible in an instructional environment that supports teachers in transitioning to a conceptual orientation of mathematics learning and teaching.

APPENDIX B

DOROTHY AND MARGARET'S GOALS

General goals for any lesson agreed upon by our group

Question asked: *What are your overall instructional goals for any lesson you teach?*

1. Focusing on good questioning strategies
 - Context specific: listening to students and thinking of questions to guide them to main idea of lesson (getting them/nudging them back on track).
 - Context specific: digging to figure out what knowledge they are drawing upon
2. Allow students to reason through problems
 - Thinking through the process & coming up with an argument
 - Making connections
 - To the main idea of the worksheet
 - To prior learning
 - To the emerging pattern
 - How it fits in the overall “big picture”
 - Giving them time to reason (wait time)
 - Having students justify their response as part of their solution process

(3) (Would be listed if curriculum did not already do it already). Try to incorporate problems from real world.

Goals for student learning for specific lessons

Question asked: *What were your goals of instruction with regards to student learning for this lesson?*

Module 2, Worksheet 1

State quantities precisely - don't use pronouns (want the object, the attribute of that object, units of measure)

Move covariation further on, mostly focus on quantities

Module 2, Worksheet 2

Have students think about what quantities are proportional, and what quantities are not proportional. A general introduction- students should see that the quantities maintain a constant ratio – using any of the three methods they think of originally.

Module 2, Worksheet 3

Discuss three ways (Ratio, constant multiple, scaling) quantities are proportional

- quantity to quantity
- changes in quantities

Module 2, Worksheet 4a

Using a linear model based on proportionality of the changes in quantities.

Module 2, Worksheet 4b

Explore linear relationships through its various representations (tables, graphs, equations).

Being attentive to quantities

Module 2, Worksheet 5

How average speed relates to constant speed and what average speed means. Focus on the idea that the average speed being equivalent to the constant speed you need to travel to cover the same distance in the same time.

Module 2, Worksheet 6

“The random one”. Doesn’t flow well or fit. Perhaps better in Module 7 or in a Module on Conics. Would be good in a conic chapter. Purpose of worksheet is obscure.

Goals: Ambiguous/obscure. Develop the equation of a circle.

Interview Questions

Flat Vista High School: May 16, 2012

1. How were professional development experiences this year most useful to you? Least useful? Suggestions for improvement? How you do envision an ideal professional development experience?
2. What are ways in which helping another teacher can help grow as a professional? What activities do you find that best do this?
3. Did contributing ideas to the lesson on the hotel task help your practice as a teacher, or not really?
4. In what ways did seeing the interaction of the hotel task play out impact you? Did you see the lesson differently after the video? If yes, how?
5. Did contributing ideas to instructor notes on the diver task help your practice as a teacher, or not really?
6. In what ways did seeing the interaction of the diver task play out impact you? Did you see the lesson differently after the video? Is yes, how?
7. Does advising a teacher on a lesson on an applet of benefit to you on your practice, or not really?
8. What activities in a professional development setting you think are most benefit that you feel would of greatest benefit to your practice? Do you have other suggestions?

9. Your instructor goals are stated here in the worksheet which are overall goals for each lesson. Were these goals the same at the beginning of the school year? If no, what did your notice changed? Can you say more?
10. Were your instructor goals the same last year before using Pathways? If no, how do you think they are different? Can you say more?
11. Although we spent time refining goals during a few sessions, do you sense that further refinement is needed, such as labeling ones that have a higher priority over others? Why or why not?
12. How do you reconcile your stated goals in light of the procedural problems, standardizing testing, and student exam scores. For example I observed some dismay over scores of an assessment having to do with inverse trigonometric functions last week, before the start of our meeting.
13. Do you have suggestions for the Precalculus curriculum to improve? Being specific is super helpful!

APPENDIX C
BELIEFS SURVEYS

Data From Participants

Date	Name/Study ID	Your school:	Gender:	23. Student 24. In class, I focus more on my course relies on their ability to	25. When students make unsuccessful attempts at solving a problem in my course it helps students to	26. I pose questions primarily to	27. For my students, making a problem is the best accomplishment through	28. When preparing time to spend more thinking about	29. My teaching focuses more on	30. When a student gives a response that perplexes me, I find it more helpful to ask questions focused around	31. I have my students work on challenging problems during class.	32. It is important that my students learn to solve novel problems on their own.	33. I try to make learning mathematics as easy as possible for my students.	34. A primary responsibility of a mathematics teacher is to show students how to work on problems so that they can complete their homework and tests.	35. I provide opportunities for students to make predictions and conjectures during class.	
May 2012	Margaret	Flat Vista	Female	4	1	1	3	4	4	1	4	6	6	3	2	6
May 2012	Dorothy	Flat Vista	Female	4	2	1	3	4	4	1	4	5	6	6	5	6
May 2012	Robert	Salt Valley	Male	3	2	2	4	3	3	2	3	5	6	4	4	6
May 2012	Margaret	Flat Vista	Female	4	1	1	3	4	4	1	4	6	6	3	2	6
May 2012	Dorothy	Flat Vista	Female	4	2	1	3	4	4	1	4	5	6	6	5	6
May 2012	Robert	Salt Valley	Male	3	2	2	4	3	3	2	3	5	6	4	4	6
May 2012	Margaret	Flat Vista	Female	4	1	1	3	4	4	1	4	6	6	3	2	6
May 2012	Dorothy	Flat Vista	Female	4	2	1	3	4	4	1	4	5	6	6	5	6
May 2012	Robert	Salt Valley	Male	3	2	2	4	3	3	2	3	5	6	4	4	6

Beliefs Survey (May 2012: Prior Version)

Instructions for contrasting alternative items are as follows:

In the following questions, choose number

1 if you completely agree with option A

2 if you do not completely agree with option A, but agree with option A more than option B

3 if you do not completely agree with option B, but agree with option B more than option A

4 if you completely agree with option B

There are 8 items in the beliefs survey that are contrasting alternative items.

Instructions for Likert scale items are as follows:

For the following questions, choose the response that is most appropriate.

1 Strongly Disagree

2 Disagree

3 Slightly Disagree

4 Slightly Agree

5 Agree

6 Strongly Agree

There are 22 items in the beliefs survey that are Likert scale items.

Questions 23, 27, 28, and 29 are identified as questions designed to give additional insight into teaching orientation.

Contrasting Alternative Questions

23. Student success in my course relies on their ability to

(A) solve specific types of problems

(B) understand key ideas of the course

24. In class, I tend to focus more time on helping students to

(A) learn to reason through problems on their own

(B) master essential skills and procedures needed for future courses

25. When students make unsuccessful attempts when solving a problem in my course it is

(A) a natural part of the problem solving process

(B) an indication of their weakness in mathematics

26. I pose questions primarily to
- (A) help my students see how to get the answer to a problem
 - (B) support my students in making sense of the problem on their own
27. For my students, making sense of a problem is best accomplished through
- (A) Knowing the sequence of steps to solve the problem
 - (B) Knowing the ideas that are the focus of the problem
28. When preparing for class, I spend more time thinking about
- (A) presenting the material so that students are prepared to complete the problems in the homework and tests
 - (B) how to engage students in making sense of and using the ideas that are the focus of the lesson
29. My teaching focuses more on
- (A) helping students understand ideas of my courses
 - (B) helping students learn how to work specific problems
30. When a student gives a response that is perplexing to me, I find it more helpful to ask questions focused around
- (A) The sequence of steps taken leading to the response
 - (B) Their insights that lead to the response

Likert Scale Questions

1. I have my students work challenging problems during class.
2. It is important that my students learn to solve novel problems on their own.
3. I try to make learning mathematics easy for my students.
4. A primary responsibility of a mathematics teacher is to show students how to work problems so that they can complete their homework and tests.
5. I provide opportunities for students to make predictions and conjectures during class.
6. I provide opportunities for my students to explain their thinking and/or solution approaches during class.

7. I actively work to help my students construct their own meaning by having them engage in activities that require them to make sense of the problem.
8. I have students reflect on the reasonableness of their responses.
9. It is important that my students develop confidence in their mathematical abilities.
10. I do not think students can solve challenging word problems without additional assistance from me.
11. I believe that the course materials for this course are appropriate for preparing students to be successful in future mathematics courses.
12. The course materials that I use are effective in supporting my students to become stronger mathematical thinkers.
13. My students' homework is graded regularly.
14. I assign homework most every class session.
15. I feel confident in my knowledge to teach mathematics.
16. I believe that I have room for new learning in my practice.
17. I regularly spend time reflecting on and adapting my instruction.
18. Listening to and observing students provides insight into how I think about mathematics.
19. When my students ask questions I try hard to understand their thinking before answering.
20. I can affect students' motivation by what I do in class.
21. I do not have time to allow my students to express their thinking during class.
22. I am comfortable when students ask challenging questions for which I don't have an answer.

Beliefs Survey (June 2013: Current Version)

Pathways - Instructor Survey
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Arizona State University

Pathways - Instructor Survey

Thank you for taking the time to fill out the survey. Please answer all questions. There is room for optional comments at the end of each section.

Your study ID:

Your school ID:

Courses you teach:

Gender:

Female Male

Please indicate the number of years you have been teaching:

0 - 1 2 - 5 6 - 10 11 - 15 16 +

Section I

For the following questions, choose the response that is most appropriate.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

1. I have my students work challenging problems during class.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

2. It is important that my students learn to solve novel problems on their own.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

3. I try to make learning mathematics easy for my students.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

4. The primary responsibility of a mathematics teacher is to show students how to work problems.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

5. I provide opportunities for students to make predictions and conjectures during class.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

Section I - continued

For the following questions, choose the response that is most appropriate.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

6. I provide opportunities for my students to explain their thinking and/or solution approaches during class.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

7. I actively work to help my students construct their own meaning by having them engage in activities that require them to make sense of problems.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

8. I have students reflect on the reasonableness of their responses.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

9. I take action in my teaching to instill confidence in my students' mathematical abilities.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

10. My students are unable to solve challenging word problems without assistance from me.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

11. I believe that my course materials are appropriate for preparing students to be successful in future mathematics courses.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

12. The course materials that I use are not effective in supporting my students to become stronger mathematical thinkers.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

13. My students' homework is graded regularly.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

14. I assign homework most every class session.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

15. I feel confident in my knowledge to teach mathematics.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

Section I - continued

For the following questions, choose the response that is most appropriate.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

16. Over the past year I have made time to continue learning in my teaching practice.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

17. I regularly spend time reflecting on and adapting my instruction.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

18. Listening to and observing students while they are working helps me be a better teacher.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

19. When my students ask questions I try hard to understand their thinking before answering.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

20. I can affect students' motivation by what I do in class.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

21. I do not have time to allow my students to express their thinking during class.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

22. I am comfortable when students ask challenging questions for which I don't have an answer.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

Any additional comments you would like to make regarding the questions in this section?

Section II

In the following questions, choose number

1 if you completely agree with option A

2 if you do not completely agree with option A, but agree with option A more than option B

3 if you do not completely agree with option B, but agree with option B more than option A

4 if you completely agree with option B

23. Student success in my course relies on their ability to

1 2 3 4

- (A) solve specific types of problems
- (B) understand key ideas of the course

24. In class, I tend to focus more time on helping students to

1 2 3 4

- (A) learn to reason through problems on their own
- (B) master essential skills and procedures needed for future courses

25. When students make unsuccessful attempts when solving a problem in my course it is

1 2 3 4

- (A) a natural part of the problem solving process
- (B) an indication of their weakness in mathematics

26. I pose questions primarily to

1 2 3 4

- (A) help my students see how to get the answer to a problem
- (B) support my students in making sense of the problem on their own

27. For my students, making sense of a problem is best accomplished by

1 2 3 4

- (A) knowing the sequence of steps to solve the problem
- (B) knowing the ideas that are the focus of the problem

28. When preparing for class, I spend more time thinking about

1 2 3 4

- (A) presenting the material so that students are prepared to complete the problems in the homework and tests
- (B) how to engage students in making sense of and using the ideas that are the focus of the lesson

Section II-continued

In the following questions, choose number

1 if you completely agree with option A

2 if you do not completely agree with option A, but agree with option A more than option B

3 if you do not completely agree with option B, but agree with option B more than option A

4 if you completely agree with option B

29. My teaching focuses more on

1 2 3 4

(A) helping students understand ideas of my courses

(B) helping students learn how to work specific problems

30. When a student gives a response that is perplexing to me, I find it more helpful to ask questions focused on

1 2 3 4

(A) The sequence of steps they took to get an answer

(B) The thinking they used to understand and respond to the problem

Any additional comments you would like to make regarding the questions in this section?

APPENDIX D

WORKSHOP – A THOUGHT REVEALING ACTIVITY

The purpose of this assignment is to help you reflect on the mathematical ideas and purpose of the first investigation and to help you think about how to use one of the Pathways animations in your class. In reflecting on the purpose of this investigation we hope that you will be better prepared to use this lesson with your students. **We also want you to gain an appreciation of how essential it is that you understand the mathematics that is the focus of each investigation and animation before using it in your class.** Gaining this knowledge is what is necessary for you to respond directly to the needs and thinking of your students when teaching.

1. What are the primary ideas being developed in this investigation?
2. Review all questions in Investigation 4. Provide at least 3 other questions you think will be useful to pose to your students as they complete this investigation.
3. Look back at Question 1-6 in Investigation 4, do all parts, and reflect on the purpose of these questions.



4. What are the key ideas of this task that you found important?

We have found that it takes some preparation to become comfortable and confident in using Pathways animations in your class. The remaining questions in this assignment are intended to help you gain this competence and confidence.

5. Go to <https://www.rationalreasoning.net/flash/candle.html> to access the candle animation.



Investigate the animation and explore its functionality.

- What ideas and reasoning abilities do you expect your students will acquire from your using this animation with your students?
 - What are your goals for student learning in this lesson? (Be specific in including the reasoning abilities and understandings you want students to acquire from this lesson.)
 - What is involved in developing these reasoning abilities and understanding these key ideas? How might students come to develop them?
 - What misconceptions might students have about these ideas?
6. Before using an animation with your students you need to think about the specific questions that you plan to ask students. What questions will you ask to:
- get your students to envision the relevant quantities and make conjectures about the situation
 - probe student understanding

- c. support students in engaging in meaning making

 - d. address possible misconceptions

 - e. help them reflect on the reasonableness of their responses
7. How will you have students share their solution approach with the class? What criteria will you use to select students, and what tools will students use (document camera, whiteboard, etc.) to share their solutions and thinking?
8. What homework questions will you select for Investigation 4? (You will be provided a link to view homework questions for Investigation 4.)
9. Summarize how you envision this lesson unfolding in your class.
-
-
10. Look back at the key ideas of the lesson you have created. Do the questions you plan to ask support the development of the student reasoning abilities and understandings you want promoted? Are there any gaps or pitfalls that you foresee? Continue refining your lesson as needed.

11. Additional space for notes and other ideas you have for implementing this lesson in your class.

12. **Teacher Reflection** (do not fill this in until you've taught the lesson): Write down what you noticed about the lesson or what a student said in class while completing the lesson that you found noteworthy, interesting, or surprising. *Do this the same day as you taught the lesson, and use this information when you refine your lesson for next semester.*

Create a camera-ready copy of your notes and thoughts that can be shared with your colleagues in the workshop tomorrow. Your notes and responses will be collected but returned. If you use additional paper, please include it with your submission.

APPENDIX E

TRANSCRIPT OF ROBERT'S GOALS

Questions for 2/5/2013:

(Sull 4.2 & 4.3)

1. What are your general goals of instruction with regards to student learning for any lesson you have?

1. One goal I have for instruction is that all students “get something” out of coming to class today. Ideally, that would be a stronger conceptual understanding of the concept being taught. For any class, the ultimate goal is that students leave with enough conceptual understanding to be successful in the next class (or course!) or in applying what they have learned to any applicable situation they might be faced with.

2. Are these goals affected by the type of lesson you have—for example, a conceptual versus skill based lesson? If yes, how are they affected? If not, how are they not affected?

2. The over-arching goal of improving student understanding remains for any lesson, regardless of the emphasis of skills vs. concepts. However, the trajectory and/or delivery method of the lesson can be affected. I visualize a concept-based lesson as having student investigation as a major portion of the activities, while a skill-based lesson is still focused on “why” certain procedures are done but there is more direct instruction of those procedures.

3. What were your goals of instruction with regards to student learning for the lesson you had today? How did the unfolding of the lesson differ?

3. My goal for today was to review skills associated with simplifying and performing operations on rational expressions. As it turned out, it took a long time to go over the homework problems and there wasn’t much time left to review the skills. We’ll finish that up tomorrow.

~~~~~  
**Questions for 2/7/2013:**

(Solving Rational Equations)

**1. What were your goals of instruction with regards to student learning for the lesson you had today?**

1. Learning goals for today
  - Students should understand what it means to find the solution to an equation
  - Students should understand the relationship between the solution to a rational equation and the values of  $x$  that make the expressions in that equation undefined.
  - Students should be able to solve a rational equation with variable denominators
  - Students should be able to check for extraneous solutions

**2. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

2. Actual lesson vs. planned lesson
  - The lesson unfolded as planned for the most part. I was able to get through the material and examples that I had planned, and I think the students had adequate practice before attempting the homework problems on their own.
  - One aspect that changed at the last minute was the idea of getting students to think about values of  $x$  for which expressions are not defined when going over the homework. I thought it might plant a seed for thinking about extraneous solutions when we got to solving equations.

**3. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

3. Noteworthy items from class
  - It’s only been 1.5 hours since class was over, but I’m having a hard time remembering specific things that students said.
  - I was happy that some students were able to see that changing the denominator of  $2-x$  to  $x-2$  would be helpful.
  - I was surprised that some students preferred the method of combining fractions on one side of the equation as a first step instead of multiplying both sides of the equation by the LCD.
  - I was happy that finding 5 different volunteers to put homework problems on the board was fairly easy.
  - I was happy that several students had a nice way to explain what it means to find the solution to an equation.

~1~

**Follow-up regarding Question 3:**

*You had said "I was surprised that some students preferred the method of combining fractions on one side of the equation as a first step instead of multiplying both sides of the equation by the LCD." What would your hypothesis be about how those students were thinking about the problem to find that combining the fractions first on either side as an initial step to be an easier (or preferred) way to solve the equation?*

I think students preferred that method because it was more similar to adding/subtracting the rational expressions that we worked on the previous day and for homework.

~~~~~  
Questions for 2/13/2013:

(Mod 7, W1)

1. What were your goals of instruction with regards to student learning for the lesson you had today?

1. What were your goals for today?

One goal was for students to gain an understanding of what it means to measure an angle. Another goal was for students to gain an understanding of what it means for an angle measure to be 1 degree. I wanted to stress the importance of thinking about an angle as an object that cuts off a certain fraction of a circle's circumference whose center is the vertex of the angle.

2. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

2. How did the lesson differ from what was planned?

I had planned to get through the activity where students investigate the meaning of a radian. We didn't get that far. Otherwise, the lesson went as planned in terms of getting through what we needed to get through. The trajectory was slightly different than what was outlined in the PowerPoint, but that's ok.

3. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

3. Was there anything surprising or noteworthy?

It shouldn't be surprising, but students are very insistent on referring to angles in terms of "the degrees in an angle." So I try to avoid that as much as possible until students have a conceptual understanding of what it means for an angle measure to be 1 degree. It was noteworthy that there were a few people who came up with the idea of using a ratio pretty early. I had to ask one girl to save her thought for later because I didn't want her to hijack the other students' opportunity to develop that understanding on their own.

~~~~~  
**Questions for 2/14/2013:**

(Mod 7, W1)

**1. What were your goals of instruction with regards to student learning for the lesson you had today?**

1. Goals for today

- Students should understand the meaning of angle measure (fraction of a circle's circumference cut off by the angle...)
- Students should understand what it means for two angles to have the same measure
- Students should understand that there is more than one type of unit that can be used to measure angles
- Students should understand the meaning of an angle measured in radians
- Students should be able to convert between different types of angle measure

**2. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

2. What unfolded differently than planned?

- This time everything went pretty much as planned... aside from going slower than intended... as usual...

**3. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

3. Noteworthy/surprising things about the lesson

- I was surprised that many students still could not communicate an appropriate conception of angle measure.
- It was interesting to hear several students mention Pi or 2Pi before they measured the circumference of their circles

*Feedback question (your insight is greatly appreciated):*

*When I've asked teachers orally in the past "What are your goals as a mathematics teacher?" I have heard the following (this list is a compilation from several teachers).*

- *Focus on good questioning strategies*
- *Letting students think and make connections*
- *Help students understand the mathematics*  
*Breaking students from traditional ways. Not all algorithms.*
- *Become better at getting at how students are thinking.*  
*Giving students time to think. Giving them more challenging questions.*  
*Try to incorporate problems from the real world.*  
*Trying to have students reason through the problems.*  
*Make learning math easy for students.*  
*Having students think about processes and coming up with an argument.*  
*Have students become better mathematical thinkers.*
- *Allow students to reason through problems.*

*In the written form, what do you think would be a better way to ask this question that conveys the meaning that could be inferred from this list of responses? (Or should the question be broken into parts?)*

*What would your response (or response list) to this question be (per your suggested modification)?*

I think when you asked me the same question (or a similar one?) on the first day, you re-phrased it using the word philosophy. Maybe it would be helpful to ask teachers what their philosophy of teaching math is. Also, when you ask what their goals are, it might be good to focus more on whether they are teaching goals or learning goals for students or both.

My philosophy would be that I believe all students can learn math and become mathematically literate. Having this belief requires that I continually evaluate my teaching practice to ensure that students are met where they are in terms of conceptual understanding and then challenged re-evaluate their own understandings by experiencing new ideas.

My teaching goals and learning goals for students would probably look like the list you provided.

~~~~~

Questions for 2/15/2013:

(Mod 7, W2)

1. What were your goals of instruction with regards to student learning for the lesson you had today?

1. Goals for today

- Students should understand what it means to measure an angle in radians
- Students should be able to convert between different types of angle measurement
- Students should understand what it means for two angles to have the same measure
- Students should understand the usefulness of measuring an angle in radians when the size of the circle is known

2. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

2. What unfolded differently than planned?

- The lesson was more slow than planned, as usual.
- I did not expect the extra questions I added in to take so long.

~3~

3. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

3. Noteworthy/surprising things about the lesson

- I was surprised that at the beginning of worksheet 2 there were still several students who couldn't get started. Ultimately, it was because they still had no conceptualized the meaning of angle measure in radians.
- I was very satisfied with Roxana's explanation for converting and measure in radians to angle measure in degrees
- I was very satisfied with Taylor's explanation of the $s = r * \theta$ formula for finding arc length.

~~~~~

**Questions for 2/20/2013:**

(Mod 7, W3)

**→ 1. What were your goals of instruction with regards to student learning for the lesson you had today?**

1. Goals for instruction

- Students should be able to identify several of the quantities that are changing
- Students should be able to describe the co-variation of the angle measure and one other quantity
- Students should be able to describe how the vertical distance changes as the angle swept out increases

**2. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

2. How did the lesson unfold

- Slow again
- One student slowed me down because she was still confused about angle measure in radians.

**3. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

3. Noteworthy/Interesting/Surprising things

- One student surprised me by saying that he was thinking of an Etch-a-Sketch when thinking about horizontal and vertical distances. We talked briefly about how the distance from point to point on an etch-a-sketch is different from the type of distance being identified in the bug-on-a-fan problem.
- I thought it was noteworthy that several students did not get confused when talking about "decreasing at a decreasing rate." This is something we spent a lot of time talking about in module 5.

~~~~~

Questions for 2/21/2013:

(Mod 7, W3)

1. What were your goals of instruction with regards to student learning for the lesson you had today?

1. Goals

- Students should recognize that the relationship between angle swept out (in circular motion) and vertical/horizontal distance measure in radian lengths is a pretty special relationship that produces the same outputs for any input for circles of any radius.
- Students be able to describe the meaning of particular input/output pairs for $f(x)=\sin(x)$ and $g(x)=\cos(x)$
- Students should be able to convert between radius lengths and length in inches/ft/etc when given the radius

2. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

2. How did the lesson unfold differently than planned

- For the most part, the lesson went as planned

3. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

3. Noteworthy/interesting/surprising

- I was surprised that one student still had a difficult time understanding what it means to measure an angle in radians when sketching the graph of $\sin(x)$. I decided to have her sketch the graph using degree measures instead which seemed to help her make a helpful connection.

~~~~~

Questions for 2/22/2013:

(Mod 7, W4)

1. *What were your goals of instruction with regards to student learning for the lesson you had today?*
2. *How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?*
3. *Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?*
4. *With regards to question 3, if your response was with respect to what a student said, what do you hypothesize the student was thinking about the task (that would be their rationale behind what they said)?*

Note: The questions to 2/22/2013 were not answered by Robert.

~~~~~

Questions for 2/27/2013:

(Mod 7, W6)

1. *What were your teaching goals for the lesson you had today?*
 1. Teaching Goals
 - To review the meaning of sine and cosine
 - To review how sine and cosine can be used to determine coordinates of a point on circle when given the angle or arc length
 - To introduce exact values of trig functions (for when theta equals $\pi/6$, etc...)
 2. *What were your goals of instruction with regards to student learning for the lesson you had today?*
 2. Student Learning Goals
 - To interpret the value of an inverse trig function
 - To use inverse trig functions to determine the angle swept out when the coordinates of a point are known
 3. *How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?*
 3. How did the lesson unfold?
 - I didn't get to the lesson. That was different than what I had planned.
 - We spent the entire class reviewing the homework assignment.
 4. *Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?*
 4. Noteworthy/interesting/surprising things
 - When one student presented a homework problem, she referred to the expression " $\cos(4.75)$ " as "cosine times 4.75." Yow! This was surprising considering the amount of time I have spent emphasizing that $\sin()$ and $\cos()$ are function names.
 - I was surprised that two of the more responsible students in class did not check online for the solutions and hints that were posted there for students.
 - Two students had been drawing triangles and using "opposite over hypotenuse..." to try and figure out the homework problems. Naturally, they were unable to justify what they were doing or make a connection between their method and the concepts we had discussed in class.

~~~~~

Questions for 3/4/2013:

(Sull, 6.1 & 6.2)

1. *What were your teaching goals for the lesson you had today?*
  1. Teaching Goals
    - To introduce the secant and cosecant functions
    - To reinforce ways to compute values of sine and cosine for common angles
  2. *What were your goals of instruction with regards to student learning for the lesson you had today?*
    2. Goals for student learning

~5~

- To make connections between finding vertical/horizontal distance of a point on a unit circle to finding vert/horiz distance on a non-unit circle.

- To use the sine and cosine functions to calculate values of secant and cosecant

**3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

3. Did the lesson unfold differently than planned

- The lesson unfolded as planned

**4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

4. Noteworthy/interesting/surprising things

- I had assigned problems from the incorrect section of the traditional textbook, so students were confused by that. Understandably so.

- Otherwise I thought the lesson went pretty smooth

~~~~~

Questions for 3/5/2013:

(Mod 7, W5)

1. What were your teaching goals for the lesson you had today?

1. Teaching goals

- To review important ideas related to sketching the graph of the sine function

- To introduce the idea of transforming the sine function to reflect a different set of quantities (e.g. time vs angle or distance in feet vs distance in radius lengths)

2. What were your goals of instruction with regards to student learning for the lesson you had today?

2. Goals for student learning

- To be able to write a function that calculates vertical distance in feet when the input is the angle swept out

- To be able to write a function that calculates vertical distance in feet given the amount of time that has elapsed

- To understand the relationship between the amount of time it takes the ball to complete revolution and the speed of the ball in radians per second

3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

3. How did the lesson unfold

- As planned

- I was pleased that almost everyone in class was able to make the correct transformation for the 3rd example problem (writing a function that finds vertical distance when the angle is swept out at a rate of 0.5 radians per second).

4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

4. Noteworthy/interesting/surprising things

- Nothing

~~~~~

**Questions for 3/7/2013:**

(Mod 7, W6)

**1. What were your teaching goals for the lesson you had today?**

1. Teaching Goals

- To review the meaning of the period of a periodic function

- To introduce transforming the sine function using a vertical shift

**2. What were your goals of instruction with regards to student learning for the lesson you had today?**

2. Goals for student learning

- Students should make a connection between the period of a function and the angular speed of the moving object

- Students should be able to transform the sine function when asked to find distance above something other than the horizontal diameter

~6~

**3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

3. How did the lesson unfold

- I did not expect to have to spend so much time discussing period.
- I had planned to do the majority of the worksheet handout during class, but was not able to even start it.

**4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

4. Noteworthy/interesting/surprising things

- Some students had assumed that the coefficient of the argument of the sine function is the same value as the period. Clearly they had not wrestled with the concept and hadn't reviewed previous work from class

**Note: Spring break from 3/11/2013-3/15/2013 - no classroom visits during this period of time**

~~~~~

Questions for 3/18/2013:

(Trans. of Trig Functions)

1. What were your teaching goals for the lesson you had today?

1. Teaching goals

- To review basic transformations of sine and cosine graphs (amplitude, period, vertical shift, reflection)
- To practice writing equations given graphs
- To practice graphing functions given the function rule

2. What were your goals of instruction with regards to student learning for the lesson you had today?

2. Goals for student learning

- None of the topics were "new" today, although it was shift in student thinking to deal with graphs of trig functions with no contextual reference.

3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

3. Did the lesson unfold differently

- I didn't get to reflections
- Otherwise It went pretty much as planned

4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

4. Surprising/interesting/noteworthy items

- Many students had forgotten the basics over spring break.
- Despite discussing the idea in depth, one student was writing the period as the coefficient of x in his function rules.
- Some students had difficulty deciding whether to use a rule based on sine or cosine.

5. Which resource will you be using for the chapter on identities chapter?

5. Identities resource

- Precalculus: Enhanced with Graphing Utilities, 5e. Sullivan, Sullivan.

6. How long have you been teaching?

How long have you been teaching high school?

How long have you been teaching at SVHS*?

6. Teaching history

- 13 years total
- 13 years high school
- 13 years at SVHS*

*** Note: Changed to reflect pseudonym**

~~~~~  
**Questions for 3/21/2013:**

(Graphing of Trig Functions)

**1. What were your teaching goals for the lesson you had today?**

1. Teaching goals
  - Mostly review of graphing trig functions

**2. What were your goals of instruction with regards to student learning for the lesson you had today?**

2. Goals for student learning

Same as yesterday:

- Students will be able to write trig equations for given graph in using both sine and cosine
- Students will be able to sketch the graph of a given trig function
- Students will be able to apply trig functions to contextual situations involving circular motion

**3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

3. How did the lesson unfold

- Just about as planned. I decided to provide answers for the worksheet problems as a time-saver since I wanted to make sure I had enough time for doing the wheels on the bus problem.

**4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

4. Noteworthy/interesting/surprising things

- It was noteworthy that most students seemed to have mastered graphing trig functions with no horizontal shift. And most of those did ok when there was a period change and horizontal shift.
- I thought it was great that a student came up with the wheels on the bus as a situation that could be modeled using sine and cosine.

~~~~~  
Questions for 3/26/2013:

(Mod 7, W8)

1. What were your teaching goals for the lesson you had today?

2. What were your goals of instruction with regards to student learning for the lesson you had today?

3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

4. Surprising/interesting/noteworthy items

- I was surprised that most students were unable to differentiate between instructions like “Evaluate $\arccos(0.4)$ ” and “Solve $\cos(\theta)=0.4$.” Even when I pointed out that there was indeed a difference between the two problems, students were unable to decide why.

→ I was surprised that the student who tried to present the first question (evaluating and explaining what $\arccos(0.4)$ represents) was unable to do so. It’s possible that she was able to get the answer, but she was clearly unable to convey any depth of meaning about the problem – she was just throwing together some of the jargon from class. How does the group decide that she’s the one who will do the talking? Did they not practice beforehand? And if so, did the other group members find her explanation acceptable or did they just try to tell her what to say? Ugh.

- I was surprised at the number of students who did not complete the homework.

- It is noteworthy that one student (Ricky) was able to illustrate the meaning of the various problems without any trouble – he did a great job.

Note: I asked Robert to answer question #4 only that day. I didn't think it made sense to answer questions 1-3 due to external events (fire drill) altering the trajectory of the intended lesson.

Questions for 3/27/2013:

(Wrap-Up Inverse Sine/Cosine & Mod 7, W9)

1. What were your teaching goals for the lesson you had today?
 1. Teaching goals
 - Review/solidify concepts related to evaluating inverse trig functions
 - Introduce the tangent function
 2. What were your goals of instruction with regards to student learning for the lesson you had today?
 2. Goals of instruction
 - Students should understand the difference between evaluating an inverse trig expression and solving a trig equation
 - Students should be able to use inverse trig functions to help them solve equations
 - Students should understand the meaning of the tangent function
3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?
 3. How did the lesson unfold
 - I had made a worksheet to review inverse trig ideas. It took a lot longer than I thought it would
 - The table of values on the tangent worksheet took forever for students to fill out
4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?
 4. Noteworthy/interesting/surprising things
 - Some students were still unable to differentiate between solving equations and evaluating inverse trig equations
 - Some students forgot how to calculate slope
 - Some students were calculating the slope between consecutive points on the circle instead of the slope between the origin and a point

~~~~~

**Questions for 3/28/2013:**

(Mod 7, W9)

1. ***What were your teaching goals for the lesson you had today?***
  1. Teaching goals
    - To introduce the tangent function
  2. ***What were your goals of instruction with regards to student learning for the lesson you had today?***
    2. Goals of instruction
      - Students should understand how x and y values can be used to calculate slope
      - Students should recognize that the values of slope between the origin and a point on the circle are periodic with respect to the angle
      - Students should be able to interpret the meaning of expressions involving the tangent function
3. ***How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?***
  3. How did the lesson unfold?
    - As expected! For once!
4. ***Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?***
  4. Noteworthy/interesting/surprising things
    - The transition from explaining the meaning of a tangent calculation seemed pretty seamless
    - It was interesting that the calculator computed  $\tan(21\pi/2)$  to be  $-1 \times 10^{13}$ , which led to good discussion
    - It was surprising that I actually got through the whole worksheet as planned.

~~~~~

Questions for 4/1/2013:

(Mod 7 Review)

1. What were your teaching goals for the lesson you had today?

1. Teaching goals
 - To review the tangent function, inverse sine and cosine, solving basic equations, and applications of writing trig functions

2. What were your goals of instruction with regards to student learning for the lesson you had today?

2. Goals for student learning
 - Students should make connections between the various functions we have been working with and between trig functions and their inverses
 - Students should determine the limitations on the value of arctan expressions

3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

3. How did the lesson unfold
 - As planned. It was mostly review and I took charge in order to make sure that we didn't get stuck on anything that would take up too much time.

4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

4. Noteworthy/interesting/surprising things
 - I was surprised at the number of students who did not complete homework over the weekend, which was another reason I wanted to take charge of reviewing the material.

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**Questions for 4/2/2013:**

(Mod 7 Review & Intro to Inverse/Reciprocal Trig Functions)

**1. What were your teaching goals for the lesson you had today?**

1. Teaching goals
  - To review ideas surrounding inverse trig functions
  - To introduce inverse reciprocal trig functions

**2. What were your goals of instruction with regards to student learning for the lesson you had today?**

2. Goals for student learning
  - Students should understand what the input and output values of an inverse trig function represent
  - Students should be able to evaluate inverse trig functions
  - Students should be able to evaluate compositions of trig functions and their inverses

**3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

3. How did the lesson unfold
  - As planned – it was mostly review

**4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

4. Noteworthy/interesting/surprising things
  - Nothing that I can remember at the moment

~~~~~

Questions for 4/11/2013:

(Identities, Sull. 7.3)

1. What were your teaching goals for the lesson you had today?

1. Teaching goals
 - To review techniques for proving identities (go over HW)
 - To introduce the even/odd properties

2. What were your goals of instruction with regards to student learning for the lesson you had today?

2. Goals for students
- Students should understand that an identity is an equation that is true for any value of the input
 - Students should use symmetry of trigonometric graphs in order to make sense out of the even/odd properties

3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

3. How did the lesson unfold
- As planned, aside from having a really short time to address even/odd properties

4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

4. Noteworthy/interesting/surprising things
- I was surprised that students seemed to do ok with the homework – I expected more complaints and more problems with understanding what to do.
 - It was interesting that one student “factored” an expression to look like $(\sin(\theta) + 1) + (\sin(\theta) + 1)$

5. One or two questions will follow, based on your responses to 1-4.

Follow-up questions:

1. In today's teaching goals, you mention "To introduce the even/odd properties". Can you clarify and say more as to what an introduction to even/odd properties means? (If you decided between alternate approaches to introduce even/odd properties, how did you choose a particular approach over another?)

There was one homework problem that required students to understand these properties but we didn't have time to talk about them in class, so I knew it would come up. I also knew that we wouldn't have much time today to talk about them, so I wanted to do a really brief investigation related to why those properties are what they are with the hope that students could investigate or think similarly about the properties we didn't get to. Using the graph of cosine was really the only approach I considered at the end of class due to time. I could have used a unit-circle approach and had a discussion about the horizontal and vertical distances, but I didn't.

2. In today's goals for students, you mention "Students should understand that an identity is an equation that is true for any value of the input". In what ways might such an understanding of identities be promoted or developed?

One way would be to have students check the truth of the statement for specific values of the input. A good way to do that would be to use the graphing calculator. Either the graph or the table of values should do the trick.

Questions for 4/12/2013:

(Sull. 7.3)

1. What were your teaching goals for the lesson you had today?

1. Teaching goals
- To review methods for verifying identities
 - To reinforce the idea of what an identity equation actually is
 - To introduce sum/difference identities

2. What were your goals of instruction with regards to student learning for the lesson you had today?

2. Goals for student learning
- Students should become competent in using simple identities to verify more complicated ones
 - Students should understand that an identity equation is an equation that is true for all values of the input variable

3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

3. How did the lesson unfold
- Mostly as planned, although I didn't have time to introduce sum/difference identities due.

4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

4. Noteworthy/interesting/surprising things

- I was surprised that one student was unable to transfer yesterday's practice to last night's homework – she thought the problems were asking her to do something completely different (i.e. solving for the variable instead of verifying the equation's truth for all values).
- One student who usually does not contribute offered suggestions more than once about how to attack problems

5. One or two questions will follow, based on your responses to 1-4.

Follow-up questions:

1. In teaching goals, you mentioned "To reinforce the idea of what an identity equation actually is". To clarify, what activities or explorations in class today do you feel helped reinforce student's understandings of this idea?

We had a class discussion about what students would expect to see in a table of inputs and outputs and we graphed both sides of the identity equation to verify that they were identical.

→ 2. How might one promote students' inclinations (or work to counter students' disinclinations) to reflect on meaning of the mathematics they are learning, such as in understanding that an identity equation is true for all values of the input variable?

One way would be to provide more time for reflection, but it's difficult to balance this with practicing skills. Friday was day 3 of the same lesson and I had planned to move on to the next set of identities. I should have built more reflection into the previous day, but clearly I was not thinking about it at the time.

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**Questions for 4/22/2013:**

(Sull. 7.7)

**1. What were your teaching goals for the lesson you had today?**

1. Teaching Goals

- To introduce solving conditional trig equations

**2. What were your goals of instruction with regards to student learning for the lesson you had today?**

2. Goals for student learning

- Students should understand the difference between identity and conditional equations
- Students should be able to solve a trig equation when restricted to a particular interval
- Students should be able to find "all" solutions to a trig equation

**3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

3. How did the lesson unfold

- As planned!

**4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

4. Noteworthy/interesting/surprising things

- Some students still had difficulty explaining what an identity equation is, although one or two students had a fantastic explanation
- The lesson overall seemed to go well.

**5. One or two questions will follow, based on your responses to 1-4.**

→ I'm having a harder and harder time differentiating between questions 1 and 2. When I'm preparing, I'm mostly focused on what I want students to learn. Then I adjust my teaching based on where they are and what they need to learn. So I feel like I don't have much in the way of teaching goals. My goal as a teacher is for students to meet their learning goals.

**Follow-up questions:**

**1. In today's goals for students, you mention "Students should understand the difference between identity and conditional equations". In what ways might such an understanding develop, be promoted, or supported for students?**

1. I suppose I could support that by using the graphing calculator like I did with the identity equations.

**2. In today's goals for students, you mention " Students should be able to find "all" solutions to a trig equation". How might students come to reflect on the meaning of the solution and develop an intuition to determine whether the expression they found representing the solution set is reasonable (or not)?**

2. I think at this point students have an understanding that we are finding the values of the variable that make the equation true. They would probably find it tedious to verify those values one by one by substituting them into the equation, but we could do it. As for developing an intuition for determining if their solution set is reasonable or not, I'm drawing a blank.

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Questions for 4/23/2013:

(Sull. 7.7)

1. What were your teaching goals for the lesson you had today?

1. Teaching goals

→ · To get students to think about what would be different when solving an equation when the argument is not just theta

· To get students to think about how they would verify the solution to an equation using a graph

2. What were your goals of instruction with regards to student learning for the lesson you had today?

2. Goals for student learning

· Students should understand how the solution(s) can be determined when the argument is not just theta

· Students should be able to determine the solution to an equation using a graph

3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

3. How did the lesson unfold

· Pretty much as planned, although I didn't get through as many examples as I had hoped

4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

4. Noteworthy/interesting/surprising things

· I was pleased with the homework problems on the board – even Jeff's solution using a calculator instead of exact values prompted some good discussion.

5. One or two questions will follow, based on your responses to 1-4.

Follow-up questions:

1. In today's goals for students, you mentioned "Students should understand how the solution(s) can be determined when the argument is not just theta". How might such an understanding develop, be promoted, or supported for students?

1. I tried to support the idea by having students consider how they might solve the equation by graphing and thinking about how the graph of something like $y=\sin(x)$ would be different from the graph of $y=\sin(2x)$ and what consequence that would have for the number of solutions that are possible. I also tried supporting the idea by having the students consider the algebraic procedure itself.

2. In today's teaching goals, you mention "To get students to think about how they would verify the solution to an equation using a graph". What might thinking about (and the process of) verifying the solution of the equation in graphical form do for a student?

2. Hopefully this will get students to make connections between the solutions to an equation and the inputs/outputs of a corresponding function (or functions).

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Questions for 4/24/2013:

(Sull. 7.8)

**1. What were your teaching goals for the lesson you had today?**

1. Teaching goals
  - To introduce methods for solving trig equations that contain multiple trigonometric expressions

**2. What were your goals of instruction with regards to student learning for the lesson you had today?**

2. Goals for student learning
  - Students should be able to use identities to assist in finding algebraic solutions to trig equations.
  - Students should be able to use a graph to assist in finding solutions to trig equations.

**3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

3. How did the lesson unfold
  - Mostly as planned, although I was a little crunched for time as usual
  - We reviewed homework problems, used the graphing calculator to make connections about solving equations, and solved several sample problems that required algebraic methods or using identities

**4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

4. Noteworthy/interesting/surprising things
  - I was frustrated that students didn't ask many questions
  - It was noteworthy that many students had difficulty remembering either of the identities that we needed in the sample problems (Pythagorean and double-angle for cosine)
  - Grades on the identities quiz were horrible. I will offer points for quiz corrections and include some additional identities questions on the equations quiz next week.

**5. One or two questions will follow, based on your responses to 1-4.**

**Follow-up questions:**

**1. Regarding noteworthy items and the identities quiz, what is your hypothesis on how might students be thinking about the questions on the quiz that lead to items being missed? (If the issues are question specific, please send me a copy of the quiz for reference.)**

1. Generally speaking, many of the students have poor algebra skills. For example, they often incorrectly square a binomial or are unable to find a least common denominator before adding or subtracting fractions. To top it off, many of them simply have not memorized the identities they needed for the quiz.

**2. Suppose if you were to try to investigate why a student (or students) missed a question on any assessment. What three questions would you want ask the student (or each student separately) that would be useful information to you as a teacher, which would reveal how they originally interpreted the problem, or how they were thinking about the problem, or what process they thought would help them solve the problem, but at the same time does not reveal your thinking on ways to address the question?**

2. Usually I begin by asking the student that very question: "What were you thinking about when you began the problem?" On the problems where students have to use identities to evaluate an expression, I usually ask students if their answer makes sense or not, hoping they will think about things like whether the answer should be positive or negative or if the value should be less than or equal to 1. If the student's problem was a particular step in an algebraic procedure, I would sometimes identify the step that was incorrect and ask the student what he/she was thinking about for that particular step.

Quiz attached.

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Questions for 4/30/2013:

(Sull. 8.1)

1. What were your teaching goals for the lesson you had today?

1. Teaching goals
 - To review/introduce methods for finding the values of missing quantities in a triangle.

2. What were your goals of instruction with regards to student learning for the lesson you had today?

2. Goals for student learning
 - Students should understand how the trig functions can be used to find the values of missing quantities in a right triangle.
 - Students should understand what was meant by the terms angle of elevation, angle of depression, and bearing.

3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

3. How did the lesson unfold
 - Mostly as expected. I made it a point to reinforce the meanings of trig functions in their relationship to vertical and horizontal distance instead of talking about opposite and hypotenuse.
 - I was able to introduce/define what is meant by the word bearing, but we didn't have enough time to do an example.

4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

4. Noteworthy/interesting/surprising things
 - Not much today...

5. One or two questions will follow, based on your responses to 1-4.

Follow-up questions:

1. In reference to Teaching Goals, which methods did you have in mind for finding the value of missing quantities in a triangle? (If you were considering between alternatives, what influenced your choice of methods during class?)

1. I was referring to the Pythagorean Theorem and the use of sine/cosine/tangent. The choice of method was based on the information provided in the problem.

2. In Goals for students learning, can you clarify what you mean by "Students should understand how the trig functions can be used to find the values of missing quantities in a right triangle?"

2. What I mean is that students should be able to set up an equation like $\sin(20^\circ) = x/10$ and use it to determine the length of one of the sides of a triangle.

~~~~~

**Questions for 5/3/2013:**

(Sull. 8.2)

**1. What were your teaching goals for the lesson you had today?**

1. Teaching Goals
  - Introduce the Law of Sines

**2. What were your goals of instruction with regards to student learning for the lesson you had today?**

2. Goals for student learning
  - Students should be able to apply the Law of Sines to find the values of unknown quantities in an oblique triangle
  - Students should understand where the Law of Sines comes from - that it's not just some random made up formula.

**3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

3. How did the lesson unfold
  - As expected. Felt a little boring.

~15~

**4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

4. Noteworthy/interesting/surprising things

· Nothing that I can remember specifically. One student had an idea about which sides and angles to use if the Law of Sines needed to be used a second time that I liked. I was going at it from an accuracy perspective, but she mentioned that it would also prevent students from working off of a possibly incorrect answer.

**5. One or two questions will follow, based on your responses to 1-4.**

**Follow-up questions:**

**1. In teaching goals, you mention "Introduce the Law of Sines", there are many alternatives available. Can you be more clear in what you meant by introduce law of sines (and if you were considering between alternatives, how you decide between them)?**

1. What I meant is that we would develop the formula. The method I used on Friday is what I have always done, although with the honors students we usually come up with the formula without using any values for the sides and angles.

**2. In how did lesson unfold, you mentioned "Felt a little boring". For whom do you feel it was boring (you, students, or both) and why do you hypothesize the lesson was boring?**

2. It was definitely boring for me and possibly the students as well. It just seemed like there was a lack of engagement.

~~~~~

Questions for 5/6/2013:

(Sull. 8.2)

1. What were your teaching goals for the lesson you had today?

1. Teaching goals

· To introduce the ambiguous case for the Law of Sines

2. What were your goals of instruction with regards to student learning for the lesson you had today?

2. Goals for student learning

· Students should understand when it is possible for one, two, or no triangle to be drawn based on the given information

· Students should be able to solve both triangles with two sets of solutions are possible

3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

3. How did the lesson unfold

· As expected – I thought it went pretty well.

4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

4. Noteworthy/interesting/surprising things

· I was satisfied that a couple of students were able to see that two solutions were possible for the first example before I showed them how

· I was happy that students volunteered to do all of the requested homework problems and that they did a great job.

5. One or two questions will follow, based on your responses to 1-4.

Follow-up questions:

1. Regarding goals for student learning, you stated "Students should be able to solve both triangles with two sets of solutions are possible". What understandings or connections must a student have (or perceive) in order for this goal to be attainable for them?

1. Students have to understand that sometimes when three pieces of information are known about a triangle, more than one triangle can be drawn with those characteristics. Furthermore, they must understand what part of the triangle can be drawn differently without changing the given characteristics.

2. In noteworthy items you mention that a couple of students were able to see that two solutions were possible for the first example prior to you showing them how. If time was not an issue, how might you have altered the trajectory to the introduction of the ambiguous case so that a majority of students in class would made the realization (on their own, or within their group) of when two solutions is/not appropriate, prior to you doing the first example. (If you had something specific or a very different task in mind, please describe in detail.)

2. I kind of eluded to it during class, but I mentioned having everyone in the class draw their own triangle with given measurements. Then they would compare their triangle with other triangles in the class and notice that either they are all identical or that some of the other triangles are not identical. I imagine having them use rulers and protractors.

~~~~~

**Questions for 5/7/2013:**

(Sull. 8.3)

**1. What were your teaching goals for the lesson you had today?**

1. Teaching goals  
· Introduce the Law of Cosines

**2. What were your goals of instruction with regards to student learning for the lesson you had today?**

2. Goals for student learning  
· Students should understand that there are some triangles for which the Law of Sines is not useful.  
· Students should understand how to use the Law of Cosines formula  
· Students should understand how the Law of Cosines has an advantage over the Law of Sines when finding an angle.

**3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?**

3. How did the lesson unfold  
· As expected, although some students have difficulty following the development of the formula.

**4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?**

4. Noteworthy/interesting/surprising things  
· Nothing that I can recall.

**5. One or two questions will follow, based on your responses to 1-4.**

**Follow-up questions:**

**1. With regards to goals for student learning, you mention "Students should understand how the Law of Cosines has an advantage over the Law of Sines when finding an angle". What advantage (or advantages) are you referring to?**

Law of cosines has an advantage in that the cosine function is able to output an obtuse angle while the sine function can only output acute angles.

**2. With regards to the lesson unfolding you mentioned "As expected, although some students have difficulty following the development of the formula." Were the difficulties you observed part of the expectation in this lesson? If so, how had you planned for these difficulties? If not, what was it about these difficulties that surprised you?**  
I think I expected some difficulty, but I didn't plan for it. I was surprised that many students had difficulty remembering how sine and cosine are related to distances and how that could be leveraged to assist in developing the formula. I should have planned for it and possibly done some sort of warm up or had the students discuss briefly in groups.

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Questions for 5/9/2013:

(Sull. 8.4 & Review)

Remark: if you find your statement of goals possibly open to alternate interpretations, follow up by a specific statement or statements to clarify what you mean.

1. What were your teaching goals for the lesson you had today?

1. Teaching goals
 - To go over the homework problems and review some main ideas from the chapter

2. What were your goals of instruction with regards to student learning for the lesson you had today?

2. Goals for student learning
 - It was basically a review day, so I would say my goal for student was for them review and synthesize what we have learned. Maybe this was best exemplified by the problem where students had to find the area of a quadrilateral.

3. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?

3. How did the lesson unfold
 - As expected

4. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?

4. Noteworthy/interesting/surprising things
 - Not much

5. There will be no specific follow-up questions today, but I will sending you today's and your prior responses this weekend (coded under a pseudonym) in anticipation of Monday's interview.

APPENDIX F
INTERVIEW PROTOCOL USED WITH ROBERT

Thank you for coming. I really appreciate your time and willingness to participate in this study.

Do you have any questions or concerns before we begin?

Describing interview process

During this interview I will be asking questions related to your planning, classroom practice, goals for teaching, and goals for student learning. Your responses will remain anonymous.

There are no right or wrong answers to the questions given. After some of your responses, I might ask for clarification or follow up with additional questions. This should not be interpreted to mean you said something wrong. I simply trying to better understand what you are saying by getting additional information.

Please note that it is absolutely okay to stop at any time during the interview for any reason.

Questions:

- Did you have an opportunity to review the summary of your responses to goals I sent you? What jumped out to you, or what did you notice?
- Describe your experiences teaching Pathways curriculum the first, second, and third times. What challenges did you face each time you taught with Pathways materials? How did you feel about teaching Pathways the first, second and third times? How did you determine how much assistance to give or not give students?
- How did your planning for class change the first, second, and third times you taught using Pathways materials? Are possible student misconceptions part of the planning phase, or are they addressed in the moment of teaching?
- What factors do you think impact whether a lesson unfolds as planned/or not?
- Mod 7 W3 p. 4: In 2/20/2013 Q1, Can you describe what do you mean in your goals for instruction? How might students come to think of these concepts?
- Sull 7.7 p. 12: In 4/12/2013, Follow-up Q2: You had mentioned it being difficult to balance reflection with practicing skills. Can you say more? Is outlining time for student reflection normally part of your planning process, or is time for reflection acted on in the moment of teaching?
- Sull 7.7 p. 12: In 4/22/2013, I noticed your comment having a harder time differentiating between questions 1 and 2. Can you say a little more?

- Sull 7.7 p. 13: In 4/23/2013, your teaching goals looked different than other teaching goals you had stated in the past. Can you say a little more? Was outlining how students may come to think about these concepts part of your planning process that day?
- You had mentioned in the process of scheduling this interview that you did not feel trigonometry was awesome for students, or for you on some of the days. Can you say more?
- Regarding your comment of 3/26/2013, p. 8: Why do students not reflect or push for justification?
- Do you think the goal of focusing on student reasoning and meaning making are contradictory or complementary to getting through content?
- How do you see your role as a teacher of mathematics?
- If you had to rank the teacher's goals on p. 3, which are 5 goals you agree with most?
- Are there any comments or suggestions you'd like to make that you feel would help the Pathways project, or a question you'd like to answer that you wish I would have asked?

APPENDIX G
CODED GOALS FOR STUDENT LEARNING

Robert

Observed Goals - Robert

Lesson Label	Lesson Date	TGSL 0	TGSL1	TGSL2	TGSL3	TGSL4	TGSL5	Researcher's comments
Mod 7, W1	13-Feb					x	x	1st goal: (desired way of thinking described via clarification statement), 2nd goal: level 4
Mod 7, W1	14-Feb		x			xxx	x	1st: level 5 (desired way of thinking described), next: 3: level 4, last goal: 1
Mod 7, W2	15-Feb		x			xxx		1st: level 4, 2nd: level 1, last 2: level 4
Mod 7, W3	20-Feb			xxx				All three refers to specific actions desired of students in this lesson: level 2
Mod 7, W3	21-Feb		x	x			x	1st: level 5 (desired way of thinking described), 2nd: level 2, 3rd: level 1
Mod 7, W4	22-Feb							No response given this day.
Mod 7, W6	27-Feb				x	x		1st: level 4, 2nd: level 3
Sull, 6.1 & 6.2	4-Mar				x	x		1st: level 4, 2nd: level 3
Mod 7, W5	5-Mar			xx				First 2: level 2, last: level 4
Mod 7, W6	7-Mar		x			x		1st: level 4, 2nd: level 1
Trans. of Trig Functions	18-Mar		x					Using "deal with graphs of trig functions with no contextual reference" as the goal.
Graphing of Trig Functions	21-Mar		x	xx				First 2: level 2, last: level 1
Mod 7, W8	26-Mar							Unannounced fire drill used up most of class time. Therefore I did not ask goals that day.
Wrap-Up Inverse Sine/Cosine & Mod 7, W9	27-Mar				x	xx		1st: level 4, 2nd: level 3, 3rd: level 4
Mod 7, W9	28-Mar					xx	x	1st: level 4, 2nd: level 5 (desired way of thinking described), 3rd: level 4
Mod 7 Review	1-Apr		x			x		1st: level 4, 2nd: level 1
Mod 7 Review & Intro to Inverse/Reciprocal Trig Functions	2-Apr			xx		x		
Total		0	7	10	3	17	4	41
Percent		0.0%	17.1%	24.4%	7.3%	41.5%	9.8%	100.0%

Lesson Label	Lesson Date	TGSL 0	TGSL1	TGSL2	TGSL3	TGSL4	TGSL5	Researcher's comments
Identities, Sull. 7.3	11-Apr				x		x	1st: level 5 (desired way of thinking described), 2nd: level 3
Sull. 7.3	12-Apr				x		x	1st: level 3, 2nd: level 5 (desired way of thinking described)
Sull. 7.7	22-Apr		xx					1st: level 4, last 2: level 1
Sull. 7.7	23-Apr				x	x		1st: level 4, 2nd: level 3
Sull. 7.8	24-Apr			xx				Both goals indicate desired methods for solving trigonometric equations.
Sull. 8.1	30-Apr				xx			Both goals imply desired ways of thinking, but the what in what sense is unarticulated.
Sull. 8.2	3-May				x	x		1st: level 3, 2nd: level 4
Sull. 8.2	6-May			x			x	1st: level 5 (desired way of thinking described), 2nd: level 4
Sull. 8.3	7-May				x	xx		1st: level 4, 2nd: level 3, 3rd: level 4
Sull. 8.4 & Review	9-May				x			To synthesize means to use known methods to find solutions to novel problems.
Total		0	2	1	8	7	3	21
Percent		0.0%	9.5%	4.8%	38.1%	33.3%	14.3%	100.0%

Dorothy and Margaret

Observed Goals - Dorothy and Margaret

Lesson Label	Lesson	TGSL 0	TGSL 1	TGSL 2	TGSL 3	TGSL 4	TGSL 5	Researcher's comments
Quantities and Co-variation of Quantities	Mod 2, W1		x	x				Two distinct goals. First goal rated level 2 because stating is not a method of math-it's something you do. Second goal rated 1 - too vague.
Introduction to Proportionality	Mod 2, W2						x	This is about getting students to think in certain ways, focusing on idea the constant ratio to see whether quantities are proportional or not.
Proportionality of Quantities and Changes in Quantities	Mod 2, W3				x			This is about methods students should use (level 3)
Constant Rate of Change & Linearity	Mod 2, W4a				x			This is about methods students should use (level 3)
Applying ideas of Constant Rate of Change & Linearity	Mod 2, W4b		x		x			Two distinct goals. This is about methods students should use to explore linear relationships (level 3). The other goal is vague.
Exploring Average Speed	Mod 2, W5						x	This is about getting students to think about average rate of change in a certain way, in terms of its relationship to constant speed.
The Distance Formula and the Equation of a Circle	Mod 2, W6	x	x					Since Dorothy and Margaret had different goals, each is reported.
Total		1	3	1	3	0	2	10
Percent		10.0%	30.0%	10.0%	30.0%	0.0%	20.0%	100.0%

APPENDIX H

IRB APPROVALS AND CONSENT FORMS



EXEMPTION GRANTED

Marilyn Carlson
Mathematics and Statistical Sciences, School of
480/965-6168
MARILYN.CARLSON@asu.edu

Dear Marilyn Carlson:

On 8/26/2014 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	A Study of Collaborative Refinement to Planning and Goals on Classroom Interactions
Investigator:	Marilyn Carlson
IRB ID:	STUDY00001437
Funding:	Name: NSF-EHR-DUE: Division of Undergraduate Education; Grant Office ID: 001081900, Funding Source ID: 943360412,
Grant Title:	001081900;
Grant ID:	001081900;
Documents Reviewed:	<ul style="list-style-type: none">• Planning and Goals Study - Consent Form, Category: Consent Form;• Planning and Goals Study - Protocol, Category: IRB Protocol;• Teacher Beliefs Survey - Pathways Instructor Survey, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions);• Pre Study Interview Protocol, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions);• Teacher Preflection and Reflection Protocol, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions);• Post Classroom Observation Protocol, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions);

	<ul style="list-style-type: none">• Post Study Interview Protocol, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions);• Planning and Goals Study - Teacher Recruitment Form, Category: Recruitment Materials;• Pathways Phase II grant (copy), Category: Sponsor Attachment;
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The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (1) Educational settings, (2) Tests, surveys, interviews, or observation on 8/26/2014.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

Sincerely,

IRB Administrator

cc: Frank Marfai
Frank Marfai

To: Marilyn Carlson
Engineerin

From:  Mark Roosa, Chair
Soc Beh IRB

Date: 08/23/2013

Committee Action: **Renewal**

Renewal Date: 08/23/2013

Review Type: Expedited F7

IRB Protocol #: 1108006730

Study Title: Pathways to calculus:Disseminating and Scaling a Professional Development Model for Precalculus Teaching

Expiration Date: 08/30/2014

The above-referenced protocol was given renewed approval following Expedited Review by the Institutional Review Board.

It is the Principal Investigator's responsibility to obtain review and continued approval of ongoing research before the expiration noted above. Please allow sufficient time for reapproval. Research activity of any sort may not continue beyond the expiration date without committee approval. Failure to receive approval for continuation before the expiration date will result in the automatic suspension of the approval of this protocol on the expiration date. Information collected following suspension is unapproved research and cannot be reported or published as research data. If you do not wish continued approval, please notify the Committee of the study termination.

This approval by the Soc Beh IRB does not replace or supersede any departmental or oversight committee review that may be required by institutional policy.

Adverse Reactions: If any untoward incidents or severe reactions should develop as a result of this study, you are required to notify the Soc Beh IRB immediately. If necessary a member of the IRB will be assigned to look into the matter. If the problem is serious, approval may be withdrawn pending IRB review.

Amendments: If you wish to change any aspect of this study, such as the procedures, the consent forms, or the investigators, please communicate your requested changes to the Soc Beh IRB. The new procedure is not to be initiated until the IRB approval has been given.

Please retain a copy of this letter with your approved protocol.

Permission to Take Part in a Human Research Study

Page 1 of 2

A Study of Collaborative Refinement to Planning and Goals on Classroom Interactions

I am a graduate student under the direction of Professor Marilyn Carlson in the School of Mathematical and Statistical Sciences at Arizona State University. I am conducting a research study to learn how refinements to the lesson planning process influences goals and classroom interactions.

I am inviting your participation, which will involve ten collaborative lesson planning sessions using Pathways materials, observations of these selected lessons, and a post lesson meeting to debrief. Each component will last approximately an hour. We will also have a pre-study interview and a post-study interview, each which will take approximately an hour. You have the right not to answer any question, and to stop participation at any time. The duration of this study will be approximately six weeks.

Your participation in this study is voluntary. If you choose not to participate or to withdraw from the study at any time, there will be no penalty. At the completion of this study, you will receive a remuneration of \$500. No remuneration will be given if you withdraw from the study prior to its completion.

Although there is no direct benefit to you, possible benefits of your participation are different ways to think about lesson planning that you may find useful in your teaching practice after this study has ended. There are no foreseeable risks or discomforts to your participation.

To protect confidentiality, all participants in the study will be assigned a study id and pseudonym. All data collected from each participant will be identified by that study id. The results of this study may be used in reports, presentations, or publications but your name will not be used. Your responses will be confidential.

I would like to audio record or video record our sessions. This includes the pre and post study interview, the lesson planning meetings, your classroom lessons associated with this study, and the post lesson meetings. Neither the interviews, the meetings, or the classroom lessons will be recorded without your permission. Please let me know if you do not want the interviews, the meetings, or the classroom lessons to be recorded; you also can change your mind after the study starts, just let me know.

Excerpts of video recordings that include you, or audio recordings that include your voice, may be used at scientific meetings and in relevant university courses, but your name will not be used. These recordings will be digitized and kept on a secure server, only accessible to personnel involved in this study. These recordings will be kept for up to 3 years. Afterwards, the recordings will be securely erased.

If you have any questions concerning the research study, please contact the research team at:

Principal Investigator	Co-Investigator
Marilyn P. Carlson	Frank Marfai
Marilyn.Carlson@asu.edu	Frank.Marfai@asu.edu

If you have any questions about your rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the ASU Office of Research Integrity and Assurance, at (480) 965-6788.

Permission to Take Part in a Human Research Study

Page 2 of 2

By signing below you are agreeing to be part of the study.

Name:

Signature:

Date:

By signing below you are agreeing to be videotaped and have excerpts of the video recordings used at scientific meetings and in relevant university courses.

Name:

Signature:

Date:

Document Revision Date: September 11, 2014

A Study of Collaborative Refinement to Planning and Goals on Classroom Interactions

RECRUITMENT SCRIPT

I am a graduate student under the direction of Professor Marilyn Carlson in the School of Mathematical and Statistical Sciences at Arizona State University. I am conducting a research study to learn how refinements to the lesson planning process influences goals and classroom interactions.

I am recruiting individuals to work with collaboratively in planning lessons that would be class tested. You will be completing a survey and participating in an interview, which will take approximately 1 to 1½ hours of your time. We will then meet to collaborate and plan ten lessons using the Pathways course materials, which you would then teach. Afterwards we would meet to talk about the lesson and your experiences teaching it. The lesson planning meetings and post lesson meetings are each expected to take about 1 hour of your time. At the end of this study, you would be participating in an interview, which would take approximately 1 to 1½ hours of your time.

Your participation in this study is voluntary. If you have any questions concerning the research study, I can be reached at frank.marfai@asu.edu.

APPENDIX I
STUDY ARTIFACTS

Beliefs Survey

Pathways - Instructor Survey
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Arizona State University

Pathways - Instructor Survey

Thank you for taking the time to fill out the survey. Please answer all questions. There is room for optional comments at the end of each section.

Your Study ID:

4328

Your School ID:

Courses you teach:

Calculus AB, Honors Pre Calculus, Integrated Math

Gender:

Female

Male

Please indicate the number of years you have been teaching:

0 - 1

2 - 5

6 - 10

11 - 15

16 +

Section I

For the following questions, choose the response that is most appropriate.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

1. I have my students work challenging problems during class.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

2. It is important that my students learn to solve novel problems on their own.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

3. I try to make learning mathematics easy for my students.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

4. The primary responsibility of a mathematics teacher is to show students how to work problems.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

5. I provide opportunities for students to make predictions and conjectures during class.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

Section I - continued

For the following questions, choose the response that is most appropriate.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

6. I provide opportunities for my students to explain their thinking and/or solution approaches during class.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

7. I actively work to help my students construct their own meaning by having them engage in activities that require them to make sense of problems.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

8. I have students reflect on the reasonableness of their responses.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

9. I take action in my teaching to instill confidence in my students' mathematical abilities.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

10. My students are unable to solve challenging word problems without assistance from me.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

11. I believe that my course materials are appropriate for preparing students to be successful in future mathematics courses.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

12. The course materials that I use are not effective in supporting my students to become stronger mathematical thinkers.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

13. My students' homework is graded regularly.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

14. I assign homework most every class session.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

15. I feel confident in my knowledge to teach mathematics.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

Section I - continued

For the following questions, choose the response that is most appropriate.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

16. Over the past year I have made time to continue learning in my teaching practice.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

17. I regularly spend time reflecting on and adapting my instruction.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

18. Listening to and observing students while they are working helps me be a better teacher.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

19. When my students ask questions I try hard to understand their thinking before answering.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

20. I can affect students' motivation by what I do in class.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

21. I do not have time to allow my students to express their thinking during class.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

22. I am comfortable when students ask challenging questions for which I don't have an answer.

Strongly Disagree 1 2 3 4 5 6 Strongly Agree

Any additional comments you would like to make regarding the questions in this section?

#3 is difficult for me to answer. I like to challenge my students, but I also then try to break problems down in chunks so that concepts are easier to grasp

Section II

In the following questions, choose number

1 if you completely agree with option A

2 if you do not completely agree with option A, but agree with option A more than option B

3 if you do not completely agree with option B, but agree with option B more than option A

4 if you completely agree with option B

23. Student success in my course relies on their ability to

1 2 3 4

- (A) solve specific types of problems
(B) understand key ideas of the course

24. In class, I tend to focus more time on helping students to

1 2 3 4

- (A) learn to reason through problems on their own
(B) master essential skills and procedures needed for future courses

25. When students make unsuccessful attempts when solving a problem in my course it is

1 2 3 4

- (A) a natural part of the problem solving process
(B) an indication of their weakness in mathematics

26. I pose questions primarily to

1 2 3 4

- (A) help my students see how to get the answer to a problem
(B) support my students in making sense of the problem on their own

27. For my students, making sense of a problem is best accomplished by

1 2 3 4

- (A) knowing the sequence of steps to solve the problem
(B) knowing the ideas that are the focus of the problem

28. When preparing for class, I spend more time thinking about

1 2 both 3 4

- (A) presenting the material so that students are prepared to complete the problems in the homework and tests
(B) how to engage students in making sense of and using the ideas that are the focus of the lesson

Section II-continued

In the following questions, choose number

- 1 if you completely agree with option A
- 2 if you do not completely agree with option A, but agree with option A more than option B
- 3 if you do not completely agree with option B, but agree with option B more than option A
- 4 if you completely agree with option B

29. My teaching focuses more on

1 2 both 3 4

- (A) helping students understand ideas of my courses
- (B) helping students learn how to work specific problems

30. When a student gives a response that is perplexing to me, I find it more helpful to ask questions focused on

1 2 3 4

- (A) The sequence of steps they took to get an answer
- (B) The thinking they used to understand and respond to the problem

Any additional comments you would like to make regarding the questions in this section?

Pre-study Task

Study ID: 4328

Pre Study Interview Lesson Planning Questionnaire

A prototype of a mathematical investigation being worked on by the Pathways course materials team is included here. Read through the investigation carefully.

Work through the activity through the lens of a student first and finish it completely.

We will talk about this lesson through the lens of a teacher when we meet tomorrow. It will include talking about the key ideas of the lesson, questions you might ask, and your goals for this lesson (if you were to teach it).

Formatting of this prototype hasn't been finalized; feel free to include additional sheets of the paper if you feel there is not enough space.

Tentative Placement: Pathways Precalculus (Carlson, Oehrtman, Moore), Module 2.

Neil and Cameron live on opposite sides of town connected by a long road. They are studying for a test and agree to meet at a library located somewhere between them on the same road.

1. When Neil and Cameron start heading toward each other, Cameron is 5 miles away from the library, while Neil is 6 miles away from the library.

- a. Draw a diagram that represents the situation described in the space below.



- b. Describe the quantities in your diagram. Can you think of others that might be important to know?

how fast each walks (varying)
how long it takes to travel to library (varying)
how far each is from Lib. when they start (constant)

- c. Write one question you would be interested in answering with regard to this situation. What are some quantities you would need to know in order to answer your question?

Who will get there first?
Speed they walk.
When they began to walk.

Cam - he started closer & went faster

2. Suppose that we knew Cameron traveled at a constant speed of 25 miles per hour and Neil traveled at a constant speed of 20 miles per hour.

a. Whom do you conjecture will arrive at the library first, or do you think will they arrive at the library at the same time? Explain why you believe this is true.

No Calculations?

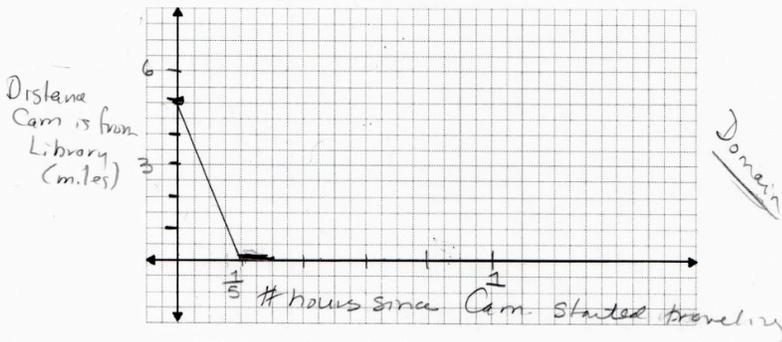
Cam
 $\frac{5m}{25mph} = \frac{1}{5} hr$

N
 $\frac{6m}{20mph} = \frac{3}{10} hr$

Did they leave at the same time?

Cam will get there first if they start at the same time

b. Plot the relationship between distance Cameron is from library and time elapsed, for the duration of time that both Neil and Cameron are travelling.



c. Determine who arrives at the library first (or if they tie) using mathematics. Show your work. Explain why your process is correct.

Cameron - see (a)
 he arrived after $\frac{1}{5}$ hour (12 min)
 Neil arrived after $\frac{3}{10}$ hour (18 min)

$\frac{20 \text{ mile}}{1 \text{ hr}} = \frac{6 \text{ m}}{x \text{ hrs}}$

d. Were you surprised with your findings? Why or why not?

No

e. Do your findings make sense? Why or why not?

Neil started further away & went slower.

3. Let's change the situation. Let's suppose that Neil traveled at a constant speed of 25 miles per hour and Cameron traveled at a constant speed of 21 miles per hour.

a. Whom do you conjecture will arrive at the library first, or do you think will they arrive at the same time? Explain why you believe this is true.

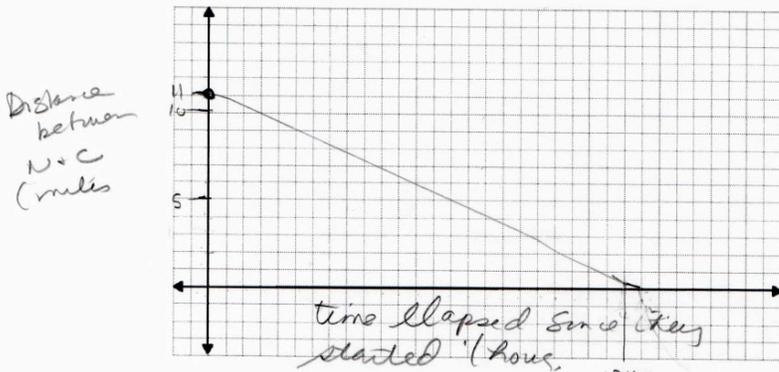
Neil 4 mph diff very close

$$D = RT$$

b. Determine who arrives at the library first (or if they tie) using mathematics. Show your work. Explain why your process is correct.

Neil $\frac{6m}{25 \text{ mph}}$.24 hrs Cam $\frac{5m}{21 \text{ mph}}$.238

c. Plot the relationship between distance Cameron and Neil are from each other and time elapsed, for the duration of time until Neil and Cameron meet each other at the library.



Please use function approximating 4 mph.

d. Were you surprised with your findings? Why or why not?

No

e. Are your findings reasonable? Why or why not?

Yes

f. Would your findings to this question change if Cameron travelled at a constant speed of 20 miles per hour? Why or why not? How do you know for sure?

Yes.
 $\frac{6}{20} = .3$ $\frac{5}{20} = .25$
now Neil first

Post-study Task

Key idea: think carefully about input & output

Study ID: 4328

Goal: Review

Post Study Interview Lesson Planning Questionnaire

A prototype of a mathematical investigation being worked on by the Pathways course materials team is included here. Read through the investigation carefully.

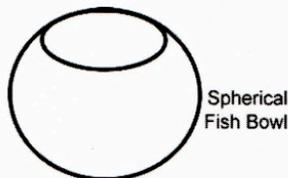
Work through the activity through the lens of a student first and finish it completely.

We will talk about this lesson through the lens of a teacher when we meet next time. It will include talking about the key ideas of the lesson, questions you might ask, and your goals for this lesson (if you were to teach it).

Formatting of this prototype hasn't been finalized; feel free to include additional sheets of the paper if you feel there is not enough space.

Tentative Placement: Pathways Precalculus (Carlson, Oehrtman, Moore), Module 5.

It takes 20 full pitchers of water to fill an empty spherical fish bowl in the diagram below to the top. Suppose that the fish bowl is 10 inches high and can hold 10 quarts of water.



Each pitcher holds $\frac{1}{2}$ qt.

1.

- a. Write one question you would be interested in answering with regard to this situation. What are the quantities you would need to know in order to answer your question?

How does height increase with each pitcher?

I would need to know height of water as each pitcher of water is added.

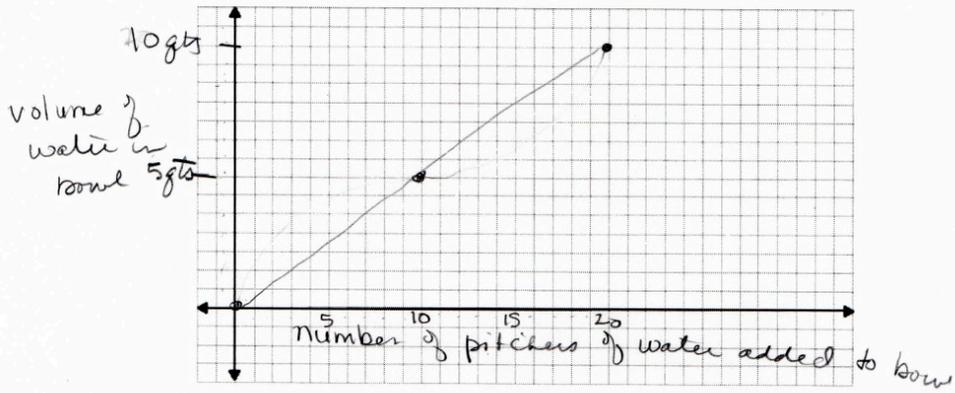
- b. Imagine pouring full pitchers of water into the fish bowl that originally is empty. As the number of pitchers of water poured into the fish bowl increases, how does the volume of water in the fish bowl change?

Explain your thought process.

The volume increases with each pitcher because water is going in and not going out.

For every pitcher poured in, volume increases by $\frac{1}{2}$ qt.

- c. Draw a graph that relates a possible relationship between the number of pitchers of water poured into the fish bowl and the volume of water (measured in quarts) in the fish bowl. Explain your thought process.



- (Additional questions....)

* Make a Table

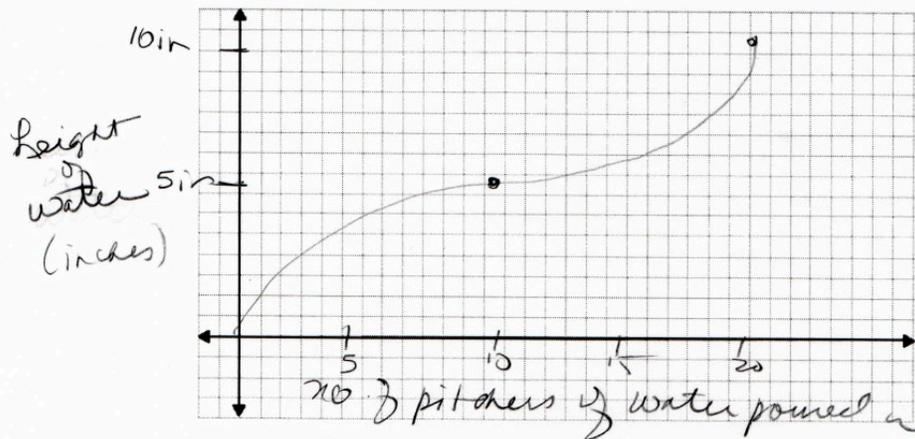
no. of pitchers	Volume
1	.5qt
2	1qt
3	1.5qt
...	...

2.

- a. Imagine repeating the process again and pouring full pitchers of water into the fish bowl that was originally empty. As the number of pitchers poured into the fish bowl increases, how does the height of water (in inches) in the fish bowl change? Explain your thought process.

height increases, at a rate that decreases for first 10 pitchers, then increases

- b. Draw a graph that relates a possible relationship between the number of full pitchers of water poured into the fish bowl and the height of water in the fish bowl (in inches). Explain your thought process.



- c. Using the graph you created in part (b), would the point (10, 20) be associated with your graph? Why or why not?

no.

the max height would be 10 inches

- d. Using the graph you created in part (b), would the point (20, 10) be associated with your graph? Why or why not?

yes - when you have poured in 20 pitchers the height would be 10 inches.

- e. Explain what the point (10, 5) would mean on your graph. Would having (10,5) on the graph make sense in the context of this situation? Why or why not?

yes
 When you have poured in 10 pitchers
 the height is 5 inches

o (Additional questions....)

3. You put the Spherical Fish Bowl away and are given a second container. It is a large empty bottle and is labeled "The Mystery Bottle".

- a. Write one question you would be interested in answering with regard to this situation. What are the quantities you would need to know in order to answer your question?

at what rate is the height of water changing?
I would need to know height at each number of pitchers.

- b. Starting with an empty bottle, you carefully start pouring full pitchers of water into the Mystery bottle. You record the relationship between the number of pitchers of water poured into the bottle and the change in height of water in the Mystery Bottle (in inches) in the table below.

Number of pitchers of water in bottle	Change in height of water in bottle, in inches
0	0
1	1.5
2	3
3	4.5
4	6

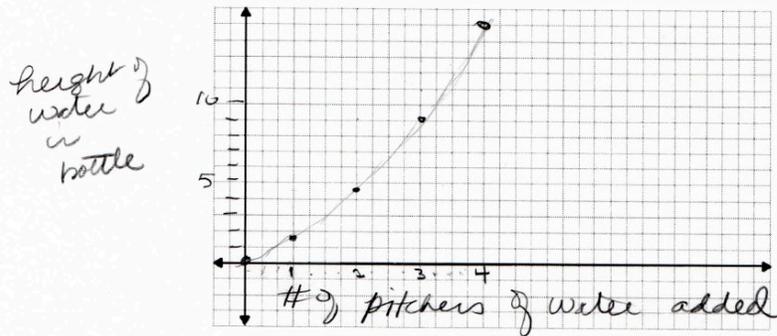
(Continued from previous page...)

What pattern do you notice in how the change in height of water in the bottle with respect to the volume of water in the bottle changes, when considering water poured into the bottle in one full pitcher increments? Explain your thought process.

The change in height is increasing by $1\frac{1}{2}$ inches for each pitcher.

The bottle must be getting narrower since for each pitcher of water the height increases a larger amount.

- c. Draw a graph that relates a possible relationship between the number of pitchers of water poured in the Mystery Bottle and the height of water in Mystery Bottle (in inches). Explain your thought process.



- d. Describe how the height of the water and volume of water in the bottle vary together for equal amounts of volume of water added to the bottle.

The height increases at an increasing rate.

- e. Describe how the volume of water and rate of change of height of water in the bottle vary together.

As volume increases height increases at an increasing rate

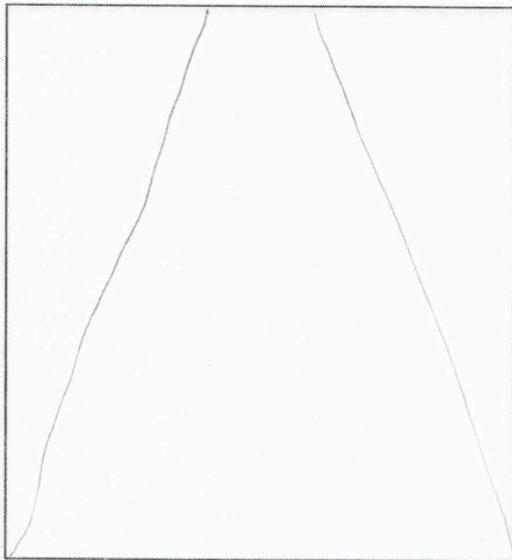
- f. Are there parts of the graph in part (c) that are concave up, concave down, a combination of both, or neither? Explain your reasoning.

because height is increasing at an increasing rate

- g. Is the graph you created in part (c) linear, quadratic, or some other type of graph? Explain your thought process.

because the rate at which the rates change is constant

- h. In the box below, draw a careful sketch of the possible bottle that could be the Mystery Bottle.



- i. How might the graph in part (c) be different if the Mystery Bottle had started partially filled?

*the y-intercept would be different,
but shape of graph would
be the same*

- (Additional questions...)

APPENDIX J
INTERVIEW PROTOCOLS

Pre-study Interview

Study ID: 4328

Thank you for coming. I really appreciate your time and willingness to participate in this study.

Do you have any questions or concerns before we begin?

Describing interview process

During this interview I will be asking questions related to your planning, classroom practice, goals for teaching, goals for student learning, and your thoughts about the key ideas of a mathematical task. Your responses will remain confidential.

There are no right or wrong answers to the questions given. After some of your responses, I might ask for clarification or follow up with additional questions. This should not be interpreted to mean you said something wrong. I simply trying to better understand what you are saying by getting additional information.

Please note that it is absolutely okay to stop at any time during the interview for any reason.

Questions:

General questions:

1. Some teachers post their daily goals visibly, while others do not. What are your thoughts on this? (Follow-up: To satisfy administrative requirements? List of topics?)
2. What criteria do you use to see your personal goals are met in a lesson? What are your personal goals? (Follow-up : When is it okay to move on? When is student work deemed appropriate?)
3. What are your general goals of instruction with regards to student learning for any lesson you have?
4. Are these general goals affected by the type of lesson you have-for example, a conceptual versus skill-based lesson? (Follow up depending on yes or no answer.)
5. What does the question “What were your goals for student learning in this lesson?” personally mean to you? (Follow-up: How do you interpret “what were your goals for student understanding”? Does the question seem neutral, or like a question implying a judgment?)
6. What does the question “What were your goals for teaching this lesson?” personally mean to you?

7. What does the question “What were your goals for interacting with students in this lesson?” personally mean to you?
8. Describe how you plan for a lesson with course materials you have not used before.
9. Describe how you plan for a lesson with course materials you have used before in the past.
10. An instructor survey, and a prototype of a mathematical investigation being worked on by the Pathways course materials team has been given to you. Read through the investigation carefully. Work through the activity through the lens of a student first. Then look at it from the lens of a teacher. When we meet tomorrow we will talk about it.

(A copy of the Beliefs Survey and the Pre-Study Mathematical Task will be shared with the participant that they can write on and annotate. Pause the interview and continue in the next session. Add clarification questions below, based on notes from interview thus far.)

Clarification questions:

11. What does “objective” mean to you? What does the word “goal” mean to you?
12. Is student understanding gauged differently in a conceptual versus a skill-based lesson for you?
13. What evidence do you use for yourself to say: okay, I feel comfortable about going on (with the lesson)?

Questions related to the Pre-study task:

14. Yesterday I shared a prototype of a mathematical investigation we worked on. Walk me through the lesson, from the lens of a student. *(Ask follow-up questions as needed to help characterize ways of thinking about the mathematical task from the participant.)*
15. Now we’re going to switch to the teacher’s perspective. What are the primary ideas being developed in this investigation? (Follow up: can you say more about what you mean by...?)
16. You have reviewed all the questions in this activity. Provide at least 3 other questions you think will be useful to pose to your students if they were asked complete this investigation. (Follow up to better model participant’s possible MKT with regards to activity.)

17. If you look back at this investigation and you already completed all the parts, what were the key ideas of this particular task (as you reflect on all the questions) that you think were important?
18. With regards to the tasks in this lesson, do you imagine other ways that students would do these?
19. Before using this investigation with your students, let's say you were thinking of specific questions you were planning to ask students. So what questions would you ask to:
 - a. get your students to make conjectures about the situation and envision the relevant quantities in the situation
 - b. probe students in conceptualizing how the quantities in the situation are related
 - c. hold students accountable for expressing their meanings
 - d. address possible misconceptions
 - e. help them reflect on the reasonableness of their responses
20. If you were to teach this lesson, what mathematical goals would you set for student learning? Are there other goals you would have for this lesson, either mathematical or non-mathematical?

Post-study Interview

Study ID: 4328

Thank you for coming. I really appreciate your time and willingness to participate in this study.

Do you have any questions or concerns before we begin?

Describing interview process

During this interview I will be asking questions related to your planning, classroom practice, goals for teaching, goals for student learning, and your thoughts about the key ideas of a mathematical task. Your responses will remain confidential.

There are no right or wrong answers to the questions given. After some of your responses, I might ask for clarification or follow up with additional questions. This should not be interpreted to mean you said something wrong. I simply trying to better understand what you are saying by getting additional information.

Please note that it is absolutely okay to stop at any time during the interview for any reason.

Questions:

General questions:

1. The award on the wall says ‘outstanding achievement by your students on an advanced placement test?’ Is this first time for recognition? (Follow-up: What percentage were from your Honors Precalculus previous year? What percentage of Precalculus students move on to Calculus?)
2. What characterizes a successful student in math? What is your definition of a successful math student? What aspects of your teaching do you focus on that helps students be successful?
3. What is your teaching philosophy? How do you see your role as a teacher of mathematics? (Follow up: What is foundational in Calculus? How does Calculus relate to Precalculus? What is foundational in Precalculus? How are honors and non-honors sections of Precalculus different?)
4. How long did it take you to plan a lesson the first, second, third time you taught with the Pathways curriculum? How has your planning changed?

(Share current version of Lesson Preflection and Lesson Debrief Protocols, and the feedback given by participant regarding protocol).

About questionnaires:

5. This is regarding the Preflection questionnaire. So you found question number 6 to be most relevant. Can you tell me what did you find most relevant about the question? This is the one about “Before using this lesson with your students what questions would you ask...” You mention that also you found question 2 to be redundant after looking at it, so I’m curious.
6. Can you clarify what you meant by “yes” to question 1a (of the feedback questionnaire)? This is the one that asks “does the type of lesson (conceptual or skill based) affect the relevance of particular questions”.
7. You had mentioned in feedback that relevance of questions in the Preflection questionnaire are different in conceptual and skill-based lessons. Included with the questionnaire are two example lessons. Can you walk me through these? (*Use observed lessons #2 [conceptual] and #10 [skills-based]. Use participant’s responses to better understand and clarify what she means about relevance.*)
8. In question #9 of Preflection questionnaire (*what do you plan to do to hold all students in class accountable*), you mentioned finding the “all” part intimidating. Can you say more?
9. In question #7 of Debrief questionnaire, you mentioned that the key ideas of a lesson and goals for student learning of the lesson are the same (that the question is redundant). Can you tell me more?
10. Based on your earlier feedback, I have made some modifications to the [Preflection] questionnaire. A prototype of a mathematical investigation being worked on by the Pathways course materials team has been given to you. Read through the investigation carefully. Work through the activity through the lens of a student first. Then look at it from the lens of a teacher. When we meet tomorrow we will talk about it.

(A copy of the a revised version of the Lesson Preflection and Reflection protocol and Post-Study Mathematical Task will be shared with the participant that they can write on and annotate. Pause the interview and continue in the next session. Add clarification questions below, based on notes from interview thus far.)

Clarification questions:

11. You mentioned question 8 in the Debrief questionnaire helpful but difficult (*what pedagogical moves did she make*). Can you clarify?
12. You mentioned not understanding question 5 in the Debrief questionnaire (*how might the understandings suggested by goals be promoted/developed*). Can you clarify?

13. You had mentioned that your lesson planning has changed over time. Because you said that the one that you did on October 27th (*which was the polynomial function of higher degree, 11th lesson observed*) was probably the closest to how you normally plan lessons when it's not a Pathways lesson. But you had also mentioned that way you plan your regular lessons has changed over time as well. You said you used to put more questions in, and this became less, so I wanted to understand that a little a bit more about what you mean.

General questions:

14. What about during collaboration, what was different, in terms of planning lessons? Like when you're working with a partner? What was the same?
15. During our collaboration, you had mentioned not liking the partial growth factor lesson from Pathways. Can you tell me more? (*Share observed lesson #4 with participant, which is Module 4 Investigation 5.*)
16. After these lesson collaborations, are there parts of the Lesson Preflection Questionnaire that you plan to incorporate into your lesson planning during the semester? Let's say you're on your own. Would you keep any of these, or would you use it only in a collaborative setting?
17. What did you find helpful about (revised current form of) the Preflection questionnaire, and what did you find not so helpful about it?
18. What criteria do you use to see your personal goals are met in a lesson? What are your personal goals?
19. What are your general goals of instruction with regards to student learning for any lesson you have?
20. What does the question "What were your goals for interacting with students in this lesson?" personally mean to you?
21. What does the question "What were your goals for student learning in this lesson?" personally mean to you?
22. What does the question "What were your goals for teaching this lesson?" personally mean to you?

Questions related to the Post-study task:

23. Tell me a little bit about the lesson. You can walk me through it from a student's perspective. (*Ask follow-up questions as needed to help characterize ways of thinking about the mathematical task from the participant.*)

24. Regarding the lesson planning of it from a teacher's perspective. So let's say you were to teach this lesson. What are the primary ideas being developed in this investigation? (Follow up – can you say more about what you mean by...?)
25. If you were to teach this lesson, what mathematical goals would you set for student learning? Are there other goals you would have for this lesson?
26. If you were to pick two questions from the Preflection questionnaire with regards to this lesson, which questions would you find most useful to ask? (*Follow-up with specific questions she would ask.*)
27. What would be your goals for student interactions in this lesson?
28. Are there comments or thoughts you would like to share about your experiences in this study regarding lesson collaboration?

APPENDIX K

LESSON PLANNING AND DEBRIEF PROTOCOLS

Teacher Preflection and Reflection Protocol

LESSON PREFLECTION QUESTIONNAIRE (Version 1: 8/11/2014)

To use when thinking about an initially planned lesson

1. What are the primary ideas being developed in this lesson?
2. Review all questions in this lesson. Provide at least 3 other questions you think will be useful to pose to your students as they complete this investigation.
3. Look back at the questions in this lesson and complete all parts, and reflect on the purpose of each question. What are the key ideas of this task that you found important?
4. Before using this lesson with your students you need to think about the specific questions that you plan to ask students. What questions will you ask to:
 - a. get your students to make conjectures about the situation and envision the relevant quantities in the situation
 - b. probe students in conceptualizing how the quantities in the situation are related
 - c. hold students accountable for expressing their meanings
 - d. address possible misconceptions
 - e. help them reflect on the reasonableness of their responses
5. What ways of thinking about the key ideas in this lesson do you think will be expressed by students during class? How might these ways of thinking be helpful for students when learning/using related concepts? Are there possible ways of thinking that may emerge that hinders future learning?
6. How do you plan to help your students make conjectures and engage in reflection? How will this help your students develop their understandings and reasoning abilities associated with the key ideas of the lesson?
7. How will you have students share their solution approach with the class? What criteria will you use to select students, and what tools will students use (document camera, whiteboard, etc.) to share their solutions and thinking?
8. What do you plan to do to hold all students accountable for expressing his/her thinking and constructing the understandings and reasoning abilities associated with the key ideas of the lesson?
9. Summarize how you envision this lesson unfolding in your class.
10. What are your mathematical goals for student learning as you plan to teach this lesson? Are there other goals you have for this lesson? What criteria will you use to see your goals (for student learning, teaching, interacting with students) are achieved in a lesson?
11. Look back at the key ideas of the lesson you refined for your class. Do the questions you plan to ask support the development of student reasoning abilities and understandings you want promoted? Are there any gaps or pitfalls that you foresee? Continue refining your lesson as needed. You may add, remove, or modify questions on the worksheet as part of your lesson revision. Describe the modification, additions, and deletions you are making to this lesson.

(Give a blank sheet of paper to teacher with items 12 and 13 listed.)

12. Additional space is given here for notes and other ideas you have for implementing this lesson in your class.
13. **Teacher Reflection** (do not fill this in until you've taught the lesson): Write down what you noticed about the lesson or what a student said in class while completing the lesson that you found noteworthy, interesting, or surprising. *Do this the same day as you taught the lesson, and use this information when (1) you plan and refine this lesson for the next school year, and (2) to possibly inform you for tomorrow's lesson.*

LESSON PREFLECTION QUESTIONNAIRE (Version 2: Used from 9/26/2014 to end of class observations. Shared 10/3/2014 and 10/21/2014)

To use when thinking about an initially planned lesson

1. What are the primary ideas being developed in this lesson?
 - What are the key ideas of this lesson that you find important?
2. What are 3 other questions you think will be useful to pose to your students as they complete this investigation?
 - What ways of thinking do you hoping emerges from these interactions?
3. What are your mathematical goals for student learning as you plan to teach this lesson?
 - Are there other goals you have for this lesson?
 - What criteria will you use to see your goals (for student learning, teaching, interacting with students) are achieved in a lesson?
 - How might the understandings that are suggested by your goals develop or be supported for students?
4. What ways of thinking about the key ideas in this lesson do you think will be expressed by students during class?
 - How might these ways of thinking be helpful for students when learning/using related concepts?
 - Are there possible ways of thinking that may emerge that hinders future learning?
5. How do you plan to help your students make meaningful conjectures?
 - How will this help your students develop their understandings and reasoning abilities associated with the key ideas of the lesson?
 - How did you plan to incorporate student reflection back to the conjectures they made?
6. Before using this lesson with your students, think about the specific questions that you plan to ask students that will enhance their learning. What questions will you ask to:
 - get your students to make meaningful conjectures related to this lesson
 - envision the relevant quantities in the situation
 - probe students in conceptualizing how the quantities in the situation are related
 - address possible misconceptions
 - hold students accountable for expressing their meanings
 - help them reflect on the reasonableness of their responses
7. How will you have students share their solution approach with the class?
 - What criteria will you use to select students?
 - What tools will students use (document camera, whiteboard, mini-boards, etc.) to share their solutions and thinking?
8. Summarize how you envision this lesson unfolding in your class.
9. What do you plan to do to hold *all* students in class accountable for expressing his/her thinking and constructing the understandings and reasoning abilities associated with the key ideas of the lesson?
10. **Your Reflection** (do not fill this in until you've taught the lesson): Write down what you noticed about the lesson or what a student said in class while completing the lesson that you found noteworthy, interesting, or surprising. *Do this the same day as you taught the lesson, and use this information when (1) you plan and refine this lesson for the next school year, and (2) to possibly inform you for tomorrow's lesson.*

LESSON PREFLECTION QUESTIONNAIRE (Version 3: Introduced at end of first session of post-study interview: 10/31/2014, based on feedback given on 10/25/2014)

To use when thinking about an initially planned lesson

1. What are the primary ideas being developed in this lesson?
 - What are the key ideas of this lesson that you find important?

2. Before using this lesson with your students, think about the specific questions that you plan to ask students that will enhance their learning. What questions will you ask to:
 - get your students to make meaningful conjectures related to this lesson
 - envision the relevant quantities in the situation
 - probe students in conceptualizing how the quantities in the situation are related
 - address possible misconceptions
 - hold students accountable for expressing their meanings
 - help them reflect on the reasonableness of their responses

3. What are 3 other questions you think will be useful to pose to your students as they complete this investigation?
 - What ways of thinking do you hoping emerges from these interactions?

4. What are your mathematical goals for student learning as you plan to teach this lesson?
 - Are there other goals you have for this lesson?
 - What criteria will you use to see your goals (for student learning, teaching, interacting with students) are achieved in a lesson?
 - What methods do you anticipate students will use?
 - What ways of thinking do you hope emerge?
 - How might the understandings that are suggested by your goals develop or be supported for students?

5. What ways of thinking about the key ideas in this lesson do you think will be expressed by students during class?
 - How might these ways of thinking be helpful for students when learning/using related concepts?
 - Are there possible ways of thinking that may emerge that hinders future learning?

6. How do you plan to help your students make meaningful conjectures?
 - How will this help your students develop their understandings and reasoning abilities associated with the key ideas of the lesson?
 - How did you plan to incorporate student reflection back to the conjectures they made?

7. How will you have students share their solution approach with the class?
 - What criteria will you use to select students?
 - What tools will students use (document camera, whiteboard, mini-boards, etc.) to share their solutions and thinking?

8. Summarize how you envision this lesson unfolding in your class.

9. What do you plan to do to hold students in class accountable for expressing his/her thinking and constructing the understandings and reasoning abilities associated with the key ideas of the lesson?

10. **Your Reflection** (do not fill this in until you've taught the lesson): Write down what you noticed about the lesson or what a student said in class while completing the lesson that you found noteworthy, interesting, or surprising. *Do this the same day as you taught the lesson, and use this information when (1) you plan and refine this lesson for the next school year, and (2) to possibly inform you for tomorrow's lesson.*

Post Classroom Observation Protocol

LESSON DEBRIEF QUESTIONNAIRE (Version 1: 8/11/2014)

To use when reflecting on a lesson that was recently taught

1. How do you think today's lesson went?
2. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?
3. What were your teaching goals for the lesson you led today?
4. What were your goals for student learning for the lesson you led today?
5. What were your goals for student interactions in the lesson you led today? (Are there other goals you had for today's lesson?)
6. How might the understandings that are suggested by your goals develop, be promoted, or supported for students?
7. Where in this lesson did you see opportunities for students to make conjectures about a problem statement or activity, a solution to a question, or the appropriate mathematics?
8. Where in this lesson did you see opportunities for students to reflect about a question, the solution to question(s), or its relationship to prior activities/other mathematical ideas?
9. What are the key ideas of mathematics you felt were important in today's lesson?
10. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended? (Did you achieve your goals during this lesson? Which goals are these? If so, what is your evidence? If not, why?)
11. Based on your observation of how the lesson unfolded today, how have your plans for tomorrow's lesson changed (or not changed)?
12. *(One or two additional questions, specific to the classroom observation, on mathematical interactions that occurred during the lesson that are identified by the researcher for follow-up will be asked.)*

LESSON DEBRIEF QUESTIONNAIRE (Version 2: Used from 9/26/2014 to end of class observations. Shared 10/21/2014)

To use when reflecting on a lesson that was recently taught

1. How do you think today's lesson went?
2. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?
3. What were your goals for student interactions in the lesson you led today?
 - Are there other goals you had for today's lesson?
4. What were your mathematical goals for student learning for the lesson you led today?
 - What methods did you envision students would use?
 - What ways of thinking did you hope emerge?
5. How might the understandings that are suggested by your goals develop or be supported for students?
6. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?
 - Did you achieve your goals during this lesson?
 - Which goals are these?
 - If so, what is your evidence?
 - If not, why?

7. What are the key ideas of mathematics you felt were important in today's lesson?
8. What pedagogical moves did you make to hold *all* students in class accountable for expressing their thinking and constructing the understandings and reasoning abilities associated with the key ideas of the lesson?
9. Where in this lesson did you see opportunities for students to make conjectures about a problem statement or activity, a solution to a question, or the appropriate mathematics to use in a task?
10. Where in this lesson did you see opportunities for students to reflect about a question, the solution to question(s), or its relationship to prior activities/other mathematical ideas?
11. Based on your observation of how the lesson unfolded today, how have your plans for tomorrow's lesson changed (or not changed)?

LESSON DEBRIEF QUESTIONNAIRE (Version 3: Introduced at end of first session of post-study interview: 10/31/2014, based on feedback given on 10/25/2014)

To use when reflecting on a lesson that was recently taught

1. How do you think today's lesson went?
2. Was there something about the lesson or what a student said in class today that you found noteworthy, interesting, or surprising?
3. What were your goals for student interactions in the lesson you led today?
 - Are there other goals you had for today's lesson?
4. What were your mathematical goals for student learning for the lesson you led today?
 - What methods did you envision students would use?
 - What ways of thinking did you hope emerge?
5. How might the understandings that are suggested by your goals develop or be supported for students?
6. How did the lesson that unfolded differ from your planned goals, or did the lesson go as you intended?
 - Did you achieve your goals during this lesson?
 - Which goals are these?
 - If so, what is your evidence?
 - If not, why?
7. What are the key ideas of mathematics you felt were important in today's lesson?
8. What pedagogical moves did you make to hold students in class accountable for expressing their thinking and constructing the understandings and reasoning abilities associated with the key ideas of the lesson?
9. Where in this lesson did you see opportunities for students to make conjectures about a problem statement or activity, a solution to a question, or the appropriate mathematics to use in a task?
10. Where in this lesson did you see opportunities for students to reflect about a question, the solution to question(s), or its relationship to prior activities/other mathematical ideas?
11. Based on your observation of how the lesson unfolded today, how have your plans for tomorrow's lesson changed (or not changed)?

The Two Protocols: Carolyn's Views

Regarding the Lesson Reflection Questionnaire

In our conversation yesterday, I had asked for your feedback on the Reflection Questionnaire after you initially plan a lesson.

1a. What questions do you find relevant to your planning, or potentially useful in refining a planned lesson?

I like #6 -- after looking at #6 I find #2 to be redundant.

2a. For question 1a, does the type of lesson (conceptual or skill based) affect the relevance of particular questions?

yes

3a. What questions would you refine or modify in the current questionnaire?

#2, #9

4a. What questions do you find potentially helpful but difficult to answer?

Difficult for me is "how do I know my goals for student learning are achieved?" My usual comment is, I won't know until tomorrow. I know that exit cards are used by some, but frankly, I don't have the time to read exit cards from every student.

5a. What questions would you suggest either to add to or delete from this questionnaire?

#2 and #6 seem to ask the same things.

6a. Other comments or suggestions?

[No response given for this item]

Regarding the Lesson Debrief Questionnaire

In our conversation yesterday, I had asked for your feedback on the Debrief Questionnaire after you teach a lesson.

1b. What questions do you find most relevant to you, when thinking about a lesson you recently taught?

#2

#11

2b. For question 1b, does the type of lesson (conceptual or skill based) affect the relevance of particular questions?

Not #2 or 11.

3b. What questions would you refine or modify in the current questionnaire?

I don't understand #5

#3. You ask for the goals, then ask for other goals....

4b. What questions do you find potentially helpful but difficult to answer?

#8

5b. What questions would you suggest either to add to or delete from this questionnaire?

In #8 I am intimidated by the "all" in italics. I don't know that I ever achieve that, and I react negatively to the emphasis on the "all."

#7. It seems to me that the goals for student learning would be the key ideas of the lesson. This seems redundant.

6b. Other comments or suggestions?

[No response given for this item]

APPENDIX L
CAROLYN'S GOALS

Mathematical Goals for Student Learning

Observed Lesson Date	Lesson Name	Pathways Lesson	Carolyn's Goals	Ranking of Stated Goals	Rationale
1	9/16/14 Mod 4, 1.1 (The Meaning of Exponents)	Yes	That the students would recognize exponential growth, and be able to represent the exponential growth by writing a function.	TGSL1, TGSL2	Two goals stated. First rates as a TGSL1 (topic with overarching action) and TGSL2 (topic with specific action: writing a function)
			To compare the linear and exponential functions. To look at the differences. Tomorrow, although I'm going to finish this up, a little bit on this same problem [question 5]. Again, looking at the graph, looking at where they intersect, and those kind of things in numbers, number 8, where they intersect. Finish up that, look at number 10 and number 11 where we have a different situation, and they have to come up with a function and talk about the growth rate and the percent change and all that. Then what I'd like to do is to go back to comparing and have them get out the big boards. And compare, make a table of values for $y=2x$, $y=x^2$, and $y=2^x$. And have them input values from 0 to 10. And look at the changes in those three graphs as we look at how each one of them grows, and where's the growth more rapid. And then maybe throw in, not actually put it on the table, but put in things like x cubed, where would that fit, between the $2x$ or between the x squared. Or 3 to the x , where would that fit, if we're looking at how the growth patterns go.		For the first goal, it is rated as TGSL1, since the looking at the differences action is overarching. Carolyn did have something specific planned with regards to this goal, although unstated. For the last goal, it is about specific actions she wants students to do [TGSL2].
2	9/18/14 Mod 4, 1.2, cont'd (Comparing Linear and Exponential Behavior)	Yes		TGSL1, TGSL1, TGSL1, TGSL2	
			Tomorrow's lesson is very straightforward, very process oriented. It's going to be, uh, basically it, I don't know if you are looking at that lesson or not, but we're going to work a problem together, and they're going to work a problem, and I'm going to work, we're going to work a problem together, and they're going to work a problem. Because they are looking at exponential functions given a function, given a table, given a graph. So it, it's just some practice.	TGSL2	With the clarification statement what tomorrow's activity entails, Carolyn's goals for student learning this goal has been characterized is TGSL2. She stated the sequence of actions/topics she wanted to focus on with more specificity, instead of stating simply they will do practice with exponential functions [TGSL1]. Using the multiple representations as an integrated tool for learning [that is, a focus on multiple using methods of mathematics deliberately – indicative of TGSL3] was not on Carolyn's radar.
3*	9/19/14 Mod 4, 1.3 (1-unit Growth and Decay Factors, Percent Change, and Initial Values)	Yes			
			That they [the students] can express those growth factors. And that they can express a function to describe the situation they are reading about, or looking about. You know, some are from information, some are from tables, some are from graphs. So I'd like them to get a feel for approaching those problems three different ways.	TGSL3	This goal is characterized as TGSL3 since she deliberating planning to use the multiple representations to highlight the idea of exponential growth.
4	9/23/14 Mod 4, 1.5 (n-unit Growth and Decay Factors, $n \neq 1$)	Yes			
			I want them to be able to solve a logarithm without their calculator. To understand the relationship between exponential and logarithmic functions.	TGSL2, TGSL4	Goal 1 is rated as TGSL2 (topic with specific action). Goal 2 is rated as TGSL4 (ways of thinking, without way of thinking articulated) As with Robert, goals rated at TGSL4 could be ways of thinking supported through procedures, or ways of thinking through leveraging key ideas, or both.
5	9/26/14 4.8.1 (Logs)	No			
			Use the properties of logarithms to expand and condense. Use the calculator to evaluate logarithms, including change of base.	TGSL2, TGSL2	Goal 1 is rated as TGSL2 (topics with specific actions). She specifies the properties of logarithms as the method used to expand/condense logarithms. Goal 2 is rated as TGSL2. It describes the tool used to be evaluating a logarithm, including changing the base. So it a topic with specific actions.
6	9/30/14 4.8.2 (Properties of Logs)	No			
7	10/2/14 4.9.2 (Solving Log Equations)	No	Solve logarithmic equations.	TGSL1	Topic with overarching actions.
			I wanted them to store values in their calculators and then use those values in solving the problems. I wanted them to use the processes learned in the last two lessons to write exponential models and solve logarithmic equations.		Goal 1 describes a particular method being used to solve a problem using technology [TGSL2]. Writing exponential models and solving logarithmic equations is the overarching goal, using the processes learned in the last two lessons isn't a method per se but implies to use the methods learned so far. Without a clarification it is unclear which methods she was referring to. Therefore [TGSL1] a topic with overarching actions.
8*	10/3/14 4.9.3 (Modeling Exponential Growth)	No		TGSL3, TGSL1	
			To recognize a function that's decreasing at a decreasing rate and a function that is decreasing at an increasing rate.	TGSL4	This is more cognitive: getting student to way of thinking about the mathematics, without that way of thinking articulated.
9	10/21/14 Mod 5, 1.2 (Concavity and Average Rate of Change)	Yes			
			For them [students] to be able to find the zeros of any quadratic function – the real zeros – of any quadratic function. We're not going to get into the imaginary zeros until next week I think, when we look at higher degree polynomials and are able to find the imaginary zeros.	TGSL4	"To be able to" implies a way of knowing for students that allows them to find zeros, without that way being articulated. Although clarification occurs during the planning session (it is about choosing the appropriate methods, with the list of possible methods articulated), looking at the goals, it is rated as TGSL4.
10	10/23/14 5.4.1 (Zeros of Quadratic Functions)	No			
			I hope they will be able to look at a function and know what the end behavior is without looking at the graph. Would like them to be able to understand what a graph is going to look like, given the multiplicity of the zeros.		Goal 1 is a way of thinking about functions, without that way of thinking articulated. [TGSL4] Goal 2 is about getting students to think about the mathematics in certain ways [TGSL5]. Carolyn wants students to use the multiplicity of zeros as a tool to gain insight into the plotted graph. So that is a specific way of thinking about graphs. Had she said draw a graph, given the multiplicity of zeros, it is would have rated TGSL3.
11	10/27/14 5.5.1 (Polynomial Functions of Higher Degree and Multiplicity of Zeros)	No	I would like them to be able to write a function if they know its zeros.	TGSL4, TGSL5, TGSL3	Goal 3 describes a method used [knowing the zeros] to write the function rule. I characterize it as TGSL3.

Goals for Student Interactions

Observed Lesson	Ranking of Stated Mathematical Goals	Goals for student interactions
1	TGSL1, TGSL2	(Planning) Well I toyed with having the kids do this on large whiteboards, but I don't think that this is the best lesson to do that with, what do you, I think since I would really like them to have the information on that table in number 2 in their book, I would rather that they do it in their book rather than doing it together on a whiteboard. They could collaborate on it but I would like to see them not necessarily putting it on big board.
2	TGSL1, TGSL1, TGSL1, TGSL2	(Debrief) Well what I have liked to do is to get the big boards out when we made some of those graph tables at the end, I didn't have time to do that, so that goal didn't work. But I like to have them helping each other when they're working on their calculators. I guess that would be a student interaction goal that they could assist each other on how to do that.
3*	TGSL2	Not asked
4	TGSL3	(Planning) I think that the conjecture, I don't expect that even to be an individual conjecture. That they should do that as a group. The question I ask I would like them to discuss, rather than individual one. So we'll see. Some as you know, will be more vocal than others. But I'll walk around and manage that. See how they're doing. (Debrief) Well, at first I thought that I would actually ask them to do more working together than I ended up doing. I felt such a time crunch, isn't it amazing how 10 minutes can make such a difference? That, you know, it took time to pass things out and pick things up. And I had hoped that, I had first thought I would have them work together, but then, I don't know that I had any goals for necessarily for student interaction today. Interact with me.
5	TGSL2, TGSL4	(Debrief) Well I really didn't have much opportunity for student interaction as I recall. I wanted them to interact with me by you know (gesturing fingers through air), doing some drawing in the air, but I really didn't have opportunities for them to interact with each other. When I had those problems on the board at the end, some of them were discussing them with each other which was fine, but I didn't make a big deal about requiring them to do that. But sometimes they do anyway. So you know, this was more of a, as you saw, more of a straightforward lecturish kind of lesson.
6	TGSL2, TGSL2	Not asked
7	TGSL1	(Debrief) That when they were, that they would be helping each other when they have issues. Sometimes when I was at a table, if I saw that one student had done things correctly, I asked him make sure that the others at his table agreed with his work. So that I didn't have to help every single person at the table, that they could help each other. I like to do that. I say: You got it, make sure everybody here has got the same result. And they help with the calculator a lot also. With calculator issues. They'll look, they'll compare and say: oh look, you don't have the right parenthesis or whatever. When they're trying to graph something.
8*	TGSL3, TGSL1	(Written Debrief) I wanted the students to be assisting each other in setting up the problems and in using their calculators.
9	TGSL4	(Debrief) That they would work together to create a graph that was appropriate. I did hear them talking about how to label the axes, and that kind of thing, so I think when they talk about, no you can't just say time, you have to be more specific, that kind of thing, that it helps to cement what, what's expected. So I was hoping they would interact about what was appropriate on drawing the graph.
10	TGSL4	(Planning) Well you know I'm not guessing that I have a great, that I'm anticipating a lot of student interaction. You know, I would like, in fact, in their assignment I ask them to derive the the quadratic formula themselves. So one of my goals is that they are able to reproduce that proof of the derivation of the quadratic formula. So I guess my goals are not necessarily interaction with each other, but interaction with what I'm doing so that they can- can do this. So that they can use the quadratic formula. So that they can complete the square.
11	TGSL4, TGSL5, TGSL3	(Debrief) That they would use their arms to show me what the end behavior of a function would be. I think that was probably the interaction that I relied on to see how they responded to the end behavior. I guess that's student interaction, isn't it?

APPENDIX M

A PROPOSED QUESTION TO PROMOTE CONJECTURE

Question to Promote Student Conjecture

The purpose of this opening question is to make students think about the n -unit and partial unit growth factors in a concrete context.

Two accounts at *Most Amazing Deposit Bank* start at \$100. Each account compounds monthly. One account grows by a factor of 8 every 3 months, while another grows by a factor of 16 every 4 months. At the end of one year, estimate which account grew more. Explain your rationale. At the end of two months, which account do you predict will grow more? Explain your reasoning.

This question can be used to start to a conversation around n -unit and partial unit growth factors when used in conjunction with the main lesson of Module 4, Lesson 4 from the *Pathways* curriculum (Carlson, et al., 2013b).

Students might observe that growth by four 3-month factors cover the same span of time as growth by three 4-month factors, and use that observation to make a prediction. Alternatively, a misconception that the second account grows more (using reasoning of a linear pattern inappropriately: the first account growing by a factor of 32 and the second account growing by a factor of 48 in one year) may also emerge.

After completing the main trajectory of the *Pathways* lesson, students should revisit their initial conjecture to either justify or refine it. They should reflect on the opening question, and be able to justify that both the accounts grow by the same amount by leveraging the key ideas of partial unit and n -unit growth factors from the lesson.

At the end of the lesson, students will have the mathematical tools to test their initial conjectures for both the account values at the end of the one year (using the idea of n -unit growth factors) and two months (using the idea of partial unit growth factors). Students should be supported on reflecting on whether their findings were reasonable or

not and asked to justify their solution process. The two units used in this context are the time spans: one corresponds to a 3-month block of time, while the other represents a 4-month block of time. Leveraging the ideas of the lesson, a two-month block of time can be represented by a $\frac{2}{3}$ -unit growth factor (if the time unit used is 3 months) or the $\frac{2}{4}$ -unit growth factor (if the time unit used is 4 months).

APPENDIX N
A PROPOSED ACTIVITY

A Concrete Activity to Support Thinking About n -Unit Factors

The purpose of this activity is to help students think about the n -unit and partial unit growth factors in a concrete context.

A \$100 investment account grows annually by 50% by the process of compounding.

- a. Write a function that models this situation, using y to track the time elapsed in years since the account was opened.
- b. Rewrite a function that models this situation, using m to track the time elapsed in months since the account was opened.

This question can be used to support reasoning with regards of n -unit and partial unit growth factors from a concrete context. The question is designed to be used in conjunction with the main lesson of Module 4, Lesson 4 from the *Pathways* curriculum (Carlson, et al., 2013b) in the idea of n -unit and partial unit factors are further abstracted.

The variables described in the task to track the quantities are y and m , respectively, to represent years or months. It follows that $m=12y$, since twelve 1-month time intervals is equivalent to one 12-month time interval. In part (a), the function that could model the current value of the investment is $g(y) = 100(1.5)^y$, if the growth factor of the money in the account is measured in years, and the elapsed time is measured in years since the account was opened.

An analogous model in part (b) can be used to determine the current value of the investment in months, $g(m)$. Leveraging the ideas of scaling, the input for one year is scaled by $1/12$. Then the analogue of scaling (exponentiation) in the multiplicative realm is used on the yearly growth factor of 1.5. The monthly growth factor is $1.5^{1/12}$, and thus $g(m) = 100(1.5^{1/12})^m$. In this context of part (b), the growth factor of the money in the

account is measured in months, and the elapsed time is measured in months since the account was opened.

APPENDIX O
KEY IDEAS AND GOALS

Carolyn's Views of the Key Ideas of a Lesson and Its Goals

The discussion in this section reviews Carolyn's views of key ideas and her mathematical goals. The Post-study task was used to illustrate potential shifts in Carolyn's thinking about the two concepts, even though her expressed views had not changed by the end of the study.

At the beginning of the professional developmental intervention, the key ideas of a lesson and her mathematical goals for student learning meant the same to Carolyn. In the first lesson planning session of the study, her response to the question what are the key ideas of the lesson was: "Okay, well so I thought that the students should be able to recognize exponential growth and then be able to represent that growth by writing a function". Her response to the researcher's query of what were goals for student learning was: "That the students will be able to recognize a function as being exponential growth to be able to represent the exponential growth by writing a function that will describe it." Her responses to the questions were not distinguishable.

After the professional development intervention, her view seemingly did not change. After the study was complete, part of the post-study interview focused on reviewing her earlier written feedback to the questions that were present on the two protocols (*Teacher Preflection and Reflection Protocol* and *Post Classroom Observation Protocol*, referred to as the *Lesson Preflection Questionnaire* and *Lesson Debrief Questionnaire*, respectively, with the participant). Carolyn had feedback to offer regarding the question "What are the key ideas of mathematics you felt were important in today's lesson?" from the *Lesson Debrief Questionnaire*. Carolyn had written: "It seems to me that the goals for student learning would be the key ideas of the lesson. This seems

redundant.” Her feedback on this question was brought up during the post-study interview.

Excerpt 83. Carolyn’s interpretation of key ideas (Post-study interview)

Res: You had mentioned that the key ideas of a lesson and goals, the goals for student learning and the key ideas of a lesson, appear to be the same. Can you tell me more about that, what you mean?

Car: In my, well, what I want them to learn are the key ideas that I am going to teach. So yes, I think that that’s just two ways to ask the same question.

To better understand Carolyn’s views of the key ideas of a lesson versus her goals for student learning of a lesson, at the end of the study, Carolyn was asked to complete the Post-study task. Besides getting insights into her meanings for key ideas and her goals with regards to the task, it was designed to gain further understanding of Carolyn’s MKT with regards to the mathematics of the task.

Carolyn walked the researcher through her responses to the questions in the Post-study task. To more easily visualize the task in the context of Carolyn’s response, I have included the statement of the task in the figure below. I shall refer to it as the Fish Bowl Task. This task was inspired by a homework question in the chapter Carolyn had completed from the conceptual curriculum she had used (Carlson, et al., 2013b, p. 201). The entire Post-study task, with Carolyn’s responses, can be found in Appendix J.

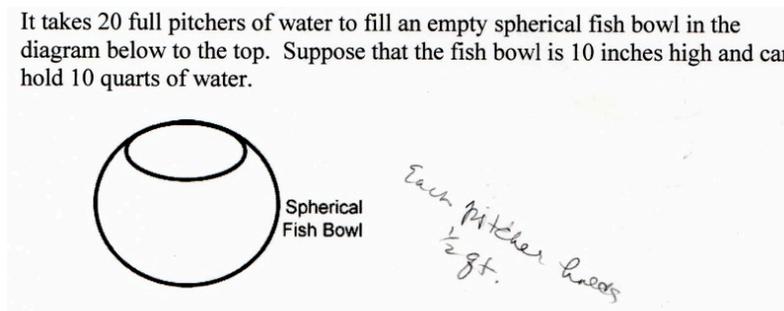


Figure 37. The Fish Bowl Task (Post-study interview)

The task itself required the participant to attend to covarying quantities. Relevant quantities Carolyn was asked to track were: the number of pitchers of water in the bowl, the number of quarts of water in the bowl, and the height of the water in the bowl. Furthermore the task was designed to be accessible to a participant who could imagine either discrete or continuous changes in the covarying quantities. Two of the questions asked Carolyn to draw a graph of the covarying quantities. Since she referred to these questions as part of her response regarding the key ideas of the prototype lesson, these have been included side by side in the figure below. Her response to Question 1, part c, is on the left side, while Question 2, part b, is on the right side of the figure.

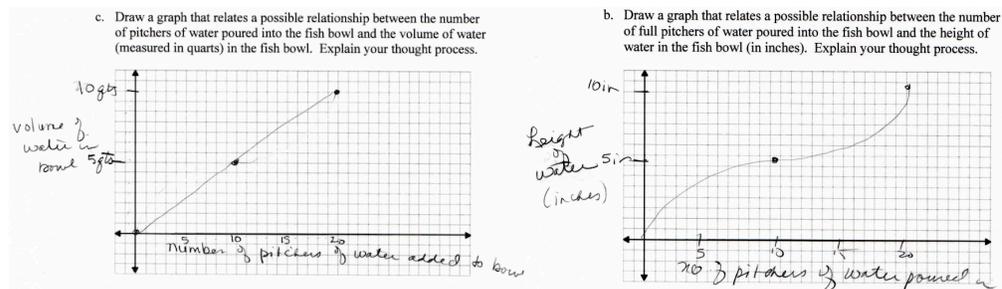


Figure 38. Carolyn’s responses to two questions in the Fish Bowl Task

From the Pre-study task in which Cameron and Neil travelled at a constant speed, the researcher found that Carolyn could think about covarying quantities that could be characterized at a Quantitative Coordination Level in the covariation framework (Carlson, et al., 2002). In response to the researcher’s query when she was asked how might students justify why graph representing a constant rate of change between quantities was linear, she replied, “They would know for equal changes in input you have correspondingly equal changes in output. They would know then that was linear.” Based on the Pre-study task, and classroom observations from the polynomials chapter in which her stated goal in one of the taught lessons was for students to “recognize a function

that's decreasing at a decreasing rate and a function that is decreasing at an increasing rate", I surmised the task would be accessible to her.

In terms of Carolyn mathematical connections with regards to the Fish Bowl Task, during her discussion of the task with the researcher, she was able to attend to the quantities under consideration, and she was able to articulate how the quantities covaried. When tasked to describe how does the volume of the water in the fish bowl change, as the number of full pitchers of water poured into the fish bowl increases, and explain her thought process, her written response was as follows: "The volume increases with each pitcher because water is going in and not going out. For every pitcher poured in, volume increases by $\frac{1}{2}$ quart." Similarly, when tasked to describe how the height of the water (in inches), as the number of full pitchers of water poured into the fish bowl increases, her response was: "Height increases at a rate that is increasing for the first 10 pitchers, then decreases." Robust meanings for proportionality were not required in order to make meaning of the key ideas of this task.

The researcher then asked during the interview what she thought were the primary ideas of the lesson, Her response is given in the excerpt below.

Excerpt 84. Carolyn's key ideas for a lesson (Post-study interview)

Res: What are the primary ideas being developed in this lesson?

Car: That I think you have to think very carefully about the input and the output when you look at functions. You have to identify, you have to make a decision what the input and output are, and then you'd, just have to be careful and I think that, um, that this (*flipping through pages in task*) will show very well. I mean I think your questions do that (*sweeping hand over questions on task*), as you talked about when those (*pointing to graph of task*) are switched. So I think if this in fact is a review lesson, then I think that that's one of the focuses about this lesson, is to be very careful about what your input and output are, because it's very easy to not sketch this first graph [*question 1, part c*] the way it is if you are not paying close attention. Because this is not something that we, in other lessons, graphed. The number of pitchers to the height was the graph virtually every time, I believe that

- we never did do this. And so kids aren't going to be thinking that this is, they're going to be thinking that it's going to look something like that [question 2, part b].
- Res: Uh huh.
- Car: So, I think that one of the goals of this lesson to just to stress the importance of being careful.
- Res: When you mean referring to input and output, what do you mean?
- Car: Input (*tracing horizontal axis of graph in question 1, part c of task*) and output (*tracing vertical axis of graph in question 1, part c of task*)

Summarizing Carolyn's response, one of the key ideas of the task was attending to two quantities that covaried. Because the covarying quantities used in the task did not have an inherent dependency, she remarked that "you have to make a decision what the input and output are". This was her recognition that there were two ways to plot the relationship between the covarying quantities, depending on how students viewed which was the 'input' quantity and which was the 'output' quantity. Furthermore she commented on the need to be careful attending to the input and output quantities, and she predicted that students might (out of habit) draw the relationship between quantities they are typically asked to graph.

The researcher then asked Carolyn about her mathematical goals for student learning, if she were to teach the lesson. Her response follows.

I think the important things here that they can sketch a graph, and have an understanding what it is they're graphing. Somebody sitting down to do this is going to put time in one of these places. You know, and you know to sort of talk about why that's not appropriate. Because somebody will do that in my class. Because isn't time always one of the two values there? You know, time and volume, or I don't know, time and number of pitchers, or something like that.

In characterizing Carolyn's mathematical goals for student learning, this goal would have been ranked at TGSL5 (getting students to think about the mathematics in certain ways). If she were to teach this lesson, she wanted her students to have an

understanding of what they were graphing, which implied that they would need to make meaning of the graph they were plotting. She identified a specific misconception that she expected some students to have, which was thinking that one of the tracked quantities was necessarily time, even though time was not a quantity being tracked in either question.

However the concluding portion of this study suggests that even though Carolyn had said that the key ideas of the lesson and mathematical goals for student learning meant the same thing to her (in same interview session as this post-study task was discussed), how she discussed these two questions was different.