

Exponential Growth and Online Learning Environments:
Designing for and studying the development of student meanings in online courses

by

Alan Eugene O'Bryan

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved November 2018 by the
Graduate Supervisory Committee:

Marilyn Carlson, Chair
Patrick Thompson
Fabio Milner
Kyeong Hah Roh
Michael Tallman

ARIZONA STATE UNIVERSITY

December, 2018

ABSTRACT

This dissertation report follows a three-paper format, with each paper having a different but related focus. In Paper 1 I discuss conceptual analysis of mathematical ideas relative to its place within cognitive learning theories and research studies. In particular, I highlight specific ways mathematics education research uses conceptual analysis and discuss the implications of these uses for interpreting and leveraging results to produce empirically tested learning trajectories. From my summary and analysis I develop two recommendations for the cognitive researchers developing empirically supported learning trajectories. (1) A researcher should frame his/her work, and analyze others' work, within the researcher's image of a broadly coherent trajectory for student learning and (2) that the field should work towards a common understanding for the meaning of a hypothetical learning trajectory.

In Paper 2 I argue that prior research in online learning has tested the impact of online courses on measures such as student retention rates, satisfaction scores, and GPA but that research is needed to describe the meanings students construct for mathematical ideas researchers have identified as critical to their success in future math courses and other STEM fields. This paper discusses the need for a new focus in studying online mathematics learning and calls for cognitive researchers to begin developing a productive methodology for examining the meanings students construct while engaged in online lessons.

Paper 3 describes the online Precalculus course intervention we designed around measurement imagery and quantitative reasoning as themes that unite topics across units. I report results relative to the meanings students developed for exponential functions and

related ideas (such as percent change and growth factors) while working through lessons in the intervention. I provide a conceptual analysis guiding its design and discuss pre-test and pre-interview results, post-test and post-interview results, and observations from student behaviors while interacting with lessons. I demonstrate that the targeted meanings can be productive for students, show common unproductive meanings students possess as they enter Precalculus, highlight challenges and opportunities in teaching and learning in the online environment, and discuss needed adaptations to the intervention and future research opportunities informed by my results.

ACKNOWLEDGMENTS

There are many people who were instrumental in helping me complete this journey and who have supported me during my academic career and throughout my entire life. First and foremost, thank you to my wife, Alissa. She is my best friend and the person who shares my joys and helps me bear my struggles. She has a spark of pure joy and an ability to find happiness in the moments of daily life. These are the things I love most about her, and her ability to make me smile every day kept me sane during this long, challenging process. I love you.

Thank you to my parents, Eugene and Joan. I know that I am lucky to have had parents who sacrificed so much of themselves to make sure that their children had all the support and love they needed to thrive. The years of attending sports practices, the late nights helping with homework, the time you spent reading to me as a child, the times you came my thesis and dissertation defenses, and all times in between. I am only successful today because of you.

I am sometimes accused of setting expectations for myself too high. To the extent that is true, it is only because I have been fortunate enough to be surrounded by people whose work I admire so much and who I want to honor by my own work. Dr. Marilyn Carlson, Dr. Pat Thompson, and Dr. Michael Tallman have tremendously impacted my personal understanding of mathematics and what it means to study others' meanings and learning. Their work is of the highest quality and inspires me to do my best. I hope they see my respect and admiration reflected in the great care I have taken to try to explain and build on their work.

To Dr. Carlson in particular I owe a huge debt. She has supported, encouraged, and challenged me through our work together over the last decade. Successfully completing this dissertation would not have been possible without her warmth, generosity, mentorship, and support, not to mention the times she calmed me down when I felt overwhelmed by graduate school and the dissertation writing process. I also owe much to Dr. Thompson, who first unlocked my passion for teaching for meaning and who intervened when I nearly left the field in 2008.

I am enormously grateful to my entire committee, including Dr. Fabio Milner and Dr. Kyeong Hah Roh, for their support and contributions to my research. No graduate student can make it through a program like this without people who are willing to selflessly give of their time and expertise to mentor graduate students. I will be forever grateful that you agreed to be with me on this journey.

Finally, thank you to all of the teachers I have worked with over the years, all of the students I have taught, and my fellow graduate students. I have learned so much from you and feel privileged to have known you. I wouldn't be here without the experiences we've shared.

TABLE OF CONTENTS

	Page
LIST OF TABLES.....	xv
LIST OF FIGURES.....	xvii
CHAPTER	
1 INTRODUCTION AND PROBLEM STATEMENT.....	1
Teaching and Learning Exponential Growth.....	2
Online Learning.....	7
Summary of Problem Statement and Research Questions.....	10
A Comment on the Three-Paper Format for Dissertation Reports.....	11
2 THEORETICAL PERSPECTIVE.....	12
Background Theory.....	12
Piaget’s Genetic Epistemology.....	13
Harel’s Duality, Necessity, and Repeated Reasoning (DNR)	
Framework.....	15
Duality.....	15
Necessity.....	16
Repeated Reasoning and What it Means to Reason.....	17
What Does it Mean to Understand?.....	19
Domain-Specific Theories.....	21
Quantitative Reasoning.....	22
Covariational Reasoning.....	24
Quantitative Reasoning and Function Representations.....	25

CHAPTER	Page
The Role of Theory in this Particular Study.....	28
Constructivism as a Theoretical Lens.....	28
Constructivism as an Instructional Design Principle.....	30
Summary and Closing Comments.....	34
3 LITERATURE REVIEW.....	36
Exponential Functions.....	37
Confrey and Smith’s Conceptual Analysis.....	37
Splitting.....	37
Covariation, rate of change, and function.....	38
Properties of exponential functions’ rates of change.....	39
Building on Confrey and Smith’s conceptual analysis.....	41
Summary and Comments.....	41
Thompson’s Conceptual Analysis.....	42
Piecewise Linear Accumulation and the Fundamental Theorem of Calculus.....	42
Reasoning about a Growth Factor in Continuous Covariation.....	44
Building on Thompson’s Conceptual Analysis.....	46
Summary.....	48
Ellis and Colleagues: An Empirically Supported Hypothetical Learning Trajectory.....	49
Students Learning Mathematics in an Online Environment.....	59
4 CONCEPTUAL ANALYSIS.....	64

CHAPTER	Page
Exponential Growth and Related Ideas from a Quantitative Perspective..	65
Flexibility in Choosing a Unit.....	65
“Elastic” Units and the Meaning of a Growth Factor.....	68
Percent Comparisons.....	70
Percent Change.....	72
Exponential Functions.....	73
Comments on the Conceptual Analysis.....	76
Settling on Some Design Principles for the Online Intervention.....	77
Repeated Reasoning [Revisited].....	77
Design Principles for Online Courses.....	79
Select Tasks within a Hypothetical Learning Trajectory.....	84
Task 1: “Elastic Ruler”.....	87
Task 2: Comparing Vine Lengths.....	89
Task 3: Identifying Growth Factors.....	90
Tasks 4 and 5: Percent Comparisons and Making Connections.....	92
Task 6: Two Consecutive Intervals.....	94
Tasks 7 and 8: Writing a Formula to Represent an Exponential Relationship.....	95
Task 9: Using Two Values to Determine the Formula for the Exponential Relationship.....	99
Closing Comments and Summary.....	99
5 METHODOLOGY.....	101

CHAPTER	Page
Overview of the Proposed Study.....	101
Design-Based Research: Integrating Theory and Practice.....	103
Online Learning of Exponential Growth: A Design-Based Research Study.....	107
Developing the Intervention.....	107
Implementation and Subjects.....	108
Data Collection from Multiple Sources.....	110
Phase I: Test Data.....	110
Phase II: Monitoring Student Interaction Data.....	111
Phase III: Clinical Interviews.....	112
Comments on Methodology.....	114
6 INTRODUCTION TO THREE PAPERS.....	115
7 PAPER 1 [LEARNING TRAJECTORIES].....	120
Students' Mathematical Experiences.....	122
Conceptual Analysis, Hypothetical Learning Trajectories, and Teaching Experiments: The Importance of Theory.....	123
Conceptual Analysis.....	124
Hypothetical Learning Trajectories.....	125
Examples of Different Uses of Conceptual Analysis.....	126
Confrey and Smith's Retrospective Conceptual Analysis: Modeling Student Reasoning.....	127

CHAPTER	Page
Images of Multiplication, Covariation, Function, Rate of Change, and Exponential Growth.....	127
Thompson’s Conceptual Analysis: Coherence of Mathematical Ideas Leading to Calculus.....	131
Quantitative and Covariational Reasoning, Rate of Change, Accumulation, and the FTC.....	132
Exponential Functions.....	134
Ellis and Colleagues: From Exploratory to Hypothetical Learning Trajectory.....	137
Summary and Theoretical Implications.....	140
A Need for Articulating Images of Long-Term Implications and Coherence.....	141
A Need for New Terminology and Articulating how Research Results are to be Interpreted.....	142
Final Comments.....	145
References.....	146
8 PAPER 2 [ONLINE LEARNING].....	150
Theoretical Perspective.....	151
Part I: A Call to Action.....	154
Online Instruction and Online Mathematics Courses.....	154
Scope of Current Research in Online Learning.....	155
Limitations in Current Research on Online Learning.....	159

CHAPTER	Page
Studying Student Learning in an Online Course as Design	
Science.....	161
The Importance of Attending to Meaning.....	161
Design-Based Research/Design Science.....	163
Design Research and Online Learning.....	165
A Call to Action – Closing Comments.....	166
Part II: Designing for and Researching Mathematical Meanings	
in Online Courses.....	167
Clinical Interviews and Teaching Experiments.....	168
Clinical Interviews.....	168
Teaching Experiments.....	170
Shared Assumptions and Theoretical Perspectives.....	172
Theoretical Considerations in Online Learning.....	174
Collecting Data to Inform Models on Student Meanings.....	175
Students Self-Reporting via Course Surveys.....	177
Looking at Course Grades and Retention Rates.....	177
Tracking Student Success in Subsequent Courses.....	178
Course Assessments.....	179
Aggregating Student Interaction Data.....	182
Validated Assessment Instruments.....	183
Using Screen Capture and Other Video to Monitor Students’	
Work as they Complete Lessons.....	184

CHAPTER	Page
Post-Lesson Interviews and Stimulated Recall.....	185
Interviewing Students as they Work through Online Lessons....	186
Clinical Interviews and Teaching Experiments Outside of the Online Environment.....	187
Commentary.....	188
Discussion.....	191
Conclusion.....	193
References.....	194
9 PAPER 3 [RESULTS].....	201
Theoretical Perspective.....	204
Exponential Functions Literature Review.....	207
Conceptual Analysis of Ideas Related to Understanding Exponential Growth.....	211
Potential Issues with Standard Definitions for Exponential Functions.....	212
Exponential Growth and Related Ideas from a Quantitative Perspective.....	215
Flexibility in Choosing a Unit and Units of Convenience.....	215
“Elastic” Units and the Meaning of a Growth Factor.....	219
Percent Comparisons and Percent Change.....	222
Exponential Functions.....	228
Measurement and Emergent Symbol Meaning.....	231

CHAPTER	Page
Study Methodology.....	235
Unit Design.....	238
Sample Tasks.....	239
Comments about Designing Tasks in iMathAS.....	245
Results.....	248
Pre-Test and Pre-Interview Results.....	248
Percent Comparison and Percent Change Tasks.....	250
Tomato Plant Tasks.....	255
Bacteria Doubling Task.....	264
Summary of Students' Meanings at the Beginning of the Course.....	270
Results from Lessons, the Post-Test, and Post-Interviews.....	272
How do Students Respond to Incorrect Attempts?.....	273
The Case of Gina, Marcus, and Shelby.....	276
How did Students Conceptualize Key Visualizations and Applets?.....	277
Persistent Guessing.....	282
Attention to Reference Quantities and a Meaning for Growth Factor, Percentages, and Percent Change.....	286
Emergent Symbol Meaning.....	292
Summary.....	297

CHAPTER	Page
The Case of John and Lisa.....	297
How did Students Conceptualize Key Visualizations and Applets?.....	298
Implications of Low Attempt Counts.....	305
Attention to Reference Quantities and a Meaning for Growth Factor, Percentages, and Percent Change.....	305
Emergent Symbol Meaning.....	308
Summary.....	310
Discussion.....	310
Teaching, Learning, and Conducting Research in the Online Environment.....	311
Exponential Growth and Related Ideas and Comparing Student Learning.....	312
Emergent Symbol Meaning.....	317
An Observation about Assessments as Measures of Student Learning.....	320
Next Steps and Future Research.....	321
Modifications Suggested by the Study.....	321
Promoting Student Reflection.....	321
Supporting an Increased Focus on Quantities and References and Supporting Students in Seeing Connections between Key Applets.....	323

CHAPTER	Page
Supporting Flexibility in Meanings for Growth Factors and Emergent Symbol Meaning.....	325
Future Research.....	328
Thinking with Magnitudes.....	328
Additional Analysis on Course Data.....	329
Continue Iterations of Reflection, Redesign, Implementation, and Study.....	330
Creating and Validating Items to Assess Students' Meanings for Exponential Growth and Related Ideas.....	331
Emergent Symbol Meaning.....	331
References.....	332
10 CONCLUSION.....	340
REFERENCES.....	345
APPENDIX	
A PRE-TEST AND POST-TEST.....	358
B STUDENT BELIEF SURVEY.....	367
C CLARIFICATION ON THOMPSON ANALYSIS.....	377
D ASU IRB APPROVAL FORM.....	379

LIST OF TABLES

Table	Page
1.1	Results from the National Assessment of Education Progress (12 th Grade).....6
1.2	Post-Secondary Enrollment in Distance Learning Courses.....8
3.1	A Table of Values for an Exponential Function of y with Respect to x39
3.2	Comparing Confrey and Smith’s Conceptual Analysis with Thompson’s Conceptual Analysis.....50
4.1	A Framework for Understanding Exponential Growth and Related Ideas.....74
5.1	iMathAS Instructor Feedback.....112
7.1	Coordinating an Arithmetic Sequence with a Geometric Sequence.....129
8.1	U.S. Post-Secondary Enrollment in Distance Learning Courses in the Fall Semester, 2012-2018.....154
8.2	Data Collection Methods for Modeling Student Meanings.....176
8.3	Student Responses to the Muffler Task181
8.4	iMathAS Instructor Feedback182
8.5	A Possible Timeline for Data Usefulness190
9.1	Coordinating an Arithmetic Sequence with a Geometric Sequence.....208
9.2	A Framework for Understanding Exponential Growth and Related Ideas...230
9.3	Students Interviewed for this Study.....237
9.4	Student Responses to the Muffler Task.....251
9.5	Student Responses to the Sale Price Task.....253
9.6	Student Responses to the Tomato Plant A Task.....255
9.7	Student Responses to the Tomato Plant B Task.....259

Table	Page
9.8 Student Performance at the End of Their Course on the Tomato Plant B Task (or an Isomorphic Task) (Madison et al., 2015).....	264
9.9 Student Responses to the Bacteria Doubling Task.....	265
9.10 Overall Post-Test Performance.....	273
9.11 How Many Lesson Tasks Required the Given Number of Attempts to Complete (out of 101).....	275
9.12 Percentage of Initially Incorrect Responses Requiring the Given Number of Attempts to Complete.....	275
9.13 Student Responses to the Muffler Task (Post).....	288
9.14 Student Responses to the Sale Price Task (Post).....	289
9.15 Student Responses to the Tomato Plant B Task (Post).....	293
9.16 Student Responses to the Tomato Plant A Task (Post).....	296
9.17 Student Responses to the Bacteria Doubling Task (Post).....	296
10.1 Results from the National Assessment of Educational Progress (12 th Grade).....	342

LIST OF FIGURES

Figure		Page
2.1	A Task I Assimilate to My Average Rate of Change Scheme.....	19
3.1	A Split as an Equal Number of Copies of an Original Amount (Left) or Partitioning into Equal-Sized Parts (Right) (Confrey, 1994).....	37
3.2	A Sequence of Bars with Equal Length Ratios (Confrey & Smith, 1994, p. 159).....	40
3.3	Focusing on the Length in Excess of the Previous Bar (Confrey & Smith, 1994, p. 160).....	40
3.4	A Piecewise Linear Accumulation Function.....	43
3.5	The Function Converges to an Exponential Function as the Number of Times the Principle is reset (n) Increases without Bound.....	44
3.6	An Approximation of the Visual Display for Thompson’s Applet for Three Different States.....	46
3.7	An Approximation of the Display for Ellis et al.’s (2012, 2013, 2015, 2016) Jactus Applet.....	52
3.8	Approximating a Student’s Drawing (Ellis et al., 2012, p. 102).....	56
4.1	Measuring Relative Sizes.....	66
4.2	An “Elastic Ruler” that Changes with a Varying Quantity so that the Measurement Remains Constant.....	68
4.3	Measuring the Height’s Magnitude on January 1 of this Year Using the Height’s Magnitude on January 1 of Last Year as the Measurement Unit.....	71

Figure	Page
4.4 Measuring the Magnitude of the Height on January 1 of this Year Using One Percent of the Height on January 1 of Last Year as the Measurement Unit...	72
4.5 Measuring the Change in Height Using One Percent of the Height on January 1 of Last Year as the Measurement Unit.....	73
4.6 My Quantitative Structure for Understanding an Exponential Model.....	74
4.7 Task 1 of Instructional Sequence for Exponential Growth and Related Ideas (O’Bryan, Sander, & Carlson, 2017).....	88
4.8 Task 2 of Instructional Sequence for Exponential Growth and Related Ideas (O’Bryan et al. 2017).....	90
4.9 Task 3 of Instructional Sequence for Exponential Growth and Related Ideas (O’Bryan et al. 2017).....	91
4.10 Task 4 of Instructional Sequence for Exponential Growth and Related Ideas (O’Bryan et al. 2017).....	92
4.11 Task 5 of Instructional Sequence for Exponential Growth and Related Ideas (O’Bryan et al. 2017).....	94
4.12 Task 6 of Instructional Sequence for Exponential Growth and Related Ideas (O’Bryan et al. 2017).....	96
4.13 Task 7 of Instructional Sequence for Exponential Growth and Related Ideas (O’Bryan et al. 2017).....	97
4.14 Task 8 of Instructional Sequence for Exponential Growth and Related Ideas (O’Bryan et al. 2017).....	98

Figure	Page
4.15 Task 9 of Instructional Sequence for Exponential Growth and Related Ideas (O’Bryan et al. 2017).....	100
5.1 Design-Based Research’s Continuous Development Cycle.....	104
7.1 For Many Students, Topics (Represented as Rots) Within Courses (Represented as Squares Containing Topics) are Isolated as Opposed to Being Connected by Strong Themes Both Within a Course and Across Courses.....	122
7.2 We Want Students to Conceptualize Strong Connections between Topics and Ideas within a Course as Well as Powerful Themes and Ways of Thinking that Connect Ideas across Many Courses	122
7.3 Parts of a Teaching Experiment	126
7.4 Two Different Conceptualizations of an Eight-Split (Confrey, 1994).....	128
7.5 Ratios Such as $512/8$ and $4096/64$ are Equal if an Equal Number of Splits Separate the Values	128
7.6 Confrey and Smith’s Recommendations can be Thought of in Terms of "Projecting Forwards".....	131
7.7 As the Total Mixture Volume Changes, the Volumes of Water and Concentrate Change so that they are Always the Same Relative Size and the Same Relative Proportion of the Mixture’s Volume	133
7.8 Piecewise Linear Accumulation Functions and the Corresponding Piecewise Constant Rate of Change Functions.....	134

Figure	Page
7.9	Another Piecewise Linear Accumulation Function and Piecewise Constant Rate of Change Function.....135
7.10	Thompson’s Recommendations can be Thought of in Terms of "Projecting Backwards"136
7.11	An Approximation of Ellis et al.’s Jactus Applet.....137
7.12	Ellis et al.’s Recommendations Focus on a Particular Topic (a Small Set of Ideas) within a Specific Course Built on a Particular Set of Assumptions for Students’ Background Knowledge140
8.1	Two Examples of Common Textbook Definitions for Exponential Function, Synthesized from Reviewing Ten College Algebra and Precalculus Textbooks Published between 2000 and 2012.....161
8.2	Design-Based Research’s Continuous Development Cycle165
8.3	Thompson’s (2013) Model of Meaningful Conversations (p.64).....173
8.4	A model of Communication between the Designer and Students in Online Courses175
8.5	A Task Given to University Precalculus Students taken from Madison, Carlson, Oehrtman, and Tallman (2015).....180
9.1	A Piecewise Constant Rate of Change Function and the Corresponding Piecewise Linear Accumulation Function where the Rate of Change Over Each Interval is Proportional to the Function’s Value at the Beginning of the Interval.....209

Figure	Page
9.2 Two Examples of Common Textbook Definitions for Exponential Function.....	212
9.3 Lisa’s Example of an Exponential Function.....	213
9.4 Measuring Relative Sizes.....	217
9.5 An “Elastic” Unit Ruler that Changes with the Size of a Varying Quantity so that the Measurement Remains Constant.....	220
9.6 A Dynamic Comparison of the Tree’s Height at Two Moments One Year Apart.....	221
9.7 Measuring the Height’s Magnitude on January 1 of this Year Using the Height’s Magnitude on January 1 of Last Year as the Measurement Unit...	224
9.8 Measuring the Magnitude of the Height on January 1 of this Year using One Percent of the Height on January 1 of Last Year as the Measurement Unit.....	225
9.9 Visualizing the Magnitude and Direction of a Change in Height.....	226
9.10 Measuring the Change in Height using One Percent of the Height on January 1 of Last Year as the Measurement Unit.....	227
9.11 The Measurement Process can be Visualized in a Graphical Representation of the Relationship between the Tree’s Height and Elapsed Time (in Years) since Some Chosen Reference Time.....	227
9.12 My Quantitative Structure for Understanding an Exponential Model.....	228

Figure	Page
9.13 A Task Designed to Support Students in Conceptualizing Necessary Properties of a Varying Length and Varying Ruler Size such that the Measurement is Constant.....	239
9.14 Students use a Sliding Interval to Visualize the Implications of a Constant Relative Size Measurement between Two Instances of a Quantity and to Estimate the Growth Factor’s Value.....	241
9.15 A Task Designed to Help Students Reason about Relative Size Measurements and Percentage Measurements between Two Instances of a Quantity and to Motivate Calculations for Evaluating These Measurements when the Quantity is Itself Measured in Some Fixed Unit...	242
9.16 A Task Designed to Help Students Understand the Meaning of Relative Size Measurements and Percentages, to Motivate Calculations for Evaluating these Measurements, and to Promote Imagery that Helps Students Reflect on the Reasonable of the Measurements they Determine.....	243
9.17 An Interactive Applet for Students to Reflect on the Relationship between One-Unit and Two-Unit Growth Factors and to Think about These as Measurements with Particular Attention Paid to what is being Measured and what is the “Unit Ruler”.....	244

Figure	Page
9.18 A Visualization to Help Support the Meaning of Growth Factors for Larger Intervals as Measurements and to Motivate Processes for Calculating the Values of Exponential Functions at Various Values of the Independent Quantity.....	244
9.19 The Muffler Task (Madison et al., 2015).....	250
9.20 Gina’s Work on the Muffler Task.....	251
9.21 The Sale Price Task (Madison et al., 2015).....	253
9.22 Lisa’s Response to the Sale Price Task.....	254
9.23 The Tomato Plant A Task (Madison et al., 2015).....	255
9.24 Marcus’s Work for the Tomato Plant A Task.....	256
9.25 Lisa’s Work for the Tomato Plant A Task.....	257
9.26 The Tomato Plant B Task (Madison et al., 2015).....	259
9.27 Shelby’s work for the Tomato Plant B Task.....	260
9.28 Students may Produce the Formula $h = 7 + 0.13t$ as a Literal Translation of What They Think They are being Asked to Represent	263
9.29 The Bacteria Doubling Task (Madison et al., 2015).....	264
9.30 Gina’s Initial Work for the Bacteria Doubling Task Prior to Erasing Her Formula	266
9.31 Gina’s Final Work for the Bacteria Doubling Task	267
9.32 Lisa’s Work for the Bacteria Doubling Task	268
9.33 The Version of the Unit Ruler Applet Used in the Post-Interviews (GeoGebra).....	277

Figure	Page
9.34 The Version of the Sliding Interval Applet Used in the Post-Interviews (GeoGebra).....	278
9.35 A Task Asking Students to Define Two Functions with the Given Ordered Pair Solutions.....	283
9.36 Shelby’s Answers to a Review Task at the Beginning of Lesson Five.....	285
9.37 Shelby’s Written Work on the Muffler Task in the Post-Interview.....	290
9.38 A Fictitious Student Example to Assess Marcus’s Meaning for Exponential Growth.....	291
9.39 Written Work for Gina’s Response to the Tomato Plant B Task in the Post-Interview.....	295
9.40 Three Representations of an Exponential Function Including John’s Annotation as He Discussed Them.....	300
9.41 A Question that Assesses Flexibility and Connections between Growth Factors over Different-Sized Intervals.....	303
9.42 Lisa’s Written Work on the Tomato Plant B Task During the First Part of the Post-Interview.....	306
9.43 Lisa’s Written Work on the Tomato Plant B Task During the Second Part of the Post-Interview.....	307
9.44 Lisa’s Diagram She Used to Justify Her Solution to the Muffler Task.....	308

CHAPTER 1

INTRODUCTION AND PROBLEM STATEMENT

Cobb (2007) argued that mathematics education “can be productively viewed as a design science, the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes” (p. 7). His perspective was not entirely new, however. For example, Thompson (1985) described the importance of considering learners as situated within a trajectory through a curriculum and wrote that the primary “aim of mathematics education is to promote mathematical thinking” (p. 189). A curriculum creates opportunities for students to construct a particular type of mathematical knowledge, under the teacher’s guidance, with the assumption that the knowledge students construct is never an exact copy of the knowledge the teacher intended they construct (or possesses herself). When the divergence appears large, revision and redesign are a natural part of crafting learning opportunities for students. Thompson (2013) demonstrated that a strong system of meanings is critical for teachers (and curriculum designers) and that effective lessons must *require* students to construct and apply a similarly strong system of meanings to be successful. A weak or incoherent system of meanings creates more space for students to construct incorrect or unhelpful meanings and for the teacher (or designer) to remain unaware of students’ constructions.

While there are many important open questions related to the teaching and learning of mathematics, one important research area seeks to describe a coherent set of powerful themes that guide student learning and teacher decision making throughout all levels of schooling and to describe powerful ways of understanding particular ideas that do work for students and fit productively into the larger coherent network of themes. For

example, Thompson (2008) suggested that mathematics from arithmetic through calculus can be organized into three general strands: the mathematics of quantity, the mathematics of variation, and the mathematics of representational equivalence. These themes can inform ways of understanding individual topics and guide instructional approaches while emphasizing deep connections across topics that reinforces a meaningful understanding of the larger themes.

In this dissertation study I sought to better understand how students develop productive meanings for exponential growth and related ideas within an online Precalculus course designed to build from and support reasoning about the key themes Thompson (2008) outlined. In the following pages I will outline my motivation for these decisions and specify my exact research questions.

Teaching and Learning Exponential Growth

Mathematics educators and mathematicians continue to debate the merits of covariational images of function relationships compared to correspondence images for grade school and undergraduate learners. For both students and teachers, understanding exponential functions seems particularly challenging regardless of how they conceptualize function relationships. An image of exponential growth grounded in repeated multiplication does not generalize well to evaluating an exponential function over its entire domain and students (and teachers) with such an image struggle to recognize situations that should be modeled with an exponential function and to understand the connection between their image and the closed-form representation of exponential functions (Davis, 2009; Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016;

Presmeg & Nenduardo, 2005; Ström, 2008). A focus on repeated multiplication may overlook supporting images of growth factors as measurements of relative size and ignores issues of continuity and interpolation in evaluating the function at non-integer inputs. On the other hand, situations well-modeled by exponential growth are often described discretely (such as providing an annual percent change) and developing smoothly continuous reasoning for a function relationship with these characteristics requires coordinating several sophisticated ideas and may not support the development of common formulas.¹

The Common Core State Standards for Mathematics encourage reasoning about exponents and exponential functions. Standards related to these topics are spread across multiple domains and throughout the middle school and high school grade bands (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). A selection of these standards are listed below.

CCSS.MATH.CONTENT.8.EE.A.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.

CCSS.MATH.CONTENT.HSF.LE.A.1.A Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

CCSS.MATH.CONTENT.HSF.LE.A.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

CCSS.MATH.CONTENT.HSF.LE.A.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

This emphasis is consistent with the important role exponential function reasoning plays in calculus, differential equations, complex analysis, modeling, interpreting common

¹ I explain these ideas in more detail in Chapters 2, 3, and 5.

measurements such as pH, decibel levels, and the Richter scale, and understanding the relationships between measurements related by orders of magnitude (Castillo-Garsow, 2010; Confrey, 1994; Ellis, Ozgur, Kulow, Williams, and Amidon, 2012, 2015; Weber 2002, 2002a). While these standards generally describe using mathematical notation to represent products with repeated factors or the values of an exponential function, they do not explicitly outline the imagery associated with powerful understandings of exponential growth, useful ways of understanding the meaning of a constant growth factor beyond its use in calculating values, or how to support students in thinking about continuous exponential growth over some domain.

By their nature, standards documents often lack this kind of support since they are usually designed to be lists of performance objectives for students at different ages. Without debating what standards should or should not address, the lack of such guidance is especially problematic for exponential growth where the standards authors tacitly convey the belief that understanding properties of natural exponents leads non-problematically to understanding exponential growth and exponential models in general. This assumption is not true for many students.² For example, only about 35% of college precalculus students who took Carlson, Oehrtman, & Engelke's (2010) multiple choice Precalculus Concept Assessment at the end of their courses could identify the difference between the growth rates for the functions $p(t) = 7(2)^t$ and $p(t) = 7(3)^t$ (Research and Innovation in Mathematics and Science Education, 2007). Weber (2002, 2002a) provides further evidence of students' impoverished understandings of exponential functions. He

² The data described in the following examples was collected prior to the Common Core State Standards. However, exponential functions and related ideas such as percent comparisons and percent change have been a key part of states' grade school standards for decades.

found that most college algebra and precalculus students receiving rules-based instruction could not provide an adequate justification for exponential and logarithmic properties only a few weeks after the topics were covered in class, nor could these students provide a rationale for why functions like $f(x) = (\frac{1}{2})^x$ are decreasing. Finally, Table 1.1 shows U.S. 12th grader performance on several assessment items from the National Assessment of Educational Progress (National Center for Education Statistics). The results indicate that U.S. students are not developing useful and lasting meanings for growth patterns and percent change typically associated with exponential functions. But difficulties in understanding exponential growth are by no means limited to students. Ström (2008) and Davis (2009) reported that many teachers' meanings for exponential functions and related ideas are similarly weak and unproductive. They both note that few studies exist exploring the link between teachers' meanings for exponential growth and the meanings that students construct from classroom experiences.

It is common for teachers and textbooks to define an exponential function as a relationship of the form $f(x) = ab^x$ for real numbers a and b and to discuss evaluating the function at integer inputs using a metaphor of repeated multiplication. For example, the value of $f(3) = ab^3$ could be thought of as “ a times the product of $b \cdot b \cdot b$ ” which is consistent with a meaning of exponents students encounter in earlier grades. However, thinking of exponential growth in terms of repeated multiplication becomes more challenging when evaluating a function for rational inputs, such as $f(3.2) = ab^{3.2}$. One must conceptualize rational exponents and their properties in order to assimilate the idea of repeated multiplication to rational inputs like reinterpreting $ab^{3.2}$ in a form such as $a(b^{1/10})^{32}$ where the repeated multiplication involves the factor $b^{1/10}$ instead of b . Such a

Table 1.1

Results from the National Assessment of Educational Progress (12th Grade)

<p>2009 Assessment [M1899E1]</p> <p>The population P of a certain town is given by the equation $P = 50,000(1 + r)^t$, where r is the annual rate of population increase and t is the number of years since 1990.</p> <p>(a) What was the population in 1990? Answer: _____</p> <p>(b) In 2001 the population was 100,000. What was the annual rate of population increase? Answer: _____</p> <p>Incorrect: 38% of students Partial 2: 1% of students Partial 1: 46% of students *Correct 9% of students Omitted: 8% of students Off task: 2% of students</p>
<p>2005 Assessment [M133801]</p> <p>A car costs \$20,000. It decreases in value at the rate of 20 percent each year, based on the value at the beginning of that year. At the end of how many years will the value of the car first be less than half the cost? Answer: _____ years Justify your answer.</p> <p>Incorrect: 60% of students Partial 2: 5% of students Partial 1: 5% of students *Correct 26% of students Omitted: 4% of students</p>
<p>2005 Assessment [M127001]</p> <p>The number of bacteria present in a laboratory sample after t days can be represented by $500(2)^t$. What is the initial number of bacteria present in the sample?</p> <p>A. 250 [10% of students] *B. 500 [33% of students] C. 750 [5% of students] D. 1,000 [46% of students] E. 2,000 [4% of students] [2% of students did not answer]</p>

conceptualization requires a constantly updating recognition of the base for exponentiation. Even this modification, however, is inadequate to justify evaluating $f(x)$ for irrational x or for conceptualizing how the algebraic representation $f(x) = ab^x$ encapsulates the relationship between all values in the domain and range.³ This helps account for Ellis et al.'s (2012) observation that “exponential growth appears to be challenging to represent for both students and teachers, and it is difficult for teachers to both anticipate where students might struggle in learning about exponential properties and develop ideas for appropriate contexts that involve exponential growth (Davis, 2009; Weber, 2002)” (p.94).

Online Learning

Nearly 30% of students at post-secondary institutions in the Fall 2015 semester enrolled in at least one distance-learning course (predominantly via online delivery), and both the number of post-secondary students enrolled in any distance-learning courses and the number of post-secondary students enrolled exclusively in distance-learning courses have consistently increased each year even while the overall post-secondary enrollment decreased (U.S. Department of Education, National Center for Education Statistics, 2013, 2014, 2015, 2016, 2017, 2018). See Table 1.2. Most colleges and universities explain their growing distance-learning course offerings in terms of providing a public service, namely increasing access to university resources and programs. In addition, they cite student demand for such courses as a key factor in expanding distance-learning programs.

³ Functions like $g(x) = 2x+1$ don't necessarily have the same issue. For example, I can think of the function's value as twice the value of x for all real numbers x .

Table 1.2

Post-Secondary Enrollment in Distance Learning Courses

semester	students enrolled in post-secondary courses	students enrolled in any post-secondary distance-learning course	post-secondary students enrolled exclusively in distance-learning courses
Fall 2012	20,642,819	5,444,701 (26.38%)	2,638,653 (12.78%)
Fall 2013	20,375,789	5,522,194 (27.10%)	2,659,203 (13.05%)
Fall 2014	20,207,369	5,750,417 (28.46%)	2,824,334 (13.98%)
Fall 2015	19,977,270	5,954,121 (29.80%)	2,871,788 (14.38%)
Fall 2016	19,988,204	5,965,813 (29.85%)	2,874,870 (14.38%)

Amidst this growth in distance learning, Xu and Jaggars (2011, 2013) completed two large-scale studies that challenged an assumption at the heart of online learning: that the quality of educational opportunity in an online environment is approximately equivalent to in-person environments. They found that, all other factors being equal, the average post-secondary student’s likelihood of completing an online course is seven percent less than his likelihood of completing a comparable face-to-face course and that, even if the student completes the course, the average grade is 0.3 grade points lower. Among students from disadvantaged populations the numbers are even more skewed. Moreover, students enrolled in online courses in their study were, on average, better prepared for the courses than students taking in-person options and reported higher levels of self-motivation. Thus, “descriptive comparisons are likely to underestimate rather than overestimate the gap between online and face-to-face performance outcomes” (Xu & Jaggars, 2013, p. 55). These recent results are more damning than meta-analyses finding no significant impact of the learning environment on student success in the infancy of online learning (Swan, 2003; Twigg, 2003; Means, Toyama, Murphy, Bakia, & Jones, 2009).

Based on these findings, is it irresponsible for colleges and universities to continue expanding, or even offer, distance-learning options? There is little doubt that distance-learning courses provide an important service to students by providing them with options for completing coursework even if they do not live close to schools or have responsibilities that preclude regular attendance in scheduled class sessions. Distance-learning also allows colleges to potentially recruit more students and provide additional pathways for students to complete their degree requirements in a timely manner. Rather than throw the baby out with the bathwater by eliminating such programs, post-secondary institutions must examine the quality of learning opportunities in their distance-learning courses. Xu and Jaggars (2013) describe qualitative research comparing online courses with their in-person equivalents. In the majority of cases professors creating an online course convert the comparable face-to-face courses to online versions without changing the approach or content. The activities and homework remain the same while in-class lectures are converted to video lectures or text. Even ignoring the initial quality of the in-person lessons, not redesigning the course with the format in mind is professionally questionable. However, most colleges and universities do not require professors to attend training in online course design even when highly skilled support professionals exist on campus (Xu & Jaggars, 2013).

For researchers who want to study the effects of online learning environments, or for colleges and universities who want to continue to offer or expand their online course offerings, it is vital that they first develop, study, and refine courses designed explicitly to exist in an online environment. Furthermore, it is important that these courses are not designed around ineffective passive activities where students are expected to watch

lectures, take notes, and copy the methods in rote practice problems. Such methods do not work well for in-person instruction and seem to be even less effective for online courses (Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt, & Wenderoth, 2014; Xu & Jaggars, 2013).

Summary of Problem Statement and Research Questions

Ellis et al. (2012) call for research to improve our “understanding of how to foster students’ learning about exponential growth, and for identifying more effective models of instruction on exponential functions” (p. 94). I believe that the first step in this process is to complete a detailed conceptual analysis (Glaserfeld, 1995; Thompson, 2008) on the topic and test and refine a hypothetical learning trajectory (Simon, 1995). This provides a foundation so that future research builds from clearly stated learning goals relative to the images and mathematical reasoning we hope students construct. A shared vision, or at least a set of common understandings, can help all researchers working on the problem communicate clearly with one another and better frame their contributions within the mutual goal of improving the quality of instruction and, consequently, student learning about exponential functions. By situating my research within an online Precalculus course, I hoped to contribute to new ways of designing distance-learning opportunities for students that support students in successfully constructing powerful mathematical understandings instead of undermining the quality of their mathematical experiences. Answering the following research questions contributed to these goals.

RQ1: What meanings do students have for exponential growth and related ideas before and after completing an online Precalculus course design to develop and leverage quantitative and covariational reasoning?⁴

RQ2: What features of the online course, specific lesson activities, and/or components in the hypothetical learning trajectory appear to support or hinder students in developing productive meanings for exponential growth and related ideas?

A Comment on the Three-Paper Format for Dissertation Reports

This dissertation report follows the three-paper formatted approved by ASU's graduate college. Chapters 6-10 include these papers in their entirety as well as an introduction and conclusion focusing on common themes within the papers and discussing their contributions to the field. Since these papers are intended to stand on their own for submission to publications, it was often necessary to repeat sections of Chapters 1-5 within Chapters 6-10. Therefore, the reader may notice duplicated text sections and/or duplicated tables and figures across these chapters. In addition, the reference list for each paper is contained within the chapter containing that paper (Chapters 7-9). The reference list at the end of the dissertation document covers Chapters 1-6, 10.

⁴ See Chapter 2 for a detailed description of quantitative and covariation reasoning.

CHAPTER 2

THEORETICAL PERSPECTIVE

[W]e have found it necessary to attribute mathematical realities to students that are independent of our own mathematical realities. By “independent of our own” we mean that we attribute mathematical concepts and operations to students that they have constructed as a result of their interactions in their physical and sociocultural milieu. In this attribution, we remain aware that we may not, and probably cannot, account for students’ mathematics using our own mathematical concepts and operations. Although our attribution of mathematical realities to students is a conceptual construct, it is grounded in the mathematical realities of students as we experience them. (Steffe & Thompson, 2000, p. 267)

Background Theory

A *background theory* (or *theoretical perspective*) serves “to constrain the types of explanations we give, to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem” (Thompson, 2002, p. 192). It provides a lens to focus a researcher’s attention on a subset of the variables present in any research setting (Cobb, 2007; Tallman, 2015; Thompson, 2000, 2002). Background theories provide guidance on what aspects of the learning process must be explained, inform the characteristics of those explanations, and perhaps give insights into features of experiences that might effectively guide students in developing powerful mathematical meanings, but they do not specifically address “ways of thinking, believing, imagining, and interacting that might be propitious for students’ and teachers’ mathematical development” (Thompson, 2002, p. 194). This is the role of *domain-specific theories* such as quantitative reasoning, which I will discuss later in this chapter.

Piaget's Genetic Epistemology

Piaget's genetic epistemology was an attempt in part to explain the origins and substance of knowledge within biological entities who cannot directly access any kind of external reality (Piaget, 1977). An individual organizes her experiences within schemes that include triggers (stimuli that *assimilate* to the scheme), *actions* associated with the stimuli, and some expectation of what outcome the action(s) will produce as a progression towards a goal or need not currently met (Glaserfeld, 1995; Piaget, 1971; Piaget & Inhelder, 1969). Actions include "all movement, all thought, or all emotions that respond to a need" (Piaget, 1967, p. 6). *Interiorization* is the process of reconstructing and organizing these actions so that an individual can reproduce the actions mentally without needing to repeat the corresponding physical actions while *internalization* describes an assimilation of a situation to a scheme with all of the meanings conferred by that scheme (Thompson, 1994c).

To assimilate to a scheme is to provide meaning and the scheme's entire contents, implications, inferences, connections, and actions encompass that meaning (Johnckheere, Mandelbrot, & Piaget, 1958). All experiences involve attempts to assimilate stimuli to a scheme. However, sometimes the actions associated with the scheme do not produce an outcome that the individual expects or he becomes aware of discontinuities in his scheme and the features of the situation. This causes a *perturbation* for that individual (or a state of *disequilibrium*) and "the original scheme is *accommodated* [emphasis added] by differentiating between conditions and subsequent implications of assimilation" (Thompson, 1994c, p. 183), a new scheme develops, or the individual resolves the issue in other ways (such as ignoring it) to achieve *equilibrium*.

Each action carries with it a context in which the action takes place, the action itself, and the action's product. It is possible for an individual to *differentiate* the action from its context and product and to project the action to a level of mental representation (Tallman, 2015; Thompson, personal communication, March 14, 2015). Both Thompson and Tallman used the sine function to exemplify the process. When a student examines a graph of the sine function, he might pick key points and know that these somehow relate to locations on a circle. However, he might not recognize a varying angle measure and might not differentiate the act of moving along the circle from defining a specific value for sine. However, if the student's attention is drawn to coordinating the actions of moving along a circle to trace out an arc subtended by an angle and to then measuring a vertical distance in units of the radius, then he can eventually differentiate the action from the result and create a mental representation of the coordinated actions. At this point he no longer *needs* the physical actions associated with evaluating the sine function to know that a sine value exists for every angle measure and to understand the value's meaning.

Once the action is represented mentally, the individual can reorganize his representations and make connections among projected actions. For example, he might coordinate projections of the sine function, cosine function, and arc length to create more robust connections between the schemes for each idea (Tallman, 2015; Thompson, personal communication, March 14, 2015). The entire process of differentiating, projecting, and coordinating actions is *reflecting abstraction*, and operating with this coordination at the level of representation is *reflected abstraction* (Piaget, 1977;

Thompson, personal communication, March 14, 2015).⁵ Reflecting abstraction is the engine that drives productive accommodations in a person's schemes and "results in increasingly organized and refined cognitive schemes...[that inject] coherence into systems of organized actions" (Tallman, 2015, p. 72).

Harel's Duality, Necessity, and Repeated Reasoning (DNR) Framework

Harel (2007, 2008a, 2008b, 2008c) and Harel and Sowder (1998, 2005) outlined a framework for mathematics instruction and learning utilizing Piaget's genetic epistemology as a background theory. The DNR framework emphasizes the role of duality, necessity, and repeated reasoning in learning and teaching mathematics.

Duality. A *mental act* describes cognitive activities "such as interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving" (Harel, 2008c, p. 2). *Products* are the results of engaging in a mental act and are often associated with observable behaviors, and each mental act has certain cognitive *characteristics*. Harel (2007, 2008a, 2008b, 2008c) calls the product and character of a mental act a *way of understanding* and a *way of thinking* respectively and insists that the two shape and define each other.⁶ Harel

⁵ It is possible that the individual does not differentiate an object's property generated from some action from the action itself. Piaget (1977) calls this *pseudo-empirical abstraction*. Tallman (2015) provides an excellent example. A student might produce an angle by rotating a terminal ray, and during this process an arc is traced out some distance from the angle's vertex. The individual generates the subtended arc through the action of rotation, and if he does not see a property of subtended arc length in angles not produced from his own action of rotating a terminal ray then his notion of subtended arc length is a pseudo-empirical abstraction. Evidence of reflecting abstraction includes a student's ability to imagine the process of recreating a given angle through rotation to talk about subtended arc length without performing the action.

⁶ I am aware that Thompson and Harel have modified some key constructs within the DNR framework in upcoming papers. I chose not to address those modifications here since I did not have access to the articles.

commonly used the example of an empirical proof scheme⁷ to demonstrate his meaning. An empirical proof scheme is a way of thinking associated with the mental act of proving. It shapes the products generated by the mental act (proofs based on a finite number of examples). However, experience justifying an argument with finitely many examples is critical in the original development and strengthening of the way of thinking.

Necessity. For Harel, problem solving is the catalyst for all learning. In order to learn (that is, for an individual to modify existing schemes or develop new schemes), there must be a problem that the individual cannot solve using her current knowledge (schemes). Here “problem” is a highly subjective notion (Harel, 2013). For example, mathematicians conceptualized the Fundamental Theorem of Calculus as a solution to the challenge of bridging rate of change and accumulation problems (Thompson, 1994a). College students taking introductory Calculus courses rarely see the theorem as a solution to a particular problem they have conceptualized. Instead, it is often just another rule to memorize. A problem conceptualized by the individual defines an *intellectual need*, and Harel contrasted intellectual needs from *psychological needs*, which are the “motivational drives to initially engage in a problem and to pursue its solution” (Harel, 2008b, p. 898). He argued that psychological needs are rarely addressed in domain-specific theories but are quite important. Students uninterested in engaging with a mathematical situation or persisting in the face of challenges will not progress in modifying their schemes through repeated reasoning that supports reflecting abstraction.

⁷ An empirical proof scheme describes the belief (not necessarily conscious) that, when proving a mathematical fact or theorem, a finite number of valid examples is sufficient justification.

Repeated reasoning and what it means to reason. Reasoning is “the process of forming conclusions, judgments, or inferences from facts or premises; or the reasons, arguments, proofs, etc., resulting from this process” (dictionary.com, 2017). Therefore, an adequate description of reasoning (particularly in mathematics) involves an explanatory theory for how a person interprets a task at hand, both in terms of the information present and a goal, and moves towards that particular goal. Genetic epistemology (Piaget, 1971, 1977), expanded as radical constructivism (Glaserfeld, 1995), provides a useful perspective on reasoning.

If reasoning is “the process of forming conclusions, judgments, or inferences from facts or premises,” then when an individual encounters what he perceives to be a set of premises or facts, the triggered scheme provides those premises with meaning. This scheme is an organization of actions, expected outcomes, goals, memories of prior experiences, and even other schemes. The individual carries out the actions associated with the triggered scheme to accomplish the perceived goal. This is consistent with describing a way of knowing as “The particular meaning students give to a term, sentence, or text, the solution they provide to a problem, or the justification they use to validate or refute an assertion” (Harel & Sowder, 2005, p. 29). Assimilation to a scheme and connecting the task at hand to ways of knowing tied to the triggered scheme describes mechanisms behind “the process of forming conclusions, judgments, or inferences from facts or premises” in general while respecting the internal and individual character of reasoning.

Harel (2008c) wrote that “Students must practice reasoning in order to internalize, organize, and retain ways of understanding and ways of thinking” (p. 21). Note that what

Harel meant by “practice” is very different from common student activities in mathematics courses. In most classrooms students practice computations presented to them in the form of algorithms, formulas, and theorems. Even if students develop proficiency with these computations they tend to develop a view of mathematics as sets of rules and procedures to learn emphasizing accurate calculations. An alternative view is that learning involves developing justifications for mathematical actions and the student’s role is to produce his own rules and algorithms while developing more general ways of thinking (Harel & Sowder, 2005). Cooper (1991) coined the phrase *repetitive experience* to describe the latter as a way of contrasting these two beliefs about mathematics.

To Cooper, talking about practice included a tacit assumption that what students practiced (from the teacher’s perspective) was in fact what they learned (Thompson, 1991). However, as Thompson pointed out, this assumption betrays an observer-centric focus because what students actually learn from practice often differs from teachers’ intentions. Using the phrase *repetitive experience* serves as a reminder that “what we want repeated is the constitution of situations in ways that are propitious for generalizing assimilations, accommodation, and reflection” (Thompson, 1994c, p. 227). In developing and refining their schemes, students must have residual memories of having engaged in some specific reasoning before. Stable understandings that persist beyond momentary insights or intuition require repeated constructions of the same or similar reasoning (Thompson, 2013). This process of repeated construction is critical for the processes of internalization and reflecting abstraction (Cooper, 1991; Harel, 2008c, 2013; Thompson, 1991, 1994c, 2013). As stable understandings develop, students no longer have to

reconstruct particular reasoning each time they encounter situations assimilated to that scheme. Instead, they can fluently and effortlessly apply their knowledge (Harel, 2008a).

What Does it Mean to Understand?

From a constructivist standpoint, both *meaning* and *understanding* refer to the components of an individual's scheme (Thompson, 2013). An example involves the problem statement in Figure 2.1.

John leaves Phoenix in a car at 9:30 am to drive to San Diego (a distance of 355 miles) for a meeting with a business client. His meeting is at 1:00 p.m. Will John arrive at his meeting on time?

Figure 2.1. A task I assimilate to my average rate of change scheme.

When I read this problem statement, I interpret it as a question about average rate of change, meaning that I assimilate the context to my scheme of average rate of change, which includes a network of actions, expected results of acting, and related schemes (such as schemes for speed and constant speed). To respond to the question I would determine the constant speed John would need to travel to cover exactly 355 miles in exactly 3.5 hours and compare this result to personal experiences of driving, knowledge of speed limits, and/or an understanding of a car's physical limitations. However, none of these issues are inherent to the problem – I bring all of this to table via the schemes to which I assimilate the problem statement. When someone else reads the prompt, there are a number of possibilities including 1) the individual does not have a scheme for average rate of change and thus assimilates the problem statement to some other scheme, 2) the individual assimilates the context to his scheme for average rate of change, but his scheme is organized differently from mine and includes different actions, expectations,

and related schemes and thus he reasons about the context in an entirely different way, or 3) the individual possesses a scheme for average rate of change but this context is not assimilated to that scheme (for that individual, this is not a question about average rate of change at all). The main point is that describing specific problems or contexts as being “about” any particular mathematical idea is inappropriate. If I say the context in Figure 2.1 is a problem about average rate of change it means that I have assimilated the context to my average rate of change scheme, and my actions and reasoning are dependent on my scheme’s contents and structure. It does not mean that the context triggers an average speed scheme for another person.

This creates two primary challenges in trying to describe what a student understands or the meanings she constructs. First, without direct access to anyone else’s thoughts, any claims about a person’s understanding derive from models of that person’s schemes constructed based on her observable actions (Glaserfeld, 1995; Steffe & Thompson, 2000; Thompson, 2013). For example, presenting the problem statement from Table 3 to a student and observing the student’s actions, responses, comments, body language, and so on provides evidence for generating an initial model of the student’s triggered schemes. Ideally a researcher refines his initial model through additional tasks, especially questions that probe at the modeled scheme’s limits. The best working model is one that appears to explain all of the subject’s observable actions, but it is important not to mistake the model of the scheme with the scheme that exists in the individual’s mind. Second, even choosing a set of ideas to research (like exponential functions) betrays a researcher-centered orientation because there is no guarantee that students will assimilate any of the activities to a scheme the researcher would recognize as

encompassing exponential growth as he understands it. Therefore, a researcher must constantly be attentive to how students appear to have understood a given task or activity.

Domain-Specific Theories

Background theories provide guidance on what aspects of the learning process must be explained, inform the characteristics of those explanations, and perhaps give insights into features of experiences that might effectively guide students in developing powerful mathematical meanings, but they do not specifically address “ways of thinking, believing, imagining, and interacting that might be propitious for students’ and teachers’ mathematical development” (Thompson, 2002, p. 194). This is the role of domain-specific theories such as quantitative reasoning. For example, genetic epistemology as a background theory does not provide explicit insights into how a person might develop productive meanings for an accumulation function. Mathematics education researchers must analyze examples of individuals engaging in mathematical reasoning as well as generate and test hypotheses about the kinds of reasoning that will benefit students’ mathematical development to create domain-specific theories that do address such issues (Thompson, 1991, 2002).

In the following sections I describe quantitative and covariational reasoning, two interconnected domain-specific theories related to the development of algebraic and functional reasoning. They are theories designed to help “describe mathematical understandings we hope students will have and to describe understandings they do have, and [to describe] how students might express their understandings in action or communication” (Thompson, 2002, p. 195). Radical constructivism and genetic

epistemology provide the background framework for quantitative and covariational reasoning and thus dictate the commitments to which their tenets must adhere. In particular, the researchers who developed these theories take seriously the following features of radical constructivism and genetic epistemology (Thompson, 2002). 1) Humans have no direct access to any reality external to themselves. They can only access stimuli from their senses resulting in neurological activity that they can isolate from the stream of such activity, re-present from memory, or imagine. 2) A person does not have direct access to the thoughts or beliefs of any other person. 3) Anything someone “learns” from experience is based on her individual constructions even when an outsider attempts to direct these experiences.

Quantitative Reasoning

Engaging in *quantitative reasoning* involves conceptualizing a situation to form a quantitative structure that organizes relevant *quantities* and *quantitative operations* (Thompson, 1988, 1990, 1993, 1994c, 2011, 2012). A *quantity* is a measurable attribute of some object that exists in an individual’s mind, and the individual’s conceptualization of quantities he deems relevant generates a space of implications for his reasoning within any given mathematical situation (Smith and Thompson, 2007; Thompson, 1994c). For example, Moore and Carlson (2012) asked students to produce a graph relating a person’s total distance traveled while riding a Ferris wheel and the rider’s height above the ground. Most students’ graphs displayed incorrect concavity, leading Moore and Carlson to conclude that those students did not conceive of the quantities *amount of change of the*

total distance traveled and the *amount of change of the rider's height above the ground*, and therefore could not account for them when producing their graphs.

Quantification schemes are vital for truly conceptualizing a quantity because “[i]t is in the process of quantifying a quality that it becomes truly analyzed” (Thompson, 1990, p. 5). Initial conceptions of a quantity might involve *gross quantification* (Piaget, 1965; Saldanha & Thompson, 1998), or what Thompson, Carlson, Byerley, & Hatfield (2014) described as *awareness of size*, whereby the attribute's size is loosely understood relative to experiential observations. For example, a person might conceptualize the amount of force it takes to move an object in a way that allows her to recognize larger or smaller force requirements, but she might not have a quantification scheme capable of producing consistent numerical values to represent a force's magnitude.

Extensive quantification is the most elementary example of “the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship...with its unit” (Thompson, 2011, p. 37) and describes quantification schemes for directly measurable quantities (such as length) using any unit that has “the same nature as the quantity being measured” (Wildi, 1991, p. 58). The measurement scheme's sophistication can range from a straightforward process of reproducing the quantity's magnitude by iterating the unit to an appreciation of the reciprocal relationship between the quantity's measurement in some unit and the size of the unit along with an understanding that the quantity's magnitude is independent of the choice of unit (Thompson et al., 2014).

As an individual conceptualizes various quantities in a situation it is possible for him to recognize relationships between these quantities and to identify “a new quantity in

relation to one or more already conceived of quantities” (Thompson, 1994c, p. 185). A quantity of this type is an *intensive quantity* formed by conceptualizing a *quantitative operation*. A quantitative operation describes the mental operations of comparison between and/or coordination of previously conceptualized quantities (Thompson, 1990, 1994c, 2011). The result is a new quantity that cannot be directly measured. Instead, its quantification scheme depends on the quantitative operation from which it arose (Johnson, 2014; Moore, 2010; Piaget, 1968; Schwartz, 1988; Simon & Placa, 2012; Thompson, 1990; Thompson et al., 2014). Examples of intensive quantities include multiplicative comparisons, rates of change, and per-capita measurements.

Covariational Reasoning⁸

An individual may draw distinctions between quantities that vary and those that remain constant when conceptualizing a situation. If she sees the situation as composed of quantities that change together and attempts to coordinate their variation, then she is engaging in *covariational reasoning* (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Like quantitative reasoning, covariational reasoning is about how an individual conceptualizes a situation, and the manner in which an individual thinks about variation and covariation has implications for how she understands function relationships, representations, and connections between representations.

⁸ Note that the definition and descriptions I provide here are not universally accepted by all researchers who use the term “covariational reasoning” in their work. However, this description is consistent with quantitative reasoning as explained in the previous pages. See Chapter 3 for more information about alternative definitions of covariational reasoning.

Sophisticated covariational reasoning involves linking two continuously varying quantities to create a *multiplicative object*, a unification that combines the attributes of both quantities simultaneously (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017). This coordination often begins with respect to experiential time as exemplified by Johnson (2012) and explained by Thompson (2012). Students in Johnson's study coordinated the varying volume and height of water in a bottle by experiencing a corresponding change in each relative to experiential time, in this case the time that elapses as someone poured an amount of water into the bottle. Eventually a student can separate her physical experiences and her conceptualization of the relationship so that she sees time as a measurable quantity distinct from either volume or height. Instead of changes in experiential time causing variation in these quantities, she links the quantities' magnitudes via *conceptual time*, meaning that she anticipates that at all moments there will be values for the height and volume of water in the bottle, that corresponding values are inextricably linked in a specific way, and that if she thinks about changes in one quantity (over some imagined interval of time) then she is explicitly aware of corresponding changes in the other quantity over the same interval of imagined time (Thompson, 2011; Thompson & Carlson, 2017).

Quantitative Reasoning and Function Representations

The way in which an individual conceptualizes variation and the degree to which he has formed a multiplicative object by uniting two covarying quantities in conceptual time have implications for his understanding of function representations, connections between representations, and meanings for algebraic representations and arithmetic

calculations. However, discussing multiple representations of functions is problematic if we emphasize representations independent of the individual conceptualizing what the representations are meant to model (Moore & Thompson, 2015; Thompson, 1994b). Thompson (1994b) said put it best when he wrote that “the core concept of ‘function’ is not *represented* by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance” (p. 36). Conceptualizing quantitative structures and coordinating changes in quantities within these structures is foundational for building rich and powerful images of quantitative relationships so that issues of representation are natural extensions of an individual’s reasoning in context and connections between representations derive from his understanding of an invariant relationship characterized in different ways. Repeated reasoning about relationships within the quantitative structure provides the means for writing arithmetic or algebraic expressions representing the value of a quantity with respect to values of other quantities in the structure (Thompson, 2011).

Thompson and Carlson (2017) argued that an awareness that something remains invariant even as two quantities’ values change in tandem is the foundation for a productive meaning of *function* based on “reasoning about constrained variation” (p. 449). Coordinating continuous quantitative covariation is different from examining a table of values or plotting a few points and connecting them with a smooth curve. It involves specific attention to what one is measuring and, when tracking how the quantities change together, attending not just to the values before and after variation but *at all moments during the variation*. If someone has formed a multiplicative object composed of two covarying quantities and conceptualizes a graph as an emergent trace of

the relationship between the quantities' values, then any meanings or generalizations he conceptualizes are properties of the covariation and are independent of the specific representations being generated (Moore & Thompson, 2015). This is a vital way of thinking that may allow someone to connect a relationship conceptualized quantitatively, the corresponding values of those quantities produced through quantification and numerical operations, an algebraic representation relating sets of corresponding measures, and the graph produced by allowing the quantities to covary while mentally linked as a multiplicative object. A table of values then may represent specific pairs of values determined by "freezing" the covariation some finite number of times and noting the quantities' corresponding values.

Perhaps the most challenging connection among traditional function representations is between algebraic formulas and graphs developed as emergent traces of covariation. Formulas often reinforce correspondence imagery of the type commonly represented by the "input-output" function machine. Ellis (2007, 2011) demonstrated that attention to conceptualizing quantitative structures facilitated students in flexibly viewing an algebraic formula relating numbers of rotations for different gears as modeling both a correspondence rule as well as a description of how the quantities changed together. Thompson (1994b) and Oehrtman, Carlson, and Thompson (2008) wrote that this flexibility requires a process conception of the algebraic formula. An individual with a process view of a formula is capable of seeing it as more than a means for producing a result based on prescribed calculations. The individual possesses a confidence that the formula represents the value of one quantity in terms of the values of other quantities, that the formula is self-evaluating (in the sense that the value exists independent of

actually needing to perform the calculations), and that if she runs through a continuum of values in the function's domain that the formula produces all corresponding function values. In this way a rule of correspondence can be integrated into a covariational image of the function as the rule that relates the values of quantities joined in a multiplicative object as they vary in tandem.

To summarize, an individual must first conceptualize quantities in a situation, organize them into quantitative structures, and develop quantification schemes in order to produce and make sense of numerical values representing the quantities' measures. Reasoning about relationships within the quantitative structure then suggests reasonable arithmetic operations to evaluate intensive quantities and, eventually, algebraic expressions to represent one quantity' value with respect to the values of other quantities. Uniting two quantities in thought and coordinating their values as they change together allows someone to develop graphical representations of the relationship as an emergent trace, and generalizations derived from the experience are likely to be conceptualized as properties of the covariation independent of any one particular representation. In addition, conceptualizing the quantitative structure and reasoning covariationally is the foundation for a productive meaning of function as constrained covariation and is the meaning behind the "something" being represented by tables, formulas, and graphs.

The Role of Theory in this Particular Study

In this dissertation study I will rely on the background theories described in this chapter in two specific ways: 1) as a theoretical lens and 2) as a guiding principle for task

and lesson design along with domain-specific theories related to learning about functions and exponential growth.

Constructivism as a Theoretical Lens

Conceding that students' mathematical realities differ from my own means that, at best, I can only create models for students' schemes based on observable behavior. Since this dissertation study does not entail a highly interactive teaching experiment, my opportunities to probe individual students' schemes may be limited.⁹ Therefore, the intervention itself must be designed to elicit as much potential evidence of how students assimilate tasks and how their schemes are structured. One method will be to leverage a careful conceptual analysis of the ideas I want to promote through the intervention to reveal places where students could construct unintended meanings for these ideas. I can then structure questions and/or responses to questions students can select that might indicate characteristics of various categories of ways of knowing. A second method is to analyze monitoring data (like average number of attempts and average time spent on each attempt) relative to cornerstone tasks in the intervention for evidence of the ideas students struggled with. This can generate hypotheses for common ways of knowing students possessed coming in to the course and/or developed in earlier activities. Analyzing homework and assessment data might then indicate accommodations students made to their schemes during the intervention. A failure to attend to the issues described (such as assessing students on procedural skills and assuming that correct responses indicated certain scheme structures) could result in a lot of data that reveals few useful conclusions.

⁹ See Chapter 5 for specific details of my proposed study.

Constructivism as an Instructional Design Principle

Knowing that students must construct their own meanings and knowing some of the mechanisms by which this construction occurs via individual activity suggests the kinds of tasks and activities that can provide opportunities for students to construct productive mathematical meanings. Thompson (1985) provides a compelling list of key design principles for curricula and a persuasive argument for each grounded in radical constructivism. His list is provided below, followed by a summary for why each principle is included.

A mathematics curriculum should

1. Be problem based.
2. Promote reflective abstraction.
3. Contain (but not necessarily be limited to) questions that focus on relationships.
4. Have as its objective a cognitive structure that allows one to think with the structure of the subject matter.
5. Allow students to generate feedback from which they can judge the efficacy of their methods of thinking. (Thompson, 1985, p. 200)

Mathematical “problems” do not exist outside of the individual. The problem only exists in the manner in which it is conceptualized by that individual. When Thompson (1985) says that a mathematics curriculum should be based on problems, what he means is that a designer “must make explicit the nature of the knowledge that [he hopes] is constructed and make a case that the chosen activities will promote its construction” (p. 192). Tasks should be chosen so that their completion is not routine for students, and furthermore the tasks should elicit and call attention to students’ patterns of reasoning while working through them. This provides the potential for students to make abstractions by reflecting on what remains invariant across a variety of problems and to differentiate, project, and coordinate mental actions (the process of reflecting

abstraction). This is important because, as mentioned previously, reflecting abstraction is the engine that drives productive accommodations in a person's schemes and "results in increasingly organized and refined cognitive schemes...[that inject] coherence into systems of organized actions" (Tallman, 2015, p. 72). Put another way, "the key to the development of operative thought is...reflective abstraction" (Thompson, 1985, p. 196).

Operative thought involves mental representations of actions and consideration of the consequences of those actions that allows students to make propitious decisions about what to do next, and "allows them to see what they might do next in relation to what has already taken place" (Thompson, 1985, p. 194).¹⁰ Thus, operative thought implies the person is exercising a level of coordination and control over her reasoning. Furthermore, "once a student has created a structure of operations, he or she may reflect on the current state of affairs into that structure, and think in terms of possibilities: What would happen *if I did (or did not) do this?*" (Thompson, 1985, p. 197). A key aspect of operative thought is the conservation of relationships within a system regardless of variations in what is being considered. Thompson (1985) argues that promoting the development of operative thought is (or should be) the ultimate goal of mathematics instruction. In order for students to develop an understanding for the conservation of relationships within a system, they must engage in repeated reasoning about these relationships in ever-increasingly sophisticated ways that support the mathematical structure of the ideas at hand.

¹⁰ Operative thought is contrasted with *figurative thought*, which describes a person's thinking within a certain scheme and manifests itself in relation to that scheme as an inability to "go beyond the elements of a problem to a network of relationships" (Thompson, 1985, p. 195).

The final component of Thompson's design principles is that students must generate feedback that allows them to reflect on their thinking. When people interact with physical objects, the objects' behavior provides instant feedback that facilitates scheme accommodations (what we might call "learning"). For example, children's schemes for weight might initially involve the idea that larger objects weigh more than smaller objects. But eventually children must encounter objects to which this relationship does not hold. The physical property of the objects assures that this will be the case, and the child cannot stubbornly refuse to acknowledge the fact. The child must make accommodations to his scheme. However, it is also common for children's multiplication schemes to include the idea that multiplication results in a product that is larger than either of the two initial factors ("multiplication makes bigger") because early experiences with multiplication are usually restricted to products involving natural number factors. While I acknowledge that there are ways to physically represent the factors and product of multiplication, the structure of multiplication schemes involve mental actions. There is often nothing to "push back" against errors in mental actions, such as an emerging intuition that multiplication always produces larger numbers.¹¹ Thompson's proposed solution is to create situations where students are asked to make predictions based on their current thinking and for computer software to allow them to enter their prediction and either see for themselves whether their prediction is accurate or to compare their

¹¹ Even when students acknowledge that multiplication does not always produce larger numbers, it is still possible for the intuition to dominate their reasoning about multiplication in subtle and insidious ways if the accommodation to their multiplication scheme involves allowing contradictions to the rule to exist as special cases but the rule itself to generally hold. Evidence of this kind of accommodation might include the tendency for students to multiply smaller magnitude numbers in a word problem or divide a larger magnitude number by a smaller magnitude number regardless of what those numbers represent within the quantitative structure of the situation.

prediction with what the outcome should be (according to the program). This immediate feedback can replicate a “push-back” similar to the way that the physical properties of the world push back against children as their schemes related to physical properties develop. It’s interesting to note that Thompson recognized very early how technology is uniquely suited to perform this role in facilitating effective mathematical learning experiences. Over 30 years since his 1985 paper, many students still do not experience the kinds of feedback-generating explorations Thompson described despite advances in technological sophistication or the general increase in teachers’ and students’ access to technology in the classroom. That is, they do not get to experience the mathematics “pushing back” against their attempts to reason about particular ideas.

Thompson’s five design principles are an excellent foundation for creating lessons in an online learning environment consistent with the background and domain-specific theories described in this chapter, and as much as practicable I will leverage these design principles in completing my conceptual analysis and hypothetical learning trajectory in Chapter 4. An online mathematics course provides both unique opportunities and tremendous challenges relative to these principles. Thompson is careful to point out that his curricular examples are not created with the intention that they are independent from a skilled instructor. No matter how carefully a teacher designs tasks or interactive software and no matter how many times the teacher has used the tasks, the obstacles students encounter, the hypotheses they generate, and the ways in which they attempt to express their reasoning vary. Leading conversations, asking the right questions, and making all manner of productive pedagogical moves takes a skilled instructor with a strong personal understanding of the relationship structure the problems are intended to support and a

clear idea of *how* the problems support students in developing similar cognitive structures. In addition, some students have negative affective reactions to lessons that don't include explicit methods for producing expected answers. The online environment typically replaces the instructor with pre-programmed lessons students move through autonomously. An instructor might be attached to the class to oversee grades and to answer student questions, but that instructor is not actively leading lessons or addressing students' emotional responses in the moment. These realities complicate the ability to provide high-quality mathematical learning experiences in an online course since all tasks, responses, comments, and follow-up questions must be pre-programmed. On the other hand, the high level of technology integration means that it might be easier to build a feedback process into every lesson where students expect to explore ideas with applets and to draw important conclusions from these interactions. A key part of the purpose of this dissertation study is to better understand how to successfully implement this design principle in online courses.

Summary and Closing Comments

In this chapter I outlined constructivism as a background theory that influences my perspective for designing an online intervention for teaching exponential growth and related ideas and for analyzing data gathered from student behaviors while participating in the intervention. The domain-specific theories of quantitative and covariational reasoning provide particular insights into ways of thinking that can support productive ways of knowing exponential growth while also providing general instructional goals and connections that promote coherent understandings throughout the intervention and the

course as a whole. In the next chapter I provide a detailed literature review related to teaching and learning exponential growth and features of online courses that impact students' learning experiences.

CHAPTER 3

LITERATURE REVIEW

Building sophisticated quantitative reasoning skills for the majority of students is not a one or two year program; it requires development throughout the elementary and middle school years. (Smith & Thompson, 2007, p. 43)

As I outlined in Chapter 1, many students and teachers lack robust meanings for exponential growth and percent change. One likely reason is that common initial models of exponential growth as repeated multiplication do not generalize well to reasoning about an exponential function's behavior over its entire domain. A thorough search of the literature revealed two different conceptual analyses describing potentially productive understandings of exponential growth in the context of supporting students' covariational reasoning about functions. In this chapter I review Confrey and Smith's and Thompson's conceptual analyses and the implications of each for supporting productive student meanings of exponential growth and related ideas. Leveraging Confrey and Smith's and Thompson's perspectives is most productive when one carefully attends to their differing conceptual analyses and the broader contexts in which they make their recommendations for conceptualizing exponential growth. For example, they each have unique meanings for covariation, function, and rate, and their distinct images for these ideas are motivated by different learning goals and begin with different initial assumptions about an exponential function's characteristics. Non-critical readings of their perspectives can suggest areas of agreement where none exist simply because they use the same common mathematical terminology to describe different underlying imagery and meanings.

Exponential Functions

Thinking about functions from a quantitative and covariational perspective entails conceptualizing two quantities that vary, uniting them as a multiplicative object, and reflecting on what remains invariant as they change together. There are two different features of a situation that suggest an exponential function as an appropriate model. If the values of one quantity have a constant growth factor (or constant percent change) for all changes of some fixed size for a second quantity, then an exponential model is appropriate. Alternatively, if the rate of change of one quantity with respect to a second quantity is proportional to the value of the first quantity at all moments, then an exponential model is appropriate. Focusing on one feature or the other has implications for how learning may unfold and the implications for understanding future mathematical ideas.

Confrey and Smith's Conceptual Analysis

Splitting. Confrey (1994) described *splitting* as a useful metaphor for thinking about multiplication and called it a *primitive action* similar to counting because it can form the basis of a numerical system. A *split* involves imagining equal-size copies of an original amount or partitioning an amount into equal-sized parts. See Figure 3.1.

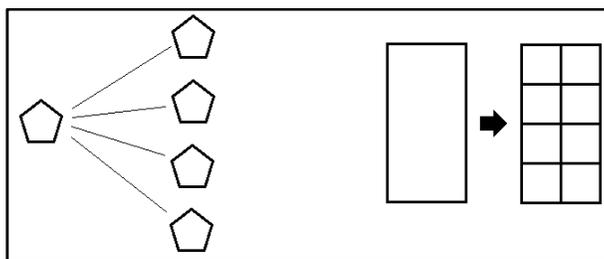


Figure 3.1. A split as an equal number of copies of an original amount (left) or partitioning into equal-sized parts (right) (Confrey, 1994).

Confrey then defined multiplication as the result of some n -split and division as examining one of the equal parts of a split relative to the whole.¹² Confrey (1994) and Confrey and Smith (1995) argued that splitting serves as the basis for an alternative number system distinct from the rational numbers but with identical density where ratios instead of differences are the natural means of comparison and ratios are equivalent if they compare values separated by the same number of splits. For example, a sequence of “two-splits” is 1, 2, 4, 8, 16, ... and the ratios $\frac{16}{2}$ and $\frac{32}{4}$ are equivalent because they each compare values separated by three two-splits.

Covariation, rate of change, and function. Confrey (1994) and Confrey and Smith (1994, 1995) rely on a covariational meaning for *function*, although their definition differs from that of Thompson and his colleagues. They defined *covariation* as a process of coordinating successive values of two variables. For example, when given ordered sets of values for two variables x and y , students engage in covariational reasoning when they coordinate movement “from y_m to y_{m+1} ... with movement from x_m to x_{m+1} ” (Confrey, 1994, p. 33).¹³ A function relationship is then “the juxtaposition of two sequences, each of which is generated independently through a pattern of data values” (1995, p. 67). Students are expected to identify patterns that emerge through repeated actions during this coordination, which forms the basis for generalizing properties and identifying

¹² Confrey (1994) gave the following example. If you cut a cake into four equal parts, you can conceptualize the number of pieces after the split (“4”) or the size of one piece of cake relative to the size of the entire cake (“1/4”).

¹³ Note that the use of “movement” here does not indicate that Confrey and Smith imagined a smooth variation from one value to another while imagining that the quantity also takes on all values in between. Their choice of coordinating sequences as their primary metaphor suggests that replacing “movement” with “a jump” would not affect their intended meaning.

operations that define the relationship between values of each variable. For example, a *rate* can be thought of as a unit-per-unit comparison where *unit* describes what remains constant in a repeated action (Confrey, 1994). This led them to define the notion of a constant multiplicative rate of change as a relationship with constant multiplicative changes in one quantity coordinated with constant additive changes in another quantity and *function* as “the juxtaposition of two sequences, each of which is generated independently through a pattern of data values” (Confrey & Smith, 1995, p. 67). Thus, an exponential function is a function with a constant multiplicative rate and derives from coordinating an arithmetic and geometric sequence as shown in Table 3.1.

Table 3.1

A Table of Values for an Exponential Function of y with Respect to x

x (values in an arithmetic pattern)	y (values in a geometric pattern)
0	1
1	4
2	16
3	64
4	256

Properties of exponential functions’ rates of change. Exponential functions play an important role in calculus and differential equations primarily because they have the unique property that the rate of change of the function at any point is proportional to the function’s value (expressed symbolically as $dy/dx = ky$ for some constant k). Confrey and Smith do not begin with this property as a fundamental assumption of what it means for a function relationship to be exponential, but they do conjecture how one might

develop the idea.¹⁴ Students could envision multiplicative growth comparisons using bars as shown in Figure 3.2. Each bar is 1.4 times as long as the previous bar, but a person can also focus on the length *in excess* of the previous bar. See Figure 3.3.

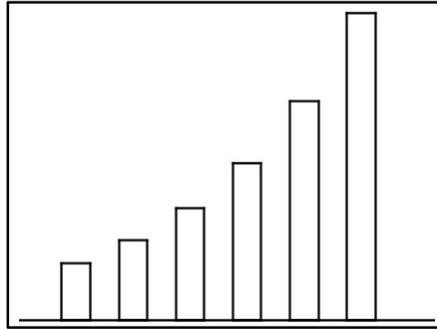


Figure 3.2. A sequence of bars with equal length ratios (Confrey & Smith, 1994, p. 159).

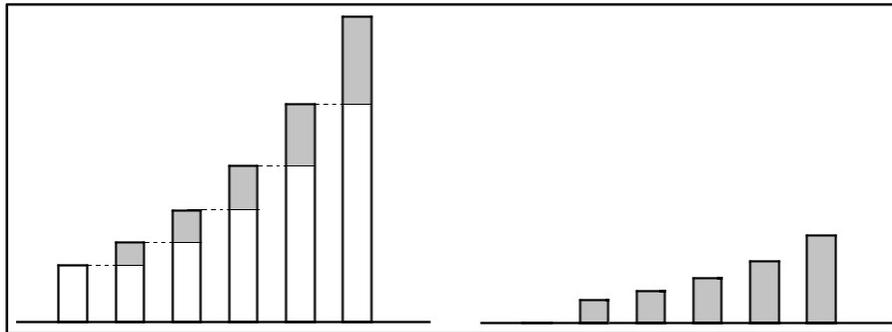


Figure 3.3. Focusing on the length in excess of the previous bar (Confrey & Smith, 1994, p. 160).

These excess lengths each represent 40% of the length of the previous bar, and each increase is also 1.4 times as long as the previous increase. “[I]t is only a short step to argue that the additive change (shaded areas) will always be the same proportion of the whole bar...[which becomes the property] that the derivative of any exponential function is directly proportional to the function itself” (p. 160).¹⁵ The value of e can emerge by

¹⁴ It is more accurate to say that their analyses can support students in understanding that the average rates of change of an exponential function over consecutive equally-sized intervals are proportional to the value of the function at the beginning of the interval when the geometric growth structure is assumed.

¹⁵ Given how their definition of rate differs from conventional meanings, it is not clear how they intend students to transition from one meaning to the other or how they expect students to understand the meaning of a derivative or derivative function.

letting $f(x) = a^x$, noting that the rate of change function is $f'(x) = k \cdot a^x$ for some constant k , and then seeking the value of a such that $k = 1$.

Building on Confrey and Smith's conceptual analysis. Ström (2008) expanded Confrey and Smith's idea of *interpolation* for exponential functions (determining intermediate function values that maintain the splitting structure of the function) by introducing the notion of *partial growth factors* to describe multiplicative changes for exponential function values that differed depending on the size of non-overlapping intervals subdividing the function's domain. For example, if an exponential function has a one-unit growth factor of two, then when the domain is divided into intervals one unit long, the function value at the end of any interval is two times as large as the function value at the end of the previous interval. If the domain is divided into intervals of length 0.1, then the function value at the end of any interval is $2^{0.1}$ (or $\sqrt[10]{2}$) times as large as the function value at the end of the previous interval. Thus, $2^{0.1}$ is the 0.1-unit growth factor.

Summary and comments. For Confrey and Smith an *exponential function* is the coordination of a counting and splitting structure and is based on one of the two primary characteristics of an exponential function (that the values of one quantity have a constant growth factor or constant percent change for all changes of some fixed size for a second quantity). There are several implications for this conceptual development. First, by building from basic notions of doubling, tripling, halving, etc., their development leads quite naturally to recognizing the appropriateness of exponential models in contexts where growth factors are easy to identify, and the coordination imagery for function relationships likely supports productive transitions to writing formulas representing generalized calculations for producing the values of an exponential function. However,

relying so heavily on the image of a multiplicative structure formed by subdividing the domain into equal-sized non-overlapping intervals is likely to promote discrete function reasoning as opposed to seeing a relationship as an emergent coordination of continuously covarying quantities. Thinking about partial growth factors, rates of change, or average rates of change in this context requires knowing the values at the beginning and end of an interval, so “in-between” values in the continuous relationship, if they are conceptualized at all, come along as afterthoughts similar to connecting discrete points on a graph to fill in the general behavior of a function.

Thompson’s Conceptual Analysis

Piecewise linear accumulation and the Fundamental Theorem of Calculus.

Thompson’s (2008) development of exponential growth is part of a larger systematic and coherent approach to thinking about functions grounded in quantitative and covariational reasoning that he argued is necessary for thinking about the Fundamental Theorem of Calculus (FTC) as a relationship between rate of change and accumulation (Thompson, 1994c, 2008).¹⁶ Therefore, he suggested defining exponential functions as relationships where the rate of change of one quantity with respect to a second quantity is proportional to the first quantity’s value at all times, which is consistent with the second key property of exponential functions I described earlier. He described his conceptual analysis relative to a simple interest model.

Suppose a bank account earns simple interest updating continuously. The value of an account earning $(100r)\%$ annual interest on the original principle is a linear function of

¹⁶ See Appendix A.

the elapsed time since the initial deposit. If the bank updates its policy so that the principle used to calculate interest updates periodically, then the account balance is a piecewise linear function (and its rate of change function is a step function). Throughout any given interval the account value's rate of change with respect to elapsed time is constant and proportional to the account balance at the start of the interval. See Figure 3.4.

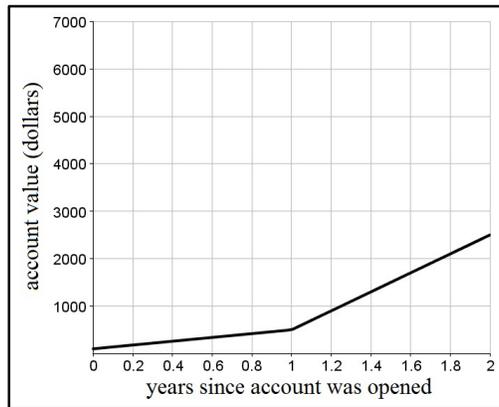


Figure 3.4. A piecewise linear function with a rate of change proportional to the function value at the beginning of each interval. The relationship is defined by $f(x) = P(1+r)^c + r[P(1+r)^c](x-c)$ for $c < x \leq c+1$, $c \in \mathbb{N}$ (Thompson, 2008).

If the principle for calculating interest updates more frequently, then the model of the account balance with respect to time converges to an exponential function. See Figure 3.5.

Thompson's image of exponential growth depends heavily on conceptualizing rate of change as a homogeneous relationship between the changes in two quantities. Over infinitesimal intervals of time a function's rate of change is constant (or can be assumed to be constant) and thus change and accumulation occur simultaneously. An individual conceptualizing exponential growth in this way is capable of seeing an

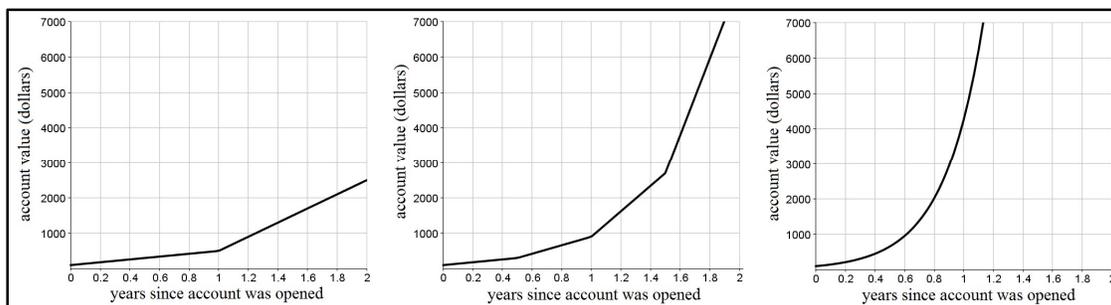


Figure 3.5. The function converges to an exponential function as the number of times the principle is reset (n) increases without bound. The formula is

$f(x) = P\left(1 + \frac{r}{n}\right)^{\lfloor nx \rfloor} + \frac{r}{n}P\left(1 + \frac{r}{n}\right)^{\lfloor nx \rfloor} (x - \lfloor nx \rfloor)$ which converges to $f(x) = Pe^{rx}$ as n increases without bound (Thompson, 2008).

exponential function emergently as a coordination of two continuously covarying quantities. In addition, the notion of a constant growth factor as a relative size comparison of the function value at the beginning and end of any interval is a natural extension of understanding how the accumulation occurred. Finally, someone thinking about all functions as piecewise linear accumulation functions over infinitesimal intervals gives them a solid foundation for thinking about the FTC in a powerful way (Thompson, 1994a, 2008). However, this understanding of exponential growth “accentuates the characteristic property of exponential functions (rate of change being proportional to the value of the function) at the expense of the intuition of doubling, tripling, etc. that comes from the idea of splitting” (Thompson, 2008, 42) and thus might be hard to develop in the context of situations where the emphasis is on constant growth factors or percent change (such as half-life or models developed from data).

Reasoning about a growth factor in continuous covariation. Thompson acknowledged that his conceptual analysis is not easily applied in situations where the multiplicative structure is given (such as repeated doubling) and that “there is something inherently discrete” (personal communication, April 9, 2015) in conceptualizing a

relationship initially defined relative to a constant growth factor. However, a focus on splitting as the foundation for the meaning of growth factor and Confrey and Smith's images of covariation and rate of change are potentially problematic for students' long-term mathematical development. For example, their image of covariation corresponds to what Thompson and Carlson (2017) call *coordination of values* or *chunky continuous covariation*. Castillo-Garsow (2010, 2012) and Castillo-Garsow, Johnson, and Moore (2013) demonstrated that chunky thinking is inherently problematic in many situations and that chunky thinking is not a gateway to thinking about continuous smooth covariation, which is a way of thinking that supports, among other things, a meaningful understanding of the FTC.¹⁷

Thompson designed a didactic object¹⁸ to support students in attending to the meaning of growth factor over some interval as a relative size comparison of two instances of one quantity while still holding in mind continuous covariation as they imagined running through a continuum of values in the domain. The object is an applet allowing the user to define a growth factor of any size and then vary the value of the quantity represented on the horizontal axis of the graph. The applet shows the corresponding magnitude of the second quantity as well as the magnitude at the end of an interval of a given size. See Figure 3.6.

During a class discussion Thompson draws students' attention to the relative lengths of the segments, using the length of the left segment as the measurement unit for

¹⁷ See Appendix A.

¹⁸ A *didactic object* is “‘a thing to talk about’ that is designed with the intention of supporting reflective mathematical discourse” (Thompson, 2002, p. 198).

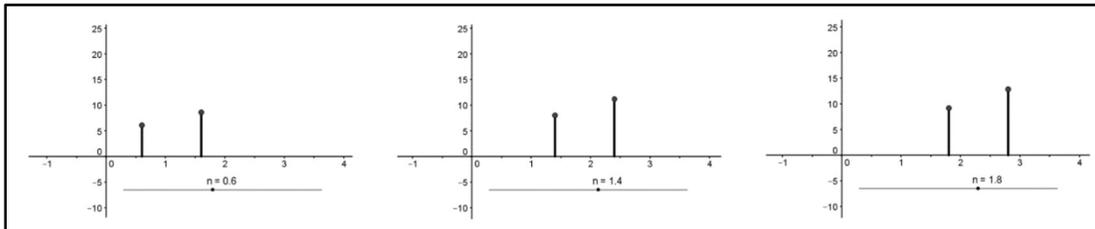


Figure 3.6. An approximation of the visual display for Thompson’s applet for three different states. Note that as the slider moves, the x -value at the beginning of the interval changes through a slide (not a jump) but the width of the interval does not change.

the length of the right segment. Although the segments’ lengths vary, their relative size remains invariant as the interval slides to the right. Also, note that students can discuss the meaning of the growth factor without paying particular attention to the specific values of the function at these points. This supports students’ development of a quantitative meaning for growth factor and de-emphasizes calculational meanings while still providing a context for deriving reasonable calculations to evaluate function values or growth factors. In addition, Thompson can support students in imagining the length of the segment growing from the first length to *become* the second length as it slides through the interval (personal communication, April 9, 2015). This imagery can support students in developing algorithmic approaches to calculate function values and building meanings for rational exponents emerging from coordination of smooth continuous covariation.

Building on Thompson’s conceptual analysis. Castillo-Garsow (2010, 2012; Castillo-Garsow, Johnson, & Moore, 2013) provided empirical support for Thompson’s conceptual analysis. In his dissertation study, Castillo-Garsow (2010) created a teaching experiment with two high school freshman algebra students (Tiffany and Derek) using Thompson’s simple interest approach to build a meaning for exponential growth. While coordinating the account value with elapsed time, and while coordinating the relationship

between the account's growth rate and the account's value, Tiffany consistently described change occurring in "chunks". She talked about equal-sized changes in time and the resulting jump in the account value, and when pressed to focus on what was happening within one of these intervals, her language and written work suggested that she thought of subdividing the interval into smaller "chunks" and producing corresponding account values in a discrete pattern (like a sequence).¹⁹ Derek, on the other hand, made it clear that he was thinking about time passing continuously and that this meant that both the account value and the rate of growth were constantly updating by "flowing" from one value to the next. Castillo-Garsow et al. (2013) claim that the distinction indicates that the two students' images of change differed. "When creating the graph, Tiffany conceived the account value as changing *after* a specified interval of time. In contrast, Derek conceived the account value as changing *with* continuous changes in time" (p. 33). These contrasting images of change, first noted by Saldanha and Thompson (1998) and termed "chunky variation" and "smooth variation" respectively by Castillo-Garsow (2010), had important implications for the students' ability to reason about exponential growth.

Castillo-Garsow et al. (2013) argue that encouraging chunky variation thinking is common in U.S. mathematics instruction. Teachers and students change an independent variable in equal-sized steps, calculate the corresponding dependent variable values, graph these pairs as points, and then connect the points with a smooth curve. While the curve is meant to approximate the ordered pairs left unevaluated, at least three issues arise. First, students often never explicitly reason about the behavior of the function

¹⁹ Based on Castillo-Garsow's description, Tiffany's reasoning appears fairly consistent with Confrey and Smith's recommended ways of understanding exponential growth.

within these intervals, commonly resulting in students thinking of graphs as curves passing through points but identifying the explicitly plotted points as the only ones that lie on the curve (Thompson & Carlson, 2017). Second, this approach makes it difficult to support images of continuous changes in quantities because students first imagine the change occurring in chunks, evaluate the results of these changes, and then (perhaps) “fill in” values between. Thus, values of the quantity are determined and conceived of out of temporal order and the individual’s initial conceptions of the function relationship are not generated through dynamic coordination of covarying quantities. Third, the size of the chunks is determined first, so the chosen interval size impacts the model’s accuracy since a person can easily overlook important features and behavior not captured by the intervals’ endpoints. In Castillo-Garsow’s study, Tiffany did not connect the points she plotted. Castillo-Garsow conjectured that this was because she was unfamiliar with both the context being considered and the type of function she was building. When students are familiar with a function family from repeated examples they often “fill in” function behavior between discrete points even if they do not have a sophisticated understanding of how the graph represents a coordination of the values for two covarying. This behavior is not ideal and differs from imagining an emergent function correspondence generated by coordinating variation in two quantities.

Summary. Thompson’s work builds from a theory of quantitative reasoning (identifying quantities, quantification, and creating structured relationships among quantities). When the individual conceptualizes two quantities that change in tandem, an image of the function relationship derives from coordinating the quantities’ values such that at all moments of conceptual time each quantity’s magnitude or measurement

persists as they vary together. Thompson argues that this is a productive meaning for function both developmentally and for supporting images of change and accumulation that make the FTC conceptually accessible to students. Thompson develops the exponential function by defining a relationship between two quantities where the rate of change of one quantity with respect to the second quantity is proportional to the value of the first quantity and creates a function to track the first quantity's accumulation by applying homogeneous rates of change over small intervals. When the size of the intervals tends to zero, the accumulation function converges to an exponential function. In cases where a growth factor is given (such as knowing a half-life or doubling time for some quantity), Thompson doesn't want students to imagine coordinating discrete ordered pairs for the function as one might when examining a table of values. Instead, he wants students to imagine that a growth factor or percent change comparison refers to what remains invariant as two quantities smoothly vary in tandem. That is, constant percent change refers to a constraint on how two quantities covary.

Table 3.2 summarizes the key aspects of the two conceptual analyses for exponential growth.

Ellis and Colleagues: An Empirically Supported Hypothetical Learning Trajectory

Ellis and her colleagues studied Confrey and Smith's and Thompson's conceptual analyses and generated a hypothetical learning trajectory and accompanying set of activities designed to support students in developing personal meanings for the following ideas.

Table 3.2

Comparing Confrey and Smith's Conceptual Analysis with Thompson's Conceptual Analysis

	Confrey and Smith	Thompson
<i>Covariational reasoning involves...</i>	...coordinating values of two discrete sequences containing the values of varying quantities.	...conceptualizing a multiplicative object uniting two varying quantities as they change in tandem in a smooth, continuous way.
<i>A function represents...</i>	...a patterns that emerges from coordinating two independent sequences.	...a relationship derived from conceptualizing constrained covariation of two quantities.
<i>A unit is...</i>	...whatever remains constant in a repeated action.	...part of the quantification process. When a person wants to measure some attribute of an object (a quantity) he chooses a fixed size of that attribute for comparison (a unit of measure). A quantity's measurement represents a relative size comparison of the quantity's and unit's magnitudes.
<i>A rate of change is...</i>	...a unit-per-unit comparison deriving from coordinating successive values in two independent sequences.	...the proportional relationship between the changes in two quantities' values as they covary.
<i>Multiplicative reasoning...</i>	...emerges from performing splits (either making an equal number of copies of an original amount or breaking an original amount into equal-sized parts). Multiplication involves conceptualizing the number of pieces after the split compared to the original number of pieces.	...describes conceptualizing a coupling of two already-conceptualized quantities so that the new object is understood in relation to the original quantities. Applied to "multiplication" contexts, one conceptualizes a product ab as a copies of b , where ab is a times as large as b and b times as large as a while understanding that a is $1/b$ times as large as ab and b is $1/a$ times as large as ab .
<i>What is the initial assumption for developing an exponential function?</i>	An exponential function develops from first coordinating geometric and arithmetic sequences and then considering interpolations (what numbers must be placed in between the values of each sequence to preserve the structure of each sequence) to create or imagine coordinated sequences with the same density as the rational numbers.	The function's rate of change over some tiny interval is proportional to the function's value at the beginning of the interval.
<i>What is the motivation for choosing this initial assumption?</i>	Studying students' primitive actions led them to develop novel meanings for <i>function</i> and <i>rate of change</i> that build from common student conceptions. Applying the meanings of function and rate of change to exponential growth creates a natural starting point for defining exponential functions in terms of coordinating splitting and counting worlds by juxtaposing a geometric sequence with an arithmetic sequence.	Conceptualizing the FTC in such a way that its results are intuitively obvious involves imagining homogeneous rates of change applied over tiny intervals so that change and accumulation occur simultaneously. Applying this imagery to exponential functions involves first conceptualizing a function whose rate of change over tiny intervals is proportional to the function value at the beginning of each interval, allowing accumulation to occur, and deriving the function relationship that results by letting the interval lengths decrease towards 0.

1. The period of time x for the y -value to double (or increase by the growth factor b) is constant, regardless of the value of a or b .
2. There is a constant ratio change in y -values for each constant additive change in corresponding x -values.
3. The percentage growth in y is always the same for any same Δx ...
4. The value of $f(x + \Delta x)/f(x)$ is dependent on Δx .
5. The constant ratio change in y -values is dependent on both the growth factor b and on Δx in the following manner: $y_2/y_1 = b^{x_2-x_1}$. This relationship will hold even when $\Delta x < 1$. (Ellis et al., 2015, p. 140)

In designing their teaching experiment, Ellis et al. anticipated, and later confirmed, that students' initial models for exponentiation involved an informal image of repeated multiplication. Ellis et al. wanted students to leverage covariational reasoning to build a more robust image of exponential growth. Their image of covariational reasoning might be described as "Confrey and Smith covariation with continuity". Ellis et al. wanted students to focus on coordinating multiplicative changes in one quantity with additive changes in another quantity so that students could appreciate that the ratio $f(x + \Delta x)/f(x)$ was constant whenever Δx was a given constant value and that more generally the value of $f(x + \Delta x)/f(x)$ depended on the value of Δx . Leveraging Carlson et al. (2002), Castillo-Garsow (2012), Saldanha and Thompson (1998), and Thompson and Carlson (2017), Ellis et al. expressed their desire for students to think about the coordination of two quantities happening continuously and smoothly. They wanted students to experience a situation where it was more natural to see a relationship where one quantity had values for all possible values of a second quantity and they recognized that many situations leveraging splitting imagery involved more static thinking. Therefore, the context at the core of their teaching experiment involved a plant (called a Jactus) growing continuously over time.

Ellis et al. built a Geogebra applet of the Jactus with a height that varied exponentially with elapsed time. A user can smoothly move a point along the horizontal axis of a plane (representing a varying time elapsed measured in weeks) and the Jactus will “grow” so that its height matches a particular exponential growth pattern. The user can also change the initial plant height and the weekly growth factor. See Figure 3.7. Note that the applet displays the time elapsed and the plant’s height as the user slides the plant horizontally.

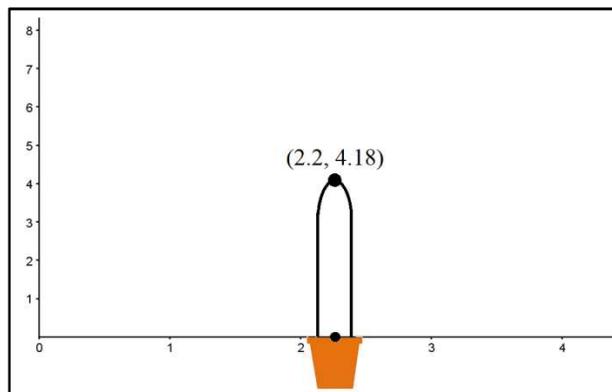


Figure 3.7. An approximation of the display for Ellis et al.’s (2012, 2015, 2016) Jactus applet.

Students initially interacted with the applet and recorded measurements in order to discuss how the height changed. Then Ellis et al. asked students to draw pictures of the plant at various points in time in order to help the students conceptualize and distinguish the quantities *plant height* and *time elapsed*. Eventually students were asked to make drawings and anticipate measurements when only partial information was present (for example, when some number of weeks’ data was omitted) and to reason in general terms about what would happen to the plant height over time if they knew the growth factor but no actual measurements. All of the activities shared the broader goals, inspired by

Carlson et al.'s (2002) covariation framework, of supporting students in flexibly coordinating the ratios of plant heights for different changes in time elapsed.

Ellis et al. were quite successful in a number of ways. Students working through the activities exhibited three key shifts in their thinking (with some variation from student to student). First, students shifted from thinking only about repeated multiplication of the y -values without worrying about how the corresponding x values changed to coordinating variation in both values. This was a key shift that allowed students to develop algebraic representations for the plant's height at different moments. Second, students were able to move from just coordinating ratios of y values for $\Delta x = 1$ to ratios of y -values for larger changes in x , including reunitizing so that they could imagine such changes in x as a single change instead of a sequence of unit changes. Third, students shifted to being able to represent ratios of y values for values of Δx less than 1. “[These] results...offer a proof of concept that even with their relative lack of algebraic sophistication, middle school students can engage in an impressive degree of coordination of co-varying quantities when exploring exponential growth” (Ellis et al., 2012, p. 110).

My understanding of how they imagined this learning trajectory eventually paying off for students was that students could come to understand b^x as both the possible height of a plant at some moment in time and as representation of a (multiplicative) change in height from an initial height to the height after x weeks.²⁰ With this understanding, the “initial” function value is a scalar for a certain class of behavior. This may support students in constructing meaning for the algebraic representation $f(x) = ab^x$. Rather than

²⁰ To fully achieve this goal might require what Weber (2002) described as a process view of exponential expressions in order for a student to have confidence that b^x has a value and a meaning even for irrational x .

imagining repeated multiplication used to evaluate $f(x)$ at some input x (with all of the issues inherent in that reasoning), students can conceptualize a meaning consistent with the order of operations used when evaluating the function. First, evaluate b^x (the ratio of $f(x)$ to $f(0)$). Second, scale $f(0)$ by a factor of b^x to determine $f(x)$. This view allows for a more general model $f(x) = f(h) \cdot b^{x-h}$ where $(h, f(h))$ is any ordered pair for the function and $f(x) = ab^x$ is simply a special case of this representation. O’Bryan and Carlson (2016) demonstrated that this kind of orientation towards developing meaningful formulas through linking the meaning of expressions to informal reasoning and attention to how the order of operations mirrors the steps in a reasoned solution was powerful for both teachers and students.

While students showed impressive gains, Ellis et al. observed noteworthy hurdles students had to overcome to make progress towards their key learning goals. Students’ initial models were so strongly tied to repeated multiplication to produce different plant heights that it was difficult to make students explicitly attend to changes in time as a quantity to which they must attend and to focus on coordinating changes in the plant’s height with changes in elapsed time. Furthermore, while students were able to utilize the notation b^m to represent m factors of b , “repeated multiplication as iteration remained a strong focus for their actions and inscriptions...[and thus] they had difficulty generalizing their thinking to a gap that was arbitrarily large or small” (Ellis et al., 2015, p. 146).²¹

Ellis et al.’s conceptual analysis is very much in line with Confrey and Smith’s and Ström’s conceptual analyses and recommendations with the exception that they recognize from Thompson’s and Castillo-Garsow’s work the importance of having

²¹ Note that this was a key finding for Ström (2008) as well.

students conceptualize situations with an image of continuous, dynamic covariation instead of static values of some number sequence. That being said, viewing their teaching experiment through the lens of Thompson's and Castillo-Garsow's work might explain some of the observed student behaviors.

In Thompson's theory of quantitative reasoning, quantities must first be conceptualized before they are quantified, and developing a quantification process capable of producing reliable measures with specific interpretations is a key step in building productive meanings. In their applet design, Ellis et al. provided the plant height measurements relative to fixed linear units, and their students could thus immediately create tables of values and begin to perform calculations that reinforced initial images of repeated multiplication. Since the first growth factors used were simple numbers, students easily recognized that they could, for example, multiply a height by two to produce the height one week later. Thus, multiplication by two was a calculation to produce heights in a sequence. Students did not need to conceptualize an intensive measurement of relative size and consider how to produce the measurement. Their focus was not on the quantitative operation at all, only on calculations to produce the numbers observed on the applet. Ellis et al. seemed to acknowledge this issue. When they didn't specify a growth factor they observed students were more successful in "describ[ing] plants as growing in such a way that depicted a sense of continuous scaling or stretching" (Ellis et al., 2015, p. 150). When students were given the growth factor (or, presumably, when the growth factor was obvious), the students described the behavior in terms of repeated multiplication.

This might explain why some students developed relatively strong correspondence views early in the teaching experiment that allowed them to develop algebraic formulas for the relationship between height and time elapsed but that constrained their thinking when it came to using more basic covariational reasoning to answer straightforward tasks. These students were so grounded in the relationship being defined by calculations on given measurements that they did not have a strong mental image of the relevant relationships from which to draw. At one point during their efforts to help students conceptualize time as a quantity in its own right to which they must attend, Ellis et al. asked students to make drawings of the situation and gave them incomplete information (one week's data was skipped). They shared a student's drawing that resembled the diagram in Figure 3.8.

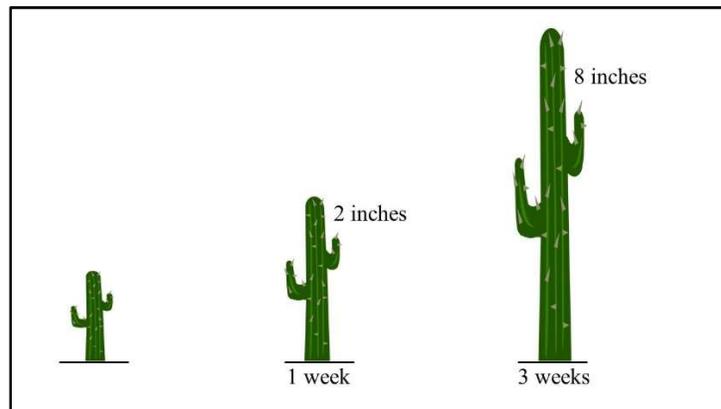


Figure 3.8. Approximating a student's drawing (Ellis et al., 2012, p. 102).

Ellis et al. did note that the student did not accurately draw the relative heights but they did not find it overly problematic since the student annotated the drawing using the correct measurements. However, the drawing may indicate that the student did not have a meaning for the growth factor beyond a number by which to multiply to produce another number. That is, this student does not appear to have been attending to relative height as a

measurement of two instances of the quantity's magnitude. Thus, from Thompson's point of view, we might question initial activities that reinforce calculational meanings for a growth factor while ignoring other potentially useful meanings, such as the measure of a conceptualized intensive quantity.

Ellis et al. acknowledge that Thompson's conceptual analysis differs from their own and wrote that they consciously chose not to use his approach with their subjects.

However, they stated

We were interested in developing a situation in which the notion of proportional rate of change would arise naturally. We have found that adopting a rate of change perspective can be accessible even for beginning algebra students in middle school, particularly if they have opportunities to explore situations that encourage students to construct meaningful relationships between quantities (Ellis, 2007, 2011a, 2011b). (Ellis et al., 2012, p. 96)

Within their work I could not find an exact definition of *rate of change* that would justify this statement nor an example of how their work built to this conclusion. They appeared to use the notion of a constant multiplicative rate of change from Confrey and Smith's conceptual analyses but it was implicit.

As Ellis et al. also point out, just having a context with a continuous relationship does not guarantee that students will conceptualize the situation in that way, nor does it guarantee that they will engage in smooth continuous variation reasoning. Castillo-Garsow (2012) was very clear that chunky thinking does not naturally lead into smooth thinking, so instructional trajectories designed to eventually support smooth variation reasoning but that begin by explicitly supporting chunky thinking put the cart before the horse. Ellis et al. wrote that "Returning to visual models of growing plants via Geogebra was merely a backdrop to encourage students to create descriptions of growth that did not

rely on discrete repeated multiplication actions” (Ellis et al., 2015, p. 150). So their use of the applet helped orient students to the problem, but reasoning with continuous variation was not at the heart of the activities in which students engaged. Students leveraged chunky covariational thinking almost exclusively throughout the various activities and the activities seemed designed to promote and make use of this reasoning.²² Thompson (2012) recognized that students in Ellis et al.’s study seemed not to reason about smooth variation while thinking about growth factors, and only one student appeared to try to build an interpolation model consistent with smooth variation reasoning but only when the student explicitly stopped talking about growth factors.

I want to make clear that I do not mean the above comments to express a negative opinion of Ellis et al.’s work. Quite the contrary. Teaching and learning ideas related to exponential growth is extremely challenging. Their study supported students in making many impressive shifts in their understanding of exponential growth and helped students successfully complete tasks that other researchers highlight as challenging. Furthermore, their work provides empirical support for the argument that attending to how two quantities change together, regardless of the specific definition of covariation being leveraged, does eventually support more productive understandings of correspondence rules. Ellis et al. are engaged in design-based research whereby researchers try to impact student learning in the messy and chaotic realities of real classroom environments and then try to learn from the intervention’s implementation and results to improve future

²² Ellis et al. wrote that “Smooth continuous variation entails imagining the variation of a quantity’s value as its magnitude increases in bits while anticipating that within each bit the value varies smoothly (Thompson & Carlson, in press). We consider this way of thinking to be a challenging goal for middle school participants – in particular for exponential growth – due to how growth is defined geometrically” (2015, p. 152).

interventions. Looking at their work critically using Thompson's conceptual analysis as a lens is an attempt to consider alternative explanations for observed student behavior to help inform my own hypothetical learning trajectory. A learning trajectory, I must emphasize, that will owe much to Ellis et al.'s work.

Before closing this chapter, I will briefly summarize the literature related to learning mathematics in an online environment. I will use the collective recommendations from this body of literature, if possible, to inform my intervention's structure and substance.

Students Learning Mathematics in an Online Environment

Research related to learning in online mathematics courses, or meta-analyses that include studies involving mathematics courses, can generally be grouped into three broad categories (with some overlap): 1) studies examining the impact of online learning on student engagement and emotions, 2) comparative analyses of student performance in online vs. face-to-face courses, and 3) attempts to discover and prescribe best practices for creating online courses. I will briefly summarize the conclusions in each of these categories before detailing important considerations not yet explored.

Researchers looking at students' emotional states while engaged in online coursework justify their studies with the premise that emotional states impact cognition, memory, creativity, motivation, and attention as well as how flexible students are in applying cognitive strategies (Isen, 2000; Kim, Park, & Cozart, 2014; Levine & Pizarro, 2004; O'Regan, 2003). Brinkman, Rae, & Dwivedi (2007) and O'Regan (2003) attribute negative student emotional reactions (such as frustration, anxiety, and embarrassment)

mostly to unreliable technology, challenges in navigating poorly designed web pages, stress related to falling behind or missing deadlines, inconsistent response times from instructors, not being confident with technology, or feeling ashamed when their work is publicly compared to classmates' work. On the other hand, positive emotional responses (such as enthusiasm, excitement, and pride) derived from autonomy, overcoming fears of technology, being impressed by new technology, or excitement at having access to a course from a remote location. Glass and Sue (2008) showed that active participation led to better and more persistent understanding, and several studies found that students were more actively engaged in an online course (Chen, Lambert, & Guidry, 2010; Glass & Sue, 2008; Hu & Kuh, 2001; Kuh & Hu, 2001; Laird & Kuh, 2005). They typically measured this using student surveys and/or monitoring the time students spent working on homework.

Some studies reported that student achievement was higher in online environments compared to traditional face-to-face classrooms. Students in Chen et al.'s (2010) study self-reported that online learning environments provided them with a better overall learning experience and improved their understanding of how to apply their learning in practical contexts. Hughes, Mcleod, Brown, Maeda, and Choi (2007) showed that students in online secondary math courses performed better than students in traditional settings on the Assessment of Algebraic Understanding, a test intended to align with the National Council of Teachers of Mathematics (2000) algebra standards. Nguyen and Kulm (2005) and Nguyen, Hsieh, and Allen (2006) demonstrated that students will spend more time on homework and complete more practice problems in online environments and that this correlated with improved achievement. They credited

the increase in students' willingness to practice, including repeating assignments to attain higher scores, to the immediate feedback computers provide. However, in many studies online courses did not produce a strong, statistically significant improvement in overall student learning (Barbour, Brown, Waters, Hoey, Hunt, Kennedy, & Trimm, 2011; Brinkman et al., 2007; Kim et al., 2014; Taylor, 2002). Some researchers argued that this is because both online and face-to-face instruction have strengths and weaknesses, and the benefits of online learning (those characteristics that might improve student performance) are balanced against the characteristics that might negatively impact performance and that educators have yet to embrace best pedagogical practices in online environments (Hughes & Mcleod, 2007). But it is still a fact that several meta-analyses demonstrated no significant positive impact for students taking online courses compared to those taking traditional courses (Hannafin, Orrill, Kim, & Kim, 2005; Means, et al., 2009; Swan, 2003; Twigg, 2003).

The final category of studies focused on developing best practices for online course design (Coomey & Stephenson, 2001, Engelbrecht & Harding, 2004; Glass & Sue, 2008; Hopper & Harmon, 2000; Kramarski & Gutman, 2006; McDuffie & Slavit, 2003; Myers, 1999; O'Regan, 2003; Rimmershaw, 1999, Stiles, 2000; Swan, 2003; Taylor, 2002; Wadsworth, Husman, Duggan, & Pennington, 2007). Synthesizing all of their advice produced the following general recommendations.

- Courses should be easy to navigate with a low bar of technical know-how needed to interact with the site and with careful routine checking to fix issues with courses such as broken links.

- Directions to students must be clear, including course requirements, deadlines, and grading rubrics, and the instructor must be easy to contact.
- The course should contain the ability for students to access practice with prerequisite skills and to self-assess their progress at any time. In addition, students should receive immediate feedback on their work whenever possible.
- Assignments should be clearly useful and be well-integrated into the flow of the course.
- The course content should include multiple formats and representations with flexible means for students to achieve course objectives.
- Activities should require a mixture of high and low cognitive levels.
- Features of the course should be designed to take advantage of the strengths of technology and autonomous learning. They should be more than just a replication of an in-person lecture course accessed via a website.
- The course must somehow engage the learner socially. The learner must feel like he is part of a community of learners and not isolated.
- The course should include questions or tasks that prompt metacognitive reflection and promote self-monitoring strategies.

While these are sound recommendations and many of them may apply to designing courses for in-person instruction, there is no focus on the ways in which the researchers intended students to think about the mathematical ideas contained within each course.

All of the studies I reviewed were based on classical experimental design testing variables such as “classroom setting” but never considering the mathematical content as potentially problematic. Therefore, this line of research offers no insight into students’

scheme development or accommodation in an online course relative to specific mathematical ideas or features of the online environment that support or inhibit these accommodations.

CHAPTER 4

CONCEPTUAL ANALYSIS

Our observation is that how students understand a concept has important implications for what they can do and learn subsequently. While this observation is neither new nor breathtaking, it is rarely taken seriously. To take it seriously means to ground the design of curricula and teaching on careful analyses of what we expect students to learn and what students do learn from instruction.

Careful analyses of what students learn means more than creating a catalog of their behaviors or strategies you hope they employ. They also entail tracing the implications that various understandings have for related or future learning. (Thompson & Saldanha, 2003, p. 95)

Thompson (2008) defined conceptual analysis as a description of “what students must understand when they know a particular idea in various ways” (p. 42) and outlined various uses for conceptual analysis. In this chapter I begin by describing my conceptual analysis for percent comparisons, measuring and representing percent change, growth factors, and understanding exponential functions with a focus on 1) outlining ways of knowing that could be beneficial for students’ mathematical development and 2) analyzing the coherence in meanings among some set of ways of knowing. Note that I consider this conceptual analysis as a body of ideas that I conjecture, but do not know for certain, will benefit students in having a powerful and coherent understanding of ideas I view as related to exponential growth. Part of my final analysis will involve assessing the degree to which students developed meanings in line with my goals and, if they did, the degree to which these meanings were useful. The second half of this chapter includes a hypothetical learning trajectory, including key tasks, which I conjecture will provide opportunities for students to construct productive meanings for the ideas described in my conceptual analysis. My main goal in this chapter is to follow Thompson’s (1985) advice for designing curricula by clearly articulating the meanings I intend students to construct

and to propose a particular activity sequence has the potential to support these constructions.

Exponential Growth and Related Ideas from a Quantitative Perspective

A growth factor is a measurement relating two instances of the same quantity separated by some additive change in a second quantity. As such, understanding *growth factor* relies on having a meaning for measurement in general as a multiplicative comparison and to understand that 1) a quantity's magnitude is independent of the magnitude of units used to measure it, 2) measurements indicate a reciprocal relationship between the quantity's magnitude and the unit's magnitude, and 3) changing the unit of measure changes the measurement value in an inversely proportional manner (Thompson et al., 2014). These understandings are also key to understanding percent comparisons and for connecting growth factor to percent comparison and percent change.

Flexibility in Choosing a Unit

In the previous section I discussed the relationship between a quantity's measurement and the magnitudes of the quantity and the unit. In my descriptions the unit was always chosen arbitrarily. This is common in measurement systems. For example, the lengths called "1 foot" or "1 meter", the volume called "1 cup", or the mass called "1 kilogram" are all chosen arbitrarily. However, another method for choosing measurement units is to rely on convenient magnitudes deriving from the context under consideration. For example, Tallman (2015) carefully lays out the reasoning involved in using a circle's radius as the measurement unit for arc lengths on that circle. While we are free to choose

a different unit (so the choice of unit can still be considered arbitrary), the size of the radius itself is not arbitrary in the same manner because its length is an inherent property of the related circle. Thus, something about the situation and measurement process dictates the unit's magnitude.

Suppose I measured a tree's height at two different moments in time and that each time I cut a piece of string with a length equal to the tree's height at the moment I measured it (call these lengths l_1 and l_2). When I lay out the pieces of string next to each other I can think of each string's length as a measurement unit for describing the other string's length. See Figure 4.1.

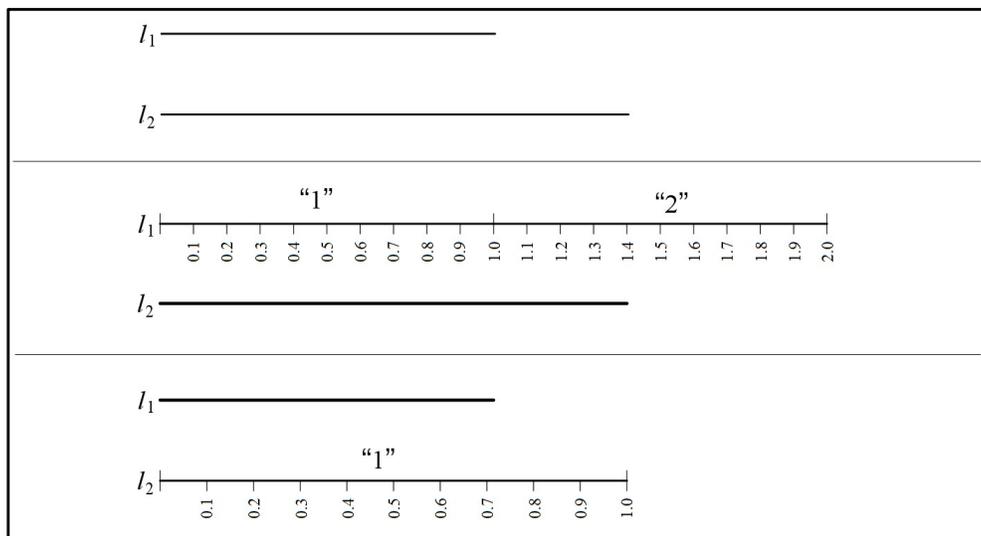


Figure 4.1: Measuring relative sizes. (top) The two string lengths representing the tree's height at two moments. (middle) Using the first height as the measurement unit, the second height has a measure of 1.4 (the second height is 1.4 times as large as the first height). (bottom) Using the second height as the measurement unit, the first height has a measure of about 0.7 (the first height is about 0.7 times as large as the second height).²³

²³ This is what Lamon (1994) describes as *norming* within a quantification scheme and is the key way of thinking that allows someone to see a number such as $2/3$ as representing different amounts of a quantity depending on conceptualizing the meaning of "1" in the context.

These two ways of generating measurements for each string length use units that are not arbitrary because they rely on magnitudes of the quantity that exist in the context at hand, although choosing either magnitude as the measurement unit (over the other magnitude or over any other choice of unit) is arbitrary. Note that the resulting measurements involve two instances of the same quantity and do not require measuring either instance in some standard unit and comparing these measurements.

I chose to represent the tree's height at the two moments in time using the lengths of two pieces of string for two interrelated reasons. First, doing so simplifies the process of holding in mind the magnitude of a quantity at two distinct moments in time. This is something lacking in Ellis et al.'s (2012, 2013, 2015, 2016) activities involving the Jactus plant. Even though students in their study could vary the height of the plant by varying the time elapsed since first measuring the plant, they could not visualize two heights of the plant at two different moments in time simultaneously. Instead, students focused on comparing measurements of the Jactus's height in some standard unit recorded in a table. This leads to my second reason for using two lengths of string. When only one instance of the quantity's magnitude is visible at a given time, it may be less natural to use a different instance of the same quantity as the measurement unit and more natural to use a standard unit (such as feet) for measuring the magnitude. Numerical representations for the quantity's measurement at multiple instances (such as a representation of multiple values in a table) provides an opportunity for students to generalize the relationship between instances as the product of performing a calculation rather than conceptualizing a relative size. It might also be easier for someone to overlook coordinating these measurements with changes in a second quantity. For example, the tree measurements

took place one year apart. Therefore, saying that the second height is 1.4 times as large as the first height is relatively meaningless in the context without recognizing when the two measurements were taken and therefore the interval of time over which the tree was growing between the measurements. Heights chosen from different moments or with a different amount of elapsed time between them would affect the measurement and its interpretation.

“Elastic” Units and the Meaning of a Growth Factor

A person with the understandings already described is positioned to consider the question, “What are the characteristics of a unit such that, as the quantity’s magnitude changes, the quantity’s measure in that unit never changes?” For the measurement of a varying quantity to remain constant, the unit must change so that its magnitude and the quantity’s magnitude always have the same relative size. In Figure 4.2 we see three instances of a varying quantity’s magnitude and a measurement unit that changes in tandem so that the quantity’s measure is always 2.5.

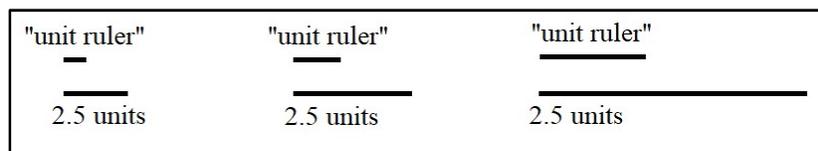


Figure 4.2. An “elastic” ruler that changes with a varying quantity so that the measurement remains constant. Contrast this with a rigid ruler where the length of the unit remains constant and the measurement changes as the quantity varies.

This is how Tallman (2015) conceptualizes radian measure for an angle with a fixed size.

As the size of the circle centered at the angle’s vertex varies, the subtended arc length varies. If we want the measurement of that subtended arc to be fixed (and thus the angle’s measure to be constant), then our measurement unit must be proportional to the circle’s

circumference. The circle's radius is a natural choice for such a unit, although it is not the only choice.

Returning to the tree height example, imagine that the tree grows such that its height at any given time is always the same measurement when using the height one year prior as the unit. What can we conclude about how the tree grows? Assuming the relative size measurement in my running example, at any moment in time the tree's height is always 1.4 times as large as the tree's height exactly one year prior. Equivalently, at any moment in time the tree's height one year in the future will always be 1.4 times as large as its current height. If we allow time to pass such that the tree's height changes, then the tree's height one year earlier/later must also change so that the two magnitudes have a constant relative size of 1.4 (using the earlier height to measure the later height). Note that there are two measurements held constant here: the relative size of the tree heights at two different moments in time and the interval of time between the moments when the tree has the two corresponding heights.

Combining all of the ideas discussed so far allows for the following potentially powerful meaning for *growth factor of a quantity (over some interval of change for a second quantity)*. Coordinating two covarying quantities (call them A and B , with magnitudes represented as $\|A\|$ and $\|B\|$ respectively²⁴) united as a multiplicative object over some interval of conceptual time sees their magnitudes vary from $(\|A_1\|, \|B_1\|)$ to $(\|A_2\|, \|B_2\|)$. The *growth factor of B* over the interval of conceptual time where A varies

²⁴ Here I use the notation and meaning from Thompson (2011), Thompson et al. (2014), and Wildi (1991) where $\|A\|$, the magnitude of quantity A , “is the size of an object having the attribute that is being measured and that is taken to have one unit of that attribute” (Thompson et al., 2014, p. 5).

from $\|A_1\|$ to $\|A_2\|$ is the multiplicative comparison of $\|B_2\|$ to $\|B_1\|$, or $\|B_2\|$ measured in units of $\|B_1\|$, or the value of k such that $\|B_2\| = k \cdot \|B_1\|$. The value of k is a constant across all choices of unit for measuring B provided that the unit is consistent for both measurements. Therefore, if a represents the possible values for the varying quantity A (measured in some appropriate unit) and b represents the possible values for the varying quantity B (measured in some appropriate unit), then the growth factor of B over the interval from $a = a_1$ to $a = a_2$ (an interval of size $a_2 - a_1$) is $\frac{b_2}{b_1}$.

Percent Comparisons

One percent of a quantity's magnitude (at some moment) is a magnitude having the same attribute as the quantity such that $\|A\| = 100 \cdot \|\text{one percent of } \|A\|\|$. In other words, the original quantity's magnitude measures 100 in units of one percent of the quantity's magnitude. Note that I wrote "one percent of a quantity's magnitude (at some moment)" and not just "one percent" because conceptualizing the meaning of "one percent" is again a norming process and has no fixed size across all contexts or even within the same context. For example, consider the following statement where "one percent" is used to reference two different volumes of water: "The current volume of water in the reservoir is 40% of its maximum capacity and is down 15% from the volume at the beginning of the year."

One percent of a quantity's magnitude at some moment is a suitable unit for measuring other quantities of the same quality or even additional instances of the same quantity as it varies. The resulting measurement is a *percentage* value. For two quantities

A and B representing the same quality of objects, measuring $\|B\|$ as a percentage of $\|A\|$ is to think of the value k such that $\|B\| = k \cdot \|\text{one percent of } \|A\|\|$. Similarly, for two instances of quantity A , measuring $\|A_2\|$ as a percentage of $\|A_1\|$ is to find the value k such that $\|A_2\| = k \cdot \|\text{one percent of } \|A_1\|\|$.

In order to demonstrate the reasoning I described I return to the example of measuring a tree at two different moments in time. Suppose that the measurements were taken on January 1 of last year and January 1 of this year. The growth factor over this one-year period is the magnitude of the height on January 1 of this year using the height on January 1 of last year as the measurement unit. See Figure 4.3.

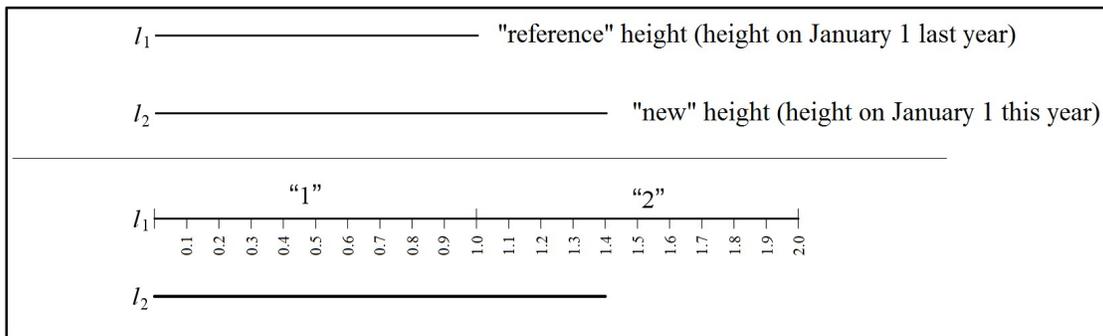


Figure 4.3. Measuring the height’s magnitude on January 1 of this year using the height’s magnitude on January 1 of last year as the measurement unit.

The new height is 1.4 times as large as the reference height. Measuring the new height as a percentage of the reference height is to measure it in a unit that is $\frac{1}{100}$ times as large, and thus the measurement value is 100 times as large. The new height is 140% of the reference height. See Figure 4.4.

Percent Change

Determining a *percent change* in one quantity's value across two instances requires conceptualizing a more complicated quantitative structure. First, the individual must conceptualize a varying quantity within a situation. Second, she must conceptualize two instances of the quantity with some implied or explicit order, the magnitude of the

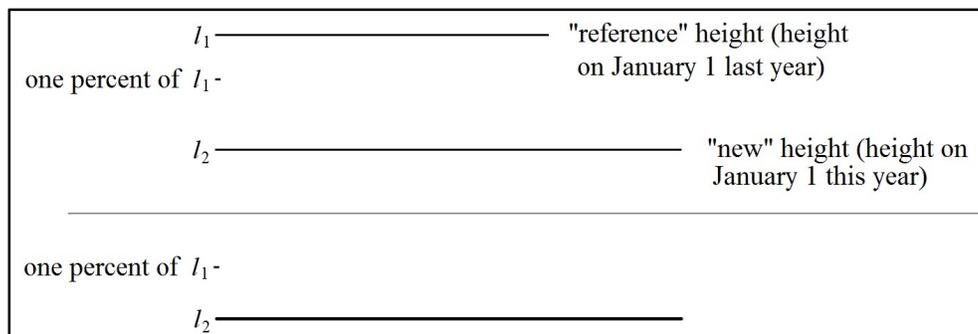


Figure 4.4. Measuring the magnitude of the height on January 1 of this year using one percent of the height on January 1 of last year as the measurement unit. The measurement unit is $\frac{1}{100}$ times as large as the unit in Figure 3, so the new measurement is 100 times as large as the previous measurement.

quantity for each instance, and an additive comparison of these magnitudes. This additive comparison, however, must be like a vector quantity with both a magnitude and a direction in order to capture whether the change is an increase or a decrease. Third, she imagines measuring the additive comparison as a percentage of one of the two magnitudes. So the percent change from $\|A_1\|$ to $\|A_2\|$ can be thought of as the value of k such that $\|\|A_2\| - \|A_1\|\| = k \cdot \|\text{one percent of } \|A_1\|\|$.

In most instances the percent change in one quantity is described with reference to an interval of change for another quantity in order to provide clarity within the given context. For example, reporting a population change of 4.8% is not very informative without describing the time period over which the change occurred. This coordination

involves covariational reasoning including an awareness of the interval of change for the second quantity and perhaps the size of this interval as an additive comparison.

The change in the tree's height from the reference height to the new height (occurring over a one-year period) is an additive comparison that captures both a magnitude and a direction of change. The percent change in the tree's height is the measurement of this change using one percent of the reference height as the measurement unit. See Figure 4.5.

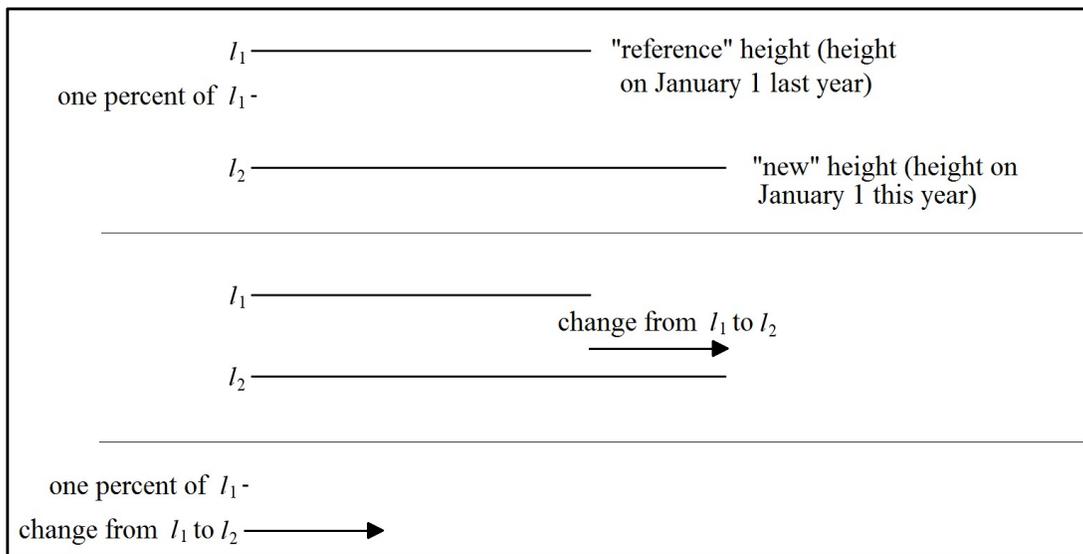


Figure 4.5. Measuring the change in height using one percent of the height on January 1 of last year as the measurement unit.

Exponential Functions

Many students enrolled in the Precalculus course under construction are either taking their final math class or will go on to enroll in a course using the *Precalculus: Pathways to Calculus* (Carlson, Moore, & Oehrtman, 2016) curriculum. Based on this assumption, the image of “exponential function” I intend students to construct is as

follows. An exponential function f of y with respect to x can be defined as a relationship where the relative size of $f(x_2)$ and $f(x_1)$ is constant whenever $\Delta x = x_2 - x_1$ is constant throughout the function's domain. The relative size measurement is the value of the Δx -unit growth factor. See Figure 4.6 for a diagram of the quantitative structure for understanding an exponential relationship based on my conceptual analysis. The framework in Table 4.1 describes the key ways of understanding exponential growth

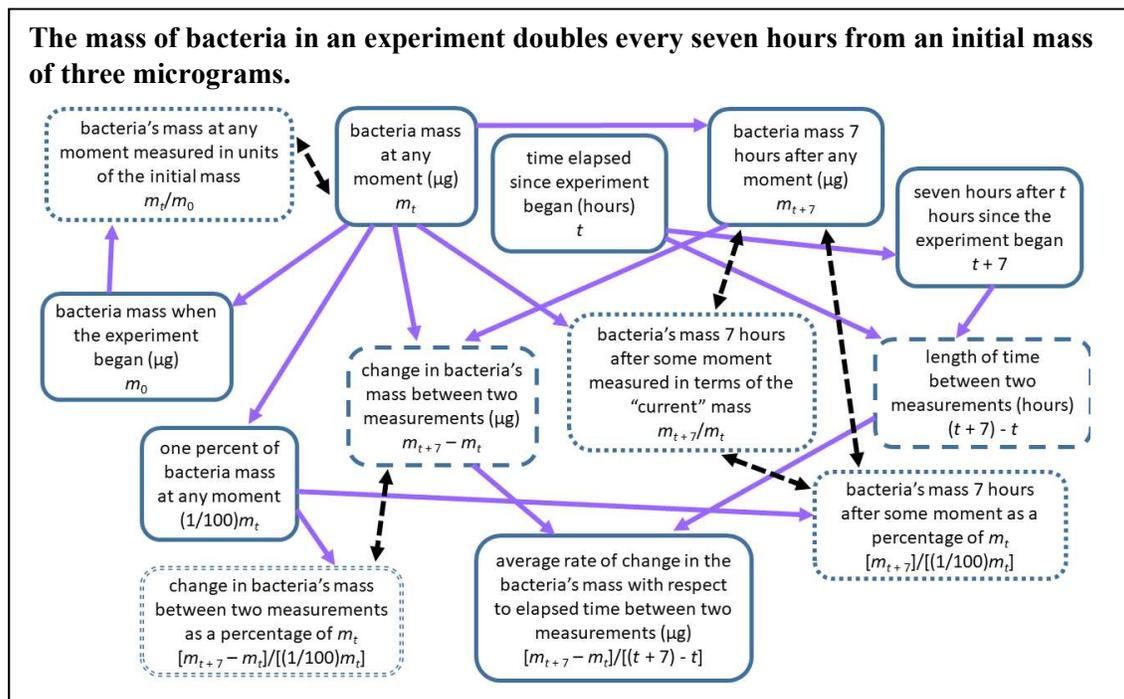


Figure 4.6. My quantitative structure for understanding an exponential model. Note that single-headed arrows imply a hierarchy of meanings such that conceptualizing the quantity at the tail of the arrow likely is a prerequisite for conceptualizing the quantity at the head of the arrow. Dashed boxes indicate a change in a quantity while dotted boxes indicate relative size comparisons between two instances of a quantity. Dashed, double-headed arrows indicate the same quantities measured in different units.

and its related ideas. Note that this framework was influenced by Ström's (2008) final framework for understanding exponential growth, Ellis et al's (2016) description of their learning goals for students, and my conceptual analysis outlined in this chapter.

Table 4.1

A Framework for Understanding Exponential Growth and Related Ideas

<p>MCM: Multiplicative comparisons as measurement systems, which refers to comparing the value of two quantities multiplicatively (using the value of one quantity as a “measurement stick” with which to report the measurement of the other quantity). The measurement reported is a quotient. This can include comparing two values of the same quantity at two different moments when the quantity was measured.</p> <ul style="list-style-type: none"> • MCM-1: Assuming comparisons between the same types of quantities, a multiplicative comparison of two quantities reports the value of one quantity using the magnitude of the other quantity as the measurement unit. • MCM-2: If the unit of measure is scaled, the measure of a quantity changes in a specific way. If the new unit of measure is k times as large as the previous unit of measure, then the quantity’s measurement in the new unit is $1/k$ times as large as its measurement reported in the previous unit. • MCM-3: Two instances of the same quantity can be compared multiplicatively using one instance of the quantity as the measurement unit for the second instance. • MCM-4: If the two quantities are proportional, or if the two instances of the same quantity maintain a relative size as they quantity changes, then the measurement of one in units of the other is constant even as the quantities change. <p>MCP: Multiplicative comparisons reported in terms of percentages. This is a slight modification to the MCM in that we are not using $1/100$ of the value of one quantity as the “measurement stick” instead of the full value of that quantity.</p> <ul style="list-style-type: none"> • MCP-1: A <i>percentage</i> is measurement that uses $1/100$ of the value of some quantity as the measurement unit. • MCP-2: Building on MCM-3, if a quantity’s value is used as a measurement unit, then using this quantity as the reference for a <i>percentage</i> measurement will yield a value 100 times as large. • MCP-3: A percentage comparison between two instances using one instance as the measurement unit will always be 100% larger than the measurement of the <i>change</i> in the quantity using the same unit. <p>MCPC: Leveraging MCP thinking to reason about the change in a quantity’s value (so the change in a quantity’s value is measured using $1/100$ of the quantity’s starting value as the “measuring stick”).</p> <ul style="list-style-type: none"> • MCPC-1: Measuring a percent change involves 1) additively comparing two measurements of the same quantity in some order and 2) multiplicatively comparing the additive change to $1/100$ of the value of the reference quantity’s value. • MCPC-2: The percent change is 100 times as large as the multiplicative comparison between the change in value and the value of the same reference quantity. • MCPC-3: When applying repeated constant percent changes, the reference value for determining the change in value updates at the end of each interval. • MCPC-4: Equal percent changes over different intervals produces non-constant absolute changes over those intervals. <p>MCEF: Thinking about an “exponential function” as an invariant relationship between two continuously co-varying quantities x and $y = f(x)$.</p> <ul style="list-style-type: none"> • MCEF-1: Within the domain of a function relationship, we can hold in mind two instances of the same quantity over a constant interval of the second quantity. • MCEF-2: The one-unit growth factor is a measurement determined by using one instance of a quantity as the unit ruler for measuring a second instance of the same quantity occurring when a second quantity varies by one unit. This measurement is constant even as the quantities vary. • MCEF-3: For every choice of c there exists a constant d such that $\frac{f(x+c)}{f(x)} = d$ as x varies continuously throughout the domain. [When $c = 1$, then the constant is the one-unit growth factor typically given as “b” in the formula $y = ab^x$]. • MCEF-4: If b is the one-unit growth factor, then b^c is the c-unit growth factor (i.e., the ratio $\frac{f(x+c)}{f(x)}$ for all x). • MCEF-5: If b is the one-unit growth factor, then b^x is the ratio $\frac{f(x)}{f(0)}$ and $f(x) = ab^x$ is one representation of the formula for the relationship where $f(0) = a$.
--

Comments on the Conceptual Analysis

Throughout this conceptual analysis and examples I maintained a consistent focus on three questions at the heart of quantitative reasoning: 1) What am I measuring? and 2) What is my measurement unit? 3) What does the value of my measurement represent? This attention ensured that the values I produced had clear referents within a quantitative structure because my measurement schemes derived from the ways in which I conceptualized the quantities and relationships between quantities. Such a focus is different from common instructional goals I have observed with in-service teachers. For example, teachers typically describe a growth factor as the number you multiply by some number of times to evaluate a formula, the percent comparison is the result of taking the growth factor's value and moving the decimal point two places (such as changing 1.05 to 105), and the percent change is this value minus 100. Such a treatment encourages the tendency for students to “employ numerical operations that have no quantitative significance” (Thompson, 2011, p. 38). Notions of measurement are absent from this treatment as are any conceptual foundations for the operations. This may explain why, for example, only about 35% of college precalculus students who took Carlson et al.'s multiple choice Precalculus Concept Assessment at the end of their courses could identify the difference between the growth rates for the functions $p(t) = 7(2)^t$ and $p(t) = 7(3)^t$ (Research and Innovation in Mathematics and Science Education, 2007). Reasoning quantitatively in the manner that I described may support students' ability to reason about percent comparisons and percent change regardless of the context or the way in which a task is presented and provides a conceptual foundation for reasoning about exponential growth.

Settling on Some Design Principles for the Online Intervention

Before outlining the specific tasks within a hypothetical learning trajectory based on my conceptual analysis, I will discuss the general principles I followed in designing the online unit.

Repeated Reasoning [Revisited]

In Chapter 1 I described Harel's meaning for repeated reasoning (or repetitive experience) and noted that "what we want repeated is the constitution of situations in ways that are propitious for generalizing assimilations, accommodation, and reflection (Thompson, 1994c, p. 227). Thompson (1991) explained one aspect of why this is so critical. In the course of a single year, students are often expected to learn (or memorize) numerous algorithms, definitions, and terms. Seen by students as disconnected facts dictated by a mathematical authority figure, the expectation is ludicrous. However, if students are engaged in repeated experiences where they are expected to develop algorithms in the course of developing stable meanings, then they build schemes that contain within them these algorithms along with powerful mental imagery supporting the algorithms connected to related schemes. This helps make problem solving routine, supports students seeing mathematics as a structured discipline, and increases students' retention of mathematical knowledge. Steffe (1996) noted that the constructs of genetic epistemology and radical constructivism suggest that only knowledge useful to the individual develops and persists. If an instructor can design activities that require specific meanings useful to students' mathematical development and can support students' successful engagement in those activities, then students stand the greatest chance of

developing meanings in line with the instructor's goals and the knowledge is more likely to persist. Furthermore, Thompson et al. (2014) describe how *functional accommodations* to a scheme often occur when someone is first learning a new idea. The accommodations are fragile and often do not persist. To achieve stable ways of thinking means the "person has developed a pattern for utilizing specific meanings or ways of thinking in reasoning about particular ideas" (p. 14), and this only occurs through repeated reasoning about those ideas.

It is also worth mentioning that repeated *quantitative* reasoning is likely to contribute to students transferring their knowledge across settings and applied contexts. Transfer is a notoriously slippery notion in education research, made more challenging because of how differently transfer is described by researchers working within different background theories. Lobato and her colleagues (Lobato, 2006; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002) defined transfer from the individual's perspective instead of focusing on transfer relative to the types of problems students can solve. They described transfer occurring when the student recognizes that a way of thinking can resolve problems they conceptualize in different settings. In contrast, traditional work on transfer relies on the researcher's determination about what aspects of tasks are invariant. As I outlined previously, for a way of thinking to develop and become incorporated within a student's scheme(s), the student must engage in repeated reasoning. Once students have stable meanings then tasks assimilated to that scheme (including what might be considered different tasks from an outsider's perspective) tap into the same "space of implications" (Thompson et al., 2014, p. 13) including the same patterns of reasoning. In

other words, students exhibit transfer when they assimilate what (to the observer) are different problems to the same scheme and thus conceive of them as the same problem.

The notion of transfer also helps explain the potential source of students' success (or failure) in mathematical modeling in applied contexts. Thompson (2011) defines a *mathematical model* as “a generalization of one’s understanding of a situation’s inner mechanics—of ‘how it works’” (p. 51). Students practicing repeated quantitative reasoning develop schemes relative to quantitative relationships within problem contexts. Students predisposed to engage in quantitative reasoning in applied contexts conceptualize situations to form quantitative structures. Recognizing potential similarities in the structures they form compared to other contexts or to general mathematical ideas (such as piecewise functions, area of a geometric figure, etc.) facilitates assimilation of problems to the same scheme. Thus, the space of implications within the scheme support modeling situations using mathematical tools and facilitates connections across applications.

Design Principles for Online Courses

Since the research literature provides little guidance on online course design from a cognitive learning standpoint,²⁵ it seems logical to begin with the assumption that theoretical perspectives that do work in understanding student learning in traditional classroom environments are applicable in an online environment. Thus, I used genetic epistemology and radical constructivism as background theories and Harel’s DNR

²⁵ Thompson (1985) being a notable exception even though his work was not framed within an online learning environment.

framework and Thompson's theory of quantitative reasoning as domain-specific theories to inform my instructional design choices including the following.

- Students construct their own meanings. Therefore, a designer cannot expect that students will learn what they intend if lessons are designed using a “telling” model of direct instruction. Lessons must engage students in interactions that encourage accommodating their schemes in productive ways through reflecting abstracting.
- Lessons must devote time to helping students develop an intellectual need for learning certain ideas by helping them conceptualize a problem the new meanings might solve. In addition, the designer should make sure that students can explore contexts and conceptualize quantities and relationships between quantities prior to expecting students to solve a variety of problems related to the context.
- Lesson tasks and homework should focus on repeated reasoning (repetitive experience) over rote practice and designers should explore ways to help students recognize invariance in their reasoning across contexts and problems related to specific ways of knowing.
- Course design should take seriously the duality principle and not just focus on supporting disconnected ways of knowing. It is important to promote coherent ways of thinking, such as quantitative and covariational reasoning throughout the course. Furthermore, the designer should make an effort to help students recognize and reflect on the invariance in their ways of thinking across different topics and problem contexts.

The research literature does provide a nice checklist for designing features of the course to minimize distractions that could interfere with student learning. For example, designers should work to make the course easy to navigate and check it carefully to ensure that there are no broken links and that the course runs on various computer platforms using a variety of web browsers. Designers should also use a variety of formats within lessons such as videos, text, applets, and questions rather than just posting hour-long lectures into a web page and asking students to watch them. However, there are still unresolved issues worthy of study and to which I must be sensitive within the scope of a study looking at students' mathematical development. What length is appropriate for an online lesson (both in terms of number of questions, videos, applets, etc. as well as the expected time commitment for completing a lesson)? What level of detail and differentiation should I include when programming feedback for student responses? What is the appropriate balance between length and repeated reasoning that will maintain student engagement and promote productive accommodations? What is the appropriate length for text and videos within a lesson? Any study focusing on students' development and modification of schemes related to some set of mathematical ideas must use their best judgement in answering these questions during the design process. Researchers (and course designers) must also devote some time to monitoring the efficacy of their decisions and modify their materials accordingly if they feel that general design decisions are undermining their primary goal of improving student learning of important mathematical ideas.

As a final comment, I will elaborate on two issues: student engagement and repeated reasoning. What does it mean for a student to be "engaged" in a lesson? There

are two ways to think about this issue. A common synonym for student engagement is *participation*. Some educators may tacitly assume that if students participate in a lesson (pay attention to lectures, complete assigned problems, avoid off-task behavior like talking or texting, etc.) that they will learn. Even if true, this assumption is insensitive to *what* students might be learning. Thompson (1991) argued that what a teacher intends students to be practicing (and thus learning) is not isomorphic to what students actually practice (and thus learn). Without being sensitive to instructional objectives in terms of desired ways of thinking and knowing, examining student engagement considers nothing more than “time on task”. There are many ways to increase students’ time on task that do not necessarily improve learning. For example, a course designer can infuse course videos with humor to increase the time students spend watching videos without any guarantee that this leads to improved learning.

Defining student engagement in terms of students’ activity in conceptualizing problem contexts, differentiating their actions from the results of those actions, and modifying cognitive schemes through various types of abstraction is more productive. Designing for this kind of engagement means taking seriously interiorization, internalization, repeated reasoning’s role in developing and modifying schemes, in particular remaining aware that students’ personal contributions to the lessons are the driving force in their learning. For example, if I want students to have a quantitative meaning for a growth factor of 1.4, I need to design the lesson so that students have repeated opportunities to conceptualize using the magnitude of a quantity in one instance to measure the magnitude of that quantity in another instance. This can involve videos and texts demonstrating this idea, but it must also involve interactive applets where the

students use applet states to demonstrate their understandings, estimate growth factors from visual displays of two magnitudes, and derive methods for calculating growth factors based on their own emerging imagery. Thus, designing for student engagement in cognitive terms involves clearly describing the kinds of imagery the designer wants students to develop, creating an applet, task, or question that encourages students to develop this imagery through their own activity, and designing assessments that test students' meanings and not just calculational proficiency.

If repeated reasoning is important for student learning, how do we integrate it as a key design principle? What are the most important considerations? A starting point is to be clear about the meanings we hope students construct prior to designing any aspect of a lesson, unit, or course (Thompson, 1991). This is best accomplished through a detailed conceptual analysis so that a designer knows the kinds of meanings and imagery that are likely to support students in modifying their schemes in productive ways. Next, the designer needs to identify or create tasks that promote recognition of invariant reasoning while also providing enough variety of contexts, given information, and complexity so that developed schemes are reasonably robust (Harel, 2008c). As much as possible, student activity should drive the development of algorithms and formulas, and practicing procedures should build from and consistently reference and reinforce corresponding conceptual foundations. In short, the practice we provide students should focus on engaging them in recognizing invariant reasoning across problem contexts rather than rote answer-getting where what “they practice [is] ignoring such things as context, structure, and situation” (Thompson, 1991, p. 269).

Select Tasks within a Hypothetical Learning Trajectory

I wrote the conceptual analysis in this chapter to unfold in a manner consistent with how activities and investigations would unfold to support the productive meanings for exponential growth and related ideas outlined in Table 5. A very brief overview of the hypothetical learning trajectory for the unit is given below with the acknowledgement that “[b]uilding sophisticated quantitative reasoning skills for the majority of students is not a one or two year program; it requires development throughout the elementary and middle school years” (Smith & Thompson, 2007, p. 43). Knowing that many students likely have not experienced rich explorations of quantities and have not engaged in sophisticated quantitative reasoning tempers my expectations for the learning goals in the course. However, I remain optimistic that many of these goals are within reach of most students in some form if the course as a whole is coherent and designed to build towards these goals.

1. Students develop basic fluency in quantitative and covariational reasoning beginning with a recognition of quantities in situations, differentiating between varying and constant quantities, and loosely coordinating covarying quantities (for example, recognizing that as one quantity increases another quantity decreases). These are key ways of thinking to develop and support throughout the course to provide coherence among topics and to build a meaningful image of function relationships.
2. Students understand a measurement as a “times as large as” comparison between the magnitude of a quantity and a measurement unit having the same quality. An explicit focus on quantification schemes helps students refine their images of quantities. In addition, this is an important prerequisite image for thinking about a norming process

and for giving meaning to the value of a growth factor as a multiplicative comparison between a quantity's magnitude at two different moments.

3. Students differentiate the value of a quantity from a change in the value of the quantity and thus conceptualize them as separate but related quantities. This is an introduction to quantitative operations and to conceptualizing measurement schemes for intensive quantities. It will also be important for students to distinguish between and relate quantities, changes in quantities, and multiplicative comparisons within a quantitative structure in order to conceptualize a percent change.
4. Students understand norming processes as the foundations for multiplicative comparisons. Students must be able to flexibly choose reference magnitudes for a quantity as a measurement unit for describing the size of other quantities or the size of other instances of the same quantity in order to make sense of growth factors and percentages. In addition, encouraging flexibility in how students think about choosing a unit is likely to improve their understanding of measurement in general (see (2)) and the sophistication of their quantification schemes. Note that this is the point at which Thompson's didactic object can be used productively to promote useful imagery of growth factors as multiplicative comparisons in continuous exponential functions.
5. Students apply norming processes to understand the meaning of one percent and percentage measurements relative to specific quantities. This is an extension of (4) that also emphasizes the inverse proportional relationship between a quantity's measure and its unit (smaller units yield larger measures and larger units yield smaller measures).

6. Students measure the change in a quantity as a percentage relative to the value of the quantity at the beginning of an interval, thus building a quantitative meaning for percent change. Again, Thompson's didactic object can be used productively at this stage by drawing a horizontal line at the height of the left line segment and using this as a reference to "see" the difference between the magnitudes in order to focus attention on how to measure this change.
7. Students compare and relate growth factors over different intervals. For example, if a radioactive substance has a half-life of 600 years, then every change of 1,200 years yields a mass that is one-fourth of its previous mass. Students also experience the usefulness of determining a growth factor of one quantity that corresponds with a change of "1" in the other quantity (measured in whatever unit they choose). I envision using Thompson's didactic object here with some modifications. Instead of one interval of a given length there are multiple consecutive intervals of the same length and the corresponding magnitudes of the quantity at the ends of each interval are displayed as segments that all move in tandem as the slider changes. Tasks prompt students to measure the lengths of all of the line segments relative to the length of a single line segment and to justify the relationships between measurements.
8. Students build a process conception for the expression $b^{\Delta x}$ as representing the growth factor for any interval of size Δx (where b is the growth factor for an interval of size 1). This generalization of the work in (7) can support the eventual process conception of the formula $f(x) = ab^x$ as relating all values of two quantities in an exponential function that goes beyond thinking about repeated multiplication of a 1-unit growth factor.

9. Students build a process conception for b^x as the growth factor relating any function value $f(x)$ to $f(0)$ (i.e., over an interval of size x), and thus understand $f(0) \cdot b^x$ as the value of $f(x)$ for all x .

The following nine tasks are sample key exercises within a trajectory designed to support student in constructing the meanings outlined in my conceptual analysis. The unit as a whole is much longer, with opportunities for repeated reasoning, for examining similar ideas in different ways, and includes instructional videos, text, and student feedback that are not included here. But these select tasks capture the essence of the activities in which students will engage while taking the course.²⁶

Task 1: “Elastic Ruler”

In modules prior to the one dedicated to exponential functions we designed activities to promote students’ conceptualizations of intensive quantities’ values as representing the relative size of two quantities’ magnitudes and the value of the intensive quantity in terms of using one magnitude as a “unit ruler” for measuring the other magnitude. For example, in a linear function the constant rate of change can be productively thought of as the size of Δy measured with Δx as the “unit ruler”. Likewise, the scale factor relating corresponding sides in similar triangles can be thought of as the lengths of the sides in one triangle using the corresponding sides in the other triangle as the “unit rulers”. I do not claim that this is the most sophisticated way of conceptualizing

²⁶ Note that I purposely only chose tasks where the 1-unit growth factor is larger than 1. Relationships with a 1-unit growth factor between 0 and 1 are included in the module but I chose not to address them here to maintain a tighter focus.

the values of intensive quantities²⁷, however this way of thinking can do work for students throughout the course while representing a reasonable learning goal for students within a single course.

In addition, it is possible for the value of the intensive quantity to be constant as the compared quantities vary. Again, constant rate of change is an excellent example since the relative size of Δy and Δx is constant as the values of the quantities vary. Understanding the meaning of a Δx -unit growth factor requires building on the reasoning students used when learning about constant rate of change with some subtle but important differences. The value of the growth factor is a measurement of the relative size of two instances of the same quantity even as that quantity varies, but that these two instances must be measured at some specific interval of change in the second quantity. To support this understanding, I start with exercises similar to the task in Figure 6 (adapted from Tallman's (2015) activity on necessary properties of units for measuring angles).

Each of the following diagrams shows a magnitude for a varying quantity along with a ruler for measuring the quantity. For each, determine if the given ruler will produce a constant measurement value for the varying quantity.

a. Will the given ruler (shown below) produce a constant measurement for the varying quantity?



Does the measurement value remain constant?

In the online investigation there are multiple iterations of this task. The length of the top segment varies as does the length of the “elastic ruler”. In some cases there is a proportional relationship and in other cases there is a different kind of relationship.

Figure 4.7. Task 1 of Instructional Sequence for Exponential Growth and Related Ideas.

²⁷ See Thompson et al. (2014) for more information on various schemes for thinking with magnitudes of quantities.

We used similar diagrams earlier in the course to facilitate students in constructing the meaning of a measurement as a relative size comparison between the magnitude of a quantity and the magnitude of a unit. In addition, similar diagrams with varying components were also used in earlier lessons to support students in reflecting on the impact on the measurement of a quantity when 1) the quantity varies but the unit is fixed, 2) the unit varies but the quantity is fixed, and 3) both the quantity and the unit vary. These exercises are additional opportunities for students to reflect on (3). The task is primarily designed to support students in pseudo-empirically abstracting the property that the measurement of a varying quantity is constant when the measurement changes to always be the same proportion of the quantity's magnitude. In these activities students can determine if specific examples display this property by looking at the measurements produced by specific pairs of a co-varying quantity and unit of measure.

Task 2: Comparing Vine Lengths

Task 1 encouraged students to determine measurements for quantities as a relative size comparison to a unit and to coordinate this measurement as the quantity and unit co-vary relative to experiential time. Eventually the goal is for students to explicitly control this coordination in terms of conceptual time, which I conjecture is necessary for holding in mind two instances of the same quantity separated by some interval of a second quantity (that is not experiential time) and imagining their relative sizes as constant. Task 2 shown in Figure 4.8 (and similar tasks) reverses the process and provides students with a measurement of relative size for two instances of the same quantity and asks students to modify an applet state that would produce the given measurement.

Over any 1-day period, the magical vine grows so that it is always 1.8 times as large as its length at the beginning of the period.

The following diagram represents the vine's length 2.3 days since it started growing. Using the purple X, set the vine's length 3.3 days since it started growing.

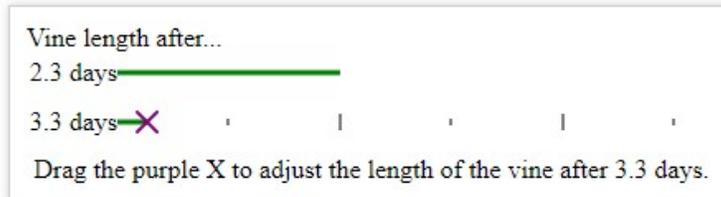


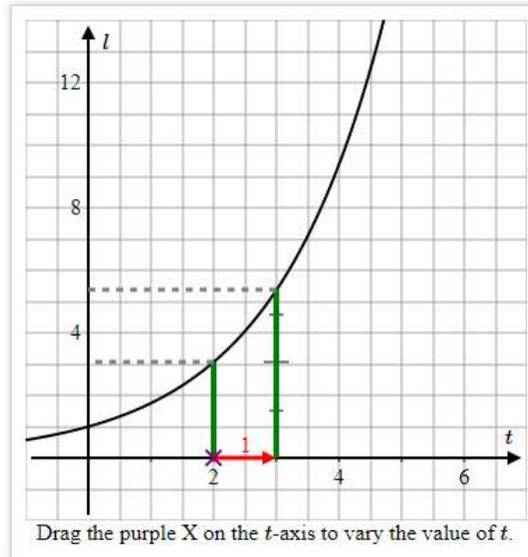
Figure 4.8. Task 2 of Instructional Sequence for Exponential Growth and Related Ideas.

I conjecture that repeated reasoning with this reverse process will help students construct the actions involved in maintaining (and measuring) a constant relative size of two instances of a quantity as mental operations. These tasks also include the potential for students to attend to intervals of a second quantity over which we compare instances of the first quantity. Note also that both Task 1 and Task 2 require students to reason about quantities' magnitudes without having those magnitudes each explicitly measured in a common unit (such as feet).

Task 3: Identifying Growth Factors

Task 3 (see Figure 4.9) represents a transition to making more explicit a coordination of two instances of one quantity over an interval of a second quantity and requiring students to extend the action of measuring the relative size of two instances of a quantity to a graphical representation of the relationship. In addition, students need to express their meaning for the resulting measurement. I designed part (b) to encourage students to reflect on their activity and the meaning of the measurement produced in part

Let's imagine a new vine with a different growth factor. Let $l = f(t)$ represent the length of the vine (in feet) t days after it started growing.



- The 1-day growth factor is approximately $b =$ _____.
- If b represents the 1-day growth factor, which of the following is true? Select all that apply.
 - The vine's length doubles every b days.
 - No matter what the change in time elapsed since the vine began growing, the vine's new length is b times as large as its previous length.
 - When the number of days the vine has been growing changes by 1 day, the vine's new length is b times as large as its previous length.
 - When the number of days the vine has been growing changes by 1 day, the vine's new length changes by b feet.
 - The vine's length changes at a constant rate of b feet per day.

Figure 4.9. Task 3 of Instructional Sequence for Exponential Growth and Related Ideas.

(a). I conjecture that repeated reasoning with similar tasks will support students in refining their coordination of the mental actions involved in measuring relative sizes of two instances of a quantity and holding this relative size constant as the quantity varies.

Tasks 4 and 5: Percent Comparisons and Making Connections

Task 4, along with the accompanying videos and similar tasks, is designed to support students in abstracting the relationship that measuring the same magnitude with a ruler that is $\frac{1}{100}$ times as large results in a measurement that is 100 times as large. See Figure 4.10.

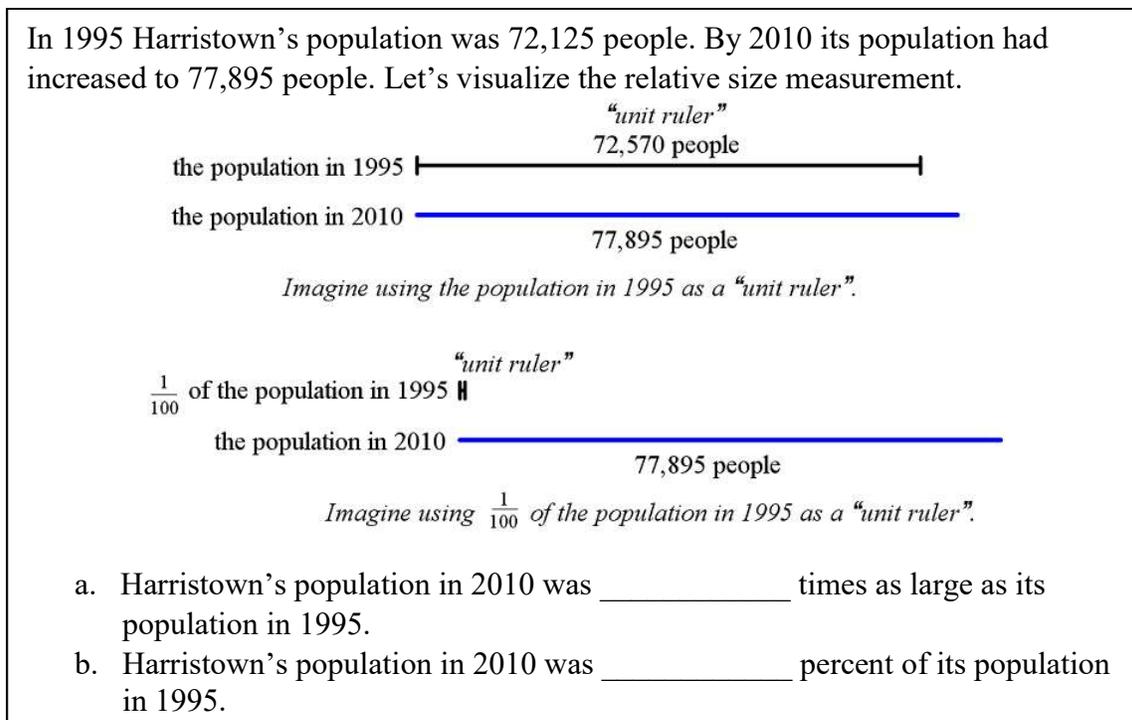


Figure 4.10. Task 4 of Instructional Sequence for Exponential Growth and Related Ideas.

Prior to this point in the module the tasks encouraged students to leverage the imagery of measuring a quantity's magnitude with a "unit ruler" by comparing their relative sizes but without having these two magnitudes measured in a separate standard unit. In this task students could estimate the relative size of the town's population at two different moments, but to improve their accuracy (and thus to get the answer correct) they must divide the population measurements (each expressed in a number of people) to determine their relative size. It is possible that students can complete these tasks by making a

pseudo-empirical abstraction based on the pattern of results from calculating a growth factor and converting between a growth factor and percent comparison. This abstraction would involve noticing that moving the decimal point two places converts between the two numerical values. To reflectively abstract the relationship between a multiplicative comparison and a percent comparison involves coordinating the mental actions of producing measurements using two different units, comparing the relative size of the units, predicting the relationship between the measurements based on the relative size of the units, and knowing that the relative size of the measurements holds regardless of the size of the reference quantity. In an online environment with no direct interaction with the students it can be difficult to look at student responses to exercises such as Task 4 and build an accurate model of students' schemes. See Chapter 5 for a discussion of how I will use clinical interviews to improve my ability to model specific students' schemes at the end of the instructional sequence.

Task 5 (see Figure 4.11) and similar tasks are designed to address the potential issue just raised. By not asking students to provide the products of calculations, but rather to interpret the quantity being measured and the unit of comparison, tasks such as these encourage students to reflect on measurements as the result of processes. Similar to the relationship between Task 1 and Task 2, Task 5 is a reversal of Task 4. I conjecture that this task could support important reflection through which students differentiate between the results of converting a growth factor to a percent comparison (or a change to a percent change) and the process of changing the measurement scheme for quantities to produce different measurements for the same quantity.

The city of Trenton received 60 inches of rain in 2014. In 2015, the city received 69 inches of rain.

year	Trenton annual rainfall (in inches)
2014	60
2015	69

For each of the following statements, determine which quantity is being described and what “unit ruler” is used for making the measurement.

The annual rainfall in 2015 was 1.15 times as large as the annual rainfall in 2014.

“1.15” measures [Select Answer] and the “unit ruler” is [Select Answer].

The annual rainfall in 2015 was 115% of the annual rainfall in 2014.

“115” measures [Select Answer] and the “unit ruler” is [Select Answer].

The annual rainfall increased by 9 inches from 2014 to 2015.

“9” measures [Select Answer] and the “unit ruler” is [Select Answer].

The increase in the annual rainfall was 0.15 times as large as the annual rainfall in 2014.

“0.15” measures [Select Answer] and the “unit ruler” is [Select Answer].

The annual rainfall increased by 15% from 2014 to 2015.

“15” measures [Select Answer] and the “unit ruler” is [Select Answer].

Figure 4.11. Task 5 of Instructional Sequence for Exponential Growth and Related Ideas.

Task 6: Two Consecutive Intervals

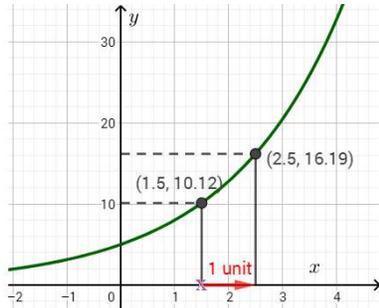
Consistent with the targeted meaning for exponential growth described earlier, I conjecture that students need to develop a process image of $b^{\Delta x}$ as the value of the ratio

$\frac{f(x + \Delta x)}{f(x)}$ for all x (with b as the 1-unit growth factor) in order to have a productive understanding of the expression ab^x as representing all of the possible output values of the function as x varies. I designed Task 6 (see Figure 4.12), and related parts of the lesson and tasks, to encourage students to reflect on the relationship between 1-unit growth factors and 2-unit growth factors as well as how to represent the values of the growth factors using mathematical notation. Repeating this reasoning for arbitrary interval sizes may help students generalize the value of $b^{\Delta x}$ as the ratio of $\frac{f(x + \Delta x)}{f(x)}$ for all x (even if the generalization is based on a pseudo-empirical abstraction).

Tasks 7 and 8: Writing a Formula to Represent an Exponential Relationship

Task 7 (see Figure 4.13) and related tasks requires students to coordinate the various meanings and imagery constructed in previous tasks to collect the required information necessary to represent an exponential relationship using a mathematical formula. I hypothesize that students who are not successful with this task have not constructed the meanings outlined in my conceptual analysis. It is possible for students to follow procedures to accurately complete this task without having constructed the meanings I intend. The degree to which this occurs, and the degree to which it is possible to complete this task within the lesson trajectory without having constructed the intended meanings, will hopefully be revealed through clinical interviews (see Chapter 5).

The following graph and table represent a function f where y varies exponentially with respect to x . The 1-unit growth factor is 1.6.



x	$y = f(x)$
-1	3.125
0	5
1	8
2	12.8
3	20.48

Drag the purple X on the x -axis to vary the value

Complete the following statements.

- The value of the ratio $\frac{f(x+1)}{f(x)}$ is always _____ for all values of x .
- The value of the ratio $\frac{f(x+2)}{f(x+1)}$ is always _____ for all values of x .
- The 1-unit percent change is _____ %.
- The 2-unit growth factor is _____.
- Which of the following is a true statement about this exponential function? Select all that apply.
 - The 2-unit growth factor is the product of the 1-unit growth factor times itself. That is, the 2-unit growth factor is $(1.6)(1.6)$ or $(1.6)^2$.
 - The 2-unit growth factor is two times as large as the 1-unit growth factor. In other words, the 2-unit growth factor is $2(1.6)$.
 - Knowing the 1-unit growth factor does not give us enough information to determine the 2-unit growth factor. We would need more points.
 - The 2-unit growth factor is the same as the 1-unit growth factor because all growth factors are constant for an exponential function.
- Which of the following represents the value of the 2-unit growth factor using mathematical notation? Select all that apply.
 - For all x , $2(f(x))$ represents the value of the 2-unit growth factor.
 - For all x , $\frac{x}{x-2}$ represents the value of the 2-unit growth factor.
 - For all x , $\frac{f(x+2)}{f(x)}$ represents the value of the 2-unit growth factor.
 - For all x , $\frac{x+2}{x}$ represents the value of the 2-unit growth factor.
 - For all x , $\frac{f(x)}{f(x-2)}$ represents the value of the 2-unit growth factor.

Figure 4.12. Task 6 of Instructional Sequence for Exponential Growth and Related Ideas.

The population of a town increased exponentially over a period of 10 years. The given table shows the town's population at various moments.

number of years since the beginning of 2005	population of the town $p_{\text{town 1}} = f(t)$
0	4,520
1	4,701
2	4,889
3	5,084
5.25	5,553
6.25	5,776

- What was the 1-year growth factor for the town's population? _____
- What was the 1-year percent change in the town's population? _____
- What is the formula for function f that represents the population of the town in terms of the number of years since the beginning of 2005? _____
- The population of a different town also increases exponentially over the same period of time. Use the given table to write the formula for function g that models this town's population in terms of the number of years since the beginning of 2005?

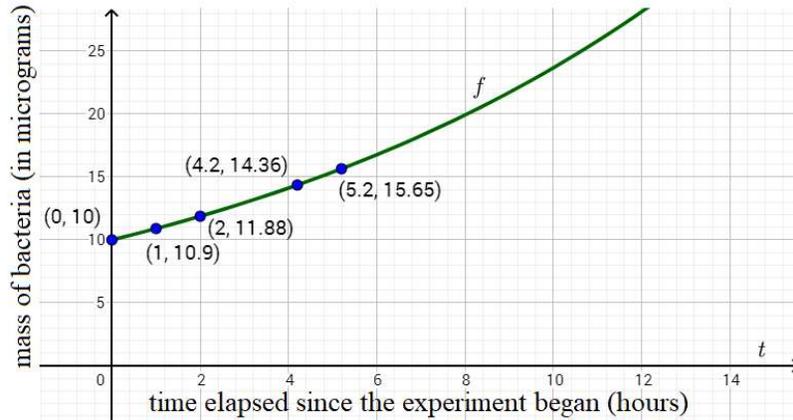
number of years since the beginning of 2005	population of the second town $p_{\text{town 2}} = f(t)$
0	6,884
1	7,022
2	7,162
4.6	7,541
5.6	7,691

Function formula: _____

Figure 4.13. Task 7 of Instructional Sequence for Exponential Growth and Related Ideas.

Task 8 (see Figure 4.14) involves repeated reasoning with the goals in Task 7 with the addition of asking students to compare the behavior of two exponential functions with the same initial conditions and different growth factors. As mentioned earlier in this chapter, many students struggle to compare the behavior of functions with different growth factors, so I designed this task to determine the degree to which the learning trajectory I designed supports students in constructing meanings for growth factor and for

The following graph represents the mass of bacteria (in micrograms) in a petri dish in a laboratory experiment in terms of the number of hours since the experiment began.



- What was the 1-hour growth factor for the bacteria's mass? _____
- What was the 1-hour percent change in the bacteria's mass? _____
- What is the formula for function f that represents the bacteria's mass (in micrograms) in terms of the number of hours since the experiment began? _____
- The experiment was run a second time with a different set of nutrients for the bacteria. In the second experiment, the bacteria's mass (in micrograms) in terms of the number of hours since the experiment began could be modeled by $g(t) = 10(1.11)^t$. Which of the following describes how the results of the second experiment compared to the results of the first experiment? Select all that apply.
 - The initial mass of the bacteria was smaller in the second experiment than in the first experiment.
 - The initial mass of the bacteria was larger in the second experiment than in the first experiment.
 - The mass of bacteria increased by more each hour in the second experiment compared to the first experiment.
 - In both experiments the mass of the bacteria became 10 times as large for each hour that passed.
 - The 1-hour percent change was 2% larger in the second experiment compared to the first experiment.

Figure 4.14. Task 8 of Instructional Sequence for Exponential Growth and Related Ideas.

the symbolic representation of an exponential function relationship that addresses this challenge.

Task 9: Using Two Values to Determine the Formula for the Exponential Relationship

Task 9 (see Figure 4.15) continues repeated reasoning relative to the goals of Tasks 7 and 8 without giving students the value of $f(0)$. I conjecture that students who do not have operational understandings of growth factor and exponential functions similar to the meanings in my conceptual analysis will perform poorly on this task. Students need to have schemes for these ideas that include the ability to identify information they are missing in order to represent the relationship as well as the ability to develop and carry out a plan for determining this missing information. Alternatively, students with a robust understanding of how mathematical notation can be used to represent the relationship, as well as a productive meaning for exponential growth, might define the function as $f(x) = 11(24.2/11)^{x-1}$ or $f(x) = 24.2(24.2/11)^{x-2}$. I see either of these solutions as strong evidence that students have reflectively abstracted a meaning for exponential growth that includes meanings for growth factor and mathematical representations discussed in my conceptual analysis. They are not solution methods the students are likely to find online if they are using outside resources to help them complete assignments and require a clear meaning for what quantities are represented by various parts of the expressions.

Closing Comments and Summary

In this chapter I outlined a conceptual analysis for exponential functions and related ideas that includes meanings I conjecture to be productive for students in an online Precalculus course and a hypothetical learning trajectory intended to support

For a certain exponential function f we know that $f(1) = 11$ and $f(2) = 24.2$. Use this information to answer the following questions.

- What is the 1-unit growth factor? _____
- What is the 1-unit percent change? _____
- What is the value of $f(0)$? _____
- Write the formula for function f . _____
- Evaluate $f(6.31)$. _____
- Function g is represented by the following graph. Write a formula for g .

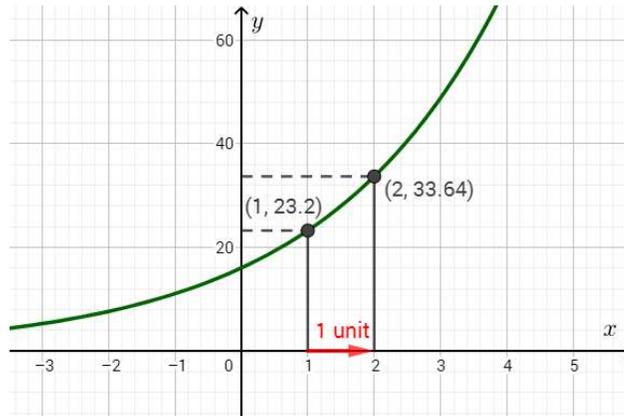


Figure 4.15. Task 9 of Instructional Sequence for Exponential Growth and Related Ideas.

students in constructing these meanings. In the next chapter I will outline a research study intended to examine what students learn from the online course that includes this hypothetical learning trajectory and how features of the course contributed to this learning.

CHAPTER 5

METHODOLOGY

The need for innovation in education is ongoing, as theories of learning and teaching lead to usable knowledge about and reform of instructional practice. (The Design-Based Research Collective, 2003, p. 8)

Q: How do you eat an elephant?

A: One bite at a time! (Author Unknown)

Overview of the Proposed Study

This dissertation study is a design-based research investigation to test a particular intervention conjectured to support students in constructing the meanings for exponential growth and related ideas described in Chapter 4 in an online Precalculus course. As a design-based study I will analyze the proposed intervention's impact holistically relative to a particular group of students working through the activities in a specific setting. Data collection will involve multiple sources and the analysis will attempt to paint a broad picture of the intervention's impact and how particular features of the setting or intervention contributed to the results. I outline the steps in a design-based research study below and will focus on stages one through six in this study.

1. Consult research literature to hypothesize learning goals, activities, and features of the environment that are likely to be productive (see Chapters 2 and 3).
2. Perform a specific conceptual analysis relative to the targeted mathematical ideas and use this conceptual analysis to inform a hypothetical learning trajectory (see Chapter 4).

3. Design the intervention's specific details (see Chapter 4).
4. Collect data from multiple sources during the intervention's implementation.
5. Analyze the data with the goals of

understand[ing] the real-world demands placed on designs...[,]
 develop[ing] better theories of the elements of context that matter for the
 nature of learning...[, contributing to] an understanding of relevant design
 knowledge and practices as they apply to naturalistic settings...[, and
 gaining] insights about what occurs when we orchestrate complex
 interventions in messy settings. (The Design-Based Research Collective,
 2003, p. 8)
6. Broadly share the findings to contribute to the collective knowledge across all fields regarding how to reform mathematics instruction specifically and how to reform instruction in all disciplines more generally.
7. Use analysis of the results to refine the intervention for future iterations.

Before describing my proposed methodology in more detail, I will repeat my primary research questions here for reference.

RQ1: What meanings do students have for exponential growth and related ideas after completing an online Precalculus course design to develop and leverage quantitative and covariational reasoning?²⁸

RQ2: What features of the online course, specific lesson activities, and/or components in the hypothetical learning trajectory appear to support or hinder students in developing productive meanings for exponential growth and related ideas?

²⁸ See Chapter 2 for a detailed description of quantitative and covariation reasoning.

Design-Based Research: Integrating Theory and Practice

As I mentioned in Chapter 1, Cobb (2007) argued that mathematics education “can be productively viewed as a design science, the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes” (p. 7). When Cobb wrote that statement, design-based studies were a relatively new emerging line of inquiry in education research. Researchers using design-based research see themselves as enacting a balance between empirical research and theory-informed instructional design. Their

innovations embody specific theoretical claims about teaching and learning, and help us understand the relationships among educational theory, designed artifact, and practice. Design is central in efforts to foster learning, create usable knowledge, and advance theories of learning and teaching in complex settings. (The Design-Based Research Collective, 2003, p. 5)

Starting with theories of how students learn, design-based researchers generate interventions to enact in specific learning environments. They do not assume that the interventions are necessarily independent from the particular features of the setting in which they are tested, nor is it their goal to design studies that limit variables that arise in the natural course of instruction. “Indeed, such phenomena are precisely what educational research most needs to account for in order to have application to educational practice” (The Design-Based Research Collective, 2003, p. 5). Design-based research often focuses on addressing large-scale problems in education that can be complicated, messy, and challenging but that represent issues at the core of reforming students’ learning experiences – issues such as learning transfer or the impact of technology (Reeves, Herrington, & Oliver, 2005). Thus, design-based research takes a holistic view on

interventions and their outcomes to encompass and embrace the intricacies of improving students' learning experiences.

Design-based researchers' immediate goals focus on impacting student learning in a very practical way. However, they always assume that the intervention will uncover important data with implications for modifying learning theories, supporting practitioners, and designing future interventions, and they intend to share their results broadly. They begin by reviewing and/or developing theories on student learning and then designing and testing interventions that theoretical analysis suggests could productively influence student learning. Figure 16 shows a general overview of the design and analysis process.

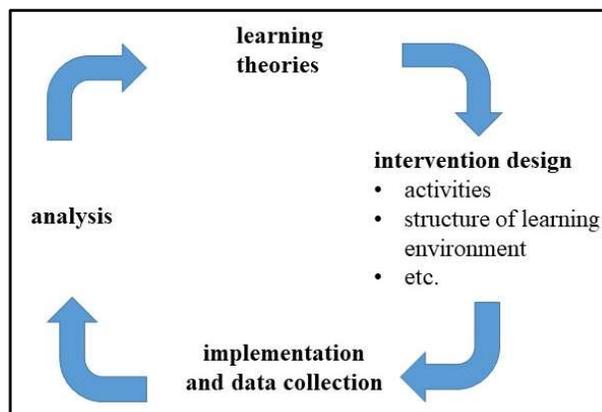


Figure 16. Design-based research's continuous development cycle. Learning theories inform the intervention design, data is collected on the intervention and then analyzed, and this analysis shapes the researcher's understanding of the original theoretical foundations of the study and impacts the design of the intervention in the next iteration (The Design-Based Research Collective, 2003).

Even though learning theories influence an intervention's design, researching its enactment can clarify exactly how features of the intervention impacted learning and how specific features of the learning environment affected the intervention's contribution to the final result. "Models of successful innovation can be generated through such work—

models, rather than particular artifacts or programs, are the goal (cf. Brown & Campione, 1996)” (The Design-Based Research Collective, 2003, p. 7). The following quote best summarizes design-based research.

[D]esign-based research methods can compose a coherent methodology that bridges theoretical research and educational practice. Viewing both the design of an intervention and its specific enactments as objects of research can produce robust explanations of innovative practice and provide principles that can be localized for others to apply to new settings. Design-based research, by grounding itself in the needs, constraints, and interactions of local practice, can provide a lens for understanding how theoretical claims about teaching and learning can be transformed into effective learning in educational settings. (The Design-Based Research Collective, 2003, p. 8)

Although the term “design-based research” was not in use at the time, Thompson (1985) provided an example of a first iteration of a high-quality design-based research.²⁹ Using general theories about how students learn and develop operative thought, Thompson generated specific theories about how students might come to understand integers and geometric transformations in particular ways that his conceptual analysis suggested could be powerful for students. He then designed classroom interventions based on these theories to structure students’ learning experiences, including specific activities within a learning trajectory and a classroom environment with particular norms. He did not assume that the intervention he created could have been implemented identically with other students, in other settings, or with a different instructor. When reflecting on the intervention’s impact, Thompson considered all factors influencing the intervention’s success in a holistic way, and his conclusions helped him refine his theory about how students “who do develop mental operations in regard to algebra” achieve this feat while other students do not (Thompson, 1985, p. 222). But his analysis was not

²⁹ I discussed this study in greater detail in Chapter 2.

removed from the intricate ways in which teacher knowledge and practices, features of activities, the structure of the learning environment, or students' prior experiences all influenced student learning in his intervention.

All scientific enterprises, whether they be in the hard sciences or social sciences, should seek valid, reliable outcomes free from personal bias. In design-based research and similar traditions the researcher is positioned as both the intervention's designer and its evaluator. There is no way to completely remove the potential for personal bias inherent in this situation, but there are ways to help mitigate the potential for biased results (The Design-Based Research Collective, 2003). One method is to collect multiple data sources in order to provide a better overall image of the intervention's impact and to minimize the chance that the final analysis overlooks potentially critical collateral impacts or contributing factors. A second method is to use as large of a sample size as is reasonable in the context and to collect multiple data points for each subject. Third, researchers should take detailed notes on complications and responses to these complications arising during the course of an intervention to strengthen claims about the reasons for specific outcomes. Fourth, since a basic assumption of design-based research is that it involves an iterative process, the researcher should attempt to collect similar data over the course of multiple iterations. A conclusion's validity strengthens when initial observations and hypotheses are integrated and tested "result[ing] in increasing alignment of theory, design, practice, and measurement over time" (The Design-Based Research Collective, 2003, p. 7).

Online Learning of Exponential Growth: A Design-Based Research Study

This dissertation study is based on the principles of design-based research. Informed by constructivist learning theories and a thorough conceptual analysis for learning ideas related to exponential growth, I will 1) design and implement an intervention to support students in learning these ideas in an online environment, 2) collect information about the implementation from multiple data sources, and 3) analyze the data in order to refine my foundational theories related to student learning about exponential functions in an online course and to form conclusions about features of the intervention and/or student or setting characteristics that impacted the results. I will not initially be concerned about constraining the variables involved in the study, instead embracing the notion that attempting to impact student learning in authentic learning environments is inherently messy and complex, but it is within these very environments that we must act to impact students' mathematical experiences. Therefore, this study considers the intervention

holistically...as enacted through the interactions between materials...and learners. Because the intervention as enacted is a product of the context in which it is implemented, the intervention is the outcome (or at least an outcome) in an important sense. (The Design-Based Research Collective, 2003, p. 5)

I outline the phases and components of my study in the following sections.

Developing the Intervention

My intervention involves a module (or unit) on percent comparisons and exponential functions situated within a fully online Precalculus course. The course is designed to engage students in developing and applying quantitative and covariational

reasoning as they progress through common topics traditionally included in a Precalculus course.³⁰ I have completed a conceptual analysis of particular ways of understanding percent comparisons and exponential growth that I conjecture will be powerful for students while also being practical given the students' expected background knowledge and experiences. The module includes approximately 10 investigations, each designed to require 30-60 minutes to complete (although I will focus this research on the first seven investigations that explicitly cover exponential growth, percentage measurements, and percent change). The investigations include homework tasks designed to reinforce and practice the ideas in each investigation with a goal of taking most students 20 minutes or less to complete. Therefore, I designed the module as a whole to take about 10 days (two weeks of the course) with students expected to complete one investigation per day (roughly one hour of work). See Chapter 4 for more information about designing the intervention, including my conceptual analysis and a discussion of key tasks within my hypothetical learning trajectory.

Implementation and Subjects

In Spring 2018 I recruited approximately $N = 70$ students from a Precalculus section at a large university in the Southwestern United States using *Precalculus Online* (O'Bryan et al., 2018).³¹ I did not require subjects to meet any particular criteria except (1) the students enrolled in the course and (2) the student consented to the collection and

³⁰ These topics include, but are not limited to, solving equations, linear functions, systems of linear equations, exponential functions, and quadratic functions.

³¹ Note that if multiple universities choose to pilot the course, then I will consider increasing the N -size for this initial implementation as well as recruit students from multiple universities. In addition, future iterations of the intervention are very likely to feature much larger N -sizes.

use of his or her demographic and performance data. All of these students took part in all phases of data collection as outlined later in this section. I also selected a subset of these students ($N = 5$) for clinical interviews. These five students represented a range of scores on the pre-test. All five students participated in two clinical interviews (Clement, 2000) focused on select items from the pre-test. The pre-interviews occurred during the first two weeks of the course and post-interviews on the same questions occurred during the last month of the course.

I also recruited to recruit approximately $N = 70$ students from face-to-face Precalculus using *Pathways Precalculus* (Carlson, Oehrtman, Moore, 2018), a research-based curriculum designed to promote similar learning goals as the online course we created. I will assess these students' performance on the same concept instrument, the demographics/beliefs instrument, and end of course survey used with the first group of subjects. This data is not designed to provide a statistical comparison because the students are a convenience sample with no guarantee that such comparisons would be valid. Rather, since *Pathways Precalculus* has been proven to significantly improve students' scores on the PCA exam (Carlson, Oehrtman, & Engelke, 2010), a validated instrument assessing students' preparation for Calculus, I wanted a rough comparison of how students using each curriculum responded to tasks related to modeling, linear functions, and exponential functions. Large deviations in scores might suggest areas where our online intervention needs improvement.

Data Collection from Multiple Sources

The data collection will involve three phases and will include mathematics assessments, beliefs surveys, monitoring student progress, and conducting clinical interviews.

- Phase I: Test Data
- Phase II: Monitoring Student Interaction Data
- Phase III: Clinical Interviews

Phase I: Test data. In Phase I subjects completed two tasks within the first week of their courses. The first task was a pretest that included key mathematical ideas and important skills that students will learn during the course (see Appendix A). The assessment included general modeling tasks, tasks covering linear functions and related ideas, and tasks covering exponential functions and related ideas.

The second task was an online form requesting demographic information that also surveyed students' beliefs and attitudes about mathematics. See Appendix B.³² Students completed the beliefs survey again at the end of the course. Recall that Harel (2008b) argued that we must help students see an intellectual need in learning mathematical ideas but we must also address psychological needs, or the motivation and confidence that drive persistence in engaging with problems and seeing them through to the solutions. Students who do not engage with mathematical situations, whether because they lack interest, they do not believe in the importance of what they are learning, or they lack confidence, are unlikely to modify their schemes through repeated reasoning that

³² Note that the three papers resulting from this study did not report data from the beliefs survey, but this data is likely to be included in future reports.

supports reflective abstraction. In addition, student confidence, attitude, and beliefs impact retention rates and student persistence in STEM courses and majors (Byars-Winston, Estrada, Howard, Davis, & Zalapa, 2010; Graham, Frederick, Byars-Winston, Hunter, & Handelsman, 2013; Hutchison, Follman, Sumpter, & Bodner, 2006). I believe it is accurate to say that addressing students' psychological needs is a necessary but not sufficient part of providing higher quality mathematical learning experiences. Therefore, it is important to monitor how the intervention impacts students' beliefs about mathematics and self-confidence. As Spangler (1992) noted, "Students' learning experiences are likely to contribute to their beliefs about what it means to learn mathematics. In turn, students' beliefs about mathematics are likely to influence how they approach new mathematical experiences" (p. 19).

Phase II: Monitoring student interaction data. Phase II involved monitoring students' interactions with the intervention lessons in a very unobtrusive manner. The course is coded in iMathAS (Lippman), and this platform provides instructors and course designers with aggregate feedback on student progress including the average score for students who attempt each problem, the average number of attempts, and the average time spent on each question per attempt. See Table 5.1. Data on average scores and average number of attempts can reveal the difficulty level of various tasks within the learning trajectory. This data pointed to tasks that students found more difficult, and analysis of these tasks can reveal key transitions in reasoning or key constructions in understanding targeted meanings. Alternatively, poor performance on tasks may reveal errors in instructional design that can be corrected for later iterations. Using multiple

Table 5.1
iMathAS Instructor Feedback

#	Question	Grade	Average Score Attempted	Average Attempts (Regens)	% Incomplete	Time per student (per attempt)	Clicked on Help
1	Mod 2 :: Computing Δx	Grade	99%	1.06 (0)	0%	43.13 sec (40.59 sec)	N/A
2	Mod 2 :: Meanings for Δx	Grade	97%	1.13 (0)	0%	3.46 min (3.08 min)	N/A
3	Mod 2 :: CROC, relating Δy and Δx	Grade	85%	1.13 (0)	0%	3.23 min (2.15 min)	N/A
4	Mod 2 :: Rental Car Cost, cost of surcharge + find additional value	Grade	96%	1.06 (0)	0%	2.45 min (2.18 min)	N/A
5	Mod 2 :: Rental Car Cost, proportional?	Grade	75%	1.63 (0)	0%	1.65 min (1.02 min)	N/A
6	Mod 2 :: car distance, compute average speed	Grade	76%	1.63 (0)	0%	2.7 min (1.66 min)	N/A
7	Mod 2 :: Meaning of average speed	Grade	93%	1 (0)	6.3%	5.42 min (4.78 min)	N/A

data sources to analyze the intervention provided a context for making a distinction between the potential reasons for students struggling with certain tasks.

Phase III: Clinical interviews. A *clinical interview* (Clement, 2000) involves asking a subject to answer a number of open-ended questions intended to generate observable behavior a researcher can use to generate models of his or her reasoning.³³ Clement describes two broad categories of clinical interviews called *generative* and *convergent* approaches. In a generative approach, the researcher creates a detailed interview protocol including tasks and various lines of follow-up questions based on the possible types of responses students might give. His goal is to create a hypothesis about mental processes and reasoning that, if a student had them, would reasonably explain the

³³ Clement's methods require that the tasks be well-structured before the interview, which includes thinking about branching paths that the interviewer can follow based on student responses. The same protocol is used with each student interviewed with minimum deviation except to push students to make their responses clearer. The purpose of a clinical interview is not to teach a subject or to get her to answer questions in a particular way. The main goal is to create a model for the subject's schemes and reasoning at one particular moment in time.

subject's observable behaviors. In a generative process it is important that the researcher makes multiple passes through the data to refine and improve his model based on two criteria. First, that the model explains the observable behaviors better than alternative models and, second, that the model is general enough to be useful in a wider range of situations beyond the task at hand. A generative approach is most useful when little is known about student reasoning in the area of interest since it is well-suited to creating initial frameworks for a given idea based on in-depth analysis of a relatively small number of case studies.

A researcher using a convergent approach focuses on frameworks developed out of generative studies and typically code interviews using these frameworks. His goal is to support the viability of the models generated previously and perhaps categorize participants within a framework with the assumption that their coded observable behavior suggests the underlying mental processes described in the framework. Such a study might generate patterns or connections not observed in the previous generative studies, and these patterns could serve as research questions for a new round of interviews beginning with more generative studies.

The clinical interviews in my dissertation served a purpose more in line with a convergent approach. The pre-interviews used select tasks from the pre-test (three general modeling tasks and many of the tasks covering exponential functions and related ideas). The post-interviews used the same tasks plus some follow-up tasks. I used the pre-interviews to model students' mathematical meanings for ideas such as exponential functions, percentage measurements and percent change, and algebraic formulas. I used the post-interviews to assess the degree to which students' meanings shifted during the

course and to determine how students' interpreted key applets and visualizations used throughout the unit. Altogether, data from these interviews led to refinements in my images of how students reason about exponential growth and related ideas coming into a Precalculus course and hypotheses about features of the course or tasks that contributed to students' ways of understanding these ideas.

Comments on Methodology

I based my conclusions primarily on retrospective analysis (Simon, Saldanha, McClintock, Akar, Watanabe, & Zembat, 2010) to characterize how students' conceptions of key mathematical ideas changed as a result of the intervention. My conceptual analysis and hypothetical learning trajectory provided an initial framework for analyzing the data, but where the data suggested alternative meanings and reasoning I broadened my framework to capture the nuances of how students appear to be reasoning in the course and on specific tasks and will explore the potential implications. In my final analysis I sought to provide insights regarding productive and unproductive student meanings, changes in students' understanding or beliefs during the intervention, and features of the intervention that may have supported or failed to support my learning goals suggested by the data.

CHAPTER 6

INTRODUCTION TO THREE PAPERS

While there are many open questions related to the teaching and learning of mathematics, one productive research area seeks to describe overarching ways of thinking to guide student learning, teacher decision-making, and curriculum development. As one example, Thompson (2008b) suggested that mathematics from arithmetic through calculus can be organized into three general themes: the mathematics of quantity, the mathematics of variation, and the mathematics of representational equivalence. These themes can inform productive ways of understanding individual topics and guide instructional approaches while emphasizing deep connections across topics that reinforce a meaningful understanding of the larger themes, such as quantitative reasoning or seeing factoring, simplifying, and combining like terms as generating equivalent representations.

Standards writers, curriculum designers, researchers, and individual teachers typically set student learning goals that are only tacitly connected via general mathematical ideas (Smith & Thompson, 2007; Thompson, 2008a, 2008b, 2013). Supporting students in constructing productive mathematical meanings and an image of mathematics as a coherent, interconnected body of important ideas takes sustained effort over long periods of time, yet “mathematicians and mathematics teachers are too eager to condense rich reasoning into translucent symbolism. They are too eager to get on to the ‘meat’ of the topic, namely methods for answering particular types of questions” (Thompson, 2008a, p. 31-32). Sadly, supporting coherent mathematical experiences for students is a casualty of this tendency despite the fact that targeting a coherent system of

meanings and organizing teacher training and instructional design to support this goal characterizes the most successful education systems in the world (Tucker, 2011).

Coherence and a focus on students' mathematical meanings are the common themes uniting the three papers in this dissertation study. In Paper 1 ("Uses and Advances in Conceptual Analysis and Learning Trajectories for Studying and Supporting Student Learning: The Case of Exponential Functions") I discuss conceptual analysis (Glaserfeld, 1995; Thompson, 2008a) and its role in learning trajectory research. Conceptual analysis is a tool that, among other things, is used to analyze and describe potentially productive and unproductive ways of understanding important mathematical ideas as well as the degree of "fit" between ways of understanding different ideas. Comparing the uses of and conclusions derived from conceptual analysis in three different approaches to teaching exponential functions highlights different ways that conceptual analysis can contribute productively to developing and testing hypothetical learning trajectories. Critical to both interpreting and participating in this work should be a focus on coherence. I argue that those conducting learning trajectory research should be clear about their expectations for how the learning goals in their work connect to other ideas within a course and, potentially, in future courses. Similarly, a review of learning trajectory research suggests a need for researchers to explain how their learning goals are situated within a coherent long-term set of learning progressions. This allows the field to situate and better understand the rationale for recommendations deriving from different studies. This perspective then shifts us away from viewing alternative recommendations as competing. The conversation can instead focus not on whether researchers agree or

disagree but about the usefulness of particular learning goals within specific visions for students' mathematical progressions.

In Paper 2 (“Inattention to Students’ Mathematical Meanings in Online Learning Research: Initial Steps Toward a New Research Focus and Methodology”) I discuss the current focus of online learning research and note that, thus far, few (if any) studies have looked at the meanings students construct as a result of online mathematics instruction. This is an important untapped research area considering that nearly 30% of post-secondary students in the Fall 2016 semester enrolled in at least one distance-learning course (typically delivered online) and that most academic leaders view increasing enrollment in online learning as a key component in their long-term growth strategies (Allen & Seaman, 2015; U.S. Department of Education, National Center for Education Statistics, 2017). Therefore, Paper 2 argues for a new research focus centered around studying the meanings students construct within online courses and presents initial thoughts about research methodologies that might allow researchers to gather evidence of (a) students’ mathematical meanings and (b) how features of online courses either support or fail to support students in constructing targeted mathematical meanings.

In Paper 3 (“Exponential Growth and Related Ideas: Examining Students’ Meanings and Learning in an Online Precalculus Course”) I provide a conceptual analysis describing potentially productive ways of understanding exponential functions and related ideas grounded in images of measurement and discuss how these meanings could fit into a coherent Precalculus course. Student interviews and assessments are used to characterize students’ meanings for key ideas at the beginning and near the end of an online course designed to support students in developing the meanings outlined in my

conceptual analysis. I also discuss initial evidence suggesting (a) aspects of the lessons that successfully supported students in developing at least some of the intended meanings, (b) areas for improvement in the online lessons, and (c) challenges inherent to supporting students in constructing the intended meanings within the online environment. I hope that the three included papers contribute to mathematics education in several productive ways. I want to increase attention on the need for high-quality research focusing on the meanings students possess and the meanings they construct as a result of instruction. This includes characterizing students' systems of meanings and the implications of those meanings and not just characterizing whether those meanings align with expected or canonical understandings (Thompson, Carlson, Byerley, & Hatfield, 2014). Such work is absent from research on online courses and online learning environments. It is also important in learning trajectory research, which could include analyses of students' meanings, the implications of those meanings, and how targeted meanings fit into a broader coherent set of mathematical progressions, not just describing targeted learning objectives and assessing students' performance relative to those objectives. Finally, I want to provide an example of using conceptual analysis to suggest potentially productive ways of understanding a particular set of mathematical ideas and how those ways of understanding may connect to other ideas in a course to create the potential for students to acquire essential reasoning abilities and cross cutting conceptions (e.g., a function graph as emerging from an image of covarying quantities) and thus see the course as coherent. In particular, my analysis attempts to begin unpacking what it might look like to emphasize *thinking with magnitudes* (Thompson et al., 2014) as a

learning goal and empirically explore the challenges, implications, and productivity inherent in targeting this overarching theme in a Precalculus course.

CHAPTER 7

PAPER 1 [LEARNING TRAJECTORIES]

Uses and Advances in Conceptual Analysis and Learning Trajectories for Studying and Supporting Student Learning: The Case of Exponential Functions

Alan E. O'Bryan

Arizona State University

***Abstract:** In this theoretical paper I discuss conceptual analysis of mathematical ideas relative to its place within cognitive learning theories and research studies. In particular, I highlight specific ways mathematics education research uses conceptual analysis and discuss the implications of these uses for interpreting and leveraging results to produce empirically tested learning trajectories. From my summary and analysis I develop two recommendations for the cognitive researchers developing empirically supported learning trajectories. 1) A researcher should frame his/her work, and analyze others' work, within the researcher's image of a broadly coherent trajectory for student learning and 2) that the field should work towards a common understanding for the meaning of a hypothetical learning trajectory, which might include developing new terminology related to different stages of the process.³⁴*

Keywords: Conceptual Analysis, Cognitive Research, Learning Trajectories, Exponential Functions

³⁴ Thank you to Michael Tallman for several long discussions that helped me clarify my thinking for this paper.

Cobb (2007) argued that mathematics education “can be productively viewed as a design science, the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes” (p. 7). This requires researchers to leverage scientific methods to inform design (at the instructional, curricular, or institutional level) with a focus on positively impacting student learning.

Thus, a useful way to characterize cognitively-oriented research goals is the production of empirically tested learning trajectories that provide opportunities for students to construct productive ways of reasoning about mathematical ideas within a coherent trajectory spanning their entire mathematical careers. *Conceptual analysis* (Thompson, 2008a) plays an important role in this work, yet researchers are not always explicit about how they use conceptual analysis, nor are they clear about how conceptual analysis of mathematical ideas contributes to both the design and refinement of interventions contributing to the broader goal of advancing knowledge in the field.

In this paper I discuss conceptual analysis of mathematical ideas relative to its place within cognitive learning theories, highlight different ways that conceptual analysis is used in specific cognitively-oriented research studies (all focused on the idea of geometric or exponential growth), and explore how these uses of conceptual analysis contribute to learning theory research. I conclude the paper by leveraging my analysis to suggest two important focal points for future work in producing and testing hypothetical learning trajectories, including an increased emphasis on researchers’ images of the broader trajectory of students’ mathematical experiences and the importance of a common understanding for constructs such as *hypothetical learning trajectory*.

Students' Mathematical Experiences

Stigler & Hiebert (2009) describe most U.S. mathematics lessons as incoherent and focused on low-level mathematical skills and ideas. Students progressing through such a system are likely to develop an image of mathematics as a series of disconnected algorithms and formulas to memorize framed within disconnected courses. See Figure 7.1.

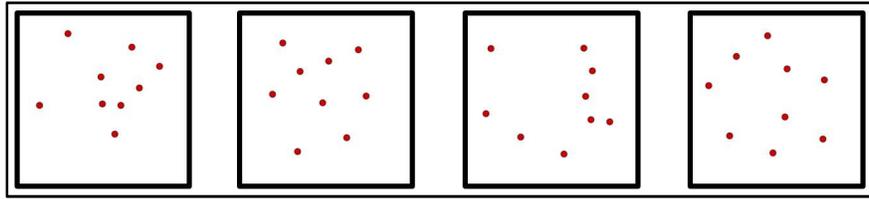


Figure 7.1. For many students, topics (represented as dots) within courses (represented as squares containing topics) are isolated as opposed to being connected by strong themes both within a course and across courses.

Thompson (2008a) commented that "coherence of a curriculum (intended, implemented, or experienced) depends upon the fit of meanings developed in it ... [and] [t]he lack of attention to meaning, I believe, is at the root of many problems that become visible only later in students' learning" (p. 32). For students to experience a coherent curriculum means that they conceptualize connections between ideas and topics within a course in addition to conceptualizing larger themes that connect topics and ideas across multiple courses. See Figure 7.2.

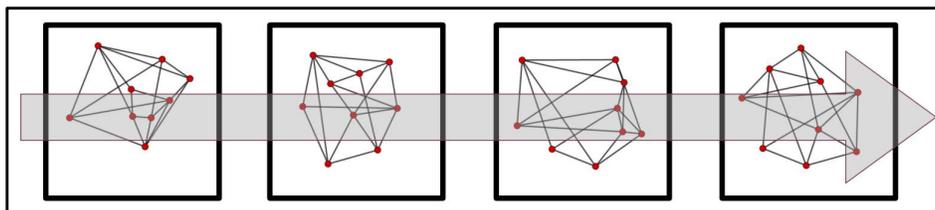


Figure 7.2. We want students to conceptualize strong connections between topics and ideas within a course as well as powerful themes and ways of thinking that connect ideas across many courses.

Supporting students in seeing the coherence within and across mathematics courses demands that curriculum designers and instructors first clearly articulate for themselves the general ways of reasoning mathematics instruction should support and how potential meanings for particular mathematical ideas fit within these goals. This is the starting point for learning trajectory research, or research that seeks to develop, analyze, and refine lessons and task sequences that effectively support students in constructing productive meanings for mathematical ideas. Such work is most impactful when it: i) begins by establishing clear intended learning goals relative to essential meanings and ways of thinking entailed in learning an idea, ii) describes how essential meanings and ways of thinking fit in the broader scope of students' mathematical experiences, and iii) includes an analysis of the meanings students do construct within an enacted learning trajectory. Results from this work has the capacity to positively impact student learning by fostering improvements in instruction, curriculum, and the coherence of state and national standards (Clements & Samara, 2004; Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016; Sztajn, Confrey, Wilson, & Eddington, 2012).

Conceptual Analysis, Hypothetical Learning Trajectories, and Teaching

Experiments: The Importance of Theory

Conceptual analysis focuses on defining mental activity characterizing both real and epistemic individuals' meanings, and as such derives from general constructivist principles. diSessa and Cobb (2004) and Thompson (2002) both describe theoretical perspective hierarchies starting from broader background theories like Piaget's (1971) genetic epistemology to more narrow domain-specific theories that "entail the conceptual

analysis of a significant disciplinary idea...with the specification of both successive patterns of reasoning and the means of supporting their emergence” (diSessa & Cobb, 2004, p. 83). Background theories serve “to constrain the types of explanations we give, to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem” (Thompson, 2002, p. 192). Domain-specific theories address “ways of thinking, believing, imagining, and interacting that might be propitious for students’ and teachers’ mathematical development” (p. 194).

That conceptual analysis originated from radical constructivism has implications for its character and purpose. A description of what it means to understand a mathematical idea should be phrased in terms that reflect a researcher’s epistemology, and not in a faint or elusive way. This explains why conceptual analysis, as defined by Glasersfeld (1995), Steffe & Thompson (2000), and Thompson (2008a), is a description of cognitive states and processes. Grounding conceptual analysis in descriptions of mental actions and schemes³⁵ attunes researchers to important ways of understanding foundational ideas that influence students’ abilities to construct and leverage productive images of sophisticated ideas articulated by a researcher’s learning goals and *hypothetical learning trajectory* (Simon, 1995).

Conceptual Analysis

Thompson (2008a) defined conceptual analysis as a description of “what students must understand when they know a particular idea in various ways” (p. 42) and outlined

³⁵ My meaning for scheme is best described by Thompson, Carlson, Byerley, and Hatfield (2014). They wrote that “[s]chemes are organizations of mental activity that express themselves in behavior, from which we, as observers, discern meanings and ways of thinking. Scheme is a theoretical construct that we impute to individuals to explain their behavior” (p. 10).

four uses: 1) to build models of students' thinking by analyzing observable behaviors, 2) to outline ways of knowing potentially beneficial for students' mathematical development, 3) to outline potentially problematic ways of knowing particular ideas, and 4) to analyze coherence in meanings among some set of ways of knowing. From a Piagetian-constructivist perspective, understandings are organizations of mental actions, images, and conceptual operations. Describing an understanding—either actual or intended—therefore involves specifying its constituent mental actions, images, and operations. Conceptual analysis provides clarity on the mental actions that characterize particular understandings, their potential origins, and their implications for subsequent mathematical learning. Conceptual analysis does not produce a list of mathematical facts or specific performance objectives. Conceptual analysis is about articulating the cognitive processes that characterize particular understandings and is thus a basis for task design and shapes researchers' identification of students' mathematical thinking and learning. Thus, conceptual analysis is a form of theory itself—an operationalization of what diSessa and Cobb (2004) call an *orienting framework* in the context of mathematics education research.

Hypothetical Learning Trajectories

There is currently no consensus definition for *hypothetical learning trajectory*. Most descriptions are built from Simon's (1995) original definition as “[t]he consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage” (p. 133). Ellis et al. (2016) provide a thorough literature review regarding researchers' differing meanings when they discuss learning trajectories.

I will not repeat their analysis here. However, Simon’s (1995) original intention was to emphasize “hypothetical” in the term *hypothetical learning trajectory* because, as the name indicates, they should be framed as hypotheses to be tested in empirical studies. Steffe and Thompson’s (2000) teaching experiment methodology was designed to support creating, assessing, and refining hypothetical learning trajectories informed by a conceptual analysis. Teaching experiments have three parts, and different uses of conceptual analysis contribute uniquely to each part (see Figure 7.3).

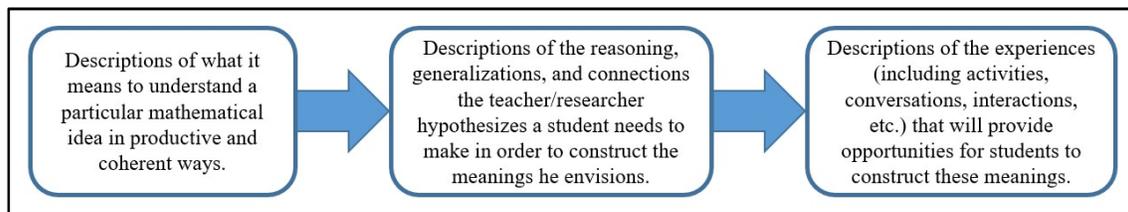


Figure 7.3. Parts of a teaching experiment. Note that articulating the three parts of the teaching experiment align well with Simon’s (1995) original description of the components of a hypothetical learning trajectory.

Testing hypothetical learning trajectories requires that each of the three components of a hypothetical learning trajectory (the goal, the activities, and the student reasoning) should be clearly articulated in enough detail so that during a teaching experiment, and in retrospect, it is possible for the researcher to provide empirical support for accepting or rejecting any part of the hypothesis. Thinking in these terms, we can clarify how the results from different research studies contribute to the goal of creating empirically tested mathematical learning trajectories.

Examples of Different Uses of Conceptual Analyses

Since researchers’ contributions to learning trajectory research depend on how they apply the methods of conceptual analysis, their conceptual analyses constitute and should be viewed as an interpretive lens to make sense of their data. Further, it provides

others a window into the authors' perspectives and ways of conceptualizing particular mathematical ideas, thus allowing others to better leverage and interpret their work. The following three examples will help to illustrate this point. Each are drawn from compelling, influential research related to the teaching and learning of exponential growth.

Confrey and Smith's Retrospective Conceptual Analysis: Modeling Student Reasoning

Confrey (1994) and Confrey and Smith (1994, 1995) developed robust descriptions for students' images of multiplication, ratio, covariation, function, and rate based on retrospective conceptual analysis of teaching interviews. By studying student thinking as they worked through tasks like paper folding and predicting future values for an item retaining 90% of its value each year, Confrey and Smith identified productive student meanings for multiplication, rate of change, and function that often differed from conventional meanings (such as conceptualizing multiplication as repeated addition or functions as injective mappings). By carefully modeling students' schemes, Confrey and her colleagues described productive images they conjectured could be a powerful foundation for understanding exponential growth.

Images of multiplication, covariation, function, rate of change, and exponential growth. Confrey (1994) described thinking about multiplication via *splitting*. A *split* is the action of creating equal copies of an original amount or breaking apart an original amount into equal-sized pieces, and multiplication is the result of some

n -split on an original whole while division involves examining one of the equal parts of the split relative to the whole. See Figure 7.4.

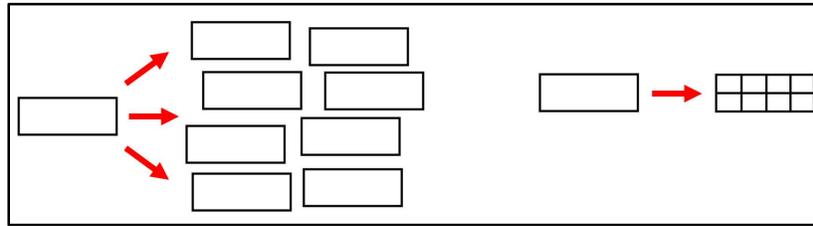


Figure 7.4. Two different conceptualizations of an eight-split (Confrey, 1994). On the left the split is conceptualized as making eight identical copies of an original amount and on the right the split is conceptualized as breaking apart an original amount into eight equal-sized pieces.

In this way students have the potential to conceptualize multiplication and division and their inverse relationships simultaneously. Ratios rather than differences are then the natural means of comparison when conceptualizing a split and equivalent ratios correspond to comparisons of values separated by an equal number of splits. See Figure 7.5.

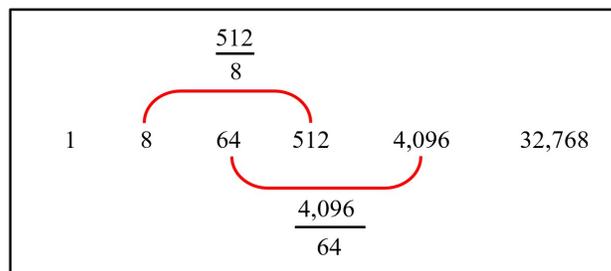


Figure 7.5. Ratios such as $512/8$ and $4096/64$ are equal if an equal number of splits separate the values.

Confrey and Smith (1994, 1995) described students engaging in *covariational reasoning* when coordinating splits and defined *covariation* discretely as a process of synchronizing successive values of two variables. A function relationship is then “the juxtaposition of two sequences, each of which is generated independently through a pattern of data values” (1995, p. 67) with specific function characteristics emerging from

repeated actions during this coordination. Table 7.1 shows an arithmetic sequence for x values coordinated with a geometric sequence for y values.

Table 7.1

Coordinating an Arithmetic Sequence with a Geometric Sequence

	x	y	
+3	2	15	x2
+3	5	30	x2
+3	8	60	x2
+3	11	120	x2
+3	14	240	x2

Confrey and Smith argued that students reasoning covariationally developed notions of *rate* that differed from conventional definitions. Some students coordinating arithmetic and geometric sequences to reason about exponential growth described the relationship as having a constant rate of change, meaning that thinking about rates as a ratio of additive differences is not an inevitable choice for students. They proposed defining *rate* in a way that respects students’ intuitions. A *rate* is a unit per unit comparison where *unit* refers to what remains constant in a repeated action (Confrey, 1994). Thus, changes (and rates) can be conceived of additively or multiplicatively. Confrey and Smith argued that coordinating repeated addition to move through an arithmetic sequence with repeated multiplication to move through a geometric sequence is a productive foundation for understanding exponential growth. In other words, exponential functions are functions with a constant *multiplicative* rate of change (such as “times two per plus three” for the relationship shown in Table 7.1). A more complete image of the relationship emerges by interpolating values within each sequence through

coordination of arithmetic and geometric means thus producing a domain and range with the density of the rational numbers.

Confrey and Smith's work modeled students' constructed schemes from empirical data and theorized about the utility of specific meanings for multiplication, covariation, function, and rate of change for understanding exponential growth. This kind of retrospective conceptual analysis is very useful for characterizing spontaneous and productive student reasoning about specific tasks, including novel ways of thinking potentially not emphasized in curricula. Confrey and Smith were not focused on generating detailed learning trajectories (that is, they did not outline specific task sequences for specific student populations as testable hypotheses for a teaching experiment, at least not in their reports),³⁶ nor did they describe the implications for their specific meanings on understanding a wide array of more advanced mathematical ideas students will encounter in the future (e.g., derivatives or the Fundamental Theorem of Calculus (FTC)). Their work was limited to modeling students' meanings for mathematical ideas within a narrow scope of mathematical tasks and considering implications of these meanings for what they conceived as related ideas. These statements are not criticisms; rather, the intent is that they clarify how Confrey and Smith used conceptual analysis in their work and highlight their study's goals and scope, thus helping others understand their data and conclusions with respect to how they contribute to learning trajectory research.

³⁶ Weber (2002a, 2002b) and Ström (2008) both studied the implications of Confrey and Smith's conceptual analysis, as did Amy Ellis and her colleagues. I will say much more about Ellis et al.'s work later in this paper.

Based on this analysis, I classify their use of conceptual analysis in relation to learning trajectory research as "projecting forwards". They described how some students spontaneously reasoned about specific tasks and conjectured how that reasoning could build a coherent trajectory forward to develop useful ways of thinking and understandings for learning related topics (from Confrey and Smith's point of view). I visualize this in Figure 7.6.

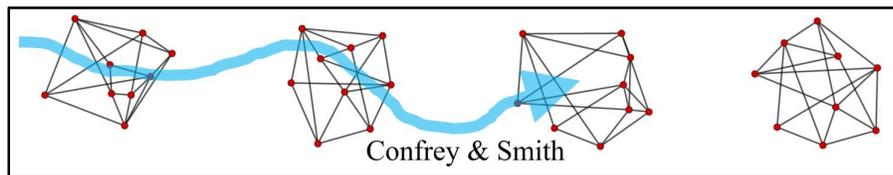


Figure 7.6. Confrey and Smith's recommendations can be thought of in terms of "projecting forwards". Starting with models of how students reasoned about particular tasks they conjectured how that reasoning might be leveraged in learning about ideas they see as related. I visualize the arrow emerging from left to right and the path it traces runs through ideas Confrey and Smith describe as connected to splitting and multiplicative rate of change.

Thompson's Conceptual Analysis: Coherence of Mathematical Ideas Leading to Calculus

Thompson's (1994a) unpacking of the key ideas in calculus, particularly the FTC, motivated and informed his conceptual analysis for exponential growth (Thompson, 2008a). Thompson imagined a broadly coherent trajectory for students' mathematical experiences focused on quantitative reasoning, covariational reasoning, and representational equivalence that could unite most topics from grade school mathematics through calculus (Thompson, 2008b). Thus, his conceptual analysis considers exponential functions as just one of many opportunities for students to develop and apply particular ways of thinking including coordinating the variation in two quantities, conceptualizing a

rate of change as measurement of how quantities change together, viewing function relationships as emerging through coordination of covarying quantities (including function values as emerging through continuous accumulation according to specific rates of change), and classifying functions by noticing what remains invariant as the quantities vary (Thompson, 1994a; Thompson & Carlson, 2017).

Quantitative and covariational reasoning, rate of change, accumulation, and the FTC.

Thompson's meanings for covariation, function, and rate of change are different from Confrey and Smith's meanings because his goals are different. His work is grounded in *quantitative reasoning*, which describes conceptualizing a situation to form a quantitative structure that organizes relevant *quantities* (measurable attributes) and *quantitative operations* (new quantities representing a relationship between other quantities) (Thompson, 1988, 1990, 1993, 1994b, 2011, 2012). If someone sees a situation as composed of quantities that change together and attempts to coordinate their variation, then she is engaging in *covariational reasoning* (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Sophisticated covariational reasoning involves linking two continuously varying quantities to create a *multiplicative object*. A multiplicative object is a conceptualization formed in the anticipation of keeping track of both quantities as they vary and maintaining that unity in the face of perceiving their individual variation (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017).

Thompson (1994a, 1994b) and Thompson and Thompson (1992) outline an image of constant rate as a proportional correspondence of two smoothly covarying quantities. When one quantity's magnitude changes by *any* amount, the other quantity's magnitude changes proportionally. Thompson often refers to a constant rate of change as a *homogeneous rate of change*, intentionally invoking imagery associated with homogeneous mixture problems (Kaput & West, 1994; Harel, Behr, Lesh, & Post, 1994). For example, if a liquid is made up of two parts (say, water and orange juice concentrate), then as the mixture is poured into a container the volumes of water and concentrate change together such that they are always the same relative size and the same relative proportion of the mixture's volume. See Figure 7.7.

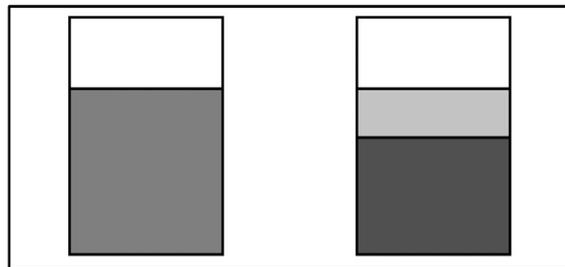


Figure 7.7. As the total mixture volume changes, the volumes of water and concentrate change so that they are always the same relative size and the same relative proportion of the mixture's volume. To capture this idea, imagine the mixture being poured into the container so that the volume changes smoothly and continuously.

Thinking about “homogeneous rates of change” allows change and accumulation to occur simultaneously for all real numbers and focuses attention not just on the values before and after variation but *at all moments during the variation* (Castillo-Garsow, 2012; Castillo-Garsow, Johnson, and Moore, 2013; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Thompson claims that this is akin to Newton's image of rate that allowed him to conceptualize the relationship between accumulation and rate of change expressed formally in the FTC (Thompson, 1994a, 2008a). Over small intervals,

Newton imagined that any two covarying quantities change together in a proportional correspondence. This can be modeled by a piecewise constant rate of change function and its corresponding piecewise linear accumulation function. The FTC describes how these two functions are related as the interval sizes tend to zero (see Figure 7.8).

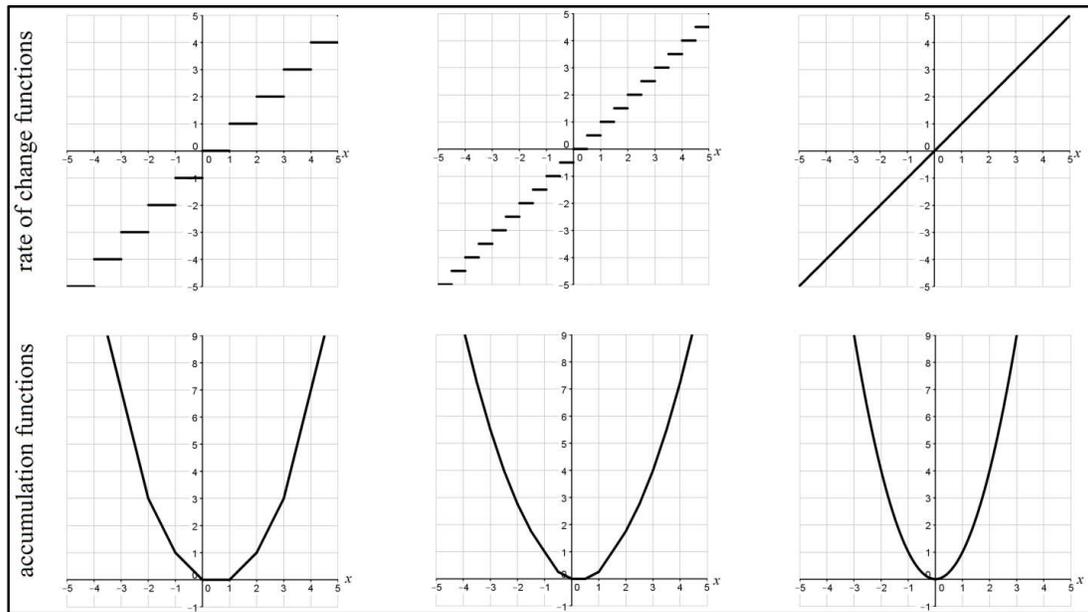


Figure 7.8. Piecewise linear accumulation functions and the corresponding piecewise constant rate of change functions.

Exponential functions. Building from his images of constant rate of change and the FTC, Thompson’s (2008a) conceptual analysis involved thinking about classifying functions based on similarities in their rate of change functions and imagining functions as emerging through accumulation. Specific to exponential functions, he conceptualized a relationship with a rate of change on some interval that is always the same proportion of the function’s value at the beginning of the interval. As the interval size decreases, the piecewise linear accumulation function converges to a function modeled by $f(x) = Pe^{rx}$ where r is the constant of proportionality relating the function’s rate of change and its value. See Figure 7.9.

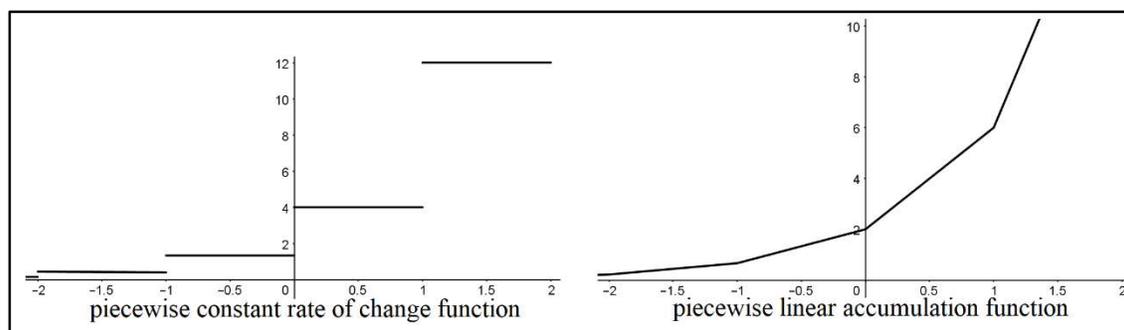


Figure 7.9. Another piecewise linear accumulation function and piecewise constant rate of change function. In this case there is a proportional relationship between the function's rate of change within any interval and the function's value at the beginning of the interval. When the interval size tends to zero, the accumulation function converges to the familiar exponential function.

In this development, the defining characteristic of an exponential function is not related to the geometric structure in its pattern of outputs. An exponential function is a relationship where the rate of change is proportional to the function's value at any point in the domain. Again, Thompson (1994a, 2008a) argued that this way of understanding allows a person to conceptualize both change and accumulation as happening simultaneously. This makes it natural to imagine the function value growing continuously and producing outputs for all real number inputs. Thompson argues that this is a potentially consistent and coherent way of reasoning about all function relationships that eventually supports a productive operational understanding of the FTC.

Much like Confrey and Smith, Thompson's work is not a detailed hypothetical learning trajectory for learning specific ideas according to Simon's (1995) description.³⁷ Thompson's conceptual analysis is part of a broader, idealized web of ideas stretching from students' first mathematical experiences through calculus. In Thompson's conceptual analysis, exponential growth is related to repeated multiplication almost by

³⁷ Castillo-Garsow (2010) did produce a learning trajectory and empirical study based on this conceptual analysis.

coincidence and is not the foundational meaning. That is, characteristics such as a constant doubling time (or half-life) or a constant Δx -unit growth factor emerge through reflecting on emergent relationships. This differs from Confrey and Smith's work where these properties are an integral part of the definition of exponential functions.

If Confrey and Smith's work can be described as "projecting forwards", I think about Thompson's recommendations as "projecting backwards".³⁸ The ways of understanding key ideas in calculus inform his recommendations for ways of thinking about most ideas prior to calculus (including exponential functions). See Figure 7.10.

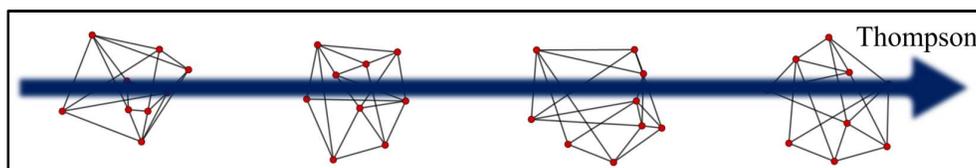


Figure 7.10. Thompson's recommendations can be thought of in terms of "projecting backwards". Starting with a conceptual analysis for potentially productive meanings for the FTC, Thompson discusses ways students might conceptualize ideas of function, rate of change, magnitude, exponential growth, etc. that are potentially coherent and that lay the foundation for eventually understanding the FTC in productive ways. I imagine the arrow as being drawn sweeping from right to left. The arrow is also drawn "fuzzy" to convey his attention more on broader ways of thinking over specific ways of understanding at the level of individual lessons and units.

Thus, Thompson's conceptual analysis relative to exponential growth is situated within a broader, idealized web of ideas and meanings that he conjectures provides an opportunity for students to conceptualize their mathematical experiences as largely coherent (Thompson, 2008b). This is a useful lens for understanding his body of work relative to quantitative and covariational reasoning in general and exponential growth in particular.

³⁸ Please see Appendix C before referencing or citing my characterization of Thompson's conceptual analysis for an important commentary.

Ellis and Colleagues: From Exploratory to Hypothetical Learning Trajectory

Ellis and her colleagues (Ellis, Ozgur, Kulow, Williams, & Amidon, 2012, 2015; Ellis et al., 2016) mostly leveraged Confrey and Smith's images of covariation, rate, and exponential growth to construct a rough exploratory learning trajectory surrounding a single context. Ellis et al. extended and clarified how Confrey and Smith's ideas might productively support students' understanding of exponential relationships and chose a situation where they conjectured students could easily justify that the function's domain and range were not restricted to a set of discrete values.³⁹

They built a Geogebra applet showing the image of a plant (the Jactus) with a height that varied exponentially with elapsed time. The applet's user can vary the elapsed time by sliding the plant along the horizontal axis and its height updates in real time. The applet also displays the time elapsed and the plant's height as an ordered pair as the user slides the plant horizontally. See Figure 7.11.

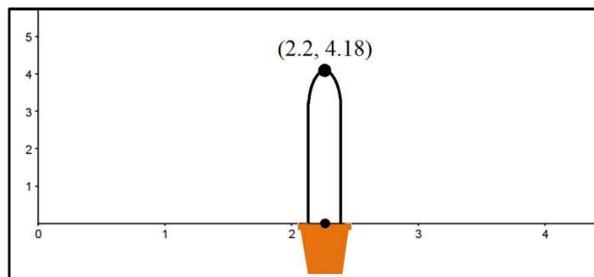


Figure 7.11. An approximation of Ellis et al.'s Jactus applet.⁴⁰

³⁹ Through their choice of task, Ellis and her colleagues attempted to bridge the meanings developed in Confrey and Smith with recommendations from Thompson's work in continuous covariation. Their expectation was that situating explorations of exponential growth in a context where students could explore continuous growth would address what their conceptual analysis revealed as potential pitfalls in focusing on discrete variation.

⁴⁰ Note that Kuper (2018) built on Ellis et al.'s work and explored how a similar applet might support students in developing productive meanings for logarithmic functions. O'Bryan (2018) also used a modified version of this applet to support students in seeing growth factors as a measurement of one instance of a quantity in units of a second instance of the same quantity.

In designing their study, Ellis et al. anticipated, and later confirmed, that students' initial models for exponentiation involved an informal image of repeated multiplication. Ellis et al. wanted students to leverage covariational reasoning to build a more robust image of exponential growth focused on coordinating multiplicative changes in one quantity with additive changes in another quantity. With this understanding, students could conceptualize b^x as both the possible height of a plant at some moment in time and as representing a (multiplicative) change in height. Their eventual goal was for students to develop a meaning for $b^{\Delta x}$ as the ratio of any two outputs $f(x_1 + \Delta x)$ and $f(x_1)$ where b is the 1-unit growth factor. Students working through their activities exhibited key shifts in their thinking that included increased attention to how the two quantities changed together over intervals of varying size and “[these] results...offer a proof of concept that even with their relative lack of algebraic sophistication, middle school students can engage in an impressive degree of coordination of co-varying quantities when exploring exponential growth” (Ellis et al., 2012, p. 110).

Ellis et al. used conceptual analysis in three ways. First, they further unpacked Confrey and Smith's conceptual analysis of exponential growth as students might construct it from images of coordinating additive and multiplicative changes. Second, they modified and updated their exploratory learning trajectory and tasks throughout the study based on emerging models of students' schemes. These analyses, coupled with retrospective analysis on the empirical data, allowed them to craft highly detailed descriptions of students' meanings at various points in time and how those meanings developed through interactions with tasks and teaching interventions (Ellis et al., 2016). The result is the foundation for a powerful hypothetical learning trajectory. Ellis et al.

now have empirical grounding for theories on how students might come to construct specific meanings related to exponential growth and related ideas. The refinements from the exploratory research and their model for how students construct specific meanings in specific contexts can now be a clearly articulated scientific hypothesis for systematic testing.

Ellis et al.'s study is an example of critical work in developing empirically tested learning trajectories and demonstrates how initial exploratory work in understanding students' scheme construction can be refined and expanded to contribute to learning trajectory research. However, as they note, "Our learning trajectory is an attempt to characterize the nature of the evolution of students' thinking in a particular instructional setting" (2016, p. 153) and is thus only one of many possible learning trajectories. One key underdefined component of Ellis et al.'s publications is a clear description of how their learning goals for students fit into a broader image of students' mathematical experiences. This includes addressing how intended meanings may support students in reasoning about other ideas within the same course and how the intended meanings establish a foundation for students to understand important mathematical ideas in future courses. They likely thought of this as beyond the scope of their research study. However, this leaves the reader to conjecture about most aspects of the broader trajectory Ellis et al. might have in mind, and this might make it difficult for other researchers or curriculum designers to know how to leverage Ellis et al.'s activity and learning trajectory in their own work.

Ellis et al.'s use of conceptual analysis supported their development of a teaching experiment with particular learning goals, initial tasks to support those learning goals,

refinements of the goals and tasks in the process of instruction, and retrospective analysis on the meanings students developed during their intervention. They grounded their work in particular characterizations for students' background knowledge of exponents and repeated multiplication and sought to impact a certain population or age level of students with the goal of supporting specific meanings for exponential relationships and growth factors. See Figure 7.12.

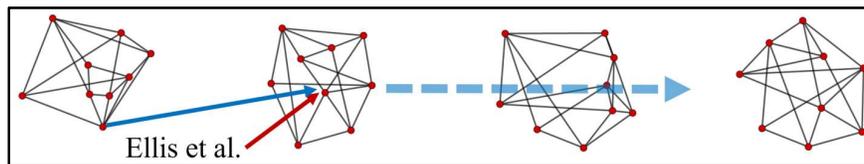


Figure 12. Ellis et al.'s recommendations focus on a particular topic (a small set of ideas) within a specific course built on a particular set of assumptions for students' background knowledge. Their work is loosely situated within a trajectory with general meanings for concepts like function, rate of change, etc., but Ellis et al. do not elaborate on specific ways that their learning goals prepare students to better understand other mathematical ideas.

Summary and Theoretical Implications

“Performing a conceptual analysis” meant something different in each of the examples I discussed (although it is important to emphasize that it was never simply a list of topics for students to learn). Reflecting on how conceptual analysis is used and how the results and recommendations of three researchers/research teams might contribute to developing and testing hypothetical learning trajectories suggests two salient issues for cognitive researchers to consider in this area.

A Need for Articulating Images of Long-Term Implications and Coherence

It is tempting to consider recommendations from Confrey and Smith, Thompson, and Ellis et al. as representing competing sets of meanings such that other researchers or curriculum designer must decide which approach is “best” (or decide upon an amalgam of two or more approaches). However, viewing the recommendations in this light overlooks an important truth. Each recommendation is situated within the researchers’ images of students existing within a larger trajectory of mathematical experiences (even if this image is not articulated in their publications), and as such the recommendations and results should not be considered outside of this context. For example, Thompson situates his meanings for exponential growth and related ideas in what he conceptualizes as a broadly coherent web of ideas that eventually support students in constructing intuitive understandings of the FTC and other more advanced ideas. Considering whether or not to build lessons to support students in developing his intended meanings for exponential functions within a trajectory of learning experiences with different ultimate goals is to miss the point of his conceptual analysis. The merits of seemingly competing learning goals cannot be compared in a vacuum and must be considered within the larger context of creating coherent mathematical experiences for students across topics and courses.

When designing or analyzing learning trajectory research, it will be useful to others if researchers articulate how students might leverage the specific ways of understanding in future learning, how these ways of understanding can be supported in earlier learning, and how the ways of understanding we want to promote connect to other ideas within the same course. If researchers would focus on communicating this vision,

and demand others do the same, this may lead to more productive collaborations and conversations grounded in our collective effort to improve student learning. I suspect that over time, as we gain increasing clarity both on how students develop productive meanings for ideas in elementary mathematics and we better understand meanings that are fundamental to students' success in higher levels of mathematics and other STEM fields, that these discussions will be the impetus for convergence in our images of productive learning trajectories spanning students' entire mathematical careers.

A Need for New Terminology and Articulating How Research Results are to be Interpreted

A teaching experiment is a method of testing a research hypothesis (a carefully detailed hypothetical learning trajectory) informed by conceptual analysis that considers the degree to which (and aspects of) tasks and interactions that promote specific learning goals. Most of the research described in this paper does not satisfy the criteria of a formal teaching experiment because the empirical work, when present, tended to be exploratory in nature. However, each of the examples contribute to the goals of cognitively-oriented mathematics education research in powerful ways. Confrey and Smith described students' schemes related to repeated multiplication based on spontaneous reasoning about particular mathematical tasks. Ellis et al. further unpacked these schemes and, based on retrospective analysis of empirical data, produced a well-defined hypothetical learning trajectory for specific meanings using specific tasks that now has the clarity and specificity necessary to be a scientific hypothesis. Thompson's work takes a broader view

and suggests ways of understanding exponential growth situated within a coherent body of mathematical ideas extending well beyond a single topic.

Currently there is no consensus on the exact meaning of a hypothetical learning trajectory. This might be due to how applying conceptual analysis in different ways contributes uniquely to learning trajectory research as discussed in this paper. In addition, reflecting on the research I described suggests that the field may benefit from greater clarity in defining different types of learning trajectories with the definitions influenced by the role of conceptual analysis. A potential starting point follows.

- *Guiding Framework* – Descriptions of the overarching ways of thinking that connect topics and ways of understanding within a broad, coherent trajectory of students’ mathematical experiences. For example, ways of thinking about covariation, relative size, rate of change, proportional reasoning, etc. would be included in a guiding framework. A guiding framework informs details of learning trajectories for specific mathematical ideas at specific grade levels or courses by situating that work within the larger imagined path that student learning may follow. Conceptual analysis is critical in analyzing the coherence in meanings within the framework.
- *Exploratory learning trajectory* – Conceptual analysis (either based on a researcher’s analysis of mathematical ideas or based on empirical data) can suggest potentially useful ways of understanding particular ideas. A researcher then creates tasks and a rough exploratory trajectory for gathering empirical data on how students reason about learning tasks in specific settings. Since the enacted

- learning trajectory is continually modified based on modeling students' emerging meanings, the stated learning is not yet a fully articulated scientific hypothesis.
- *Enacted learning trajectory* – An actual learning trajectory unfolds informed by the exploratory learning trajectory but modified through interactions between a researcher/teacher and students. Conceptual analysis is used retrospectively to describe how students' schemes changed as a result of their mathematical activity and how specific features of the learning tasks supported (or failed to support) students in developing the intended meanings.
 - *Hypothetical learning trajectory* – This describes a specifically stated research hypothesis outlining targeted mental actions and schemes, specific tasks and a task sequence, and descriptions of how those tasks will contribute to students developing the intended meanings. The teaching experiment that tests this hypothetical learning trajectory seeks to accept or reject particular aspects of the hypothesis and will ultimately result in refinement. Conceptual analysis is critical to the design of the learning trajectory and retrospectively in analyzing outcomes in the more formal teaching experiment. Note that I intend a narrower use of this term than is currently in practice. Initial iterations of learning trajectories that are still subject to significant modification are not yet at the level of a stable scientific hypothesis. I suggest reserving the term *hypothetical learning trajectory* for more stable descriptions of learning trajectories that have already been through multiple refinement iterations in particular settings and are ready for more systematic testing of the relatively stable sequence of tasks and lessons. For example, I suggest that Ellis et al.'s work culminated in articulating a hypothetical learning

trajectory. Their papers described exploratory learning trajectories and analysis on enacted learning trajectories.

- *Empirically supported learning trajectory* – After potentially several rounds of refinement and testing with exploratory and hypothetical learning trajectories, a researcher can articulate an empirically supported learning trajectory. This is the stage at which it may be appropriate to compare the implications or results of different learning trajectories for particular mathematical ideas.

Any of these categories could describe work that is narrow in scope (focused on a particular mathematical lesson or set of lessons) or grander in scope (focused on students' learning across units, an entire course, or a set of courses). Researchers' questions of interest and how they use conceptual analysis dictate the type of learning trajectory they are developing and studying, and the scope of their work dictates their contribution to the field from models of students' schemes relative to particular ideas to coherent mathematical experiences across many topics and grade levels.

Final Comments

As researchers, we are obligated to not only produce scientifically-valid findings but also to communicate our work in ways that allow others to leverage our results to advance the collective mission of our design science. Being more explicit about the role of conceptual analysis in our work and having greater clarity on how our learning trajectory research contributes to design research can help us make our findings more useful and relevant. It is also vital that we not lose sight of the role that conceptual analysis can play in the systematic advancement of knowledge of productive learning

trajectories within courses and across grades. Research that provides clarity about productive learning trajectories has the potential to become a powerful model for advancing curriculum and instruction to be more effective in supporting student learning of key mathematical ideas, both within a course and across courses spanning multiple years.

I hope that my articulation of how different uses of conceptual analysis are relevant to developing different categories of learning trajectories facilitates relevant and productive communication among cognitively-oriented, qualitative mathematics education researchers working to develop and test learning trajectories.

References

- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 35(2), 352-378.
- Castillo-Garsow, C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth*. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, 33(3), 31-37. Chicago.
- Clements, D.H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical thinking and learning*, 6(2), 81-89.
- Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 1-38). Greenwich, CT: Information Age Publishing.
- Confrey, J. (1994). Multiplication and Exponential Functions. *The development of multiplicative reasoning in the learning of mathematics*, 291-330.

- Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in mathematics*, 26(2-3), 135-164.
- Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in mathematics education*, 66-86.
- diSessa, A. A., & Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *The journal of the learning sciences*, 13(1), 77-103.
- Ellis, A. B., Ozgur, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An exponential growth learning trajectory: Students' emerging understanding of exponential growth through covariation. *Mathematical Thinking and Learning*, 18(3), 151-181.
- Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: the case of the Jactus. *WISDOMe monographs*, 2, 93-112.
- Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior*, 39, 135-155.
- Glaserfeld, E. v. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.
- Kuper, E. G. (2018). Sparky the saguaro: Teaching experiments examining students' development of the idea of logarithm. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- O'Bryan, A. E. (2018). Exponential growth and related ideas: Examining students' meanings and learning in an online precalculus course. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Piaget, J. (1971). *Genetic epistemology*. New York, NY: W. W. Norton & Company Inc.
- Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America*. Raleigh, NC: North Carolina State University.

- Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for research in mathematics education*, 114-145.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education*. Dordrecht, The Netherlands: Kluwer.
- Stigler, J. W., & Hiebert, J. (2009). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. Simon and Schuster, Chicago.
- Ström, A. (2008). *A case study of a secondary mathematics teacher's understanding of exponential function: An emerging theoretical framework*. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher*, 41(5), 147-156.
- Thompson, P. W. (1988). Quantitative concepts as a foundation for algebra. In M. Behr (Ed.). *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* Vol. 1 (pp. 163-170). Dekalb, IL.
- Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebraic. Center for Research in Mathematics & Science Education: San Diego State University.
- Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, 25(3), 165-208.
- Thompson, P. W. (1994a). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, 26(2-3), 229-274.
- Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.
- Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel

- (Eds.), *Symbolizing and Modeling In Mathematics Education*. Dordrecht, The Netherlands: Kluwer.
- Thompson, P. W. (2008a). *Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education*. Plenary paper delivered at the 32nd Annual Meeting of the International Group for the Psychology of Mathematics Education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. SÈpulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 45-64). MorÈlia, Mexico: PME.
- Thompson, P. W. (2008b). *One approach to a coherent K-12 mathematics. Or, it takes 12 years to learn calculus*. Paper presented at the Pathways to Algebra Conference, June 22-25, Mayenne, France.
- Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education*. WISDOMe Monographs (Vol. 1, pp. 33- 57). Laramie, WY: University of Wyoming.
- Thompson, P. W. (2012). Advances in research on quantitative reasoning. In R. Mayes, R. Bonillia, L. L. Hatfield & S. Belbase (Eds.), *Quantitative reasoning: Current state of understanding* WISDOMe Monographs (Vol. 2, pp. 143-148). Laramie, WY: University of Wyoming Press.
- Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.
- Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: An hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing.*, WISDOMe Monographs (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming. <http://bit.ly/1aNquwz>.
- Thompson, P. W., & Thompson, A. G. (1992, April). *Images of rate*. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA.
- Weber, K. (2002a). Developing Students' Understanding of Exponents and Logarithms.
- Weber, K. (2002b). Students' Understanding of Exponential and Logarithmic Functions.

CHAPTER 8

PAPER 2 [ONLINE LEARNING]

Inattention to Students' Mathematical Meanings in Online Learning Research:

Initial Steps Toward a New Research Focus and Methodology

Alan O'Bryan

Arizona State University

***Abstract:** Post-secondary student enrollment in online courses is increasing, yet little is known about the mathematical meanings students construct while engaged in online coursework. Prior research tested the impact of online courses on measures such as student retention rates, satisfaction scores, and GPA. But this data does not provide a complete picture of student learning in this context. In particular it does not assess the meanings students are constructing for mathematical ideas researchers have identified as critical to their success in future math courses and other STEM fields. This paper discusses the need for a new focus in studying online mathematics learning and calls for cognitive researchers to begin developing a productive methodology for examining the meanings students construct while engaged in online lessons.*

Keywords: Online Learning, Cognitive Research, Research Methodology

“[S]tudents’ mathematical learning is the reason our profession exists. Everything we do as mathematics educators is, directly or indirectly, to improve the learning attained by anyone who studies mathematics. Our efforts to improve curricula and instruction, our efforts to improve teacher education, our efforts to improve in-service professional development are all done with the aim that students learn a mathematics worth knowing, learn it well, and experience value in what they learn. So, in the final analysis, the value of our contributions derives from how they feed into a system for improving and sustaining students’ high quality mathematical learning.” (Thompson, 2008a, pg. 31)

Studying online learning is increasingly important considering the growing prevalence of online courses in post-secondary education. Prior research has focused on the impact of the online environment on student satisfaction, retention rates, and GPA. Little attention has been paid to the meanings students construct as a result of online lessons or the features of online lessons that support students' construction of meanings researchers know to be critical in STEM courses and careers. As Thompson (2013) points out, "if we intend that students develop mathematical understandings that will serve them as creative and spontaneous thinkers outside of school, then issues of meaning are paramount" (p. 61). Thus, studying students' construction of meaning in online courses is a potentially rich area of cognitive research that could provide critical feedback for future iterations of and validation for these interventions.

In Part I of this paper I summarize the current body of research regarding online learning and the opportunities and importance of studying the meanings students construct while engaged in online courses. In Part II I summarize current research methodologies designed to model students' mathematical meanings and discuss implications in applying these methodologies to study student learning in online environments. I conclude with a discussion of possible data researchers might collect for modeling students' meanings and briefly highlight the importance of triangulating from multiple data sources and the unique challenges in studying online learning.

Theoretical Perspective

As Cobb (2007) pointed out, researchers choose theoretical perspectives best-suited to address their research questions. Research on student learning in the online

environment has thus far focused on the concerns of school administrators using classical experiment methodologies. They compare student GPAs, retention rates, satisfaction, and exam scores for students in online courses and students in comparable face-to-face courses. Thus, these studies focus on a student as “a statistical aggregate that is constructed by combining measures of psychological attributes of the participating students” (Cobb, 2007, p. 16). Cobb calls this a *collective individual* and it is to this abstract individual that a researcher attributes an intervention’s impact. This work is important for ensuring that students’ academic progress is not negatively impacted by substituting online courses for comparable face-to-face courses. However, it is also critical that we understand if students are constructing mathematical meanings that we know are important for success in future math courses and STEM fields. This kind of work requires a different theoretical perspective and a different kind of study, one that “account[s] for specific students’ and teachers’ mathematical reasoning and learning” (Cobb, 2007, p. 19).

I leverage radical constructivism (Glaserfeld, 1995) as the foundation for my image of students’ mathematical meanings, what it means to study them, and why this focus is important. A student’s mathematical meanings (a) are entirely internal to the student and (b) refer to the complex web of understandings, imagery, connections, etc. that make up the scheme to which a student assimilates a given stimulus (such as a particular math problem, graph, or animation). A researcher interested in a student’s mathematical meanings must attempt to model them by reflecting on the student’s observable behaviors and theorizing a set of meanings that, if the student possessed them, would best explain those behaviors (Thompson, 2013). Radical constructivism is an

extension of Piaget's (1977) genetic epistemology. For Piaget, an individual organizes her experiences within schemes that include triggers (stimuli that *assimilate* to the scheme), *actions* associated with the stimuli, and some expectation of what outcome the action(s) will produce as a progression towards a goal or need not currently met (Thompson, Carlson, Byerley, & Hatfield, 2013; Glasersfeld, 1995; Piaget, 1971; Piaget & Inhelder, 1969). Actions include "all movement, all thought, or all emotions that respond to a need" (Piaget, 1967, p. 6). An individual that experiences a stimulus assimilates that stimulus to a scheme that provides meaning, with the scheme's entire contents, implications, inferences, connections, and actions encompassing that meaning (Johnckheere, Mandelbrot, & Piaget, 1958). All experiences involve attempts to assimilate stimuli to a scheme and thus provide those experiences with meaning.

Thus, *meaning* refers the components of an individual's scheme (Thompson, 2013). This creates two primary challenges in trying to describe what a student understands or the meanings she constructs. First, without direct access to anyone else's thoughts, any claims about a person's understanding derive from models of that person's schemes constructed based on her observable actions (Glasersfeld, 1995; Steffe & Thompson, 2000; Thompson, 2013). Second, choosing a set of ideas to research (like exponential functions) betrays a researcher-centered orientation because there is no guarantee that students will assimilate any of the activities to a scheme the researcher would recognize as encompassing exponential growth as he understands it. Therefore, a researcher must be attentive to how students appear to have understood a given task or activity, the possible structure of that person's triggered scheme, and "the space of

implications” (Thompson et al., 2014, p. 13) for a person assimilating a stimulus to the hypothesized scheme.

Part I: A Call to Action

Online Instruction and Online Mathematics Courses

Over 26% of U.S. students enrolled in post-secondary courses in the Fall 2012 semester registered for at least one distance-learning course (predominantly delivered online), and post-secondary enrollment in distance-learning courses during the fall semester have since increased even while the overall post-secondary enrollment decreased (U.S. Department of Education, National Center for Education Statistics, 2013, 2014, 2015, 2016, 2017, 2018). See Table 8.1.

Table 8.1

U.S. Post-Secondary Enrollment in Distance Learning Courses in the Fall Semester, 2012-2018

semester	students enrolled in post-secondary courses	students enrolled in any post-secondary distance-learning course	post-secondary students enrolled exclusively in distance-learning courses
Fall 2012	20,642,819	5,444,701 (26.38%)	2,638,653 (12.78%)
Fall 2013	20,375,789	5,522,194 (27.10%)	2,659,203 (13.05%)
Fall 2014	20,207,369	5,750,417 (28.46%)	2,824,334 (13.98%)
Fall 2015	19,977,270	5,954,121 (29.80%)	2,871,788 (14.38%)
Fall 2016	19,988,204	5,965,813 (29.85%)	2,874,870 (14.38%)

Growth in online enrollment is spurred by both student demand and academic leaders’ views of these courses (Allen & Seaman, 2011; Xu & Jaggars, 2013). The majority of academic leaders believe that the quality of online courses is at least as good as in-person courses and also believe that students are at least as satisfied with their online courses

compared to in-person courses (Allen & Seaman, 2011). In addition, about 71% of these leaders say that online learning represents a key part of their long-term growth strategies (Allen & Seaman, 2015). There is also a common assumption that online courses provide a reduction in long-term costs associated with teaching post-secondary courses which increases their attractiveness (Xu & Jaggars, 2013).

There is little doubt that online courses provide students with options for completing coursework regardless of where they live or if circumstances preclude regular attendance in face-to-face classes. These courses can also help colleges recruit more students and provide additional pathways for students to complete degree requirements. However, online learning is a relatively new phenomenon in education and little is known about the meanings students construct while interacting with online lessons or how best to leverage technology to promote specific learning goals. This is critical information if online courses continue to be a large part of students' post-secondary educational experiences.

Scope of current research in online learning. Research related to online learning (including online math courses) or meta-analyses that include studies involving mathematics courses can generally be grouped into four broad categories (with some overlap): 1) studies examining the impact of online learning on students' emotional states, 2) attempts to describe key characteristics of students who are successful in online courses, 3) comparative analyses of student performance in online vs. face-to-face courses, and 4) attempts to discover and prescribe best practices for creating online courses.

Researchers studying students' emotional states while engaged in online coursework claim that emotional states impact cognition, memory, creativity, motivation, and attention as well as how flexibly students apply cognitive strategies (Isen, 2000; Kim, Park, & Cozart, 2014; Levine & Pizarro, 2004; O'Regan, 2003). Brinkman, Rae, and Dwivedi (2007) and O'Regan (2003) attribute negative student emotional reactions (such as frustration, anxiety, and embarrassment) mostly to unreliable technology, challenges in navigating poorly designed web pages, stress related to falling behind or missing deadlines, inconsistent response times from instructors, not being confident with technology, or feeling ashamed when their work is publicly compared to classmates' work. On the other hand, positive emotional responses (such as enthusiasm, excitement, and pride) derived from autonomy, overcoming fears of technology, being impressed by new technology, or excitement at having access to a course from a remote location.

Students successful in online courses tend to be self-motivated with above-average technology and communication skills (Dabbagh, 2007). Berenson, Boyles, and Weaver (2008) used emotional-intelligence measurement surveys to score students on their ability to regulate their feelings and needs and found that emotional intelligence made up 11% of the variance among students' GPAs in online courses. Other studies linked low success rates with poor time management, low self-discipline, and low self-motivation (Lim and Kim 2002; Muilenburg & Berge, 2005; Park and Choi 2009; Waschull 2005; Yukselturk and Bulut 2007).

Researchers making a comparative analysis of student performance in online courses vs. in-person courses arrive at a wide variety of conclusions. Some studies reported that student achievement was higher in online environments compared to

traditional face-to-face classrooms. Students in Chen et al.'s (2010) study self-reported that online learning environments provided them with a better overall learning experience and improved their understanding of how to apply their learning in practical contexts. Hughes, Mcleod, Brown, Maeda, and Choi (2007) showed that students in online secondary math courses performed better than students in traditional settings on the Assessment of Algebraic Understanding, a test intended to align with the National Council of Teachers of Mathematics (2000) algebra standards. Nguyen and Kulm (2005) and Nguyen, Hsieh, and Allen (2006) demonstrated that students will spend more time on homework and complete more practice problems in online environments and that this correlated with improved achievement. They credited the increase in students' willingness to practice, including repeating assignments to attain higher scores, to the immediate feedback computers provide.

However, in many studies online courses did not produce a statistically significant improvement in student performance compared to in-person versions of the same course (Barbour, Brown, Waters, Hoey, Hunt, Kennedy, & Trimm, 2011; Brinkman et al., 2007; Kim et al., 2014; Taylor, 2002). Some researchers argued that this is because both online and face-to-face instruction have strengths and weaknesses, and the characteristics of online courses that might improve student performance are balanced against characteristics that might negatively impact performance and that educators have yet to embrace best pedagogical practices in online environments (Hughes & Mcleod, 2007). Several meta-analyses demonstrated no significant positive impact for students taking online courses compared to those taking traditional courses (Hannafin, Orrill, Kim, & Kim, 2005; Means, et al., 2009; Swan, 2003; Twigg, 2003). In some cases, researchers

found that enrolling in an online course had a negative impact on student success compared to enrolling in a traditional in-person course. Xu and Jaggars (2011, 2013) completed two large-scale studies on students enrolled in online courses and showed that, all other factors being equal, the average post-secondary student's likelihood of completing an online course is seven percent less than his likelihood of completing a comparable face-to-face course and that, even if the student completes the course, the average grade is 0.3 grade points lower. Among students from disadvantaged populations the numbers are further skewed. Moreover, students enrolled in online courses were, on average, better prepared for the courses than students taking in-person options and reported higher levels of self-motivation. Thus, "descriptive comparisons are likely to underestimate rather than overestimate the gap between online and face-to-face performance outcomes" (Xu & Jaggars, 2013, p. 55). Cao and Sakchutchawan (2011) found that MBA students in online courses had similar success rates compared to students taking in-person courses but that their course satisfaction ratings were significantly lower while other studies found that attrition rates can be 10% to 50% higher in online courses compared to similar in-person courses (Moody 2004; Park 2007; Park and Choi 2009; Tirrell and Quick 2012).

The final category of studies focuses on developing best practices for online course design (Coomey & Stephenson, 2001, Engelbrecht & Harding, 2004; Glass & Sue, 2008; Hopper & Harmon, 2000; Kramarski & Gutman, 2006; McDuffie & Slavit, 2003; Myers, 1999; O'Regan, 2003; Rimmershaw, 1999, Stiles, 2000; Ruey, 2010; Swan, 2003; Taylor, 2002; Wadsworth, Husman, Duggan, & Pennington, 2007). A synthesis of their findings produces a list of general recommendations including that courses should be

easy to navigate with a low bar of technical know-how needed to interact with the site, course requirements and deadlines should be clear, the instructor should be easy to contact, the course should somehow encourage social interactions among learners, and that the course should include opportunities for students to reflect on their learning. The advice they provide is intended to increase the likelihood that students can successfully complete an online course, maximize their performance, and minimize sources of emotional discomfort.

Taken as a whole, the body of research related to online learning in general (and math courses specifically) addresses administrator-level concerns such as maintaining or improving passing rates, GPA, and retention as institutions shift traditionally in-person courses to an online format. But there is a key focus yet to be explored.

Limitations in current research on online learning. Xu and Jaggars' (2013) review of qualitative studies that compared online courses with their in-person equivalents revealed that the authors of online courses typically converted the comparable face-to-face course with little or no change to the approach or content. The activities and homework remained the same with in-class lectures converted to video lectures or text. These findings suggest that most online courses are not intentionally leveraging potential advantages of an online environment to support students' construction of desirable meanings while also not considering likely disadvantages. For example, asking students to view pre-recorded lectures removes the opportunity for students to participate in class discussions during the lesson and ask questions in the moment. On the other hand, students completing work in an online math course often receive immediate feedback on their submitted work. Substituting pre-recorded lectures

with interactive lessons that ask students to answer carefully scaffolded questions with immediate feedback interspersed with text, animations, applets, and brief summary videos positions the student as a more active participant in the learning process. Furthermore, novel lesson and homework tasks such as those that require students to submit applet states as their response can potentially support the development of and assess productive understandings of mathematical ideas.⁴¹

I am unaware of any online learning study where the quality of mathematical meanings students develop is of primary interest. This leaves critical research questions unaddressed. (a) Are students learning mathematics worth knowing (are the meanings they develop consistent with meanings that researchers know to be productive in future math courses, STEM fields, and real-life applications)? (b) Are they constructing deep, flexible, and coherent meanings for the targeted ideas (and what counts as evidence for this)? (c) What aspects of the online lesson(s) support students' construction of important targeted meanings? Student GPA and retention rates shed little insight into these questions and self-reporting by students is not reliable when it comes to their depth of learning (Ke & Xie, 2009). Cases where data on course assignments and tests are used to make comparisons follow a classical research design where only the course environment is varied. This typically leaves the mathematical meanings at the heart of instruction

⁴¹ To illustrate this idea, consider Tallman's (2015) description of productive meanings for the sine function. The function accepts as its argument an angle measure in radians (which is a measurement of the subtended arc on a circle centered at the angle's vertex measured in units of the circle's radius) and produces a vertical displacement describing the directed distance above the circle's horizontal diameter where the angle's terminal ray intersects the circle also measured in units of the circle's radius. In an online lesson or assignment, students could be given an applet with a circle centered at an angle's vertex. Students must rotate the terminal ray until it demonstrates an angle measuring θ radians such that $\sin(\theta) = 0.6$. Submitting the question determines whether the chosen applet state meets the required conditions up to an acceptable level of error.

unexamined. A focus on students' meanings is a critical missing piece that should inform the initial design and iterations of online math courses.

Studying Student Learning in an Online Course as Design Science

The importance of attending to meaning. Thompson (2013) demonstrated that a weak or incoherent system of meanings (held by an instructor or present in lesson design) creates more space for students to construct incorrect or unhelpful meanings and for the teacher (or designer) to remain unaware of students' constructions. For example, most College Algebra and Precalculus textbooks define exponential functions relative to an algebraic representation, and this representation is simply given to students on the first page of the unit on exponential functions with only shallow or no development. See Figure 8.1.

An exponential function is denoted by $f(x) = a^x$ where the base a is positive real number and $a \neq 1$.

An exponential function f is a function of the form $f(x) = ab^x$ where a is a positive real number and the base b is a positive real number with $b \neq 1$.

Figure 8.1. Two examples of common textbook definitions for exponential function, synthesized from reviewing ten College Algebra and Precalculus textbooks published between 2000 and 2012.

If this is the common way of presenting what it means for a function to be exponential then students working in these classes are likely to be grounded in a relatively weak set of meanings for the nuances of exponential growth and imagery of what remains invariant in an exponential situation as two quantities change in tandem. This provides space for students to develop unhelpful or incorrect meanings for exponential growth and related ideas. Consider that in a recent study I interviewed five university Precalculus students

who had previously taken (and passed) Precalculus in high school and I asked them what it means for a function to be exponential. I received a variety of responses including any function that increases (as long as the values get large), any function with a graph that curves up, and a function with an algebraic representation that either has “something to the x , or x to the something.” It is not clear to me how any of these meanings for an exponential relationship would help students recognize situations that could be appropriately modeled by an exponential function or attend to features of graphical or tabular representations that would allow them to distinguish exponential functions from many other types of relationships. In fact, there is little “mathematical” about their definitions and little to suggest productive meanings for broader mathematical ideas like covariational reasoning or interpreting graphical representations.

Thus, an important research area involves describing a coherent set of powerful general meanings and reasoning [what Harel (2008a, 2008b) calls *ways of thinking*] and to describe powerful *ways of understanding* (Harel, 2008a, 2008b) particular ideas that fit productively into the larger coherent network of themes. Thompson (2013) points out that students’ meanings shape their interactions with the mathematical tasks in which they engage, and it is their individual meanings we hope to affect through instructional interventions. We have a greater chance of supporting students in constructing important meanings if we are explicit in the meanings we want to encourage and if we are aware that those meanings cannot be transmitted but must be constructed by students. However, Thompson (2013) laments that this focus is uncommon in mathematics education research.

What I find more troubling is the rarity of research in mathematics education that takes the issue of mathematical meaning seriously. Research that is ostensibly on knowing or understanding, whether the context is teaching or learning, too often examines performance instead of clarifying the meanings students or teachers have when they perform correctly or the meanings they are working from when they fail to perform correctly. Neither correct performance nor incorrect performance says anything about the nature of a person's system of meanings that expresses itself therein. This is not to say that no research considers students' or teachers' meanings. Rather, it is too rare. (p. 78)

Thompson did not direct his observations toward research on student learning in online courses, but his comments are quite appropriate in this context. If researchers are interested in exploring the meanings students construct, they must first carefully consider their learning goals for the course, consider the coherence among ideas and learning goals across the course, develop research-informed learning trajectories to inform the design of individual lessons and units, and develop a plan for gathering evidence on the meanings that students do develop while engaging with these lessons. I will return to this in Part II, but for now I repeat that this focus is currently absent from research in online learning despite calls for an increased focus on design research in mathematics education (for example, Cai, Morris, Hohensee, Hwang, Robison, & Hiebert, 2017).

Design-based research/design science. Cobb (2007) argued that mathematics education “can be productively viewed as a design science, the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes” (p. 7). His perspective was not entirely new, however. For example, Thompson (1985) described the importance of considering learners as situated within a trajectory through a curriculum and wrote that the primary “aim of mathematics education is to promote mathematical thinking” (p. 189). A curriculum creates opportunities for students to construct a particular type of mathematical knowledge under

the teacher's guidance and with the assumption that the knowledge students construct is never an exact copy of the knowledge the teacher intended they construct (or possesses herself). Revision and redesign are a natural part of crafting learning opportunities for students based on feedback collected during the act of teaching. Researchers engaged in design studies see themselves as enacting a balance between empirical research and theory-informed instructional design. Their

innovations embody specific theoretical claims about teaching and learning, and help us understand the relationships among educational theory, designed artifact, and practice. Design is central in efforts to foster learning, create usable knowledge, and advance theories of learning and teaching in complex settings. (The Design-Based Research Collective, 2003, p. 5)

Starting with theories of how students learn, design-based researchers generate interventions to enact in specific learning environments to impact student learning in a practical way. However, they always assume that the intervention will uncover important implications for modifying learning theories, supporting practitioners, and designing future interventions. They begin by reviewing and/or developing theories on student learning and then designing and testing interventions that theoretical analysis suggests could productively influence student learning. Figure 8.2 summarizes design-based research's continuous development cycle. Even when learning theories influence an intervention's design, researching its enactment is needed to clarify exactly how features of the intervention impacted learning and how specific features of the learning environment contributed to the final result. "Models of successful innovation can be generated through such work—models, rather than particular artifacts or programs, are the goal (cf. Brown & Campione, 1996)" (The Design-Based Research Collective, 2003, p. 7). "Models of successful innovation can be generated through such work—models,

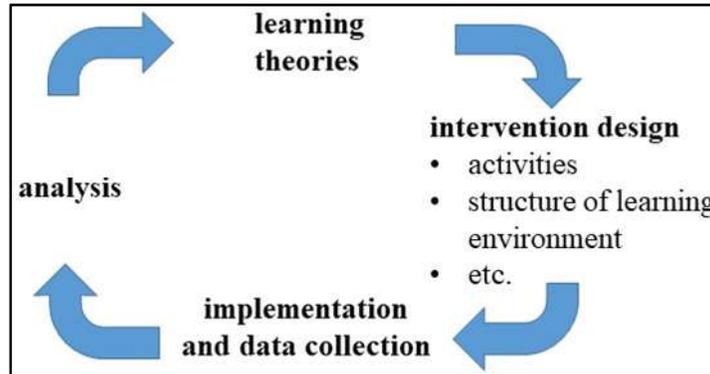


Figure 8.2. Design-based research’s continuous development cycle. Learning theories inform the intervention’s design, implementation data is collected and analyzed, and this analysis shapes the understanding of the original theoretical foundations of the study and impacts future iterations (The Design-Based Research Collective, 2003).

rather than particular artifacts or programs, are the goal (cf. Brown & Campione, 1996)”

(The Design-Based Research Collective, 2003, p. 7). The following quote best

summarizes design-based research.

[D]esign-based research methods can compose a coherent methodology that bridges theoretical research and educational practice. Viewing both the design of an intervention and its specific enactments as objects of research can produce robust explanations of innovative practice and provide principles that can be localized for others to apply to new settings. Design-based research, by grounding itself in the needs, constraints, and interactions of local practice, can provide a lens for understanding how theoretical claims about teaching and learning can be transformed into effective learning in educational settings. (The Design-Based Research Collective, 2003, p. 8)

In design research, the investigator is positioned as both the intervention’s designer and its evaluator with the expectation of enacting multiple iterations of implementation, study, and refinement where the “implementation itself becomes the source of information that guides refinements in learning opportunities” (Cai et al., 2017, p. 345).

Design research and online learning. Researching the development of student meanings in an online course is a perfect environment for applying design research

principles. Online courses are generally designed as a single entity that will make up the entirety of students' learning opportunities. Thus, researchers have near total control over the design and implementation of these courses. This lessens sources of variability such as differences in instructor experience or knowledge and can enable researchers to have a greater focus on how specific features of the course impact student learning. The key focus for conducting design research in online environments then is to identify student learning goals and to develop a reasonable plan for collecting evidence about the meanings that students construct in a way that productively informs future iterations.

A Call to Action – Closing Comments

Researchers in mathematics education have identified important mathematical meanings that teachers and curriculum designers should target as well as complexities that arise in supporting students' construction of these meanings (for example, Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Oehrtman, & Engelke, 2010; Castillo-Garsow, 2010; Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012, 2015; Moore, 2010, 2012, 2014; Moore, Paoletti, & Musgrave, 2014; Tallman, 2015; Thompson, 1990, 1993, 1994, 2008a, 2008b; Thompson & Carlson, 2017). Researchers are beginning to apply this work to design targeted interventions at the course level, such as Carlson, Oehrtman, and Moore's (2018) *Project Pathways* Precalculus curriculum, Thompson's (2016) *Project DIRACC* calculus curriculum, or *Realistic Mathematics Education* curricula grounded in work by Treffers (1978, 1987) and Freudenthal (1973, 1983, 1991) and refined and expanded by researchers around the world. It is important for this work to extend into online courses

since they represent an increasingly large part of students' post-secondary experiences. In particular, (a) online courses should be designed to support important meanings we know are vital for STEM courses and careers, (b) researchers must carefully attend to the meanings students construct while working through the online lessons, and (c) researchers need to gather data on how lesson features either support or fail to support the construction of productive meanings. This focus is currently missing from research on online learning yet is critical for the design and improvement of high-quality online learning experiences.

Part II: Designing for and Researching Mathematical Meanings in Online Courses

Stigler and Hiebert (2009) argued that throughout the 20th century a gap formed between teaching and research practices in the U.S. They, along with Steffe and Thompson (2000), described key repercussions from this evolution. First, classroom teachers did not use instruction to build models of student thinking and act on these models to make key instructional decisions while planning or implementing lessons. Second, researchers, being removed from the classroom, based their work on classical experimental design where they attempted to identify possible variables in learning, control for some variables while changing others, and compare the results of different treatments. Thus far I have argued for viewing online learning through the lens of design research. I further claim that the current body of literature related to online learning predominantly emphasizes the classroom environment as a variable to be tested while overlooking the mathematical meanings students construct while engaged in online lessons. This produces results that are useful for academic leaders making decisions

about course offerings at the institution level but does not contribute to a design-based iterative improvement cycle for these courses (Cobb, 2007).

In the second half of the paper I review established methodologies for studying student meanings, consider implications of applying those methodologies to online learning, and discuss triangulating various data to help researchers model the meanings students construct as a result of online instruction.

Clinical Interviews and Teaching Experiments

Within constructivism there are well-articulated research methodologies for studying student meanings, including meanings at some moment [*clinical interviews* (Clement, 2000)] and meanings as they evolve through interactions with an instructor [*teaching experiments* (Steffe & Thompson, 2000)]. Both methodologies are grounded in Piaget's genetic epistemology (1971) and rely on modeling students' mathematical meanings based on observable behaviors. Their theoretical perspective assumes that a student's activity is driven by the way he conceptualizes a task (not the way a teacher or researcher conceptualizes the task). Clinical interviews and teaching experiments produce evidence of how a student conceptualizes a task at hand so that a researcher can model an individual's scheme of meanings relative to that conceptualization. Therefore, trying to model students' mathematical meanings requires more care and planning than simply judging if students can successfully obtain answers to given mathematical tasks.

Clinical interviews. Clinical interviews (Clement, 2000) involve open-ended questions intended to generate observable behavior relative to problem solving and reasoning, including student explanations and justifications for their work, with the goal

of modeling her current thinking relative to particular mathematical ideas. Clement describes two broad categories of clinical interviews called *generative* and *convergent* approaches. In a generative approach, the researcher creates a detailed interview protocol including tasks and various lines of follow-up questions based on the possible types of responses students might give. The goal in a generative approach is to get evidence of student thinking on the table and then generate hypotheses about mental processes and reasoning that, if a student had these, would represent a viable explanation for the observable behaviors. A researcher often goes through multiple passes and revisions until the most viable hypothesis survives (the one that best explains the observable behaviors while still being general enough to be useful in a wider range of situations). Convergent approaches focus on frameworks developed out of generative studies and typically code interviews using these frameworks. The researcher's goal is to support the viability of the models previously generated and perhaps categorize participants' meanings within a framework with the assumption that their coded observable behavior suggests their underlying mental processes. These studies might generate patterns or connections not observed in previous generative studies, and these patterns could serve as research questions for a new round of interviews beginning with more generative studies.

Clinical interviews are vital to cognitively oriented research because they

include the ability to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques. These abilities are especially important because of Piaget's discovery that people have many interesting knowledge structures and reasoning processes that are not the same as academic ones—they have alternative conceptions and use nonformal reasoning and learning processes. Mapping this “hidden world” of indigenous thinking is crucial for the success of instructional design. Students cannot help but use their own prior conceptions and reasoning processes during

instruction, and these have strong effects on the course of instruction. (Clement, 2000, p. 547)

Students bring their own unique meanings to any learning experience, and the makeup of these meanings is outside the instructor's control. Effective instructional design and enactment requires attending to the meanings students bring to the lesson, including drawing out and recognizing common meanings (both productive and unproductive) students might possess based on observable behaviors and planning instruction to support productive shifts in those meanings.

Teaching Experiments. Teaching experiments (Steffe & Thomson, 2000) build on the same foundation as clinical interviews by acknowledging that students have different mathematical conceptions from the teacher/researcher and that these meanings are not directly accessible to an observer – they can only be modeled via a conceptual analysis of observable behaviors. However, teaching experiments carry the expectation (or hope) that students learn something through interactions with the teacher/researcher. Results of teaching experiments help researchers develop and test domain-specific theories of learning mathematical ideas and thus help to inform future teaching and curriculum design.

Steffe and Thompson describe two categories of teaching experiments – exploratory teaching and teaching that tests a specific research hypothesis. Exploratory teaching involves sustained interaction with students where the researcher seeks to “become thoroughly acquainted, at an experiential level, with students’ ways and means of operating in whatever domain of mathematical concepts and operations are of interest” (Steffe & Thompson, 2000, p. 274). That is, the researcher attempts to build models for

how students think about certain mathematical ideas including, but not limited to, the potential origins of their thinking, sources of barriers to learning ideas, and key distinctions between students in how they reason about what, from the researcher's point of view, are similar tasks.

After engaging in exploratory teaching or building from the results of previous exploratory teaching, the researcher may recognize a significant issue worthy of further study. At this point the researcher develops a hypothesis to test and performs a conceptual analysis to plan interventions and test the hypothesis. The initial plan is a rough working outline that he can scrap or adapt on the fly based on interactions with students. As students assimilate given tasks to their personal schemes they produce products of their reasoning. The researcher analyzes these products to build a model of schemes that best account for the students' behaviors, comments, and persistent errors (or *essential mistakes*). The researcher then tests the validity of his emerging model with follow-up questions and tasks.

Once a researcher believes that he can think like the students, he can create "an itinerary of what they might learn and how they might learn it" (Steffe & Thompson, 2000, p. 280). The goals he formulates inform future teaching experiment sessions and task design to support students in modifying their existing schemes independently. This involves presenting tasks that the researcher hypothesizes will be assimilated to existing schemes, but that cannot be addressed within those schemes as they currently stand, and designing ways of interacting with these tasks that may promote accommodations to the triggered schemes. Throughout this process, the researcher maintains a focus on student's reasoning rather than the researcher's learning goals for the student. This focus enables

the researcher to regularly develop and test hypotheses about the student's current thinking, what tasks are within reach given his model of the student's schemes, and what tasks are beyond the student's current understandings assuming his model is valid. The teaching agent will develop, alter, and discard several hypotheses and potential models of students' mathematics during the course of any given teaching episode.

Shared assumptions and theoretical perspectives. Central to both clinical interview and teaching experiment methodologies is the researcher's goal of understanding students on their terms and modeling students' meanings to better understand their structure and the implications and boundaries of those meanings. A researcher's models emerge from extensive direct interactions with students as the researcher develops and tests several hypotheses about the meanings they possess. Steffe & Thompson (2000) describe a student's *essential mistakes* as key data in modeling their thinking. Students bring their own unique meanings to any task they encounter, and the way they conceptualize the task and the meanings that students apply to address the task constrain their responses. Since a student's mathematical meanings are relatively stable over the short-run, exploring the occurrence and boundaries of a student's incorrect reasoning helps the researcher model the student's meanings at that point in time. Within a teaching experiment, the gradual reduction of essential mistakes is key evidence that students are modifying and reorganizing their schemes (and thus learning).⁴²

Thompson (2013) described a theoretical model for how individuals attempt to engage in a meaningful conversation as part of his description for a researcher's mindset

⁴² Clement (2000) did not use the term *essential mistakes* in his framework, but examining students' essential mistakes is a useful goal when conducting clinical interviews as well.

when engaging with students while attempting to model their thinking. In a conversation where individuals try to convey meaning, they each understand that they are speaking to a person with potentially different meanings and thus try to frame their utterances and process what they hear based on models of the other person. In effect, each person is communicating with their model of the other individual. The intention of a meaningful conversation is to engage in communication until there is no evidence to suggest that either participant has been misunderstood. See Figure 8.3.

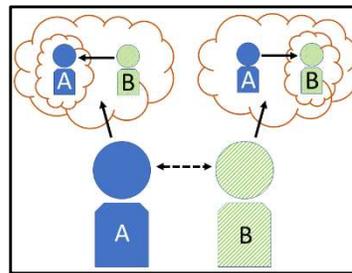


Figure 8.3. Thompson's (2013) model of meaningful conversations (p.64).

Thus, when a researcher engages with a student in a clinical interview or teaching experiment, the goal is to continually generate hypotheses about the student's reasoning and to focus the interaction to test these hypotheses. For example, the researcher might pose questions that would generate a specific type of response if the student does indeed reason in the way the researcher hypothesizes. In this way the researcher is addressing the interaction to a conceptual model of the student and revising the model based on the actual responses the student provides. This perspective is also useful for conceptualizing the design and research of online courses.

Theoretical Considerations in Online Learning

Teaching in an online learning environment, from a constructivist standpoint, shares the same foundational assumptions as teaching in a traditional learning environment. First, that students are individual, active constructors of their own knowledge and develop meanings that may not match either the teacher's intended learning goals or her perceptions of what students learned. Second, despite this awareness, the teacher is responsible for establishing learning goals relative to ways of thinking and knowing that she believes will be advantageous for her students and for developing lessons that "create a particular dynamical space, one that will be propitious for individual growth in some intended direction, but will also allow a variety of understandings that will fit with where individual students are at that moment in time" (Thompson, 2002, p. 194). Third, that these learning goals and lessons are designed and adapted from her (or others') experiences with students reasoning about the same mathematical ideas and her conceptual analysis of those ideas. In designing lessons intended to convey particular meanings, the researcher creates activities with an image of an *epistemic student* in mind – a generalized student "that encapsulates aspects of individual students' understandings that appear, from the [researcher's] perspective, prevalent among a population of students" (Tallman, 2015, p. 37). But each student in the class possesses his own individual meanings, and the researcher gains insights, and may adapt her epistemic model, by analyzing data produced by students when interacting with the text, questions, tools, etc. in the online environment. Each student is individually interpreting the tasks and features of the lesson and responding based on these

conceptions as well as his model of what the course program expects from him. See Figure 8.4.

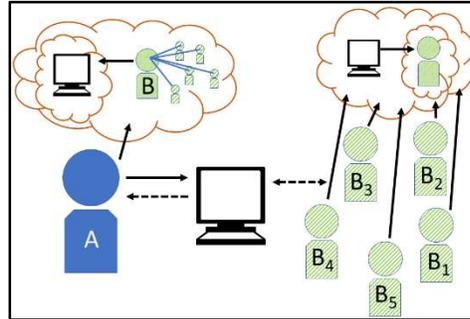


Figure 8.4. A model of communication between the designer and students in online courses. Here B represents an epistemic student.

This perspective is important because it highlights how issues of design impact research on students' meanings as well as the importance of studying how students conceptualize features of the online lesson and the meanings they actually construct. This data produces more accurate epistemic models, informs improvements in lesson design, and is the basis for future iterations of the course. As we shift to be more attentive to the meanings students construct while engaged in online lessons we are left to determine the combination of data that best supports our goals.

Collecting Data to Inform Models on Student Meanings

Best practices in design research suggest collecting multiple sources of data about individual students (and the class as a whole) to inform conclusions on an intervention's impact (The Design-Based Research Collective, 2003). This advice is important for modeling student meanings in online courses (either as they exist at some point in a course or as they evolve across lessons) precisely because there is an inherent tension between the methods that yield the best data and those that respect the reality of the

learning environment. Table 8.2 shows a list of data collection methods a researcher could use to study learning in an online environment. This list is extensive but not necessarily exhaustive.

Table 8.2

Data Collection Methods for Modeling Student Meanings

Data Collection Method for Modeling Student Meanings	
↑ Information Relative to Student Meanings Less Information	students self-reporting via course surveys
	looking at final course grades and retention rates
	tracking student success in subsequent courses
	course assessments (midterms, finals, etc.)
	aggregate student interaction data
	research-validated assessment instruments
	using screen capture and other video to monitor students' work as they complete lessons
	post-lesson interviews and stimulated recall
	interviewing students as they work through tasks in online lessons
↓ More Information	clinical interviews or teaching experiments using tasks, lessons, videos, applets, etc. outside of the online environment
	↓ More Interference Interfering with the Typical Online Lesson Experience

The collection methods are organized very roughly from those that provide less useful information for modeling the meanings that students construct to those that provide the most useful information. Note that data generation methods that make it easier to model students' meanings also tend to most alter how the student is experiencing the intervention so that what is being studied is no longer a typical experience of a student in the course. This may introduce potential issues in generalizing results. For this report two points are worth emphasizing. First, triangulation of multiple data sources improves the quality of models and the generalizability of research results as many researchers have

noted (i.e., Schoenfeld, 2000). This seems especially true given the complications in gathering data on students' meanings in a course where there is less inherent personal interaction between instructors and students. Second, it is an open question what combination of data is best for balancing these concerns. In the sections that follow I highlight some salient points to consider about data collected from each category.

Students self-reporting via course surveys. Surveys are often used to efficiently collect measures of students' self-confidence, persistence, and satisfaction. These surveys can suggest sources of negative emotional reactions which is important because these reactions are likely to interfere with students successfully meeting mathematical learning goals. Negative reactions (whatever their source) might also correlate with higher withdrawal rates that can make the course less likely to be continued by academic leaders. However, Kauffman (2015) noted that student surveys are often poorly validated, so researchers should always seek to support survey results with other data they collect. In addition, surveys are not accurate in capturing the quality of student learning. Ke and Xie (2009) showed that students in their study self-reported deep learning on course objectives but that analysis of students' contributions to course discussions did not support that claim.

Looking at course grades and retention rates. Student grades may provide feedback on whether a course is appropriately challenging while retention rates often correspond with the kinds of affective data captured in surveys. If students are dissatisfied with a course for any reason (challenge level, lack of interaction with the instructor or other students, difficulties with technology, or poor website design) it will likely manifest with higher withdrawal rates. But both measurements can vary based on a

number of factors that have little to do with the meanings students construct as a result of learning opportunities in an online course. Course grades and retention rates, like students' self-reports on course surveys, produce no data on the meanings students constructed during the course. In instances when a large portion of the course grade is based on assignments that explicitly require demonstration of particular conceptual meanings for mathematical ideas it is possible to use this data to make inferences about students' meanings. Even in this case, however, the data is unlikely to suggest how particular features of the course contributed to supporting students in constructing these meanings and thus provides little guidance to inform future iterations of the course.

Tracking student success in subsequent courses. Collecting information on student success in future courses may indicate whether students are constructing meanings that serve them well in future courses. This data is also useful for demonstrating to administrators that the course is generally successful and that ongoing study and refinement is a worthwhile investment of time and resources. However, a researcher must be cautious in reading too much into the data in terms of the meanings students construct. For example, suppose a researcher follows Thompson and Carlson (2017) and believes that it will be useful for students heading into Calculus to understand a function's graph as traced out in an emergent way through a systematic coordination of how the magnitudes of two varying quantities change in tandem and designs his Precalculus course to support students in constructing this meaning. The fact that students are later successful in Calculus does not confirm either that students developed this intended meaning or that constructing that meaning contributed to their success. Student success in later courses may depend on future instructors' experience and expectations,

changes in delivery method (for example, switching from an online to in-person course or vice versa), and even self-selection bias (which students choose to enroll in future courses or what courses they select). Thus, tracking student success in future courses may give a general sense of an intervention's success without providing specific feedback to inform improvements in the course relative to the meanings students constructed.

Course assessments. Course exams assess objectives instructors expect students to master and their content informs the focus of daily lessons. But Reeves (2006) reported that course assessments are often poorly aligned with stated course objectives no matter the subject area. Tallman et al. (2016) conducted a study to characterize assessments in calculus courses across the U.S. and that the vast majority of Calculus I final exams assessed mostly students' recall of facts and their procedural proficiency as opposed to students' meanings for fundamental ideas or their ability to apply these meanings. While their study focused only on Calculus I final exams, it is not a stretch to assume that analysis of the midterms in these courses would yield similar results or that examining assessments in College Algebra and Precalculus courses might arrive at the same conclusion. Clement (2000) refined and published his clinical interview methodology in part because he found that course tests were not well-suited for gathering information on student thinking. He wrote that "[b]ecause tests are almost always written from the point of view of the teacher and are designed to detect standard forms of academic knowledge, they can fail to detect key elements in students' thinking" (p. 54).

Without specific attention to the goal and question composition of course exams the quality of data they produce relative to modeling students' meanings will be low. Complicating the issue is the fact that students may answer differently depending on the

question format (such as open-ended vs. multiple choice) or the context or representation used to present the question (such as a word problem vs. a graph vs. a table). From the researcher's point of view, a set of questions might all test the same or similar ideas, but a student's answer to one question within that set often does not predict how they will answer other questions from that set. Carlson, Oehrtman, and Engelke (2010) noted this phenomenon while validating their Precalculus Concept Assessment (PCA) instrument and Thompson (2016) and Byerley and Thompson (2017) also described the same trend while validating the Aspire instrument to measure teachers' mathematical meanings. I also noticed this tendency in a recent study. I asked 65 Precalculus students to complete a 25-question multiple-choice assessment and interviewed five students about their responses to eight select tasks afterwards (for a total of 40 pairs of responses to compare) without providing the multiple-choice selections. Twenty-two of the 40 responses the students provided differed between the multiple-choice and open-ended versions of these tasks, sometimes dramatically so. Figure 8.5 shows one such task and Table 8.3 shows the answers students provided on the multiple-choice assessment (marked MC) and during the interview (I). Four of five students (pseudonyms used) answered differently when given possible answer choices.

The cost of replacing the exhaust muffler on your car is currently \$195. The previous time that you had the same replacement done, the cost was \$131. What is the percent increase in your repair bill (rounded to the nearest percent)?

- a. 33%
- b. 49% [correct response]
- c. 64%
- d. 67%
- e. 149%

Figure 5. A task given to university Precalculus students taken from Madison, Carlson, Oehrtman, and Tallman (2015).

Table 8.3

Student Responses to the Muffler Task

answer choice	Class Totals	Gina	Marcus	Shelby	John	Lisa
a. 33%	32%	MC		I		
b. 49% [correct]	45%				MC, I	I
c. 64%	12%	I		MC		
d. 67%	6%					MC
e. 149%	5%		MC			
164%			I			

Furthermore, during interviews it was clear that even students who answered the question correctly held only algorithmic meanings for percent change and percentage comparisons. At one point during their solution justifications all five students stated that they needed to take the result of a calculation and move the decimal point two positions to the right. However, none of the students could provide a mathematical justification for this process.

Thus, common course assessments are unlikely to provide an accurate characterization of students' meanings and may encourage researchers to mistake performance for indications of particular reasoning. This is especially true if researchers are crafting an intervention yet leaving course exam creation up to instructors (or course coordinators). In this case the chances are high that the results of those assessments will not yield reliable evidence of students' meanings. If researchers design the assessments it might be easy to fall into the same trap given traditional images for the kinds of assessment items used to measure course objectives. Course assessments can still serve a useful purpose such as verifying that students have mastered certain expected procedural

skills or even as initial steps in beginning to identify and hone items for a future validated assessment instrument, but researchers should always be wary of drawing too many conclusions about students' meanings based on performance tasks.

Aggregating student interaction data. Online course environments often produce aggregated data on specific lessons or items within lessons. For example, I helped to design an online course using iMathAS (Lippman), and this platform provides instructors and course designers with data on average item scores, average number of attempts, and the average time spent per attempt. See Table 8.4.

Table 8.4

iMathAS Instructor Feedback

#	Question	Grade	Average Score Attempted	Average Attempts (Regens)	% Incomplete	Time per student (per attempt)	Clicked on Help
1	Mod 2 :: Computing Δx	Grade	99%	1.06 (0)	0%	43.13 sec (40.59 sec)	N/A
2	Mod 2 :: Meanings for Δx	Grade	97%	1.13 (0)	0%	3.46 min (3.08 min)	N/A
3	Mod 2 :: CROC, relating Δy and Δx	Grade	85%	1.13 (0)	0%	3.23 min (2.15 min)	N/A
4	Mod 2 :: Rental Car Cost, cost of surcharge + find additional value	Grade	96%	1.06 (0)	0%	2.45 min (2.18 min)	N/A
5	Mod 2 :: Rental Car Cost, proportional?	Grade	75%	1.63 (0)	0%	1.65 min (1.02 min)	N/A
6	Mod 2 :: car distance, compute average speed	Grade	76%	1.63 (0)	0%	2.7 min (1.66 min)	N/A
7	Mod 2 :: Meaning of average speed	Grade	93%	1 (0)	6.3%	5.42 min (4.78 min)	N/A

Data on average scores and average number of attempts may reveal the difficulty level of various tasks within a learning trajectory. Analyzing these tasks could support hypotheses of key meanings students must construct within a learning trajectory or may help to identify common gaps in students' background knowledge or common unproductive meanings students possess. However, once again caution is necessary. In the course we

created we noticed many sources of variability in this type of data. For example, we had instances where one or more students walked away from the computer mid-question for as much as 24 hours which severely skewed the average time per attempt for that item. We have also observed that many students exhibit the following behavior while working on an exercise containing multiple parts. The student attempted only the first part of a question prior to submitting the entire question for grading. She repeated this until she was notified that her answer to the first part was correct. She then attempted the second part of the question and submitted the entire question for grading. She repeated this until she was notified that her answer to the second part was correct. This pattern continued until all parts were answered correctly. We are not necessarily interested in limiting this behavior, but its common occurrence drastically impacted our data.

Validated assessment instruments. There are many sources of assessment validation. For example, assessments might be validated as predictive of future student success or validated to avoid bias relative to demographic differences. Carlson et al. (2010) developed the PCA instrument to assess students' understanding of the conceptual foundations for calculus including rate of change, function, function composition, and proportionality. Literature on students' struggles in calculus informed their choice of ideas, and they followed a multi-step validation process to finally arrive at a stable 25-question multiple choice assessment such that students' selection of particular answer choices are highly correlated with the meanings revealed during over 300 clinical interviews and thousands of student exam responses. They demonstrated that the PCA's stable version is a reasonable assessment for students' readiness for calculus since "77% of the students scoring 13 or higher on the 25-item test passed the first-semester calculus

course with a grade of C or better while 60% of the students scoring 12 or lower failed with a D or F or withdrew from the course” (Carlson et al., 2010, p. 140). Thompson (2015) provides another example of a validation process in creating and refining the Aspire instrument for modeling teachers’ mathematical meanings. Carefully validated assessments can be a more reliable indication of students’ meanings at some moment and can thus provide evidence of shifting meanings over time. Such assessments might not provide information on how specific features of lessons supported students in constructing those meanings, but they can be administered repeatedly to detect shifts in learning among the population of students completing the course.

None of the data collection methods up to this point are likely to alter students’ experiences within a course. Filling out course surveys and taking exams are a normal part of students’ participation in post-secondary classes. Furthermore, tracking students into future math courses or monitoring course grades and retention rates require no active participation from the student. The remaining methods, however, can potentially alter the very thing a researcher wants to study despite providing data that is better at capturing individual students’ meanings.

Using screen capture and other video to monitor students’ work as they complete lessons. It can be very difficult to integrate data collection methods that capture student meanings into the flow of an online lesson. Methods such as screen capturing, which can track every keystroke a student makes, might provide useful insights into the meanings students are constructing as they interact with specific tasks while progressing through the course. By analyzing each calculation and attempt, a researcher can hypothesize how students are conceptualizing tasks and the meanings they try to leverage

to complete them. Looking for *essential mistakes* (Steffe & Thompson, 2000), or mistakes that tend to persist, can produce powerful insights. Noticing when essential mistakes decrease in frequency can inform hypotheses about when key shifts in meaning occurred for specific students. Eye-tracking technology might add an additional layer of information by indicating where students direct their attention when observing an animation or reading text in between moments of entering information into the system.

If the researcher refrains from talking with students during the sessions or answering questions during the recording, this is the least intrusive way to gather detailed information on how individual students interact with online lessons. However, the fact that students would feel obliged to complete entire lessons in one sitting (a requirement they might not have to meet when completing lessons at home) and the fact that they know they are being observed might change their behavior or their attentiveness to aspects of the online lessons.

Post-lesson interviews and stimulated recall. After a student completes a lesson, a researcher can interview that student about his experiences and meanings using the clinical interview methodology. The researcher may ask the student to summarize their understanding of the lesson's goals, describe what he learned, solve novel tasks related to the ideas in the lesson, or revisit parts of the lesson and describe how he interpreted a text excerpt, applet, animation, or exercise. This is likely to produce rich data about how a student interpreted features of the lesson, how the student's meanings may have changed as a result of instruction, and the role of specific lesson components in supporting this change. This data collection method alone may not drastically change the student's experience with the particular lesson on which he is interviewed, although the

additional interactions with an instructor, additional prompts to reflect on his learning, and the fact that his attention will be pointed to what the researcher views as key moments in a lesson may change how that student interacts with future lessons relative to the general population of students that enroll in the course.

Interviewing students as they work through online lessons. Instead of interviewing a student after completing a lesson, a researcher can interview a student as he works through a lesson. The researcher can ask the student to explain his interpretation of lesson tasks, videos, applets, and text, to explain the reasoning behind his solution processes, and to reflect on why certain answers were correct or incorrect. Superficially this may seem like the closest parallel to a teaching experiment within an online course. But unlike in a teaching experiment, here the researcher would not have the ability to adapt the flow of the lesson, the questions, or the tasks in response to hypothetical models of the student's mathematical meanings, at least not within the flow of the online lesson (this would need to be done by interjecting additional tasks for the interviewee). These interactions are likely to produce rich data that can help a researcher model a student's meanings and perhaps capture moments where his meanings changed or evolved.

However, the requirement that a student continuously reflects on and monitors his thinking throughout the lesson will result in the student interacting with a lesson in ways that are different from how other students interact with the same lesson while alone. An important example of this occurred when we piloted our online precalculus course. We designed lessons with careful scaffolding to support students in developing meanings we believe are important. We conceptualized all aspects of our lessons as being integral to this process, including students reading the given text, interacting with the given applets,

answering the questions in the indicated order, and watching short conceptually-oriented summary videos placed at key points in each lesson. At the end of the course we noticed that the overall watch count for the videos was very low, and in an end-of-course survey we asked students if they watched the videos while completing lessons. Most students said that they never watched the videos or only watched the videos if they could not complete certain exercises. If we asked a student to watch all videos while working through the lessons as part of this interview process that alone would have ensured that the student's experience in that lesson differed from the experience of typical students in the course and may have complicated our understanding of how students constructed meanings while interacting with the lessons. With this in mind, if a researcher wants to interview students working through a lesson it might be useful to either refrain from requiring students to complete parts of the lessons they choose to skip or to conduct comparative studies for students who do or do not use particular features of a lesson.

Clinical interviews and teaching experiments outside of the online environment. Researchers might want to assess student interpretations of certain contexts, learning trajectories, interactive applets, videos, and so on outside of the context of the online course. Conducting clinical interviews and teaching experiments outside of the online environment can help shape hypothetical learning trajectories and lesson design prior to the pilot following the methodologies described by Clement (2000) and Steffe and Thompson (2000). Clinical interviews prior to and at the end of the course can also provide important triangulation supporting conclusions about what meanings students constructed. Beyond the specific intervention at hand, comparing student learning on lessons designed to support specific meanings in and out of the online

environment would benefit the field by helping shape our understanding of the effect size of the instructor and face-to-face student interactions compared to the online environment in leveraging the same set of activities to support students in constructing a particular set of meanings. Note that it can be helpful to conduct clinical interviews with students before and/or after they participate in other activities such as completing lessons while screen capture technology monitors their inputs. This may assist a researcher in hypothesizing about the source of essential mistakes and/or recognizing key shifts in meaning when those essential mistakes begin to disappear.

Commentary. The list of data and data collection methods I described highlights a wide variety of information that could be useful for researchers as they try to model the meanings students construct while interacting with online lessons. Ideally researchers would identify multiple categories of data to assist in triangulation (supporting conclusions derived from analyzing one data source with conclusions derived from analyzing other data sources).

My analysis of these categories and data collection methods raises a few questions that researchers will need to consider as we work to refine methodologies for studying student learning in online courses. First, I have been careful to highlight data collection methods that create learning environments different from the experience of typical students in the course. By doing so I do not want to imply that such methods yield “bad data”. They do, however, introduce the potential that a researcher’s presence has an impact on student learning separate from the online lesson and its features. How large is that impact? And how can we separate the researcher’s impact from the lesson’s impact? Steffe & Thompson (2000) talk about a student’s *essential mistakes* as persistent errors

that derive from the schemes triggered when a student conceptualizes a task. It takes a long time and many opportunities to engage in repeated reasoning for students to modify their schemes such that those essential mistakes decrease or disappear. Thus, on the one hand we might find that essential mistakes persist regardless of the researcher's presence. On the other hand, the researcher's presence may prompt more frequent and deeper reflection and thus support students in reducing essential mistakes more quickly than for the general population of students enrolled in the course. Clarity on these issues will likely only emerge from repeated attempts to model students' meanings in online courses across various studies.

Different categories of data and data collection methods may contribute more or less useful information depending on the current state of the design research process. In designing an online course, the researcher will start off by making assumptions about the meanings students might have as they start the course, the meanings the researcher will target during instruction, how to design lessons to support students in constructing these meanings, and how to assess the intervention's effectiveness. This leads to a first draft of the intervention and a pilot run to test the initial design. Analyzing aspects of the pilot pushes the study along the design research cycle that can (and should) be repeated multiple times to improve the intervention. When is each category of data most useful to a researcher? Table 8.5 includes one possible answer with solid dots indicating the periods where the data is most useful and open dots indicating periods where the data might still be collected for triangulation but could warrant less attention.

Table 8.5

A Possible Timeline for Data Usefulness

	pre-intervention	pilot	early revision iterations	iterations moving towards course stability
student surveys		●	○	○
final course grades/retention rates		●	●	○
tracking future success			●	●
course assessments		●	●	○
student interaction data		●	○	
validated assessments	●	●	●	●
screen capture		●	○	○
post-lesson interviews		●	●	○
mid-lesson interviews		●	●	
clinical interviews/teaching experiments set outside of the online environment	●			●

Clinical interviews and teaching experiments outside of the online environment can help to shape hypothetical learning trajectories and lesson design for the initial pilot intervention. Clinical interviews might also be useful in later interventions to support the results of validated instruments as a measure of students’ meanings at the end of the course. Gathering survey data on student satisfaction, monitoring course grades and withdrawal rates, and monitoring student performance on non-validated assessments may be part of all iterations but are far more important during the first iterations of the course. This is because researchers must respect the constraints under which academic leaders operate and what they value. These leaders will not approve additional iterations if students cannot pass the course or students are highly dissatisfied with their experience. Close monitoring of students working through lessons and attempts to model the meanings they construct produces data that is extremely useful during early iterations of

the course. This data may demonstrate a need to drastically overhaul lessons, units, or general course design and contributes to refining hypothesis about how to support students in constructing particular meanings. However, the course is likely to become increasingly stable in later iterations (with survey results, course grades, etc. also becoming relatively stable), and validated assessment instruments might become the key piece of data for recognizing shifts in learning and comparing student learning across various similar courses. Table 4 is not meant to be a definitive set of recommendations, but rather represents an acknowledgement that, when faced with such a wide variety of data options, researchers should think carefully about the most useful sources of data at various stages of their study in order to prioritize their time and resources.

Discussion

We face unique challenges in modeling students' mathematical reasoning in online courses in that current research methodologies require interactions between the researcher and students that do not match the typical experiences for students in these courses. This calls for careful reflection on the kinds of data we are collecting on student performance and reasoning and how the various data sources can generate evidence that together provide an accurate characterization of how students conceptualize the tasks, features, and ideas in the online lessons and what meanings they actually construct. Such information can inform future iterations of the course to better introduce perturbations that challenge common unproductive meaning students may have or might develop and to scaffold lessons to support opportunities for students to construct more productive meanings we know to be important for future mathematics and STEM courses.

I did not address *adaptive learning* in this paper – a key buzzword in the field of online learning today. The U.S. Department of Education (2013) defined an adaptive learning system as one that

can dynamically change to better suit the learning in response to information collected during the course of learning rather than on the basis of preexisting information such as the learner's gender, age, or achievement test score. Adaptive learning systems use information gained as the learner works with them to vary such features as the way a concept is represented, its difficulty, the sequencing of tasks, and the nature of hints and feedback provided. (p. 27)

I do not have room in this paper to fully unpack this definition as it pertains to mathematics courses or to describe a full analysis of current online courses that claim to be adaptive. My experiences with most products on the market that promote adaptive features is that they adapt relative to students' procedural skills and proficiency. For example, if a student fails to solve an equation accurately within a given problem the student is given more practice with solving an equation. If the student only fails to solve equations when constant values within expressions are fractions, students are assigned additional practice performing arithmetic with fractions. Moreover, marketing materials for these products primarily tout their impacts on passing and persistence rates, course grades, and exam grades. None of these measurements capture detailed information on the actual meanings students develop as a result of instruction or the depth of their understanding relative to those meanings.

If the designers of these courses are not researching the meanings students develop as a result of instruction (which requires more than testing procedural skills and proficiency), then these cannot be factors in their adaptation process. I return once again to Thompson's (2013) salient comments.

Research that is ostensibly on knowing or understanding, whether the context is teaching or learning, too often examines performance instead of clarifying the meanings students or teachers have when they perform correctly or the meanings they are working from when they fail to perform correctly. Neither correct performance nor incorrect performance says anything about the nature of a person's system of meanings that expresses itself therein. (p. 78)

Studies of student learning in both in-class and online settings can benefit by placing more emphasis on the meanings students are constructing. The findings that emerge from this focus will enable course designers to adapt their lessons to produce greater learning and more robust meanings of a course's key ideas in students. We must take care that we do not fall into the trap of taking measurements such as course grades, retention rates, or exam scores as sufficient evidence that students are developing deep understandings of important ideas.

Conclusion

Thus far, research on student learning in online courses has been limited to what Cobb (2007) classifies as *experimental psychology* with the goal of “assess[ing] the relative effectiveness of alternative curricular and instructional approaches” (p. 15). Results of these studies are useful to academic leaders in charge of making institution-level decisions. However, they provide little feedback about the meanings students construct while engaging with online mathematics courses or features of those activities that best support construction of useful meanings. As post-secondary online course enrollment increases, it is vital that (a) these courses are designed to support students in constructing essential meanings, (b) researchers characterize the meanings students do construct while working through the online lessons, and (c) researchers gather data on

how lesson features support the construction of productive meanings. Such information supports the iterative cycle of generating high-quality courses that better support students in both overcoming unproductive meanings and constructing more productive meanings of important mathematical ideas.

References

- Allen, I. E. & Seaman, J. (2011). Going the Distance: Online Education in the United States, 2011 [online]. Sloan Consortium website. Available at: http://sloanconsortium.org/publications/survey/going_distance_2011. Retrieved August 8, 2018.
- Allen, I. E. & Seaman, J. (2015). Grade level: Tracking online education in the United States [online]. Babson Survey Research Group website. Available at: <https://www.onlinelearningsurvey.com/reports/gradelevel.pdf>. Retrieved August 8, 2018.
- Berenson, R., Boyles, G. & Weaver, A. (2008). Emotional intelligence as a predictor for success in online learning. *International Review of Research in Open & Distance Learning*, 9(2), pp. 1-16.
- Brinkman, W. P., Rae, A., & Dwivedi, Y. K. (2007). Web-based implementation of the Personalised System of Instruction: A case study of teaching mathematics in an online learning environment. *International Journal of Web-Based Learning and Teaching Technologies*, 2(1), 39-69.
- Byerley, C., & Thompson, P. W. (2017). Teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior*, 48, pp. 168-193.
- Cai, J., Morris, A., Hohensee, C., Hwang, S., Robison, V., & Hiebert, J. (2017). Making classroom implementation an integral part of research. *Journal for Research in Mathematics Education*, 48(4), 342-347.
- Cao, Y. & Sakchutchawan, S. (2011). Online vs. traditional MBA: an empirical study of students' characteristics, course satisfaction, and overall success. *Journal of Human Resources and Adult Learning*, 7(2), pp. 1-12.

- Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, 28(2), 113-145.
- Chen, P. S. D., Lambert, A. D., & Guidry, K. R. (2010). Engaging online learners: The impact of web-based learning technology on college student engagement. *Computers & Education*, 54(4), pp. 1222-1232.
- Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. *Handbook of Research Design in Mathematics and Science Education*, pp. 547-589.
- Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, 1, pp. 1-38). Greenwich, CT: Information Age Publishing.
- Coomey, M., & Stephenson, J. (2001). Online learning: it is all about dialogue, involvement, support and control-according to the research. *Teaching and Learning Online: Pedagogies for New Technologies*, pp. 37-52.
- Dabbagh, N. (2007). The online learner: characteristics and pedagogical implications. *Contemporary Issues in Technology and Teacher Education*, 7(3), pp. 217-226.
- The Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher*, pp. 5-8.
- Engelbrecht, J., & Harding, A. (2004). Combining online and paper assessment in a web-based course in undergraduate mathematics. *Journal of Computers in Mathematics and Science Teaching*, 23, pp. 217-232.
- Freudenthal, H. (1973). *Mathematics as an educational task*. Reidel Publishing, Dordrecht.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Reidel Publishing, Dordrecht.
- Freudenthal, H. (1991). *Revisiting mathematics education. China lectures*. Kluwer, Dordrecht.
- Glaserfeld, E. v. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.

- Glass, J., & Sue, V. (2008). Student preferences, satisfaction, and perceived learning in an online mathematics class. *MERLOT Journal of Online Learning and Teaching*, 4(3), pp. 325-338.
- Hannafin, M., Orrill, C., Kim, H., & Kim, M. (2005). Educational technology research in postsecondary settings: Promise, problems, and prospects. *Journal of Computing in Higher Education*, 16(2), pp. 3-22.
- Harel, G. (2008a). DNR perspective on mathematics curriculum and instruction, Part I: focus on proving. *ZDM*, 40(3), 487-500.
- Harel, G. (2008b). A DNR perspective on mathematics curriculum and instruction. Part II: with reference to teacher's knowledge base. *ZDM*, 40(5), 893-907.
- Hopper, K. B., & Harmon, S. W. (2000). A multiple-case study of exemplary internet courses. *Education at a Distance*, 14(9).
- Hu, S., & Kuh, G. D. (2001). Does the degree of campus "wiredness" matter? *Education Policy Analysis Archives*, 9.
- Hughes, J. E., McLeod, S., Brown, R., Maeda, Y., & Choi, J. (2007). Academic achievement and perceptions of the learning environment in virtual and traditional secondary mathematics classrooms. *The American Journal of Distance Education*, 21(4), pp. 199-214.
- Isen, A. M. (2000). Positive affect and decision making. In M. Lewis & J. Haviland-Jones (Eds), *Handbook of Emotions* (2nd ed.). (pp. 417-435). New York: Guilford.
- Kim, C., Park, S. W., & Cozart, J. (2014). Affective and motivational factors of learning in online mathematics courses. *British Journal of Educational Technology*, 45(1), pp. 171-185.
- Kramarski, B., & Gutman, M. (2006). How can self-regulated learning be supported in mathematical E-learning environments? *Journal of Computer Assisted Learning*, 22(1), pp. 24-33.
- Kuh, G. D., & Hu, S. (2001). The relationships between computer and information technology use, selected learning and personal development outcomes, and other college experiences. *Journal of College Student Development*, 42(3), 217.

- Laird, T. F. N., & Kuh, G. D. (2005). Student experiences with information technology and their relationship to other aspects of student engagement. *Research in Higher Education, 46*(2), pp. 211-233.
- Levine, L. J. & Pizarro, D. A. (2004). Emotion and memory research: a grumpy overview. *Social Cognition, 22*(5), pp. 530–554.
- Lim, D. H. & Kim, H. (2002). Motivation and learner characteristics affecting online learning and learning application. *Journal of Educational Technology Systems, 31*(4), pp. 423-439.
- Madison, B. L., Carlson, M., Oehrtman, M., & Tallman, M. (2015). Conceptual Precalculus: Strengthening Students' Quantitative and Covariational Reasoning. *The Mathematics Teacher, 109*(1), 54-59.
- McDuffie, A. R., & Slavit, D. (2003). Utilizing online discussion to support reflection and challenge beliefs in elementary mathematics methods classrooms. *Contemporary Issues in Technology and Teacher Education, 2*(4), pp. 446-466.
- Means, B., Toyama, Y., Murphy, R., Bakia, M., & Jones, K. (2009). Evaluation of evidence-based practices in online learning: A meta-analysis and review of online learning studies. *US Department of Education*.
- Moody, J. (2004). Distance education: why are the attrition rates so high? *The Quarterly Review of Distance Education, 5*(3), pp. 205-210.
- Muilenburg, L. Y. & Berge, Z. L. (2005). Student barriers to online learning: a factor analytic study. *Distance Education, 26*(1), pp. 29-48.
- Myers, M. (1999). Investigating information systems with ethnographic research. *Communications of the AIS, 2*(4), 1.
- Nguyen, D. M., Hsieh, Y. C. J., & Allen, G. D. (2006). The impact of web-based assessment and practice on students' mathematics learning attitudes. *The Journal of Computers in Mathematics and Science Teaching, 25*(3), 251.
- Nguyen, D. M., & Kulm, G. (2005). Using web-based practice to enhance mathematics learning and achievement. *Journal of Interactive Online Learning, 3*(3), pp. 1-16.
- O'Regan, K. (2003). Emotion and e-learning. *Journal of Asynchronous learning networks, 7*(3), pp. 78-92.

- Park, J. (2007). Factors related to learner dropout in online learning. In F. M. Nafukho, T. H. Chermack & C. M. Graham (Eds), *Proceedings of the 2007 Academy of Human Resource Development Annual Conference*, 25, pp. 125-128.
- Park, J. & Choi, H. J. (2009). Factors influencing adult learners' decision to drop out or persist in online learning. *Journal of Educational Technology & Society*, 12(4), pp. 207-217.
- Piaget, J. (1971). *Genetic epistemology*. New York, NY: W. W. Norton & Company Inc.
- Reeves, T. C. (2006). How do you know they are learning? The importance of alignment in higher education. *International Journal of Learning Technology*, 2(4), pp. 294-309.
- Rimmershaw, R. (1999). Using conferencing to support a culture of collaborative study. *Journal of Computer Assisted Learning*, 15(3), 189-200.
- Schoenfeld, A. H. (2000). Purposes and methods of research in mathematics education. *Notices of the AMS*, 47(6), pp. 641-649.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education*. Dordrecht, The Netherlands: Kluwer.
- Stiles, M. J. (2000). Effective learning and the virtual learning environment. EUNIS 2000: Towards virtual universities: Proceedings of the European university information system 2000 conference, pp. 171-180.
- Swan, K. (2003). Learning effectiveness: what the research tells us. In J. Bourne & J. C. Moore (Eds.), *Elements of quality online education, practice and direction*, pp. 13-45.
- Tallman, M. A. (2015). An examination of the effect of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Tallman, M. A., Carlson, M. P., Bressoud, D. M., & Pearson, M. (2016). A characterization of calculus I final exams in US colleges and universities. *International Journal of Research in Undergraduate Mathematics Education*, 2(1), pp. 105-133.

- Taylor, R. W. (2002). Pros and cons of online learning—a faculty perspective. *Journal of European Industrial Training*, 26(1), pp. 24-37.
- Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing and Modeling in Mathematics Education*. Dordrecht, The Netherlands: Kluwer.
- Thompson, P. W. (2008a). *Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education*. Plenary paper delivered at the 32nd Annual Meeting of the International Group for the Psychology of Mathematics Education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. SÈpulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 45-64). MorÈlia, Mexico: PME.
- Thompson, P. W. (2008b). *One approach to a coherent K-12 mathematics. Or, it takes 12 years to learn calculus*. Paper presented at the Pathways to Algebra Conference, June 22-25, Mayenne, France.
- Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education*, pp. 57-93. New York: Springer.
- Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byereley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *Journal of Mathematical Behavior*, 48, 95-111.
- Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing*. WISDOMe Monographs (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming.
- Tirrell, T. & Quick, D. (2012). Chickering's seven principles of good practice: Student attrition in community college online courses. *Community College Journal of Research and Practice*, 36(8), pp. 580-590.
- Treffers, A. (1978). Wiskobas doelgericht [Wiskobas goaldirected]. IOWO, Utrecht.
- Treffers, A. (1987). Three dimensions. A model of goal and theory description in mathematics instruction – the Wiskobas project. D. Reidel Publishing, Dordrecht.
- Twigg, C. A. (2003). Models for online learning. *Educause review*, 38, pp. 28-38.

- U.S. Department of Education, National Center for Education Statistics. (2018). *Digest of educational statistics, 2016, Table 311.15*. U.S. Department of Education, National Center for Education Statistics. Washington, D. C.
- U.S. Department of Education, Office of Educational Technology. (2012). *Expanding evidence approaches for learning in a digital world*. U.S. Department of Education, Office of Educational Technology. Washington, D.C.
- Wadsworth, L. M., Husman, J., Duggan, M. A., & Pennington, M. N. (2007). Online mathematics achievement: Effects of learning strategies and self-efficacy. *Journal of Developmental Education, 30*(3), 6.
- Waschull, S. B. (2005). Predicting success in online psychology courses: Self discipline and motivation. *Teaching of Psychology, 32*(3), pp. 190-192.
- Xu, D., & Jaggars, S. S. (2011). The effectiveness of distance education across Virginia's community colleges: Evidence from introductory college-level math and English courses. *Educational Evaluation and Policy Analysis, 33*(3), pp. 360-377.
- Xu, D. & Jaggars, S. (2013). The impact of online learning on students' course outcomes: Evidence from a large community and technical college system. *Economics of Education Review, 37*, pp. 46-57.
- Yukselturk, E. & Bulut, S. (2007). Predictors for student success in an online course. *Journal of Educational Technology & Society, 10*(2), pp. 71-83.

CHAPTER 9

PAPER 3 [RESULTS]

Exponential Growth and Related Ideas: Examining Students' Meanings and Learning in an Online Precalculus Course

Alan E. O'Bryan
Arizona State University

Online courses play an increasingly prominent role in post-secondary students' academic experiences, yet past research on student learning in the online environment has not focused on the meanings students develop while enrolled in online mathematics courses. In addition, an important research area involves identifying coherent systems of meanings that might be productive for supporting student success in STEM courses and careers and creating and studying learning trajectories and interventions designed to support students in constructing those meanings. This paper reports results from an online university Precalculus course designed around measurement imagery and quantitative reasoning as themes that unite topics across units. In particular, I focus on the unit for exponential functions and related ideas (such as percent change and growth factors). I provide a conceptual analysis guiding its design and discuss pre-test and pre-interview results, post-test and post-interview results, and observations from student behaviors while interacting with lessons. I demonstrate that the targeted meanings can be productive for students, show common unproductive meanings students possess as they enter Precalculus, highlight challenges and opportunities in teaching and learning in the online environment, and discuss needed adaptations to the intervention and future research opportunities informed by my results.

Keywords: exponential functions, online learning, conceptual analysis, undergraduate mathematics

The truth is that in many areas of the subject, mathematics has as much to do with computation as writing has to do with typing. Algorithms, rules, and drill are certainly not unimportant..., but our mathematical problems result more from insufficient exposure to mathematics as a way of thinking and a set of intricately connected higher-level skills than from an inability to compute. (Paulos, 2001, p. xiii)

Online learning is a relatively new phenomenon in education, but statistics indicate that it plays a large role in students' post-secondary experiences. In Fall 2016 almost 30% of students enrolled in post-secondary courses were enrolled in at least one distance-learning course (usually delivered online) and over 14% were enrolled exclusively in distance-learning courses (U.S. Department of Education, 2018). Most academic leaders view online course quality as equal to or better than the quality of face-to-face courses and see increasing online enrollment as a key part of their long-term growth plans (Allen & Seaman, 2011, 2015). In O'Bryan (2018b, in preparation) I summarize current research on how the online learning environment impacts student success and demonstrate a need for careful analysis of the mathematical meanings students develop while enrolled in an online course, particularly their meanings for ideas proven critical for success in future mathematics and other STEM courses. As one example, Oehrtman, Carlson, and Thompson (2008) describe challenges inherent in students' developing a productive meaning for function and the consequences when students fail to develop these meanings. They observe that "[s]tudents who think about functions only in terms of symbolic manipulations and procedural techniques are unable to comprehend a more general mapping of a set of input values to a set of output values"

(p. 151). On the other hand, imagining functions as a self-evaluating process is a necessary foundation for thinking about functions as modeling dynamic relationships between pairs of co-varying quantities. This more dynamic image of function relationships is at the heart of understanding the key ideas of Calculus (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson, 1994a; Thompson & Carlson, 2017). Studies that focus on retention rates, student GPA, and students' affective responses to the online environment provide little insight on how students conceptualize important ideas like the meaning of function at the end of their course. This data also provides few insights into how specific aspects of online lessons may support students in constructing productive meanings for key ideas.

In this report I describe results from the most recent iteration of an online Precalculus course (O'Bryan, Carlson, & Sander, 2018) designed to support students in developing quantitative and covariational reasoning and that leveraged images of measurement and relative size as common themes connecting topics across units. In particular, I focus my attention on ideas related to exponential functions. I chose this focus for two reasons. First, exponential functions play an important role in calculus, differential equations, complex analysis, modeling, and interpreting common measurements such as pH, decibel levels, and the Richter scale, yet developing productive understandings for exponential growth seems exceptionally challenging (Castillo-Garsow, 2010; Confrey, 1994; Davis, 2009; Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016; Ellis, Ozgur, Kulow, Williams, and Amidon, 2012, 2015; Ström, 2008; Weber 2002, 2002a). Second, ideas related to exponential growth are an excellent backdrop for discussing the learning goals we established for our online course and ways

of reasoning that I hypothesize provide a coherent foundation for understanding ideas throughout Precalculus.

I organized this paper into three main parts. The first part includes my theoretical perspective, literature review, and conceptual analysis of reasoning abilities and meanings for understanding exponential functions and related ideas. Included in this section is a discussion of two general mathematical ideas we targeted throughout the course (*emergent symbol meaning* and *measurement imagery*) that could help students develop productive meanings for the calculations, algorithms, and formulas central to topics across a Precalculus course. The second part analyzes student pre-test and pre-interview data to characterize the reasoning and meanings students possess entering a college Precalculus course. The third part discusses observations from analyzing student behaviors when interacting with lessons and reports results from analyzing post-test and post-interview data. This analysis informed my characterization of the meanings students possessed at the end of the course. Based on this analysis I discuss modifications to my learning trajectory and implications for these results on future iterations of the course. I also discuss potential areas for future research my results suggest.

Theoretical Perspective

Background theories provide guidance on what aspects of the learning process must be explained, inform the characteristics of those explanations, and perhaps give insights into features of experiences that might effectively guide students in developing powerful mathematical meanings. I leverage radical constructivism (Glaserfeld, 1995), building from Piaget's (1977) genetic epistemology, as the foundation for my image of a

student's mathematical meanings, for shaping my hypotheses relative to students' meanings and learning, and for informing the focus and features of lessons in the intervention. Thus, constructivism serves as both (a) a theoretical lens and (b) a guiding principle for task and lesson design along with domain-specific theories related to learning about functions and exponential growth.

Piaget's genetic epistemology was an attempt in part to explain the origins and substance of knowledge within biological entities who cannot directly access any kind of external reality (Piaget, 1977). Individuals organize their experiences within *schemes*, and they make sense of new stimuli by *assimilating* those experiences to existing schemes that include associated actions and expectations for the results of those actions in making progress towards a perceived goal (Glaserfeld, 1995; Piaget, 1971; Piaget & Inhelder, 1969). Piaget broadly interpreted actions to include "all movement, all thought, or all emotions that respond to a need" (Piaget, 1967, p. 6). Assimilation to a scheme provides meaning to the stimulus based on the scheme's contents (which includes connections to other schemes) (Johnckheere, Mandelbrot, & Piaget, 1958). Learning occurs when "the original scheme is *accommodated* [emphasis added] by differentiating between conditions and subsequent implications of assimilation" (Thompson, 1994c, p. 183).

Thompson et al. (2014) argued that it is productive to think about a student's understanding relative to some mathematical idea (such as proportional reasoning) as a "cloud", or a sort of Venn diagram, that emphasizes how "at every moment in the child's development of proportional reasoning, any two aspects of proportional reasoning entails some common ways of thinking while at the same time involving ways of thinking that

are unique to themselves” (p. 14). This perspective is useful for both thinking about designing learning opportunities and studying student learning. A given mathematical idea (such as proportional reasoning) might involve coordinating meanings for other ideas (such as measurement, variation, scaling, and fractions). Supporting students in constructing useful meanings for each of the related ideas might require slightly different approaches because there are unique elements to each, but similarities may also be leveraged in ways that reinforce connections and common ways of thinking to support growth in related understandings. In addition, modeling an individual student’s reasoning means theorizing about how he might understand related ideas that explains his observable behaviors. Evidence informing these theories include differences in how the student responds to tasks that a researcher sees as leveraging similar ideas but the student does not.

Knowing that students construct their own meanings and knowing some of the mechanisms by which this construction occurs via individual activity suggests the kinds of tasks and activities that may support students in constructing productive mathematical meanings. Thompson (1985) provides a list of key design principles for curricula that includes a focus on relationships and a mechanism for students to test their reasoning and generate feedback. Thompson argued that a key goal of mathematics instruction should include a focus on the conservation of relationships within a system, a notion that he and others expanded upon within theories of quantitative and covariational reasoning (i.e., Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson, 1990, 2011, 2012; Thompson & Carlson, 2017). As quantities in a system vary together, certain elements remain consistent, and reflection about these consistencies yields important insights about

function behavior and function families. Thompson argued that technology’s greatest contribution to lessons is its potential to create opportunities for students to make predictions based on their current thinking and for software to generate feedback about these predictions. These experiences may help encourage students to reflect on their thinking and can generate opportunities for students to modify their schemes in productive ways. I will return to this idea later in the report.

Exponential Functions Literature Review

In this section I briefly review the body of literature related to student learning about exponential functions. For a more complete review and analysis, see O’Bryan (2018a, 2018c, in preparation).

Confrey and Smith (1994, 1995) argued that it is productive for students to think about a function relationship as a process that synchronizes the values of two variables that change together (*covariational reasoning*). Then any given function is formed by “the juxtaposition of two sequences, each of which is generated independently through a pattern of data values” (1995, p. 67). Specific characteristics of a function (or function family) emerge through coordinating repeated actions. For example, an exponential growth function results from coordinating values in an arithmetic sequence with values in a geometric sequence as shown in Table 9.1. From this image of functions, they define *rate* as a comparison between what remains constant in each varying quantity (Confrey, 1994). Thus, exponential functions have a constant *multiplicative* rate of change (such as “times three per plus two” in Table 9.1). Weber (2002a, 2002b) provided empirical

Table 9.1

Coordinating an Arithmetic Sequence with a Geometric Sequence

	x	y	
	4	5	
+5	9	15	x3
+5	14	45	x3
+5	19	135	x3

support that this meaning for function and exponential growth can be productive for students, as did Ström (2008), who fleshed out key understandings related to interpolating values within the coordinated sequences.

Thompson (2008a) also suggested that covariational reasoning is a critical component of students understanding function relationships in general and exponential functions specifically, although his image of covariational reasoning is different from Confrey’s and Smith’s (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). When students conceptualize a situation they can identify *quantities* (attributes of an object the student imagines as measurable) (Thompson, 1988, 1990, 1993, 1994b, 2011, 2012). When two such quantities are conceptualized such that the student imagines that these quantities change in tandem, then he can link them together to form a *multiplicative object*, or a mental unification whereby the individual is aware of how both quantities change in tandem and anticipates tracking this relationship (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017). In Thompson’s conceptual analysis, all rates of change (ratios of additive changes in two quantities’ values) are constant, at least over small intervals, and function values can emerge as the result of smooth, continuous accumulation for piecewise linear functions. Functions can then be classified by similarities in their rate of change functions, such as defining

exponential functions as the class of functions where the rate of change over some interval is proportional to the function value at the beginning of the interval. See Figure 9.1.

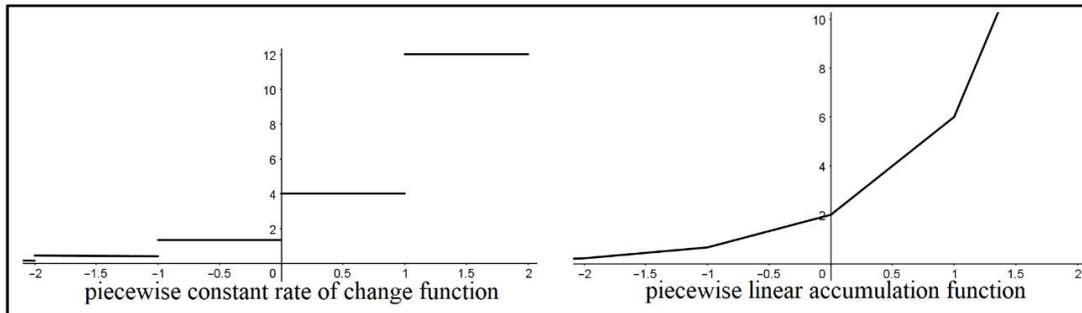


Figure 9.1. A piecewise constant rate of change function and the corresponding piecewise linear accumulation function where the rate of change over each interval is proportional to the function’s value at the beginning of the interval.

Decreasing the interval size produces an accumulation function that converges to a continuous exponential function. Castillo-Garsow (2010, 2012) developed a teaching experiment based on Thompson’s conceptual analysis of exponential growth. His results helped demonstrate the utility in imagining functions as emerging through coordinating two quantities that vary together in a smooth, continuous manner (as opposed to other imagery, such as variation occurring in “chunks” where the student does not attend to variation within specific intervals, only the values at the end of intervals over which variation occurred).

Ellis et al. (2012, 2015, 2016) built on Confrey and Smith’s definitions of covariation, rate, and exponential growth to create a teaching experiment with middle school students intended to support students in conceptualizing b^x as the height of a plant at some point in time as well as its multiplicative change in height over x units of time. Their lessons made extensive use of a GeoGebra applet showing a plant (the Jactus) that

grew according to a chosen 1-unit growth factor as the user slid the plant along the horizontal axis (representing elapsed time). They chose this visualization in light of Thompson's and Castillo-Garsow's arguments for the importance of students reasoning with smooth continuous covariation and the potential pitfalls of focusing on discrete variation. Their work has thus far produced insights into how students without sophisticated meanings for function relationships might advance their reasoning relative to exponential growth, including conceptualizing a meaning for the value of exponential functions at non-natural values in the domain. Kuper (2018) adapted their applet to support students in developing quantitative structures that give meaning to the inputs and outputs of logarithmic functions as emerging from linked covariation of quantities.

Both Confrey and Smith and Ellis and her colleagues grounded their work in images of repeated multiplication while Thompson's conceptual analysis relied on exponential growth emerging through coordination of a quantity that accumulates in a piecewise linear fashion as a second quantity increases with rates of change over specific intervals proportional to the function value at the beginning of the interval. My own conceptual analysis for exponential functions, and the resulting learning trajectory and lessons we designed, owes much to the work of Confrey and Smith, Thompson, and Ellis et al. However, the details differ in important ways due to differences in setting, students, and specific learning goals we selected for the course as described in the subsequent sections. Analysis of pre-test and pre-interview data also raises questions about whether or not university Precalculus students tend to conceptualize repeated multiplication as the foundation for exponential growth situations, especially when those situations are presented in terms of constant percent change.

Conceptual Analysis of Ideas Related to Understanding Exponential Growth

In the sections below I summarize the key meanings I intended to support within lessons on percentages, percent change, and exponential growth. My student learning goals differed somewhat from the goals in the research previously described primarily because of my attempt to design for coherence across an online university Precalculus course. As I discuss at the end of this section, our goals of promoting *emergent symbol meaning* and a meaning for relative size comparisons made up a *guiding framework* (O’Bryan, 2018c, in preparation) informing the conceptual analysis and initial drafts for the unit learning trajectories. It is worth emphasizing that I therefore assume that students’ meanings for these ideas developed (to the extent they did change) within learning opportunities throughout the course and not necessarily solely as a result of activity within any single unit.

Conceptual analysis is a description of “what students must understand when they know a particular idea in various ways” (Thompson, 2008a, p. 42), two uses of which include (a) outlining meanings that might benefit students’ mathematical development and (b) analyzing the coherence across potentially related meanings. My conceptual analysis of exponential growth and related ideas constitutes a body of ideas that I conjecture will benefit students’ mathematical development. My main goal within for conceptual analysis is to follow Thompson’s (1985) advice for designing curricula by first clearly articulating the meanings I intend students to construct and using this to inform an activity sequence with the potential to support students in constructing these targeted meanings.

Potential Issues with Standard Definitions for Exponential Functions

In Calculus and beyond, an exponential relationship is defined as a function with an instantaneous rate of change proportional to the function value at all points in the domain, but it is uncommon to expect this definition within Algebra and Precalculus courses. Figure 9.2 synthesizes definitions in ten algebra and precalculus textbooks published between 2000 and 2012.

An exponential function is denoted by $f(x) = a^x$ where the base a is a positive real number and $a \neq 1$.

An exponential function f is a function of the form $f(x) = ab^x$ where a is a positive real number and the base b is a positive real number with $b \neq 1$.

Figure 9.2. Two examples of common textbook definitions for exponential function.

The definition boxes containing these statements all appeared on the first page of the first section for the unit with little development. Several of the textbooks additionally emphasized that exponential functions are transcendental functions (as if that should mean something important to students). The examples within these sections did not discuss the relationship between an exponential function's value and its rate of change. Instead, the common approach was to highlight repeated multiplication of a growth factor as the foundation for exponential functions and the algebraic formula as a generalization of the repeated multiplication process.

The definitions above are problematic for several reasons. First, the definitions are entirely symbolic, which provides no guidance for recognizing an exponential function represented in other forms. Second, despite the presence of variables in the definitions, there is no explicit mention of varying quantities, and thus no discussion of what feature in the relationship remains invariant as related quantities co-vary. Third,

Thompson (2013) argued that a weak system of meanings (held by an instructor or present in lesson design) opens up enormous space within which students can construct incorrect or unhelpful meanings. He also emphasized that a lack of clarity in the meanings students should construct during a lesson decreases the chance that instructors (or designers) will recognize the meanings students do construct (including unintended meanings). I am unclear what meanings students should take away from one of the definitions in Figure 2 or how either definition helps students recognize when exponential functions are appropriate models for a given graph, data set, or verbal description or helps them understand what remains invariant in a relationship as two quantities change together. From a student's point of view, the emphasis seems to be about memorizing what an exponential function "looks like" in algebraic form.

The following two interview excerpts from this study show student responses to the question, "What makes a function exponential?" Both students said they had learned about exponential growth in previous classes and Lisa had taken and passed Calculus AB as a senior in high school.

- 1 Lisa: I believe it's just a function that, like, the answers are gonna increase. Like
2 if you graph it, is the best way I visualize it, is they're gonna increase
3 exponentially, so they're gonna start out gradually and really quickly [*she*
4 *draws Figure 9.3*]...

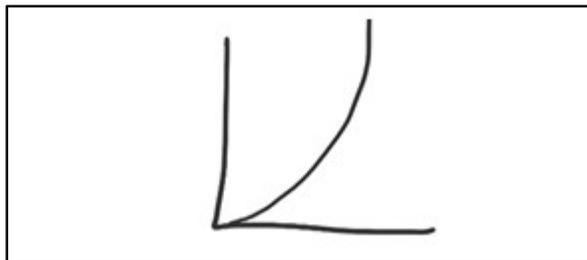


Figure 9.3. Lisa's example of an exponential function.

5 AO: Okay.
6 Lisa: ...move up. So the answers are gonna be like closer and closer together
7 for, like, amounts. I guess would be the best way to describe that.
8 AO: Okay. So any kind of graph that, any kind of function that does that
9 behavior [*points to the graph Lisa drew*]...
10 Lisa: Yeah.
11 AO: ...is going to be exponential?
12 Lisa: Yeah.
13
14
15 Marcus: Yeah. Um. I would say uh. I wanna, I wanna say it, it definitely has
16 something to do with um greater numbers. I'm not too sure. Something
17 along those lines.
18 AO: Okay.
19 Marcus: Something like that. I know because if you put it in exponential form.
20 Well, I think I'm on to something [*laughs*].

If instruction on and definitions for exponential functions are based on a relatively weak system of meanings, then as Thompson (2013) argued it is more likely that students construct unhelpful or weak meanings for important mathematical ideas and for instructors to fail to recognize the meanings that students did construct. The student responses recorded above (which were representative of responses by the five interview subjects in this study) suggest that students often construct meanings for exponential growth based on some combination of an image of a graph (Lisa), a function with increasing values (Lisa and Marcus), or a vague recollection of an algebraic representation (Marcus). Potential implications for students with one of these meanings include categorizing any increasing function with positive concavity or any function with an exponent in its algebraic representation as exponential. It is doubtful that using these meanings to classify exponential functions provides much mathematical substance with which a student can then draw additional conclusions.

Exponential Growth and Related Ideas from a Quantitative Perspective

My primary learning goal for students is that they conceptualize an exponential function as a function with a constant growth factor or constant percent change over all equal-sized intervals of the domain. I intend that *growth factor* be understood as a measurement relating any two instances of the same quantity separated by some fixed additive change in a second quantity and not be restricted to a factor in repeated multiplication that produces specific output values. As such, understanding *growth factor* relies on having a meaning for measurement in general as a multiplicative comparison and to understand that (a) a quantity's magnitude is independent of the magnitude of units used to measure it, (b) measurements indicate a reciprocal relationship between the quantity's magnitude and the unit's magnitude, and (c) changing the unit of measure changes the measurement value in an inversely proportional manner (Thompson et al., 2014). Then conceptualizing a meaning for b^x with respect to constant percent change involves imagining repeatedly growing by a constant percent change on an interval and seeing that this constant percent change can also be expressed relative to powers of the one-unit growth factor. In the sections that follow I will fully unpack my meanings for these ideas and how they form a basis for defining and recognizing exponential relationships regardless of the form in which they are expressed.

Flexibility in choosing a unit and units of convenience. Thompson et al. (2014) discuss images of measurement students might possess and the implications of each scheme. *Extensive quantification* is the most elementary example of “the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship...with its unit” (Thompson,

2011, p. 37) and describes quantification schemes for directly measurable quantities (such as length) using any unit that has “the same nature as the quantity being measured” (Wildi, 1991, p. 58). The measurement scheme’s sophistication can range from a straightforward process of reproducing the quantity’s magnitude by iterating the unit (size as a *Measure magnitude*) to an appreciation of the reciprocal relationship between the quantity’s measurement in some unit and the size of the unit (size as a *Steffe magnitude*) to an understanding that the quantity’s magnitude is independent of the choice of unit along with an anticipation for how changes to the measurement unit affect the measurement value (size as a *Wildi magnitude*) (Thompson et al., 2014). I will discuss the implications for these schemes as I unpack my intended meaning for the growth factor of an exponential function.

Arbitrary conventions dictate the size of many common units we use such as feet or gallons. However, conceptualizing non-standardized units based on convenient magnitudes within a context can be very productive. As one example, Tallman (2015) carefully lays out the reasoning involved in using a circle’s radius as the measurement unit for arc lengths on that circle. Radius length is a convenient “unit ruler” for measuring arc length primarily because its length is an inherent property of the circle and its size varies proportionally with the circle’s circumference. There are other instances where we might use rulers of convenience (rather than standardized measurement units). Suppose I measured a tree’s height at two different moments in time and that each time I cut a piece of string with a length equal to the tree’s height (call these lengths l_1 and l_2). When I lay out the pieces of string next to each other I can think of each string’s length as a potential

measurement unit without needing to measure these lengths in any standard unit, and the resulting measurement can provide useful information about the situation. See Figure 9.4.

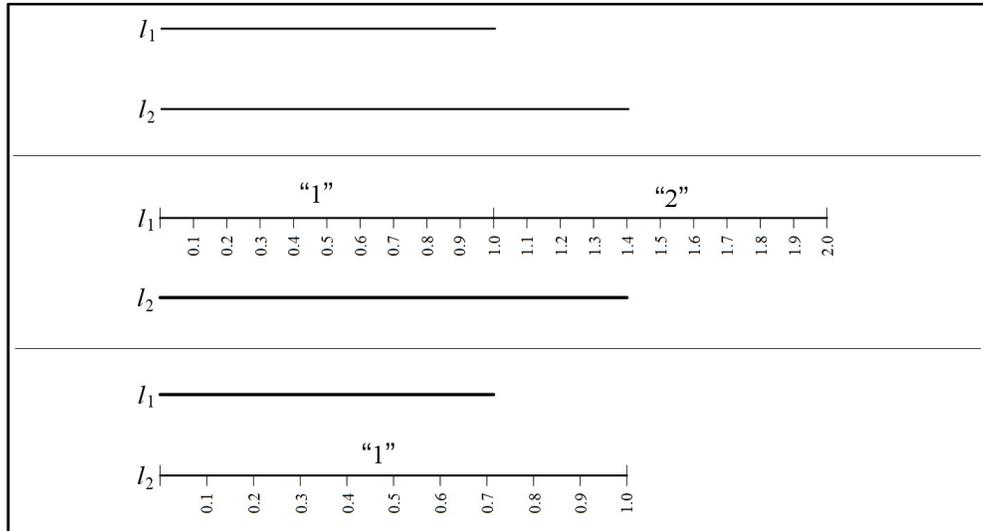


Figure 9.4: Measuring relative sizes. (top) The two string lengths representing the tree’s height at two moments. (middle) Using the first height as the measurement unit, the second height has a measure of 1.4 (the second height is 1.4 times as large as the first height). (bottom) Using the second height as the measurement unit, the first height has a measure of $10/14$, or about 0.7 (the first height is $10/14$ times as large as the second height).⁴³

Thompson et al. (2014) discuss different ways that students might conceptualize the goal and meaning of measurements. If I think of the length measurement for one string produced by counting iterations of the other string, I am thinking about size as a Measure magnitude (an additive meaning for measurement). Potential implications of this measurement scheme include confusion when comparing the measurement of a quantity’s size in different units, such as thinking a distance measured as 3,218 meters might be longer than a distance of 12 miles because the measurement value is larger. Thinking of size as a Steffe magnitude assumes that the measurement value communicates a relative

⁴³ This is what Lamon (1994) describes as *norming* within a quantification scheme and is the key way of thinking that allows someone to see a number such as $2/3$ as representing different amounts of a quantity depending on conceptualizing the meaning of “1” in the context.

size comparison between the magnitude being measured and the unit's magnitude. If a piece of string is 17 feet long, then its length is 17 times as long as a length we call "1 foot", which in turn is $1/17$ times as long as the string length (so "1 foot" measures $1/17$ when the string length is used as the measurement unit). This measurement scheme is necessary to see the relationship between measurements for the string lengths represented in Figure 4. If l_2 is 1.4 times as long as l_1 (its measurement is 1.4 in units of l_1 's magnitude) then l_1 is $1/1.4$, or $10/14$, times as long as l_2 (its measurement is $10/14$ in units of l_2 's magnitude).

I will continue to build on this example, but for now I want to highlight some important points. This notion of a "ruler of convenience" may be a helpful conceptualization within many topics in Precalculus (including angle measure as I noted earlier). I also hypothesize that children's early introduction to the idea of measuring magnitudes with convenient rulers could be key to supporting images of the relationship between the size of the unit and the measurement value, the meaning of equivalent measurements in different units, and the important role of standard units in both science and everyday life. This would be an interesting research question to explore. I also want to emphasize that I chose to represent the tree's height at the two moments in time using the lengths of two pieces of string because doing so simplifies the process of holding in mind the magnitude of a quantity at two distinct moments in time, which is critical for thinking of an exponential function's growth factor as a measurement. This is something lacking in Ellis et al.'s (2012, 2015, 2016) activities involving the Jactus plant. Even though students in their study could vary the height of the plant by varying the time elapsed since first measuring the plant, the applet did not allow them to simultaneously

represent two heights of the plant at two different moments in time. When only one instance of the quantity's magnitude is visible at a given time, it may be less natural to use a different instance of the same quantity as the measurement unit and more natural to use a standard unit (such as feet) for measuring the magnitude. Numerical representations for the quantity's measurement at multiple instances (such as a representation of multiple values in a table) may encourage students to generalize the relationship between instances as the product of performing a calculation rather than conceptualizing a relative size. It might also be easier for someone to overlook coordinating these measurements with changes in a second quantity. For example, the tree measurements took place one year apart. Therefore, saying that the second height is 1.4 times as large as the first height without attending to the interval of time over which the tree was growing between the measurements reduces the measurement's usefulness.

“Elastic” units and the meaning of a growth factor. A person with the understandings already described is positioned to consider the question, “What are the characteristics of a unit such that, as the quantity's magnitude changes, the quantity's measure in that unit never changes?” For the measurement of a varying quantity to remain constant, the unit must change so that its magnitude and the quantity's magnitude always have the same relative size. In Figure 9.5 we see three instances of a varying quantity's magnitude and a measurement unit changing in tandem so that the quantity's measure is always 2.5.

Returning to the tree height example, imagine that the tree grows such that its height at any given time is always the same measurement when using the height one year prior as the unit. What can we conclude about how the tree grows? Assuming the relative

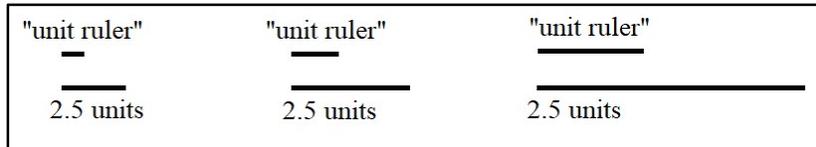


Figure 9.5. An “elastic” unit ruler that changes with the size of a varying quantity so that the measurement remains constant. Contrast this with a rigid ruler where the unit size remains constant and the measurement changes as the quantity’s size varies.

size measurement in my running example, at any moment the tree’s height is always 1.4 times as large as the tree’s height exactly one year prior. Equivalently, at any moment in time the tree’s height one year in the future is 1.4 times as large as its current height (and we can also say that the tree’s height at any moment is 10/14 times as large as its height one year in the future). If we allow time to pass such that the tree’s height changes, then the tree’s height one year earlier/later must also change so that the two magnitudes under comparison have a constant relative size. Note that there are two measurements held constant here while the actual tree height changes: (1) the relative size of the tree heights at two different moments in time and (2) the change in time under consideration.

Combining all of the ideas discussed so far allows for the following potentially powerful meaning for *growth factor of a quantity (over some interval of change in a second quantity)*. Coordinating two covarying quantities (call them A and B , with magnitudes represented as $\|A\|$ and $\|B\|$ respectively⁴⁴) over some interval of conceptual time sees their magnitudes vary from $(\|A_1\|, \|B_1\|)$ to $(\|A_2\|, \|B_2\|)$. The *growth factor of B* over the interval of conceptual time where A varies from $\|A_1\|$ to $\|A_2\|$ is the

⁴⁴ Here I use the notation and meaning from Thompson (2011), Thompson et al. (2014), and Wildi (1991) where $\|A\|$, the magnitude of quantity A , “is the size of an object having the attribute that is being measured and that is taken to have one unit of that attribute” (Thompson et al., 2014, p. 5).

multiplicative comparison of $\|B_2\|$ to $\|B_1\|$, or $\|B_2\|$ measured in units of $\|B_1\|$, or the value of k such that $\|B_2\| = k \cdot \|B_1\|$. The value of k is a constant across all choices of unit for measuring B . Therefore, if a represents values for the varying quantity A (measured in some appropriate unit) and b represents values for the varying quantity B (measured in some appropriate unit), then the growth factor of B over the interval from $a = a_1$ to $a = a_2$ (an interval of size $a_2 - a_1$) is b_2 / b_1 . Visualizing this in the tree height example involves imagining holding the time interval size between measurements constant and the strings changing in length such that l_2 is always 1.4 times as long as l_1 . Graphing the relationship between the tree's height and time elapsed, any two heights measured one year apart will have a constant relative size. We can imagine this dynamically with a sliding 1-year interval and comparing the magnitude of the heights at the beginning and end of the interval (Patrick W. Thompson, personal communication, April 9, 2015).

Figure 9.6 shows three screen captures of a computer applet demonstrating this idea.

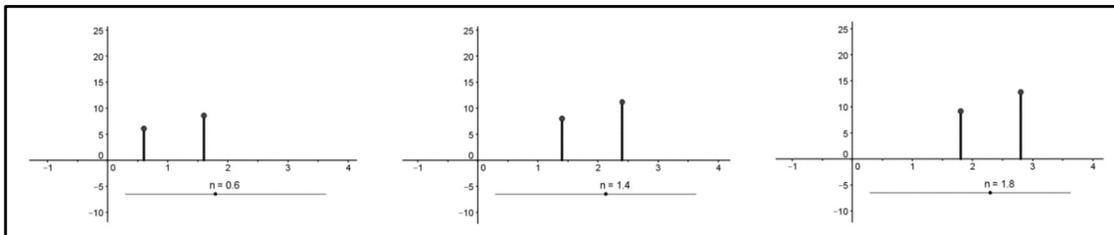


Figure 9.6. A dynamic comparison of the tree's height at two moments one year apart. The tree's height always measures 1.4 when using its height one year earlier as the measurement unit. This visualization is based on an applet designed by Patrick W. Thompson (personal communication, April 9, 2015) and is a modification of Ellis et al.'s (2012, 2015, 2016) Jactus animation. The reader is encouraged to imagine the beginning of the interval sliding smoothly to the right and visualize what remains invariant and what changes based on this action. Thompson also emphasizes attending to how the segment on the left changes and "becomes" the segment on the right as the interval slides.

If students conceptualize the ideas represented in Figure 9.6 it may support them in thinking of the growth factor over some interval as a relative size comparison of two instances of one quantity while still holding in mind continuous covariation as they imagine running through a continuum of values in the domain. The growth factor is the measure of the function value's magnitude at the end of the interval in units of the function value's magnitude at the beginning of the interval. Note that we can discuss the growth factor without attending to specific function values. Doing so could support students' development of a quantitative meaning for growth factor as a measurement and deemphasize calculational meanings while still providing a context for deriving reasonable calculations to evaluate function values or growth factors if the function values are known (or can be calculated). Working with the visualization involving an elastic ruler (or the same concept represented graphically as in Figure 6) also provides context for discussing how the absolute difference between the two instances of the tree's height separated by one year (1) becomes larger as the tree grows in height, (2) is always the same relative size compared to the tree's height at the beginning of the interval (0.4 in this case), and (3) why the measurement of the change in tree height over one year is always one less than the measurement of the tree's height at the end of the interval when each are measured in units of the tree's height at the beginning of the interval.

Percent comparisons and percent change. The choice to model a real-world relationship with an exponential function is often based on recognizing constant percent changes in one quantity for equal additive changes in a second quantity. For example, coordinating the growth of a savings account based on a constant APY or exploring the long-term implications of a population growing by 3% per year are common scenarios

modeled with exponential functions. Most students, however, do not conceptualize percentage comparisons and percent change as measurements as I demonstrate later in this report.⁴⁵ Thinking about percentages as measurements involves careful attention to what is being measured and the measurement unit. Consider the statement, “The median sales price for a home in Springfield is up 4% over the median sales price from a year ago and up 18% from the median sales price at this time ten years ago. This is the highest median sales price in the state, 5% higher than any other city.” The meaning of “one percent” is (1) different in every instance both in absolute size measured in some currency and (2) references a different quantity in each case. Students who lack clarity in the reference quantity (those who cannot regularly answer the question “Percent of what?” when reading such statements) will likely struggle to interpret and differentiate between various percentage and percent change measurements.

One percent of a quantity’s magnitude (at some moment) is a magnitude having the same attribute as the quantity such that $\|A\| = 100 \cdot \|\text{one percent of } \|A\|\|$. In other words, the original quantity’s magnitude measures 100 in units of one percent (or 1/100) of the quantity’s magnitude. Note that I wrote “one percent of a quantity’s magnitude (at some moment)” and not just “one percent” because conceptualizing the meaning of “one percent” involves a re-norming process and has no fixed size across all contexts or even in the same context if quantities are varying. Using percent measurements helps communicate relative sizes to provide context for interpreting important features of a

⁴⁵ Lack of awareness of percentages as measurements is not restricted to undergraduate students. In a recent seminar I asked graduate student TAs in mathematics and mathematics education PhD programs what the word “percent” means. Only one of eight knew that it meant something similar to “by the hundred”. The other seven expressed that they did not realize the word had a meaning indicating it was a kind of measurement unit.

situation. As one example, it is common for home sellers to do a price reduction if their house has not sold after some amount of time. A \$10,000 reduction in price means something quite different for a home currently listed for \$2.7 million compared to a home currently listed for \$65,000. Describing the price reductions as changes of -0.37% and -15.38% respectively communicates useful information in this context.

One percent of a quantity's magnitude at some moment is a suitable unit for measuring other quantities of the same quality or even additional instances of the same quantity as it varies. The resulting measurement is a *percentage* value. For two quantities A and B representing the same quality of objects, measuring $\|B\|$ as a percentage of $\|A\|$ is to think of the value k such that $\|B\| = k \cdot \|\text{one percent of } \|A\|\|$. Similarly, for two instances of quantity A , measuring $\|A_2\|$ as a percentage of $\|A_1\|$ is to find the value k such that $\|A_2\| = k \cdot \|\text{one percent of } \|A_1\|\|$.

Returning to the tree height example, suppose that measurements were taken on January 1 of last year and January 1 of this year. The growth factor over this one-year period is the magnitude of the height on January 1 of this year using the height on January 1 of last year as the measurement unit. See Figure 9.7.

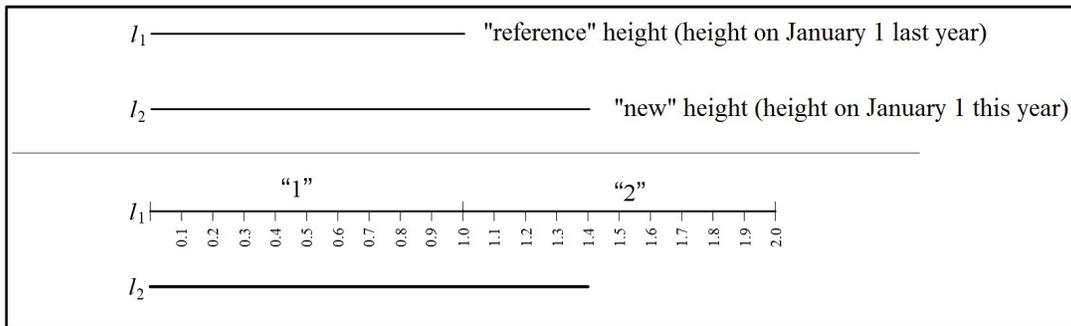


Figure 9.7. Measuring the height's magnitude on January 1 of this year using the height's magnitude on January 1 of last year as the measurement unit.

The new height is 1.4 times as large as the reference height. Measuring the new height as a percentage of the reference height is to measure it in a unit that is $\frac{1}{100}$ times as large, and thus the measurement value is 100 times as large. The new height is 140% (140 times as large as $1/100$ of the length of l_1) of the reference height. See Figure 9.8.

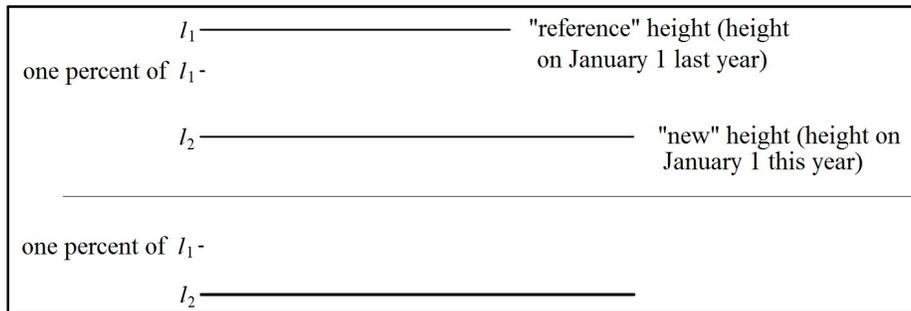


Figure 9.8. Measuring the magnitude of the height on January 1 of this year using one percent of the height on January 1 of last year as the measurement unit. The measurement unit is $\frac{1}{100}$ times as large as the unit in Figure 3, so the new measurement is 100 times as large as the previous measurement.

A robust understanding of the ideas just described entail an image of size as a *Wildi magnitude* (Thompson et al., 2014). The magnitude of what we are measuring does not change when the measurement unit changes in size but the measurement value updates in a way that is inversely proportional to the relative size of the new unit compared to the old unit. If a quantity's magnitude is measured to be 10 in unit A, and if unit B is $\frac{3}{5}$ times as large as unit A, then the quantity's magnitude will be $\frac{5}{3}(10)$ measured in unit B (and a person who understands this relationship can anticipate this result without needing to perform a second measurement in the new unit). Changing the measurement unit from the tree's height on January 1 of last year to one percent of the tree's height on January 1 of last year involves measuring the same quantity's magnitude (the tree's height on January 1 of this year) using a unit that $1/100$ times as large as the original unit. Thus, we can anticipate that the measurement value is 100 times as large.

This is the conceptual underpinning for the algorithm of moving the decimal point two places when converting back and forth between relative size measurements and percentage measurements.

Determining a *percent change* in one quantity's value across two instances requires conceptualizing a more complicated quantitative structure. First, the individual must conceptualize a varying quantity within a situation. Second, she must conceptualize two instances of the quantity with some implied or explicit order, the magnitude of the quantity for each instance, and an additive comparison of these magnitudes. This additive comparison, however, must be like a vector quantity with both a magnitude and a direction to capture whether the change is an increase or a decrease. See Figure 9.9.

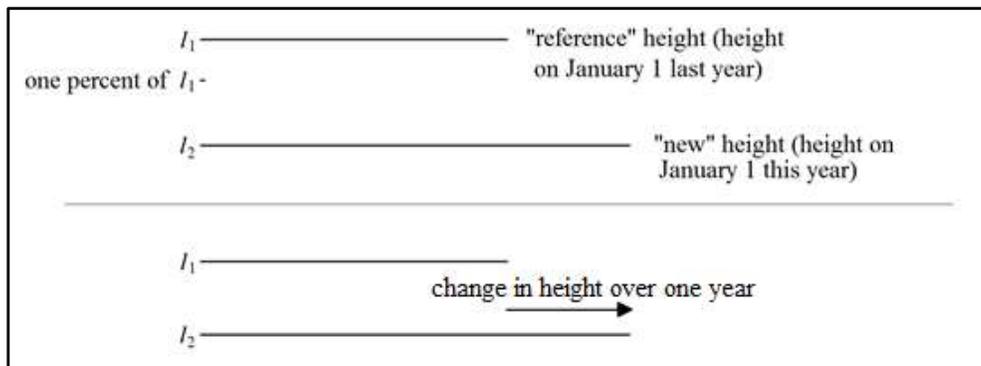


Figure 9.9. Visualizing the magnitude and direction of a change in height.

Third, she imagines measuring the additive comparison as a percentage of one of the two magnitudes (typically the magnitude representing the baseline from which the change occurred). So the percent change from $\|A_1\|$ to $\|A_2\|$ can be thought of as the value of k such that $\| \|A_2\| - \|A_1\| \| = k \cdot \| \text{one percent of } \|A_1\| \|$.

In most instances the percent change in one quantity is described with reference to an interval of change for another quantity to provide clarity within the given context. For

example, reporting a population change of 4.8% is not very informative without describing the time period over which the change occurred. The change in the tree's height from the reference height to the new height (occurring over a one-year period) is an additive comparison that captures both a magnitude and a direction of change. The percent change in the tree's height is the measurement of this change using one percent of the reference height as the measurement unit. See Figure 9.10.

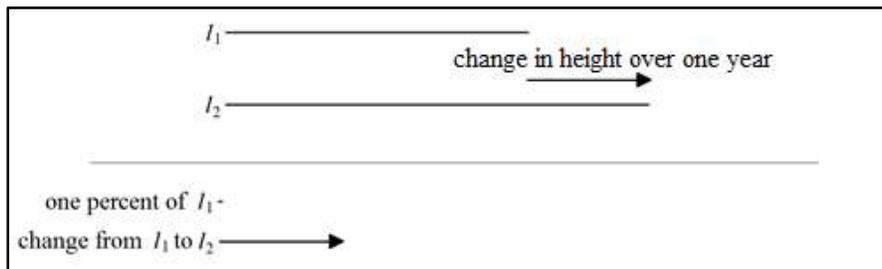


Figure 9.10. Measuring the change in height using one percent of the height on January 1 of last year as the measurement unit.

We can visualize these relationships as they appear in the graph of the tree's height compared to elapsed time (in years) since some chosen reference time. The magnitude of the tree's height at t_1 can be used to measure the tree's height at t_2 and the change in height over the time period from t_1 to t_2 . See Figure 9.11. A percentage comparison or percent change measures the tree's height at t_2 and the change in height

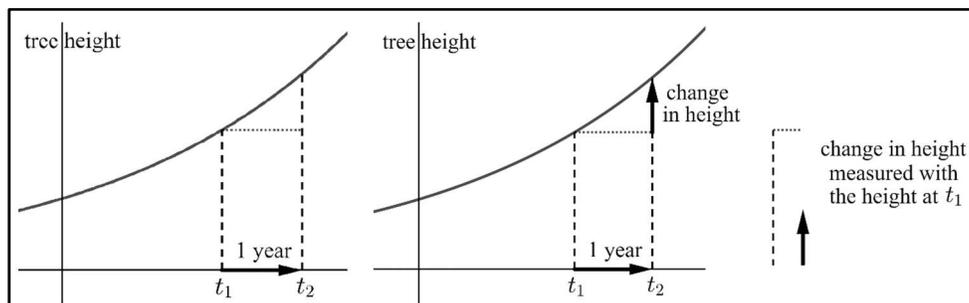


Figure 9.11. The measurement process can be visualized in a graphical representation of the relationship between the tree's height and elapsed time (in years) since some chosen reference time.

respectively using $1/100$ of the magnitude of the tree's height at t_1 as the measurement unit.

Exponential functions. Pulling together the meanings just described, an exponential function f can be defined as a relationship where the relative size of $f(x_2)$ and $f(x_1)$ is constant whenever $\Delta x = x_2 - x_1$ is constant throughout the function's domain. This relative size measurement is the value of the Δx -unit growth factor. In addition, exponential functions are relationships with a constant percent change in the dependent quantity over all equally-sized intervals of the domain. These relatively compact statements contain within them the intricate imagery described in the previous sections.

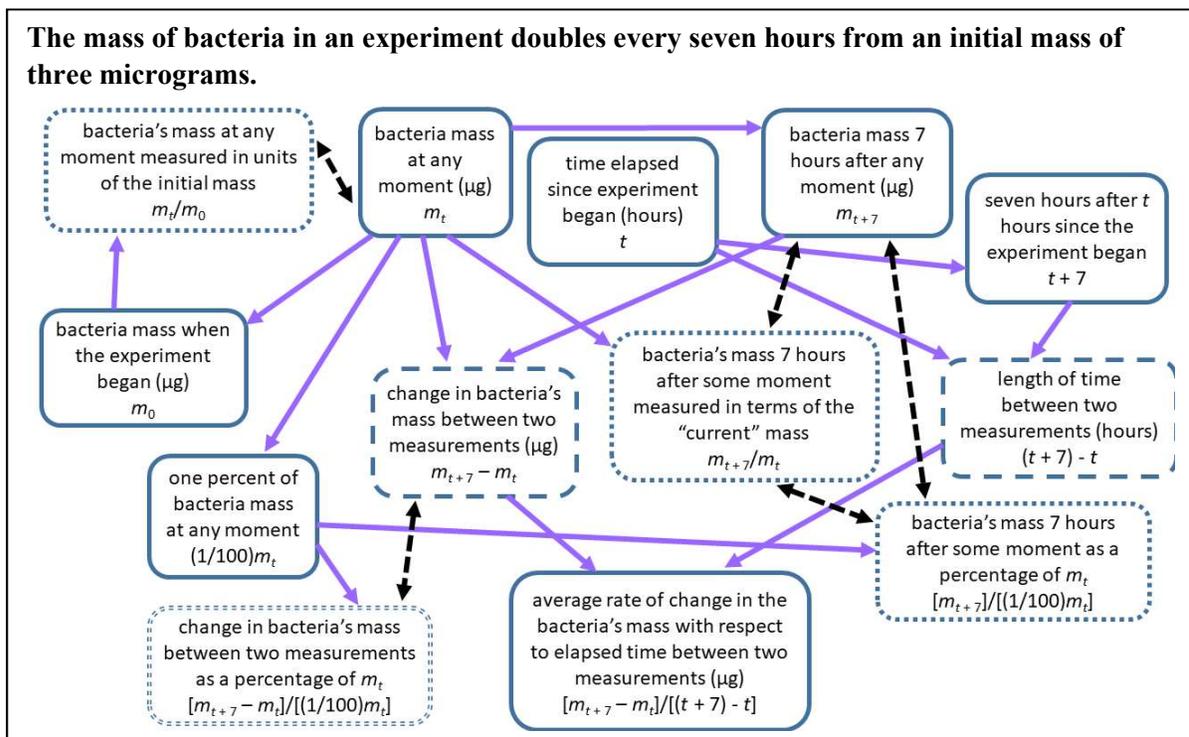


Figure 9.12. My quantitative structure for understanding an exponential model. Note that single-headed arrows imply a hierarchy of meanings such that conceptualizing the quantity at the tail of the arrow likely is a prerequisite for conceptualizing the quantity at the head of the arrow. Dashed boxes indicate a change in a quantity while dotted boxes indicate relative size comparisons between two instances of a quantity. Dashed, double-headed arrows indicate the same quantities measured in different units.

To get a sense for the complexity of these ideas I have included a diagram of the quantitative structure I imagine when generating a mental model for a task involving modeling with an exponential function. See Figure 9.12.

A person who has conceptualized the relationships described may be positioned to construct the meaning for the algebraic representation of an exponential function f recommended by Ellis et al. (2016). If b is the 1-unit growth factor, then $b^{\Delta x}$ is the Δx -unit growth factor for any value of Δx . If we imagine $x = 0$ as our reference point, then for any value of x (say $x = n$), the change in x away from $x = 0$ is $\Delta x = n - 0 = n$ and the growth factor from $x = 0$ to $x = n$ is b^n and $f(n) = ab^n$ where $f(0) = a$.

The framework in Table 9.2 describes the key ways of understanding exponential growth and related ideas as described in my conceptual analysis. Note that this framework was partially influenced by Ström's (2008) final framework for understanding exponential growth and Ellis et al's (2016) description of their learning goals for students. It is also worth mentioning that the various elements of the framework are not to be understood as separate ideas that exist independently. Students' meanings for the ideas within the framework are interconnected in ways that would be difficult, if not impossible, to isolate. The main benefit of a framework such as this is to clarify my learning goals for lesson and unit design, identify the meanings I intend to assess or promote with certain tasks, and focus my attention during data analysis to inform interpretations of student behaviors.

Table 9.2

A Framework for Understanding Exponential Growth and Related Ideas

<p>MCM: Multiplicative comparisons as measurement systems, which refers to comparing the value of two quantities multiplicatively (using the value of one quantity as a “measurement stick” with which to report the measurement of the other quantity). The measurement reported is a quotient. This can include comparing two values of the same quantity at two different moments when the quantity was measured.</p> <ul style="list-style-type: none"> • MCM-1: Assuming comparisons between the same types of quantities, a multiplicative comparison of two quantities reports the value of one quantity using the magnitude of the other quantity as the measurement unit. • MCM-2: If the unit of measure is scaled, the measure of a quantity changes in a specific way. If the new unit of measure is k times as large as the previous unit of measure, then the quantity’s measurement in the new unit is $1/k$ times as large as its measurement reported in the previous unit. • MCM-3: Two instances of the same quantity can be compared multiplicatively using one instance of the quantity as the measurement unit for the second instance. • MCM-4: If the two quantities are proportional, or if the two instances of the same quantity maintain a relative size as they quantity changes, then the measurement of one in units of the other is constant even as the quantities change. <p>MCP: Multiplicative comparisons reported in terms of percentages. This is a slight modification to the MCM in that we are not using $1/100$ of the value of one quantity as the “measurement stick” instead of the full value of that quantity.</p> <ul style="list-style-type: none"> • MCP-1: A <i>percentage</i> is measurement that uses $1/100$ of the value of some quantity as the measurement unit. • MCP-2: Building on MCM-3, if a quantity’s value is used as a measurement unit, then using this quantity as the reference for a <i>percentage</i> measurement will yield a value 100 times as large. • MCP-3: A percentage comparison between two instances using one instance as the measurement unit will always be 100% larger than the measurement of the <i>change</i> in the quantity using the same unit. <p>MCPC: Leveraging MCP thinking to reason about the change in a quantity’s value (so the change in a quantity’s value is measured using $1/100$ of the quantity’s starting value as the “measuring stick”).</p> <ul style="list-style-type: none"> • MCPC-1: Measuring a percent change involves 1) additively comparing two measurements of the same quantity in some order and 2) multiplicatively comparing the additive change to $1/100$ of the value of the reference quantity’s value. • MCPC-2: The percent change is 100 times as large as the multiplicative comparison between the change in value and the value of the same reference quantity. • MCPC-3: When applying repeated constant percent changes, the reference value for determining the change in value updates at the end of each interval. • MCPC-4: Equal percent changes over different intervals produces non-constant absolute changes over those intervals. <p>MCEF: Thinking about an “exponential function” as an invariant relationship between two continuously co-varying quantities x and $y = f(x)$.</p> <ul style="list-style-type: none"> • MCEF-1: Within the domain of a function relationship, we can hold in mind two instances of the same quantity over a constant interval of the second quantity. • MCEF-2: The one-unit growth factor is a measurement determined by using one instance of a quantity as the unit ruler for measuring a second instance of the same quantity occurring when a second quantity varies by one unit. This measurement is constant even as the quantities vary. • MCEF-3: For every choice of c there exists a constant d such that $\frac{f(x+c)}{f(x)} = d$ as x varies continuously throughout the domain. [When $c = 1$, then the constant is the one-unit growth factor typically given as “b” in the formula $y = ab^x$]. • MCEF-4: If b is the one-unit growth factor, then b^c is the c-unit growth factor (i.e., the ratio $\frac{f(x+c)}{f(x)}$ for all x). • MCEF-5: If b is the one-unit growth factor, then b^x is the ratio $\frac{f(x)}{f(0)}$ and $f(x) = ab^x$ is one representation of the formula for the relationship where $f(0) = a$.
--

Measurement and Emergent Symbol Meaning

Throughout my conceptual analysis and examples I maintained a consistent focus on three questions at the heart of quantitative reasoning: (1) What am I measuring? (2) What is my measurement unit? (3) What does the value of my measurement represent? This attention ensured that the values I discussed had clear referents within a quantitative structure because my measurement schemes derived from the ways in which I conceptualized the quantities and relationships between quantities. Such a focus is different from common instructional goals I have observed in Precalculus courses. For example, instructors typically describe a growth factor as the value by which you multiply some number of times to evaluate a formula, the percent comparison is the result of taking the growth factor's value and moving the decimal point two places (such as changing 1.05 to 105), and the percent change is this value minus 100. Such a treatment encourages the tendency for students to “employ numerical operations that have no quantitative significance” (Thompson, 2011, p. 38). Notions of measurement are absent from this treatment as are any conceptual foundations for the operations. This may explain why, for example, only about 35% of college Precalculus students who took Carlson et al.'s multiple choice Precalculus Concept Assessment at the end of their courses could identify the impact of updating a function definition from $p(t) = 7(2)^t$ to $p(t) = 7(3)^t$ (Carlson, Oehrtman, & Engelke, 2010).

Reasoning about measurement in the manner I described may support students' understanding and application of percent comparisons, percent change, growth factors, and exponential growth regardless of the context or the way in which a task is presented. But beyond that, a focus on relative size and measurement were key learning goals that

informed many lessons and units in the course. Quantification is the process that generates measurements and working with these measurements and relationships between measurements then allows an individual to engage in mathematical reasoning such as exploring and predicting how changes to a situation impact those measurements or reasoning about aspects of situations that remain invariant even as quantities' values change (Thompson, 2011). Yet Thompson et al. (2014) show the complexity inherent in measurement schemes based on different levels of reasoning with magnitudes suggesting that productive images of both extensive and intensive measurements are likely to emerge only from concerted attention over long periods of time and in a variety of contexts.⁴⁶ Ideally early mathematics courses would have initiated this process, but Thompson et al. do not find this to be the case, and pre-interviews in this study support that conclusion. Many other researchers have documented students' challenges in constructing productive meanings for quotient, intensive quantities, ratios, and related ideas (i.e., Johnson, 2014; Moore, 2010; Piaget, 1968; Schwartz, 1988; Simon & Placa, 2012). Yet Thompson et al. (2014) hypothesize that "children's development of algebraic reasoning and calculus reasoning is strongly dependent upon their abilities to think with magnitudes" (p. 9). Thus, we attempted, to the best of our ability within the boundaries of our programming environment, to design units on proportionality, linear functions, rational functions, exponential functions, and trigonometry to center around thinking with and about magnitudes and relative magnitudes. Our hope was that such a focus might support students in seeing coherence across ideas in the course as well as provide enough

⁴⁶ An extensive quantity is one that can be directly measured (like a length) while an intensive quantity is a quantity conceptualized via a mental operation of comparison between and/or coordination of previously conceptualized quantities (like a rate of change).

sustained focus and repeated reasoning for students to progress to higher levels of thinking with magnitudes and gain clarity on how measurement processes are central to making sense of mathematical contexts.

Thompson (2011) describes four dispositions that would allow a student to use algebra representatively to reflect her quantitative reasoning in a context. First, the student has a disposition to represent solutions without simplifying the calculations into a single value. Second, the student has a disposition to conceptualize new quantities within a situation and integrate those quantities into the situation to draw additional conclusions. Third, the student has a disposition to reason with abstract units. Recognizing a consistent relative size measurement of one tree height in terms of another tree height measured one year earlier even as those heights change and regardless of the unit used to measure the heights is an example of this idea. Finally, the student has a disposition to reason about magnitudes as described in Thompson et al. (2014). The key idea Thompson intends to communicate is that these dispositions all reflect a student's expectation that mathematics is representational. This might include expectations such as the following.

- The student expects that operations he performs will have “quantitative significance” (Thompson, 2011, p. 38). That is, the operations he performs are motivated by an intention to determine the value of a specific quantity that he believes is important to advance towards a conceptualized goal.
- The student expects that expressions and formulas reflect an underlying quantitative structure. The student also expects that he can unpack expressions and formulas to reveal the quantitative structure that motivated its original creation.

- The student expects that the order of operations used in evaluating an expression or solving an equation reflects the hierarchy of quantities within a conceptualized quantitative structure.

To condense this, I will say that a student operating with these expectations as he develops expressions and algebraic representations within a mathematical task is engaging in *emergent symbol meaning*.⁴⁷ In O’Bryan and Carlson (2016) we provided examples of this and demonstrated that a professional development intervention supporting a middle school teacher in developing emergent symbol meaning (although we did not use that term at the time) positively impacted the quality of discourse in her classroom while using a conceptually-oriented algebra curriculum. As I will demonstrate in my results section, students without these expectations struggle to produce accurate mathematical models and interpret the implications of the mathematical structures in models they do generate. For example, the majority of students in this study wrote $h = 7 + 0.13t$ to model a seven-inch plant growing by 13% per week. Despite acknowledging that 13% was communicating something about height (and being able to accurately calculate a plant’s height two weeks in the future given a weekly percent change in an earlier task), students expressed no hesitation in writing and using this model. Only one of five students recognized this reasoning was flawed after initially employing it.

⁴⁷ This name intentionally parallels the term *emergent shape thinking* (Moore & Thompson, 2015) describing an image of a relationship’s graph emerging as a “trace in progress...with the trace being a record of the relationship between covarying quantities” (p. 785). Students that reason with emergent shape thinking are not only able to imagine graphs as generated by emergent traces but are also able to imagine already-created graphs as having been originally generated emergently and can work to think about relationships between co-varying quantities that would have produced such a trace. By using the term “emergent symbol meaning” I intend to suggest a similar way of thinking about symbolic expressions and conceptualizations of quantities and quantitative relationships.

On the other hand, O'Bryan and Carlson (2016) showed that, at least in one study, it was possible to promote the development of these expectations through activities like comparing how different orders of operations for evaluating the same quantity reflect different underlying ways of understanding the situation. In other words, drawing attention to the meanings and motivations for every step in evaluation and solution processes within several specific activities successfully shifted the teacher to expecting that this would be the case in other situations. I suspect that this could be true for students as well, thus we designed the course to support students in developing these expectations and engaging in emergent symbol meaning. In addition to directly asking students to justify the meaning of parts of expressions and the results of intermediate calculations in solution processes, we leveraged the ability of iMathAS to recognize equivalent solutions in any form to encourage students to represent their answers using unevaluated expressions. As one example, students could represent a price of \$43.99 as a percentage of a price \$38.99 using $100(43.99/38.99)$ instead of calculating the value and entering the approximate answer 112.8238. The course instructor encouraged this explicitly by trying to coach this behavior and implicitly by maintaining a precision level for accepted correct answers that was easier to meet with exact expressions than rounded calculations.

Study Methodology

This iteration of the intervention included a 70-student section of Precalculus at a large public university in the Southwest United States. The online materials (lessons, homework, quizzes, and online forums and instructor messaging) served as the primary source of content for the course with course exams proctored in person every two weeks.

The course instructor set office hours during the regularly scheduled weekly class times in the scheduled class location and we gave students the option to switch sections on the first day of class if they preferred a face-to-face Precalculus class. Students were then asked to complete a 25-question pre-test during the first class meeting that focused on assessing aspects of quantitative reasoning, linear functions and related ideas, and exponential functions and related ideas. Several of the assessment questions were tasks from the APCR and CCR exams (Madison, Carlson, Oehrtman, & Tallman, 2015). I chose these tasks because mathematicians have determined that they are aligned with College Algebra and Precalculus course goals and because I intend to contribute to ongoing validation and refinement attempts relative to these assessments. Students completed the same assessment near the end of the course as well. In addition, we designed the course using the iMathAS platform (Lippman) which saves all student attempts to each lesson task, homework question, and assessment item and provides aggregate data on these items including average number of attempts and average time spent on each item. Of the 70 students who initially enrolled in the course, 68 students started course assignments, 65 students completed the pre-test, 60 completed the course, 58 completed the post-assessment, and 54 earned a C or better in the course.

We invited a subset of students enrolled in the course to participate in additional clinical interviews. My study involved five such students: Gina, Marcus, Shelby, John, and Lisa (all pseudonyms). I selected the students based on their responses to the pre-test. I wanted students with a range of scores, both overall and on the items assessing exponential functions and related ideas. See Table 9.3.

Table 9.3

Students Interviewed for this Study

	Gina	Marcus	Shelby	John	Lisa
Overall pre-test score (out of 25)	9	8	16	21	15
Score on general modeling questions (out of 6)	3	4	4	6	5
Score on linear relationship questions (out of 9)	4	1	7	7	7
Score on exponential relationship questions (out of 10)	2	3	5	8	3
Year in school	Sophomore	Sophomore	Freshman	Freshman	Junior
Most recent math class and grade	College Precalculus (D)	College Algebra (B) ⁴⁸	High School Calculus (A)	High School Precalculus (A)	Geometry for Artists (A) ⁴⁸
Major	Medical Studies	Interior Design	Microbiology	Fine Arts	Interior Design

I interviewed each student using a subset of the pre-test items during the first two weeks of the course and then again using the same items (and some additional tasks and questions) during the last month of the course. Each interview lasted 60-90 minutes, and I opted to conduct the interviews without providing the multiple-choice options for each task. Two of these students (John and Marcus) also agreed to work a subset of unit lessons while having their screens recorded and then participated in a brief post-interview. These sessions lasted approximately 60 minutes. Students were paid \$20 for each interview and each lesson they completed.

⁴⁸ Lisa informed me that as a high school senior she completed Calculus AB. Marcus said that he completed Precalculus as a high school senior.

Students enrolled in two other 35-student Precalculus sections at the same university also completed both the pre- and post-test. Students in these sections used the *Pathways Precalculus* (Carlson, Oehrtman, & Moore, 2018) student workbook, a research-based curriculum that routinely supports higher scores and larger gains on the PCA than other Precalculus course materials (McNicholl, Frank, Hogenson, Roat, & Carlson, under review). This was a sample of convenience and was not intended as a statistical comparison. Rather, the online intervention was based on the same research as the *Pathways* materials and targets similar mathematical meanings. We used the data from these small in-class sections to determine if our course was achieving roughly similar outcomes on specific learning goals. As the intervention enters later iterations we will conduct statistically valid comparisons with students in a variety of Precalculus courses.

Unit Design

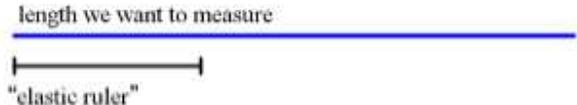
Our unit on exponential functions and related ideas contains approximately 170 tasks across seven lessons and the corresponding homework assignments. Due to space limitations, I briefly discuss key features of the unit using a few sample tasks and highlight critical reasoning abilities and understandings these tasks were intended to develop. Students had multiple opportunities to reason through similar tasks and variations on these tasks.

Sample Tasks

I wrote the unit based on my hypothesis of how to support student learning, as described in my conceptual analysis. An early task (Figure 9.13) focused on developing students' ability to measure one varying quantity (the length of a line) using a varying magnitude as the unit of measure [MCM-1, MCM-3].

Each of the following diagrams shows a magnitude for a varying quantity along with a ruler for measuring the quantity. For each, determine if the given ruler will produce a constant measurement value for the varying quantity.

a. Will the given ruler (shown below) produce a constant measurement for the varying quantity?



Does the measurement value remain constant?

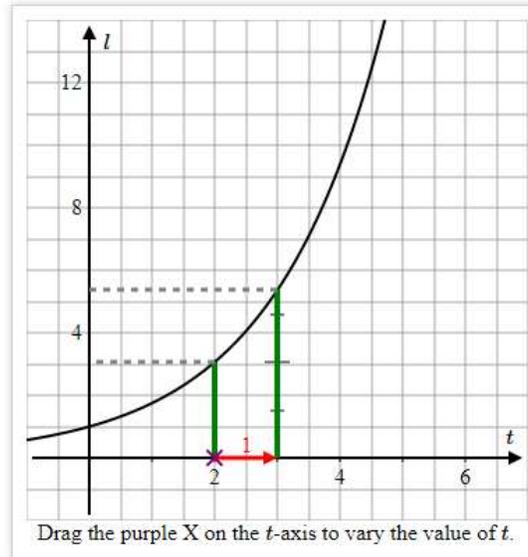
Figure 9.13. A task designed to support students in conceptualizing necessary properties of a varying length and varying ruler size such that the measurement is constant (O'Bryan et al., 2018).

In the online version the image is dynamic such that the line length to be measured and the elastic ruler length vary in a looping animation. Students respond to a variety of prompts, including cases where the relationship is proportional and cases where the relationship is not proportional. The purpose of this task is to support students in visualizing multiplicative comparisons in terms of measurement [MCM-1] and generalizing the proportional relationship between a varying quantity to be measured and a measurement unit such that the measurement value remains constant [MC-4]. This task is similar to tasks Tallman (2015) designed to explore meanings for measuring angles using subtended arc lengths and appropriate units for performing these measurements.

I later asked students to think about situations where one instance of a quantity is used to measure a second instance of the same quantity [MCM-3], such as in the tree height example I described, and the implications of maintaining this relative size between instances even as the quantity varies [MCM-4]. If the two measurements are always taken at the beginning and end of some fixed change in time then we can visualize the relationship graphically as shown in the task in Figure 9.14. Students can slide the interval in the applet and visualize the measurement process throughout the visible part of the function's domain by holding in mind the two dependent quantity magnitudes [MCEF-1] and imagining one as the unit of measure for the second [MCEF-2]. A variety of tasks used this applet including tasks asking students to estimate or calculate growth factors and determine if a constant growth factor existed for given functions. Later in the module students explored the implications for growth factors between 0 and 1.

Since many real-world scenarios modeled by exponential growth are presented and discussed in terms of percent change, understanding concepts of percentages and percent change were integral to the unit. Figure 9.15 presents a task designed to support students in visualizing the meaning of relative size comparisons as measurements [MCM-1] and the impact on the measurement value when the measurement unit becomes $1/100$ times as large [MCM-2, MCP-2], which produces a percentage comparison [MCP-1]. We leveraged diagrams like Figure 9.15 to motivate the calculations involved in converting between relative size comparisons and percentage comparisons.

Let's imagine a new vine with a different growth factor. Let $l = f(t)$ represent the length of the vine (in feet) t days after it started growing.



- The 1-day growth factor is approximately $b =$ _____.
- If b represents the 1-day growth factor, which of the following is true? Select all that apply.
 - The vine's length doubles every b days.
 - No matter what the change in time elapsed since the vine began growing, the vine's new length is b times as large as its previous length.
 - When the number of days the vine has been growing changes by 1 day, the vine's new length is b times as large as its previous length.
 - When the number of days the vine has been growing changes by 1 day, the vine's new length changes by b feet.
 - The vine's length changes at a constant rate of b feet per day.

Figure 9.14. Students use a sliding interval to visualize the implications of a constant relative size measurement between two instances of a quantity and to estimate the growth factor's value (O'Bryan et al., 2018). As mentioned previously this applet is based on a suggestion from Patrick W. Thompson (personal communication, April 9, 2015) and is a modification of Ellis et al.'s (2012, 2015, 2016) Jactus animation.

Complete the following statements.

a. What is the *change* in the price per gallon from the original price to the new price?

\$

b. The *change* in the price per gallon from the original price to the new price is how many times as large as the original price?

times as large

c. The *change* in the price per gallon from the original price to the new price is what percent of the original price?

%

Figure 9.16. A task designed to help students understand the meaning of relative size measurements and percentages, to motivate calculations for evaluating these measurements, and to promote imagery that helps students reflect on the reasonableness of the measurements they determine (O’Bryan et al., 2018).

Figures 9.17 and 9.18 show tasks intended to help students develop a meaningful understanding of the effects of exponential growth over different-sized intervals. First, I wanted students to focus on thinking about function values across multiple consecutive intervals as reasoning about the growth factor and percent change as measurements where the “unit ruler” always updates to reference the function value at the beginning of the interval [MCPC-3]. I also wanted students to realize that they could discuss a growth factor for any interval size [MCEF-3] and to consider the effect on the relationship between growth factors over different intervals [MCEF-4]. Eventually I intended for

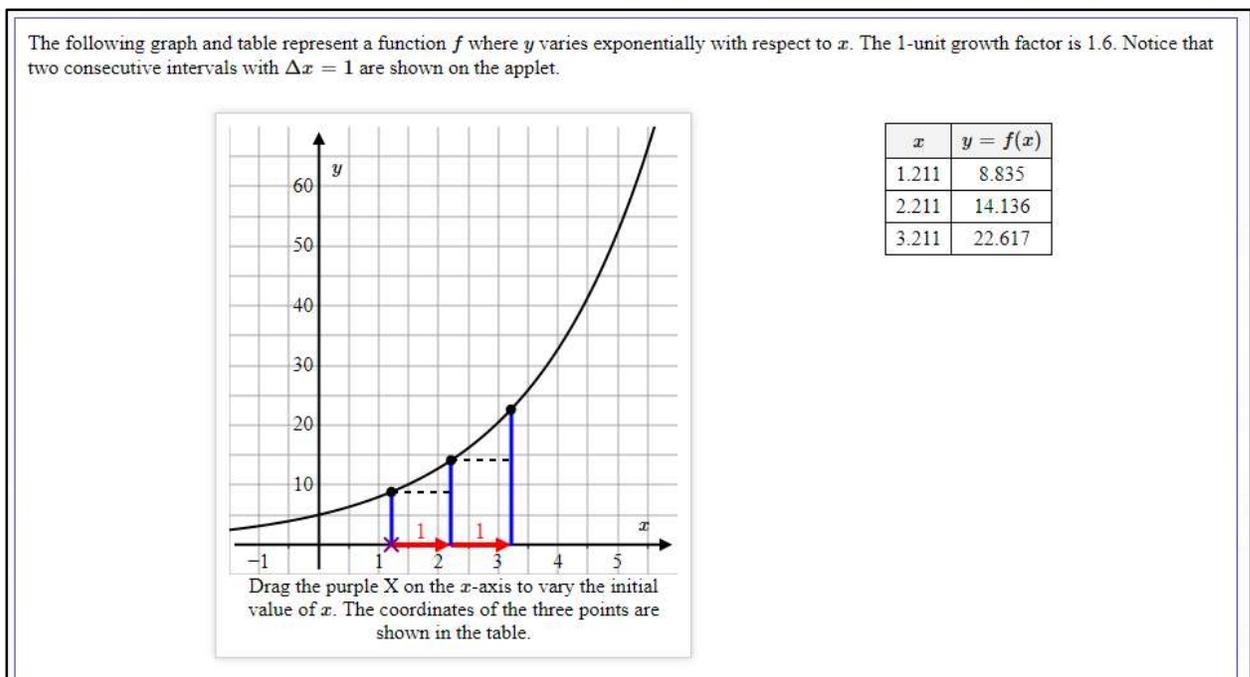


Figure 9.17. An interactive applet for students to reflect on the relationship between one-unit and two-unit growth factors and to think about these as measurements with particular attention paid to what is being measured and what is the “unit ruler” (O’Bryan et al., 2018).

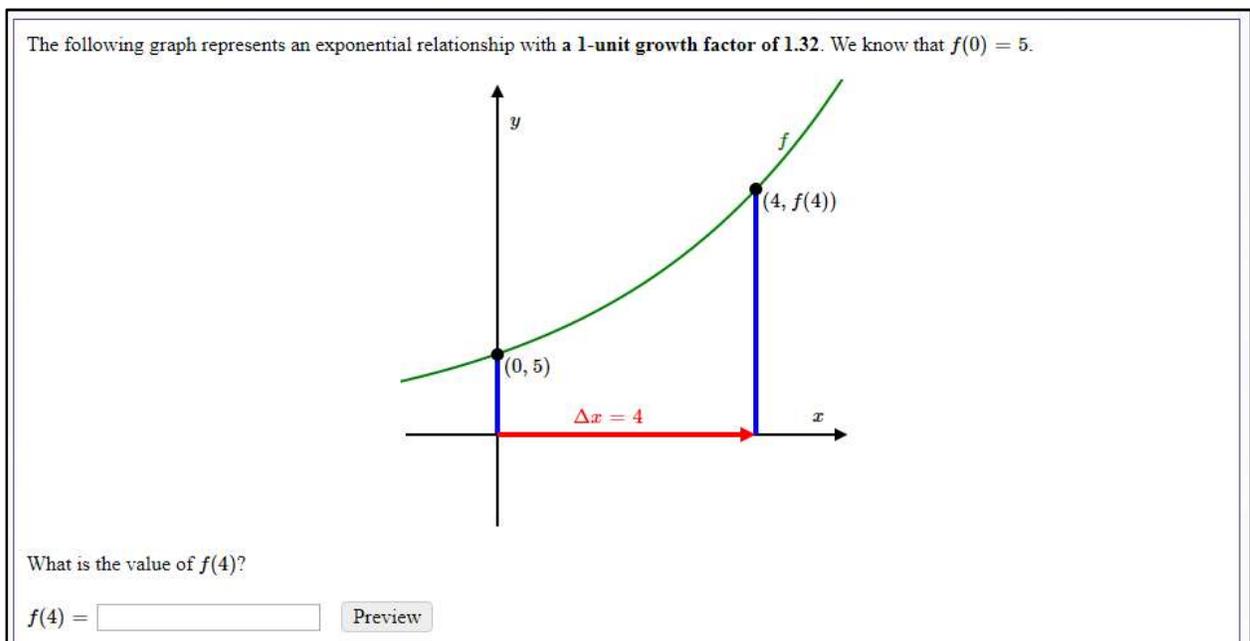


Figure 9.18. A visualization to help support the meaning of growth factors for larger intervals as measurements and to motivate processes for calculating the values of exponential functions at various values of the independent quantity (O’Bryan et al., 2018).

students to see the value of $b^{\Delta x}$ as the Δx -unit growth factor if b is the one-unit growth factor [MCEF-4] so that they conceptualize the algebraic formula $f(x) = ab^x$, as evaluated according to the order of operations, as scaling the value of $f(0)$ by the x -unit growth factor to determine the value of $f(x)$ [MCEF-5]. The task in Figure 17 was an interactive applet and follow-up questions encouraged students to slide the intervals and consider relationships between function values for changes in one unit and two units in the independent quantity. The task in Figure 18 was intended to support the equivalence of imagining the growth over four one-unit changes in x and one four-unit change in x using the corresponding growth factors and for interpreting the different growth factors in terms of measurements.

Comments about Designing Tasks in iMathAS

There are a variety of positive aspects to lesson and task design in the iMathAS (Lippman) environment. The system allows for a wide variety of question types (such as algebraic statements, strings, numerical values, multiple-choice and multiple-answer, matching, and many more) and it is possible to program applets within questions such that applet states can be submitted as solutions. The program is also designed to recognize equivalent representations for solutions when desired. We encouraged students to enter their answers as expressions that would calculate the value they intended, such as entering “ $100((46.2-38.1)/38.1)$ ” as the percent change from a value of 38.1 to a value of 46.2. As Thompson (2011) argued, this practice may help students eventually develop a more productive meaning for algebraic representations. The immediate feedback the program provides to students, including hints and comments a designer can program into

a question that are triggered by correct responses, general incorrect responses, or specific incorrect responses, allows a designer to create lessons that are responsive to student answers. Finally, the system records all iterations of students' responses to lesson and homework tasks which provides designers and researchers information about sources of student difficulties and how students are interpreting specific tasks.

However, designing online lessons, particularly using iMathAs, introduces a number of challenges. iMathAS was originally designed to be an assessment platform as opposed to a platform for programming lessons and courses. Thus, each lesson must be programmed as an assessment introducing some limitations in lesson and task design. Each assessment allows text, images, and gifs to appear outside of assessed questions but features like interactive applets must be programmed within assessed questions. One drawback to this is that a designer can never include interactive applets as diagrams within the flow of a lesson that students can modify and explore without separating those diagrams into their own question box and requiring some input from students that must be assessed. This becomes a more significant issue when coupled with precision considerations. During early introductions to specific ideas a designer might want to shift students' attention away from specific values and calculations and get them to think more generally about certain relationships. But to allow interactive exploration, to hold students accountable for interacting, or to provide students feedback on their work requires the designer to program a graded question. Since the computer must assess the answers relative to some precision level, we often found it challenging to design tasks we were confident would shift students' attention away from focusing on calculations since the computer (1) needed some input to grade and (2) graded students on their answers'

precision. These limitations may increase the likelihood that students can complete lessons with a calculational image of lesson goals.

When we did create tasks that allowed for approximately accurate responses we ran into two problems. For example, I created a series of tasks that asked students to set both a length to be measured and a unit ruler length using interactive sliders to demonstrate a given measurement value. This required that I decide on a reasonable margin of error. If the margin of error was too large it was possible for students to easily guess correct answers without developing intended meanings. If the margin of error was too narrow, then students who did understand the point of the task often complained that they could not get the answer correct because of slight errors in their estimated relative sizes. In early stages of our course design we tested a variety of innovative task types with students. As one example, we gave students a diagram of a situation and asked them to drag written quantity descriptions or variable expressions onto the diagram to indicate what those quantities or values represented. We found that these tasks successfully improved students' ability to conceptualize quantities in a situation and develop meaningful algebraic expressions. But we then encountered programming barriers in attempting to replicate these tasks in the iMathAS environment.

Another challenge of lesson and task design in the online environment is the lack of interaction between a teacher and students during the learning process. A skilled instructor can build and test models of a student's meanings in the moment and react in ways she predicts will effectively support that student in shifting to more productive meanings for given mathematical ideas. We had to pre-program all lessons and task sequences based on our image of an epistemic Precalculus students progressing through

the course. All students worked through the same lessons with little ability for the program to recognize the meanings students possessed and adapt to address them in the way that a skilled instructor could adapt learning experiences in face-to-face settings. If significant misalignments exist between the imagined epistemic students and the actual students in the class then lesson effectiveness can suffer. Reprogramming tasks and lessons in a timely manner was impossible within this study, and thus leveraging emerging insights we gained had to be delayed until future course iterations.

Results

I present my results in two parts. In the first section I detail key observations from pre-interviews with five students including summarizing common reasoning I observed for students entering the course. In the second section I discuss some initial evidence from students' lesson interactions and post-test and post-interview data to characterize students' meanings for the ideas outlined in my conceptual analysis at the end of the course.

Pre-Test and Pre-Interview Results

The pre-interviews revealed several common student conceptions regardless of their pre-test scores. When students were asked to generate formulas to model the relationship between the values of two quantities there was no evidence that students engaged in emergent symbol meaning (that is, they did not describe the quantities and relationships they intended to represent prior to performing calculations or writing expressions and formulas and they rarely justified their calculations and representations

in terms of quantities). Instead, interviewees tended to write down or circle all numbers and variables listed in a problem statement and then either connect them using operations suggested by key words they identified in the problem statement or test various possible combinations of calculations using these numbers and variables until they produced an answer they deemed acceptable. This echoes observations by other researchers such as Smith and Thompson (2007) and Sowder (1988) and appeared across all tasks I used in the interviews. Some students also displayed a consistent lack of attention to variables' meanings by using the same variable to represent the values of multiple quantities within a situation. These tendencies demonstrate students' lack of expectation that they are generating symbolic representations that reflect relationships between quantities' values in a quantitative structure. Furthermore, students entering the course demonstrated little awareness of the importance of reference quantities for thinking about, predicting, or interpreting the results of relative size and percentage comparisons. For example, students tended not to conceptualize the need to clearly answer the question, "Percent of what?" when reasoning about percent change or percentage comparisons. Because of this they often performed calculations or generated representations using incorrect reference quantities. Furthermore, students held vacuous meanings for "percent" resulting in their inability to mathematically justify actions such as moving the decimal point two places when given a percentage value prior to using it in calculations. Finally, students' meaning for "more" or "increase" were strongly connected to the operation of addition, and this consistently made it challenging for students to represent exponential relationships. In the following sections I provide evidence for these observations across various tasks and discuss their implications for student reasoning.

Percent comparison and percent change tasks. Figure 9.19 shows the Muffler task (Madison et al., 2015). I chose this task, along with the Sale Price task that follows it, primarily to assess students' attention to the reference quantity implied in making percent comparisons [MCP-2] or determining percent change [MCPC-1]. While these questions are answerable without the full detailed meaning for a percentage measurement described in my conceptual analysis, I hypothesized that students' answers would provide insight into their level of attention to reference quantities for percentage and percent change measurements.

<p>The cost of replacing the exhaust muffler on your car is currently \$195. The previous time that you had the same replacement done, the cost was \$131. What is the percent increase in your repair bill (rounded to the nearest percent)?</p> <ul style="list-style-type: none">a. 33%b. 49% [correct response]c. 64%d. 67%e. 149%
--

Figure 9.19. The Muffler task (Madison et al., 2015).

Table 9.4 shows pre-test and pre-interview results for this question. Since I did not provide students with the multiple-choice options during the pre-interview, I use solid dots to represent student responses on the pre-test and “I” to represent pre-interview responses. For interview responses I indicate the final answer students provided without my interference. If students changed their answer as a result of follow-up discussion and activity I prompted I did not note their new solutions in the results.

Note that only John provided the same response on the pre-test as he did in the pre-interview. The following excerpt from my interview with Gina demonstrates her lack of clarity in what is being measured and the meaning of the measurement when

Table 9.4

Student Responses to the Muffler Task

answer choice	Intervention Class (Pre)	Pathways In-Person Class (Pre)	Gina (Pre-test)	Marcus (Pre-test)	Shelby (Pre-test)	John (Pre-test)	Lisa (Pre-test)
a. 33%	32%	44%	•		I		
b. 49% **	45%	29%				•, I	I
c. 64%	12%	12%	I		•		
d. 67%	6%	6%					•
e. 149%	5%	7%		•			
no final answer	0%	3%					
164%				I			

discussing percentage and percent change measurements. Her written work is shown in Figure 9.20.

The image shows handwritten work on a piece of paper. On the left, there is a subtraction problem: $195 - 131 = 64$. The numbers are written in a slightly messy, cursive style. To the right of this, there is a division problem: $64 \div 100 = 0.64$. The result 0.64 is written, and then 64% is written next to it. There are some additional scribbles and a small arrow pointing to the right.

Figure 9.20. Gina’s work on the Muffler task.

- 1 Gina Um, so, when you first got, got it done, it was 131 dollars [*she writes “1st*
- 2 *= \$131”*]. And then you just got it done and it charged you 195 [*she*
- 3 *writes “- = \$195”*]. So I would subtract 195 minus 131 [*she writes “195 –*
- 4 *131 /”*] and this is four, six [*she writes “64”*]. And then I’d say if it’s, uh,
- 5 out of a hundred [*she writes “/ 100”*] it’d be 64 percent [*she uses a*
- 6 *calculator to compute “64/100” and gets 0.64*]. Yeah, point six four
- 7 percent [*she writes “.65”, then changes it to “.64%”*]
- 8 AO: Okay.
- 9 Gina: increase.
- 10 AO: So, sa-, suppose somebody says they’re not quite sure why, when you
- 11 subtract, why you would divide 64 by a hundred for this. Um. What would
- 12 you tell them?
- 13 Gina: So, since it’s, this is your current [*she underlines “\$195”*], what you’re

14 currently charged, and this was the first time you did it [*she makes a mark*
15 *next to "\$131"*], I'm trying to find the difference between these two [*she*
16 *writes ">" next to "64"*] and then since it's divided by a hundred, that's
17 how the percent in-
18 AO: So if you have any number and you want to turn it into a percent, you just
19 divide it by a hundred?
20 Gina: Mm-hmm [*yes*].
21 AO: Okay.

Gina correctly identified that the difference in prices was an important value for describing the price increase (lines 1-4), but she did not conceptualize either \$131 or 1/100 of \$131 as units of comparison for subsequently measuring this change. Instead, Gina incorrectly converted the difference to a percentage measurement by dividing the difference by 100 (lines 5-6). Gina is conflating two processes here, namely making a relative size comparison and converting a percentage measurement to a relative size comparison. Her final answer (0.64) is a relative size measurement between 64 and 100, although Gina labeled this value as a percent. Gina does not conceptualize answering the implicit question, "Percent of what?" while formulating her solution process. Instead, her meanings for percentage and percent change measurements involve calculational processes (such as dividing a given value by 100) that do not involve clearly identifying reference quantities for performing relative size comparisons. Marcus demonstrated similar deficiencies in clearly justifying his calculations and solutions relative to what was being measured and its unit of measure. He initially determined the difference between the prices and then interpreted that difference (\$64) as a percentage measurement instead (64%), although he then claimed that the percent increase was 164% because "100 percent's 131" and "164 percent of 131 is 195".

Figure 9.21 shows the Sale Price task (Madison et al., 2015). The important difference between the Muffler task and the Sale Price task is that the reference quantity for the percentage measurement (the computer’s retail price) has an unknown value. Therefore, students need to conceptualize that 20% (or 80%) is in reference to an unknown price and then decide how to use the given information to determine that unknown amount. Students’ responses are recorded in Table 9.5.

Jim bought a computer on sale at a 20% discount off of the original price. If Jim’s sale price was \$440.00, what was the original price of the computer?

- a. \$352.00
- b. \$366.67
- c. \$528.00
- d. \$550.00 [correct response]
- e. \$2200.00

Figure 9.21. The Sale Price task (Madison et al., 2015).

Table 9.5

Student Responses to the Sale Price Task

answer choice	Intervention Class (Pre)	Pathways In-Person Class (Pre)	Gina (Pre-test)	Marcus (Pre-test)	Shelby (Pre-test)	John (Pre-test)	Lisa (Pre-test)
a. \$352.00	9%	15%					
b. \$366.67	3%	10%					
c. \$528.00	49%	42%	●, I	●, I	●		●, I
d. \$550.00 **	38%	26%				●, I	
e. \$2200.00	0%	0%			I		
no final answer	0%	8%					

Four out of five students I interviewed, and about half of students tested, chose the answer consistent with using the sale price as the reference quantity for the 20% measurement. Two observations are worth mentioning. First, the students in the interview did not hesitate in their responses. That is, the four students who provided an incorrect

answer did not express reluctance to link 20% with the sale price even though “20” was a measurement using 1/100 of the retail price as the unit. Some of the students checked their answer and verified that \$528.00 could not be the correct sale price but resolved the issue by stating that they did something wrong in the verification process or ignoring the discrepancy and choosing to keep their answer regardless. Second, reasoning about this situation involves conceptualizing a quantity whose value is not presented in the problem statement and then imagining a process acting on this quantity. The most common student response in the interviews was to rewrite 20% as 0.20, multiply this by 440, then add the result to 440. See Figure 9.22.

$$\begin{array}{r}
 \$440 \\
 \times .2 \\
 \hline
 88 \\
 \\
 440 \\
 + 88 \\
 \hline
 528.
 \end{array}$$

Figure 9.22. Lisa’s response to the Sale Price task.

In the interviews it seemed clear that this was driven by students’ expectations that tasks can be solved by performing operations using the given values in the problem statement. Since only 20% and \$440.00 were given in the problem, students expected that the solution would involve performing an operation involving 20% (or 0.20) and \$440.00, and multiplication seemed like the best candidate based on their experience working with percentages. Correctly completing this task (with understanding) requires conceptualizing a quantity’s value and a relative size measurement that are both implicit

in the problem statement – an original retail price and a measurement of the sale price or discount in units of the retail price – and not just performing an operation with a combination of given values.

Tomato plant tasks. Figure 9.23 shows the Tomato Plant A task (Madison et al., 2015). I chose this task to assess students’ ability to recognize the need to update the reference quantity from one week to the next when applying a constant percent change over multiple intervals [MCPC-3] and noticing that the absolute increase per week is not constant [MCPC-4]. That is, the growth in the first week depended on the plant’s initial height while its growth in the second week depended on the plant’s height at the end of the first week. Table 9.6 shows pre-test results for this task.

A tomato plant that is 4 inches tall when first planted in a garden grows by 50% each week during the first few weeks after it is planted. How tall is the tomato plant 2 weeks after it was planted?

- a. 5 inches
- b. 6 inches
- c. 8 inches
- d. 9 inches [correct response]
- e. 12 inches

Figure 9.23. The Tomato Plant A task (Madison et al., 2015).

Table 9.6

Student Responses to the Tomato Plant A Task

answer choice	Intervention Class (Pre)	Pathways In-Person Class (Pre)	Gina (Pre-test)	Marcus (Pre-test)	Shelby (Pre-test)	John (Pre-test)	Lisa (Pre-test)
a. 5 inches	2%	3%	I				
b. 6 inches	3%	1%					
c. 8 inches	40%	37%		●, I			
d. 9 inches **	54%	51%			●, I	●, I	●, I
e. 12 inches	2%	7%	●				

Marcus's solution to this task is particularly interesting (see Figure 9.24).

Handwritten work for the Tomato Plant A task. On the left, a vertical line with three segments is drawn. The top segment is labeled "4 in.", the middle segment is labeled "> 50%", and the bottom segment is labeled "4 in.". To the right of this diagram, the equation $4 + 2 \cdot \frac{1}{2} = 8$ is written in red ink.

Figure 9.24. Marcus's work for the Tomato Plant A task.

- 1 Marcus: I know that, I know my answer is eight, but I don't think this, this'll all
- 2 work, this is not gonna work out [*laughs*]. Um cuz I mean
- 3 AO: So tell me why it's gotta be eight.
- 4 Marcus: Um cuz 50 percent of four is two.
- 5 AO: Okay.
- 6 Marcus: So you got two. Um. And then, he's, it's two weeks gone by so times that
- 7 by two. That's four. And four plus four is eight.

It is possible that students who conceptualize a constant change in height each week are unaware of the conventional meaning for a constant percent change. When Marcus drew the diagram in Figure 9.24 he seemed to intentionally indicate a constant change in height each week. However, writing " $4 + 2 \cdot (1/2) = 8$ " (see Figure 9.24) is telling relative to tendencies I noticed across students and tasks. Marcus knew this statement was false, but as I will demonstrate later it is highly representative of how all five students attempted to algebraically model exponential relationships, including those students who clearly did understand the need to update the reference value each week in the Tomato Plant A task. In other words, Marcus knew that he wanted the height after two weeks to be eight inches, but this task also triggered an expectation that the answer would be of the form $a + bt$ where a is the initial height (in inches), b is the growth per week (generated by converting the given percentage value to decimal or fraction) and t is the time elapsed in weeks. Even after seeing that this approach failed to produce the

“correct” value here, this expectation would appear repeatedly for Marcus and other students.

Lisa’s response to the Tomato Plant A task represented the most common reasoning among interviewees. She recognized the need to update the reference value each week and focused on determining the increase in height she needed to add for that week. However, her meaning for percentages is entirely calculational. See Figure 9.25 for her written work.

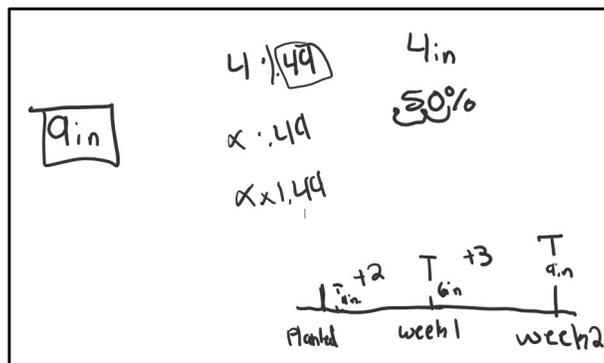


Figure 9.25. Lisa’s work for the Tomato Plant A task.

- 1 Lisa: [As she reads the problem, she writes “4 inches” and “50%”.] Okay. So I
 2 always draw for these, so if it started um [she draws a horizontal line
 3 segment and writes “planted” at the bottom left] planted and you starting
 4 ugh four inches [she draws a vertical line segment and labels its length “4
 5 in.”]. Um. And it grows by 50 percent so 50 percent of four would be two.
 6 Um, so you would add two to it [she writes “+2”]. So after week one [adds
 7 the label “week 1”] it would be six inches [she draws a vertical segment
 8 and labels its length “6 in.”] and then 50 percent of that would be three so
 9 I add three to that [she writes “+3”] and then after week two [adds the
 10 label “week 2”] it would be the nine inches [draws a vertical segment and
 11 labels its length “9 in.”]. So I would say nine inches [writes “9 in” and
 12 draws a box around it].
 13 AO: So the fifty percent was kind of a nice number. Like you just immediately
 14 saw
 15 Lisa: Yeah. [laughs]
 16 AO: that was half. Um, what if, what if the problem said it grew by 49 percent
 17 each week? How
 18 Lisa: Um.

19 AO: does that change what you're...
20 Lisa: I would just, I would still just do like four times the point four nine [*writes*
21 "4.49"] to figure out how much you would be adding and then same with
22 whatever that is times the point four nine [*writes "x.49"*] to figure out
23 how much would be adding to the third week.
24 AO: So how, why is 49 percent, why did you use point four nine for 49
25 percent?
26 Lisa: Oh.
27 AO: Like if you had to explain to somebody why that would make sense.
28 Lisa: Um it's just easiest for multiplication to convert it to a decimal where you
29 move it over two spots [*she scribbles to indicate moving a decimal place*
30 *two positions where she had written "50%"*]
31 [...]
32 AO: Um, but why, why does moving the decimal makes sense? Why does that
33 work to talk between percents and decimals?
34 Lisa: I hon-, I probably couldn't accurately explain that. That's just something I
35 guess you always were taught and so it's just something I immediately do
36 without thinking about it. I honestly probably couldn't figure out,
37 remember the exact reason why [*laughs*].

Lisa's justification for moving the decimal was that "it's just easiest for multiplication" (line 28). There was no indication in her answer that she viewed 49% and 0.49 as related measurements using different units (the height at the end of the first week being 49 times as long as 1/100ths of four inches compared to 0.49 times as long as four inches) and why one of them is more appropriate as a factor within a product involving the plant's height at some moment. The conversion process was not linked to a conceptualization of a measurement process (lines 34-37). Rather, it was just a necessary step for performing calculations in contexts involving percentages.

I chose the Tomato Plant B task (Madison et al., 2015), shown in Figure 9.26 to assess students' ability to algebraically represent exponential growth (such as the relationship in the Tomato Plant A task) as a generalized process [MCEF-5]. However, this task can assess many ideas in the framework since its completion requires

understanding many other ideas such as the need to determine a monthly growth factor [MCEF-2], converting percent change into a growth factor [MCP-2 and MCP-3], and understanding the impact of constant percent change over many consecutive intervals [MCPC-3 and MCPC-4]. Table 9.7 shows pre-test results for this question in all classes and among the interviewees.

José plants a 7-inch tomato plant in his garden. The plant grows by about 13% per week for several months. Which formula represents the height h of the tomato plant (in inches) as a function of the time t in weeks since it was planted?

a. $h = 7(0.13)^t$
 b. $h = 7 + 1.13t$
 c. $h = 7(1.13t)$
 d. $h = 7(1.13)^t$ [correct response]
 e. $h = 7 + 0.13t$

Figure 9.26. The Tomato Plant B task (Madison et al., 2015).

Table 9.7

Student Responses to the Tomato Plant B Task

answer choice	Intervention Class (Pre)	Pathways In-Person Class (Pre)	Gina (Pre-test)	Marcus (Pre-test)	Shelby (Pre-test)	John (Pre-test)	Lisa (Pre-test)
a. $h = 7(0.13)^t$	28%	37%					
b. $h = 7 + 1.13t$	14%	13%				•	
c. $h = 7(1.13t)$	5%	4%			•		
d. $h = 7(1.13)^t$ **	20%	24%					
e. $h = 7 + 0.13t$	34%	20%	•, I	•, I	I		•, I
No final answer		2%				I	

In the interview, four of five students claimed that $h = 7 + 0.13t$ was the appropriate model for this relationship. The fifth student (John) did not complete the task, but he initially wrote “7 + 0.13” before recognizing that he did not know how to express his answer. It is interesting to note that all three of the students who correctly answered

the Tomato Plant A task failed to answer this task correctly either on the pre-test or in the pre-interview, and that two of those students (Lisa and Shelby), along with Marcus and Gina, failed to recognize that the answer they did provide could not be correct. The following excerpt is from my interview with Shelby and Figure 9.27 shows her written work.

$$h = 7 + (0.13)t$$

$$h = 7 + (.13)(2)$$

$$h = 7 + .26$$

$$\begin{array}{r} 7 + .26 \\ \hline 33 \end{array}$$

Figure 9.27. Shelby’s work for the Tomato Plant B task.

- 1 Shelby: Okay so we're trying to find the height [*she underlines "height h" in the*
2 *problem statement*] and then t is gonna be a variable [*she underlines "t in"*
3 *in the problem statement*]. So height [*she writes "h="*] and then we start
4 with seven [*she writes "7"*]. So we start at seven, so that's just gonna keep
5 increasing [*she writes "+" after h = 7*] so and then put a plus. Um [*she*
6 *writes "(0.13)t" after h=7+*]. Yeah, I'm gonna go with that.
- 7 AO: Okay, so like after two weeks how tall is the tomato plant and does that
8 seem to make sense with your...
- 9 Shelby: Let's find out [*she writes "h=7+(.13)(2)"*].
10 [...]
- 11 Shelby: [*She writes out and computes the product "13·2=26" and then enters*
12 *".13x2" into her calculator and gets 0.26*]. Yeah, point [*she writes ".26"*].
13 [*long pause*] I mean we could move that over and that might make sense
14 [*she indicates the decimal point in 0.26, then she writes "7+" plus in front*
15 *of .26*] cuz that would not make sense. So would you, I'm, I'm pretty
16 positive that this is right [*she indicates the statement h=7+(0.13)t*].
- 17 AO: Okay.
- 18 Shelby: I'm pretty positive [*laughs*]. Um. So would this just be [*she writes*
19 *"7+.26"*] three three [*she writes "h=33"*]. I don't know [*laughs*].
- 20 AO: So you're not thinking that seven point two six, if you add the seven and
21 the point two six, that that doesn't make sense?
- 22 Shelby: I don't know. I don't think so. So wait, okay, if it was two weeks after it's
23 been planted. Me, it, you now that probably might make sense because in

24 two weeks if it only grows 13 percent, this would be, like more than 50
25 percent [*indicates the statement $h=33$*] so I don't think that's right [*she*
26 *scratches out the last two lines of work*].
27 AO: Okay.
28 Shelby: I think that's right [*she underlines the statement $h=7+.16$*].

In lines 11-16 Shelby produced a value that she did not think was correct, so she considered moving the decimal point two places to get an answer “that might make sense”. Ultimately, her decision for selecting her final answer was because the alternative solution possibility she generated did *not* make sense (lines 22-26, 28). Students’ responses to this task and the Bacteria doubling task (discussed in the next section) perturbed me the most during analysis because, at first, these responses seemed to be at odds with the reasoning the same students exhibited in the Tomato Plant A task. The students who correctly recognized the need to update the reference value in the earlier task produced representations that did not preserve this property in the new task. However, further analysis yields a possible explanation.

In completing the Tomato Plant A task, all students focused intently on how much height to add each week (see the excerpts from Marcus’s and Lisa’s interviews). Regardless of whether the student conceptualized the need to update the reference value each week, the notion of beginning with a height of four inches and adding changes in height to produce new values was key to their solution methods. That is, addition was the operation associated with the process of finding new heights. In this case, all five students produced (or started to produce) responses in the interview of the form $a + bt$ with a representing the initial height (in inches), b representing the growth per week (generated by converting the percentage value given to a decimal value), and t representing the number of weeks elapsed. Only one of the five students (John) recognized that the

solution $h = 7 + 0.13t$ does not reflect a valid quantitative structure in the interview, although he did select a similar answer on the pre-test.⁴⁹ Here $0.13t$ must represent a number of weeks (13/100 of the elapsed time in weeks), but this value is added to a height (in inches). This suggests that at least four of the five students did not expect that the expressions they wrote as they constructed their model should reflect the values of quantities they conceptualized and wanted to relate. In justifying their solution, students routinely interpreted $0.13t$ as a height (or change in height) in inches. Not only does this solution reflect the belief that addition is the required operation to increase a quantity's value, but it also reflects students' inattention to reference values when discussing percentages (i.e., clearly answering the question, "Percent of what?") as well as their expectation that solutions are formed by combining the values, variables, and operations explicitly given in a problem statement without identifying values or operations that might be more implicit. The value 1.13 appears in the correct formula, yet this number is not explicitly given in the problem statement. Furthermore, there are no key words that students might more readily associate with exponentiation if we understand that students associate "growth" with addition.

It is possible that producing the response " $h = 7 + 0.13t$ " is a result of performing a literal translation of a goal statement similar to "show that the height starts at 7 inches and increases by 13% per week" without having an expectation that the formula and its order of operations needs to be constructed to respect a valid quantitative structure. If the students have a procedural meaning for percentages ("when I see a percentage, I am

⁴⁹ John wrote " $h = 7 + (.13)$ " but then stopped. After pausing for a moment, he said that this was not correct but that he was not sure how to write an acceptable model.

supposed to move the decimal two places”) as opposed to thinking of a percentage as the result of a measurement process, if they believe that formulas are just instructions for calculations as opposed to representations of one quantity’s value in terms of another quantity’s value within a quantitative structure, and if they have been trained to circle or underline numbers, variables, and key words as a solution strategy to answering “word problems”, then the solution shown in Figure 9.28 seems quite reasonable.

①
②
③
④
⑤
 the height starts at 7 inches and increases by 13% per week

①
②
④
⑤
 $h = 7 + 0.13t$
③

Figure 9.28. Students may produce the formula $h = 7 + 0.13t$ as a literal translation of what they think they are being asked to represent.

I considered the possibility that students’ lack of familiarity with exponential functions led to an increased tendency to use linear functions to represent exponential relationships. In other words, the tendencies described reflect the topics to which they had been exposed in earlier classes rather than fundamental issues with their conceptualizations of exponential relationships. This explanation could be valid, but there are reasons to suspect additional causes. First, every student I interviewed had taken Precalculus or Calculus in the past and thus must have received exposure to exponential function relationships, not to mention exposure to percentages and percent change over many years of schooling. Second, the ways of reasoning were so consistent that they suggest a fundamental way of understanding integral to most students’ schemes relative to tasks an observer would classify as examples of exponential growth. Third, these questions (or versions identical to them with the values changed) were included in

preliminary iterations of the APCR and CCR exams given to students at the end of their College Algebra, Precalculus, and Calculus I courses (Madison et al., 2015). The results, shown in Table 9.8, suggest that students struggle to see the algebraic form of an exponential function as an accurate model for relationships where one quantity changes by a constant percent for equal changes in a second quantity. Future research should examine if the tendencies I observed in the pre-interviews help explain these results for each of these populations.

Table 9.8

Student Performance at the End of Their Course on the Tomato Plant B Task (or an Isomorphic Task) (Madison et al., 2015)

	Number of students	Percent correct
College Algebra	677	15%
Precalculus	122	14%
Calculus I	1021	33%

Bacteria doubling task. Figure 9.29 shows the Bacteria Doubling task (Madison et al., 2015). I selected this question to compare student responses with the Tomato Plant B task since, in this case, the growth was described in terms of doubling instead of percent change. I was curious how this would impact the form of the answers students selected or provided. Table 9.9 shows pre-test results for this question in all classes and among the interviewees.

A culture containing 14 bacteria doubles every day. Which of the following formulas represents the number of bacteria, p , in the culture after t days?

- a. $p = 14 + 2t$
- b. $p = 28t$
- c. $p = 14(1.2)^t$
- d. $p = 14^{2t}$
- e. $p = 14(2)^t$ [correct response]

Figure 9.29. The Bacteria Doubling task (Madison et al., 2015).

Table 9.9

Student Responses to the Bacteria Doubling Task

answer choice	Intervention Class (Pre)	Pathways In-Person Class (Pre)	Gina	Marcus	Shelby	John	Lisa
a. $p = 14 + 2t$	11%	22%	I				
b. $p = 28t$	3%	1%					
c. $p = 14(1.2)^t$	3%	4%					
d. $p = 14^{2t}$	23%	27%		•			•
e. $p = 14(2)^t$ **	60%	42%	•		•	•	
No final answer		4%				I	I
$p = 14(2)(t)$				I	I		

It is worth noting that none of the students I interviewed provided the same response in their pre-interview as they did on the pre-test. Also, I chose not to group $p = 28t$ and $p = 14(2)(t)$ as the same response for several reasons. No student who provided $p = 14(2)(t)$ as their answer simplified this to the equivalent form $p = 28t$, and $p = 14(2)(t)$ was the most common response provided in the interview yet $p = 28t$ was selected least often on the multiple choice pre-test.⁵⁰ Students' expectation that solutions should result from combining numbers and variables explicitly listed in the problem might contribute to this result. Fourteen and two (in the form of the word "doubling") are both given in the problem statement in some form while 28 is not, and $p = 28t$ and $p = 14(1.2)^t$ are the only answer choices that do not solely use combinations of p , t , 14, and two, and those are the least-selected options on the pre-test.

⁵⁰ This strongly suggests that if this question is used in the future it might be more productive to replace the option $p = 28t$ with $p = 14(2)(t)$.

In the interview students often initially expected that the formula should involve addition similar to their solutions to the Tomato Plant B task. I provide an example from Gina's interview.

14 2(t)
~~p = 14~~ + 2(t)

Figure 9.30. Gina's initial work for the Bacteria Doubling task prior to erasing her formula.

- 1 Gina: Okay so they already contains 14 [she writes "14"] and then it's doubling
2 every day. So, um [she writes "2(t)"]. So just like the other one it'd be,
3 um, the number of bacteria [she writes "p="] equals 14 [she writes "14"]
4 because it already has 14...
- 5 AO: Alright.
- 6 Gina: at the beginning and then you add, um the [she writes "+"] total [she
7 writes "2t" after the plus sign] of what it doubled. This is the days [she
8 underlines "t" in the statement $p = 14 + 2(t)$]. Um. Yeah this is the days
9 and then [trails off]

At this point I asked Gina to consider how many bacteria would be present after two days and she realized that her formula " $p = 14 + 2(t)$ " produced a value that was too small because the result was less than 28. This is evidence that Gina could at least conceptualize that she needed to double the value 14 after one week, but also that she was not initially conceptualizing how this might be communicated accurately with a mathematical expression. She erased " $p = 14 + 2(t)$ " and started again. See Figure 9.31.

$$14 \quad 2(t)$$

$$= 14 \times 2(t) -$$

$$= 14 + 2(t) - 18$$

$$= 14 -$$

Figure 9.31. Gina's final work for the Bacteria Doubling task.

- 10 Gina: Okay, so um what I'm think, cuz it wouldn't make sense to add 14. So if it
 11 were already, for like the first day it doubled it'd be 14 times two [*she*
 12 *writes "14x2"*].
 13 AO: Okay.
 14 Gina: Not add two. Um [*she writes an equal sign to the left of 14x2*]. So say it's,
 15 yeah. So [*she writes "(t)" after =14x2*].
 16 AO: So 14 times two t?
 17 Gina: Mm-hmm [*"yes"*].
 18 AO: So then after two days...
 19 Gina: Oh wait no [*laughs*].
 20 AO: No? How do you, how do you know...
 21 Gina: Cuz that'd be too much. Cuz that, it would be, um, yeah. It would be like,
 22 it'd be like um. It'd be saying like there's 14 times, actually [*long pause*].
 23 Hmm. [*long pause*] Okay. Yeah, never mind. I think this is, because if you
 24 do the adding one [*she writes "+2(t)"*]
 25 AO: Hm-hmm [*"yes"*].
 26 Gina: You're gonna be adding 14 to each consecutive, and then this is, since it is
 27 doubling, um, you just, yeah. In my mind I'm just saying that to go with
 28 this one [*indicates the statement =14x2(t)*] cuz with this one [*indicates the*
 29 *statement 14+2(t)*] for each day you'd be adding 14.⁵¹
 30 AO: Okay.
 31 Gina: It doesn't really make sense because they already started at 14 and if you'd
 32 go, mmm, say it was like two days, this is four, 14 plus four is 20, 20, 18.
 33 And then, yeah this would be 18 [*writes "=18" after 14+2(t)*] and this 14
 34 [*writes "14" below the last line*]. It didn't really double. It just, you just
 35 added four.

⁵¹ I am not clear on what Gina meant by "each day you'd be adding 14" since that is not what her formula represents, nor does it match her description of evaluating the formula.

Again, Gina’s solution seemed not to be driven by an attention to the relationship between quantities but was “a matter of ungrounded debate about choosing numbers and operations” (Smith & Thompson, 2007, p. 108). Gina finally settled on the representation “ $= 14 \times 2(t)$ ” instead of “ $p = 14 + 2(t)$ ” only because the latter produced values that were too small, not because the former seemed to accurately capture her conceptualization of the situation (lines 26-35). Note that in those same lines Gina twice refers to “adding 14” using language that indicates she might be conceptualizing repeated addition by 14 linked with the representation “ $14 + 2(t)$ ”. However, I think that these are misstatements since her description of that possible solution in lines 31-35 do not show her reasoning about repeated addition by 14.

Lisa’s initial response to this task is also worth sharing because it reflects the tendency for students to not incorporate values in their formulas that are not explicitly listed in the problem statement.

Handwritten work for the Bacteria Doubling task. The work shows several equations and calculations. At the top, $t = 1$ and 14 are written. Below, there are several equations: $P = 14^t$, $P = (2 \cdot 14)^t$, $P = 14 + 2^t$, $P = 14 + 2^3$, $P = 14 + 12$, $P = 26$, $P =$, $28 = 14$, a diagram with three vertical bars of increasing height labeled 1 , 2 , 3 and 28 above them, and a calculation $(14 \cdot 2) = 28$ with a crossed-out $(14 \cdot 2) = 28$ below it.

Figure 9.32. Lisa’s work for the Bacteria Doubling task.

- 1 Lisa: [She writes 14, then p, and then t as she reads the problem statement.]
- 2 Okay so it doubles every day, 14, so after one day it’s gonna double to
- 3 twenty eight. So p that’s what I’m searching for [she writes “P=”] after t
- 4 days. I wanna say it’s just like [she writes “14”].

Lisa's first attempt only included p , 14, and t , the three variables and values given in the problem statement. Lisa's second attempt (" $P = 14 + 2^t$ ") was based on the realization that she had not accounted for "doubling" in her formula. Again, an increase was interpreted as a need to perform addition even though she conceptualized at least one doubling of the initial 14 bacteria (line 3 and lines 22-23). I hypothesize that the presence of an exponent in these answers resulted from a comment she made at the end of the Tomato Plant B task. She stated that she thought that she remembered she had to use an exponent in these situations but was not certain why.

- 5 Lisa: At t it's gonna double every, yeah, see, no, that's not right. That's
6 definitely not right. [*She crosses out what she wrote.*] Um. It doubles.
7 That's the important part [*she draws a box around the word "doubles" in*
8 *the problem statement*]. After t days. [*long pause*] This is gonna be the
9 initial that it has, so after, when it starts it has 14 [*she writes "P=14"*]
10 kinda like the last problem [laughs] and then it's gonna double after every
11 day and then [*she writes "+2"*]. See if, I'm about to try the same exact
12 answer. So it'll be times two [*she adds "t" as an exponent*] but I don't
13 think that's right. But I wanna plug something in just to see [laughs].
14 AO: Okay.
15 Lisa: Um so if it was three days [*she writes "t = 3"*] then it's gonna be two to
16 the third so it'll be p equals 14, two to the third [*she writes "P=14+2^3"*].
17 Going to be p equal 14 plus 12. [*she writes "P=14+12"*]. It's going to be
18 p equals 26 [*she writes "P=26"*]. And then if I do it my favorite way
19 [laughs]
20 AO: [laughs]
21 Lisa: drawing it out [*she draws a horizontal segment, then a short vertical*
22 *segment and labels it 14*]. So it's gonna start with 14. And then it's gonna
23 double, yeah so it's immediately gonna be 28 the next day. So, that can't
24 be right either [*she crosses out her work*]. So p equals [*she writes "P="*]
25 [*long pause*] that would be [*she writes "(2·14)^t"*]. A lot of solving in math
26 tends to be me doing something wrong and then checking it with numbers
27 and then [laughs and then she trails off].

In her final comments (lines 25-27) Lisa essentially admits what I had suspected throughout her interview, namely that her efforts to generate solution processes and

algebraic representations is based more on a “guess and check” method of trying different combinations of numbers, variables, and operations stated or suggested by the problem statement. It is also likely that Lisa (and most of the other students I interviewed) did not have a productive meaning for exponential expressions like 2^t and thus these expressions were not available to them while reasoning about the tasks as they conceptualized them. Even if this is true, however, students’ solution behaviors still indicate that they do not necessarily expect their algebraic representations to build through conceptualizing intermediate relationships between quantities within a quantitative structure. Students with this expectation would describe what they intended to communicate with their representation and then acknowledge that they did not have a way to represent the quantities or relationships they conceptualized. Instead, students in the pre-interview tended to generate the algebraic representation first and only later (and sometimes only after being prompted) would they check that their model might make sense. Typically their criterion for this judgment was whether the model produced reasonable values (from the student’s point of view) and not whether its structure reflected their image of the situation.

Summary of students’ meanings at the beginning of the course. Responses to the pre-test and pre-interview tasks suggest that students entering a college Precalculus share many common ways of reasoning about modeling tasks, percent change and percentage comparisons, and exponential functions. Students entering Precalculus tend to view the creation of algebraic models as a process of using combinations of operations with given variables and explicitly stated values. Students particularly struggled to produce accurate solutions and models when they needed to conceptualize quantities,

operations, or measurements not expressed explicitly in the problem statement (such as needing to represent a 13% per week increase in some quantity with the factor 1.13 in the algebraic model). The students were not usually inclined to test the reasonableness of their solutions, but when I prompted them to do so they sometimes recognized that their models were incorrect. When this happened, the common justification was that the models were incorrect because they did not produce expected values and not because they somehow failed to accurately represent relationships within a quantitative structure that they had conceptualized (although students sometimes insisted that a solution was correct even when they acknowledged that it did not produce correct values). Students also tended to not see quantities' values as resulting from an explicitly conceptualized measurement process, so when working with percentages they often did not clearly identify the reference quantity (the answer to the question, "Percent of what?") and their meaning of percentages was entirely procedural ("when I see a percentage, I move the decimal point two places"). Taken together, these descriptions of meanings typically held by students entering Precalculus suggest steep challenges in supporting Precalculus students in constructing the meanings described in the framework in Table 9.2. None of the meanings described in Table 9.2 appear to describe the reasoning an instructor might expect from a typical Precalculus student at the beginning of the course.

Much of the current research on students' meanings for exponential growth is grounded in building on images of repeated multiplication, which is reasonable considering common meanings for exponents and the presence of an exponent in the algebraic representation for exponential functions. However, I did not see evidence that students were inclined to conceptualize repeated multiplication as part of a solution

process for evaluating and representing situations typically modeled by exponential functions, especially when those situations were framed in terms of constant percent change. In future iterations I will need to reflect on whether lessons and tasks throughout the course must be revised to better address these common unproductive meanings. Other researchers working in this area might also benefit from considering the implications of these findings for their own learning trajectories relative to supporting students in conceptualizing connections between percent change and growth factors for exponential functions.

Results from Lessons, the Post-Test, and Post-Interviews

We administered the post-test to all students within the final month of the course and each post-interview occurred within approximately one week of the post-test. Student performance on the post-test was significantly higher than on the pre-test, both overall and on tasks covering exponential functions and related ideas. See Table 9.10, which includes post-test scores as well as the increases relative to the pre-test scores for the same groups/students. However, post-interviews suggest more modest gains relative to my learning goals for many students. To some degree this was expected considering this research was conducted on an intervention that is early in the refinement phase, the complexity involved in fully unpacking ideas related to exponential growth (see Figure 9.12), and the fact that students generally did not enter the course with any of the targeted meanings outlined in my framework in Table 9.2. We know that students' schemes tend to persist and that it takes sustained effort over long periods of time to support shifts in meanings. Additionally, as Thompson et al. (2014) argued by using a cloud metaphor to

Table 9.10

Overall Post-Test Performance

	Intervention Class Average (Post) 58 students	Pathways Class Average (Post) 68 students	Gina	Marcus	Shelby	John	Lisa
Overall post-test score (out of 25)	17.74 (+5.77) (71%)	15.35 (+5.28) (61%)	16 (+7)	21 (+13)	21 (+5)	24 (+3)	23 (+8)
Score on general modeling questions (out of 6)	4.58 (+1.15) (76%)	3.80 (+0.84) (63%)	4 (+1)	5 (+1)	5 (+1)	5 (-1)	6 (+1)
Score on linear relationship questions (out of 9)	6.16 (+1.69) (68%)	5.77 (+1.95) (64%)	5 (+1)	6 (+5)	8 (+1)	9 (+2)	7 (+0)
Score on exponential relationship questions (out of 10)	7 (+2.94) (70%)	5.74 (+2.44) (57%)	7 (+5)	10 (+7)	8 (+3)	10 (+2)	10 (+7)

describe students' schemes for mathematical ideas, learning may occur unevenly throughout constituent parts of a student's scheme and related schemes. Furthermore, behaviors indicating shifts in students' meanings may be revealed when responding to some tasks and not revealed when responding to others.

In this section I report results of shifts in student learning based on pre- and post-test scores and provide evidence that less productive meanings persisted for some students. I report results from analyzing data that suggest how students conceptualized critical visualizations and applets designed to support my learning goals and how students interacted with lesson tasks that may help explain some differences in learning outcomes.

How do students respond to incorrect attempts? Thompson (1985) discussed the important role technology can play in students' learning by providing feedback that allows students to reflect on their thinking. I looked for evidence that this occurred for students working through lessons in this study. iMathAS saves all student submissions so

I was able to examine each lesson to view all submitted attempts in order, the total number of attempts, and the average time between attempt submissions. While examining this data I considered three things. (1) I looked for “spikes” in the number of attempts on tasks throughout the module. That is, I looked for tasks with high attempt counts and examined the answers record to hypothesize whether the student appeared to be guessing or making concerted progress towards targeted reasoning. (2) I looked to see if the average number of attempts tended to decrease over the course of the lessons. While new aspects of ideas were continually introduced throughout the unit, I had designed the lessons to support and build from the same foundational meanings, and thus a decrease in the average number of attempts could suggest that earlier lessons and tasks had prompted students to reflect on and modify their meanings in productive ways. (3) I looked for evidence that the imagery I intended for students to internalize and use as tools in their reasoning process for these ideas appeared in their post-interviews or, alternatively, if students engaged in guess-and-check style behavior in those interviews.

Table 9.11 shows how many tasks within the seven lessons required the indicated number of attempts for each student to solve. Note that the seven lessons included 103 total tasks, but two tasks were removed from this analysis because the required level of precision for the answers was unclear, leading to abnormally high attempt counts for all students. Table 9.12 shows the percentage of tasks initially answered incorrectly requiring certain numbers of attempts to successfully complete.

This data alone is not conclusive in showing that incorrect responses either did or did not prompt students to reflect on their thinking. In these five cases, however, post-interviews confirmed that students with higher attempt counts on lesson tasks shared

Table 9.11

How Many Lesson Tasks Required the Given Number of Attempts to Complete (out of 101)

<i>number of attempts</i>	1	2	3	4-6	7-10	11-15	16-20	21+
Shelby	31	13	10	24	14	5	4	0
Gina ⁵²	35	18	12	14	7	4	1	0
Lisa	70	18	5	7	1	0	0	0
Marcus ⁵²	34	10	9	23	13	2	1	2
John	54	18	11	14	4	0	0	0

Table 9.12

Percentage of Initially Incorrect Responses Requiring the Given Number of Attempts to Complete

	2-3	4-6	7+	attempted but did not finish ⁵²
Shelby	33%	34%	33%	0%
Gina	45%	21%	19%	15%
Lisa	74%	23%	3%	0%
Marcus	28%	34%	27%	10%
John	61%	30%	9%	0%

certain characteristics, including a general inattentiveness to quantitative meanings for calculations, expressions, and formulas and clear indications of guessing on many tasks. This data does not definitively show that high attempt counts are either a cause or a symptom of characteristics like weak quantitative meanings, but it is clear they are related. I discuss this connection more in the rest of my results.

My findings from analyzing five interviewees are organized into two groups based on similarities in four areas: (1) student meanings for the key applets and

⁵² Gina did not complete ten of the tasks, averaging 7.8 attempts prior to moving on. Marcus did not complete seven of the tasks, averaging 5.3 attempts prior to moving on.

visualizations employed in unit lessons, (2) student attempt counts on lesson tasks, (3) the degree to which students referenced quantities in discussing and working with percentages, percent change, and growth factors, and (4) the degree to which students connected calculations, expressions, and formulas to quantities in the situation. Organizing the students into two groups provides important insights that inform key conclusions in this study and suggest some initial modifications that may improve the intervention's impact in subsequent iterations.

The case of Gina, Marcus, and Shelby. If we take post-test results and final exam scores⁵³ as measures of learning, then Gina, Marcus, and Shelby all successfully met unit and course objectives. The three students averaged 8.3 out of 10 on post-test items covering exponential growth and related ideas and scored an average of 19.3/25 on the entire post-test. Marcus and Shelby also earned an A on the final exam and Gina earned a B. My analysis of post-interview data, however, demonstrated that these assessment scores did not reveal student learning relative to some of the meanings targeted in instruction. My analysis of the post-interview data further revealed common behaviors and meanings held by these three subjects (Gina, Marcus, and Shelby). (1) They struggled to describe ideas conveyed by visualizations and applets used throughout the module and how different applets might be used to convey similar mathematical ideas; (2) they tended to have higher attempt counts on lesson tasks and displayed clear indications of using “persistent guessing” to complete certain tasks; (3) they tended not to identify clear referents while working with percentages and percent change and had

⁵³ The final exam was a multiple-choice assessment covering content from the entire course.

unproductive meanings for growth factor and exponential function; and (4) they typically did not describe their calculations, expressions, and formulas in terms of quantities and quantitative relationships. In the next few sections I show evidence for these claims and discuss the implications.

How did students conceptualize key visualizations and applets? As part of this initial reflection on aspects of the lessons that seemed to support (or fail to support) students in constructing the meanings I intended, I focused my analysis on how students used and made sense of the applets that appear in various lessons. One set of applets included variations on the Unit Ruler applet shown in Figure 9.33.

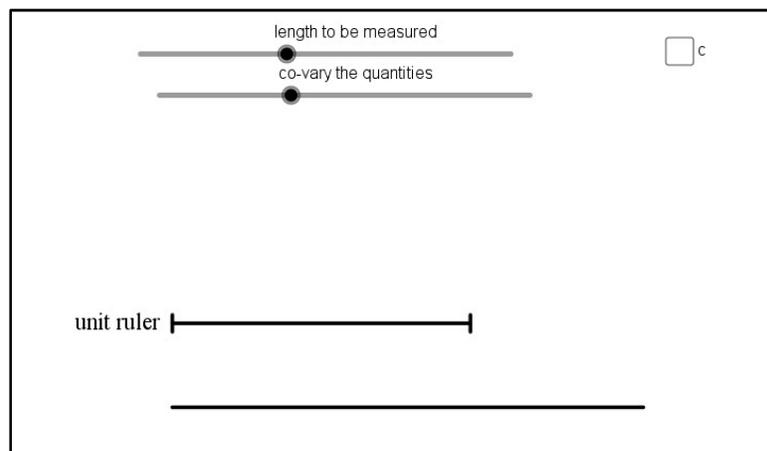


Figure 9.33. The version of the Unit Ruler applet used in the post-interviews (GeoGebra).

Within lessons I asked students to modify either the “unit ruler” length or the length to be measured to demonstrate a given relative size measurement. I placed such tasks throughout the course, including in units on proportionality, constant rate of change, and exponential growth, with the intention that students conceptualize a relative size of two magnitudes that is invariant across changes in those quantities. I then designed lessons to leverage this idea to motivate algebraic and graphical representations of relationships between quantities and calculations for evaluating relative size measurements. I also

included animations showing the two segments' lengths changing in tandem and asked if the measurement of the length of one segment remained constant if the length of the other segment was the unit ruler. I described these as “elastic ruler” explorations because the absolute size of the unit was not constant. Note that the version in Figure 34 has more functionality than students could usually control within a single task. A user can move the “length to be measured” slider to vary the bottom segment length only or move the “co-vary the quantities” slider to change both segment lengths while maintaining a constant relative size.

The second category of interactive applets and animations were variations on the Sliding Interval applet shown in Figure 9.34.

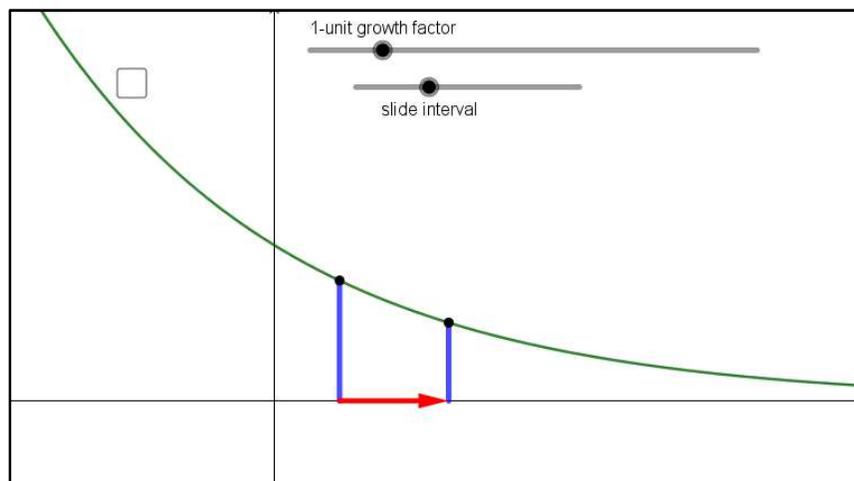


Figure 9.34. The version of the Sliding Interval applet used in the post-interviews (GeoGebra).

Again, this version has more functionality than students could usually control within a single task because we never allowed students to vary the one-unit growth factor with a slider. In this version a user can move the “1-unit growth factor” slider and the one-unit growth factor (where “one-unit” refers to the magnitude of the horizontal vector) varies between zero and four. Note that the applet does not display the exact value for the one-

unit growth factor or the function values at any point. A user can also move the “slide interval” slider and the horizontal vector shifts back and forth along the horizontal axis.

During the post-interviews I asked four of the five students to experiment with the applets and then to explain the key ideas they believe the applets were designed to help communicate as well as the connection between the two applets.⁵⁴ Both Shelby and Marcus expressed confusion or surprise when they moved the “1-unit growth factor” slider on the Sliding Interval applet. Marcus said, “Oh, sh*t [...] Okay, I’ve never seen one like *that* [*emphasis his*]” after first moving the slider. Shelby said, “Whoa. Huh,” and then later said, “this one is throwing me off” when referring to moving the “1-unit growth” slider. Marcus in particular became fixated on this slider and focused on actions such as finding positions for the “1-unit growth factor” slider that made two graphs appear to be horizontal reflections across the vertical axis. I eventually had to ask him not to move that slider and just focus on one single exponential function with a one-unit growth factor of approximately 1.5. Only when working with a specific example could Marcus describe estimating the one-unit growth factor value by comparing the lengths of the two vertical segments [part of MCM-3], although he did so without sliding the interval or discussing it as a general property that always holds [MCEF-2]. When I moved the interval, he was able to explain why a constant growth factor does not produce a constant difference between two values [MCPC-4]. Although he could estimate the unit growth factor from the graph by talking about relative size, when he discussed the meaning of the growth factor he routinely described what he would add as he moved

⁵⁴ I did not ask Gina to describe her understanding of the applets because we ran out of time during her interview. I did not ask Marcus about the Unit Ruler applet for the same reason.

from the function value at the beginning of the interval to the function value at the end of the interval. For example, in his descriptions he talked about the length on the left being “100%” and then showed that he would need to add 50% to get to the second value. He tended not to use measurements like 1.5 or 150% in describing how he thought about the relationship between two function values at the beginning and end of the sliding interval [MCP-3 and MCM-3], although this was true to some degree for all five students depending on the task.

I presented the Unit Ruler applet to Shelby and asked her to describe what ideas it could help demonstrate. After some time spent exploring the applet’s features and struggling to explain the general idea she wanted to convey, she eventually varied the length to be measured until it was half as long as the unit ruler length and said,

1 Shelby: X is point five times as big as the unit ruler. And so even as you vary the
2 quantities [*she moves the slider “co-vary the quantities”*] this one [*she*
3 *points to the bottom segment*] is still gonna be point five times as tall, I
4 mean as big [*she moves the slider “co-vary the quantities” again*] as the
5 unit ruler no matter how big they are.

Much like Marcus, Shelby appeared to need specific examples to convey her ideas. Once she had that example, she was able to explain how the relative size of the two lengths could remain constant even as the lengths changed in this context [MCM-4].

I next presented the Sliding Interval applet and asked Shelby to describe the ideas it was meant to convey. After sliding the interval back and forth along the horizontal axis, she said that one idea it demonstrates is that the one-unit growth factor can change because the *steepness of the graph between the endpoints of the interval changed as she slid the interval*. I emphasized parts of her reasoning because it represents a significant departure from the meanings I intended students to construct. Namely, it demonstrates a

lack of awareness of the growth factor as a constant measurement of relative size for two varying quantities or two instances of the same varying quantity [MCEF-2]. It is possible that she misspoke when she said “one-unit growth factor” and intended to say something like “the difference between the output values”. But the rest of her response suggests that Shelby did not make clear distinctions between various quantities and their measurements.

I next asked Shelby if the two applets demonstrate similar ideas.

- 6 Shelby: Um, I mean th-, they’re linked, they’re both, they’re showing different
7 ways of like demonstrating proportionality I guess. Um. The other one
8 was more linear, this one is more graphical, I. Um [*pause*]. I, I am very
9 bad at explaining things.
- 10 AO: So are they demonstrating the same idea or are they demonstrating
11 different ideas?
- 12 Shelby: Can I look on the other one?
- 13 AO: Yeah, yeah.
- 14 Shelby: [*She brings up the unit ruler applet, then returns to the sliding interval
15 applet, then there is a long pause*]. I, um, I think, you know, I think you’re
16 right that they’re demonstrating the same idea. Uh, I think it’s just this one
17 is throwing me off [*she points to the slider labeled “1-unit growth factor”
18 and then moves it back and forth*]. So this would be like, this one, the unit
19 ruler [*she brings back up the unit ruler applet*] here [*she returns to the
20 sliding interval applet, points at the slider “1-unit growth factor” and then
21 at the vertical segment on the right*] I believe, and so, I remember you
22 guys saying that like if the unit ruler got smaller, then, it would get, like
23 this would get bigger [*she points her pen back and forth between the tops
24 of the two vertical segments*].
- 25 AO: So which, what’s the unit ruler in this diagram? Like does the unit ruler
26 appear someplace?
- 27 Shelby: Um. I don’t know. I fe-, I wanna say that the unit ruler’s this [*she points to
28 the vertical segment on the right*], but that doesn’t seem right. It might be
29 [*she points to the horizontal vector*], it might be this because we aren’t
30 able to vary the unit ruler [*she returns to the unit ruler applet*]
- 31 AO: Mm-hmm [*yes*].
- 32 Shelby: we’re only able to vary the, the length to be measured. So this is probably
33 the unit ruler [*she points to the horizontal vector*] cuz it gives us the one-
34 unit growth factor, and then this [*she moves the slider “1-unit growth*

35 *factor*”] is what can be changed, uh, this is what we’re measuring, this
36 [*she moves her pen back and forth between the tops of the two vertical*
37 *segments*] is what we’re measuring.

When I asked Shelby if the two applets demonstrated similar ideas, she reacted as if she had not considered this possibility (lines 6-16). This response is plausible considering that she wasn’t aware that the length of the horizontal vector could be varied independently using the applet controls (lines 28-33), and then concluded that the horizontal vector represented the unit ruler since its length didn’t vary. Shelby also said, “I remember you guys saying that like if the unit ruler got smaller, then, it would get, like this would get bigger [*she points her pen back and forth between the tops of the two vertical segments*]” (lines 21-24). I suspect she is referring to efforts to support the idea that when the measurement unit size varies the measurement value varies in an inversely proportional manner [MCM-2]. However, it is not clear how she sees that idea as helpful in this context, and I regret that I did not follow up with her. Shelby’s descriptions of her conceptualization of the ideas I intended to convey with the Sliding Interval applet reflect her consistent lack of attention to reference quantities and the meanings for the calculations she performed throughout the post-interview.

Persistent guessing. When Gina, Marcus, and Shelby incorrectly responded to lesson tasks they tended to engage in persistent guessing. When Shelby and Marcus initially answered a task incorrectly they required seven or more attempts to provide an acceptable answer about 30% of the time. For Gina the figure was about 20%, with another 15% of such questions never correctly completed after an average of 7.8 attempts. Several observations suggest that many of these high attempt counts resulted from repeated guessing. Two examples follow.

I used screen-capture to record Marcus’s attempt to respond to the task in Figure 9.35. This task is used in a lesson comparing linear and exponential behavior with the expectation that comparing and contrasting these relationships could bring each idea into sharper focus. There are several reasonable ways to complete this task, but part (b) is intended to assess whether students recognized that a one-unit growth factor b is the ratio of two instances of the same quantity [MCEF-1 and MCEF-3] and whether they see $g(0)$ as the value such that $g(0) \cdot b = 1.5$ so that they can generate an algebraic representation using this information [MCEF-5].

Two ordered pairs for a function are given below. Notice that as you enter answers to the tasks below, your answers will appear in the graph.

a. If the function containing these ordered pairs is a linear function, what is the formula defining the relationship?

$f(x) =$

b. If the function containing these ordered pairs is an exponential function, what is the formula defining the relationship?

$g(x) =$

Question 4. Points possible: 2
 Unlimited attempts.
[Message instructor about this question](#)
[Post this question to forum](#)

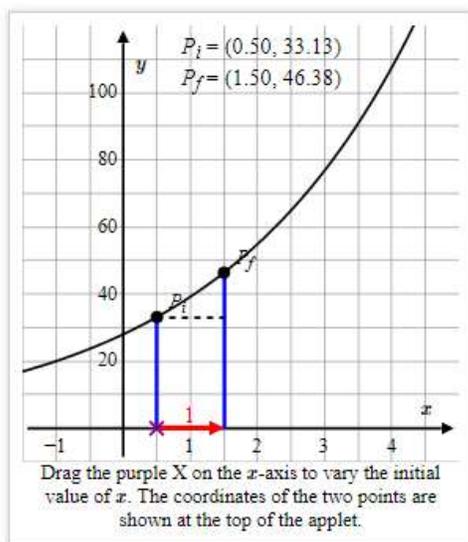
Question ID: 2094
 License

Figure 9.35. A task asking students to define two functions with the given ordered pair solutions. Note that in this task the student can see the graphs of the functions they define, thus they are able to predict a correct answer before submitting. This resulted in attempt counts that did not necessarily reflect students’ actual number of attempts.

Marcus defined 13 different linear functions before producing a correct response, and his final answer was “ $f(x) = 2.25x + 1.5 - 2.25$ ”, which he arrived at by noticing that the line shifted when he added or subtracted a value to the end of his formula. He went through several iterations of testing different numbers until he found one that produced a graph passing through the two points. He then defined almost 20 different exponential functions using various combinations of the values in the ordered pairs, operations, and the variable x (including adding and subtracting constants to the end of the formula as he did in part (a)) before giving up and moving on to the next task. Moreover, each attempt occurred in relatively rapid succession, suggesting that Marcus was not taking the time to reflect on his reasoning and was instead essentially guessing and hoping to land on an acceptable formula. Prior to attempting this task Marcus watched one of the short videos in the lesson. During his numerous attempts to complete the task, however, he did not do any scratch work, nor did he leave the page and review past lessons or homework assignments.

Figure 9.36 shows a summary of Shelby’s work on a task at the beginning of lesson five of seven. This task was intended as a review of students’ work in previous lessons in preparation for building on the main unit ideas to develop a general algebraic formula to model exponential functions (that is, this was not developing or assessing new content in that lesson). Shelby correctly answered five of the six parts on her first attempt (as seen in the “Previous Attempts” section at the bottom of Figure 9.36). Her only mistake was to enter “140” instead of “40” as the one-unit percent change. Shelby then produced *19 additional attempts* changing only this answer before finally arriving at the correct response (40%). Her attempts, in order, were 140, 240, 1.40, 139, 239, 299, 399,

The following graph and table represent a function f where y varies exponentially with respect to x .



x	$y = f(x)$
-1	20
0	28
1	39.2
2	54.88

a. What is the value of $f(2)$?

$f(2) =$

b. What is the value of $f(3)$?

$f(3) =$

c. What is the 1-unit growth factor for this function?

d. Whenever x changes by 1, the function output becomes *what percent* of its previous value?

e. What is the 1-unit percent *change* for this function?

%

f. Using the applet, estimate the value of x such that $f(x) = 70$.

$x \approx$

Question 1: out of 6 in 20 attempt(s) [Add Feedback](#)

Quick grade: [Full credit](#)

Previous Attempts: 1: 54.88; 76.832; 1.4; 140; 140; 2.74 2: 54.88; 76.832; 1.4; 140; 240; 2.74 3: 54.88; 76.832; 1.4; 140; 140; 2.74 4: 54.88; 76.832; 1.4; 140; 139; 2.74 5: 54.88; 76.832; 1.4; 140; 239; 2.74 6: 54.88; 76.832; 1.4; 140; 299; 2.74 7: 54.88; 76.832; 1.4; 140; 399; 2.74 8: 54.88; 76.832; 1.4; 140; 499; 2.74 9: 54.88; 76.832; 1.4; 140; 199; 2.74 10: 54.88; 76.832; 1.4; 140; 240; 2.74 11: 54.88; 76.832; 1.4; 140; 1.4; 2.74 12: 54.88; 76.832; 1.4; 140; 39.9; 2.74 13: 54.88; 76.832; 1.4; 140; 29.9; 2.74 14: 54.88; 76.832; 1.4; 140; 49.9; 2.74 15: 54.88; 76.832; 1.4; 140; 2; 2.74 16: 54.88; 76.832; 1.4; 140; 200; 2.74 17: 54.88; 76.832; 1.4; 140; 100; 2.74 18: 54.88; 76.832; 1.4; 140; 140; 2.74 19: 54.88; 76.832; 1.4; 140; 14; 2.74

Average time per submission: 38.6 seconds

[Use in Msg](#) (Question ID: 2058)

Figure 9.36. Shelby's answers to a review task at the beginning of lesson five.

499, 199, 240, 1.4, 39.9, 29.9, 49.9, 2, 200, 100, 140, 14 and finally 40. Examining her response patterns reveals an apparent tactic: if she gets the percentage value incorrect then the answer is likely to be wrong because she forgot to adjust the answer by 100 or needed to move the decimal place.

These two examples suggest that these students interpreted correct responses as the goal of lessons as opposed to developing certain meanings. When the students produced incorrect answers, persistent guessing was an acceptable technique to eventually earn credit for the task. Students *could* respond to incorrect answers by reviewing the lesson, reviewing past lessons, drawing diagrams to work out their reasoning, or engage in other behaviors. That these students did not choose these alternative behaviors means that the automatic feedback provided in the online environment did not appear to prompt them to reflect on their reasoning (at least not in many circumstances) but instead allowed, perhaps even encouraged, repeated guessing. After all, it is quite possible that it took Shelby less time to produce 20 solution attempts through guessing than by taking the time to review prior lessons or rework prior tasks.

Attention to reference quantities and a meaning for growth factor, percentages, and percent change. Gina, Marcus, and Shelby all continued to display some of the same challenges seen in their pre-interviews relative to reasoning about percentages and percent change. They routinely struggled to recognize what quantity was being referenced and how it was being measured and, in many cases, persisted in their behavior of trying various combinations of calculations involving given numbers to produce possible answers as opposed to first articulating what they intended to calculate or

represent. Gina’s work on the Muffler and Sale Price tasks in the post-interview demonstrate implications of a lack of attention to quantitative meanings for operations and expressions.

Gina began the Muffler task by writing and evaluating the difference “ $195 - 131 = 64$ ” (she later added “\$” in front of 64 when I asked her what she had calculated). She then stated, “I kinda remember how to do this” and used a calculator to perform the following calculations in order.⁵⁵

- $64/195 = 0.328$
- $\text{ans} * 100 = 32.8$
- $195/64 = 3.047$
- $\text{ans} * 100 = 304.7$
- $195/131 = 1.489$

Note that all of these calculations are either quotients using pairs of values from the difference statement she wrote or a product of one of these quotients and 100. After performing the entire set of calculations and reviewing them, she declared that the price increase was 32.8%. I asked her what that value she was measuring and to what it was being compared. This question prompted her to calculate $131/195 = 0.672$, at which point she changed her answer to an increase of 67%. I asked her, “67 percent of what?” to which she responded “131”. For reference, Table 9.13 shows post-test results for all students. Recall that interview responses shown in these results represent students’ final

⁵⁵ Gina used an on-screen calculator that displayed a history of her previous calculations and thus did not write down any of these statements. The rounding here is mine.

answers prior to any intervention by the interviewer that may have caused them to change their solution.

Table 9.13

Student Responses to the Muffler Task (Post)

answer choice	Intervention Class (Post)	<i>Pathways</i> In-Person Class (Post)	Gina (Post-test)	Marcus (Post-test)	Shelby (Post-test)	John (Post-test)	Lisa (Post-test)
a. 33%	16%	16%	I				
b. 49% **	62% (+17%)	47% (+18%)		●, I		●, I	●, I
c. 64%	2%	5%					
d. 67%	7%	14%					
e. 149%	14%	19%	●		●, I		

Gina’s work on the sale price task was very similar. She began by writing 20% and \$440.00 then said that she needed to convert 20% to 80% “since you subtracted 20 percent of the like initial price to get the sale price, like in order to get it back [...] to the same price I would multiply by point eight.” She then performed the following series of calculations in order.

- $440 \times 0.8 = 352$
- $440 / 0.8 = 550$
- $440 / 0.2 = 2200$
- $440 \times 0.2 = 88$

After examining these calculations, she stated that \$352 was deducted from the initial price to get the sale price, so the initial price was $\$440 + \$352 = \$792$. Table 9.14 shows that Gina answered this question differently on the post-test.

Table 9.14

Student Responses to the Sale Price Task (Post)

answer choice	Intervention Class (Post)	<i>Pathways</i> In-Person Class (Post)	Gina (Post-test)	Marcus (Post-test)	Shelby (Post-test)	John (Post-test)	Lisa (Post-test)
a. \$352.00	9%	19%					
b. \$366.67	2%	0%					
c. \$528.00	19%	41%					
d. \$550.00 **	69% (+31%)	35% (+9%)	•	•, I	•, I	•, I	•, I
e. \$2200.00	2%	5%					
\$792.00			I				

While solving these tasks, Gina never drew a diagram or referred to any of the visualizations from the unit lessons. She also never justified her calculations as she performed them. It was clear that her primary solution method involved performing operations with as many combinations of values given in the problem or that she had identified as potentially important and deciding which result seemed most reasonable (although her criteria for deciding this was not always revealed and seemed to change quickly if I asked her follow-up questions). Considering that she performed the calculation $131/195 = 0.672$ and reported this result as a 67% increase of 131, I have little confidence that she could explain the result of most calculations she performed in these contexts in terms of what was being measured and the reference unit for the measurement [MCM-1 and MCM-3, or MCP-2 for understanding the role of multiplying by 100 to produce a percentage comparison from a relative size comparison, or MCPC-1 for justifying percent change calculations]. Note that after she provided her solutions, I asked her if she could draw diagrams to represent the relationships between the prices in each problem and how the percentage values and percent change values were related. I

interpreted her reaction and the ensuing discussion as indicating that she did not routinely draw diagrams like those displayed in the module lessons while working on her own. This could be because she did not understand what they were trying to convey, she did not find them useful, or perhaps she found persistent guessing a quicker method of eventually getting a correct answer to most tasks.

Shelby claimed that the repair cost in the Muffler task increased by “about 150%” (from \$131 to \$195), but immediately second-guessed herself and said that the answer might be “flipped”. She thus tested the ratio $131 / 195 = 0.67$. When increasing 131 by 67% did not yield 195 she returned to her original answer. Her work is shown in Figure 9.37.

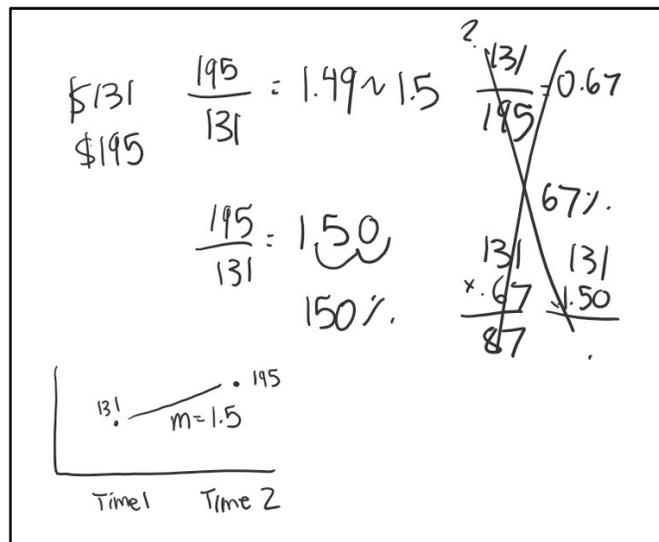


Figure 9.37. Shelby’s written work on the Muffler task in the post-interview.

It was very common for Shelby and Marcus to choose answers that were “off” by 100 on course exams and in the post-interview because they tended not to carefully distinguish between a percentage as a measure of relative size [MCP-1] and a percent change as a measurement of relative size for the difference in two values [MCPC-1]. I wanted to see

if Shelby would recognize her error if she drew a diagram, so I asked her to create a diagram to demonstrate what 150% represented in this context. She produced the graph shown in Figure 9.37 and said that the slope of the line was 1.5. This response indicates a lack of productive meanings for percentage and percent change as measurements but also a lack of productive meaning for slope (constant rate of change). I suspect they both derive from her inattention to the meanings of her calculations in terms of producing the values of quantities she has conceptualized and intends to represent.

Time ran short on my interviews with both Gina and Shelby and I was unable to ask them directly about how they would define “exponential function” in their post-interviews. However, I did present the graph in Figure 9.38 to Marcus and asked him how he would respond to a student who said something like the following. “This graph shows that the U.S. national debt increased exponentially from 2000 to 2016 because it was increasing really, really fast.”

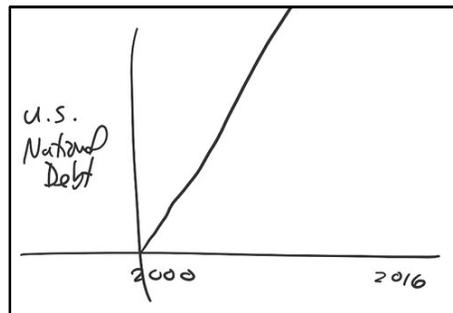


Figure 9.38. A fictitious student example to assess Marcus’s meaning for exponential growth.

- 1 Marcus: Um, so I mean for that, that problem, um, I would say that it is looking for
- 2 exponential growth, um just because that number is getting higher and
- 3 higher and higher. It’s moreso how the way you look at it. If you were to
- 4 look at it in a decaying factor like the, like the increase in price but you
- 5 were using negatives,
- 6 AO: Mm-hmm [yes].
- 7 Marcus: instead of like the positives, of like it getting bigger and bigger, using the

- 8 kind of like the percent change, moreso um, that graph, graph would be
9 downward, and I would say that would be decay.
- 10 AO: Okay.
- 11 Marcus: But for this, I would say that's exponential growth.
- 12 AO: So their description is fine? Their, their use of the term exponential is fine
13 in that case?
- 14 Marcus: I would say yes.
- 15 AO: Okay.
- 16 Marcus: For that problem, most definitely.

This exchange provides further evidence of Marcus' impoverished conception of exponential growth. He does not appear to recognize the need to explore how the quantities being represented by the graph are related and change together as a way of classifying a function as exponential or not. Admittedly there were no numbers on the axes, but the fact that he did not see that specific values or comparisons were needed suggests that his general meaning for exponential functions, at least when represented graphically, is tied to "increasing".

Emergent symbol meaning. Students who engage in emergent symbol meaning have an expectation that the symbols they write and the calculations they represent on paper are communicating relationships between quantities' values. They see each part of an expression as representing values of new quantities in relation to the values of other quantities. Their expectation that symbols and expressions have quantitative referents also allows them to look at given calculations, expressions, and formulas and expect that they can make sense of the relationships between quantities thus represented.

These behaviors and expectations were not generally present during the pre-interviews with incoming Precalculus students. Instead, students exhibited behaviors such as associating addition with any mention of an increase in value and appearing to perform literal transcriptions of task goals into symbolic representations without attending to the

quantities' values represented by the expressions they wrote. Students employed techniques like circling all variables and numbers explicitly given in a problem statement and then generating calculations or models using only those values and variables with some combination of operations. Thus, they often failed to include necessary values in their models that were not explicitly given in the problem statement. The combination of these ways of reasoning explain why most students wrote $h = 7 + 0.13t$ to model the tomato plant height in terms of elapsed time in weeks for the Tomato Plant B task at the beginning of the course.

For the Tomato Plant B task on the post-test, 23% of students in the intervention class chose models consistent with associating “increasing” with addition and/or choosing a model that could result from a literal translation of the task goal. See Table 9.15, answer choices (b) and (e).

Table 9.15

Student Responses to the Tomato Plant B Task (Post)

answer choice	Intervention Class (Post)	Pathways In-Person Class (Post)	Gina (Post-test)	Marcus (Post-test)	Shelby (Post-test)	John (Post-test)	Lisa (Post-test)
a. $h = 7(0.13)^t$	10%	10%					
b. $h = 7 + 1.13t$	9%	17%	I	* ⁵⁶			
c. $h = 7(1.13t)$	9%	4%					
d. $h = 7(1.13)^t$ **	59% (+39%)	60% (+36%)	●	●	●	●	●
e. $h = 7 + 0.13t$	14%	9%			* ⁵⁶		

Student interviews confirmed that these meanings remained strong among some students for the Tomato Plant B task (although less so in the Bacteria Doubling task). Shelby first wrote “ $f(t) = 7 + 0.13t$ ” after reading the problem statement, but then said, “Oh, wait, this

⁵⁶ These represent the students’ initial responses during the interview. Both students recognized the error on their own and changed their answers to match their response on the post-test, but their initial models are worth noting. I discuss this in more detail later in this results section.

is exponential” and changed her response. Her next response was to write “ $f(t) = 7(0.13)^t$ ”, updating it to “ $f(t) = 7(1.13)^t$ ” only after evaluating $f(2)$ and recognizing that a height of 0.1183 inches was not a reasonable value. Marcus initially wrote “ $h = 7 + 1.13t$ ” and explained that $1.13t$ represented “thirteen percent of the already made seven-inch tomato plant.” As soon as he said this, however, he changed his answer to a correct algebraic model. While these solutions may demonstrate some progress for Marcus and Shelby relative to recognizing the quantitative meaning for expressions and formulas they write, the fact that both students initially used linear models for their answers even after completing the entire unit on exponential growth supports how deeply imbedded these tendencies and meanings are for many students. I would be interested to interview many students from this course and see if the models they produce would, over time, revert to one of these linear forms.

Gina’s responses to the Tomato Plant A and B tasks and the Bacteria Doubling task showed that her meanings for exponential growth and her expectation for the goal of generating mathematical models did not noticeably change during the course. She continued to produce models consistent with interpreting an increase as requiring addition and, as long as her models showed that the dependent values increased as the independent value increased, she believed that her models were reasonable. She did not consider the meaning of her intermediate calculations nor did she discuss growth factors or percent comparisons as measurements with specific reference quantities within her solutions. The following excerpt is taken from her solution to the Tomato Plant B task and shows her producing and justifying $h = 7 + 1.13t$ as a model for the plant’s height in terms of the number of elapsed weeks.

1 Gina: [She draws a vertical segment and writes “7 in” next to it as she reads the
 2 problem statement.] So, it’d be [pause]. Since it’s asking for the height
 3 [she writes “h=” and underlines “h” in the problem statement] and like in
 4 terms of t [she underlines “t” in the problem statement] of weeks since it
 5 was already seven inches [she writes “7” after h] when it were first
 6 planted I’m just gonna go ahead and add the seven. And then if it’s
 7 growing, so it’d be one point, and if you make it into a decimal it’d be [she
 8 writes “+ 1.13(t)” after h = 7]. So, if I wanna, it’d give me, [she uses a
 9 calculator to evaluate “1.13x0”] yeah, okay. I don’t know, yeah it would
 10 be, I feel like this would be, it’s growing at like the, like this is the rate, or
 11 like it’s growing by 13 percent. Um, one point one three, cuz like I said if
 12 you add one point one three, and then, um, like it’s asking like for
 13 whatever week [she underlines “t” in her formula] like this week, say if it
 14 was like two weeks, um, and then it’s [she writes “(1.13)(2)”], so like, this
 15 is the amount that it grew. So [she uses a calculator to evaluate “1.13x2”]
 16 it grew, in two weeks it grew two point two six [she writes “2.26 in”]
 17 inches, but since it was already seven inches when it like first got planted
 18 [she uses a calculator to evaluate “2.26 + 7”] his total height [she writes
 19 “h = 7 +” in front of 2.26 in] would be nine point two six inches. So this
 20 would be the function [she draws a box around “h=7+1.13(t)”] that I
 21 would use to find like the total of his height of whatever week they give
 22 me.

$$h = 7 + 1.13(t)$$

$$(1.13)(2)$$

$$h = 7 + 2.26 \text{ in}$$

$$9.26 \text{ in}$$

Figure 9.39. Written work for Gina’s response to the Tomato Plant B task in the post-interview.

In the Tomato Plant A task (not shown here), Gina followed a similar process that was equivalent to developing and evaluating the model $h = 4 + 1.5t$. Table 9.16 shows Gina’s and others’ responses to this task in both the post-test and post-interview.

Table 9.16

Student Responses to the Tomato Plant A Task (Post)

answer choice	Intervention Class (Post)	Pathways In-Person Class (Post)	Gina (Post-test)	Marcus (Post-test)	Shelby (Post-test)	John (Post-test)	Lisa (Post-test)
a. 5 inches	2%	4%					
b. 6 inches	3%	3%					
c. 8 inches	12%	34%					
d. 9 inches **	83% (+23%)	57% (+6%)	•	•, I	•, I	•, I	•, I
e. 12 inches	0%	1%					
7 inches			I				

In justifying her interview answer she said she had to use 1.5 because if she used 0.5 that would indicate a decrease. While this is true for growth factors in exponential models, she was essentially claiming that $h = 4 + 0.5t$ is a decreasing function. In the Bacteria Doubling task Gina created the pattern “14, 28, 56” after reading the problem but attempted to model the number of bacteria using the expression “ $14 + 2t$ ”. However, she tested this model and recognized that it did not produce the expected results. She subsequently gave up on the task. See Table 9.17.

Table 9.17

Student Responses to the Bacteria Doubling Task (Post)

answer choice	Intervention Class (Post)	Pathways In-Person Class (Post)	Gina (Post-test)	Marcus (Post-test)	Shelby (Post-test)	John (Post-test)	Lisa (Post-test)
a. $p = 14 + 2t$	5%	6%					
b. $p = 28t$	0%	0%					
c. $p = 14(1.2)^t$	3%	6%					
d. $p = 14^{2t}$	9%	17%					
e. $p = 14(2)^t$ **	83% (+23%)	71% (+29%)	•	•, I	•	•, I	•, I
No final answer			I				
$f(t) = 14(1.5)^t$					I		

Throughout the post-interview Gina displayed little awareness of how the expressions that comprise her model may or may not represent valid relationships between quantities in the situation.

Summary. Gina, Marcus, and Shelby displayed some variation in their progress relative to the meanings I intended students to conceptualize. However, looking at commonalities among these three students help clarify a few key findings. The unit did support these students in making significant gains on the post-test relative to recognizing correct solutions to tasks involving modeling exponential functions and reasoning with percent change. However, these assessment results did not reveal some persistent challenges. Gina, Marcus, and Shelby all continued to struggle to varying degrees in clearly identify reference quantities for making percentage and percent change measurements, and their conceptualizing addition as the operation needed to represent any increase in a quantity's value interfered with their ability to produce accurate models for exponential relationships. In addition, a lack of expectation that various parts of algebraic models represented specific quantities' values and relationships between quantities made it difficult for the students to consistently reproduce their responses from the multiple-choice post-test. This also perhaps contributed to high attempt counts and persistent guessing on lesson tasks since the students viewed accurate models more in terms of producing acceptable answers as opposed to representing relationships between quantities they conceptualized within a quantitative structure.

The case of John and Lisa: positive shifts relative to key learning goals. John and Lisa also had strong post-assessment scores, correctly answering all post-test items covering exponential growth and related ideas, averaging of 23.5/25 on the entire post-

test, and both earning an A on the final exam. However, their post-interviews show that they made greater shifts relative to targeted meanings and they displayed common behaviors and meanings including the following. (1) They confidently manipulated applets used in the unit and were able to explain key connections between the applets as well as how those applets could be used to demonstrate mathematical ideas; (2) they had lower attempt counts on lesson tasks and more frequently completed tasks correctly on the first attempt; (3) they tended to identify clear referents when working with percentages and percent change when asked and had relatively productive meanings for growth factor and exponential function; and (4) they were generally able to describe their calculations, expressions, and formulas in terms of quantities and quantitative relationships.

How did students conceptualize key visualizations and applets? Both John and Lisa had little trouble manipulating the Unit Ruler and Sliding Interval applets to demonstrate key ideas, and several exchanges throughout the post-interview suggested that they found the visualizations useful in their reasoning.

I showed John the unit ruler applet and, after exploring it briefly, he explained how he could measure the length of the bottom segment as a percentage of the unit ruler length regardless of whether the length to be measured was longer, shorter, or the same length as the unit ruler [MCM-1].⁵⁷ For each case he adjusted the applet to demonstrate the scenario and described the relationships without emphasizing specific values. He then

⁵⁷ While John said “percentage” here, I saw no evidence in the post-interview that either John or Lisa thought about percentage measurements as using a different “unit ruler” than direct relative size measurements. My current hypothesis is that, despite developing more productive meanings for several mathematical ideas, both John and Lisa still think of a ratio value and percentage comparison as essentially the same except that they move the decimal to report one or the other.

described how the length to be measured will always be the same percentage of the unit ruler as he moved the “co-vary the quantities” slider [MCM-4]. I next asked John how this applet helps us understand exponential functions without showing him the Sliding Interval applet.

1 John: So, any change, your previous one is always, so your y one, say your unit
2 ruler, cuz that’s what you’re measuring your y^2 off of, it’s always gonna
3 be the same percentage of your second one, or vice versa. Um. So, even if
4 you start out in like the beginning of your graph and you’ve got little baby
5 measurements [*he moves the “co-vary the quantities” slider until both*
6 *lengths are very short*] to further on [*he moves the “co-vary the*
7 *quantities” slider until both segments are near their maximums lengths*]
8 it’s, if you divide them it’s always gonna be the same.

Although John said, “if you divide them, it’s always gonna be the same,” (line 8) he did not have numerical measurements of any lengths in this applet. I interpreted his invoking division here as a substitute for a mental image of making a relative size comparison between two instances of a quantity (lines 1-2) [MCM-3] and not just thinking about performing a calculation. Throughout the post-interview he was able to estimate the values of growth factors even without values and explained how he could correctly reinterpret and represent the Muffler task as a percent decrease from the higher cost if that was requested. It is also worth emphasizing that John described how the relative size of the two instances of a quantity can hold as that quantity varies while also invoking variation in a second quantity (lines 3-8) [MCEF-2]. He seemed to be imagining the Sliding Interval applet even while working with the Unit Ruler applet.

Instead of displaying the Sliding Interval applet next, I instead presented Figure 9.40 showing three different representations for the same exponential function.

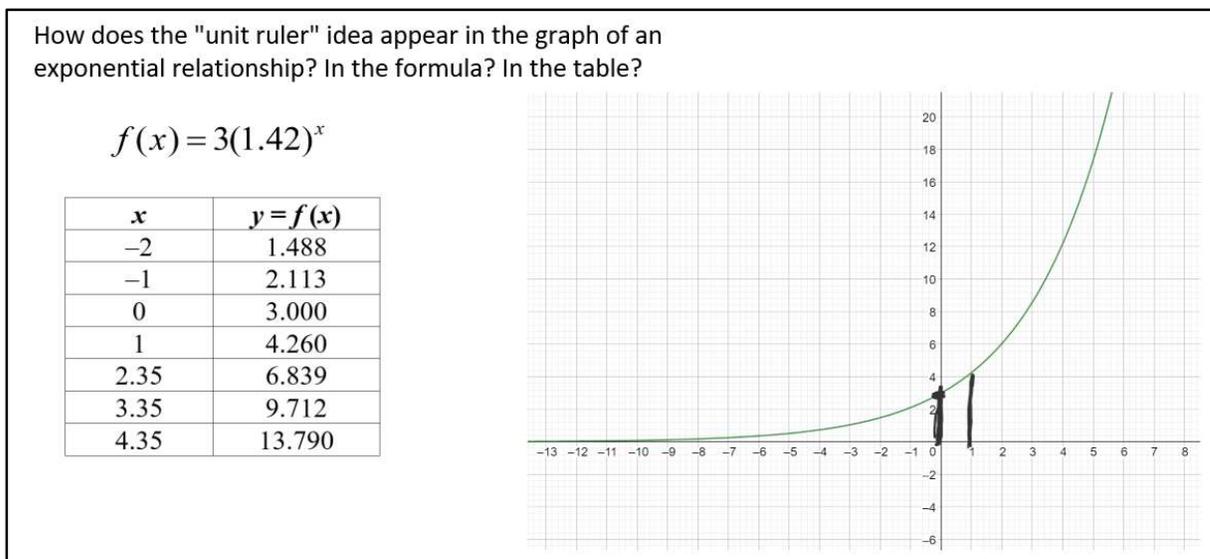


Figure 9.40. Three representations of an exponential function including John’s annotation as he discussed them.

I asked John how the ideas he just described show up in each of the representations.

Since other students had struggled to describe a meaning for the unit ruler in various representations, I wanted to see how John connected his explanations in the Unit Ruler applet to these “static” representations and if he would spontaneously invoke images of co-variation.

9 John: So, say if this is like my unit ruler [*he draws a vertical segment from (0,0)*
 10 *to (0,3)*], so I just start at zero. Um, and you wanna look at your one [*he*
 11 *draws a vertical segment from (1,0) to (1, 4)*], for, if, so like if you’re
 12 sliding to covary, these are always gonna be the same percentage if they, if
 13 you keep moving them back and forth [*he indicates that he imagines*
 14 *sliding the one-unit interval*], this is always gonna be, um, four point two
 15 six divided by three times as large as your initial one, whatever point you
 16 are on the graph.

John’s response is significant for three reasons. First, he created an interval and represented two dependent function values as magnitudes (lines 9-11), then described sliding the interval and attending to what remained invariant (lines 11- 13) [MCEF-1 and MCEF-2]. Second, he seamlessly connected the table of values and graphical

representations and described his one-unit growth factor as $4.36/3$ (lines 14-15) without needing to evaluate this ratio with a calculator. I thus interpreted that he was imagining this as the value for the measurement he had in mind and not just a calculation. Third, he described this value as the one-unit growth factor “whatever point you are on the graph” (lines 15-16). In other words, he did not anticipate needing to recalculate the growth factor as the function values updated. He was confident the expression $4.36/3$ was the one-unit growth factor and was constant even as the quantities’ values changed. In his initial answer he did not mention the algebraic representation, so I asked him how he sees the unit ruler idea represented in the formula.

17 John: So, it’s just the one point four two is, oh, you have a hundred percent of
18 your initial, so like your unit ruler, and then your length is forty-two
19 percent of your unit ruler added to a hundred percent of your unit ruler and
20 then your three is just your reference, where you’re starting.

John described the meaning of the various components of the algebraic formula [MCEF-3], although again his exact meaning of “percentage” is not clear.

Lisa’s explanations for the ideas conveyed by the applets were quite similar to John’s so I will not include long segments of her interview. Two brief excerpts are worth sharing because they highlight something that Lisa and John discussed that differed from Marcus’s and Shelby’s responses. In this first part she is interacting with the Sliding Interval applet.

1 Lisa: Yeah, so like on every single graph, like this obviously is different from
2 this [*she moves the “1-unit growth factor” slider and leaves it at a value*
3 *of about 1.1 or 1.2*]
4 AO: Yeah. But whenever you lock that one in-
5 Lisa: Yeah, whenever I lock that one in, that’s always gonna be the same
6 growth factor [*she is moving the “sliding interval” slider as she talks*].

The data (lines 1-6) supports that Lisa recognized that changing the “1-unit growth factor” slider creates different relationships, but that locking in that value indicates what remains invariant as the independent and dependent values vary [MCEF-2]. In the next part Lisa is interacting with the Unit Ruler applet.

7 Lisa: [*She moves the “co-vary the quantities” slider*].] So it always just, it just
8 depends on like what fraction, like if you know that your measurement is
9 point seven five of the unit ruler, but your unit ruler is you know this long
10 [*she moves the “co-vary the quantities” slider to increase the length of*
11 *both segment*], then it’s always gonna be point seven five, but if your unit
12 ruler is this long [*she move the “co-vary the quantities” slider to decrease*
13 *the length of both segments*] it’s gonna look a lot closer. So it’s like the
14 same thing just in a different format. Just like if it’s double [*she moves the*
15 *“length to be measured” applet so that the bottom segment is twice the*
16 *length of the unit ruler segment*] [...] If you know it’s double, and, of the
17 unit ruler, and the unit ruler gets longer, it’s gonna be double that, so it’s
18 gonna look a lot bigger than if it was a small thing. Just like on the graph.

In lines 7-9 she made a similar acknowledgement in the case of the Unit Ruler applet, noting that she could fix the relative size and that this will remain constant as the two lengths change in tandem [MCM-4]. She then explained how this applied to two specific examples (lines 11-13 and lines 14-18) where she not only talked about this relative size but discussed the implications for how the difference in values appears different as the lengths vary even though the relative size is constant [MCPC-4].

While Shelby and Marcus struggled to describe mathematical ideas that the applets might convey unless they locked in single examples, both John and Lisa spontaneously discussed growth factors and relative size comparisons either in general terms or using a variety of examples to demonstrate different possible circumstances.

This suggests that John and Lisa may have connected the ideas across various tasks to

develop more generalized meanings for ideas like “growth factor” while Marcus and Shelby focused more on applying solution methods to single problems.

I asked John and Lisa the question shown in Figure 9.41 to assess the flexibility in their meanings for growth factor and percent change over different-sized intervals. Their responses are given.

Increasing exponential functions have a constant doubling time based on the 1-unit percent change. For example, if the price of milk increases by 3% per year, then its price doubles every 23.5 years.

Why does it make sense that a given exponential function has a constant doubling time no matter what specific input values we are considering?

Figure 9.41. A question that assesses flexibility and connections between growth factors over different-sized intervals.

- 1 Lisa: I mean I guess to me it makes sense just because um, just like you have
2 like the unit rulers or what not, where you talk about like the growth
3 factor, how you can take it at one per every year or you can max out. You
4 can say like every four years this is always going to be the factor. So I
5 guess it's just making that large scale, to me, is like every, you know that
6 it's gonna be doubled every 23 point five years, so it's just taking the
7 original unit measurement of like one year and just making it incredibly
8 large scale. Um, and just like it wouldn't matter where on the graph you
9 would put like negative one to zero, or five to six, you know that's always
10 going to be the same, so it's the same idea just on a larger scale.

Lisa expressed an understanding that the independent interval size under consideration does not matter for the general property of a constant relative size between two dependent values to remain constant (lines 3-4) [MCEF-3], and then described how this applied to the given scenario based on using the given time interval (23.5 years) and the relative size (2) (lines 6-8). She also describes her understanding of how these relationships hold throughout the function's domain (lines 8-10) [MCEF-2]. John described many of the same meanings as Lisa. John's description, however, included more detail about how an

increase of 3% per year eventually will cause the value to double (lines 1-5 in the following excerpt).

- 1 John: Um, because if you have a specific um percent change per year, or per
2 whatever your um unit is, um, eventually after so many changes in x it's
3 gonna be, um, double whatever you started with or double whatever
4 you're referencing as your starting point, um, because it's always, it's a
5 constant change per year.
- 6 AO: Okay. So why would it, why would I expect, as I start to slide that
7 interval, the twenty-three point five year interval,
- 8 John: Mm-hmm [*yes*].
- 9 AO: I start to slide it, why should I expect that the doubling relationship should
10 be maintained?
- 11 John: Because the ratio after applying three percent for twenty-three point five
12 years, per year, um and if you calculated that that is for any point, um, it's
13 gonna be double whatever your initial is, the percent, the ratio's gonna
14 stay the same. It's the same as if, finding a ratio between just one change
15 in x , once you find, instead of it being a three percent change it's then a,
16 uh, a hundred percent change. Yeah. [*laughs*] Then you can find that
17 across any, any points on the graph.
- 18 AO: Okay.

Both John and Lisa demonstrated flexibility in thinking about the interval size over which comparisons in function values are made. They each acknowledged that if a quantity grows by three percent per year that it will grow by a different percentage over longer periods of time, and that whatever that growth factor is (including two) would apply over any equal length of time elapsed [MCEF-3]. This suggests that the sliding interval applet and similar visualizations in the module was a very useful tool for supporting John and Lisa in developing flexible and productive meanings for growth factors. Not only could they reason with and explain one-unit growth factors as measurements, but they generalized this reasoning to intervals of any size and at any point in the function's domain.

Implications of Low Attempt Counts. Lisa and John more often answered questions correctly on the first attempt and tended to have lower numbers of attempts when their responses were incorrect. While I had clear evidence that Gina, Marcus, and Shelby tended to engage in persistence guessing, the data suggests that John and Lisa did not utilize this tactic, at least not to the same extent. I think much more research and a larger sample size is needed to make stronger claims about the sources of lower attempt counts, such as making claims about whether John and Lisa responded to system feedback on incorrect attempts by reflecting on their work and perhaps reconceptualizing the question and context. However, the data supports that John and Lisa did construct more productive meanings (both compared to their pre-course meanings and compared to the other students I interviewed) for many ideas targeted in instruction, and that those meanings appeared useful in helping them more efficiently and successfully complete a range of tasks. This is likely connected to their tendency to have lower attempt counts, but I do not have enough data to definitively explain the connection.

Attention to reference quantities and a meaning for growth factor, percentages, and percent change. During the post-interview, John and Lisa both consistently held in mind the reference quantities for relative size comparisons, percentages, and percent change (that is, the answer to the questions, “Compared to what?” or “Percent of what?”).⁵⁸ Lisa’s solution to the Tomato Plant B task is shown, which is similar to John’s response to the same task. This excerpt picks up after she wrote down the solution “ $h = 7(1.13')$ ” and I asked her to justify her answer.

⁵⁸ Again, the only exception is that they do not often differentiate percentages as measurements with a different unit compared to relative size comparisons between quantities. They appear to use the ideas interchangeably without ever mentioning a reason for moving the decimal place.

- 1 Lisa: Every time I think of like, every time I get these kind of questions, I
 2 literally just view it as the little like plant or bar, that's not a tomato, but
 3 it's gonna work [*she draws a vertical segment with a flower on top and*
 4 *writes "7" next to it between brackets that extend to the top and bottom of*
 5 *the figure*].
- 6 AO: That's fine.
- 7 Lisa: Alright. [*laughs*] Um, like I view it as just like a little week by week
 8 growth. Just these lines, like [*she draws two more similar figures to the*
 9 *right, each one taller than the previous figure*], so I know, one, that it's
 10 never gonna get smaller cuz it grows. Dit-, it hasn't wilted yet. But, um,
 11 yeah, and that these are gonna get like exponentially larger [*emphasis*
 12 *hers*], so like eventually it's gonna be like this [*she draws a fourth figure*
 13 *that is taller than the previous three*], so it's not gonna be the equal
 14 distance, so it's just gonna get like really long [*emphasis hers*] every
 15 single time [*she draws a fifth figure that is taller than the previous four*
 16 *figures. See Figure 9.42 for her written work thus far.*]

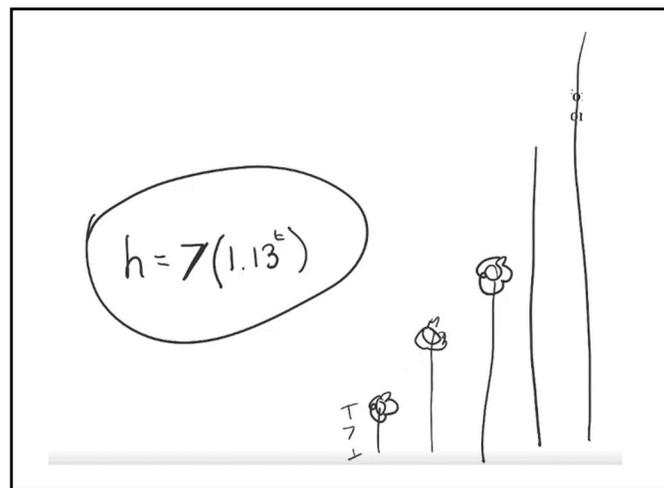


Figure 9.42. Lisa's written work on the Tomato Plant B task during the first part of the post-interview.

- 17 AO: Okay. And where would somebody see the point one three in the, in these
 18 drawings that you did?
- 19 Lisa: I mean really it's, it would just be like [*she draws a horizontal segment*
 20 *beneath all of the figures*] this open space here [*she indicates the area*
 21 *between the tops of the two figures furthest to the left*]. Especially, I guess
 22 the best way to view would just be of course in the first original pair [*she*
 23 *draws a box around the two figures furthest to the left and writes "1.13"*
 24 *between the two figures*]. Um, so this would be point one three here of
 25 seven, so it's just like this distance [*emphasis hers, she draws a bracket on*
 26 *the figure second from the left that indicates the difference in heights for*

27 *the two figures she boxed*]. Does it equal, yeah. So it'd be like that
 28 distance, or whatever from the top of the flower to the, the, like the, the
 29 stem growth in this example, it'd be like this empty space here. [*She*
 30 *repeats indicating the difference in heights between each consecutive*
 31 *figure. See Figure 9.43.*] for every single one.

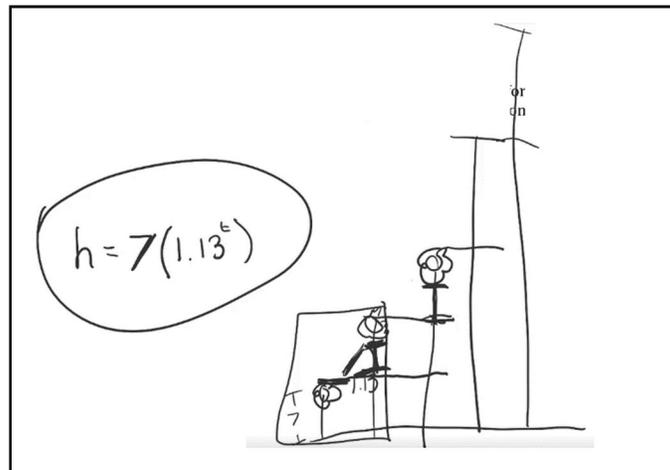


Figure 9.43. Lisa's written work on the Tomato Plant B task during the second part of the post-interview.

32 AO: Okay. And where do they see the one part of that. So that's where the
 33 point one three comes from, but where's the one in your diagram?
 34 Because you've got the one point one three.
 35 Lisa: Um, so it'd just be the top, like wherever the top of the previous one hits
 36 [*she indicates the horizontal segment extending from the top of the first*
 37 *figure*]. So because that would make it one hundred percent of the original.
 38 And then, so, well I guess technically the whole flower would be the one
 39 point one three compared to the original, but this, like the growth
 40 [*indicates the difference in height between the second and third figures*],
 41 would be the point one three almost, like the difference. Yeah, okay so I
 42 described that wrong. So this would
 43 AO: No, I think you described that right.
 44 Lisa: Okay, I was like, and then the one would just be wherever it's equal
 45 [*indicates the height of the second figure*] so all of this would be the one
 46 and then this would be the extra [*indicates the difference in height*
 47 *between the second and third figures*], the point one three, so the whole
 48 it's the point one point, one point one three. Yeah. Yeah.

Lisa's solution suggests that she has a clear image of a growth factor as a measurement of one instance of a quantity using another instance as the measurement unit (lines 21-24,

35-39) [MCM-3], and that she can differentiate between measuring values of a quantity and changes in the quantity using this reference unit (lines 24-31, 39-41) [differentiate between MCM-3 and MCPC-1]. She also clearly linked these measurement values to her algebraic representation [MCEF-3] since she described her meanings for these values after producing the model. After explaining her solution to the Tomato Plant B task Lisa returned to the Muffler task and demonstrated that a similar way of reasoning justifies the calculations and solution to that task. She drew the image in Figure 9.44, using plant heights to substitute for prices, and discussed how the difference in the repair costs was identical to discussing the stem growth from one instance to the next (that is, measuring only the quantity's new magnitude in excess of the previous magnitude), and that the measurement is reported in terms of the magnitude before the change.

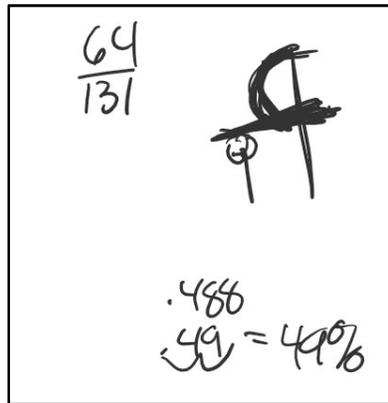


Figure 9.44. Lisa's diagram she used to justify her solution to the Muffler task.

Emergent symbol meaning. It is challenging to recognize emergent symbol meaning in interviews on tasks that are at least somewhat familiar to the interviewees such as modeling exponential functions after completing a unit designed to teach this idea. In these cases, the students are more likely to be relying on familiar algebraic structures to produce solutions in expected forms as opposed to authentically deriving

expressions and formulas by conceptualizing a novel situation, exploring the relationships between quantities, and building the formula to reflect the relationships they identify. However, Lisa’s work in the Tomato Plant B task shown in the previous section suggests that, at the very least, she has a productive meaning for the various components of the formula she produced. An example from John’s post-interview is also worth sharing.

In the Tomato Plant A task John was the only student who spontaneously represented the plant height as an exponential expression rather than calculating 50% of the starting value and adding this to the original height, then repeating the process using the height at the end of the first week. He wrote “ $y = 4 \times (1.5)^2$ ” [“ x ” *representing multiplication*] as his answer and then he determined that value with a calculator. He first evaluated $(1.5)^2$ to get 2.25 and multiplied this by four. I asked him if 2.25 was an important number to understand or if it just represented an intermediate step in the calculation process.

- 1 John: “It’s the rate of change for a, or for, um, two changes in x . Um. For this
- 2 graph and then you apply it to your, to your starting y .”

Although he called it a “rate of change”, this was the only time he used that phrase while describing his reasoning on these tasks. I believe he intended to say “growth factor”, in which case he was able to interpret and justify the steps in his evaluation process meaningfully in terms of relationships between quantities. This tended not to be a feature of Gina’s, Marcus’s, or Shelby’s evaluation and solution processes. This is also the foundation for MCEF-4 and MCEF-5, which describe understanding the meaning of b^x in the formula $f(x) = ab^x$ as the x -unit growth factor and the value of the ratio $f(x) / f(0)$, a

meaning I hypothesize is critical if students are to see quantitative significance in the order of operations for evaluating exponential functions.

Summary. All five students I interviewed had relatively high post-test scores and earned an A or B on the course final exam. Post-interviews with John and Lisa, however, demonstrated that they made larger gains relative to my learning objectives, including being able to clearly articulate reference quantities for making percentage and percent change comparisons and being able to justify their solution methods and algebraic models with respect to quantities they conceptualized in the situation. John's and Lisa's flexibility in using the applets that appeared within the unit, describing the ideas the applets could be used to demonstrate, and describing how similar ideas are represented in each applet clearly differentiated their reasoning from that of Gina, Marcus, and Lisa, as did their repeated reference to the imagery displayed in these applets within their solution methods across the post-interview. It is clear that John and Lisa found these applets and related imagery to be powerful tools in supporting their reasoning about exponential growth and related ideas and this raises questions about how to better support all students in interacting with these applets and making connections between the ideas they are intended to support.

Discussion

This study makes important contributions to research in online learning, student thinking and learning about exponential functions, percentages, and percent change, and the utility of promoting students' quantitative reasoning skills. In these final sections I

discuss implications of my results and map out future research opportunities and next steps for this work.

Teaching, Learning, and Conducting Research in the Online Environment

An online mathematics course provides both opportunities and challenges for designing high-quality student learning experiences. Many teaching experiment studies (e.g., Moore, 2014; Ellis et al., 2016) demonstrate how certain meanings may be productive for students, how certain tasks and didactic objects might be leveraged to facilitate conversations that promote those meanings, and how aspects of specific learning trajectories may have supported students in making connections between different ideas. It remains an open question as to how this research based and refined online instructional intervention (course) compares with in-class precalculus courses using different curriculum and/or instructional approaches. I conjecture that instructors in face-to-face settings who have clear meanings for a course's key ideas (Thompson, 2013), are oriented to make sense of and support student thinking, and who use course materials that are similarly designed to support students' development of essential reasoning abilities and understandings may better support student learning compared to a fully online course. For example, that instructor can engage students in emergent conceptual discussions, pose novel questions to respond to models of students' thinking, or foster effective collaborative exploration among small student groups. I also conjecture that an online course based in research on student learning of targeted ideas and developed using design research will likely result in larger gains in student learning than a course taught by a procedurally-focused instructor using a procedurally-focused

curriculum. By engaging in focused research over time online instruction may even evolve to be better at anticipating the diverse thinking that students present than many teachers. In fact, there is evidence that mathematics instructors often possess weak meanings of key ideas in precalculus level courses they are teaching (e.g., Musgrave and Carlson, 2017) and little interest in understanding or acting on student thinking (Bas and Carlson, 2018).

Exponential Growth and Related Ideas and Comparing Student Learning

Students in this intervention demonstrated significant growth from the pre-test to the post-test. These gains suggest that the online course led to improvements in students' ability to respond to tasks assessing students' ability to recognize accurate models for exponential relationships and to reason about percent change and repeated constant percent change. Furthermore, students demonstrating the largest shifts relative to targeted learning goals demonstrated flexibility in applying meanings for growth factors and relative size comparisons across a variety of tasks and situations. Students entered the course with vague meanings for exponential functions, such as viewing exponential growth as relating to “bigger numbers” or having a graph with a certain general shape. There was little evidence that students' meanings were mathematically productive—the meanings they described and the behaviors they exhibited suggested that there was little substance carried with these meanings that could productively support reasoning about situations involving exponential growth, constant percent change, and percentage measurements. This was also true for some students at the end of the class, but other students did develop productive meanings for exponential functions and growth factors

that suggest potentially powerful ways of reasoning about these relationships even if it also revealed challenges in designing lessons to support students in constructing these meanings.

Researchers studying student learning of exponential growth and related ideas (e.g., Confrey, 1994; Confrey & Smith, 1994, 1995; Ellis et al., 2012, 2015, 2016; Ström, 2008) grounded their work in images of repeated multiplication and/or assumed that students' background knowledge involved characterizing exponential growth as repeated multiplication. Thus, their focus on productive meanings for “growth factor” emerged from this foundation. However, it was uncommon for these researchers to focus on expressing exponential growth as a constant percent change in a dependent quantity relative to uniform changes in an independent quantity. My data suggests that supporting students in seeing the connection between meanings for growth factors as expressed in these studies and constant percent change is nontrivial. Students tended to focus on the amount to add when reasoning about percent change and repeated constant percent change. This carried forward into their reasoning about representing repeated percent change. As one example, most students produced the formula $h = 7 + 0.13t$ to model the height of a plant after t weeks assuming that its initial height is 7 inches and it grows by 13% per week. Contributing to this tendency is the common student belief that their answers should reflect some combination of the variables and numbers explicitly given in a problem statement. Thus, students often overlook the need to account for quantities and quantity values that are not directly mentioned, such as not including the growth factor 1.13 in their algebraic models since neither 1.13 nor 113% are explicitly given in the problem statement or when one student wrote $P = 14^t$ to model a doubling function when

only 14 and the two variable names were directly stated in the problem text. Finally, these incorrect models reveal a general lack of student awareness for how the expressions and formulas they write should represent the relationship between values of quantities they have conceptualized in the situation. If students tend to view percent change in terms of repeated addition (even if they recognize a need to update the reference value for determining the value to add when performing calculations), if they struggle to routinely consider and answer the question, “Percent of what?” when reasoning about percent change, and if they tend not to reflect on the quantitative meaning for components of the models they produce, then students are unlikely to generalize meanings they might develop within learning trajectories such as that proposed by Ellis et al. (2016) when presented with descriptions of exponential growth as constant percent change.

I hypothesized that a focus on measurement as the source for the values used to describe quantities and as a visualization for thinking about relative size comparisons could help students connect ideas across lessons and units and could be a productive foundation for thinking about exponential growth and related ideas. Evidence suggests that this hypothesis might hold for students like John and Lisa who internalized this imagery and used it to justify their mathematical processes. However, the overall course design and the lessons and tasks I created did not support this internalization for all students. Three of the students I interviewed showed uneven development in their meanings relative to my learning goals despite significant improvements in their ability to answer assessment questions in the course. I hypothesize that the differences between the two groups is best explained by the degree to which the students engaged in reflection on their meanings during lessons.

Thompson (1985) contrasted *figurative thought* and *operative thought*. Figurative thought refers to a person's thinking within a certain scheme and manifests itself in relation to that scheme as an inability to work beyond the task at hand and reason about more general relationships and connections. Operative thought involves mental representations of actions and consideration of the consequences of those actions that allows students to make propitious decisions about next steps in their reasoning process and how those steps connect to conclusions already made. Thus, operative thought implies the person is exercising a level of coordination and control over her reasoning. Furthermore, "once a student has created a structure of operations, he or she may reflect on the current state of affairs into that structure, and think in terms of possibilities: What would happen *if* I did (or did not) do this?" (p. 197). A key aspect of operative thought is the conservation of relationships within a system regardless of variations in what is being considered.

I conjecture that the student grouping I noticed reflects evidence of students engaging in reflective thought vs. operative thought within tasks in the post-interview. Gina, Marcus, and Shelby tended to focus on features of a specific problem and struggled to think more generally about how certain relationships were preserved across contexts, different representations, and variations in the kinds of information provided within a task. For example, Marcus and Shelby both had difficulties explaining general ideas that could be supported by the Unit Ruler and Sliding Interval applets unless they were discussing only a single established example. Consistencies in their work across tasks were due more to general solution strategies such as circling all numbers and variables in a problem statement and fitting them together in some string of calculations rather than

seeing commonalities in quantitative relationships and regularities in mathematical reasoning across contexts. John's and Lisa's work in the post-interview paints a different picture, perhaps best encapsulated by their explanations for the ideas that the Unit Ruler and Sliding Interval applets could help demonstrate. Both students described mathematical ideas conveyed within each applet, connected the two applets by describing how ideas in each applet could be visualized within the other applet, and communicated the application of those ideas across multiple examples and in general terms. Furthermore, the meanings they conveyed and the visualizations they described appeared in their reasoning and solutions across multiple tasks in the post-interview.⁵⁹

Thompson (1985) argued that promoting the development of operative thought should be the goal of mathematics instruction. The differences I identified between the two groups of students provides a good example of why this is important. But for students to develop an understanding for the conservation of relationships within a system, they must engage in repeated reasoning about these relationships in increasingly sophisticated ways that support the mathematical structure of the ideas at hand and they must reflect on their reasoning in ways that highlight consistencies in reasoning across various tasks and scenarios. That Gina, Marcus, and Shelby did not appear to engage in reflection on their meanings and common features of their reasoning within the unit lessons potentially explains much of the differences in learning outcomes and informs some needed modifications to future iterations of the intervention.

⁵⁹ It is worth noting that I presented the applets to students near the end of each interview after they had described their reasoning and justified their solutions to all tasks in the post-interview.

Emergent Symbol Meaning

O'Bryan and Carlson (2016) demonstrated that professional development training for a teacher (Tracy) focused on the following ideas helped her shift to pay attention to her students' meanings for the calculations they performed and the expressions and formulas they produced: (1) attending to and writing out in words the quantities she constructed and quantified in each step in reasoning through a solution process, (2) considering how to represent each of those quantities with notation using nested mathematical expressions, (3) considering how the order of operations for evaluating models she produced mirrored the steps in her original reasoning process, (4) noticing how a mathematically equivalent model that follows a different order of operations may produce the same values but have different underlying justifications and produce values for different quantities during the evaluation process, and (5) attempting to explain how a person might be reasoning about a situation when presented with the mathematical model he produced. After focusing on these ideas during training, Tracy began creating her own lesson activities to replicate the experiences for her students, such as asking students to write out in words the steps in their reasoning, write out those steps in symbols, and then explain what quantity's value was being calculated at each step in an evaluation or solving process. She also routinely prepared alternative solutions to lesson tasks so she could present them and ask questions such as, "How must this student have defined his variables for this model to be accurate?" Observations of her lessons showed that students were increasingly attentive to the quantitative meaning for calculations they performed and the models they produced. It was this study that first highlighted the power of what we are now calling *emergent symbol meaning*.

We describe *emergent symbol meaning* relative to students' expectations when symbolizing mathematical reasoning or interpreting algebraic statements and calculations. Students engaging in *emergent symbol meaning* have an expectation that the calculations they perform and components of expressions and formulas they write represent a process of quantifying and representing new quantities in relation to other quantities in a situation. They also expect that the order of operations involved in evaluating an expression reflects the steps in their reasoning process and that this insight, when applied to already-written expressions and formulas, can help them unpack the meanings, reasoning, and conceptualized quantities that someone used to generate the statement. Note that the term "emergent symbol meaning" is intended to echo Moore and Thompson's (2015) *emergent shape thinking* construct referring to a way of conceptualizing graphing as visualizing a trace emerging from coordinating magnitudes or values of two co-varying quantities. A person with this image is also positioned to view already-produced graphs as having emerged from such a trace, and attempting to reproduce the trace uncovers important insights about the modeled phenomenon.

My study highlighted the important role of *emergent symbol meaning* as a cross-cutting way of thinking and novel research construct worthy of additional research. Moore and Carlson (2012) demonstrated that students' initial conception of quantities in a situation heavily influenced the mathematical representations they produced to model relationships between those quantities. Evidence from this study suggests that, even when students have conceptualized certain quantities or processes, representing those algebraically requires that the students attend to how each aspect of their model reflects relationships between quantities and a process of representing new quantities. As one

example, in the Tomato Plant A task most students were able to correctly determine a plant's height after two weeks given its initial height and the weekly percent change in height. However, in attempting to model the same phenomenon algebraically in the Tomato Plant B task, students produced models such as $h = 7 + 0.13t$ with calculations and an order of operations that did not reflect their solution process in the Tomato Plant B task. Similar results occurred in the Bacteria Doubling task where students could determine how many bacteria should be present at the end of one or two weeks but struggled to produce algebraic models for a general doubling process. Students likely lacked a strong meaning for exponential notation entering the course, and it is fair to assume that students might not have had this notation available to represent a relationship that they had conceptualized. There is a more significant implication of these results, however. Four of the five students who produced, or started to produce, the model $h = 7 + 0.13t$ failed to recognize that this did not reflect the relationship that they intended to represent. It is one thing for a student to acknowledge that they do not have appropriate notation to represent a certain process. It is quite another for students to generate an incorrect model, have no awareness that what they produced contains calculations and produces values for quantities not present in their previous solution methods, and to declare the model acceptable. That this occurred so frequently demonstrates that students entering Precalculus generally do not engage in emergent symbol meaning when generating algebraic models. It also suggests that supporting students in shifting their expectations relative to the meanings for components of algebraic models is a worthy instructional goal and one that might positively impact students' success in mathematics.

An Observation about Assessments as Measures of Student Learning

In this study student answers on multiple-choice items often differed from their responses to the same tasks in interviews. In the pre-test and pre-interviews on several tasks that required modeling with algebraic representations, working with exponential functions, and reasoning about percent change, students provided identical answers only 45% of the time. During the post-test and post-interview the number increased to a little over 70%. The most likely reason for this discrepancy between assessment and interview responses is that it is easier to recognize an appropriate model when presented than it is to create one from scratch. Many students in this study judged the appropriateness of their models in interviews based on whether the models produced final values that matched what students might have expected or that seemed reasonable. When presented with a choice of models as multiple-choice options, students could test each option until they found one that produced certain expected values (such as correctly doubling a value after one week) or could at least eliminate answers that produced values that seemed unreasonable. These strategies are less available when the models are not provided. However, even correct responses to tasks either on the assessment or in interviews did not necessarily reflect that a student possessed the most productive conceptual meanings for the underlying ideas. For example, all of the interviewed students held procedural meanings for “percent” (something like “when I see a percent value I move the decimal point two places before doing calculations”) including students who answered questions correctly involving percentages and percent change. Some of these ways of thinking may be so deeply ingrained in past experiences that changing students’ schemes will be

challenging for any intervention, but it is also likely that this intervention requires modifications to better support shifts in students' meanings.

Overall, these findings suggest a need for assessment validation methods to attend to the underlying meanings and ways of thinking that research literature has revealed to be critical for student learning and using ideas. Features of the tasks, the question format (such as multiple choice vs. open-ended), and the choice of values explicitly given in problem statements may contribute as much to students' performance on these items as the meanings students hold for the mathematical ideas we intend to assess. Any research study or analysis that relies on student assessment performance must consider how an assessment's form and question selections may impact the data collected.

Next Steps and Future Research

There are a number of implications of this study both in terms of modifications needed to the intervention and productive future lines of inquiry.

Modifications Suggested by the Study

Promoting student reflection. Thompson (1985) explained that operative thought develops when students reflect on what remains invariant across a variety of problems and when they can differentiate the results of their reasoning and the reasoning that produced those results. This is the process whereby schemes are altered, allowing students to spontaneously and productively use these mental actions in novel (but similar) problem contexts. Task design can help foster experiences where students might engage in this behavior, and certainly more work is needed to assess the effectiveness or

ineffectiveness of tasks and task sequences within this unit to create lessons that better promote student reflection. I am certain that a variety of tasks within the lessons could be improved to better support students in reflecting on their reasoning and making connections in how they reasoned across different situations. However, I also believe that students have certain dispositions influencing whether they engage in this reflection when the opportunity arises. Thompson argued that integrating technology into lessons and tasks provides students opportunities to test hypotheses, gain immediate feedback on these hypotheses, and draw important conclusions from these interactions. But this study showed at least some students respond to this feedback by engaging in persistent guessing, and the students I interviewed who did this frequently were the students who struggled most to make connections between ideas in the unit and who often failed to provide a reasonable mathematical justification for their reasoning and solutions.

Thus, future iterations need to test modifications that deter high attempt counts while still allowing students space to test hypotheses and receive feedback. For example, we could attempt to program lessons to prevent students from submitting more than five attempts to a single question during any five-minute period. If students know about this limitation, it may encourage them to think more carefully about the task and the question being asked before submitting their first attempt, and they may take more time to reflect on their reasoning and their solution before submitting additional attempts. A small-scale study could investigate how this change impacts student behaviors, expectations, motivation, and learning prior to scaling this change to an entire class. We can also compare alternative methods for reducing persistent guessing such as awarding lower lesson scores for higher attempts or programming follow-up questions for key tasks that

ask the students to pick reasons from a list explaining why they believe their answer was wrong. It will be important also to attend to potential collateral effects of these modifications. For example, it is possible that one of these changes might cause higher student frustration levels and lower persistence rates because students find it more tedious and time-consuming to complete lessons.

Supporting an increased focus on quantities and references and supporting students in seeing connections between key applets. Reducing persistent guessing might help change students' expectations and behaviors for engaging with course tasks, but it does not address issues such as students' tendency to conceptualize a given visualization or applet in ways other than what the designer intended. In particular, it seemed that students' ability to see the connection between the Unit Ruler applet and the Sliding Ruler applet, as well as use the applets to describe general ideas in exponential growth, was integral to their ability to reason productively in tasks throughout the post-interview. Four initial changes to the unit are likely to pay large dividends in helping students develop productive ways of thinking when using these applets and identifying important quantities and reference values.

First, students need to be held accountable for drawing diagrams to represent their image of the relationship between quantities in a given situation, such as the relationship between an original price, a sale price, and a discount amount both in dollars and percentages. During this iteration we struggled to integrate more open-ended tasks in the iMathAS environment due to programming/assessment limitations. However, these diagrams seem to be useful tools that facilitate students' reasoning and construction of productive meanings. Therefore, we need to work to create and program tasks where

students must draw diagrams to support their solutions. Pilot studies also showed that asking students to drag quantities' descriptions onto diagrams and drag mathematical expressions onto diagrams to demonstrate what quantities they evaluate/represent positively impacted students' ability to identify quantities in a situation and model those quantities using various representations. This study highlights the importance of finding a way to program such tasks within the iMathAS environment and integrate them within our lessons.

Second, the interactive applets representing variations on the Unit Ruler and Sliding Interval applets should be included within more tasks, and students should be asked to not only solve the given tasks but alter and submit applet states to demonstrate their solution. For example, if students are given two populations and asked to compare their relative size, we could require students to not just provide the value but to also set the Unit Ruler applet to a state that shows their approximate relative size. We may also include more prompts for students to interpret what a quotient value (that represents a relative size of say two populations) represents. My data suggest that many students were not fluent in interpreting a quotient as representing how many times as large the numerator value is than the denominator value, nor did they see the quotient as representing the value of the numerator measured in units of the denominator.

Third, tasks explicitly linking variations of the Unit Ruler applet to variations of the Sliding Interval applet should be included that focus on commonalities in the ideas they can demonstrate. Doing so is not just a way to help students more productively respond to that question in the post-interview. The connection between the ideas demonstrated by the applet is key to developing the reasoning I targeted as evidenced by

students' responses in the post-interview. In particular, we need to develop tasks that include both applets within the same question set with interactions between the two representations necessary to solve the task. For example, students might use the Unit Ruler applet to estimate the relative size of two instances of a quantity and then modify the growth factor in the Sliding Interval applet to represent the same relative size. This leads into the final modification I will discuss.

Fourth, we should introduce tasks that encourage students to examine the impact of changes in the growth factor for an exponential model and how those changes are reflected in various representations of the function relationship. The post-interview data suggests that the unit did not contain enough opportunities for students to think about the impact of changing the one-unit growth factor. Not explicitly asking students to compare the impacts of different growth factors in a given situation likely contributed to uneven results in students' ability to connect meanings for growth factors across various tasks and representations.

Supporting flexibility in meanings for growth factors and emergent symbol meaning. Kuper (2018), in a teaching experiment to support students constructing productive meanings for logarithmic functions, found that students struggled to see the equivalence of repeated multiplication by some set of factors and multiplying by the product of those factors. For example, students could not fluently recognize and use the fact that multiplication by two followed by multiplication by three is equivalent to multiplication by six. This is a significant finding for planning interventions to support students in understanding logarithms and logarithm properties and also suggests instructional goals for teaching exponential growth. My study examined students'

meanings for growth factors as measurements of relative size between two instances of a quantity and demonstrated challenges in students conceptualizing a single multiplicative comparison, let alone the results of multiple consecutive comparisons.

O'Bryan and Carlson (2016) showed that it was useful for students to expect that the algebraic models they produce reflect features of their reasoning process, such as the order of operations used to evaluate the model reflecting a hierarchy in the quantities they conceptualized. This expectation helped them develop productive meanings for symbolic representations. Relative to an exponential function, this frames how students might need to think about exponential function formulas such that they fit within this general way of thinking. Given

$f(x) = ab^x$, evaluating $f(n)$ for some real number n involves a particular order of operations. (1) Evaluate b^n (call this value c). (2) Evaluate $a \cdot c$. Following the order of operations here creates a slightly different interpretation compared to an image of growth determined by multiplying a by b , then the result of that by b , and so on (i.e., repeated multiplication by b). Instead, understanding the calculations in the order of operations requires having a meaning for b^n as the value of some quantity such that a times that value has meaning as well (in this case, b^n is the n -unit growth factor, which is then used to scale a to determine $f(a)$). As Kuper demonstrated, however, it is nontrivial for students to see multiplying by the value b^n and multiplying by n factors of b as equivalent. My study suggests cases where exponential growth is communicated as a repeated constant percent change in a dependent quantity relative to some constant additive change in the independent quantity introduce additional complications. Students tend to strongly link percent change with addition, and thus they need specific support to

see the connection between percent change and growth factors and flexibly convert between them by relying on productive meanings of measurement before they can then be supported in seeing the equivalence of and connection between using growth factors for different interval sizes to evaluate the same final product.

While I did target the meanings discussed in the previous paragraph within the unit, and at least one student (John) twice during interviews described a flexible meaning for growth factors by describing both what b and b^n represent in an exponential context consistent with those meanings, most students did not show evidence that they developed these intended meanings. I believe more progress is possible after making other revisions already described, but specific attention to developing a meaning for b^n is also necessary. In the next iteration I foresee two focal points for revisions. First, applets simultaneously showing comparisons across multiple one-unit intervals might need to include more specific information directing students to attend to various comparisons. For example, a toggle slider can be programmed so that moving the slider highlights different measurements across different intervals and links the measurement values to the one-unit growth factor both numerically and symbolically. The applet can be imbedded in questions that require students to position the applet in different states to represent given comparisons and can ask them to enter expression representing various growth factors in relation to the one-unit growth factor. This may help support students in visualizing b^n as a measurement of relative size over some interval but also as the cumulative effect of n consecutive b -tuplings (using the language introduced by Kuper (2018)). Second, more emphasis is needed on representing repeated percent change, such as seeing $4(1.15)$ as a value that is 15% greater than four, $4(1.15)^2$ as a value that is 15% greater than $4(1.15)$,

and so on while also comparing $4(1.15)^2$ to 4. Creating tasks that require students to represent their solutions as expressions that they *could* simplify to produce a numerical answer might further support students in develop the intended meanings. In this iteration we encouraged students to represent their calculations in iMathAS, but many students used separate calculators and then entered only numerical answers to tasks. It might be that it is only within contexts requiring students to reason about, represent, and attempt to generalize repeated percent change that they make meaningful connections between percent change and growth factors.

Future Research

Future research and follow-up work can strengthen and extend conclusions from this study. I foresee at least five productive avenues for future research extending from the work in this study.

Thinking with magnitudes. Thompson et al. (2014) argued “that children’s development of algebraic reasoning and calculus reasoning is strongly dependent upon their abilities to think with magnitudes” (p. 9) but acknowledged that this is merely a hypothesis since “research on mathematics learning has not attended to issues of magnitude” (p. 9). They call for more work in this area, including performing conceptual analysis to reveal mathematical ideas where thinking with magnitudes might be important and investigating how levels of thinking with magnitudes influence learning and teaching specific ideas. My conceptual analysis of exponential functions provides a detailed description of how thinking with magnitudes might contribute to productive meanings for exponential growth and related ideas. However, I did not collect data explicitly designed

to: (1) characterize the levels of thinking with magnitudes students engaged in for various tasks, (2) characterize the level of thinking with magnitudes a given student was capable of and how this may have impacted their success internalizing intended meanings, or (3) chart the impact of lessons and tasks on supporting students in developing more sophisticated ways of thinking with magnitudes in general. Pursuing this in future studies related to this intervention could provide insights that inform future iterations of the course but also contribute to further development of Thompson et al.'s framework.

Additional analysis on course data. The current study generated far more data than could be analyzed and described in this report. Within the seven lessons and associated homework assignments in the exponential functions sections we have all students' responses to every attempt for every task, including the order in which series of incorrect responses were entered. This amounts to over 11,000 sets of responses and several times as many individual attempts across all tasks for approximately 65 students in this section alone. This data set could provide important insights into questions including but not limited to the following.

- What were the most common initial attempts to each task? The most common incorrect initial attempts? What might this suggest about how students are interpreting each task?
- Did students' responses reflect incorrect or unproductive meanings identified in the pre-interviews? If not, is the unit currently designed to recognize whether students are attempting to leverage unproductive reasoning (beyond simply telling them their answer is incorrect)? If so, is there evidence to suggest that

certain tasks or lessons contributed to a decrease in students' applications of this reasoning?

- When students produced a high number of attempts to a question, was there a recurring pattern to their guesses? If so, what might that tell us about their thinking?
- Are there “lynchpin” tasks within the lessons that seem highly predictive of how students respond to later tasks? If so, does analyzing those tasks provide any insights into productive ways of understanding ideas, challenges preventing students from making certain connections, or common ways that students interpret tasks or visualizations in unintended ways?

Continue iterations of reflection, redesign, implementation, and study. This intervention iteration demonstrated the usefulness of certain mathematical meanings while reasoning about tasks related to exponential growth and percent change but also revealed challenges in supporting those meanings. Much work is needed to continue to shape the course into a higher-quality learning experience for students, and this will require many iterations of studying student behaviors and learning within the course, generating new theory based on our observations, hypothesizing and testing adaptations, and starting the cycle again. Of particular interest will be: i) adapting and studying the impact of new approaches for supporting students in conceptualizing and relating quantities in a problem context; ii) analyzing the effectiveness of a general focus on measurement and relative size as a key theme uniting a variety of mathematical topics and ideas; iii) supporting and studying efforts to support students in applying quantitative reasoning across lessons; iv) supporting and studying efforts to support students in

making connections across lessons and across units. The results of this work will not only help to create a better intervention but can also contribute new theory in areas such as learning trajectory research, online learning, and quantitative reasoning.

Creating and validating items to assess students' meanings for exponential growth and related ideas. Several of the tasks used in this study were written for inclusion in broadly disseminated formative assessments that could be used as course exams, readiness or placement tests, or to measure the impacts of intervention studies. To that end it is important to understand how students respond to these tasks, how certain answer choices reflect specific ways of thinking about targeted ideas, and using the best choice of distractor such that the questions best capture the variety of ways students might be thinking about the targeted ideas. Item validation requires repeated iterations of task redesign and data collection until our data supports that all items assess what we intend and item distractors capture common student thinking (e.g., Carlson et al., 2010). I will continue to use, refine, and test refinements of items in this study to examine the impacts of future modifications to the course as well as contribute to the important work of generating validated assessment items for assessing students' meanings for specific mathematical ideas, beginning with ideas of percent, percent change, growth factor and exponential growth models. This will extend past research that produced broad assessments of key ideas of a course (e.g., PCA, CCR).

Emergent symbol meaning. I described *emergent symbol meaning* relative to students' expectations that calculations performed and algebraic representations generated reflect connections to reasoning patterns emerging from attending to quantities in a situation and relationships between quantities. I also described students with this

image having a similar expectation that they can analyze algebraic representations, including the order of operations used to evaluate the expression/formula, to unpack the reasoning that generated those representations. O’Bryan and Carlson (2016) demonstrated that activities supporting such a focus can productively help both teachers and students engage in more productive discourse about algebraic representations and increase their attention on the quantities and relationships they intend to model. However, more work is needed to flesh out this construct. Much like Moore and Thompson (2015) explored students’ covariational reasoning in generating graphs and were able to flesh out differences between and implications of static vs. emergent shape thinking, researchers focusing on how students think about symbolic representations in modeling contexts are likely to uncover nuances influencing how students see algebraic representations as representative of their reasoning and how they see these representations as constructed systematically by linking the constructed order of operations and a hierarchy of quantities within a quantitative structure. Studying students across many age levels, courses, and with respect to different mathematical ideas while seeking to promote the expectations described as emergent symbol meanings may reveal benefits of this focus and generate new insights into how coherent themes (like emergent symbol meaning, emergent shape thinking, etc.) may help students connect their learning experiences across topics and courses and promote their success in STEM courses and careers.

References

Allen, I. E. & Seaman, J. (2011). Going the Distance: Online Education in the United States, 2011 [online]. Sloan Consortium website. Available at: <http://sloanconsortium.org/>

- publications/survey/going_distance_2011. Retrieved August 8, 2018.
- Allen, I. E. & Seaman, J. (2015). Grade level: Tracking online education in the United States [online]. Babson Survey Research Group website. Available at: <https://www.onlinelearningsurvey.com/reports/gradelevel.pdf>. Retrieved August 8, 2018.
- Bas, S. and Carlson, M. P. (2018). Observable manifestations of a teacher's actions to understand and act on student thinking. In (Eds.) A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown, *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*, San Diego, CA, pp. 1093-1101.
- Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education, III. Issues in Mathematics Education*, 7, 115-162.
- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.
- Carlson, M. P., Oehrtman, M. C., & Moore, K. C. (2018). *Precalculus: Pathways to calculus (7th ed.)*. Rational Reasoning, LLC. Phoenix, AZ.
- Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, 28(2), 113-145.
- Castillo-Garsow, C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth*. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Castillo-Garsow, C. (2012). Continuous quantitative reasoning. *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context*, 2, 55-73.
- Confrey, J. (1994). Multiplication and Exponential Functions. *The development of multiplicative reasoning in the learning of mathematics*, 291-330.
- Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in mathematics*, 26(2-3), 135-164.

- Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in mathematics education*, 66-86.
- Davis, J. D. (2009). Understanding the influence of two mathematics textbooks on prospective secondary teachers' knowledge. *Journal of mathematics teacher education*, 12(5), 365-389.
- Ellis, A. B., Ozgur, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An exponential growth learning trajectory: Students' emerging understanding of exponential growth through covariation. *Mathematical Thinking and Learning*, 18(3), 151-181.
- Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: the case of the Jactus. *WISDOMe monographs*, 2, 93-112.
- Ellis, A. B., Özgür, Z., Kulow, T., Williams, C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior*, 39, 135-155.
- Geogebra. (n.d.). Geogebra Classic [computer software]. Retrieved from <http://geogebra.org>.
- Glaserfeld, E. v. (1995). Radical constructivism: A way of knowing and learning. London: Falmer Press.
- Johneckheere, A., Mandelbrot, B. B., & Piaget, J. (1958). La lecture de l'expérience [Observation and decoding of reality]. Paris: P. U. F.
- Kuper, A. E. (2018). Sparky the saguaro: Teaching experiments examining students' development of the idea of logarithm. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Lippman, D. (n.d.). iMathAS [computer software]. Retrieved from <http://imathas.com/>.
- Johnson, H. L. (2014). Images of intensive and extensive quantity: A framework for reasoning about change in covarying quantities. In *Epistemic algebraic students: Emerging models of students' algebraic knowing*. WISDOMe Monographs (Vol. 4, pp. 267-280). Laramie, WY: University of Wyoming. 267-280.
- Madison, B. L., Carlson, M., Oehrtman, M., & Tallman, M. (2015). Conceptual Precalculus: Strengthening Students' Quantitative and Covariational Reasoning. *The Mathematics Teacher*, 109(1), 54-59.

- McNicholl, T., Frank, K., Hogenson, K., Roat, J., & Carlson, M. (under review). Improving student success and supporting student meaning-making in large-lecture precalculus classes.
- Moore, K. C. (2010). The role of quantitative reasoning in precalculus students learning central concepts of trigonometry (Unpublished doctoral dissertation). Arizona State University, Tempe.
- Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, 45(1), 102-138.
- Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. E. Infante, K. Keene & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*, pp. 782-789. Pittsburgh, PA: RUME.
- Musgrave, S., & Carlson, M. P. (2017). Understanding and advancing graduate teaching assistants' mathematical knowledge for teaching. *The Journal of Mathematical Behavior*, 45, 137-149.
- O'Bryan, A. E. (2018a). Exponential growth and online learning environments: Designing for and studying the development of student meanings in online courses. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- O'Bryan, A. E. (2018b, in preparation). Inattention to Students' Mathematical Meanings in Online Learning Research: Initial Steps Toward a New Research Focus and Methodology
- O'Bryan, A. E. (2018c, in preparation). Uses and Advances in Conceptual Analysis and Learning Trajectories for Studying and Supporting Student Learning: The Case of Exponential Functions
- O'Bryan, A. E. & Carlson, M. P. (2016) Fostering teacher change through increased noticing: Creating authentic opportunities for teachers to reflect on student thinking. In (Eds.) T. Fukawa-Connelly, N. Infante, M. Wawro, and S. Brown, *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education*, Pittsburgh, Pennsylvania, pp. 1192-1200.
- O'Bryan, A. E., Carlson, M. P., & Sander, G. K. (2018). *Precalculus Online*. Gilbert, AZ: COSMa Learning, Inc.

- Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 150-171). Washington, DC: Mathematical Association of America.
- Paulos, J. A. (2001). *Innumeracy: Mathematical illiteracy and its consequences*. New York: Holt-McDougal.
- Piaget, J. (1967). *Six psychological studies*. New York: Random House.
- Piaget, J. (1968). *On the development of memory and identity*. Worcester, MA: Clark University Press.
- Piaget, J. (1971). *Genetic epistemology*. New York, NY: W. W. Norton & Company Inc.
- Piaget, J. (1977). *Psychology and epistemology: Towards a theory of knowledge*. New York, NY: Penguin Books.
- Piaget, J., & Inhelder, B. (1969). *The psychology of the child*. New York, NY: Basic Books.
- Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America*. Raleigh, NC: North Carolina State University.
- Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 41-52). Reston, VA: National Council of Teachers of Mathematics.
- Simon, M. A., & Placa, N. (2012). Reasoning about intensive quantities in whole-number multiplication? A possible basis for ratio understanding. *For the Learning of Mathematics*, 32(2), 35-41.
- Smith, J. & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). New York: Erlbaum.
- Sowder, L. (1988). Children's solutions of story problems. *Journal of Mathematical Behavior*, 7, 227-238.

- Ström, A. (2008). *A case study of a secondary mathematics teacher's understanding of exponential function: An emerging theoretical framework*. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Tallman, M. (2015). An examination of the effect of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Thompson, P. W. (1985). Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula. In E. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 189-243). Hillsdale, NJ: Erlbaum.
- Thompson, P. W. (1988). Quantitative concepts as a foundation for algebra. In M. Behr (Ed.), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* Vol. 1 (pp. 163-170). Dekalb, IL.
- Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebraic. Center for Research in Mathematics & Science Education: San Diego State University.
- Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, 25(3), 165-208.
- Thompson, P. W. (1994a). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, 26(2-3), 229-274.
- Thompson, P. W. (1994b). Students, functions, and the undergraduate mathematics curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1* (Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.
- Thompson, P. W. (1994c). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

- Thompson, P. W. (2008a). *Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education*. Plenary paper delivered at the 32nd Annual Meeting of the International Group for the Psychology of Mathematics Education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. SÈpulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 45-64). MorÈlia, Mexico: PME.
- Thompson, P. W. (2008b). One approach to a coherent K-12 mathematics. Or, it takes 12 years to learn calculus. Paper presented at the Pathways to Algebra Conference, June 22-25, Mayenne, France.
- Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education. WISDOMe Monographs* (Vol. 1, pp. 33- 57). Laramie, WY: University of Wyoming.
- Thompson, P. W. (2012). Advances in research on quantitative reasoning. In R. Mayes, R. Bonillia, L. L. Hatfield & S. Belbase (Eds.), *Quantitative reasoning: Current state of understanding WISDOMe Monographs* (Vol. 2, pp. 143-148). Laramie, WY: University of Wyoming Press.
- Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education*, pp. 57-93. New York: Springer.
- Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.
- Thompson, P. W, Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing. WISDOMe Monographs* (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming.
- U.S. Department of Education, National Center for Education Statistics. (2018). *Digest of educational statistics, 2016, Table 311.15*. U.S. Department of Education, National Center for Education Statistics. Washington, D. C.
- Weber, K. (2002a). Developing students' understanding of exponents and logarithms. *Proceedings of the Annual Meeting of the North American Chapter of the*

International Group for the Psychology of Mathematics Education, 1-4, 1019-1027.

Weber, K. (2002b). Students' understanding of exponential and logarithmic functions. Murray, KY: Murray State University. (ERIC Document Reproduction Service No. ED477690).

Wildi, T. (1991). *Units and conversions: A handbook for engineers and scientists*. New York: IEEE Press.

CHAPTER 10

CONCLUSION

Thompson (2013) demonstrated that a strong system of meanings is critical for teachers (and curriculum designers). He argued that a weak or incoherent system of meanings creates more space for students to construct incorrect or unhelpful meanings and for the teacher (or designer) to remain unaware of students' constructions. Understanding exponential functions seems particularly challenging for students, which is not surprising given the complexity inherent in fully unpacking critical features for such relationships and evidence that many curricula overlook this complexity when introducing the concept (see Paper 3). Several previous studies on exponential growth worked from the assumption that students conceptualize exponential relationships via images of repeated multiplication (Confrey, 1994; Confrey & Smith, 1994, 1995; Ellis et al., 2012, 2015, 2016). This introduces two broad areas of conceptual difficulties. First, interpreting the meaning of and motivation for operations in evaluating exponential functions, determining growth factors, and solving exponential equations involves reasoning about products, quotients, ratios, and relative size, and many researchers have demonstrated students' difficulties with these ideas (i.e., Harel & Behr, 1995; Johnson, 2015; Kaput & West, 1994; Simon & Placa, 2012; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998; Steffe, 2010; Thompson et al., 2014). Second, an image of exponential growth grounded in repeated multiplication does not generalize well to evaluating that function over its entire domain and students (and teachers) often struggle to connect their image of repeated multiplication and the closed-form representation of exponential functions (Davis, 2009; Ellis et al., 2012, 2015, 2016; Presmeg &

Nenduardo, 2005; Ström, 2008). Kuper (2018) recently demonstrated that students have difficulties conceptualizing that multiplication by ab is equivalent to multiplication by a followed by multiplication by b . This result helps to explain why students might struggle to see the value of b^x as a ratio of $f(x)$ to $f(0)$ for any x , a key learning goal for Ellis et al. (2012, 2015, 2016) and myself.

Student performance on tasks researchers classify as related to exponential growth demonstrate the impoverished nature of student images of these relationships. For example, only about 35% of college precalculus students who took Carlson, Oehrtman, & Engelke's (2010) multiple choice Precalculus Concept Assessment at the end of their courses could identify the impact on the growth pattern when updating a function definition from $p(t) = 7(2)^t$ to $p(t) = 7(3)^t$ (Research and Innovation in Mathematics and Science Education, 2007). Weber (2002a, 2002b) found that most college algebra and precalculus students receiving rules-based instruction could not provide an adequate justification for exponential and logarithmic properties only a few weeks after the topics were covered in class, nor could these students provide a rationale for why functions like $f(x) = (\frac{1}{2})^x$ are decreasing. Table 10.1 shows U.S. 12th grader performance on several assessment items from the National Assessment of Educational Progress (National Center for Education Statistics). The results indicate that U.S. students are not developing useful and lasting meanings for growth patterns especially when expressed as a constant percent change, which is a typical way of characterizing exponential growth.

Evidence from student interviews in this study, however, points to an additional issue not reported in other studies. The current body of literature focuses attention on addressing challenges in generalizing images of repeated multiplication. *But what if*

Table 10.1

Results from the National Assessment of Educational Progress (12th Grade)

<p>2009 Assessment [M1899E1]</p> <p>The population P of a certain town is given by the equation $P = 50,000(1 + r)^t$, where r is the annual rate of population increase and t is the number of years since 1990.</p> <p>(a) What was the population in 1990? Answer: _____</p> <p>(b) In 2001 the population was 100,000. What was the annual rate of population increase? Answer: _____</p> <p>Incorrect: 38% of students Partial 2: 1% of students Partial 1: 46% of students *Correct 9% of students Omitted: 8% of students Off task: 2% of students</p>
<p>2005 Assessment [M133801]</p> <p>A car costs \$20,000. It decreases in value at the rate of 20 percent each year, based on the value at the beginning of that year. At the end of how many years will the value of the car first be less than half the cost? Answer: _____ years Justify your answer.</p> <p>Incorrect: 60% of students Partial 2: 5% of students Partial 1: 5% of students *Correct 26% of students Omitted: 4% of students</p>
<p>2005 Assessment [M127001]</p> <p>The number of bacteria present in a laboratory sample after t days can be represented by $500(2)^t$. What is the initial number of bacteria present in the sample?</p> <p>A. 250 [10% of students] *B. 500 [33% of students] C. 750 [5% of students] D. 1,000 [46% of students] E. 2,000 [4% of students] [2% of students did not answer]</p>

students do not conceptualize representing repeated multiplication as the goal of modeling these situations? The students in my study displayed a strong tendency to rely on images of repeated addition to solve tasks that I classified as related to exponential growth (such as a constant percent change in a quantity's value over equal intervals of time). My initial conceptual analysis and lesson design were based on the expectation that I would need to support shifts in students' meaning for growth factors (the number by which they would expect to multiply repeatedly). That students held very weak meanings for exponential functions was not surprising, but the meanings they apparently did hold (and the implications for those meanings) were surprising, and I am not sure that the course lessons, as written, were fully prepared to support students in productively shifting their meanings. Addressing this will be a key focus in future iterations of the unit and course.

I was motivated to focus on exponential functions in this study because I viewed it as an opportunity to begin to perform conceptual analyses and unpack learning trajectories to generate empirical evidence for the challenges and benefits of supporting students in *thinking with magnitudes* (Thompson et al., 2014). I hypothesized that this learning goal, when repeated across many topics in the course, might support students in conceptualizing an important way of reasoning that connects topics and could help the students experience a coherent mathematics course. I noticed that identifying broader learning goals and explicitly describing how those goals might support students in understanding topics throughout a course and into future courses is not always present in researchers' work with learning trajectories. This motivated me to write Paper 1, which calls for an increased focus on explicating how particular learning goals targeted in

learning trajectory research feed into a system designed to promote coherency within and across courses.

I situated my study in an online learning environment because I believe that online learning will play an increasingly prominent role in students' academic careers, yet we know little about how to design these courses to best support students in constructing productive meanings that will serve them well in future math and STEM courses. In addition, studying the meanings students do construct within these courses is conspicuously absent from current research. To ensure that the growing prevalence of online mathematics courses does not unintentionally harm students' long-term mathematical development, it is important to look beyond measurements of performance, emotional responses, student satisfaction, and retention rates as evidence for the quality and success of these courses. Addressing this call to action includes the design and implementation of teaching experiments within these environments. Analyzing the results of such studies might additionally require modifications to common research methodologies. These concerns motivated me to write Paper 2 and also describe the connections I see between Paper 2 and the other papers that comprise this dissertation.

REFERENCES

- Allen, I. E. & Seaman, J. (2015). Grade level: Tracking online education in the United States [online]. Babson Survey Research Group website. Available at: <https://www.onlinelearningsurvey.com/reports/gradelevel.pdf>. Retrieved August 8, 2018.
- Barbour, M., Brown, R., Waters, L. H., Hoey, R., Hunt, J. L., Kennedy, K., & Trimm, T. (2011). Online and Blended Learning: A Survey of Policy and Practice from K-12 Schools around the World. *International Association for K-12 Online Learning*.
- Brinkman, W. P., Rae, A., & Dwivedi, Y. K. (2007). Web-based implementation of the Personalised System of Instruction: A case study of teaching mathematics in an online learning environment. *International Journal of Web-Based Learning and Teaching Technologies (IJWLTT)*, 2(1), 39-69.
- Byars-Winston, A., Estrada, Y., Howard, C., Davis, D., & Zalapa, J. (2010). Influence of social cognitive and ethnic variables on academic goals of underrepresented students in science and engineering: a multiple-groups analysis. *Journal of counseling psychology*, 57(2), 205.
- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.
- Carlson, M. P., Moore, K. C., & Oehrtman, M. C. (2016). *Precalculus: Pathways to calculus*. Rational Reasoning, LLC. Phoenix, AZ.
- Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, 28(2), 113-145.
- Castillo-Garsow, C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth*. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Castillo-Garsow, C. (2012). Continuous quantitative reasoning. *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context*, 2, 55-73.
- Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, 33(3), 31-37.

- Chen, P.-S. D., Lambert, A. D., & Guidry, K. R. (2010). Engaging online learners: The impact of Web-based learning technology on college student engagement. *Computers & Education*, 54(4), 1222-1232
- Clement, John. (2000). Analysis of clinical interviews: Foundations and model viability. *Handbook of research design in mathematics and science education*: 547-589.
- Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 1-38). Greenwich, CT: Information Age Publishing.
- Confrey, J. (1994). Multiplication and Exponential Functions. *The development of multiplicative reasoning in the learning of mathematics*, 291-330.
- Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in mathematics*, 26(2-3), 135-164.
- Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in mathematics education*, 66-86.
- Coomey, M., & Stephenson, J. (2001). Online learning: it is all about dialogue, involvement, support and control-according to the research. *Teaching and learning online: Pedagogies for new technologies*, 37-52.
- Cooper Jr, R. G. (1991). The role mathematical transformations and practice in mathematical development. In *Epistemological foundations of mathematical experience* (pp. 102-123). Springer New York.
- Davis, J. D. (2009). Understanding the influence of two mathematics textbooks on prospective secondary teachers' knowledge. *Journal of mathematics teacher education*, 12(5), 365-389.
- The Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher*, 5-8.
- Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education*, 194-229.
- Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education*, 42(4), 308-345.

- Ellis, A. B., Ozgur, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An exponential growth learning trajectory: Students' emerging understanding of exponential growth through covariation. *Mathematical Thinking and Learning*, *18*(3), 151-181.
- Ellis, A.B., Ozgur, Z., Kulow, T., Dogan, M.F., Williams, C.C., & Amidon, J. (2013, November). Correspondence and covariation: Quantities changing together. In M.V. Martinez & A.C. Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 119-126). Chicago, IL: University of Illinois at Chicago.
- Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: the case of the Jactus. *WISDOMe monographs*, *2*, 93-112.
- Ellis, A. B., Özgür, Z., Kulow, T., Williams, C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior*, *39*, 135-155.
- Engelbrecht, J., & Harding, A. (2004). Combining online and paper assessment in a web-based course in undergraduate mathematics. *Journal of Computers in Mathematics and Science Teaching*, *23*, 217-232.
- Freeman, S., Eddy, S., McDonough, M., Smith, M., Okoroafor, N., Jordt, H., & Wenderoth, M. (2014). Active learning increases student performance in science, engineering, and mathematics. In B. Alberts (Ed.), *Proceedings of the National Academy of Sciences of the United States of America*, *111*(23), pp. 8410-8415. San Francisco, CA: PNAS.
- Glaserfeld, E. v. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.
- Glass, J., & Sue, V. (2008). Student preferences, satisfaction, and perceived learning in an online mathematics class. *MERLOT Journal of Online Learning and Teaching*, *4*(3), 325-338.
- Graham, M. J., Frederick, J., Byars-Winston, A., Hunter, A. B., & Handelsman, J. (2013). Increasing persistence of college students in STEM. *Science*, *341*(6153), 1455-1456.
- Hannafin, M., Orrill, C., Kim, H., & Kim, M. (2005). Educational technology research in postsecondary settings: Promise, problems, and prospects. *Journal of Computing in Higher Education*, *16*(2), 3-22.

- Harel, G. (2007). The DNR system as a conceptual framework for curriculum development and instruction. *Foundations for the future in mathematics education*, 263-280.
- Harel, G. (2008a). DNR perspective on mathematics curriculum and instruction, Part I: focus on proving. *ZDM*, 40(3), 487-500.
- Harel, G. (2008b). A DNR perspective on mathematics curriculum and instruction. Part II: with reference to teacher's knowledge base. *ZDM*, 40(5), 893-907.
- Harel, G. (2008c). What is mathematics? A pedagogical answer to a philosophical question. *Proof and other dilemmas: Mathematics and philosophy*, 265-290.
- Harel, G. (2013). Intellectual need. In *Vital directions for mathematics education research* (pp. 119-151). Springer New York.
- Harel, G., & Behr, M. (1995). Teachers' Solutions for Multiplicative Problems. *Hiroshima Journal of Mathematics Education*, 3, 31-51.
- Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *Research in collegiate mathematics education III*, 234-283.
- Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical thinking and learning*, 7(1), 27-50.
- Hopper, K. B., & Harmon, S. W. (2000). A multiple-case study of exemplary internet courses. *Education at a Distance*, 14(9).
- Hu, S., & Kuh, G. D. (2001). Does the Degree of Campus" Wiredness" Matter? *Education policy analysis archives*, 9, 49.
- Hughes, J. E., McLeod, S., Brown, R., Maeda, Y., & Choi, J. (2007). Academic achievement and perceptions of the learning environment in virtual and traditional secondary mathematics classrooms. *The American Journal of Distance Education*, 21(4), 199-214.
- Hutchison, M. A., Follman, D. K., Sumpter, M., & Bodner, G. M. (2006). Factors influencing the self-efficacy beliefs of first-year engineering students. *Journal of Engineering Education*, 95(1), 39-47.
- Isen, A. M. (2000). Positive affect and decision making. In M. Lewis & J. Haviland-Jones (Eds), *Handbook of emotions* (2nd ed.). (pp. 417-435). New York: Guilford.

- Johnckheere, A., Mandelbrot, B. B., & Piaget, J. (1958). *La lecture de l'expérience* [Observation and decoding of reality]. Paris: P. U. F.
- Johnson, H. L. (2012). Reasoning about quantities involved in rate of change as varying simultaneously and independently. In R. Mayes, R. Bonillia, L. L. Hatfield, & S. Belbase (Eds.), *Quantitative reasoning: Current state of understanding*, WISDOMe Monographs (Vol. 2, pp. 39–53). Laramie: University of Wyoming.
- Johnson, H. L. (2014). Reasoning about quantities involved in rate of change as varying simultaneously and independently. In L. P. Steffe, K. C. Moore, & L. L. Hatfield, (Eds.), *Epistemic algebraic students: emerging models of students' algebraic knowing*, WISDOMe Monographs (Vol. 4, pp. 267–280). Laramie: University of Wyoming.
- Heather Lynn Johnson (2015) Secondary Students' Quantification of Ratio and Rate: A Framework for Reasoning about Change in Covarying Quantities, *Mathematical Thinking and Learning*, 17:1, 64-90.
- Kim, C., Park, S. W., & Cozart, J. (2014). Affective and motivational factors of learning in online mathematics courses. *British Journal of Educational Technology*, 45(1), 171-185.
- Kaput, J. J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*, 237–287. Albany, NY: State University of New York Press.
- Kramarski, B., & Gutman, M. (2006). How can self-regulated learning be supported in mathematical E-learning environments?. *Journal of Computer Assisted Learning*, 22(1), 24-33.
- Kuh, G. D., & Hu, S. (2001). The relationships between computer and information technology use, selected learning and personal development outcomes, and other college experiences. *Journal of College Student Development*, 42(3), 217.
- Kuper, E. G. (2018). Sparky the saguaro: Teaching experiments examining students' development of the idea of logarithm. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Laird, T. F. N., & Kuh, G. D. (2005). Student experiences with information technology and their relationship to other aspects of student engagement. *Research in Higher Education*, 46(2), 211-233.

- Lamon, S. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. *The development of multiplicative reasoning in the learning of mathematics*, 89-120.
- Levine, L. J. & Pizarro, D. A. (2004). Emotion and memory research: a grumpy overview. *Social Cognition*, 22, 5, 530–554.
- Lippman, D. (n.d.). iMathAS [Computer software]. Retrieved from <http://imathas.com/>.
- Lobato, J. (2006). Alternative perspectives on the transfer of learning: History, issues, and challenges for future research. *Journal of the Learning Sciences*, 15(4), 431-449.
- Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior*, 21(1), 87-116.
- Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio as measure as a foundation for slope. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions: 2002 Yearbook of the NCTM* (pp. 162-175). Washington, DC: National Council of Teachers of Mathematics.
- McDuffie, A. R., & Slavit, D. (2003). Utilizing online discussion to support reflection and challenge beliefs in elementary mathematics methods classrooms. *Contemporary Issues in Technology and Teacher Education*, 2(4), 446-466.
- Means, B., Toyama, Y., Murphy, R., Bakia, M., & Jones, K. (2009). Evaluation of evidence-based practices in online learning: A meta-analysis and review of online learning studies. *US Department of Education*.
- Moore, K. C. (2010). The role of quantitative reasoning in precalculus students learning central concepts of trigonometry (Unpublished doctoral dissertation). Arizona State University, Tempe.
- Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *Journal of Mathematical Behavior*, 31(1), 48-59.
- Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. E. Infante, K. Keene & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*, pp. 782-789. Pittsburgh, PA: RUME.
- Myers, M. (1999). Investigating information systems with ethnographic research. *Communications of the AIS*, 2(4es), 1.

- National Center for Education Statistics. (n.d.). *NAEP questions tool*. Retrieved from <https://nces.ed.gov/nationsreportcard/itmrlsx/>.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. National Governors Association for Best Practices, Council of Chief State School Officers, Washington, D. C.
- Nguyen, D. M., Hsieh, Y. C. J., & Allen, G. D. (2006). The impact of web-based assessment and practice on students' mathematics learning attitudes. *The journal of computers in mathematics and science teaching*, 25(3), 251.
- Nguyen, D. M., & Kulm, G. (2005). Using web-based practice to enhance mathematics learning and achievement. *Journal of Interactive Online Learning*, 3(3), 1-16.
- O'Bryan, A. E. & Carlson, M. P. (2016) Fostering teacher change through increased noticing: Creating authentic opportunities for teachers to reflect on student thinking. In (Eds.) T. Fukawa-Connelly, N. Infante, M. Wawro, and S. Brown, *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics Education*, Pittsburgh, Pennsylvania, pp. 1192-1200.
- O'Bryan, A. E., Carlson, M. P., & Sander, G. K. (2017). *Precalculus Online*. Gilbert, AZ: COSMa Learning, Inc.
- Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 150-171). Washington, DC: Mathematical Association of America.
- O'Regan, K. (2003). Emotion and e-learning. *Journal of Asynchronous learning networks*, 7(3), 78-92.
- Paulos, J. A. (2001). *Innumeracy: Mathematical illiteracy and its consequences*. New York: Hill and Wang.
- Piaget, J. (1965). *The child's conception of number*. New York: W. W. Norton.
- Piaget, J. (1967). *Six psychological studies*. New York: Random House.
- Piaget, J. (1968). *On the development of memory and identity*. Worcester, MA: Clark University Press.

- Piaget, J. (1971). *Genetic epistemology*. New York, NY: W. W. Norton & Company Inc.
- Piaget, J. (1977). *Psychology and epistemology: Towards a theory of knowledge*. New York, NY: Penguin Books.
- Piaget, J., & Inhelder, B. (1969). *The psychology of the child*. New York, NY: Basic Books.
- Presmeg, N., & Nenduardu, R. (2005). An investigation of a pre-service teacher's use of representations in solving algebraic problems involving exponential relationships. In H. L. Chick & J. K. Vincent (Eds.), *Proceedings of the 29th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 105–112). Melbourne, Australia: PME.
- Reasoning [Def. 1 and 2]. (n.d.). In Dictionary.com, Retrieved January 17, 2017, from <http://www.dictionary.com/browse/reasoning?s=t>.
- Reeves, Herrington, & Oliver. (2005). Design research: A socially responsible approach to instructional technology research in higher education. *Journal of Computing in Higher Education*, 16(2), 97-116.
- Research and Innovation in Mathematics and Science Education. (2007). *Precalculus assessment (PCA) assessment items with video clips and transcripts*. Retrieved from <https://mathed.asu.edu/instruments/PCA/vH/index.shtml>.
- Rimmershaw, R. (1999). Using conferencing to support a culture of collaborative study. *Journal of Computer Assisted Learning*, 15(3), 189-200.
- Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America*. Raleigh, NC: North Carolina State University.
- Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 41-52). Reston, VA: National Council of Teachers of Mathematics.
- Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for research in mathematics education*, 114-145.

- Simon, M. A., & Placa, N. (2012). Reasoning about intensive quantities in whole-number multiplication? A possible basis for ratio understanding. *For the Learning of Mathematics*, 32(2), 35-41.
- Simon, M. A., Saldanha, L., McClintock, E., Akar, G. K., Watanabe, T., & Zembat, I. O. (2010). A Developing Approach to Studying Students' Learning through Their Mathematical Activity. *Cognition and Instruction*, 28(1), 70-112.
- Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). New York: Erlbaum.
- Sowder, J., Armstrong, B., Lamon, S., Simon, M., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education*, 1(2), 127-155.
- Spangler, D. A. (1992). Assessing students' beliefs about mathematics. *Mathematics Educator*, 3(1), 19-23.
- Steffe, L. P. (1996). Radical constructivism: A way of knowing and learning [Review of the same title, by Ernst von Glasersfeld]. *Zentralblatt für Didaktik der Mathematik* [International reviews on Mathematical Education], 96(6), 202-204.
- Steffe, L. P., & Olive, J. (2010). *Children's Fractional Knowledge*. Springer.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education*. Dordrecht, The Netherlands: Kluwer.
- Stiles, M. J. (2000) Effective learning and the virtual learning environment. In *EUNIS 2000: Towards virtual universities: Proceedings of the European university information system 2000 conference*. INFOSYSTEM 2000, Poznan, Poland, Poznan: Instytut Informatyki Politechniki Poznanskiej, 171-180.
- Ström, A. (2008). *A case study of a secondary mathematics teacher's understanding of exponential function: An emerging theoretical framework*. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Swan, K. (2003). Learning effectiveness online: What the research tells us. *Elements of quality online education, practice and direction*, 4, 13-47.
- Tallman, M. (2015). An examination of the effect of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge. Unpublished

- Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.
- Taylor, R. W. (2002). Pros and cons of online learning—a faculty perspective. *Journal of European Industrial Training*, 26(1), 24-37.
- Thompson, P. W. (1985). Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula. In E. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 189-243). Hillsdale, NJ: Erlbaum.
- Thompson, P. W. (1988). Quantitative concepts as a foundation for algebra. In M. Behr (Ed.), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* Vol. 1 (pp. 163-170). Dekalb, IL.
- Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebraic. Center for Research in Mathematics & Science Education: San Diego State University.
- Thompson, P. W. (1991). Getting ahead, with theories: I have a theory about this. In R. Underhill & C. Brown (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education, North America, Plenary Lectures* Vol. 1 (pp. 240-245). Blacksburgh, VA: Virginia Tech.
- Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, 25(3), 165-208.
- Thompson, P. W. (1994a). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, 26(2-3), 229-274.
- Thompson, P. W. (1994b). Students, functions, and the undergraduate mathematics curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education*, 1 (Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.
- Thompson, P. W. (1994c). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.
- Thompson, P. W. (2000). Radical constructivism: Reflections and directions. In L. P. Steffe & P. W. Thompson (Eds.), *Radical constructivism in action: Building on*

- the pioneering work of Ernst von Glasersfeld* (pp. 412-448). London: Falmer Press.
- Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing and Modeling In Mathematics Education*. Dordrecht, The Netherlands: Kluwer.
- Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95-114). Reston, VA: National Council of Teachers of Mathematics.
- Thompson, P. W. (2008a). *Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education*. Plenary paper delivered at the 32nd Annual Meeting of the International Group for the Psychology of Mathematics Education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. SÈpulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 45-64). MorÈlia, Mexico: PME.
- Thompson, P. W. (2008b). One approach to a coherent K-12 mathematics. Or, it takes 12 years to learn calculus. Paper presented at the Pathways to Algebra Conference, June 22-25, Mayenne, France.
- Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education*. WISDOMe Monographs (Vol. 1, pp. 33- 57). Laramie, WY: University of Wyoming.
- Thompson, P. W. (2012). Advances in research on quantitative reasoning. In R. Mayes, R. Bonillia, L. L. Hatfield & S. Belbase (Eds.), *Quantitative reasoning: Current state of understanding* WISDOMe Monographs (Vol. 2, pp. 143-148). Laramie, WY: University of Wyoming Press.
- Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education*, pp. 57-93. New York: Springer.
- Thompson, P. W., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools*, 30, 124-147.
- Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

- Thompson, P. W, Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing*. WISDOMe Monographs (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming.
- Tucker, M. S. (2011). Standing on the shoulders of giants: An American agenda for education reform. Washington, DC: National Center on Education and the Economy.
- Twigg, C. A. (2003). Models for online learning. *Educause review*, 38, 28-38.
- U.S. Department of Education, National Center for Education Statistics. (2013). *Digest of educational statistics, 2012, Table 311.15*. U.S. Department of Education, National Center for Education Statistics. Washington, D. C.
- U.S. Department of Education, National Center for Education Statistics. (2014). *Digest of educational statistics, 2013, Table 311.15*. U.S. Department of Education, National Center for Education Statistics. Washington, D. C.
- U.S. Department of Education, National Center for Education Statistics. (2015). *Digest of educational statistics, 2014, Table 311.15*. U.S. Department of Education, National Center for Education Statistics. Washington, D. C.
- U.S. Department of Education, National Center for Education Statistics. (2016). *Digest of educational statistics, 2015, Table 311.15*. U.S. Department of Education, National Center for Education Statistics. Washington, D. C.
- U.S. Department of Education, National Center for Education Statistics. (2017). *Digest of educational statistics, 2016, Table 311.15*. U.S. Department of Education, National Center for Education Statistics. Washington, D. C.
- U.S. Department of Education, National Center for Education Statistics. (2018). *Digest of educational statistics, 2016, Table 311.15*. U.S. Department of Education, National Center for Education Statistics. Washington, D. C.
- Wadsworth, L. M., Husman, J., Duggan, M. A., & Pennington, M. N. (2007). Online mathematics achievement: Effects of learning strategies and self-efficacy. *Journal of Developmental Education*, 30(3), 6.
- Weber, K. (2002a). Developing students' understanding of exponents and logarithms. Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 1-4, 1019-1027.

- Weber, K. (2002b). Students' understanding of exponential and logarithmic functions. Murray, KY: Murray State University. (ERIC Document Reproduction Service No. ED477690).
- Wildi, T. (1991). Units and conversions: A handbook for engineers and scientists. New York: IEEE Press.
- Xu, D., & Jaggars, S. S. (2011). The effectiveness of distance education across Virginia's community colleges: Evidence from introductory college-level math and English courses. *Educational Evaluation and Policy Analysis*, 33(3), 360-377.
- Xu, D. & Jaggars, S. (2013). The impact of online learning on students' course outcomes: Evidence from a large community and technical college system. *Economics of Education Review*, 37, 46-57.

APPENDIX A
PRE-TEST AND POST-TEST

Question 1

If Porter is 60 inches tall and Maddox is 45 inches tall, what does the quotient $\frac{60}{45} = \frac{4}{3}$ represent?

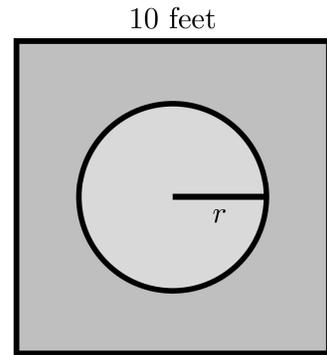
- a. Porter's height is $\frac{4}{3}$ more than Maddox's height
- b. Porter is $\frac{4}{3}$ inches taller than Maddox
- c. Porter is $\frac{4}{3}$ times as tall as Maddox
- d. Maddox is $\frac{4}{3}$ times as tall as Porter
- e. None of the above

Question 2

A landscaper is exploring the varying sizes of a circular flower-bed that can be built within a 10 ft x 10 ft square region. Let r represent the varying length (in feet) of the radius of the circular flower bed.

What are the possible values of the radius length (in feet) of the flower-bed, r ?

- a. $r = 5$
- b. $r = 10$
- c. r can vary from 0 to 5
- d. r can vary from 0 to 10
- e. r cannot vary because r is an unknown

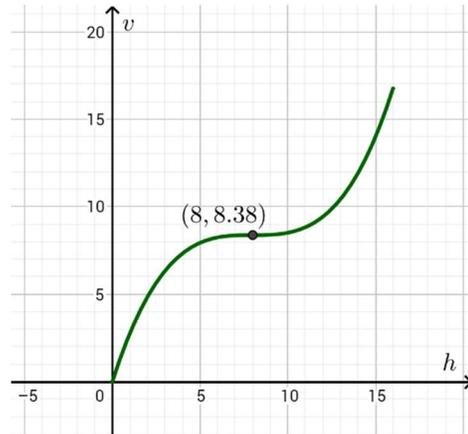


Question 3

A vase is filled with water. The graph to the right shows the volume of water in the vase in cm^3 , v , in terms of height of water in the vase in cm, h .

What does the point $(8, 8.38)$ on this graph represent?

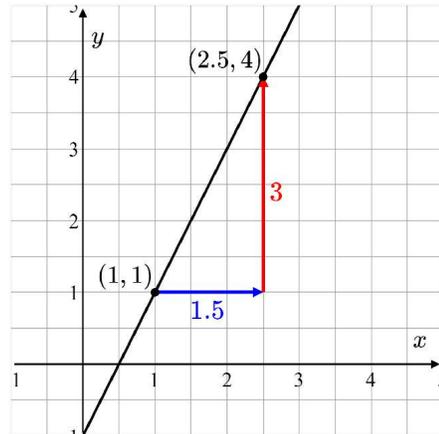
- a. 8 minutes after the vase started filling the volume of water in the vase is 8.38 cm^3
- b. 8.38 minutes after the vase started filling the height of water in the vase is 8 cm
- c. When the height of water in the vase is 8 cm the volume of water in the vase is 8.38 cm^3
- d. When the height of water in the vase is 8.38 cm the volume of water in the vase is 8 cm^3
- e. The volume of water in the vase is currently 8.38 cm^3



Question 4

The graph to the right shows a relationship between x and y . What does the 3 next to the vertical vector (arrow) represent?

- a. The value of y is 3.
- b. When $x = 1.5$, $y = 3$.
- c. When $x = 2.5$, $y = 3$.
- d. As x increases from 1 to 2.5, y increases by 3.
- e. As x increases from 1 to 2.5, $y = 3$.



Question 5

A tomato plant that is 4 inches tall when first planted in a garden grows by 50% each week during the first few weeks after it is planted. How tall is the tomato plant 2 weeks after it was planted?

- a. 5 inches
- b. 6 inches
- c. 8 inches
- d. 9 inches
- e. 12 inches

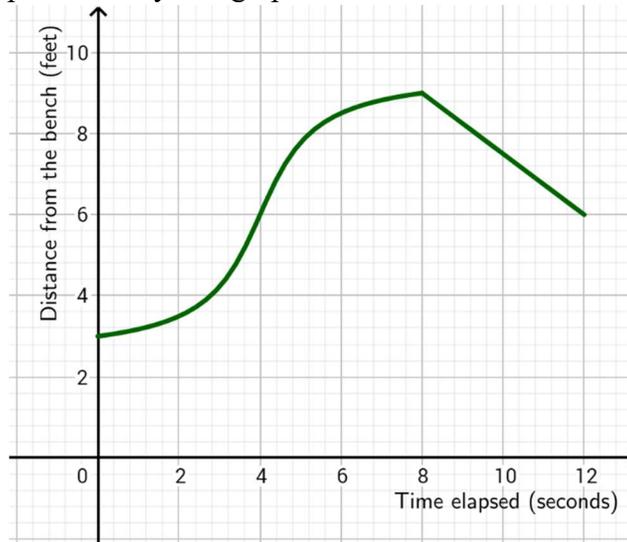
Question 6

When preparing a prescription, a pharmacy mixes 7 parts of water with 2 parts of concentrated Amoxicillin. Which formula should the pharmacy technician use to determine the number of milliliters of water n_w to add to n_a milliliters of concentrated Amoxicillin?

- a. $n_w = 7n_a + 2$
- b. $n_w = 2n_a + 7$
- c. $n_w = \frac{2}{7}n_a$
- d. $n_w = 7n_a - 2$
- e. $n_w = \frac{7}{2}n_a$

Question 7

Kirk's distance from a park bench (in feet) in terms of the number of seconds since he started walking is represented by this graph.



Which of the following statements about this situation is *false*?

- a. As the number of seconds since Kirk started walking increases from 0 to 8 seconds, Kirk's distance from the park bench is always increasing.
- b. As the number of seconds since Kirk started walking increases from 8 to 12 seconds, Kirk's distance from the park bench decreases at a constant rate.
- c. As the number of seconds since Kirk started walking increases from 0 to 4 seconds, Kirk's distance from the park bench increases by a greater amount each second.
- d. As the number of seconds since Kirk started walking increases from 4 to 8 seconds, Kirk travels a greater distance each consecutive second.
- e. After Kirk has walked for 8 seconds Kirk changes directions and begins walking toward the park bench.

Question 8

The equation of a line is given by $x + 2y = 14$. What is the slope of this line?

- a. $-\frac{1}{2}$
- b. $\frac{1}{2}$
- c. -2
- d. 2
- e. 14

Question 9

The given table shows ordered pairs for an exponential function. What is the value of y when $x = 0$?

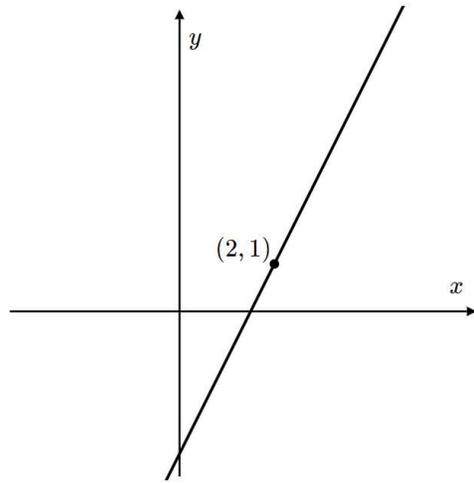
x	2	3	4
y	36	108	324

- a. -108
- b. -36
- c. 4
- d. 12
- e. 324

Question 10

The graph to the right shows how x and y are related. The constant rate of change of y with respect to x is 2. What is the value of y when $x = 3.5$?

- a. $y = 2.5$
- b. $y = 3$
- c. $y = 4$
- d. $y = 7$
- e. None of the above



Question 11

José plants a 7-inch tomato plant in his garden. The plant grows by about 13% per week for several months. Which formula represents the height h of the tomato plant (in inches) as a function of the time t in weeks since it was planted?

- a. $h = 7(0.13)^t$
- b. $h = 7 + 1.13t$
- c. $h = 7(1.13t)$
- d. $h = 7(1.13)^t$
- e. $h = 7 + 0.13t$

Question 12

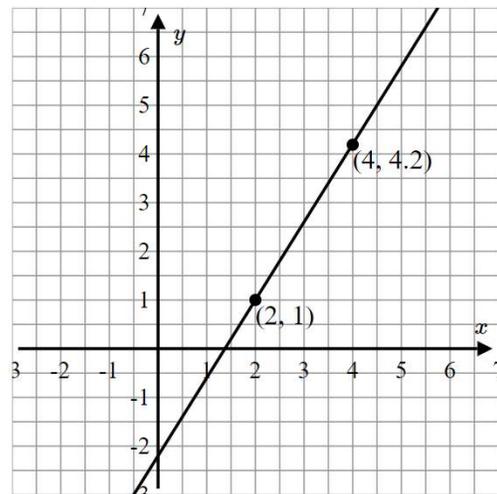
A vase with flowers is filled with 10 cups of water. The flowers in this vase drink the water at a constant rate of 1.2 cups per day. Which formula determines the number of cups of water in the vase, n , in terms of the number of days, t , that have elapsed since the vase was filled.

- a. $n = 1.2t$
- b. $n = 1.2t + 10$
- c. $n = 1.2 - 10t$
- d. $n = -1.2t$
- e. $n = 10 - 1.2t$

Question 13

Consider the line shown to the right. Which of the following is an equation for this line?

- a. $y = 1.6x - 2$
- b. $y = 1.6(x - 2) + 1$
- c. $y = 3.2x - 2$
- d. $y = 3.2(x - 2) + 1$
- e. $y = 3.2x$



Question 14

The cost of replacing the exhaust muffler on your car is currently \$195. The previous time that you had the same replacement done, the cost was \$131. What is the percent increase in your repair bill (rounded to the nearest percent)?

- a. 33%
- b. 49%
- c. 64%
- d. 67%
- e. 149%

Question 15

Suppose x and y vary together such that $y = 0.96x + 0.15$. Which of the following statements best describes what 0.96 conveys in the context of this situation?

- a. For any change in x , Δx , the change in the y is $0.96\Delta x$.
- b. For any change in y , Δy , the change in x is $0.96 \Delta y$.
- c. y increases by 96% whenever x increases by 1.
- d. Any time x increases, y increases by 0.96.
- e. None of the above.

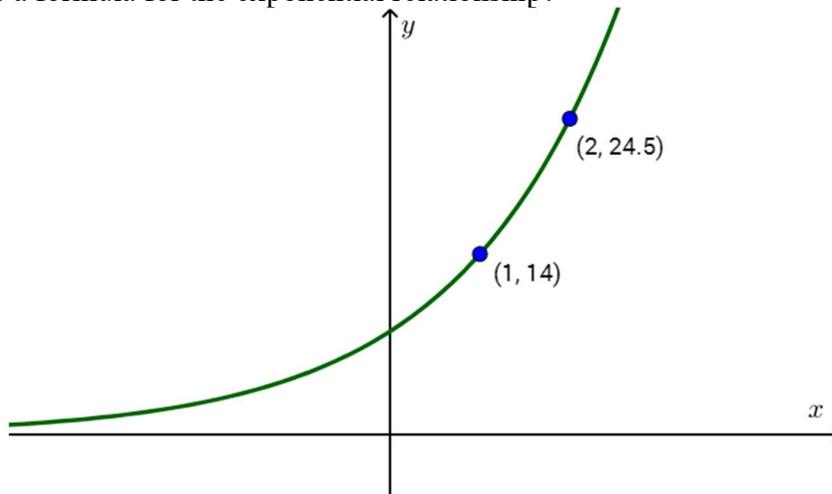
Question 16

A culture containing 14 bacteria doubles every day. Which of the following formulas represents the number of bacteria, p , in the culture after t days?

- a. $p = 14 + 2t$
- b. $p = 28t$
- c. $p = 14(1.2)^t$
- d. $p = 14^{2t}$
- e. $p = 14(2)^t$

Question 17

The two points shown are ordered pairs for an exponential function. Which of the following is a formula for the exponential relationship?



- a. $y = 8(1.75)^x$
- b. $y = 14(1.75)^x$
- c. $y = 8(10.5)^x$
- d. $y = 10.5(x - 1) + 14$
- e. $y = 10.5x + 14$

Question 18

A line has a slope of 2.5 and the point $(-2, 1)$ is on the line. Which of the following is an equation for the line?

- a. $y = 2.5x + 1$
- b. $y = 2.5(x - 2) + 1$
- c. $y = 2.5(x + 2) + 1$
- d. $y = 2.5(x - 2) - 1$
- e. $y = 2.5x$

Question 19

If $y = 3^x$ and x is increased by 2, then...

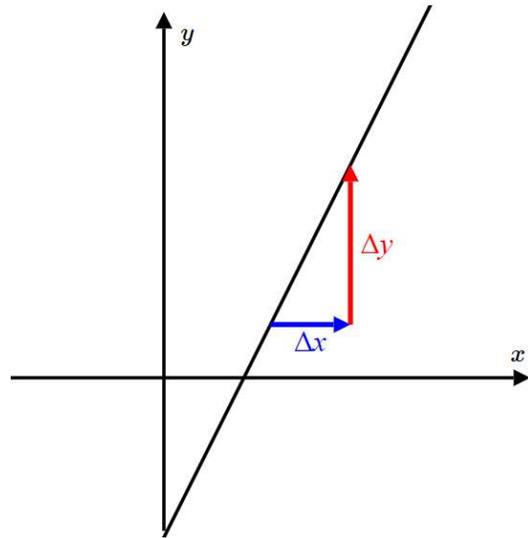
- a. y increases by 2.
- b. y increases by 9.
- c. y increases by a factor of 2.
- d. y increases by 6.
- e. y increases by a factor of 9.

Question 20

x and y vary together and are related by a constant rate of change. The graph of y in terms of x is shown to the right (the axes each have the same scale).

Use the graph to *approximate* the value of m , the constant rate of change of y with respect to x .

- a. -2
- b. -1
- c. 1
- d. 2
- e. Not enough information



Question 21

Which of the following logarithmic equations is equivalent to the statement $5^{-3} = \frac{1}{125}$?

- a. $\log_5\left(\frac{1}{125}\right) = -3$
- b. $\log_5(125) = -3$
- c. $125 \cdot \log(5) = -3$
- d. $\log_3(5) = \frac{1}{125}$
- e. $\log_{-3}\left(\frac{1}{125}\right) = 5$

Question 22

Two cars that are originally 216 miles apart start traveling toward one another at 9 am. One car travels 62 miles per hour and the other travels 68 miles per hour. Both cars travel at a constant speed. Which formula represents d , the distance between the two cars in miles, in terms of the number of hours after 9 am, t ?

- a. $d = 62t + 68t$
- b. $d = 216 - 62t$
- c. $d = 68t - 62t$
- d. $d = 216 - (68t + 62t)$
- e. $d = 216 + (68t + 62t)$

Question 23

A line has a slope of -4 and a y -intercept of -2.5 . Which of the following is an equation for the line?

- a. $y = -4x$
- b. $y = -4x + 2.5$
- c. $y = 4x + 2.5$
- d. $y = 2.5x - 4$
- e. $y = -4x - 2.5$

Question 24

Jim bought a computer on sale at a 20% discount off of the original price. If Jim's sale price was \$440.00, what was the original price of the computer?

- a. \$352.00
- b. \$366.67
- c. \$528.00
- d. \$550.00
- e. \$2200.00

Question 25

The amount of medicine (in milligrams) in a patient's bloodstream t hours since taking a pill can be modeled by $y = 300(0.85)^t$. The patient must take another pill when the amount of medicine in her bloodstream reaches 100 milligrams. How long since taking the first pill does the patient need to take the second pill?

- a. 6 hours
- b. 6.75 hours
- c. 7 hours
- d. 8.5 hours
- e. 32.6 hours

APPENDIX B
STUDENT BELIEFS SURVEY

Thank you for taking the time to fill out this survey. The completion of this survey is required for your course grade; however, you are not required to share your responses as part of a research study. If you consent to sharing your responses, your identity will not be disclosed to any party and your responses will be anonymous. All data is confidential. Therefore, consenting to sharing your anonymous responses requires no extra effort, and does not subject you to any harm. If you refuse to consent to sharing your anonymous responses, it will *not* impact your grade on this assignment. Please indicate below whether or not you consent to having your anonymous responses be used as part of a research study.

- I **do** consent to having my (anonymous) responses be part of a research study.

Background Information

Your name:

Your teacher's name:

Your student id number: *

Your school: *

Course name: *

Gender: *

Female

Male

Grade level: *

Freshman

Sophomore

Junior

Senior

Other:

What was your last mathematics course?

Algebra II

College Algebra

Precalculus

Calculus I

Other:

What grade did you receive in your last mathematics course?

A

B

C

D

F

When did you complete your last math course?

2017

2016

2015

2014

Prior to 2014

What grade do you expect (or did you receive) in this course?

A B C D F

Do you intend to take another math course? *

Yes

No

I'm not sure

Affect: Confidence, Mathematical identity and Enjoyment

I enjoy mathematics

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I see myself as a strong mathematics student

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I get frustrated when asked to complete challenging problems on my own

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I am confident in my mathematical abilities.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

If I had a choice I would not take another mathematics course

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

Students' Mathematical Methods/Practices

Making unsuccessful attempts when solving a mathematics problem is a natural part of doing mathematics.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

When trying to solve a problem that is challenging for me I persist until I find an answer.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I learn mathematics best when someone shows me steps for getting an answer.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I try to understand mathematics instead of just memorize how to get an answer.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I enjoy seeing how ideas in mathematics are connected.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I am good at working word problems.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

When reading a word problem I typically draw a picture to represent the situation.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

When working a word problem I try to make sense of the problem.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

Students' experience in CAO

I prefer to learn from a teacher in a face-to-face classroom setting.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I understand ideas better when they are taught online.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

My understanding improved as a result of watching the course videos.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

The lessons in this course were confusing.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I stayed engaged when watching the videos.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I was unable to follow the reading in the online lessons.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

The homework was aligned with the lessons.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

The homework reinforced my understanding of the content.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

The amount of homework was adequate for helping me learn the ideas.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I enjoyed working through the online lessons.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

Doing mathematics in an online environment is frustrating.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

The online instruction focused primarily on helping me understand and use ideas in the course.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I will recommend this online course to a friend.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I was able to complete the problems in this online course.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

I think the ideas in this course will be useful to me.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

What aspects of this online course did you like the most?

<text box>

What aspects of this online course do you like the least?

<text box>

What recommendations do you have for improving this online course?

<text box>

On a scale of 1-5, rate the usefulness of each component of this online course.

- 1 Not at all Useful
- 2 Slightly Useful
- 3 Moderately Useful
- 4 Very Useful
- 5 Exceptionally Useful

- ___ videos
- ___ animations and applets
- ___ written explanations in the lessons
- ___ questions I completed while working through the lessons
- ___ homework sets
- ___ quizzes

The nature of mathematics (in this course)

I see a mathematics formula as a means of representing how the values of two quantities change together.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

A variable is a letter that stands for a single unknown value.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

Solving equations in mathematics is useful.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

A function's graph represents how the values of two quantities change together.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

Solving an equation involves finding a specific input value for a formula when an output value for the formula is known.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

Evaluating a function involves finding a specific input value of the function for a specific output value.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

A variable is used to represent the varying values of some quantity.

- 1 Strongly Disagree
- 2 Disagree
- 3 Slightly Disagree
- 4 Slightly Agree
- 5 Agree
- 6 Strongly Agree

APPENDIX C
CLARIFICATION OF THIOMPSON ANALYSIS

Thompson (personal communication, November 7, 2018) explained that this characterization may not accurately capture his theory. His theory on quantitative reasoning derived originally from work with children and analyzing how meanings they developed or that were productive in various situations might also be productive in the future. Thompson's later work then traced those implications forward while working with progressively older students. It was through conceptual analysis that he was able to both look forward to consider implications of learning and look backwards to consider productive ways to build for future learning. Thompson relayed his concern that my characterization might frame his work as insensitive to children's reasoning and a "top-down" approach whereby learning goals for all levels of mathematics are dictated only by specific ways of understanding Calculus ideas.

As my work moves toward formal publication, I will correct errors in my characterization. It was not my intent to suggest that Thompson's conceptual analysis is not based on a reciprocal relationship between looking forward and looking backwards to articulate productive learning goals. It was only my intention to help emphasize why Thompson's learning goals for exponential functions are markedly different from other researchers'—namely conceptualizing function values emerging in a piecewise linear manner. This goal, at least in part, is informed by his conceptualization of the Fundamental Theorem of Calculus.

APPENDIX D
ASU IRB APPROVAL



EXEMPTION GRANTED

Mathematics and Statistical Sciences, School of
480/965-2891
MARILYN.CARLSON@asu.edu

Dear Marilyn Carlson:

On 12/11/2017 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	Studying the Effectiveness of Online Curriculum Materials in College Algebra and Precalculus
Investigator:	Marilyn Carlson
IRB ID:	STUDY00006662
Funding:	Name: National Science Foundation (NSF), Grant Office ID: TKS0202, Funding Source ID: 1323753
Grant Title:	TKS0202;
Grant ID:	TKS0202;
Documents Reviewed:	<ul style="list-style-type: none"> • Lesson Capture Interview Protocol.pdf, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • NSF Grant Award.pdf, Category: Sponsor Attachment; • CITI Course Completion Alan, Category: Other (to reflect anything not captured above); • Sample Online Lesson and Homework.pdf, Category: Participant materials (specific directions for them); • Consent Form for Interviews, Category: Consent Form; • General Consent Form (Control), Category: Consent Form; • Beliefs Survey (Post).pdf, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Recruitment Statements, Category: Recruitment Materials; • NSF Award, Category: Sponsor Attachment; • CAO Study Protocol, Category: IRB Protocol; • General Interview Protocol.pdf, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Beliefs Survey (Pre).pdf, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Concept Assessment.pdf, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Table of Contents.pdf, Category: Resource list; • General Consent Form (Treatment), Category: Consent Form;

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (1) Educational settings, (2) Tests, surveys, interviews, or observation on 12/11/2017.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

Sincerely,

IRB Administrator

cc: Grant Sander
Marilyn Carlson
Grant Sander
Alan O'Bryan