The Relationships Between Meanings Teachers Hold and Meanings Their Students Construct

by

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ABSTRACT

This dissertation reports three studies of the relationships between meanings teachers hold and meanings their students construct.

The first paper reports meanings held by U.S. and Korean secondary mathematics teachers for teaching function notation. This study focuses on what teachers in U.S. and Korean are revealing their thinking from their written responses to the MMTsm (Mathematical Meanings for Teaching secondary mathematics) items, with particular attention to how productive those meanings would be if conveyed to students in a classroom setting. This paper then discusses how the MMTsm serves as a diagnostic instrument by sharing a teacher's story. The data indicates that many teachers name rules instead of constructing representations of functions through function notation.

The second paper reports the conveyance of meaning with eight Korean teachers who took the MMTsm. The data that I gathered was their responses to the MMTsm, what they said and did in the classroom lessons I observed, pre- and post-lesson interviews with them and their students. This paper focuses on the relationships between teachers' mathematical meanings and their instructional actions as well as the relationships between teachers' instructional actions and meanings that their students construct. The data suggests that holding productive meanings is a necessary condition to convey productive meanings to students, but not a sufficient condition.

The third paper investigates the conveyance of meaning with one U.S. teacher. This study explores how a teacher's image of student thinking influenced her instructional decisions and meanings she conveyed to students. I observed 15 lessons taught by a calculus teacher and interviewed the teacher and her students at multiple points. The results suggest that teachers must think about how students might understand their instructional actions in order to better convey what they intend to their students.

The studies show a breakdown in the conveyance of meaning from teacher to student when the teacher has no image of how students might understand his or her statements and actions. This suggests that it is crucial to help teachers improve what they are capable of conveying to students and their images of what they hope to convey to future students.

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OUTLINE OF THE THREE STUDIES

My dissertation is the three-paper format from the data that the project Aspire team and I gathered: (1) U.S. and Korean teachers' responses to the MMTsm instrument that project Aspire team developed, (2) classroom observations, pre- and post-interviews with eight Korean teachers and their students, (3) classroom observation, pre- and post-interviews with one U.S. teacher and her students.

Study one is about meanings held by U.S. and Korean secondary mathematics teachers for teaching function and function notation. The purpose of study one is not to compare teachers' meanings in the two countries. Rather, I focus on what teachers in U.S. and Korean are revealing their thinking from their written responses to the MMTsm items.

Study two is about the conveyance of meaning with eight Korean teachers who took the MMTsm. The data that I gathered was their responses to the MMTSm, what they said and did in the classroom lessons I observed, pre- and post-lesson interviews with them, and pre-and post-lesson interviews with their students. I will focus on the relationships between teachers' mathematical meanings and their instructional actions as well as the relationships between teachers' instructional actions and meanings that his or her students construct.

Study three is about the conveyance of meaning with one U.S. teacher. After observing the eight Korean teachers' classroom I thought it would have been better if I had observed their lessons more. I observed two lessons and one post-lesson interview with each Korean teacher. Most of them said they wanted to adjust their lessons to help students' understandings after watching students' understanding in the post-lesson interview. However, I had no chance to witness their actual adjustments in the next lesson. Thus, I changed my methodology to observing one U.S. teacher's lesson 15 times. I witnessed not only the conveyance of meanings but also how her image of students' thinking affected her adjustments and students' understandings in depth. My focus will be the teacher's image of students' thinking and how it influenced the meanings she conveyed to students.

PAPER ONE: U.S. AND SOUTH KOREAN TEACHERS'MEANINGS FOR FUNCTION NOTATION

1. INTRODUCTION

Function plays a central role in mathematics and thus the function concept is central to school mathematics. This multifaceted concept entails ideas of variable, variation, and the co-variation of quantities, along with images of a function as an object, as a transformative process and as a relationship (DeMarois & Tall, 1996). Prior research focuses on students' or teachers' ways of thinking for function. Research shows that students and teachers predominately think of functions in terms of a rule of assignment/calculation (Even, 1993; Hitt, 1998; Sajka, 2003; Vinner, 1983).

While investigating teachers' meanings for function, teachers' use of function *notation*, or lack thereof, provides a different lens through which we might understand teachers' meanings for function. According to Trigueros and Jacobs (2008), students considering the mathematical statement $\lim_{x \to a} f(x) = f(a)$ possess dynamic images of the independent value when thinking "as x goes to a", but they think only of a static dependent value $f(a)$ rather than imagining values of $f(x)$ varying as *x varies*. In line with this, we suspected that even those teachers who think of x as varying might think of function notation $f(x)$ as a static value instead of a varying quantity. This way of thinking might prevent them from thinking of functions as relationships between covarying quantities. If our hypothesis was correct, then understanding teachers' use of function notation can give information about teachers' understanding of function.

Historically, mathematicians recognized a need to represent the values of two quantities changing in tandem without having a formula or other means of explicitly representing the relationship between them (Cajori, 1928, 1929). To resolve this quandary and convey their intended meaning, mathematicians established conventions for notations for functions through which to express functional relationships between quantities. We make a distinction between "notation for function" and "function notation"; the former includes all conventional symbolic representations of function such as $g : Z \to Z$, $f(x)=2x$, or $x \mapsto \ln(x)$, whereas we use the latter to refer specifically to Euler's modern notation[1] (i.e. $y = f(x)$). This notation, $f(x)$, is the focus of this paper.

According to Thompson and Carlson (2017), the correspondence definition for function[2] does not entail notions of variation and covariation; rather, it supports individuals in thinking about discrete values plucked from a set without a sense of continuous change. Yet the ideas of variation and covariation are foundational to the study of mathematics, even in advanced mathematics. For instance, the formal definition of the limit of a function as x approaches some value entails both variation and covariation, the latter stemming from the choice of delta depending on the value of epsilon. One way to support those key ideas is to use function notation as a representational tool without needing a specific rule of assignment. We will say someone uses function notation representationally when he uses function notation to represent the value (that might vary) of a function in relation to a value in its domain, without mentioning an actual rule of assignment.

Project Aspire developed the *Mathematical Meanings for Teaching Secondary Mathematics* (MMTsm), a 44 item diagnostic instrument designed primarily to give

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[1] Leonhard Euler used the word "function" to describe any expression consisting of a variable and some constants. He introduced the notation $y = f(x)$ in 1734 (Cajori, 1929).

[2] y is a function of x, for a given domain of values of x, whenever a precise law of correspondence between x and y can be stated clearly (Boyer, 1946, p. 13 as cited in Thompson & Carlson, 2017) .

professional developers insight into mathematical meanings with which teachers operate. The project team then categorized teachers' responses by asking ourselves "how productive would meanings we can discern from the teacher's response be for a student were the teacher to convey it?" Thus, a high level of response in the MMTsm instrument means that the teacher's meanings were potentially productive for students, because the team thought the response would benefit students when students develop that meaning.

It is worthwhile to explain what we mean by productive meanings for student learning. Being productive for student learning is not the same as either being mathematically correct. For example, a formal definition of function, "a relation between sets that associates to every element of a first set exactly one element of the second set[3]" is mathematically correct, but might not be productive for a high school student because the formal definition does not entail that the independent variable's values in the domain are varying. On the other hand, a meaning for continuous function that is a graph that can be sketched without taking off a pencil is mathematically incorrect[4], but could be productive for high school students. We will present productive meanings that allow students to build robust understanding of function notation in the theoretical perspective section.

Our thought on "how productive would meanings we can discern from the teacher' response be for a student were the teacher to convey it" allows us to discuss productive meanings for student learning. Function notation is a means to represent a function value even when a rule is unknown. However, in the U.S., we have a culture of

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[3] http:// en.wikipedia.org/wiki/Function_(mathematics)#cite_note-1

[4] For example, $y=\frac{1}{x}$ cannot be sketched without taking off a pencil, but is continuous on the domain.

labeling rules with function notation. Often times, mathematics textbooks use *f* as the name for any arbitrary function. This practice is consistent with viewing notation as a label; in this case *f* for function (Booth, 1988; Kinzel, 1999).

Substituting this into the expression for C, we have $$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$ Therefore, the equation $$C(w) = 20w^2 + \frac{180}{w} \qquad w > 0$$ expresses C as a function of w.	$f(a) = \pi r^2(t)$

Figure 1. Left-Textbook Statement (Stewart, 2008, pg. 15); Right-a Teacher's Definition of a Function Regarding Inconsistent Use of Variables.

Stewart (2008) used *P* as the name of his function dealing with the world's population and in *Figure **1***, Stewart's *C* represents the cost of building a container. In addition to textbooks, Khan Academy participates in this same culture. In Khan Academy's Function Notation series for eighth graders (Khan, 2015), Khan also tends to use *f* (even calling *f* a variable in the first video) and using label-names such as *V* for volume. To further confound students, Khan also switches between using *f* and *f*(*x*) as the name of a function. This cultural practice negates the representational power of function notation.

We hypothesized that the meanings teachers expressed in the instrument were their mathematical meanings for teaching, but this was in the project team's judgment. We did not claim that the assessment predicted teachers would express this meaning in their classrooms. Rather, teachers' responses in the MMTsm were mathematical meanings they held when they took the instrument, and we hypothesized that they would afford and constrain their instructional actions and decisions, which then would affect what students understand from the teacher's instruction. We will use teachers' MMTsm responses as an indicator of their meanings at the moment of the assessment and thereby as a starting point for analyses of teachers' instructional actions and students' conveyed meanings.

In this study, we will focus on U.S. and Korean teachers' meanings about function notation, with particular attention to how productive those meanings would be if conveyed to students in a classroom setting. We will describe what teachers are revealing about their thinking from their answers on function items. In particular, we will attend to teachers' meanings for function notation as a means to extend the research into teachers' teaching and students' learning. We will then discuss how the MMTsm serves as a diagnostic instrument by sharing a teacher's story; meanings he demonstrated in the MMTsm, meanings he expressed in his classroom, and meanings his students developed from experiencing the lesson.

2. LITERATURE REVIEW

Researchers have focused on students' and teachers' understandings of the concept of function. Sierpinska (1992) and Sfard (1991) identified obstacles and difficulties in students' understanding of the concept of function. Even (1993) reported that a large number of pre-service teachers thought "functions are equations and can always be represented by formulas" (p. 104). Similarly, Vinner (1983) found that 10th and 11th grade students think a "function is a rule of correspondence", and that "function is an algebraic term, a formula, an equation, and arithmetical manipulations"(pp. 8-9). Hitt (1998) also identified that teachers are likely to expect continuous functions to be defined by a formula.

Carlson (1998), Sajka (2003), Musgrave and Thompson (2014), and Thompson and Milner (2019) shifted focus to teachers' understanding of function notation. They found that function notation served as a label for the defining formula rather than a representation of one quantity's values in relation to another quantity's values. Booth (1988) and Kinzel (1999) both found that students think the letters used in equations and functions such as *f* or *b* are abbreviations for the names of actual objects. For example, a student might think that *3b* refers to exactly three books rather than three times the value of *b*. Greeno and Hall (1997) pointed out that different notations and symbols are "potential representations," achieving full representational status only when the student constructs them as such in various problem-solving contexts. Arcavi (1994) emphasized the need for teachers to encourage students to understand both the usefulness and applications of symbols, and to have the aim that students establish reflexive relationships between symbols and their referents.

That students view function notation as a label might explain why so many studies identify function notation as a source of students' difficulty (Carlson, 1998; Dreyfus & Eisenberg, 1982; Vinner & Dreyfus, 1989). In particular, Sierpinska (1992) highlights the problem of sloppy language surrounding function notation that allows students to think of "$f(x)$" as both the value of the function f with respect to a value of x and as a four-character name of the function.

Thinking of functions in terms of the label-rule visual structure echoes prior research showing that students often think of a function as requiring an explicit rule (Even, 1993; Sajka, 2003; Vinner, 1983). Students often think if there is no rule, there is no function. Normative use of function notation allows for communication about a functional relationship even in the case where an explicit rule is unknown. We suspect that making the representational power of function notation explicit for teachers could help them support students in developing a meaning for function beyond that of a rule. Additionally, we also believe that the representational power of function notation could resolve thinking of $f(x)$ as a label because students can understand that $f(x)$ represents a value of one quantity with respect to the value of a second quantity according to a relationship between them named f.

Many studies of students' meanings for function notation drew on procept theory (Gray & Tall, 1994). There are three components in a procept: a process that produces a mathematical object (or concept), the object or concept, and a symbol that a person can take to represent either the process or the object that the process produces. According to Gray and Tall (1994), the statement $f(x) = x^2 - 3$ tells both how to calculate the value of the function for a particular value of x and encapsulates the complete concept of the function for a general value of x. Sajka (2003) described a student's understanding of function notation, and identified sources of the difficulties based on procept theory. The student in Sajka's study thought that the letter f in $f(x)$ or $f(y)$ represents "the beginning of an equation" or "a formula of a function" (Sajka, 2003, p. 238). Sajka also reported that the student understood "$f(x)$" as the formula of a function that consists of the symbol "=", the letter f on the left hand side, and an algebraic expression with the variable x on the right hand side (Sajka, 2003, p. 239).

In this study, we offer further support for findings of other studies related to students' or teachers' difficulties in understanding of function (Carlson, 1998; Dreyfus & Eisenberg, 1982; Even, 1993; Gray & Tall, 1994; Hitt, 1998; Musgrave & Thompson, 2014; Sajka, 2003; Sfard, 1991; Sierpinska, 1992; Vinner, 1983; Vinner & Dreyfus, 1989). We then extend the literature by exploring teachers' meaning for function notation; in particular, we hypothesize that many teachers name rules rather than construct representation of functions in terms of using function notation. In exploring consequences of this study, we point to potential problems for these teachers in supporting students to develop fluency with function notation and an image of function notation as a representational tool.

3. THEORETICAL PERSPECTIVE

3.1. Meanings

Coherent mathematical meanings serve as a foundation for future learning, so it is important that students build useful and robust meanings. One way students develop meanings is by trying to make sense of what their teacher say and do in the classroom. Before discussing how meanings are conveyed in the classroom, we will explain what we mean by *meanings*. According to Piaget, to understand is to assimilate to a scheme (Skemp, 1962, 1971; Thompson, 2013; Thompson & Saldanha, 2003). Thus, the phrase "a person attached a meaning to a word, symbol, expression, or statement" means that the person assimilated the word, symbol, expression, or statement to a scheme.

A scheme is an organization of ways of thinking, images, and schemes (Thompson, Carlson, Byerley, & Hatfield, 2014). When we say *assimilate* we mean the ways in which an individual interprets and make sense of a text, utterance, or self-generated thought. According to Piaget, repeated assimilation is the source of schemes, and new schemes emerge through repeated assimilations, which early on require functional accommodations and eventually entail metamorphic accommodations (Steffe, 1991).

We focus on teachers' mathematical meanings because of their centrality in students' construction of meaning. In classrooms, students might construct their meanings from their peers, from prior schemes, from resources the teacher selects for them or resources they find on their own. However, we suspect that a main source of students' mathematical meanings lies in what teachers say and do. Students try to assimilate what the teacher says and does using their understandings of what is being taught. In doing so, the students will adjust what they say and do according to their understanding of what their teacher intends. In this sense, conversations in the classroom between a teacher and students entail mutual attempts by the teacher and students to understand each other. We suspect that teachers exert less effort in this regard than do students, and hence teachers have a greater impact on students' meanings than do students have on the teacher's meanings.

Our theory of meaning, and of ways meanings are conveyed through mutual interpretation, allows us to bridge theoretically what teachers know, what they teach, and what their students learn. While we cannot access the teachers' mathematical meanings directly, we can delimit categories of responses according to particular mathematical meanings that we discern from them. We categorize teachers' response based on meanings we believe might underlie the response based on the best available evidence of interviews and prior qualitative work. We assumed that, for the most part, meanings that teachers used to construct their responses to an item are meanings that would guide their decisions in the classroom. Our focus on teachers' meanings as a root for their actions allows us to think of meanings we think students might construct based on meanings we attribute to teachers.

3.2. Productive Meanings for Function Notation

Function notation $f(x)$ represents the function's output values for varying values of the input x in the context of defining a function. Thompson (2013b) addressed the issue of understanding of function notation by parsing the components of a function definition and how they work together to make a definition. A function defined using function notation includes the name of the function, the variable that represents a value of the input, and a rule that states how to determine the function's output with respect to a given input. It also makes clear that the function definition is the entire statement, just as a dictionary definition is the entire entry made by a word together with its meaning.

According to *Figure **2***, we can interpret the meaning of each part of a function's definition as follows:

- "u" represents an element of the domain, called an input value—the value at which to evaluate the function.
- "V" represents the relationship that ties values in the function's domain to values in its range.
- "$V(u)$" represents the output of the function V corresponding to the input u.
- "=" means "is defined as".

The entire left-hand-side of the above function definition represents the output of the function V as we use "$V(u)$" to represent the output of the function V, regardless of whether we know V's rule of assignment. In addition, the whole statement, including the left-hand-side, "=", and the right-hand-side constitutes the definition of V using function notation. A function definition is the pairing of the function name with a rule of assignment. Moreover, in the same way a word is used to represent its meaning, a function name can be used to represent the relationship it describes.

$$\overbrace{\overbrace{\overbrace{V}^{\text{Name}}\ (\ \overbrace{u}^{\text{Input}}\ )}^{\text{Output}} = \overbrace{u(13.76-2u)(16.42-2u)}^{\text{Rule for how to produce an output}}}^{\text{Function definition}}$$

Figure 2. Parts of a Function Definition via Function Notation (Thompson, 2013b)

It is worth noting that once a teacher or student establishes a function definition, he or she must reconceive of the above collection of symbols to use function notation representationally. For instance, after defining the function V in *Figure* ***2***, a teacher can call on $V(4)$ to refer to the value of V when the input has a value of 4. If it proves useful, that teacher could actually compute the value of $V(4)$ using the definition:

$$V(4) = 4(13.76 - 2(4))(16.42 - 2(4))$$

Notice, however, that in this mathematical sentence, the meaning of the equal sign is no longer "is defined as" because the teacher had already defined V. Instead, the equal sign refers to an equality of values. This means that after an individual defines a function with an explicit algebraic rule of assignment, it is valid to say that the value represented through function notation (e.g. $V(4)$) is the same as the value represented by the rule for a given input (e.g. 4(13.76–2(4))(16.42–2(4))). We wish to stress, however, that not every function is defined with an explicit algebraic rule, and hence it is powerful to think of function notation as a means for representing values of the output relative to a given input.

4. METHOD

4.1. The Development of MMTsm

Developing the *MMTsm* items and rubrics for the *MMTsm* was a multi-year process involving many researchers. Thompson (2016) explained the process of creating items and rubrics for the *MMTsm*. Here is a summary of the steps of the process:

1) Create a draft item, interview teachers using the draft item, and give the item to mathematics and mathematics educators for review.
2) Revise the item, interview teachers again.
3) Administer items to a large sample of teachers and analyze responses in terms of the meanings they reveal.
4) Retire unusable items.
5) Interview teachers to understand why they gave the response that they did.
6) Revise items, potentially using teacher responses to make items multiple-choice options.
7) Administer revised items to a large sample of teachers.
8) Develop scoring rubrics.

After a first round of data collection in 2012 from 144 teachers, the Project Aspire research team categorized the thinking revealed in the items using a modified grounded theory approach (Corbin & Strauss, 2007). The modification is that we analyzed teachers' responses with theories of the nature of mathematical meanings that were germane to particular tasks. During team discussions of rubrics and responses, team members continually asked themselves "how productive would this response be for a student if this is what the teacher actually said while teaching?" The team piloted the *MMTsm* instrument with 460 high school mathematics teachers in the U.S. After the 2013 pilot with revised items, we developed a scoring rubric for each item by grouping grounded codes into levels based on the quality of the mathematical meanings. The *MMTsm* instrument contains items that assess teachers' meanings for covariation, function, proportionality, frame of reference, rate of change, and structure sense (Byerley & Thompson, 2014; Joshua, Musgrave, Hatfield, & Thompson, 2015; Musgrave, Hatfield, & Thompson, 2015; Musgrave & Thompson, 2014; Yoon, Byerley, & Thompson, 2015).

The first author produced a Korean translation of the *MMTsm* in the summer of 2014 using the method of translation and back translation (Behling & Law, 2000; Harkness, Van de Vijver, Mohler, & fur Umfragen, 2003). The first author translated each item into Korean, and then a Korean mathematics Ph.D. student translated the items back into English. The Korean Ph.D. student was a high school mathematics teacher and wrote items for the Korean version of the practice SAT test given by the Korean Education Office. The second author reviewed the back translations and the first author made adjustments to the Korean versions. Then, the first author piloted 43 items[5] to a convenience sample of 66 Korean teachers who taught mathematics in grades 7 to 12.

Based on the pilot study in the summer of 2014, the first author made adjustment to the Korean versions. Dr. Oh Nam Kwon assisted in recruiting 366 Korean secondary teachers (264 high school[6], 102 middle school; Table **1**). The 366 Korean mathematics teachers were taking a qualification program[7] in Seoul, a suburb of Seoul, the Southwest, and Southeast South Korea. We then administered 43 items of the *MMTsm* to Korean teachers. For scoring Korean teachers' responses, the first author trained a five-person

[5] We removed two items that did not work because of differences in language. One item was added after the summer of 2014.
[6] In Korea, middle school teachers teach mathematics to grade 7th-9th students and high school teachers teach mathematics to grade 10th-12th students.
[7] In Korea, all teaches who have taught more than three years must take a qualification training program to earn "1st class" teacher certificates.

team of mathematics education Ph.D. students in a Korean university on scoring teachers' responses and conducting inter-scorer reliability (ISR) tests during training, during scoring and after scoring so that the five Korean scorers and the first author had consistent scores. Project Aspire paid for the recruitment of Korean teachers and paid stipends to the Korean scorers.

4.2. Subjects

The Project Aspire team administered the *MMTsm* to 366 SK (South Korea) middle and high school teachers and 252 US high school teachers in 2013, 2014, and 2015 (see Table **1**). The 252 US high school teachers volunteered to participate in summer professional development projects taking place in the Southwest and Midwest U.S. They took the *MMTsm* as part of their professional development program.

Table 1. *US and SK Teachers, School Level by Major*

	Math Majors	MathEd Majors	Other Major	Total
Korea High School teachers	81	175	7	263
Korea Middle School teachers	33	49	19	101
U.S. Calc+ teachers*	29	24	21	74
U.S. Calc– teachers, **	34	59	85	178
Total	177	307	132	616

* Calc+ means U.S. high school teachers who taught calculus at least once
** Calc– means U.S. high school teachers who never taught calculus
*** Two Korean teachers and one U.S. teacher did not report their degrees.

In Table **1**, "Math Majors" mean that a teacher reported having either a bachelor's or master's degree in mathematics whereas "MathEd Majors" mean that a teacher reported having either a bachelor's or master's degree in mathematics education. "Other Majors" mean that a teacher reported degree(s) that were neither mathematics nor mathematics education. We separated U.S. teachers into teachers who taught calculus at least once and teachers who never taught calculus because calculus is in the high school curriculum in

South Korea, but not in the U.S. Therefore, U.S. teachers teaching calculus have similar teaching experience to Korean high school teachers, and U.S. teachers who never taught calculus have similar teaching experience to Korean middle school teachers. Korean teachers taught for an average of 4.3 years (some switched from middle to high or vice versa during this time). The 252 high school teachers in the US taught at least one high school mathematics class (algebra and above). The Project Aspire asked the US high school teachers how many times they had taught each subject and recorded the total number of high school classes taught. On average the US high school teachers had taught 26.2 classes, which corresponds to approximately 5 or 6 years. In the following section, we present function notation items in the *MMTsm*.

4.3. Tasks

Thompson and Milner (2019) presented data on three function items, two of which are re-analyzed in this paper because two items show teachers' meanings for function notation. The first item, "Understanding Input Variables" (see *Figure 3*) is designed to reveal teachers' awareness that the letter denoting a variable in the function notation should be consistent with the letter used in the defining rule of the function.

Here are two function definitions.

$$w(t) = \sin(t-1) \text{ if } t \geq 1$$

$$q(s) = \sqrt{s^2 - s^3} \text{ if } 0 \leq s < 1$$

Here is a third function c, defined in two parts, whose definition refers to w and q. Place the correct letter in each blank so that the function c is properly defined.

$$c(v) = \begin{cases} q(\_\_\_) \text{ if } 0 \leq \_\_\_ < 1 \\ \\ w(\_\_\_) \text{ if } \_\_\_ \geq 1 \end{cases}$$

*Figure 3***.** The Item, "Understanding Input Variables" © 2015 Arizona Board of Regents. Used with Permission.

We had observed that students and teachers often think of function notation idiomatically, “*c* (*v*)” is a four-character name. If a teacher thinks of function notation as a label for the formula that is on the right hand side of a function definition, we anticipate that he or she thinks *w* must always have *t* as its input variable because *t* is a part of the function name. We categorized teachers’ responses in terms of what letters they inserted in each blank to complete the definition of *c*. The categorizations and sample responses are shown in Table **2**.

Table 2. *Categorizations and Responses to the Item “Understanding Input Variables”*

Level	Categorization	Sample teacher’s response
3	All four spaces filled with the letter *v*	$c(v)=\begin{cases} q(v) \text{ if } 0 \le v < 1 \\ w(v) \text{ if } v \ge 1 \end{cases}$
2	Filled one, two or three spaces with *v*	$c(v)=\begin{cases} q(s) \text{ if } 0 \le v < 1 \\ w(t) \text{ if } v \ge 1 \end{cases}$
1	Filled blanks with *s* and *t*	$c(v)=\begin{cases} q(s) \text{ if } 0 \le s < 1 \\ w(t) \text{ if } t \ge 1 \end{cases}$
0	- Filled something in blanks other than *s*, *t*, or *v* - The scorer cannot interpret the teacher’s response.	$c(v)=\begin{cases} q(w) \text{ if } 0 \le w < 1 \\ w(q) \text{ if } q \ge 1 \end{cases}$
IDK	I don’t know	IDK
NR	No response	

The second item, “Express the Varying Area”, is shown in *Figure 4*. The purpose of “Express the Varying Area” is to see how teachers use function notation when prompted to use function notation to represent a dynamic situation. We added “at a non-constant rate" because we anticipated that teachers would assume the radius increases at a constant rates and hence write a linear relationship such as $r = k \cdot t$.

Hari dropped a rock into a pond creating a circular ripple that spread outward. The ripple's radius increases at a non-constant speed with the number of seconds since Hari dropped the rock. Use function notation to express the area inside the ripple as a function of elapsed time.

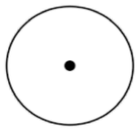
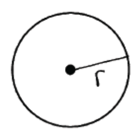


Figure 4. Express the Varying Area. © 2015 Arizona Board of Regents. Used with Permission.

The rubric for the second item focused on two features of teachers' responses: where to use function notation, and how to use variables. We separated these features because they convey different information about teachers' function schemes. Teachers' use of function notation tells us about what came to their mind when they were asked to use function notation and the defining rule is unknown. Their use of variables gives us information about their tendency to use letters to complete a function definition. The two features were scored independently.

The focus in analyzing responses to the item "Express the Varying Area" is to see whether a teacher uses function notation to represent the ripple's radius that increases at a non-constant speed, say $g(t)$, and the ripple's area, say $f(t)$, when defining a mathematical model (e.g. $f(t) = \pi\left[g(t)^2\right]$). We focused on teachers' use of function notation on both sides of the function definition to investigate teachers' representational use of function notation, even in the rule of another function definition, when the defining rule is unknown because one cannot use an explicit rule to represent a non-constant speed. We also paid attention to whether teachers used variables consistently. We categorized a response into “Used variables inconsistently” when the response has clear evidence that the teacher used variables inconsistently. A response of “I don’t know” does not have clear evidence that the teacher used variables inconsistently, so we categorized the response into “Used variables consistently”. In order to see the relationships between two dimensions (teachers’ use of function notation and teachers’ use of variables), we improved the original rubric that Thompson and Milner (2019) reported for this item. The new rubric is in Table **3**.

Table 3. *Rubric for “Express the Varying Area” Item. © 2015 Arizona Board of Regents. Used with Permission.*

	Used variables consistently	Used variables inconsistently
Level 3: Used function notation on both sides of the definition	$A(t)=\pi(r(t))^2$	$f(x)=\pi r^2(s)$
Level 2: Used function notation only in the defining rule	$\pi\left\{\int_0^t r'(t)\,dt\right\}^2$	$f(s)$ = non-constant function of t $A=\pi r^2=\pi(f(s))^2$
Level 1: Used function notation only to represent area	$A=\pi r^2$ $A(t)=\pi t^2$	 $f(r)=r^t$
Level 0: Others	$A=\pi r^2$ $=\pi(r^2)t$	$\pi r^2=\pi t^2$ ↑ seconds

Given our stance of categorizing teachers’ responses in terms of meanings rather than correctness, we were not concerned with the accuracy of the model a teacher generated. Rather, we focused only on the teachers’ use of function notation. For instance, we would consider $A(r)=2\pi r$ as a demonstration of using function notation even though the given formula does not accurately describe the area of the ripple. However, the statement $A(r)=2\pi r$ only uses function notation to represent the area and does not use function notation to represent the circle's radius in the defining rule (i.e. a function to account for the non-constant rate of change for the length of the radius with respect to the time elapsed). Thus, "$A(r) = 2\pi r$" fits Level 1 in Table **3**.

We think using function notation in both sides of the function definition in Level 3 suggests that the teacher intentionally utilizes function notation representationally – to represent a relationship between quantities whose values vary, but for which there is no known rule for the relationship. In contrast, Level 2 responses contain function notation only in the defining rule. We felt this use of function notation was more significant than if teachers only used function notation on the left to represent area (Level 1 responses) because a teacher's definition such as $\pi\left\{\int_0^t r'(t)\,dt\right\}^2$ in Table **3** conveys to students that radius is a function of elapsed time without the need for a function rule.[8] We view the use of function notation in the defining rule as indicating teachers' spontaneous use of function notation representationally.

The difference between Level 2 and Level 1 lies in whether a teacher used function notation to represent the radius' length. In many Level 1 examples, the teacher tried to create a rule to represent the non-constant rate of change, such as $r(t)=t$, instead of simply using function notation to represent that variation. We hypothesize that teachers' attempts to determine an explicit rule to describe the changing radius corresponds to a view of function as a rule and an inability to use function notation representationally. This aligns with previous research regarding meanings for function as a rule (Even, 1993; Sajka, 2003; Vinner, 1983).

[8] We ignored the fact that this teacher used "t" both as upper limit of the integral and as the variable within $r'(t)dt$.

The second scoring dimension assesses whether teachers used variables consistently. If a teacher used “t” as an independent variable in function notation such as $A(t)$, but “t” was not in the defining rule, we consider that he or she used variables inconsistently (e.g. $A(t) = \pi x^2$). The rubric for this item gave the benefit of the doubt regarding the usage of the letter “r”. In other words, if a teacher defined the area as a function of time, but still included “r” in the rule such as $A(t) = \pi r^2 t$, we interpret that this response does not contain using variables inconsistently because the teacher could have been using “r” as a parameter, not a variable. However, the rubric views “t” always as a variable to be consistent with conventional usage of “t” for varying time. Thus, the Level 1 example $f(r) = r^t$ from Table **3** was coded as using variables inconsistently.

We suspect that the prompt “use function notation” in the item’s context led teachers who used variables inconsistently to use function notation when they might not have without the prompt. In other words, they used function notation as a label when asked to, because their responses do not suggest that “f(t)” is used to represent a quantity dependent only on a varying quantity “t” .

Inter-rater agreement scoring for SK responses was conducted by having the first author and Korean scorers score 30-response subsets. Inter-rater agreement scoring for US responses was conducted by having members of the project team score 30-response subsets. “Agree” meant a perfect match in scores. Inter-rater agreement for the first item was 93.3% for SK responses (0.845 Cohen’s Kappa) and 88.0% for US responses (0.828 Cohen’s Kappa). Inter-rater agreement for the second item was 80.0% for SK responses (0.725 Cohen’s Kappa) and 96.0% for US responses (0.945 Cohen’s Kappa).

5. RESULTS

We present the distribution of teachers' responses on the two items, the relationship between the items, the relationship between the two dimensions of the second item. We also discuss interviews with teachers who took the two items and then present an example of instruction in one Korean teacher's classroom. We include the classroom example to show the potential of the MMTsm as a diagnostic instrument.

We separated analyses of U.S. and South Korean teachers' responses because the two populations were different greatly in their characteristics. We did not separate them for purposes of comparison, and we do not compare them in terms of better or worse. Rather, we simply report them separately and look within each group for relationships among responses to the items.

5.1. Results for the First Item "Understanding Input Variables"

We gave the two tasks in *Figure **3*** and *Figure **4*** to 614 secondary mathematics teachers from the US (253 high school) and South Korea (264 high school; 102 middle school). Table **4** presents results from the first item "Understanding Input Variables".

Table 4. *Results for the First Item "Understanding Input Variables"*

	Level 3	Level 2	Level 1	Level 0	I don't know	No Response	Total
Korea HS	203 (76.9%)	1 (0.4%)	14 (5.3%)	39 (14.8%)	2 (0.8%)	5 (1.9%)	264 (100%)
Korea MS	65 (63.7%)	0 (0.0%)	6 (5.9%)	19 (18.6%)	1 (1.0%)	11 (10.8%)	102 (100%)
US Calc+	32 (43.2%)	5 (6.8%)	25 (33.8%)	7 (9.5%)	3 (4.1%)	2 (2.7%)	74 (100%)
US Calc–	53 (29.6%)	7 (3.9%)	74 (41.3%)	20 (11.2%)	13 (7.3%)	12 (6.7%)	179 (100%)

* Cells contain number of respondents total and percent of row total.

Approximately 77% of SK high school teachers (203 of 264) filled the letter *v* in all four spaces whereas 43% of US Calc+ teachers (85 of 253) did. In addition, about 64% of SK middle school teachers (65 of 102) placed the letter *v* in all four blanks whereas 30% of US Calc– teachers did. It seems that both SK high and middle school teachers were more sensitive to the role of input variables (*s*, *t* or *v*). Approximately 6% of both SK high school teachers (15 of 264) and SK middle school teachers (6 of 102) filled *s* or *t* in the blanks (Level 2 or Level 1). About 41% of US Calc+ teachers (30 of 74) and 45% of US Calc– teachers (81 of 179) used *s* or *t* in at least one blank (Level 2 or Level 1), which suggests that *w* must have *t* with it because *t* is a part of the function name. US teachers seemed more likely than SK teachers to think "c(v)" was the function's name, which means that "v" in $c(v)$ is a part of the function name. To conduct statistical tests of the relationship between the items we combined Level 2 and Level 1 in Table **4**. The results of the relationship between the items will be presented in section 5.3.

Thompson (2013b) suggests a reason for the teachers' responses in Level 2 and Level 1. Teachers thought of function notation as a four-character symbol that can be replaced with a letter "y" (Thompson, 2013b). Teachers who filled the blanks with *s* or *t* might consider "$w(t)$" as one symbol because they thought they could replace "$w(t)$" with "y".

5.2. Results for the Second Item "Expressing the Varying Area"

Table **5**, Table **6**, Table **7**, and Table **8** present results from the second item "Express the Varying Area".

Table 5. *Results for the Second Item "Express the Varying Area" from SK High School Teachers*

	Level 3	Level 2	Level 1	Level 0	I don't know	No Response	Total
Use variable consistently	86	77	21	48	0	0	232
Use variable inconsistently	0	0	11	0	0	0	11
I don't know	0	0	0	0	9	0	9
No Response	0	0	0	0	0	12	12
Total	86 (32.6%)	77 (29.2%)	32 (12.1%)	48 (18.2%)	9 (3.4%)	12 (4.5%)	264 (100.0%)

* Cells contain number of respondents total and percent of row total in the last row.

Table 6. *Results for the Second Item "Express the Varying Area" from SK Middle School Teachers*

	Level 3	Level 2	Level 1	Level 0	I don't know	No Response	Total
Use variable consistently	24	15	18	23	0	0	80
Use variable inconsistently	0	2	5	0	0	0	7
I don't know	0	0	0	0	5	0	5
No Response	0	0	0	0	0	10	10
Total	24 (23.5%)	17 (16.7%)	23 (22.5%)	23 (22.5%)	5 (4.9%)	10 (9.8%)	102 (100.0%)

* Cells contain number of respondents total and percent of row total in the last row.

Table 7. *Results for the Second Item "Express the Varying Area" from US Calc+ Teachers*

	Level 3	Level 2	Level 1	Level 0	I don't know	No Response	Total
Use variable consistently	19	6	29	6	0	0	60
Use variable inconsistently	2	0	5	0	0	0	7
I don't know	0	0	0	0	6	0	6
No Response	0	0	0	0	0	1	1
Total	21 (28.4%)	6 (8.1%)	34 (45.9%)	6 (8.1%)	6 (8.1%)	1 (1.4%)	74 (100.0%)

* Cells contain number of respondents total and percent of row total in the last row.

Table 8. *Results for the Second Item "Express the Varying Area" from US Calc– Teachers*

	Level 3	Level 2	Level 1	Level 0	I don't know	No Response	Total
Use variable consistently	20	11	64	23	0	0	118
Use variable inconsistently	5	0	21	1	0	0	27
I don't know	0	0	0	0	12	0	12
No Response	0	0	0	0	0	10	10
Total	25 (15.0%)	11 (6.6%)	85 (50.9%)	24 (14.4%)	12 (7.2%)	10 (6.0%)	167 (100.0%)

* Cells contain number of respondents total and percent of row total in the last row.

Only 241 US teachers (and all 366 SK teachers) saw the "Express the Varying Area" item. The first columns in Table **5**, Table **6**, Table **7**, and Table **8** present teachers who used function notation in both sides to represent a relationship between quantities whose values vary. About 33% of SK high school teachers (86 of 264) and 28% of US Calc+ teachers (21 of 74) used function notation to represent the area and radius. Approximately 24% of SK middle school teachers (24 of 102) used function notation on both sides whereas 15% of US Calc– teachers (25 of 167) did.

Responses were scored at Level 2 if a teacher used function notation to represent the radius increasing at a non-constant speed. To conduct statistical tests of the relationship between the two dimensions we combined Level 3 and Level 2 because teachers in both levels used function notation representationally. The results of the relationship between the items will be presented in section 5.4.

Approximately 62% of SK high school teachers (163 of 264) and 36% of US Calc+ teachers (27 of 74) gave Level 3 and Level 2 responses to "Express the Varying Area" item, using function notation to represent the radius that increases a non-constant rate. About 40% of SK middle school teachers (41 of 102) and 22% of US Calc– teachers (36 of 167) responded with function notation on the right side or both sides.

In addition, about 12% of SK high school (32 of 264) and 46% of US Calc+ teachers (34 of 74) gave Level 1 responses, using function notation only to represent area. Approximately 23% of SK middle school teachers (23 of 102) and 51% of US Calc– teachers (85 of 167) responded with function notation only on the left side such as $A(t) = \pi t^2$, which indicates they used function notation to represent the area because of the prompt "use function notation" in the item. Thus, it is plausible that teachers in Level 1 used function notation like "$A(t)$" as a four-character label, only because of the prompt "use function notation".

The third rows in Table **5**, Table **6**, Table **7**, and Table **8** present 52 responses (18 SK responses and 34 US responses) that have clear evidence that the teachers used variables inconsistently. Of 18 SK teachers who used variables inconsistently, 16 responses were in Level 1, where teachers used function notation only to represent the area. Of 34 US teachers who used variables inconsistently, 26 responses were in Level 1. One example of these was $f(t) = \pi r^2$. This suggests that the 42 teachers (16 SK teachers and 26 US teachers) used function notation such as $f(t)$ as a label to replace the word "Area", since the function definition did not actually provide a rule dependent only on the varying quantity "t".

We hypothesize that teachers thought they had to use any notation when asked to use function notation from the prompt and wrote $A(t)$ to represent the area. They might consider $A(t)$ as one and $A(t)$ could replace with y, $A(t)$=y=a rule.

5.3. The Relationship Between the Two Items

With regard to the relationship between the two items, we combined Level 2 and Level 1 in the first item "Understanding Input Variables", and combined Level 3 and Level 2 in the second item "Express the Varying Area". Table **9** and Table **10** compare teachers' tendency to fill the blanks with letters to their usage of function notation. To conduct statistical tests we combined SK high school and middle school teachers, and US Calc+ and Calc– teachers.

Table 9. *Responses to "Understanding Input Variables" Compared to Responses to "Express the Varying Area" from SK Teachers*

Count	Used function notation in the defining rule	Used function notation only to represent the area	Others	Total
Filled the blanks with the letter *v*	173 (68.9%)	35 (13.9%)	43 (17.1%)	251 (100.0%)
Filled the blanks with the letter *s* or *t*	4 (28.6%)	5 (35.7%)	5 (35.7%)	14 (100.0%)
Others	27 (41.5%)	15 (23.1%)	23 (35.4%)	65 (100.0%)
Total	204	55	71	330

* Cells contain number of respondents total and percent of row total.

Table 10. *Responses to "Understanding Input Variables" Compared to Responses to "Express the Varying Area" from US Teachers*

Count	Used function notation in the defining rule	Used function notation only to represent the area	Others	Total
Filled the blanks with the letter *v*	35 (46.1%)	35 (46.1%)	6 (7.9%)	76 (100.0%)
Filled the blanks with the letter *s* or *t*	23 (24.7%)	53 (57.0%)	17 (18.3%)	93 (100.0%)
Others	5 (11.6%)	31 (72.1%)	7 (16.3%)	43 (100.0%)
Total	63	119	30	212

* Cells contain number of respondents total and percent of row total.

The results from the two items show SK and US teachers' different tendency to use function notation. Teachers' use of consistent variables and idea of function notation to represent varying quantities were more linked in SK responses than US responses. Approximately, 52% of SK teachers (173 of 330) used function notation in the defining rule *and* filled the blanks with the letter *v* on *Understanding Input Variable*. In contrast, about 17% of US teachers (35 of 212) used function notation in the defining rule *and* filled the blanks with the letter *v*.

Looking through the columns of Table **9** and Table **10**, we see that 85% of SK teachers (173 of 204) who used function notation in the defining rule in the second item filled all blanks with the letter *v*. Only 2% of SK teachers (4 of 20) who used function notation to represent the varying radius filled the blanks with *s* or *t*. In contrast, we see that US teachers showed less of a link between their use of consistent variables and idea of using function notation representationally. Approximately, 56% US teachers (35 of 63) who used function notation in the defining rule in the second item filled all blanks with the letter *v*. Among US teachers used function notation to represent the varying radius, about 37% of US teachers (23 of 63) filled the blanks with *s* or *t*.

Scanning the rows of Table **9** and Table **10**, we see that 69%of SK teachers (173 of 251) who filled the blanks with the letter *v* used function notation to represent the radius increasing at a non-constant rate. Again, we see that US teachers showed less of a link between their use of consistent variables and idea of using function notation representationally. About 46% of U.S. teachers (35 of 76) who filled the blanks with the letter *v* used function notation to represent the radius increasing at a non-constant rate, but the percent of teachers (35 of 76) who filled the blanks with the letter *v* used function notation only to represent the area. The second row of Table **10** shows that about 57% of US teachers (53 of 93) who typed “s” or “t” in the blanks used function notation only to represent the area.

The association between responses to *Understanding Input Variable* and *Express the Varying Area* from SK teachers is statistically significant ($\chi^2(4, n=330)=24.010$, p < 0.001). The association between responses to *Understanding Input Variable* and *Express the Varying Area* from US teachers is also statistically significant ($\chi^2(4, n=212)=19.037, p=.0008$), but less strong than for SK teachers. These strong associations are consistent with the hypothesis that teachers who used function notation only to represent the area thought of "f(t)" as a label because they were likely to use variables inconsistently in *Understanding Input Variables.*

5.4. The Relationship Between the Two Features of Teachers' Responses on *Express the Varying Area*

When analyzing the two dimensions' relationship in the second item, we first excluded IDK (I don't know) and NR (No response) responses from each table because IDK or NR on one dimension correlates automatically with IDK or NR on the other. Table **11** and Table **12** show the relationship between the two features of teachers' responses on *Express the Varying Area*.

Table 11. *SK Teachers' Use of Variables by Their Use of Function Notation on "Express the Varying Area"*

	Used function notation in the defining rule	Used function notation only to represent the area	Others	Total
Used variables consistently	202 (64.7%)	39 (12.5%)	71 (22.8%)	312 (100.0%)
Used variables inconsistently	2 (11.1%)	16 (88.9%)	0 (0.0%)	18 (100.0%)
Total	204	55	71	330

* Cells contain number of respondents total and percent of row total.

Table 12. *US Teachers' Use of Variables by Their Use of Function Notation on "Express the Varying Area"*

	Used function notation in the defining rule	Used function notation only to represent the area	Others	Total
Used variables consistently	56 (31.5%)	93 (52.2%)	29 (16.3%)	178 (100.0%)
Used variables inconsistently	7 (20.6%)	26 (76.5%)	1 (3.0%)	34 (100.0%)
Total	63	119	30	212

* Cells contain number of respondents total and percent of row total.

According to Table **11** and Table **12**, the two features are associated. The first column of Table **11** shows that 99% of SK teachers (202 of 204) who used function notation to represent the radius increasing at a non-constant rate used a consistent letter in the notation and the rule of the function. Similarly, the first column of Table **12** shows that 89% of US teachers (56 of 63) who used function notation to represent the radius increasing at a non-constant rate used a consistent letter in the notation and the rule of the function. In contrast, the first row of Table **11** tells us that 89% of SK teachers (16 of 18) who used variables inconsistently used function notation only to represent the area. Similarly, the first row of Table **12** tells us that 76% of US teachers (26 of 34) who used variables inconsistently used function notation only to represent the area. The association between the two features of *Express the Varying Area* from SK teachers' responses is statistically significant ($\chi^2(2, n = 330) = 71.598, p < .0001$). The association between the two features of *Express the Varying Area* from US teachers' responses is also statistically significant ($\chi^2(2, n = 212) = 7.716, p = .0211$). These statistical results are consistent with the hypothesis that teachers who used function notation only to represent the area (such as $A(t)$) think of t as a part of the function's name instead of an input variable. We hypothesized that if they thought of $A(t)$ as one symbol, they might think "A(t)=" is just another way of writing "y=".

5.5. Interviews with Three Teachers

We interviewed 17 teachers (eight SK and nine US teachers) about their responses in the two items. We present item interviews with three teachers to illustrate the spectrum of responses we found in all 17 teachers. Table **13** shows Teacher1's responses to the two items.

Table 13. *Teacher1's Responses to the Two Items*

	Response to *Understanding Input Variables*	Response to *Expressing the Varying Area*
Teacher1	Here are two function definitions. $w(t)=\sin(t-1)$ if $t\geq 1$ $q(s)=\sqrt{s^2-s^3}$ if $0\leq s<1$ Here is a third function *c*, defined in two parts, whose definition refers to *w* and *q*. Place the correct letter in each blank so that the function *c* is properly defined. $c(v)=\begin{cases} q(s) \text{ if } 0\leq s<1 \\ w(t) \text{ if } t\geq 1 \end{cases}$	$A(t) = \pi\ r^2$

Teacher1 filled the blanks with *s* or *t* in *Understanding Input Variables*. Her response to *Expressing the Varying Area* was $A(t)=\pi r^2$. She used function notation only to represent the area and used variables inconsistently. We interviewed Teacher1 to better understand why someone used variables inconsistently when defining a function.

Excerpt 1**.** Teacher1's Interview of *Understanding Input Variable*

I: What's your interpretation of this item?

T1: Um so I am looking at two functions become a piecewise defined function. Um I am thinking and looking at the two functions *w* is a function of *t* and *q* is a function of *s*. Umm… anything that relates to *q* should have *s* in the parenthesis in the blanks and then anything that relates to *w* should have *t*. Since I am wondering if these two functions are coming from elsewhere with different inputs, so $c(v)$ would be $q(s)$ and $w(t)$ combined.

Excerpt **1** tells us that Teacher1 thought $q(s)$ was one inseparable symbol instead of thinking q is a function's name and s is an input variable on *Understanding Input Variables*. Although she said q is a function of s, she thought q always accompanies with s, and viewed $q(s)$ as one entity. Teacher1 demonstrated her meanings for function notation when talking about her response to *Express the Varying Area* (see Excerpt **2**).

Excerpt 2**.** Teacher1's Interview of *Express the Varying Area*

T1: (After writing $A = \pi r^2$, and then added (t) after A) I am not sure I am satisfied with my answer, but I think I'm done. But I am not quite sure. I am not totally sure how else I would express this.

I: What's your interpretation of this item?

T1 Yes, I was thinking about rock dropping and the area of the circle I am thinking is πr^2 , the area of the circle. When I am using function notation I am thinking $A(t)$ because of the area with respect to time. The issue that I couldn't come up or couldn't figure out the rest of this was if the radius is increasing at a non-constant speed like due to time how can I make radius in terms of time? And I guess it made me like $A(t) = \pi r^2$ doesn't show the input of t, r is changing with respect to time. Would this be a situation where there would be multiple equations to represent the same situation or do I change the radius to be time, um but there are some relationships between time and radius, so I didn't feel satisfactory.

I: You feel like there is the relationship between time and radius, but you have no idea how to represent the relationship. Am I right?

T1: Yes

I: I saw you changed your answer A to $A(t)$. Could you explain why you changed your answer?

T1: Sure, first it (referring to the item) says use function notation, so I was trying to represent with function notation. It started with A. I was just noting the area of the circle which was what I was starting. Umm it says the area as a function of elapsed time, so using t to represent time. So I knew that the area was the function of time and I couldn't figure out what I wanted to do with r. I guess I could now change r to t although radius is not necessarily equal to time but there are some relationships between r and t that I want to represent. Since I don't

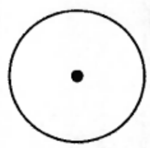
> know what's causing what relationship time and radius have. I don't exactly know how to factor time into the equation.

When taking *Express the Varying Area*, Teacher1 first wrote $A = \pi r^2$, and then added (t) after A. Her final answer was $A(t) = \pi r^2$. She used function notation only to represent the area on the left hand side, and used *t* in the function notation and *r* in the defining rule. Her statement "when I am using function notation I am thinking *A*(*t*)" and "I was just noting the area of the circle which was what I was starting" is consistent with our hypothesis that teachers who used function notation only to represent the area think of *A*(*t*) as one symbol, and *A*(*t*) is a label for the formula on the defining rule.

It is worth noting that Teacher1 expressed a need to relate radius and time and said she did not know how to do it. One reason might have been that to use function notation Teacher 1 needed a defining formula, and function notation did not serve a representational function for her without knowing a defining formula.

Teacher2's responses to the two items were in the highest levels according to the rubrics (see Table **14**).

Table 14. *Teacher2's Responses to the Two Items*

Response to *Understanding Input Variables*	Here are two function definitions. $w(t)=\sin(t-1)$ if $t\geq 1$ $q(s)=\sqrt{s^2-s^3}$ if $0\leq s<1$ Here is a third function *c*, defined in two parts, whose definition refers to *w* and *q*. Place the correct letter in each blank so that the function *c* is properly defined. $c(v)=\begin{cases} q(\underline{V}) \text{ if } 0\leq \underline{V}<1 \\ w(\underline{V}) \text{ if } \underline{V}\geq 1 \end{cases}$
Response to *Expressing the Varying Area*	Hari dropped a rock into a pond creating a circular ripple that spread outward. The ripple's radius increases at a non-constant speed with the number of seconds since Hari dropped the rock. Use function notation to express the area inside the ripple as a function of elapsed time.  $A(t)=\pi(f(t))^2$ where $f(t)$ gives the radius at time t

Teacher2 filled the blanks with *v* on Understanding Input Variables and wrote $A(t)=\pi\left(f(t)\right)^2$ where *f*(*t*) gives the radius of time *t*. We interviewed Teacher2 to see whether teachers who wrote highest-level responses demonstrate coherent meanings.

Excerpt 3. Teacher2's Interview of *Understanding Input Variable*

I: What's your interpretation of this item?

T2: Oh, I feel like I might be missing something. So the function *c* is properly defined. So, I mean this is *q*, so I would use *q* if my variable is between 0 and 1, and I would use *w* if my variable is greater and equal to 1. So, I mean I don't know I put *v*'s everywhere. (laughing) But since we are defining *c* of *v*, it seems the variable like it ought to be *v*. So the same domain restrictions would apply to *w* and *q*.

Excerpt **3** shows Teacher2's awareness that (1) the letter inside of the parenthesis represents a variable, and (2) the letter denoting the variable of a function should be consistent with the letter used in the defining rule. Her interview on the second item also displays her coherent meanings for function notation (see Excerpt **4**).

Excerpt 4**.** Teacher2's Interview of *Express the Varying Area*

I: What's your interpretation of this item?

T2: Oh, well it seems… wanting me to show that I understand that the radius isn't constant so I have to do a function for the radius. And you know obviously area equals π radius squared, but the radius is…Given that there is no more information about how radius relates to time, I thought I just need to do a function then of time.

I: That's why you wrote $f(t)$?

T2: Yes.

I: To represent the varying radius?

T2: Yes.

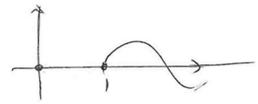
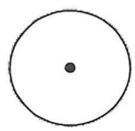
I: You wrote also $A(t)$. Could you tell me know why you wrote $A(t)$?

T2: Because it wants the area as a function of elapsed time, so I wrote A for the area and t is for time as my variable.

Excerpt **4** tells us that Teacher2 used function notation as a means to represent varying quantities in the case where an explicit rule is unknown. She also demonstrated that she was aware that t on both sides represents the input variable for the function defined.

Teacher3 filled the blanks with x in *Understanding Input Variables*, and expressed his confusion because he did not know an explicit rule for the radius in *Express the Varying Area* (see Table **15**).

Table 15. *Teacher3's Responses to the Two Items*

Response to *Understanding Input Variables*	두 함수가 다음과 같이 정의되어 있다. $w(t) = \sin(t-1)$ if $t \geq 1$ $q(s) = \sqrt{s^2 - s^3}$ if $0 \leq s < 1$  세 번째 함수 c 는 w 와 q 두 부분으로 정의되어 있다. 함수 c 가 알맞게 정의되도록 빈 칸을 채우시오. $c(v) = \begin{cases} q(\underline{x}) \text{ if } 0 \leq \underline{x} < 1 \\ w(\underline{x}) \text{ if } \underline{x} \geq 1 \end{cases}$ → 이것이 [0,1]에서 정의된 함수임에 문제가 있나...?? The teacher's scratch work: What is wrong with this for the function that is defined on [0,1]…??
Response to *Expressing the Varying Area*	메시가 연못에 돌을 던져서 생긴 원형 물결이 밖으로 퍼져나가고 있다. 원형 물결의 반지름은 메시가 돌을 던진 이후 지난 시간 (초)에 대해 일정하지 않은 속력으로 증가한다. 함수 기호를 사용하여 원형 물결 안의 넓이를 경과 시간 (초)에 대한 함수로 나타내시오.  ?? 원이 어떤 양상으로 커지는지 모르겠음. The teacher's response: ?? I don't know how the circle is increasing.

Teacher3's responses to the two items indicate his inclination of using x in function notation and finding out rules when thinking about functions. Unlike the two teachers (Teacher1 and Teacher2) I presented earlier, Teacher3's interview included looking at his original responses from a prior administration of the MMTsm. Excerpt **5** shows his reasoning on the first item.

Excerpt 5. Teacher3's Interview of *Understanding Input Variables*

I: Could you explain your response to me?

T3: Doesn't this item want me to say it doesn't matter to use different letters?

I: What does that mean?

T3: This (referring to *w* function) is defined from 1, and this (referring to *q* function) is defined from 0 to 1, so it (referring to *c* function) is defined well, isn't it? But, c is defined in two parts, so I used the same letter. Ah, it is written $c(v)$. Oh, there is v. Don't I have to fill the blanks with v?

I: Could you explain to me why you filled the blanks with x?

T3: At that time when I was taking this item? Was I in a hurry? Anyway I thought the intention of this item was…this (referring to w function) is in terms of t, this (referring to q function) is in terms of s. And the domains were connected, so I have to use one letter, my favorite letter. If I had seen v here (pointing to v in $c(v)$), I might have written v, but…Maybe I didn't see this (referring v) because I was in a hurry.

Teacher3 expressed his tendency to use x as the letter inside of the parenthesis in Excerpt **5**. He did recognize that the input variable for a function is a place holder for a value and can be changed as long as a parallel change is made in the function's rule of association. He did not recognize that the variable in a function definition could be consistent. It seemed that he was perturbed by my questions during the interview because he started to focus on v. Teacher3 also said why he did not answer the second item in Excerpt **6**.

Excerpt 6. Teacher3's Interview of *Express the Varying Area*

I: Could you explain your response to me?

T3: I didn't understand this item at all. I asked teachers who took this item as soon as it was over. They said they chose any function in terms of time. But, I thought I needed a formula that represents how the circle is increasing. The radius is increasing at a non-constant rate, so I thought this item was asking me to find out how the radius is increasing. For example… (reading the item again) Does this (pointing to "the radius is increasing at a non-constant rate") mean that t and r are not directly proportional? For example, if I think of any function, I can take $r = t^2$ or $r = \sqrt{t}$ and apply the area formula πr^2 . Can this be one of correct answers? I feel comfortable with a specific function.

Excerpt **6** tells us that Teacher3 did not answer the second item because he did not come up with a formula that represent the radius increasing at a non-constant rate. His statement in this excerpt shows that his meaning for function is a formula such as $r = t^2$ or $r = \sqrt{t}$. It seemed that Teacher3 first tried to come up with a rule when asked to use function notation. This teacher was also very explicit about how his own comfort level played into his actions. He used "x" in *Understanding Input Variables* not because it was relevant to the function definition but because "it was his favorite variable", and he wanted to find a specific rule for how radius and time varied in *Express the Varying Area* because "he feels comfortable with a specific function". Though other teachers did not indicate the same self-awareness of their own motivations, their responses indicated that many other teachers might have felt comfortable only when using specific function rules on the right-hand side of a function.

5.6. The Story of Teacher3

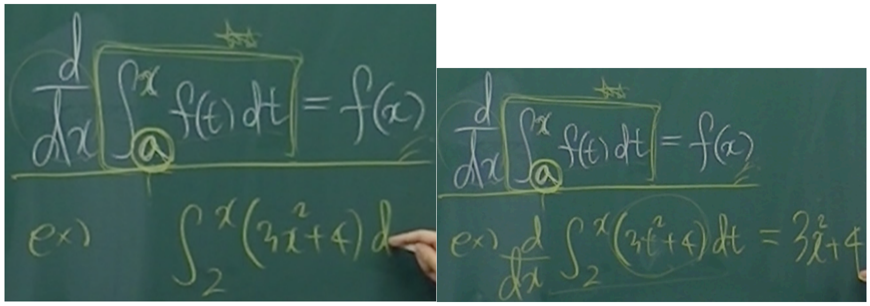
Teacher3's lesson was about the fundamental theorem of calculus. The first author interviewed Teacher3 and two students before and after the lesson. In the lesson, the teacher expressed meanings for function notation consistent with the meanings he demonstrated in the MMTsm during the lesson. Teacher3 demonstrated his tendency to use x as the letter inside of function notation in the MMTsm. During the lesson, he expressed his confusion about the roles of variables in integrals by using an incorrect variable, fixing it from the memory, and telling the students that his correction was only "for convenience". His students constructed meanings that were compatible with the teacher's.

5.6.1. Teacher3's Lesson

Teacher3 showed that he was inconsistent with his use of variables because he first wrote $\int_2^x (3x^2+4)dt$ instead of $\int_2^x (3t^2+4)dt$ in the lesson and then immediately corrected his error (see Excerpt **7**). I anticipated that students might ask about the usage of variables between "x" and "t". As I expected, one student asked him about the two variables (*x* and *t*). Teacher3 answered said that *t* was used "for convenience" (see Excerpt **7**). A more conceptually coherent response would have been, "As the value of *x* varies; the value of *t* varies from *a* to *x* for each value of *x*.

Excerpt 7. Teacher3's Lesson Explaining the Fundamental Theorem of Calculus

> T3: A function integrating *f* from *a* to *x* means the area from *a* to *x*. Interestingly, if you differentiate the function representing the area, you will get the original function *f*(*x*). Here is the first link for integrating to get the area and differentiating. When we proved the theorem [referring to $\frac{d}{dx}\int_a^x f(t)dt = f(x)$], did we use the value of *a*? We never used it. It means that *a* doesn't matter. If you plug in "1", so from 1 to *x*, then integrate and differentiate, you will get *f*(*x*). If you plug in "5", you will get the same *f*(*x*). Let's see an example. [Writing $\int_2^x (3x^2+4)dx$ and then changing *x* to *t*, then wrote $\int_2^x (3t^2+4)dt$ on the board (see the image below)] this means a function that is integrating this function [referring to $(3x^2+4)$ from 2 to *x*. If you differentiate this function [referring to $\int_2^x (3t^2+4)dt$] it goes back to the original *f*(*x*) [pointing to *f*(*x*) in $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. What is *f*(*x*)? *f*(*x*) is this [circling $(3t^2+4)$ in $\int_2^x (3t^2+4)dt$]. So, the answer is $(3x^2+4)$ as the result of differentiating. Again, this is the first link between integrating to get the area and differentiating [pointing to $\frac{d}{dx}\int_a^x f(t)dt = f(x)$].



S: Why do you have "*t*" instead of "*x*" [referring to $\int_2^x (3t^2+4)dt$]?

T3: Why did I change from *t* inside to *x*? [Pointing to $(3t^2+4)$ in $\int_2^x (3t^2+4)dt$ and $3x^2+4$]. This is a function of *x* [pointing to $\int_a^x f(t)dt$]. We wrote "t" [pointing to "t" in *f*(*t*) in $\int_a^x f(t)dt$] for convenience in order not to be confused with *x*. Without confusion in the letters.

It seems that Teacher3's scheme of meanings for *integrals* did not include roles that *x* and *t* play in it. Rather, the teacher's meaning for integral was how to write the integral and how to write an antiderivative. Teacher3's meanings for *integrals* influenced his instructional actions: (1) writing $\int_2^x (3x^2+4)dx$ and then changing *x* to *t*, then wrote $\int_2^x (3t^2+4)dt$ on the board *t*, (2) circling $(3t^2+4)$ in $\int_2^x (3t^2+4)dt$ as the answer to *f*(*x*), (3) saying that using *t* instead of *x* is for convenience.

5.6.2. The Meanings Two Students Constructed

During Post-Lesson Interview with Teacher3's students, I asked them "why do you have *f*(*x*) on the right hand side although you have *f*(*t*) on the left hand side in $\frac{d}{dx}\int_a^x f(t)dt = f(x)$?" Both students told me that the teacher said "for convenience".

Student3A then said the teacher said it looked bad if he wrote x and x such as $\int_a^x f(x)dx$ whereas Student3B told me that it would okay to write x instead of t, then wrote $\frac{d}{dx}\int_a^x f(x)dx = f(x)$. The two students seemed to understand Teacher 3's utterance, "for convenience" as implying that you could use x or you could use t, whichever was more "convenient". Student3A understood using only x makes the expressions "look bad" whereas Student3B understood using x everywhere would be fine.

Teacher3's classroom actions are consistent with our hypothesis that certain meanings are more productive in the classroom than others, and that these meanings can be characterized by responses to our items. Teacher3's inconsistent variable usage and his verbal explanations for when to use certain variables created an opening for students to interpret his actions and utterances in conflicting ways.

6. CONCLUSION

Teachers' responses to the two items in this study suggest that approximately half of all teachers held unproductive, and sometimes incoherent meanings for function notation. US teachers showed more unproductive meanings for function notation compared to SK teachers. The results show that attending to teachers' meanings for function notation revealed teachers' understanding of functions. From the first item, about 66% of US teachers' responses suggest their unawareness that the letter denoting the variable of a function should be consistent with the letter in the defining rule (see Table **4**). If teachers are indeed unaware of this convention, they probably are unaware of a reason for it.

From the second item, 77% of US teachers' responses suggest that they did not use function notation to represent varying quantities. The association between responses to *Understanding Input Variable* and *Express the Varying Area* from SK and US teachers' responses is statistically significant. The association between the two features of *Express the Varying Area* from SK and US teachers' responses is also statistically significant. Those results strongly suggest that (1) teachers held a meaning for function notation as a label to replace a word such as $A(t)$ to replace the area in the second item, (2) teachers treated function notation $A(t)$ as a variable name because they used different variables inside of function notation and in the defining rule, and (3) teachers thought that every function has to be defined by a rule. The results suggest a need for attention to ideas regarding function notation in teacher education for pre-service teachers and professional development programs for in-service teachers.

Teacher3's story is consistent with our hypothesis that the meanings teachers demonstrated in the MMTsm instrument influence meanings they express in their classroom. He expressed his propensity for the letter "x" inside of function in the MMTsm item and expressed the same in his lesson, and his students developed understandings consistent with the teacher's meaning.

We do not claim all teachers would express their meanings demonstrated in the MMTsm in their classrooms. However, the teacher's story supports our claim that teachers' responses in the MMTsm support and constrain meanings they might express during their lesson.

PAPER TWO: RELATIONSHIPS BETWEEN WHAT TEACHERS KNOW, WHAT THEY DO IN CLASSROOM, AND WHAT THEIR STUDENTS UNDERSTAND

1. INTRODUCTION

Teachers' mathematical meanings play a significant role in student learning. It is plausible that the more productive mathematical understanding a teacher holds, the more opportunities to construct robust understanding students will have. However, teachers holding productive meanings for a mathematical idea do not guarantee that their students will learn the idea coherently. In this study, I will make a distinction between mathematical meanings that teachers hold and mathematical meanings that they express during lessons.

Project Aspire developed the *Mathematical Meanings for Teaching secondary mathematics* (MMTsm), a 44 item diagnostic instrument designed primarily to give professional developers insight into mathematical meanings with which teachers operate. The project team then categorized teachers' responses by asking ourselves "how productive would meanings we can discern from the teacher's response be for a student were the teacher to convey it?" Thus, a high level of response in the MMTsm instrument means that the teacher's meanings were potentially productive for students, because the team thought the response would benefit students when students develop that meaning.

The team hypothesized that the meanings teachers expressed in the instrument were their mathematical meanings for teaching, but this was in the project team's judgment. We did not claim that the assessment predicted teachers would express this meaning in their classrooms. Rather, teachers' responses in the MMTsm were mathematical meanings they held when they took the instrument, and we hoped they might be the meanings they would convey to students. I will use teachers' MMTsm responses as an indicator of their meanings at the moment of the assessment, and then investigate the relationships between their responses in the MMTsm and meanings that they express during lessons.

In order to see how teachers' mathematical understanding influences student understanding, researchers have investigated relationships between teacher understanding and student learning. They have tried to measure teachers' mathematical knowledge, the quality of instruction, and student performance. Studies have demonstrated that there is a positive relationship between teacher knowledge and student performance as well as teacher knowledge and their instructional quality (Baumert et al., 2010; Hill, Ball, Blunk, Goffney, & Rowan, 2007). However, researchers used different instruments to measure teachers' mathematical knowledge, the quality of instruction, and student performance: an assessment for teachers' performance, measure for teachers' performance in actual instruction, and a test for student achievement. Thus, what the researchers measured as teacher knowledge is not connected with what teachers did in instruction and what students understood because they used different frameworks for viewing them.

This study is designed to provide a new lens to explain conceptual connections between what teachers know, what teachers do in the classrooms, and what students learn. I studied these connections by using the constructs of *meaning, conveyance of meaning*, *Key Pedagogical Understanding* (*KPU*) to guide observations and analyses of classroom observations and interviews with teachers and students.

In this article I focus on what meanings eight teachers conveyed to their students and aspects that played role in the conveyance of meaning from a teacher to students. I observed two lessons for each teacher and conducted Pre- and Post-Lesson Interviews with their students. The research questions that I seek to answer are (1) *What relationships are there between teachers' mathematical meanings for the ideas that they teach and their instructional actions regarding those ideas? (2) What relationships are there between teachers' instructional actions and meanings their students construct? (3) How do teachers' meanings and their images of student thinking influence their interpretation of meanings that students construct from their lesson?*

2. LITERATURE AND THEORETICAL PERSPECTIVE

Researchers holding a knowledge-primitive perspective have hypothesized types of mathematical knowledge needed in the practice of teaching and designed assessments to measure teachers' Mathematical Knowledge for Teaching (MKT). Prominent projects are *Learning Mathematics for Teaching* (Hill, 2010; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008), *Mathematics Teaching in the 21st Century* (Schmidt et al., 2011), *Teacher Education and Development Study-Mathematics* (Blömeke, Suhl, & Döhrmann, 2013; Buchholtz & Kaiser, 2013; Hsieh et al., 2011; Tatto et al., 2008; Tatto et al., 2009; Tatto & Senk, 2011; Tatto, Senk, Bankov, Rodriguez, & Peck, 2011; Wang & Tang, 2013), *Knowledge of Algebra for Teaching project* (McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012), and *Professional Competence of Teachers, Cognitively Activating Instruction, and the Development of Students' Mathematical Literacy* (Baumert et al., 2010; Kunter et al., 2007). The frameworks of these projects are designed to categorize teachers' MKT. They are not intended to describe and explain moment-by-moment events of teaching and learning or to describe and explain cumulative effects of such events on students' learning. However, the focus of this article is on understanding these moment-by-moment events both short-term and long-term, which requires another type of framework.

To address my research questions requires a theoretical perspective that takes instruction as a form of conversation between teachers and students. Conversations can be of various qualities in terms of how well participants understand each other, or how seriously each takes contributions of the other, but they are conversations nevertheless. In this regard I use a framework developed by Thompson, which is a hybrid of Bauersfeld's symbolic interactionism as specialized for mathematics classrooms, Piaget's genetic epistemology, and his and Steffe's restatement of von Foerster's cybernetics in the language of first- and second-order observers making first- or second-order models of others' thinking[9].

I use Thompson (2013a)'s constructs of *meaning* and *conveyance of meaning*, and Silverman and Thompson's (2008) construct *Key Pedagogical Understanding* (*KPU*). Thompson's definition of meaning evolved from Piaget's genetic epistemology, especially Piaget and Garcia's (1991) logic of meanings and Skemp's (1962) interpretation of Piaget's meaning of "to understand" as "assimilate to a scheme".

Thompson and colleagues (Thompson, Carlson, Byerley, & Hatfield, 2014) defined *mathematical meanings* to refer to the space of implications of an understanding. Hub and Dawkins (2018) rephrased this as a person's mathematical meaning at a moment that is the space of inferences available to the student given the student's understanding at that moment. The construct *conveyance of meaning* does not mean that students end up

[9] The point of view of a second-order observer refers to "the observer's ability through second-order consensuality to operate as external to the situation in which he or she is, and thus be the observer of his or her circumstance as an observer" (Steffe & Thompson, 2000, p. 10)

having the same meanings that a teacher holds. Rather, meanings conveyed from a teacher to his students are the meanings students construct by attempting to understand what the teacher intends. A conveyed meaning might or might not resemble the teacher's meaning. I use the six-phase model of teachers' development of KPU that Thompson developed to discuss teachers' image of student thinking as well as the extent to which they consider student thinking in making instructional decisions. My perspective on mathematical meanings, conveyance of meaning, and KPU allows me to bridge what teachers know, what they teach, and what their students learn.

2.1. Meaning

Productive mathematical meanings serve as a foundation for future learning, so it is important that students build useful and robust meanings. I use "productive meaning" in the way Thompson (2016) defined it—as a meaning a teacher holds that we judge would be productive for students' understanding and future learning were they to hold it also.

One way students develop meanings is by trying to make sense of what their teacher say and do in the classroom. Before discussing how meanings are conveyed in the classroom from a teacher to students, I will explain what Thompson (2013a) meant by *meanings*. According to Piaget, to understand is to assimilate (Richard R Skemp, 1962; Skemp, 2012; Thompson, 2013a; Thompson & Saldanha, 2003) and "assimilation is the source of schemes" (Piaget, 1977, p.70 cited in Thompson, 2013a). Thus, the phrase "a person attached a meaning to a word, symbol, expression, or statement" means that the person assimilated the word, symbol, expression, or statement to a scheme. A scheme is an organization of ways of thinking, images, and schemes. When I say *assimilate* I mean the ways in which an individual interprets and make sense of a text, utterance, or self-generated thought. According to Piaget, repeated assimilation is the source of schemes, and new schemes emerge through repeated assimilations, which early on require functional accommodations and eventually entail metamorphic accommodations (Steffe, 1991).

Thompson (2013a) said *meaning* is the space of implications of an understanding. For example, a student can understand slope as a coefficient of x because she learned "m" is slope in $y = mx + n$. This is her understanding of slope in the moment. Then, she could think about slope as "1" when first looking at $x = 1$ because 1 is the coefficient of x. This is an implication of her understanding in the moment. The students' meaning in the moment of understanding is the space of implications of that understanding.

2.2. Conveyance of Meaning

Thompson (2013a, 2016) explains that when one person intends to convey a meaning to another, the speaker's conveyed meaning is the meaning the other person constructs in attempting to understand what the speaker intends. Thompson makes a clear distinction between a teacher' intended meaning (the meaning he or she envisions students eventually having) and students' conveyed meaning (the meanings they construct in attempting to understand what the teacher intends). Thompson offers this vision of a conversation in his role of a second-order observer observing conversations involving teachers (as first- or second-order observers) and students (as first- or second-order observers).

Consider a teacher who teaches mathematical ideas to his students. A teacher has his meanings for the mathematical ideas. The teacher intends to convey the mathematical ideas to his students. In doing so, the teacher and his students are interacting and making an attempt to interpret others in class. Thompson (2013a) proposed a theory to explain how two people (person A and person B) attempt to have a conversation that leads to mutual understanding.

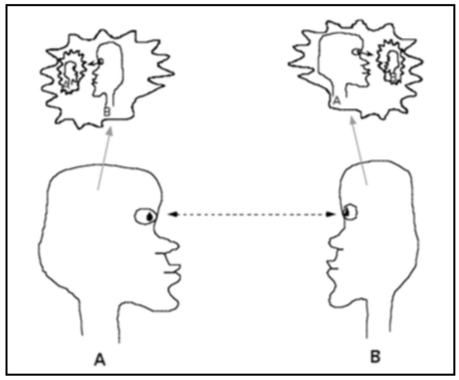


*Figure 5***.** Persons A and B Attempting to Have a Reflective Conversation (Thompson, 2013a, p. 64).

According to Thompson (2013a), person A in *Figure* ***5*** holds something in mind that he intends Person B to understand. Person A considers not only how to express what he intends to convey but also how Person B might hear Person A. If both persons are acting as second-order observers, Person A constructs his model of how he thinks Person B might interpret him and Person B does the same thing in regard to person A. Person B constructs her understanding of what Person A said by thinking of what she might have meant were she were to say it. Thus, Person B's understanding of what Person A said comes from what she knows about person A's meanings, thereby Person B's understanding of Person A's utterance need not be the same, and likely is not the same, as what Person A meant.

The intersubjective actions described in the discussion of *Figure* ***5*** would happen between two people attempting to construct second-order models of each other's intentions and meanings. But a conversation could happen with many different variations, depending on the nature of the model-of-other each constructs, which depends on the observer level of each participant and each person's personal meanings regarding the conversation's substance. Even then, the form of the conversation would still be as depicted in *Figure **5***.

2.3. Teacher's Image of Student Thinking

Silverman and Thompson (2008)'s framework explains how a teacher develops schemes that support conceptual teaching of a particular mathematical idea when she has an image of how her students might hear her statements. Silverman and Thompson referred to Key Pedagogical Understanding (KPU) to discuss teachers' image of students' thinking. Thompson (2008) described a six-phase model of teachers' development of a KPU.

Table 16. *KPU Phases (Thompson, 2008)*

Phase	Description
1	Teacher develops an understanding of an idea that the curriculum addresses. Student thinking is not an issue.
2	Oriented to student thinking, but tacitly assumes that information is all that students need, Projecting oneself by default. That is, person A presumes unthinkingly all students are A' (on the road to being A)
3	Teacher becomes aware that students think differently than teacher anticipates they do, but teacher is overwhelmed by seeming cacophony of student thinking (students in her head are B1, B2, B3, …)
4	In dealing with students' (B1, B2, B3, …) contributions: 1. Teacher begins to imagine different "ways of thinking" (epistemic students) 2. These ways of thinking are still grounded largely in teacher's ways of thinking.
5	Teacher begins to imagine how different ways of thinking among students will lead to different interpretations of what she says and does. Begins to develop a mini-theory of actions that might help students think the way teacher intends
6	Teacher adjusts: 1. Her understanding of the mathematical idea as she adjusts her image of ways students think about it. 2. Her understanding of how students might think about the idea as she adjusts her understanding of it

The KPU phases are in terms of how a teacher thinks about student thinking. A teacher in Phase 1 knows that students are human beings, but presumes unthinkingly that all he needs to share is his understanding expressed as he understands it for students to discern what he intends. In this sense, a teacher in Phase 1 constructs a first order model of students.

A teacher moves to Phase 2 when she becomes aware that students think differently than her. However, the teacher might be thinking about how she can explain mathematical ideas to students more clearly. Put another way, the teacher is projecting herself by default. The Phase 2 teacher's model of students is also a first order model in the sense that the teacher thinks the students are just like the teacher, so the teacher is using a model of her own understanding to consider how it might be made clearer. Teachers in Phase 1 or Phase 2 use a first order model of students, but the difference between Phase 1 and Phase 2 is that a teacher in Phase 2 becomes oriented to students' understanding.

Phase 3 is the initial phase of moving beyond using a first order model. A teacher in Phase 3 starts to think about students who might think differently than the teacher. Suppose one of her students wrote something on the board and the teacher realized that what the student wrote is different from what the teacher intended. Then the teacher might try to figure out what the student was thinking from what the student wrote. To do so, the teacher begins to have a second order model. The teacher becomes aware of the fact that student understanding differs from what the teacher intended. Then the teacher starts to think about how students think and what students think. Even if the teacher cannot explicitly state different thinking that students construct, just being aware of students thinking differently than the teacher is the beginning of moving towards creating a second order model. The teacher's orientation to students' thinking occurs in Phase 3 and the teacher is wondering what students are understanding when they hear the teacher, but the Phase 3 teacher does not have an image of students' images regarding what the teacher says and does.

A teacher in Phase 4 begins to imagine different understandings that students construct, but imagines these variations as different forms of her ways of thinking. A teacher in Phase 5 becomes aware that different understandings students are constructing can lead them to different interpretations of what the teacher says and does. A teacher in

Phase 4 or Phase 5 wonders about not only how students are thinking but also students' images of what the teachers said and did. However, a teacher in Phase 5 is aware that students have different ways of thinking than herself, whereas a teacher in Phase 4 is thinking of students' ways of thinking still by adjusting her own ways of thinking.

Suppose a teacher thinks slope is about relative size of changes in two variables, but speaks about slope in terms of triangles and steepness. If the teacher is in Phase 4, she might imagine students' ways of thinking about slope only in terms of triangles or steepness because her ways of speaking are grounded in triangles or steepness. However, this teacher in Phase 5 could recognize students speaking of slope as how much larger a vertical length thing is than a horizontal length. Thus the teacher in Phase 5 is ready to think, for example, "Oh, my way of speaking about triangles and steepness led students to think about comparing lengths additively. I will have them determine slope by measuring vertical change in units of horizontal change" because she thinks the new activity will support students in understanding slope in the way she intended.

Thompson (2013a)'s theory of conveyance of meaning and the KPU phases are useful to explain the evolution of interactions between teachers and students in classrooms in terms of ways they understand each other. A teacher expresses his meanings to his students by saying or doing something. Then, his students try to understand what the teacher says and does. Whatever meanings his students construct by attempting to understand what the teacher intends are the meanings the teacher conveyed to the students. A conveyed meaning might or might not be the same as the teacher's meaning, and most likely is not. Moreover, the extent to which a teacher envisions how his students might understand his actions and utterances affects his interpretations of students' actions and utterances and thereby affects what he says and does, which then further affects meanings conveyed to students.

3. METHOD

This study focused on the relationships between mathematical meanings teachers hold, their instructional actions, meanings they conveyed to students. I observed eight teachers' lessons: two consecutive lessons taught by each teacher in order to witness how teachers' mathematical meanings affected their instructional actions, and how their instructional actions influenced students' understandings. I also interviewed their students before and after lessons. By comparing students' meanings as I inferred from pre-lesson interviews to their meanings as I inferred from post-lesson interviews I concluded what they understood of their teacher's instruction–conveyed meanings from a teacher to students.

3.1. Participants

Project Aspire team developed the Mathematical Meanings for Teaching Secondary Mathematics (MMTsm), a diagnostic instrument designed to investigate mathematical meanings with which teachers operate. The eight teachers in this study took the MMTsm in 2014 or 2015, and agreed to classroom observations in 2016. The eight teachers' educational background and years of experience teaching mathematics are shown in Table **17**.

Table 17. *The Eight Teachers and Their Students*

Teacher	Background	Grade
Teacher1	3yrs (middle school), computer ed BS	Grade 8
Teacher2	4yrs (high school) math ed BS	Grade 11
Teacher3	9yrs (high school) Math ed BS and MS	Grade 10
Teacher4	8yrs (high school), Math BS, Math ed MS	Grade 10
Teacher5	3.5yrs (high school), Math ed BS	Grade 10
Teacher6	4yrs (4high school) Math ed BS	Grade 11
Teacher7	5yrs (2middle school, 3high school), math ed BS	Grade 11
Teacher8	5yrs (high school), Math ed BS and MS	Grade 11

I asked teachers to select two middle-performing students who, in their judgment, pay close attention during lessons. They selected two students (for interviews. I conducted Pre- and Post-Lesson Interviews with the two students because I wanted to witness what they understood from their teacher's lessons.

Teacher1's students were middle school students. The rest of students selected by teachers were high school students. They heeded their teachers' lessons when I observed.

3.2. Procedure

For each classroom observation I conducted separate pre-lesson interviews and post-lesson interviews with the teacher and two students. The process of this study for one teacher is shown in Table **18**.

Table 18. *The Process of Classroom Observations for One Teacher*

Day #1	Day #2	Day #3	Day #4
Get permission from a principal before Day #1 and meet the principal	Pre-Lesson 1 with a teacher	Pre-Lesson 2 with a teacher	Post-Lesson with a teacher
Give the consent form and MMTsm items to a teacher	Pre-Lesson 1 with students	Pre-Lesson 2 with students	
Ask the teacher to select two of middle performers and to collect the students' consent forms	Lesson observation 1	Lesson observation 2	
	5min Post-Lesson with a teacher	5min Post-Lesson with a teacher	
	Post-Lesson 1 with students	Post-Lesson 2 with students	

Teachers told me their lesson goals in each Pre-Lesson Interviews. They also expressed their meanings and the extent to which they were sensitive to student thinking in the Pre- and Post-Lesson Interviews and their lessons. Pre-Lesson Interviews with teachers took 10-15 minutes. Post-Lesson Interviews with teachers were two types: 5 minutes Post-Lesson Interviews right after each lesson and two hour Post-Lesson Interviews after two lessons. I conducted two hour Post-Lesson Interview with teachers after students' Post-Lesson 2 Interview because I wanted to see their image of student thinking after watching student video clips.

Students also expressed their meanings in Pre- and Post-Lesson Interviews. I conducted a Pre-Lesson Interview prior to every lesson, so the next Pre-Lesson Interview sometimes served as a Post-Lesson Interview for the previous lesson. For example, I was able to see what they understood in lesson 1 during Pre-Lesson Interview 2. I also conducted Post-Lesson Interviews after each lesson. Every Pre-Lesson Interview with students took approximately 5 minutes and each Post-Lesson Interview with students took about 30 minutes.

Mathematical topics that eight teachers taught are shown in Table **19**.

Table 19. *Mathematical Topics That Korean Teachers Taught*

Mathematical topic that they taught	Number of teachers (Name)
Quadratic function	Teacher1
Calculus- integrals of polynomials, indefinite integrals, fundamental theorem of calculus	Teacher2, Teacher7
Equation of line	Teacher3, Teacher4, Teacher5
Calculus-volume and area using integrals	Teacher6
Calculus-derivative rules	Teacher8

In order to see teachers' mathematical meanings for the ideas they taught I used items in the MMTsm (Korean version). One of items used for teachers who taught the concept of slope is in *Figure* ***6***.

> Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.
>
> Convey to Mrs. Samber's students what 3.04 means.

Figure 6. A Slope Item in the MMTsm © 2015 Arizona Board of Regents. Used with Permission

Before observing the first lesson, I conducted a pre-lesson interview with the teachers to investigate what the teacher intended students to learn. The interview questions for a teacher are shown in Table **20**, Table **21**, and Table **22**.

Table 20. *Pre-lesson Interview Questions for Teachers*

1. What is the central mathematical concept of this lesson?
2. What would you like your students to understand about the central mathematical concept?
3. Why would you like your students to understand the concept in the ways that you mentioned in the question 2?
4. What will you do to help your students to develop this understanding?
5. What do you think how your students might understand about this idea that is different than what you intend?

Table 21. *5-minutes Interview Questions for Teachers (Right After the Lesson)*

1. Do you think your lesson went on as you expected?
2. What is your sense of what students understood about the central mathematical concept?

Table 22. *Post-lesson Interview Questions for Teachers*

1. What do you think your students understood from your lessons 1 and 2?
2. (After viewing video clips from students' interviews.) What do you think these students' understood regarding the meaning of (the content of the clips)?
3. Is the understanding in the video clips consistent with what you intended? If not, what can be possible reasons for this discrepancy?
4. If you teach again, is there anything that you want to change? If so, what are these things and why do you do differently?
5. (After showing the teacher's responses to the MMTsm items.) Do you want to change your responses to these items? If so, why do you want to change?

One pre-lesson interview question was, "Do you think your students might understand slope differently than what you intend?" I asked this question to discern how a teacher thinks about student understanding before the lesson.

After every Pre-Lesson Interview with a teacher, I met the two students selected for interviews one by one. The purpose of the Pre-Lesson Interviews for students was to see their understanding of the topic covered in the upcoming lesson. I compared students' meanings demonstrated in Pre-Lesson Interviews to their meanings demonstrated in Post-Lesson Interviews to conclude what they understood from the lesson. After the lesson, I asked a student to describe what he or she learned from the lesson. The interview questions for post-lesson interview with a student are shown in Table **23**.

Table 23. *Post-lesson Interview Questions for Students*

1. What do you think was the central idea of this lesson? Why do you think so?
2. What do you think your teacher hoped you would learn from this lesson?
3. A question about mathematical ideas that would give the student an opportunity to *use* their meanings of the central idea.

I audio recorded pre-lesson interviews with teachers and students and post-lesson interviews with teachers. However, I video recorded the lessons and post-lesson interviews with students because I showed video clips of students' post-lesson interviews to their teachers. The purpose of sharing students' video clips with teachers was to provide them with opportunities to think about students' understandings and to reflect on their teaching and meanings by showing excerpts from the two student interviews that reveal how they understood central ideas of the lesson.

3.3. Analysis

My analysis focuses on my lesson observations and interviews with the teachers and the students. I discerned teachers' mathematical meanings from their responses in the MMTsm instrument, interviews with me, and their interpretation of student meaning. I also experienced their lessons as an observer, so I saw their instructional actions. I determined meanings teachers conveyed to their students by comparing students' meanings demonstrated in Pre-Lesson Interviews to their meanings demonstrated in Post-Lesson Interviews.

I discerned teachers' different KPU phases from interviews with them, their plans to adjust lessons after watching students' video clips. Teachers expressed what they thought about students' understandings and how they might adjust lessons to help students' understanding. I will describe how teachers' meanings and their KPU phases influence their instructional actions, and meanings students constructed in the next section.

4. RESULTS

Overview

I analyzed data from all eight teachers, but I will describe my observations of Teacher1, 3, and 6's lessons to illustrate the relationships between meanings teachers hold, their instructional actions and meanings students construct. *Figure 7* illustrates the overview of my observations of lessons of the eight teachers.

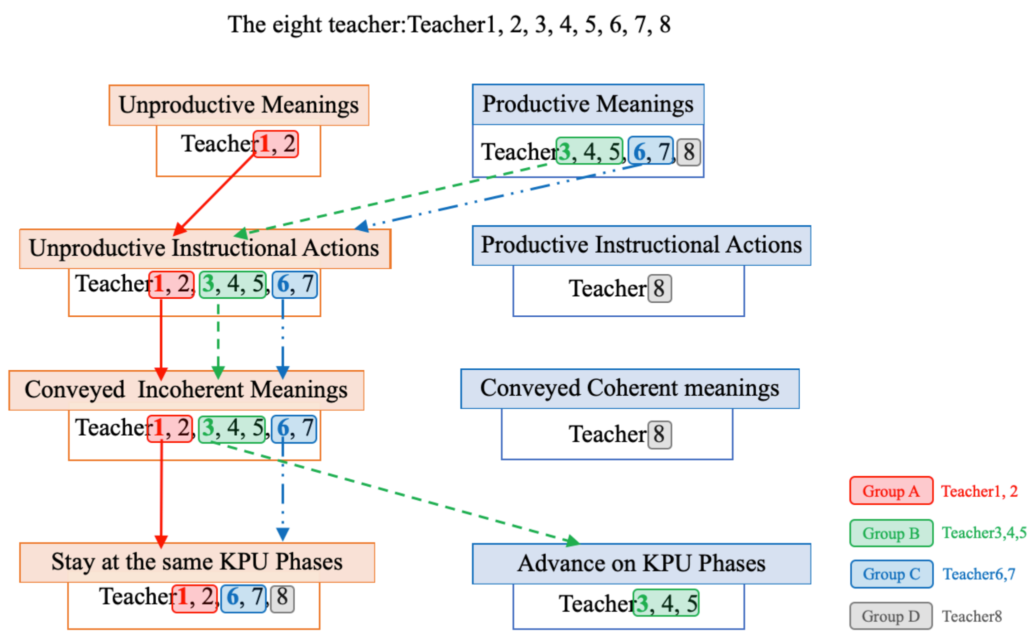


Figure 7. The Overview of My Observations of the Eight Teachers' Lessons

I categorized my observations of the eight teachers' lessons into four groups based on the patterns of behavior where teachers' meanings and their attention to student thinking influenced their instructional actions and conveyed meanings.

Group A (Teacher1 and Teacher 2) followed the same pattern of behavior where their meanings and their insensitivity to student thinking led to unproductive instructional actions and incoherent conveyed meanings. Group B (Teacher3, Teacher4, Teacher5) and Group C (Teacher6, and Teacher7) shared the same pattern of behavior where they failed to convey the productive meanings to students although they demonstrated productive meanings in the MMTsm instrument. The difference between Group B and Group C is that teachers in Group B advanced their KPU Phases whereas teachers in Group C stayed at the same KPU Phase after watching students' video clips during Post-Lesson Interview with me. Group D consists only of Teacher8 whose productive meanings and his sensitivity to student thinking led him to convey his intention to students. However, Group D (Teacher8) did not deal with any meanings when teaching the product rule. He simply focused on helping students remember the product rule. Thus, I will focus on Group A, B and C.

Below I will discuss Teacher1, Teacher3, and Teacher6 as representatives of Groups A, B and C respectively in order to illustrate characteristics of each group. I will also discuss teachers' interpretation of students' responses to my questions, which informed my image of student thinking and their own meanings.

Teacher1's meanings for the ideas that he taught (1) led him to set unclear and incomprehensible goals before the lesson, (2) led his students to develop incoherent meanings, (3) led to his inability to think about student thinking and to progress in his KPU phase.

Teacher3 demonstrated productive meanings for slope before her lessons, but she was not sensitive to how students might interpret her instructional actions. She taught the main ideas that were about the equation of line in ways that were clear to her, but her instructional actions were unclear even to an observer (me) who knew her intentions. After experiencing her lessons, her students developed incoherent understandings. However, after Teacher3 identified a miscommunication between her and students when watching the students' Post-Lesson Interviews, she noticed her student's confusion between a varying point (x, y) and a given point (x_1, y_1) from the slope formula that her student wrote. Her productive meanings allowed her to think about her own contribution to students' misunderstanding and to talk about her decision to adjust the next lessons to address the sources of students' difficulties. Her own productive meanings made her KPU Phase progress possible.

Teacher6 also demonstrated productive meanings for calculus ideas such as function and rate of change, but his inattention to student thinking led his students to understand the main idea differently than he intended. As opposed to Teacher3, Teacher6 ascribed any student understanding inconsistent with his own to student habits or mistakes. His plan to adjust the next lesson was to get rid of the example that he identified as leading students to misunderstand. Teacher6 stayed at the same low KPU Phase after watching the students' Post-Lesson Interviews.

I will describe what happened in Teacher1, 3, and 6's classrooms. I will then discuss the relationships between meanings teachers hold, their instructional actions, meanings their students construct. In this study, teachers heard what students said after their lessons by viewing the Post-Lesson Interview videos. Thus, I witnessed how teachers interpreted understandings students constructed. In doing so, I will present what I discerned from teachers' interpretation of the conveyed meanings from them to students.

4.1. What happened in Teacher 1, 3, and 6's classrooms

I will describe important events (Teacher's Pre-Lesson Interview, Teacher's lesson, Students' Pre-and Post-Lesson Interviews[10], and Teacher's Post-Lesson Interview) in the three teachers' (Teacher1, Teacher3, and Teacher6) classrooms, and my analysis of them.

4.1.1. The story of Teacher1

The main topic of Lesson 1 was quadratic functions. Lesson 2 was a continuation of Lesson 1 where students worked on problems, so the focus of Teacher1's story will be his Lesson 1.

4.1.1.1. Teacher1's Pre-Lesson 1 Interview

In Pre-Lesson 1 Interview, Teacher1 said "I think students have difficulty understanding the shapes of quadratic functions because their shapes are curves instead of lines like the shapes of linear functions, so, I would like my students to understand the shapes of $y = ax^2$ and $y = -ax^2$ as the central concept of the introductory lesson on quadratic functions." The teacher spoke of "the shapes of quadratic functions" instead of "the shapes of quadratic function's *graphs* in the Cartesian coordinate system", which indicates that the teacher equated a function with a function's graph. He never discussed how a function is a particular kind of relationship, and functions' *graphs* have shapes, and any function can be graphed with many shapes depending on the coordinate system in which it is displayed. He continued to say "I will help students understand the shapes of quadratic function by playing a video clip of bungee jumping

[10] I conducted Pre-Lesson Interviews with students before lessons, but I combined Pre- and Post-Lesson Interviews with students, and placed them after Teacher's lesson in this paper to demonstrate the conveyed meanings from a teacher to students by comparing Pre- and Post-Lesson Interviews.

(referring to $y = ax^2$ when $a > 0$) and using a bouncing ball (referring to $y = -ax^2$ when $a > 0$)."

4.1.1.2. Teacher1's Lesson 1

In Lesson 1, Teacher1 played the video clip of bungee jumping and bounced a rubber ball. He then told students, "You have to learn the trajectories of two different movements to understand quadratic function. Bungee jumping represents a concave up curve like an amusement park swinging pirate ship or a swing whereas a bouncing ball shows a concave down curve (he drew two different trajectories with his hand)." Teacher1 did not mention quantities such as the bungee jumper's distance from launch that are significant for students to determine whether a graph is concave up or down. He only asked students to distinguish the different shapes of the trajectories of both movements in space.

4.1.1.3. Teacher1's students' pre-and post-lesson interviews

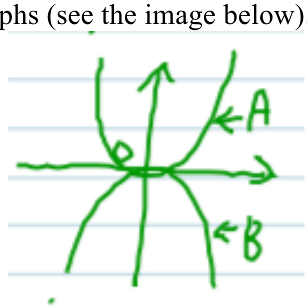
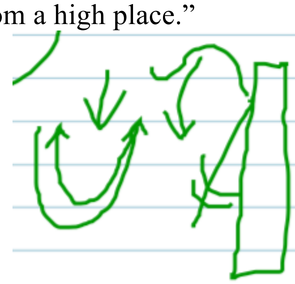
By comparing the two students' Pre- and Post-Lesson 1 Interviews, I discerned what they understood from Teacher1's lesson 1. I asked them about their meanings of a quadratic function in Pre-Lesson 1 Interview. In Post-Lesson 1 Interview, I asked them two questions: (1) What do you think was the central idea of Lesson 1? (2) What do you think your teacher hoped you would learn from Lesson 1? The two students understood Teacher1's instructional actions regarding bungee jumping and bouncing a ball differently (see Table **24** and Table **25**).

Table 24. *Summaries of Student1A's Pre- and Post-Lesson 1 Interviews*

Pre-Lesson 1 Interview	Post-Lesson 1 Interview
Question: What does a quadratic function mean to you? Student1A said what came to his mind was $y = ax^2 + bx + c$. He continued to say that the coefficient "a" affected the shape: when the value of a is increasing, the shape of the quadratic function is getting skinner because change in y is going to increase more as change in x increases.	**Question 1: What do you think was the central idea of Lesson 1?** Student1A said "The central topic of Lesson 1 was examples that look like quadratic functions (he used his hand to show a concave down curve.)." **Question 2: What do you think your teacher hoped you would learn from Lesson 1?** Student1A said "I think the teacher hoped we learned that the meaning of quadratic function is a shape that looks like this (showing a concave down curve). The shapes of bungee jumping and a bouncing ball are the same (drew the image below) because both the bungee jumper and the bouncy ball were rebounded." 

Student1A's response to my question in Pre-Lesson 1 Interview shows that he thought about quantities (change in x and change in y) and the role of the coefficient of x^2 to determine the shape of quadratic functions' graphs before Lesson 1. After experiencing Teacher1's Lesson 1, he explained quadratic functions only in terms of shapes instead of quantities that he mentioned before the lesson. Student1A's response to my question in Post-Lesson 1 Interview also suggests that he saw both the bungee jumper and the bouncy ball were rebounded, and decided that the trajectories of both bungee jumping and bouncing a ball had the shape of a concave down curve, which is true if one takes the bungee jumper's apex as a reference point.

Table 25. *Summaries of Student1B's Pre- and Post-Lesson 1 Interviews*

Pre-Lesson 1 Interview	Post-Lesson 1 Interview
Question: What does a quadratic function mean to you? Student1B said every (quadratic) function has a rule. He then drew graphs (see the image below).  	**Question 1: What do you think was the central idea of Lesson 1?** Student1B said "The central topic of the lesson was the basics of quadratic function." He then said "The teacher introduced bungee jumping and a bouncing ball to show that the shapes of quadratic functions are parabolas as an image associative technique." **Question 2: What do you think your teacher hoped you would learn from Lesson 1?** Student1B said "The teacher wanted us to understand that the shapes of bungee jumping and the bouncing ball were different. Bungee jumping is concave up (see the image below) because the bungee jumper was swinging after falling from a high place."  He said "Bouncing a ball is concave down because the ball bounced up again after falling (see the image below)." 

Student1B expressed his meanings for a quadratic function in Pre-Lesson 1 Interview: (1) having a rule with an exponent of 2, (2) a graph having a u-like shape (see drawings above). His responses to my questions in Post-Lesson 1 Interview show that he made sense of Teacher1's lesson by focusing only on shapes, and then understood bungee jumping goes with a concave up curve whereas bouncing a ball goes with a concave down graph. Student1B spoke of "the shapes of quadratic functions" instead of "the shapes of quadratic functions' *graphs*", which suggests that he were confounding the ideas of a function, a rule of association, and a graph, putting them all into one undifferentiated basket in his mind. It seemed that Student1B's meanings for quadratic functions was largely untouched, but he learned about "shapes" of bungee jumping and bouncing balls..

4.1.1.4. Teacher1's post-lesson interview

After watching the two students' Post-Lesson 1 Interviews, Teacher1 said that Student1A misunderstood his lesson when he saw Student1A saying the shapes of bungee jumping and bouncing a ball have the same concavity. Teacher1 seemed unconcerned that that Student1A attended only to tracing the bungee jumper's movement and not to quantities of time and distance. I took Teacher1's lack of concern as natural, since Teacher1 also attended only to the jumper's movement without attending to time and distance from a reference point.

Teacher1 watched Student1B's Post-Lesson 1 Interview, in which Student1B said, "Teacher1 introduced bungee jumping and a bouncing ball to show that the shapes of quadratic functions are parabolas as an image associative technique". Teacher1 said the central topic of the lesson according to Student1B was not consistent with what he intended. Teacher1 continued to say that Student1B should have said "the reason why the teacher introduced bungee jumping and a bouncing ball is he hoped students understood the shapes of quadratic functions' *graphs* from the two movements" instead of what

Student1 actually said. Contrary to Pre-Lesson 1 Interview and Lesson 1, Teacher1 spoke of “the shapes of quadratic functions’ *graphs*” instead of “the shapes of quadratic functions” and during Post-Lesson Interview with me. Teacher1 then said Student1B understood the different shapes of bungee jumping and bouncing a ball correctly, but expressed his dissatisfaction because he thought Student1B “rambled on too much” when explaining the shapes of bungee jumping and bouncing a ball.

At the end of Teacher1’s Post-Lesson 1 Interview I asked, “If you teach again, is there anything that you want to change? If so, what are these changes and why would you make them?” He answered, “Student1A seems to be unfamiliar with the shape of bungee jumping, so he thought the shapes of bungee jumping and bouncing a ball were the same. I want to emphasize the shape of bungee jumping by clearly saying that it is concave up.”

4.1.1.5. Analysis of Teacher1’s lesson and students’ interviews

Teacher1 taught the central topic (the shapes of quadratic functions) as he planned. He did not mention quantities or variables to help students think about how two movements (bungee jumping and bouncing a ball) are related to quadratic functions in Pre-Lesson 1 Interview or in Lesson 1. Teacher1’s instructional actions, which were mainly about the shapes of trajectory of two different movements (bungee jumping and bouncing a ball), influenced the meanings Student1A and Student1B developed.

Although Student1A’s original meanings for quadratic functions were about shapes, the shapes were related to quantities (change in x and change in y) before Lesson 1. He then experienced Teacher1’s lesson and thought the teacher wanted him to focus on the shapes of trajectory of two movements. Thus, Student1A tried to focus on the shapes of trajectory of two movements, and found that both the bungee jumper and the bouncy ball kept springing back. He then concluded that the trajectory of both bungee jumping and bouncing a ball was concave down. However, Student1B succeeded in identifying the different shapes of the two different movements' trajectories because his prior meanings for quadratic functions were about shapes, not quantities. What Student1B said in Post-Lesson 1 Interview was consistent with what Teacher1 intended.

During the Pre-Lesson 1 Interview and Lesson 1, I struggled to discern the significance the teacher saw in what he called "the different shapes of the two movements" (bungee jumping and bouncing a ball) because the teacher just said "the bungee jumping represented a concave up curve whereas a bouncing ball shows a concave down curve" without explaining *how* the two movements "represented" concave up or concave down curves. Student1A also had difficulties in understanding the teacher's instructional actions, and ended up understanding that both objects' trajectories were concave down curves, which was different than the teacher's intention that students see the two object's movements as associated with different shapes of graphs (one concave up and one concave down).

As I interviewed Teacher 1 before Lesson 1 and experienced his lesson, his emphasis on "the shapes of quadratic functions" confused me because functions do not have shapes. A function is a particular kind of relationship, and functions' *graphs* have shapes, and any function can be graphed with many shapes depending on the coordinate system in which it is displayed. Though Teacher1 sometimes mentioned "the shapes of quadratic functions' *graphs*" while watching students' video clips, his emphasis in his lesson was that the students focus on "the shapes of quadratic functions" instead of "a quadratic relationship between two quantities" or even "the shapes of quadratic functions in the Cartesian coordinate system" during Lesson 1. As a result, Student1B's Post-Lesson 1 Interview showed that the student was also confused about a function and its graph.

4.1.1.6. Analysis of Teacher1's post-lesson interview

In listening to the two students during their Post-Lesson Interview it seemed Teacher1 only compared what the students said to what he would have said when he judged their understandings of his lesson. He then decided that what they said in the video clips was not consistent with what he intended, because he did not hear from them an explanation that mimicked his own in-class explanation word-for-word. With Student1B he was more concerned with how long the students' explanation was than with its content. This suggests that Teacher1 was in KPU Phase 1 because he had the belief that if a student uses his own exact phrasing, then their understanding is the same as his, and vice versa.

Teacher1 decided to change his lesson in the future by giving greater emphasis to the fact that the shape of bungee jumping is concave up. His plans did not indicate any reflection on what he might have done or not done that led to his perceived miscommunication between him and the two students. Thus, his adjustment did not address sources of what he perceived as their difficulties. Teacher1 did not consider his students' thinking because he did not think, "Why did my students have different understandings than I intended?" Instead he mentioned inattentive students and unhelpful private tutorials, finally deciding that the way to help his students have the meanings he wanted was to be more emphatic in his reteach. Again, this shows that Teacher1 was in KPU phase 1.

Though Teacher1 judged that his intentions were not successfully conveyed to Students 1A and 1B, it is worth mentioning in my judgment he successfully conveyed his *meaning* to students. The meaning he conveyed was that one makes a graph of a moving object by mimicking its motion.

4.1.2. The story of Teacher3

The main topics of Teacher3's Lesson 1 were (1) how to determine the equation of line from given conditions, and (2) the differences between the general forms $ax + by + c = 0$ and $y = mx + n$. The central topic of Lesson 2 was the conditions when two lines are perpendicular and form right angles. I will focus on Teacher3's Lesson 1 because her instructional actions in the first lesson gave more insight into her meanings for slope and her image of student thinking.

4.1.2.1. Teacher3's Pre-Lesson 1 Interview

In Pre-Lesson 1 Interview, Teacher3 said the central topic of Lesson 1 was (1) how to determine the equation of line from given conditions, such as two points on the line or one slope and one point, and (2) the equation $ax + by + c = 0$ can be used to represent all lines unlike $y = mx + n$ whereas $y = mx + n$ cannot represent lines that are parallel to the *y*-axis. I asked Teacher3, "What different understandings do you think your students might have about your main idea other than what you intend?" She replied, "My students already know the basic concept of linear equation, so this idea [referring to equation of line] is one where students cannot have different understandings than I intend."

4.1.2.2. Teacher3's Lesson 1

In Teacher3's lesson, she wrote $y = m(x - x_1) + y_1$ and $ax + by + c = 0$, and then said the two equations were related to each other. She then wrote "(in the Cartesian coordinate plane) equation of line ⇔ for *x* and *y*, linear equation" (see *Figure* ***8***).

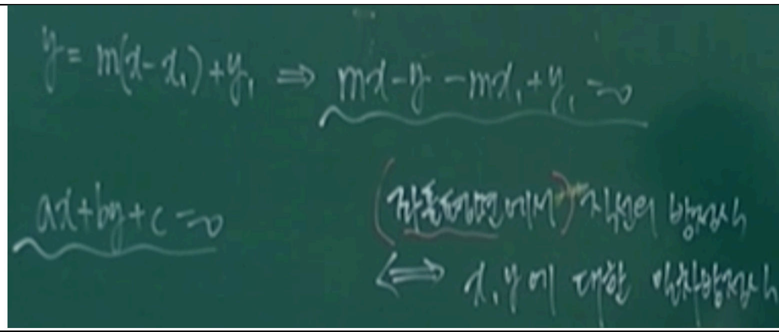
Teacher3's board work	
Translation of her work	$y = m(x - x_1) + y_1 \Rightarrow mx - y - mx_1 + y_1 = 0$ $ax + by + c = 0$ (in the Cartesian coordinate plane) equation of line ⇔ for *x* and *y*, linear equations

Figure 8. Teacher3's Writing on the Board in Lesson 1 to Explain the Two Equations

Teacher3's board work (*Figure* ***8***) confused me because she wrote "(in the Cartesian coordinate plane) equation of line ⇔ for *x* and *y*, linear equations" suggesting $y = m(x - x_1) + y_1$ (or $mx - y - mx_1 + y_1 = 0$) is equivalent to $ax + by + c = 0$. This was inconsistent with her stated goal ($ax + by + c = 0$ and $y = mx + n$ are not equivalent) for the lesson.

4.1.2.3. Teacher3's students' pre-and post-lesson interviews

During Pre-Lesson 1 Interview, Student3A said "slope means how steep a graph is, so the graph lies horizontally when *m* in $y = mx + n$ gets smaller." Student3B said "Slope is slantiness or change in *y*/change in *x*. So, if the absolute value of the slope *m* is increasing, the line would be closer to the *y*-axis." Though Student3B mentioned change in *y*/change in *x*, both students' primary meaning for slope was slantiness. Due to limited time, I was only able to ask them about their meaning for slope in the pre-lesson interviews, and did not question them about procedures to determine a line or general forms of lines.

During Post-Lesson 1 Interview with Teacher3's students, I asked about the central topics of Lesson 1. Both students said there were two central topics: (1) Determining equation of line when having (x_1, y_1) and *m* (the given slope), (2) the relationship between $ax+by+c=0$ and $y=mx+n$, which was consistent with what Teacher3 said. I then asked about the two central topics that Teacher3 planned to convey: (1) What is the equation of a line when having (x_1, y_1) and *m* (the given slope), and (2) Are $ax+by+c=0$ and $y=mx+n$ the same or different?

After experiencing Lesson 1, Student3A and Student3B had different understandings of the main points of the lesson than what the teacher intended. (see Table **26**).

Table 26. *Teacher3's Students' Understandings after Lesson 1*

Student	Issue 1: Determining equation of line when having (x_1, y_1) and *m* (the given slope)	Issue 2: Whether $ax+by+c=0$ and $y=mx+n$ are different
Student 3A	The student wrote $y_2 = m(x_2 - x_1) + y_1$.	The student said "$y=mx+n$ and $ax+by+c=0$ are different because $y=mx+n$ cannot represent $x_1 = x$."
Student 3B	The student wrote $y = m(x - x_1) + y_1$.	The student said "I knew for the first time today that $y = m(x - x_1) + y_1$ and $ax+by+c=0$ were the same."

Student3A confounded (x, y) with (x_2, y_2) when writing the equation of line through the point (x_1, y_1) with a slope of *m* because wrote he wrote $y_2 = m(x_2 - x_1) + y_1$ instead of

$y = m(x - x_1) + y_1$. Student3A understood the difference between $ax + by + c = 0$ and $y = mx + n$ in the way Teacher3 intended. Student3B determined the equation of line when having (x_1, y_1) and *m* (the given slope), but thought that the forms $ax + by + c = 0$ and $y = mx + n$ were equivalent (see *Figure* ***9***).

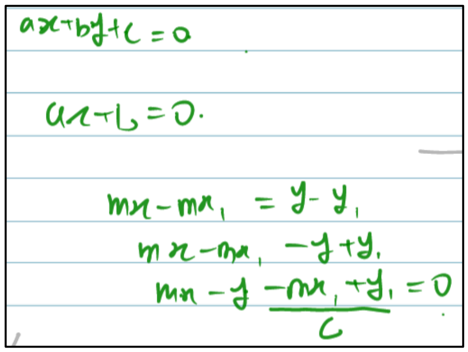


Figure 9. Student3B’s Response to Issue 2 in Post-Lesson 1 Interview

After writing $ax + by + c = 0$ and $mx - y - mx_1 + y_1 = 0$, Student3B said “*x* here (pointing to *x* in $ax + by + c = 0$) is the same as *x* here (pointing to *x* in $mx - y - mx_1 + y_1 = 0$), *y* here (pointing to *y* in $ax + by + c = 0$) is the same as *y* here (pointing to *y* in $mx - y - mx_1 + y_1 = 0$), and c is this (pointing to $-mx_1 + y_1$ in $mx - y - mx_1 + y_1 = 0$), so $ax + by + c = 0$ and $mx - y - mx_1 + y_1 = 0$ are the same.” It seemed Student3B compared coefficients in the two equations to arrive at the determination that the two equations had the same form.

4.1.2.4. Teacher3’s post-lesson interview

When watching the students’ video clips in Post-Lesson Interview with Teacher3, she was surprised as she identified inconsistencies between what she intended and what she heard from the students. She said Student3A misunderstood the difference between a varying point (x, y) and a given point (x_1, y_1) when writing the equation of line through the point (x_1, y_1) with a slope of *m*. She continued to say that Student3A seemed to think (x, y) and (x_1, y_1) were the same. Teacher3 kept trying to remember her instructional actions that might have caused Student3A's misunderstanding, and said "I remember I said putting (x_2, y_2) is okay instead of (x_1, y_1) at that moment. I think this led students to think this way." I then asked her "If you teach again, is there anything that you want to change?" She answered "If I teach again, I would say *x* and *y* in (x, y) are varying quantities whereas x_1 and y_1 in (x_1, y_1) are constants".

Teacher3 then watched Student3B's video clips and identified the student's misunderstanding when she saw the student saying, "I knew for the first time today that $y = m(x - x_1) + y_1$ and $ax + by + c = 0$ were the same." She also remembered her instructional actions and told me that she made a mistake. Teacher3 said "I remember I wrote two forms ($ax + by + c = 0$ and $y = mx + n$) together and said "they are equivalent." (She actually said "they are related to each other".) She continued to say that her instructional actions were unclear, so it was natural for students to be confused. She confessed that she had never thought students might understand two forms are the same. I asked, "If you teach this lesson again, is there anything that you want to change?" She said that if she teaches the lesson again, she would make it clear that $y = m(x - x_1) + y_1$ cannot represent $x = x_1$ (when x_1 is a constant).

4.1.2.5. Analysis of Teacher3's lesson and students' interviews

I knew from Pre-Lesson 1 Interview that Teacher3's intention was to convey that the two forms are different and that $ax + by + c = 0$ is a more general form because it can represent lines such as *x*=2 whereas the form $y = m(x - x_1) + y_1$ cannot. However, Teacher3 said during her lesson “the two equations were related to each other” and wrote on the board (“[in the Cartesian coordinate plane] equation of line ⟺ for x and y, linear equations”). I considered that her instructional actions (“being related” instead of “being different” and “⟺” indicating being equivalent) were potentially confusing and somewhat contradictory for her students. Her instructional actions were not consistent with what she intended but they were, unfortunately, consistent with understandings students developed.

Contrary to Teacher3’s assumptions that her students already knew the basic concept of linear equation, Student3A confounded a variable x with a constant x_2. Also, Teacher3’s instructional actions influenced Student3B’s understanding of the lesson—he understood that $y = m(x - x_1) + y_1$ and $ax + by + c = 0$ are equivalent, the opposite of Teacher3’s intention.

4.1.2.6. Analysis of Teacher3’s post-lesson interview

My observations of Teacher3’s classroom suggest that productive meanings teachers hold do not guarantee that they convey those meanings to their students in classrooms. Teacher3 was unaware that students’ understanding might be different than what she intended. She was aware of her own understanding, but she did not see any value in thinking about students thinking because she assumed that all her students thought as she did. This suggests that she was in KPU Phase 2 before lessons and during lessons.

Teacher3’s KPU phase 2 led her to plan and teach the lessons in ways that were clear to her without considering their clarity to students. However, Teacher3 seemed to move from KPU Phase 2 to Phase 3 as she watched students’ understandings in the video clips with me after the lessons. My conversation with her during the Post-Lesson Interview suggests she moved from KPU Phase 2 to Phase 3 for this particular lesson because she started to think how her instructional actions led to students' misunderstandings once she identified students' misunderstandings. She then mentioned her plan to consider students' ways of thinking if she teaches it again. Her productive meanings seemed to help her move from KPU Phase 2 to KPU Phase 3 quickly, during a single 90-minute session.

Teacher3 focused on what the students understood and thought about her own contribution to students' understandings. Thus, her lesson adjustment was about students' understandings. For example, Teacher3's meanings for variables helped her notice Student3A's confusion between (x, y) and (x_1, y_1). She then spoke of her plan to adjust the lesson by saying *x* and *y* in (x, y) are varying quantities whereas x_1 and y_1 in (x_1, y_1) are constants.

4.1.3. The story of Teacher6

Teacher6 said his lesson goal for Lesson 1 was for students to understand how to compute volumes of solids using integrals by (1) setting a reference axis, (2) finding cross-sectional areas instead of surface areas, (3) making the slices that are perpendicular to the reference axis. The main topic of his Lesson 2 was the same as Lesson 1, but he said he wanted to add an example with a sphere to reinforce with the main topic. I will focus on Teacher6's Lesson 2 because Lesson 2 illustrated Teacher6's concrete instructional actions although both lessons shared the same goal.

4.1.3.1. Teacher6's Pre-Lesson 2 Interview

In Pre-Lesson 2 Interview, Teacher6 said the central topic of his Lesson 2 was for students to understand that it is just a coincidence that integrating the formula for surface area of a sphere gives the same formula as for the volume of the sphere ($\int_0^r 4\pi x^2 dx = \frac{4}{3}\pi r^3$) because $4\pi x^2$ represents surface areas, not cross-sectional areas based on what they already learned from Lesson 1. He continued to say that he hoped students would not imagine collections of nested surface of a sphere as a way to compute the volume of the sphere [in Cartesian coordinates]. When I asked Teacher 6 "What different understandings do you think your students might have about your main idea other than what you intend?" he said "My students' understanding will never be different than I intend, but it is possible they do not care."

4.1.3.2. Teacher6's Lesson 2

During Lesson 2, Teacher 6 said "If you differentiate the volume of a sphere, you will get the cross-sectional area, not the surface of the sphere" (see *Excerpt **8***), intending to refer to differentiating with respect to the *x*-axis in the Cartesian system. He then talked about collections of shells of a sphere with a gesture as if molding a sphere in the air.

Excerpt 8. Teacher 6's Lesson 2 Explaining the Volume and the Surface Area of a Sphere

> T: [writing "the surface area of a sphere: $4\pi r^2$, the volume of the sphere: $\frac{4}{3}\pi r^3$" on the board] Student 1, if you differentiate the volume of a sphere, you get the surface area of the sphere?
>
> S1: Yes.
>
> T: Is this right? Really? Isn't it odd? It is not consistent with what I've said. If you differentiate the volume of a sphere, you will get the cross-sectional area, not the surface of the sphere. [Pause] Am I wrong or is this wrong [pointing to the

$4\pi r^2$ and $\frac{4}{3}\pi r^3$ on the board]? [Pause] Am I wrong? [Teacher 6 and students laugh] You guys saw this in the textbook. If you differentiate a volume, you will get a cross-sectional area. If so, why does this [referring to getting the surface area of the sphere by differentiating the volume of a sphere] work?

Student6A: A coincidence!

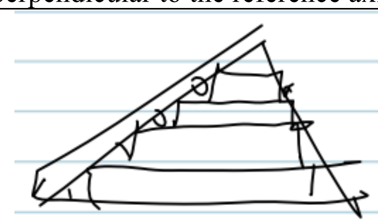
T: A coincidence? Oh, it's a good answer. You forgot about a reference axis again. [Pointing to the volume of the sphere: $\frac{4}{3}\pi r^3$] You are differentiating this [referring to the volume] with respect to the radius "r". "r" is positive, so you are computing the volume by collecting a shell, a shell, a shell, and a shell [showing a bigger and bigger shell with hands as if he was molding a ball in the air]. Of course, there is a way to compute a volume in this way. But, I told you only one thing, you should find a cross-sectional area that is perpendicular to a reference axis.

Teacher6's gesture in *Excerpt **8*** confused me as an observer because molding a sphere in the air suggested that the volume of a sphere was the sum or the collection of surface areas, which was opposed to his main goal for Lesson 2.

4.1.3.3. Teacher6's students' pre-and post-lesson interviews

During Pre-Lesson 1 Interview, both Student6A and Student6B said volumes of solids using integrals were the sum or the collection of them after cutting them in small pieces. After experiencing the teacher's Lesson 1, the two students said that making slices should be perpendicular to the reference axis to compute volumes of solids using integrals. In Post-Lesson 1 Interview, Student6A said "We can get volumes of solids using integrals, but we have to make slices perpendicular to the reference axis. Otherwise, probably there will be empty spaces (pointing to the circled area in the right image in Table **27**)." Student6A tried to understand why making slices should be perpendicular to the reference axis by postulating that non-perpendicular slices would be unable to cover all of the volumes. He failed to notice that both of his pictures would have volume that was unaccounted for until the slices become infinitesimally thin.

Table 27. *Student6A's Understanding after Lesson 1*

Student6A's drawing of making slices being perpendicular to the reference axis	Student6A's drawing of making slices not being perpendicular to the reference axis
	

Student6B also explained why making slices should be perpendicular to the reference axis to compute volumes of solids using integrals by saying "if we make slices making not being perpendicular to the reference axis, it would be equivalent to saying the area of parallelogram with the width of *a* and the slant height of *b* is *ab* (drawing a parallelogram in Figure **10**). Thus, it doesn't make sense." She tried to make a distinction between being perpendicular and being slanted using an analogy of the area of parallelogram.

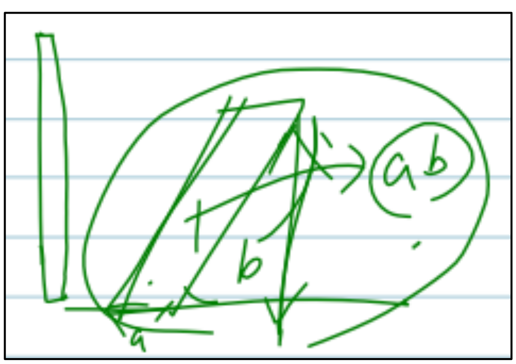


Figure 10**.** Student6B's Understanding after Lesson 1

Before Teacher6's Lesson 2, both Student6A and Student6B understood the main topic of Lesson 1 and Lesson 2 that making the slices should be perpendicular to the reference axis when computing volumes of solids using integrals. However, after experiencing the teacher's lesson 2 where the teacher gave an example of a sphere to reinforce the main topic, both Student6A and Student6B said they got the surface area of the sphere by differentiating the volume of a sphere because the volume of a sphere was the sum or the collection of surface areas.

Table 28**.** *Teacher6's Students' Understandings after Lesson 2*

Student	**Question:** What do you think about the fact that integrating the formula for surface area of a sphere gives the same formula as for the volume of the sphere?
Student 6A	The student said "collections of shells of a sphere give the volume of the sphere."
Student 6B	The student said "we can imagine cutting the sphere from the outside. If we collect the skin of an onion, we can get the volume of the sphere."

Although both Student 6A and Student6B understood that making slices should be perpendicular to the reference axis to compute volumes of solids using integrals after Lesson 1, they accepted that the volume of a sphere was the sum or the collection of surface areas. Student6A imagined collections of shells of a sphere to compute the volume of the sphere. Student 6B assimilated this by imagining an onion because she said collecting the skin of an onion would make the volume of a sphere. The two students' understandings were inconsistent with Teacher6's intention.

4.1.3.4. Teacher6's post-lesson interview

In the Post-Lesson Interview with me Teacher 6 was surprised by what the students said during Post-Lesson Interview. He first watched Student 6A's video clip. He then said that the student who answered "a coincidence" was Student 6A during the lesson, so he was astonished that Student 6A said he got the surface area of the sphere by differentiating the volume of a sphere because the volume of a sphere was the sum of the surface area. Teacher 6 ascribed misunderstanding to Student 6A by saying that the student had his own issues because the student always tried to overstate his explanation. He then asked me whether the next student (Student 6B)'s understanding was consistent with his intention. He continued to say that if Student 6B also understood correctly, Student 6A's misunderstanding was the student's fault. I then showed Student 6B's video clips where she understood the surface area of the sphere by differentiating the volume of a sphere with an onion and its skins. Teacher 6 then said both students did not understand what he intended, and started to think about why their understandings were different from what he intended. He said the students misunderstood because they focused on result of a computation (referring to getting the surface area of the sphere by differentiating the volume of a sphere) and then tried to think about a process that fits the consequence. He said he was too confident with his students because they were gifted students, but he overlooked students who were trying to make sense of his instructional actions and to justify them. He said he would get rid of the example (referring to the volume and the surface area of a sphere) because the example might help students misunderstand. He continued to say that he would say clearly that it is not true when he wanted to convey the surface area of the sphere by differentiating the volume of a sphere is a coincidence.

4.1.3.5. Analysis of Teacher6's lesson and students' interviews

I conducted Pre-Lesson 2 Interview with Teacher 6, so I knew what he wanted to convey to students. However, after listening to and watching his instructional actions, I wondered if students understood his intention was to convey that getting the surface area of a sphere from differentiating the volume with respect to radius is a coincidence. Although Student 6A answered that it was a coincidence during Lesson 2, I doubted that the student understood the teacher's intention because Teacher 6's instructional actions were not clear (not mentioning with respect to the *x*-axis in the Cartesian system and not having a clear conclusion) (see *Excerpt* ***8***).

Contrary to Teacher6's assumptions that his students' understandings will never be different than he intended, both Student 6A and Student 6B understood that they got the surface area of the sphere by differentiating the volume of a sphere because the volume of a sphere was the sum or the collection of surface areas, the opposite of Teacher6's goal of Lesson 2.

4.1.3.6. Analysis of Teacher6's post-lesson interview

Teacher6 was insensitive to how students might think about the idea he would convey before the lesson, so his instructional actions were not clear to me as an as observer. Teacher6 was unaware that students' understanding might be different than he intended. He was aware of his own understanding and the thing that is valuable for students have, but he did not see any value in thinking about students having it. Student thinking is not an object for consideration. This suggests that he was in KPU Phase 2 before lessons and during lessons.

Teacher 6 remained at KPU phase 2 even after watching students' video clips. Teacher 6 ascribed their misunderstandings to their mistakes instead of reflecting on his instructional actions by saying that students misunderstood because they focused on result of a computation (referring to getting the surface area of the sphere by differentiating the volume of a sphere) and then tried to think about a process that fits the consequence. His plan to adjust the next lesson was to get rid of the example that led to student misunderstanding instead of thinking about his own contribution to student misunderstanding or the sources of students' difficulties.

4.2. Relationships Between Meanings Teachers Hold, What They Did in the Lessons, Meanings Their Students Construct

In this section, I will discuss relationships between the meanings teachers held, what they did in their lessons, and the meanings their students constructed during the lessons. I discerned meanings teachers expressed in their responses to MMTsm items and meanings they expressed in the Pre- and Post-Lesson Interviews and the lessons I observed. I also determined the meanings students constructed by comparing students' meanings demonstrated in Pre-Lesson Interviews to their meanings demonstrated in Post-Lesson Interviews. Based on what I discerned or observed, I will describe the relationships in *Figure **11***.

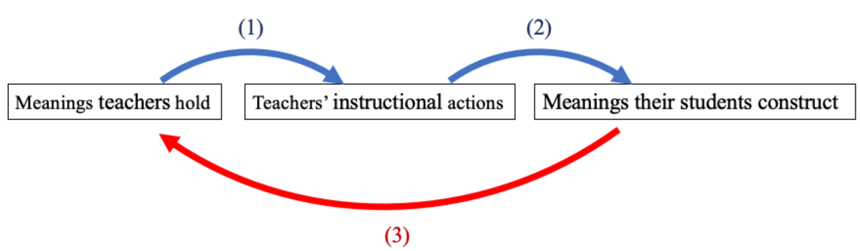


Figure 11. Relationships Between Meanings Teachers Hold, Their Instructional Actions, Meanings Students Construct

*Figure **11*** illustrates the two relationships (between meanings teachers hold and their instructional actions, and between teachers' instructional actions and meanings students construct). *Figure **11*** also shows the link between teachers' interpretation of meanings students construct and their own meanings. I will examine the three links in *Figure **11***.

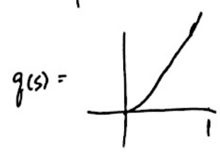
4.2.1. Teachers' Meanings, Their Instructional Actions, and Students' Meanings

I will argue meanings teachers held influenced what they wanted students to understand and the way they designed their lessons. Thus, teachers' meanings affect their instructional actions. From the three groups (Group A, B, and C), I identified that teachers' productive meaning is a necessary condition for them to convey coherent meanings to students, but not a sufficient condition.

I will explain the relationships on the basis of two teachers' mathematical meanings: (1) Teacher1 who demonstrated unproductive meanings in the MMTsm items, and (2) Teacher3 who demonstrated productive meanings in the MMTsm items.

Teacher 1's meanings led him to set unclear and incomprehensible goals. Teacher 1 responded to two tasks involving function notation in the MMTsm and Pre-Lesson Interviews with me. His meaning for function notation such as $w(t)$ or $q(s)$ that he demonstrated during a MMTsm task was that function notation is a label for the formula on the right hand side of a function definition" because he filled the blanks with "s" and "t" although the input variable of the function c was v (See Table **29**). It seemed he viewed "$w(t)$" as one inseparable symbol that could be replaced with "y".

Table 29. *Teacher1's Response in a Function Notation Item of MMTsm*

Teacher1's response to Korean version	두 함수가 다음과 같이 정의되어 있다. $w(t) = \sin(t-1)$ if $t \geq 1$ $q(s) = \sqrt{s^2 - s^3}$ if $0 \leq s < 1$   세 번째 함수 c 는 w 와 q 두 부분으로 정의되어 있다. 함수 c 가 알맞게 정의되도록 빈 칸을 채우시오. $c(v) = \begin{cases} q(\underline{s}) \text{ if } 0 \leq \underline{s} < 1 \\ w(\underline{t}) \text{ if } \underline{t} \geq 1 \end{cases}$
English version of the item	Here are two function definitions. $w(t) = \sin(t-1)$ if $t \geq 1$ $q(s) = \sqrt{s^2 - s^3}$ if $0 \leq s < 1$ Here is a third function c, defined in two parts, whose definition refers to w and q. Place the correct letter in each blank so that the function c is properly defined. $c(v) = \begin{cases} q(\underline{\quad}) \text{ if } 0 \leq \underline{\quad} < 1 \\ w(\underline{\quad}) \text{ if } \underline{\quad} \geq 1 \end{cases}$

In another item Teacher1 demonstrated his meaning for function notation as multiplication of the function name and the argument, when he selected option *c.* His response is consistent with viewing *h* as a variable for which to solve instead of the name of a function (see Table **30**).

Table 30. *Teacher1's Response to Another Function Notation Item of MMTsm*

Teacher1's response to Korean version	함수 h 는 모든 실수에서 정의된 단조 증가 함수이고, 상수 b 에 대하여 $h(b-5)=9$ 일 때, 두 점$(b, 9)$ 와 $(b-5, 9)$ 중에 어떤 것이 그래프 $y=h(x-5)$ 에 있는가? 설명하시오. *가장 알맞은 답과 설명을 고르시오.* a. b 가 9 를 함숫값으로 내는 x 값이기 때문에 $(b, 9)$ 가 그래프 위에 있다. b. $b-5$ 에서 h 의 함숫값이 계산됐고 함수가 단조증가하기 때문에$(b, 9)$ 가 그래프 위에 있다. c. h 에 대해 풀면 $h=\frac{9}{b-5}$ 이고, $y=\left(\frac{9}{b-5}\right)(x-5)$ 가 되어 $x=b$ 일 때 9 의 값이 나오므로 $(b, 9)$ 가 그래프 위에 있다. d. $b-5$ 는 x 자리이고, 9 는 y 자리이기 때문에 $(b-5, 9)$ 가 그래프 위에 있다. e. $b-5$ 에서 함숫값이 9 로 계산됐기 때문에 $(b-5, 9)$ 가 그래프 위에 있다. f. 모르겠다.
English version of the item	h is a strictly increasing **function** defined for all real numbers. $h(b-5)=9$ for some number b. Which of $(b, 9)$ or $(b-5, 9)$ is on the graph of $y=h(x-5)$ when graphed on a calculator? Explain. *Select the best answer and explanation.* a. $(b, 9)$ is on the graph because b is the input that gives 9 as the output. b. $(b, 9)$ is on the graph because $b-5$ is the input to h and the function is strictly increasing. c. $(b, 9)$ is on the graph because when we solve for h, $h=\frac{9}{b-5}$, making $y=\left(\frac{9}{b-5}\right)(x-5)$, for which $x=b$ produces a value of 9. d. $(b-5, 9)$ is on the graph because $b-5$ is in the x position and 9 is in the y position. e. $(b-5, 9)$ is on the graph because $b-5$ is the input that gives 9 as the output. f. I don't know.

Teacher1 demonstrated two unproductive and even contradictory meanings for function notation in the MMTsm: that $f(x)$ is one inseparable symbol that could be replaced with "y" , and that $f(x)$ is "f" multiplied by "x". His responses suggest that his meanings for function were about symbols and formulas instead of about relationships between two varying quantities. Teacher1's primary meaning for quadratic function (a quadratic can be either a concave up curve or a concave down curve) led him to set goals for his lesson that were based entirely on associations between an object's motion in space and graphs that mimicked that motion. The idea that graphs arise from the covariation of two quantities' values was absent from his thinking and therefore absent from his instruction.

Teacher1's instructional actions were mainly about the shapes of trajectory of bungee jumping ($y = ax^2$ when $a > 0$) and bouncing a ball ($y = -ax^2$ when $a > 0$) as he planned. He said bungee jumping "represented" a concave up curve like the movement of an amusement park swinging pirate ship, and a bouncing ball showed a concave down curve, without ever mentioning the quantities being related by a quadratic function. Teacher1's students, Student1A and Student1B, developed different understandings as they tried to make sense of the teacher's lesson. Student1A said that both bungee jumping and bouncing a ball had the shape of a concave down curve because both the person bungee jumping and the bouncy ball rebounded. Student1B focused on different shapes of the two movements and said the person bungee jumping was swinging whereas the bouncy ball kept springing back. Teacher1's lesson guided the two students to focusing only on the shapes of objects' trajectories.

Teacher1's unproductive meanings for the ideas that he taught led to his inability to think about student thinking. When he watched students' Post-Lesson Interviews, he focused on students' actions or behavior instead of their thinking. He said the students understood his lesson when he heard from students what he would have said. For example, Student1B said what the teacher intended, but Teacher 1 expressed dissatisfaction with Student1B's understanding because the student "rambled on too much".

In addition, Teacher1 did not think about contributions of his own actions to student understanding. He ascribed student statements that differed from what he would have said to mistakes, inattention, or private tutorials, but did not reflect on his instructional actions and their relationships to student understanding when he identified discrepancies between his intentions and student understanding.

What happened in Teacher3's classroom shows that teachers' productive meaning does not guarantee that they convey their meanings to students in class. I reviewed Teacher3's response to the MMTsm item on meanings for slope (see *Figure **6***). Teacher 3 demonstrated her meanings for slope, which was the relationship between a change in x and the associated change in y (see Table **31**).

Table 31. *Teacher3's Response to a Slope Item in the MMTsm*

<table>
<tr>
<td>Teacher3's response to Korean version</td>
<td>최 선생님은 기울기에 대한 생각열기 수업에서 직선의 기울기가 3.04 임을 계산하기 위해서 8.2 를 2.7 로 나누었다.
$8.2/2.7 = 3.04$

최 선생님의 학생에게 3.04 가 무엇을 의미하는지 전달하시오.

직선 상의 두 점에 대하여
x 값이 2.7만큼 증가할 때 y값이 8.2만큼 증가하는 ~~직선~~
정도의 기울기는, x값이 1만큼 증가할 때 y값이
3.04 만큼 증가하는 정도의 기울기와 ~~같고,~~ 같다.

(next page)

Part B.

최 선생님은 기울기에 대한 생각열기 수업에서 직선의 기울기가 3.04 임을 계산하기 위해서 8.2 를 2.7 로 나누었다.
.

한 학생이 3.04 의 의미를 다음과 같이 설명했다. "그건 x 가 1 만큼 변할 때마다 y 가 3.04 만큼 변한다는 것을 의미해" 최 선생님이 물었다. "만약 x 가 1 이외의 다른 수만큼 변하면 3.04 는 뭘 의미할까?"

최 선생님의 질문에 좋은 대답은 무엇일까?

~~변화율~~
x의 변화에 3.04를 곱하면 y의 변화가 도출된다.
즉 3.04는 평균변화율을 의미한다.</td>
</tr>
<tr>
<td>English version of the item and translation of her response</td>
<td>Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04. 8.2/2.7=3.04

Convey to Mrs. Samber's students what 3.04 means.

For two points on a line, the slope of ~~a line~~ if x increases by 2.7, y increases by 8.2 is the same as the slope if x increases by 1, y increases by 3.04, ~~and~~

(next page)

Part B.

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.

A student explained the meaning of 3.04 by saying, "It means that every time x changes by 1, y changes by 3.04." Mrs. Samber asked, "What would 3.04 mean if x changes by something other than 1?"

What would be a good answer to Mrs. Samber's question?

~~Rate of change~~</td>
</tr>
</table>

	The change in y will be determined by multiplying the change in x by 3.04. Thus, 3.04 means the average rate of change.

Teacher3's response to Part A[11] in the slope item shows that her primary meaning for slope was the relationship between change in x and the associated change in y. Her response to Part B conveyed that when given a probing question, her meaning for slope as a multiplicative comparison of any change in x and the associated change in y emerged. Although Teacher3 demonstrated productive meanings for slope in the MMTsm item and interviews with me, in the Post-Lesson Interview she identified inconsistencies between what she intended and what students understood. One of her students wrote $y_2 = m(x_2 - x_1) + y_1$ instead of $y = m(x - x_1) + y_1$ as the equation of a line when having (x_1, y_1) and m (the given slope) and she then realized that her students did not see x and y as variables and x_1 and y_1 as constants even though she used them in these ways. Her lack of orientation to students' mathematics before and during the lesson led her to teach the ideas in ways that turned out to be clear only to her, which led to miscommunication between her and students.

Teacher3 did not think about the possibility of students having different understandings than her intentions. She taught the main idea in ways that were clear only to her, so her instructional actions were unclear even to an observer (me) who knew her intention. For example, Teacher3 said the main idea she wanted her students to understand was that $y = ax + b$ and $ax + by + c = 0$ were different forms of linear

[11] The first part of the item is not labeled "Part A" to ensure that teachers would not look ahead to Part B before answering Part A.

equations because $ax+by+c=0$ was more general (it could represent x=constant where $y=ax+b$ cannot). However, she wrote $y=m(x-x_1)+y_1$ and $ax+by+c=0$, and then said the two forms were related to each other during her lesson. She then wrote "(in the Cartesian coordinate plane) equation of line ⟺ for x and y, linear equation". Teacher 3 showed ambiguous instructional actions: the unclear statement (being related) and the wrong notation for her goal (⟺) although she intended to convey $y=m(x-x_1)+y_1$ and $ax+by+c=0$ are different forms. After experiencing this lesson, one of Teacher3's student developed the opposite understanding that the forms $ax+by+c=0$ and $y=mx+n$ were equivalent. When she watched the student's understanding, she acknowledged that her instructional actions were unclear and they led the student to misunderstand the lesson. The fact that Teacher3 considered that her actions might have contributed to Student3B's understanding that the two forms were equivalent is consistent with my claim that she was capable of moving to KPU Phase 3.

4.2.2. The Nature of Teachers' Interpretation of Students' Responses to My questions

In this study, the teachers listened to what their students said during Post-Lesson Interviews with me by watching clips of my interviews of their students. After watching students' video clips, the teachers tried to interpret what their students said. I was able to discern teachers' own meanings as well as their image of student thinking from their interpretation of what student said.

Teacher1's interpretation of what his students said suggests that his meanings led him to focus on students' actions or behaviors instead of student thinking. Teacher1 also decided to ascribe any of the two students' understandings inconsistent with his intention to their own mistakes because they did not repeat his explanations verbatim. It seemed that his meanings led to his inability to think about student understanding and his own contribution to student understanding. Teacher1 did not demonstrate a clear understanding of the differences between a quadratic function, a parabolic graph in the Cartesian coordinate plane, and a parabolic trajectory of motion of an object in space. He confused all three in his lesson, yet when he saw his students making similar mistakes he attributed their confusion only to lack of attention or inferior after-school tutoring. Teacher1 even said "Student1B should have said the shapes of quadratic functions' *graphs*" after watching a video, but did not connect this language to his own language use during the lesson.

Teacher1's decisions to adjust his lessons were unrelated to what their students understood. His plan to adjust the next lessons did not address the sources of students' difficulties. He rather planned to change his lesson a little bit in ways that it was clearer to him, not to students. Teacher1 said he would change his lesson only by emphasizing what he wanted students to say about bungee jumping—saying that the shape of bungee jumping is concave up clearly.

As opposed to Teacher1, Teacher 3's interpretation of what her students said indicates that her productive meanings allowed her to think about her own contribution to students' understanding after watching students' Post-Lesson Interviews. Teacher3's meaning for variables helped her to notice her student's confusion between a varying point (x, y) and a given point (x_1, y_1) from the slope formula that her student wrote. Teacher3, who taught the concept of slope, tried to remember her instructional actions that might have caused her student's misunderstanding when watching students' video clips. She started to create her model of the student thinking by saying that "this student seemed to think (x,y) and (x_1,y_1) were the same. I remember I said putting (x_2,y_2) is okay instead of (x_1,y_1) at that moment. I think this led students to think this way." She then said, "If I teach again, I would say x and y in (x,y) are varying quantities whereas x_1 and y_1 in (x_1,y_1) are constants". Teacher3 attributed the discrepancy between her intended meaning and meaning the student constructed to her instructional actions.

What happened in Teacher 3 and Teacher 6's classrooms suggests that coherent meanings and intention teachers demonstrated during Pre-Lesson Interviews do not guarantee that they conveyed meanings to their students in class. They all were unaware that students' understanding might be different than they intended. They were aware of their own understanding and the thing that is valuable for students have, but they did not see any value in thinking about students having it. Student thinking is not an object for consideration. This suggests that they were in KPU Phase 2 before lessons and during lessons.

Teachers' in Group B and Group C KPU phase 2 led them to plan and teach the lessons in the ways that it was clear to them. However, teachers in Group B seemed to move from KPU Phase 2 to Phase 3 as they watched students' understandings in the video clips with me after the lessons. Post-Lesson Interview with Teacher3 suggests that she moved from KPU Phase 2 to KPU Phase 3 for the lesson because she started to think about their instructional actions that led to students' misunderstanding and mentioned her attempt to adopt students' ways of thinking if she teach again. Her coherent meanings seemed to help her move from KPU Phase 2 to KPU Phase 3 easily. For example, Teacher3's meanings for variables helped her notice Student3A's confusion between (x, y) and (x_1, y_1). She then spoke of her plan to adjust the lesson by saying x and y in (x, y) are varying quantities whereas x_1 and y_1 in (x_1, y_1) are constants.

However, all teachers who demonstrated coherent meanings and intention before the lessons improved their KPU phases. Teachers in Group C remained at KPU phase 2 even after watching students' video clips. For example, Teacher6 ascribed their misunderstandings to their mistakes instead of reflecting on his instructional actions by saying that students misunderstood because they focused on result of a computation (referring to getting the surface area of the sphere by differentiating the volume of a sphere) and then tried to think about a process that fits the consequence. His plan to adjust the next lesson was to get rid of the example that led to student misunderstanding instead of thinking about his own contribution to student misunderstanding or the sources of students' difficulties.

5. DISCUSSION OF RESULTS

My research questions were (1) What relationships are there between teachers' mathematical meanings for the ideas that they teach and their instructional actions regarding those ideas? (2) What relationships are there between teachers' instructional actions and meanings his or her students construct? (3) How do teachers' meanings and their images of student thinking influence their interpretation of meanings that students construct from their lesson?

My answer to the first two questions is that meanings teacher hold influenced their instructional actions, and teachers' instructional actions have an impact on meanings their students construct. However, teachers' productive meanings did not guarantee that they conveyed their meanings to students in class because the extent to which they were sensitive to students thinking (teachers' KPU) affected the way they taught mathematical ideas. For example, Teacher 3 taught the main idea in ways that were clear to her, but the conveyed meanings to students were inconsistent with her intention.

My answer to the third question is that teachers' meanings and their image of student thinking affected their interpretation of students' meanings. I discerned teachers' own meanings and their image of student thinking from their interpretation of what student said. Teacher1's meanings and his insensitivity to student thinking led to his inability to describe what students said in terms of students' understandings. Rather, he focused on visible actions or behaviors that students demonstrated when interpreting what students said. Teacher2 followed the same pattern of behavior where his meanings were expressed in the classroom and conveyed to students, and those meanings prevented him from learning while watching his student's interviews or progressing in his KPU phase (see *Figure 7*).

On the other hand, Teacher3's productive meanings led to her interpretation of what students said, which was about students' understandings. After hearing what students said, she became more sensitive to student thinking. Finally, Teacher3 moved her KPU phase from Phase 2 to Phase 3. Teachers4 and 5 followed the same pattern of behavior where their productive meanings were not expressed in the classroom or conveyed to students, but those meanings did allow them to learn from watching their student's interviews and progress in their KPU phase (see *Figure 7*).

The results of this study show (1) teachers' productive meaning is a necessary condition to express those meanings in classroom, but not a sufficient condition, and (2) teachers' productive meaning is a necessary condition to convey productive meanings to students, but not a sufficient condition. This is consistent with Sánchez-Matamoros, Fernández, and Llinares (2019, p. 96) that said "identifying the mathematical elements in students' answers is necessary but not sufficient to recognize different characteristics of students understanding" and "the process of interpreting students' mathematical understanding is not only determined by mathematical knowledge but also by the knowledge of students' mathematical thinking."

Although teachers' productive meanings were not a sufficient condition for conveying those meanings to students, when teachers had productive meanings those meanings helped them quickly improve their KPU phases. Teacher3 was in KPU phases 2 before the lessons because she did not think about how their students would see their lesson. However, her KPU Phase advanced after watching students' Post-Lesson Interviews. Teacher3 then talked about her decision to adjust her lessons and tried to address the sources of students' difficulties. How Teacher3 talked about her plan to adjust future lessons after hearing what students said indicates the potential relationships between teachers' adjusted meanings from their interpretation of student meaning and their future instructional actions (the dotted line (4) in *Figure **12***).

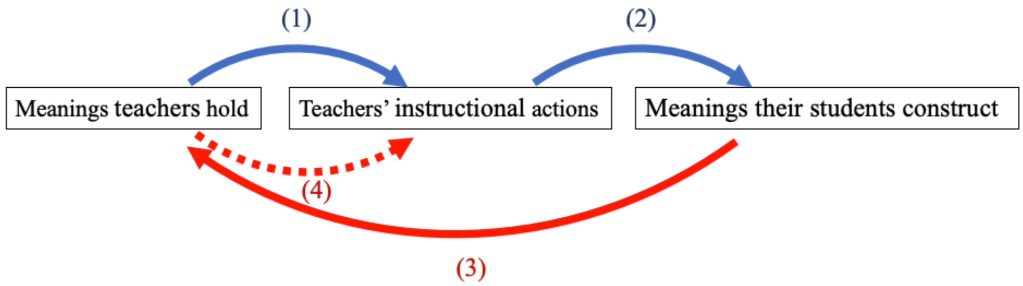


Figure 12. Added Relationships Between Meanings Teachers Hold, Their Instructional Actions, Meanings Students Construct

6. LIMITATIONS AND IMPLICATIONS

The eight teachers were not selected random and so cannot be taken as representative of all Korean middle and high school teachers, nor their students be taken as representative of all Korean students. Thus, generalization of the results is not possible. Another limitation is that I was also a participant in this study. I perturbed students by asking about their lesson and provided the teachers with opportunities to think about students' thinking and their lessons. Thus, what I observed might not be the same as what happens when teachers listen to students responding to their own questions. Teachers rarely have opportunities to listen in-depth to students' understandings as described to an outside interviewer.

It was hard to make a distinction between KPU phase 3 and 4 because I had no chance to observe how the teachers adjusted their next lessons. I will address this limitation by observing a teacher's lessons comprehensively in my next study.

The findings from this study suggest that students can develop a startlingly wide variety of meanings through interactions with their teachers, and teachers need to think ahead to how their students might understand their statements or actions. Thinking about teaching in this way means that we must also think about the most productive meanings that teachers can have for mathematical ideas, as well as teachers' image of student thinking.

By using a conceptually coherent framework to explain what happens in classrooms, this study demonstrated that to maximize the potential of their classroom teachers must a) think about productive mathematical meanings for student learning, b) have clear intentions on how to convey those meanings to their students, c) remain cognizant of the multiple ways in which students might interpret their statements and actions, and d) develop hypotheses about their students' understandings of the teacher's instruction based on expressions of their understandings in written work and classroom conversations, test those hypotheses with further tasks, and adjust their lessons accordingly. My research could therefore contribute to improving teacher preparation and professional development programs.

7. CONCLUSION

Teachers' mathematical meanings play a significant role in student learning. It is plausible that the more productive mathematical understanding a teacher holds, students will have greater opportunities to construct robust understanding. However, this study shows productive meaning a teacher holds does not guarantee that he or she could convey his or her meaning to students in class. In other words, holding productive meanings is a necessary condition to convey productive meanings to students, but not a sufficient condition.

Teachers who demonstrated productive meanings constantly experienced discrepancies between what they intended and what their students understood, and wonder why students did not understand what they tried to convey. The results described here point to a breakdown in the conveyance of meaning from a teacher to students when the teacher has no image of how students might understand his or her statements and actions. This suggests that supporting in-service teachers in deliberating on student thinking so that they can convey their meanings and what they intend to students in classroom is critical. This study also indicates that it is crucial to help pre-service teachers improve what they are capable of conveying to students and their images of what they hope to convey to future students.

This study also illustrates that teachers' productive meanings can help them advance their KPU Phases. This also suggests that teacher educators help teachers develop productive meanings first in order them to improve their image of student thinking.

PAPER THREE: RELATIONSHIPS BETWEEN WHAT TEACHERS KNOW, WHAT THEY DO IN CLASSROOM, AND WHAT THEIR STUDENTS UNDERSTAND

1. INTRODUCTION

There has been substantial interest in teachers' mathematical understanding needed for the practice of teaching ever since Shulman (cited in1986; Shulman, 1987, p. 10) distinguished between pedagogical content knowledge (PCK) and content knowledge (CK). To Shulman, PCK is one's knowledge of how to make the subject comprehensible to others whereas CK refers to a person's understanding of the subject per se.

On the basis of Shulman's distinction between CK and PCK, researchers have investigated teachers' knowledge for teaching mathematics in an effort to classify what a teacher knows in the practice of teaching (Ball, Thames, & Phelps, 2008; Blömeke et al., 2013; Buchholtz & Kaiser, 2013; Buchholtz et al., 2013; Hill, 2010; Hill & Ball, 2004; Hill et al., 2008; Hsieh et al., 2011; McCrory et al., 2012; Schmidt, Houang, & Cogan, 2002; Tatto et al., 2008; Tatto et al., 2009; Tatto & Senk, 2011; Tatto et al., 2011; Wang & Tang, 2013).

Frameworks that categorize teachers' knowledge are important for researchers to develop items for assessing teacher's mathematical knowledge based on their categorizations. However, frameworks that categorize knowledge typically have two major shortcomings: (1) "knowledge" is a primitive, undefined term; (2) with "knowledge" undefined, they cannot hypothesize mechanisms by which a teacher's knowledge influences his or her instructional actions. In addition, to explain ways in which students learn from instruction, the framework must be able to connect teachers' instructional actions with what students learn from them. This last requirement points to the need for including models of student thinking and teachers' images of student thinking in the framework.

My aim is to explain relationships among what teachers know, their in-the-moment instructional decisions, and what their students learn as a result. The relationships between what teachers know and what students understand should be explained based on a conceptually coherent narrative that connects what teachers know with what they do and what their students learn. Thus, my stance accords with Silverman and Thompson (2008). They asked the question "What mathematical understandings allow a teacher to act in these ways spontaneously? How might these understandings develop?" (Silverman & Thompson, 2008, p. 500).

In this article I focus on a calculus teacher's (Terri's) image of student thinking, how this image influenced Terri's instruction, and the meanings students built in trying to understand Terri's intentions (Terri's "conveyed meanings"). I observed 15 lessons and conducted Pre- and Post-Lesson Interviews with Terri and her students at multiple points. I used the constructs of *meaning*, *conveyance of meaning* and *Key Pedagogical Understanding* (KPU) to guide observations and analyses of classroom observations and interviews with the teacher and her students to investigate the following research questions.

1. The meanings Terri conveyed to students
    a. What are Terri's meanings for the ideas she is teaching?
    b. How do Terri's meanings influence her instructional actions?
    c. What meanings do students already possess in relation to the ideas Terri is teaching?
    d. How do those meanings influence their interpretations of Terri's utterances and actions?
2. Terri's development of Key Pedagogical Understanding (KPU)
    a. To what extent does Terri think about ways her students understand her?
    b. How does Terri's image of student thinking influence her instructional decisions?

2. LITERATURE AND THEORETICAL PERSPECTIVE

Researchers holding a knowledge-primitive perspective have hypothesized types of mathematical knowledge needed in the practice of teaching and designed assessments to measure teachers' Mathematical Knowledge for Teaching (MKT). Prominent projects are *Learning Mathematics for Teaching* (Hill, 2010; Hill & Ball, 2004; Hill et al., 2008), *Mathematics Teaching in the 21st Century* (Schmidt et al., 2011), *Teacher Education and Development Study-Mathematics* (Blömeke et al., 2013; Buchholtz & Kaiser, 2013; Hsieh et al., 2011; Tatto et al., 2008; Tatto et al., 2009; Tatto & Senk, 2011; Tatto et al., 2011; Wang & Tang, 2013), *Knowledge of Algebra for Teaching project* (McCrory et al., 2012), and *Professional Competence of Teachers, Cognitively Activating Instruction, and the Development of Students' Mathematical Literacy* (Baumert et al., 2010; Kunter et al., 2007). As already mentioned, the frameworks of these projects are designed to categorize teachers' MKT. They are not intended to describe and explain moment-by-moment events of teaching and learning or to describe and explain cumulative effects of such events on students' learning. However, the focus of this article is on understanding these moment-by-moment events both short-term and long-term, which requires another type of framework.

To address my research questions requires a theoretical perspective that takes instruction as a form of conversation between teachers and students. Conversations can be of various qualities in terms of how well participants understand each other, or how seriously each takes contributions of the other, but they are conversations nevertheless. In this regard I use a framework developed by Thompson, which is a hybrid of Bauersfeld's symbolic interactionism as specialized for mathematics classrooms, Piaget's genetic epistemology, and his and Steffe's restatement of von Foerster's cybernetics in the language of first- and second-order observers making first- or second-order models of others' thinking[12].

I use Thompson (2013a)'s constructs of *meaning* and *conveyance of meaning*, and Silverman and Thompson's (2008) construct *Key Pedagogical Understanding* (*KPU*). Thompson's definition of meaning evolved from Piaget's genetic epistemology, especially Piaget and Garcia's (1991) logic of meanings and Skemp's (1962) interpretation of Piaget's meaning of "to understand" as "assimilate to a scheme".

Thompson and colleagues (Thompson et al., 2014) defined *mathematical meanings* to refer to the space of implications of an understanding. Hub and Dawkins

[12] The point of view of a second-order observer refers to "the observer's ability through second-order consensuality to operate as external to the situation in which he or she is, and thus be the observer of his or her circumstance as an observer" (Steffe & Thompson, 2000, p. 10)

(2018) rephrased this as the space of inferences available to the student given the student's understanding at that moment. The construct *conveyance of meaning* does not mean that students end up having the same meanings that a teacher holds. Rather, meanings conveyed from a teacher to his students are the meanings students construct in attempting to understand what the teacher intends. A conveyed meaning might or might not resemble the teacher's meaning. I use the six-phase model of teachers' development of KPU that Thompson developed to discuss teachers' image of student thinking as well as the extent to which they consider student thinking in making instructional decisions. My perspective on mathematical meanings, conveyance of meaning, and KPU allows me to bridge what teachers know, what they teach, and what their students learn.

2.1. Meaning

Coherent mathematical meanings serve as a foundation for future learning, so it is important that students build useful and robust meanings. One way students develop meanings is by trying to make sense of what their teacher say and do in the classroom. Before discussing how meanings are conveyed in the classroom from a teacher to students, I will explain what Thompson (2013a) meant by *meanings*. According to Piaget, to understand is to assimilate (Richard R Skemp, 1962; Skemp, 2012; Thompson, 2013a; Thompson & Saldanha, 2003) and "assimilation is the source of schemes" (Piaget, 1977, p.70 cited in Thompson, 2013a). Thus, the phrase "a person attached a meaning to a word, symbol, expression, or statement" means that the person assimilated the word, symbol, expression, or statement to a scheme. A scheme is an organization of ways of thinking, images, and schemes. When I say *assimilate* I mean the ways in which an individual interprets and make sense of a text, utterance, or self-generated thought.

According to Piaget, repeated assimilation is the source of schemes, and new schemes emerge through repeated assimilations, which early on require functional accommodations and eventually entail metamorphic accommodations (Steffe, 1991).

Thompson (2013a) said *meaning* is the space of implications of an understanding. For example, a student can understand slope as a coefficient of x because she learned "m" is slope in $y = mx + n$. This is her understanding of slope in the moment. Then, she could think about slope as "1" when first looking at $x = 1$ because 1 is the coefficient of x. This is an implication of her understanding in the moment. The students' meaning in the moment of understanding is the space of implications of that understanding.

2.2. Conveyance of Meaning

Thompson (2013a, 2016) explains that when one person intends to convey a meaning to another, the speaker's conveyed meaning is the meaning the other person constructs in attempting to understand what the speaker intends. Thompson makes a clear distinction between a teacher' intended meaning (the meaning he or she envisions students eventually having) and students' conveyed meaning (the meanings they construct in attempting to understand what the teacher intends). Thompson offers this vision of a conversation in his role of a second-order observer observing conversations involving teachers (as first- or second-order observers) and students (as first- or second-order observers).

Consider a teacher who teaches mathematical ideas to his students. A teacher has his meanings for the mathematical ideas. The teacher intends to convey the mathematical ideas to his students. In doing so, the teacher and his students are interacting and making an attempt to interpret others in class. Thompson (2013a) proposed a theory to explain how two people (person A and person B) attempt to have a conversation that leads to mutual understanding.

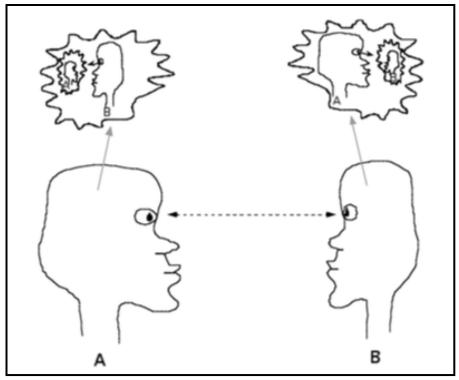


Figure 13. Persons A and B Attempting to Have a Reflective Conversation (Thompson, 2013a, p. 64).

According to Thompson (2013a), person A in *Figure **13*** holds something in mind that he intends Person B to understand. Person A considers not only how to express what he intends to convey but also how person B might hear person A. If both persons are acting as second-order observers, person A constructs his model of how he thinks person B might interpret him and person B does the same thing in regard to person A. Person B constructs her understanding of what person A said by thinking of what she might have meant were she were to say it. Thus, person B's understanding of what person A said comes from what she knows about person A's meanings, thereby person B's understanding of person A's utterance need not be the same, and likely is not the same, as what person A meant.

The intersubjective actions described in the discussion of Figure 1 would happen between two people attempting to construct second-order models of each other's intentions and meanings. But a conversation could happen with many different variations, depending on the nature of the model-of-other each constructs, which depends on the observer level of each participant and each person's personal meanings regarding the conversation's substance. Even then, the form of the conversation would still be as depicted in *Figure **13***.

2.3. Teacher's Image of Student Thinking

Silverman and Thompson (2008)'s framework explains how a teacher develops schemes that support conceptual teaching of a particular mathematical idea when she has an image of how her students might hear her statements. Silverman and Thompson referred to Key Pedagogical Understanding (KPU) to discuss teachers' image of students' thinking. Thompson (2008) described a six-phase model of teachers' development of a KPU.

Table 32. *KPU Phases (Thompson, 2008)*

Phase	Description
1	Teacher develops an understanding of an idea that the curriculum addresses. Student thinking is not an issue.
2	Oriented to student thinking, but tacitly assumes that information is all that students need, Projecting oneself by default. That is, person A presumes unthinkingly all students are A' (on the road to being A)
3	Teacher becomes aware that students think differently than teacher anticipates they do, but teacher is overwhelmed by seeming cacophony of student thinking (students in her head are B1, B2, B3, …)
4	In dealing with students' (B1, B2, B3, …) contributions: 1. Teacher begins to imagine different "ways of thinking" (epistemic students) 2. These ways of thinking are still grounded largely in teacher's ways of thinking.
5	Teacher begins to imagine how different ways of thinking among students will lead to different interpretations of what she says and does. Begins to develop a mini-theory of actions that might help students think the way teacher intends
6	Teacher adjusts: 1. Her understanding of the mathematical idea as she adjusts her image of ways students think about it. 2. Her understanding of how students might think about the idea as she adjusts her understanding of it

The KPU phases are in terms of how a teacher thinks about student thinking. A teacher in Phase 1 knows that students are human beings, but presumes unthinkingly that all he needs to share is his understanding expressed as he understands it for students to discern what he intends. In this sense, a teacher in Phase 1 constructs a first order model of students.

A teacher moves to Phase 2 when she becomes aware that students think differently than her. However, the teacher might be thinking about how she can explain mathematical ideas to students more clearly. Put another way, the teacher is projecting herself by default. The Phase 2 teacher's model of students is also a first order model in the sense that the teacher thinks the students are just like the teacher, so the teacher is using a model of her own understanding to consider how it might be made clearer. Teachers in Phase 1 or Phase 2 use a first order model of students, but the difference between Phase 1 and Phase 2 is that a teacher in Phase 2 becomes oriented to students' understanding.

Phase 3 is the initial phase of moving beyond using a first order model. A teacher in Phase 3 starts to think about students who might think differently than the teacher. Suppose one of her students wrote something on the board and the teacher realized that what the student wrote is different from what the teacher intended. Then the teacher might try to figure out what the student was thinking from what the student wrote. To do so, the teacher begins to have a second order model. The teacher becomes aware of the fact that student understanding differs from what the teacher intended. Then the teacher starts to think about how students think and what students think. Even if the teacher cannot explicitly state different thinking that students construct, just being aware of students thinking differently than the teacher is the beginning of moving towards creating a second order model. The teacher's orientation to students' thinking occurs in Phase 3 and the teacher is wondering what students are understanding when they hear the teacher, but the Phase 3 teacher does not have an image of students' images regarding what the teacher says and does.

A teacher in Phase 4 begins to imagine different understandings that students construct, but imagines these variations as different forms of her ways of thinking. A teacher in Phase 5 becomes aware that different understandings students are constructing can lead them to different interpretations of what the teacher says and does. A teacher in

Phase 4 or Phase 5 wonders about not only how students are thinking but also students' images of what the teachers said and did. However, a teacher in Phase 5 is aware that students have different ways of thinking than herself, whereas a teacher in Phase 4 is thinking of students' ways of thinking still by adjusting her own ways of thinking.

Suppose a teacher thinks slope is about relative size of changes in two variables, but speaks about slope in terms of triangles and steepness. If the teacher is in Phase 4, she might imagine students' ways of thinking about slope only in terms of triangles or steepness because her ways of speaking are grounded in triangles or steepness. However, this teacher in Phase 5 could recognize students speaking of slope as how much larger a vertical length thing is than a horizontal length. Thus the teacher in Phase 5 is ready to think, for example, "Oh, my way of speaking about triangles and steepness led students to think about comparing lengths additively. I will have them determine slope by measuring vertical change in units of horizontal change" because she thinks the new activity will support students in understanding slope in the way she intended.

Thompson (2013a) theory of conveyance of meaning and the KPU phases are useful to explain the evolution of interactions between teachers and students in classrooms in terms of ways they understand each other. A teacher expresses his meanings to his students by saying or doing something. Then, his students try to understand what the teacher says and does. Whatever meanings his students construct by attempting to understand what the teacher intends are the meanings the teacher conveyed to the students. A conveyed meaning might or might not be the same as the teacher's meaning, and most likely is not. Moreover, the extent to which a teacher envisions how his students might understand his actions and utterances affects his interpretations of students' actions and utterances and thereby affects what he says and does, which then further affects meanings conveyed to students.

3. METHOD

This study focused on how a calculus teacher's (Terri's) image of student thinking influenced Terri's instruction, and meanings she conveyed to students. I observed Terri's lessons consecutively on 15 consecutive school days in order to witness how her image of student thinking affected her adjustments and how her adjustments affected students' understandings. I also interviewed her students before and after lessons. By comparing students' meanings I inferred from pre-lesson interviews to their as I inferred in post-lesson interviews I concluded what they understood of Terri's instruction—Terri's conveyed meanings.

3.1. Participants

Terri had about 20 years of experience teaching mathematics (algebra I: 8 times, algebra II: 2 times, geometry: 12 times, calculus AB: 6 times, calculus BC: 8 times, and elementary mathematics). Terri had a MA in counseling and a BA in secondary education (mathematics major). Amy and Alex were Terri's students. They were in 12^{th} grade and taking AB calculus taught by Terri. Terri had taught Alex when he took Algebra I, but never taught Amy before.

3.2. Procedure

This study presents a subset of the data by focusing on Lessons 1-7 where Terri taught the concept and definition of derivative. On the first day of observation, Terri introduced the definition of derivative. The observations ended when she introduced power rule. Terri sometimes decided to review earlier ideas after watching video clips in students' interviews. The topic for each lesson (Lessons 1-7) is shown in Table **33**.

Table 33. *Terri's Lesson Topic for Lessons 1-7*

Lesson	Topic
1	The definition of derivative: taking derivative gives a formula and substituting any *x* value for the formula you get a number, which is slope on the curve.
2	Use the definition of derivative to find derivative via algebraic procedures
3	Algebraic strategies to find derivative with examples
4	The derivative formula gave slope at a point, then we can find a tangent line at that point.
5	Every tangent line is unique. (By this Terri meant there is a different tangent line at every point on a curve.)
6	The difference quotient in the definition of derivative came from the slope formula.
7	Decreasing at an increasing rate (slopes of tangent lines get steeper) and decreasing at a decreasing rate (slopes of tangent lines get less steep)

I asked Terri to select two middle-performing students who, in her judgment, pay close attention during lessons. Terri selected Amy and Alex (pseudonyms) for interviews. I conducted Pre- and Post-Lesson Interviews with Amy and Alex because I wanted to witness what they understood from Terri's lessons.

Terri told me her lesson goals in each Pre-Lesson Interviews. She also expressed her meanings and her image of student thinking in the Pre- and Post-Lesson Interviews and her lessons. Pre-Lesson Interviews with Terri took 10-15 minutes. Post-Lesson Interviews with Terri were two types: 5 minutes Post-Lesson Interviews right after the lesson and one hour Post-Lesson Interviews every three lessons. I conducted one hour Post-Lesson Interview with Terri the day after students' Post-Lesson Interview because I wanted to see Terri's image of student thinking after watching student video clips.

Amy and Alex also expressed their meanings in Pre- and Post-Lesson Interviews. I conducted a Pre-Lesson Interview prior to every lesson, so the next Pre-Lesson Interview sometimes served as a Post-Lesson Interview for the previous lesson. For example, I was able to see what they understood in lesson 1 during Pre-Lesson Interview 2. I also conducted Post-Lesson Interviews every three lessons. Every Pre-Lesson Interview with students took approximately 5 minutes and each Post-Lesson Interview with students took about 30 minutes. The schedule for interviews with Terri, Amy and Alex is shown in Table **34**.

Table 34. *The Schedule for the First Four Lessons (Repeated for 11 More Lessons)*

Lesson 1	Lesson 2	Lesson 3	Lesson 4
Pre-Lesson 1 with Terri	Pre-Lesson 2 with Terri	Pre-Lesson 3 with Terri	Pre-Lesson 4 with Terri
Pre-Lesson 1 with Amy & Alex	Pre-Lesson 2 with Amy & Alex	Pre-Lesson 3 with Amy & Alex	Pre-Lesson 4 with Amy & Alex
Lesson observation	Lesson observation	Lesson observation	Lesson observation
5 min Post-Lesson with Terri	5min Post-Lesson with Terri	5 min Post-Lesson with Terri	5 min Post-Lesson with Terri
		Post-Lesson 3 with Amy & Alex	
			Post-Lesson 4 with Terri

Before observing every lesson, I conducted a pre-lesson interview with the teacher to investigate what the teacher intended students to learn. Then I conducted the 5 minutes Post-Lesson Interviews right after the lesson and one hour Post-Lesson Interviews every three lessons. The interview questions for a teacher are shown in Table **35**, Table **36**, and Table **37**.

Table 35. *Pre-lesson Interview Questions for Terri*

1. What is the central mathematical concept of this lesson?
2. What would you like your students to understand about the central mathematical concept?
3. Why would you like your students to understand the concept in the ways that you mentioned in the question 2?
4. What will you do to help your students to develop this understanding?
5. What do you think how your students might understand about this idea that is different than what you intend?

Table 36. *5-minutes Interview Questions for Amy and Alex (Right after the Lesson)*

1.	Do you think your lesson went as you expected?
2.	What is your sense of what students understood about the central mathematical concept?

Table 37. *Post-lesson Interview Questions for Terri*

1.	What do you think your students understood from your lessons?
2.	(After viewing video clips from students' interviews.) What do you think these students' understood regarding the meaning of (the content of the clips)?
3.	Is the understanding in the video clips consistent with what you intended? If not, what can be possible reasons for this discrepancy?
4.	If you teach again, is there anything that you want to change? If so, what are these things and why do you do differently?

One pre-lesson interview question was, "Do you think your students might understand the definition of derivative differently than what you intend?" I asked this question to discern how Terri thought about student understanding before the lesson.

After every pre-lesson interview with Terri, I met the two students selected for interviews. The purpose of the pre-lesson interviews for students was to see their understanding of the topic covered in the upcoming lesson. I compared students' meanings demonstrated in pre-lesson interviews to their meanings demonstrated in post-lesson interviews to conclude what they understood from the lesson. After the lesson, I asked a student to describe what he learned from the lesson. The interview questions for post-lesson interview with a student are shown in Table **38**.

Table 38. *Post-lesson Interview Questions for Students*

1.	What do you think was the central idea of this lesson? Why do you think so?
2.	What do you think Terri hoped you would learn from this lesson?
3.	A question about mathematical ideas that would give the student an opportunity to *use* their meanings of the central idea.

I audio recorded pre-lesson interviews with Terri and the two students and post-lesson interviews with Terri. However, I video recorded the lessons and post-lesson interviews with students because I showed lessons to the two students when they did not remember what Terri said. I also shared video clips of students' post-lesson interviews with Terri. The purpose of sharing students' video clips with Terri was to provide her with opportunities to think about students' understandings and to reflect on her teaching and meanings by showing excerpts from the two student interviews that revealed how they understood central ideas of the lesson.

3.3. Analysis

My analysis focuses on interviews with the Terri and two students (Amy and Alex). I discerned Terri's KPU phases from interviews with her, her decisions to adjust lessons after watching students' video clips, and what she actually did in next lessons. Terri expressed what she thought about students' understandings and how she might adjust lessons to help students' understanding. I also observed what she adjusted her lessons after interviews with me. I determined meanings Terri conveyed to Amy and Alex by comparing Amy's and Alex's meanings demonstrated in Pre-Lesson Interviews to their meanings demonstrated in Post-Lesson Interviews. I will describe Terri's image of student thinking (Terri's KPU phases) as well as how her image of student thinking influenced her instructional actions, and meanings that students constructed in the next section.

4. RESULTS

I focused on two key mathematical ideas for analysis: (1) difference quotient in the definition of derivative, (2) slope (or rate of change) between two points versus slope (or rate of change) at one point. I drew upon multiple events to describe the consistency of Terri's meanings as she expressed in her instructions and the conveyance of meanings in the lessons. *Figure **14*** and *Figure **15*** illustrate the overviews of when Terri taught the two key mathematical ideas.

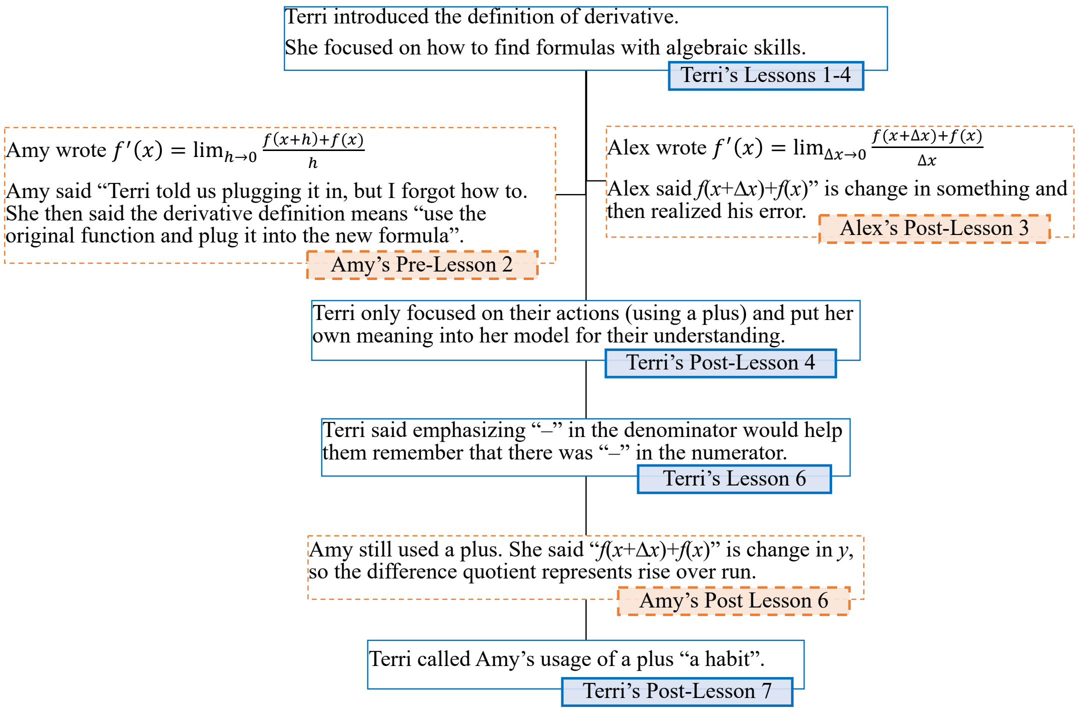


Figure 14. The Overview of Difference Quotient in the Definition of Derivative.

Both Amy and Alex used a plus sign instead of a minus sign in the definition of derivative after experiencing Terri's lessons (see *Figure **14***). When Terri watched students' video clips, she was first surprised because she thought they had strong understanding of slope. She then talked about her decisions to adjust her lessons to help students remember the derivative definition by emphasizing "-" in the denominator. Terri dismissed their use of "+" as a minor mistake or a bad habit.

Amy and Alex had difficulty in understanding slope or rate of change at one point because they thought they needed two points to think about slope or rate of change (see *Figure **15***). After watching their difficulties, Terri focused on what students did not understand when she detected students' understanding was not consistent with what she intended. Then, she planned to adjust her lesson in ways that it was clear to her such as putting in more algebraic steps or showing more visual attributes because she put her own ways of thinking into her model of students' understanding.

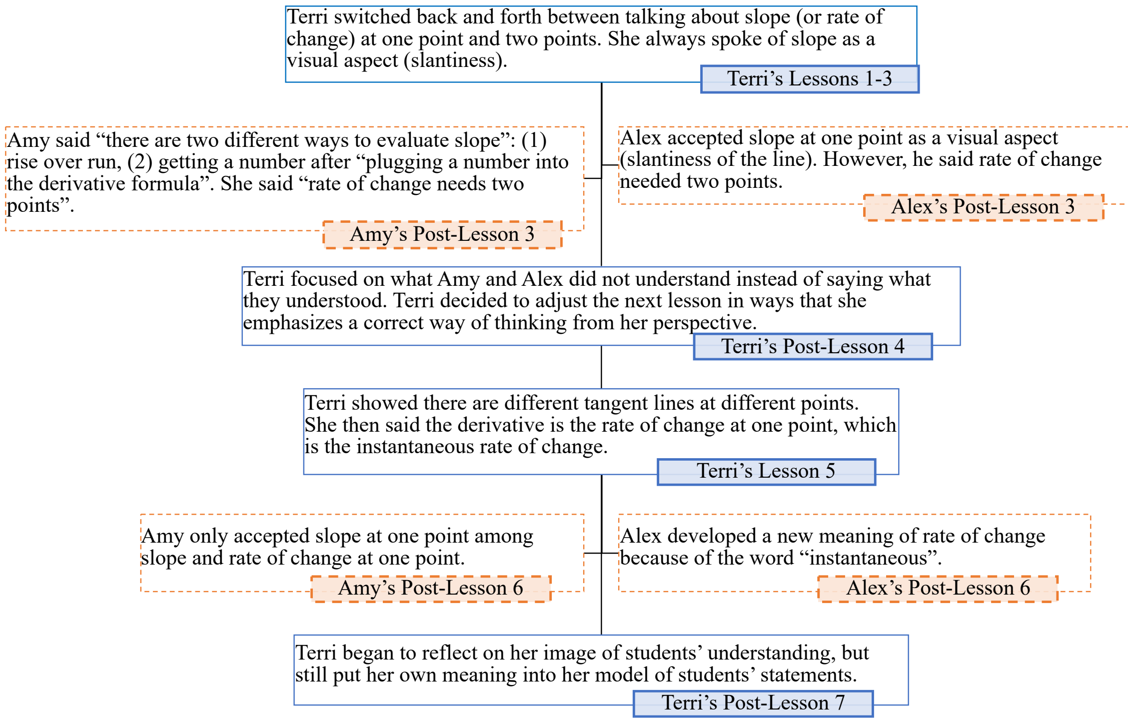


Figure 15. The Overview of Slope (or Rate of Change) Between Two Points Versus Slope (or Rate of Change) at One Point.

4.1. Difference Quotient in the Definition of Derivative (Lessons 1-6)

Overview

On the first day of observation Terri introduced the definition of derivative. The definition of derivative includes the difference quotient, so students' meanings for rate of change or slope influenced how they made sense of Terri's lesson. The Pre-Lesson Interviews with Amy and Alex suggested that their existing schemes of slope and rate of change were different. Amy's meaning for slope did not involve changes in quantities' values, instead being “going up (value of numerator) and over (value of denominator)”. Alex’s meaning for slope was the relationship between the change in *x* and the change in *y*. When Terri taught the definition of derivative, her lessons generally focused on how to find formulas using the definition of the derivative with algebraic procedures. Amy and Alex used their personal schemes for slope in making sense of what Terri said in the lessons, and their different schemes led them to understand Terri’s lessons differently.

4.1.1. Terri’s Lessons (Lessons 1-3)

In the Pre-Lesson 1 Interview, Terri said she would introduce the definition of derivative in lesson 1. Terri explained that her goal for Lessons 1-3 was for students to be able to “find a derivative” which was to use the definition $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to derive a closed form definition for. To help students find derivatives, she said “I will teach some algebra with comfortable functions such as x^2 or x^3 that students are familiar with”. Her stated goal was focused on student actions. She also expressed her meaning for derivative. She told me “the idea of derivative is actually finding the slope as they go around the curve and derivative itself is a function”, which indicates her image of derivative (tracking the slopes of tangent lines as she moves along the graph) (see *Figure **16***).

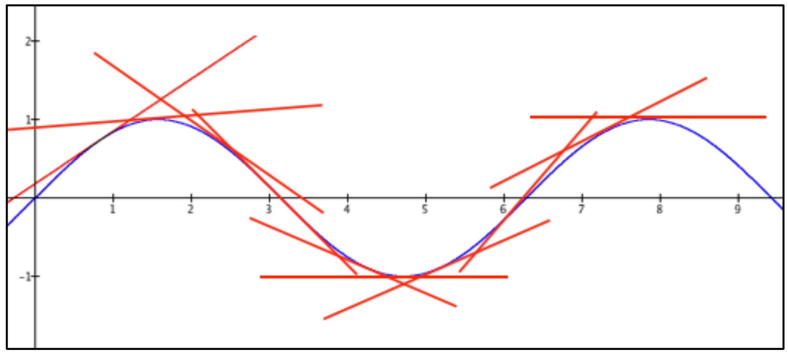


Figure 16. My Image of Terri's Meaning for Derivative

Terri taught Lesson 1 as she planned, illustrating a procedure to find derivatives of familiar functions. In the excerpt below, Terri began by connecting instantaneous velocity to instantaneous rate of change, and said "instantaneous rate of change is called $f'(x)$ and $f'(x)$ has been designated to be the derivative". She then introduced the definition of derivative and used three functions ($f(x)=x^2$, $f(x)=x^3$, and $f(x)=\sqrt{x}$) to show students how to apply the definition of derivative, and then use algebra and rules of limits to express a derivative in closed form. She referred in passing to *h* as a change in *x*, putting greatest emphasis on strategies to eliminate *h* in the denominator. Terri also tried to connect the idea of slope with the difference quotient.

Excerpt 9. Terri's Lesson 1

Terri:

$$\boxed{\lim_{\substack{h \to 0 \\ \uparrow \\ \Delta x}} \frac{f(x+h)-f(x)}{(x+h)-x}}$$

[In previous classes] Your teacher was saying "Hey, can you do *f*(*x*+3)?", "Hey, can you do *f*(*x*-4)?", "Hey, can you do *f*(*x*+*h*)?" You were kind of just plugging in. I know what you do in precalculus or algebra 2. Am I right or wrong? Okay (*nodding*).

This is gonna be new for you. And the algebra involved is just gonna be pretty challenging. But, we need this. I am gonna teach how to do it. Okay. But this formula (*pointing to the image above*) can be simplified to what?

Just one small change. I can say that *f'*(*x*), the derivative *f* with respect to *x* is gonna be the limit as *h* approaches zero. That means that my change in *x* is getting really really really small. Okay? And the only thing that is changed is the denominator. What does the denominator become? Just *h* (*writing the image below*)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let's go back and talk about it (*the algebra*). Mark? Why is making *h* zero be a problem? What if I plugged in the zero would that be a problem? (*Mark said "the denominator is zero.")*

I can't have a zero in the denominator. I can't divide by zero. Okay? So, I am gonna show you strategies. That's what they are. There are algebraic strategies. We are gonna start with a really really easy function… Before I do that just wannna get across that this is the definition of derivative (*circling what she wrote as shown above*).

Without the limit, without the limit it's not. Okay? If you don't have a limit involved that is just a slope between two points. When you say my change in *x* is getting so incredibly small, infinitesimally small that I am getting closer closer closer to the point then I can actually find what is happening instantaneously at a point, which is what especially Isaac Newton was trying to do.

Terri's statements in the excerpt above inform us of four things: (1) $f(x+h)$ is an object where she was supposed to substitute values, (2) Terri talked about the derivative as a formula, (3) Terri's main focus was on the algebraic procedures to eliminate *h* after "plugging in" and obtain a closed form derivative, (4) The only thing that changes is *h* in the denominator, and did not mention that *h* changes in the numerator as well, and (5) Terri rarely mentioned the idea of change, and did not mention change in relation to rate of change.

In Lessons 2 and 3, Terri continued to teach how to find the derivative of functions such as $f(x)=\sqrt{x}$ and $f(x)=\frac{1}{x}$. Then, she asked students to find the derivative of functions such as $f(x)=\dfrac{1}{x-1}$ and $f(x)=\sqrt{x+1}$. Terri put emphasis on the algebraic manipulations needed to get the derivatives.

4.1.2. Amy's and Alex's Understanding of the Difference Quotient in the Definition of Derivative (Pre-Lesson 2 and Post-Lesson 3 Interviews)

I conducted the Pre-Lesson 2 Interview with Amy after Lesson 1. Terri's goal for Lesson 2 was to find derivatives, so I asked Amy to find the derivative of $f(x)=\dfrac{1}{x}$. Amy first said "Terri told us plugging it in yesterday" (Lesson 1). Then, Amy wrote $f'(x)=\lim\limits_{h\to 0}\dfrac{\left(\frac{1}{x}\right)+\frac{1}{x}}{h}$, which provides strong suggestion that "$1/x$" was the "it" in "plug it in". She then said, "I kind of forgot how to". When I asked her for her meaning of $f'(x)=\lim\limits_{h\to 0}\dfrac{\left(\frac{1}{x}\right)+\frac{1}{x}}{h}$ (see *Figure 17*), she said "use the original function and plug it in into the new formula to find the limit of it and once you factor it like take out the elements of it".

What Amy meant by "use the original function and plug it into the new formula" was to substitute $\dfrac{1}{x}$ into the blanks in $\lim\limits_{h\to 0}\dfrac{(\square)+\square}{h}$ and her idea of "plugging in" seemed to mean "writing something in place of". This tells us that her meaning of the original function was just the inscription $\dfrac{1}{x}$ that consisted of 1, "—", and *x*. Amy said $\lim\limits_{h\to 0}\dfrac{(\square)+\square}{h}$ was the new formula, which means she already had a formula that was what she called the original function: the inscription “1 bar *x*”. Amy understood the definition of derivative as a new formula that gave her a new inscription.

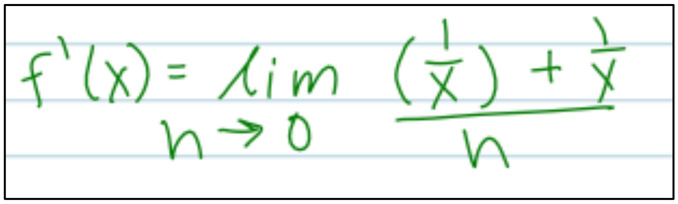


Figure 17. Amy’s Derivative Formula in Pre-Lesson 2 Interview

Amy’s Post-Lesson 3 Interview confirmed that the meanings that she was working with were “functions are formulas” and “the definition of derivative is a new formula that gave her a new inscription through a sequence of actions”.

Alex also used a plus in the numerator of the difference quotient in Pre-Lesson 2 and 3 Interviews (see *Figure **18***), but his original meaning for slope, being about a quotient of changes, led him to catch his error (see *Excerpt **10***).

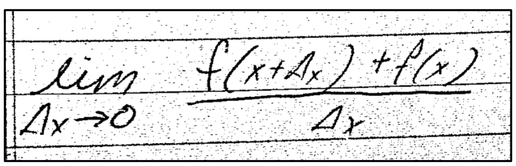


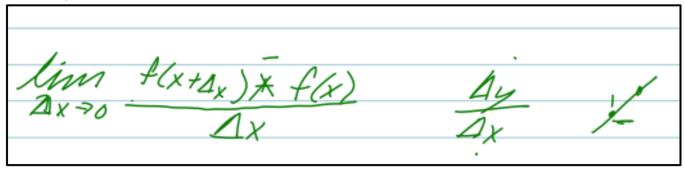
Figure 18. Alex’s Work in Pre-Lesson 2 Interview (Note That He Used “+” in the Numerator.)

In Post-Lesson3 Interview Alex said derivative and slope were the same thing. Then, he mentioned the formal definition of derivative. I asked what the formal definition of derivative is and he talked about his realization.

Excerpt 10. Alex’s Post-Lesson 3 Interview

Interviewer:	What is the formal definition?
Alex:	The derivative one. I realized during the lesson (Lesson 3) that I wrote wrong.
Interviewer:	What did you realize?
Alex:	Before I was saying that it was the limit of Δx approaches

zero as…I was saying um that it's plus. And then that led me into some difficulty during the Pre-Lesson Interview (*referring to the Pre-Lesson 2 Interview*) because you asked me to explain the top part (*circling* $f(x+\Delta x)+f(x)$). And then I realized now that once it's a minus umm… it's just the difference between the values on the points (*writing the image below*).



So it really is a slope.

Interviewer: How did you get that?

Alex: That I had wrong? Cause I was working back I was like I didn't understand what the top part meant really. I was kind of just telling you where I can come up with. It really is still finding even by the formal definition you still finding slope between two points.

Alex's statements in Line 4 and 6 suggest that he originally intended to represent changes when I asked him to explain the meaning of the difference quotient in the definition of derivative in the Pre-Lesson Interviews. He realized that he had not represented what he intended and he wanted to represent changes, as in the Pre-Lesson Interview. The nature of his realization was that he must represent change in the numerator with a difference of functions' values.

4.1.3. Terri's Thinking about Amy's Understanding of the Difference Quotient in the Definition of Derivative (Terri's Post-Lesson 4 Interview)

When Terri and I watched Amy's Post-Lesson 3 Interview and she first heard Amy saying that the central idea of the lesson was how to find derivatives by plugging functions into the derivative function, she said it was what she wanted to convey. Terri was surprised when Amy used a plus in the numerator because to Terri the numerator stands for a change in *y*. After watching Amy's interview more, Terri rethought her earlier comment and said that Amy's focus on a sequence of actions was not what she intended to convey. She said she did not know why Amy only talked about what to do, and that she needed to be more careful when she gives the definition. She was also surprised that Amy used a plus while also using the phrase "difference quotient". At first she said "Amy understands slope" because Amy gave the formula correct, but later she said Amy's understanding of slope was lacking.

Excerpt 11. Post-Lesson 4 Interview with Terri Watching Amy's Post-Lesson 3 Interview

1 I: (*showing Amy's Post-Lesson 3 Interview where she was explaining what she wrote* $\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)+f(x)}{h}}$) What do you think about her understanding?

2 T: It's a minus because it's a difference, change in y. She never got that. I don't know she does now. She maybe does. You probably can ask her the same question again. But, that's misunderstanding. Um writing a plus doesn't make any sense. It's too abstract to them. Because she is actually the one - you weren't there (*referring to an earlier lesson*). She is the one who said what the slope formula was.

3 I: Really? When you taught slope?

4 T: Yeah. $\frac{y_2-y_1}{x_2-x_1}$. She was making that connection. She was the one who said what the slope formula is. So, it's disturbing she has got a plus there. And I would not have any idea but she didn't get that.

5 I: This is not consistent with what you intend?

6 T: No. Not at all.

7 I: Do you think why there are some discrepancies?

8 T: I just don't know to be honest. I can say what I am saying. I think that $f(x+h)$ and $f(x)$ are outside around what she has ever seen before. And I think she has disconnected between…Because I taught average velocity and average rate of change which they have seen since their freshman year. Okay they have seen it since their freshman year. That doesn't mean they truly grasp it and understand it, but they have seen it. And it's always a minus. Change in y…I've written four different ways for them. Change in y over Change in x, Δy over Δx, $\frac{y_2-y_1}{x_2-x_1}$. I don't know why she doesn't have that.

9 I: Alex also wrote a plus in the difference quotient of the definition of derivative.

10 T: Did he really? Surprise.

11 I: I also asked Amy her meaning of *h*.

12 T: (*After watching Amy saying "h means you have to do more work")* That comes from something I talked about limit. If you get zero over zero it usually means you have to more work. You need to either factor or you have to. That's what she talked about. How to do, not what it means.

13 I: I also asked her what you hoped she would understand from the lesson. (*Showing Amy saying "she wants us to learn you have to get a function and put that function into the derivative function to find the derivative*) What do you think about her understanding?

14 T: Well, I think she just thinks I want her to plug stuff in and use the formula. Because as time goes on their understanding gets better.

15 I: Is this understanding in the video clip consistent with what you intended?

16 T: No. I don't know why because they are pretty good about asking questions. I think it's possible that I jumped to the definition too quickly. Um it's exactly how I do in BC, but they don't seem to have any trouble. So I told you that AB kids are not all coming to me from precalc and they are coming to me from four or five different teachers, so I am not blaming their teachers but their experience is different.

Terri's response in line 4 shows that she assumed that because Amy could recite the slope formula when asked, Amy made a connection between the slope formula and the difference quotient in the derivative definition. She also did not consider the possibility that Amy's meaning for slope was just a formula that did not refer to changes. In both instances, she assumed that Amy's meanings matched her own.

Terri decided that she would adjust a future lesson to emphasize that the numerator has a minus by writing *Figure **19***. She thought talking about the steps shown in *Figure **19*** would help her students to remember that there was a minus in the numerator by making a connection to the slope formula. She said using Δx instead of *h* might help Amy connect the difference quotient to the slope formula. She also thought emphasizing the minus in the denominator would resolve Amy's problem of putting a plus in the numerator.

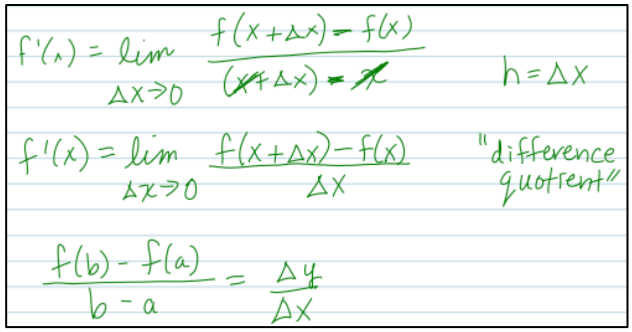


Figure 19. Terri's Adjustment for the next Lesson to Put More Emphasis on "—" Operation in the Difference Quotient

Terri only focused on Amy's action (using a plus) and put her own meaning for the difference quotient and for changes into her model for Amy's understanding. Her adjustments focused on ways to help students recall the presence of minus signs in the difference quotient, but lacked plans to convey that the numerator and the denominator of the difference quotient each represent changes, or the meaning of changes themselves. Her plans did not indicate any reflection on what she might have done or not done that led Amy to using a plus in the difference quotient of the definition of derivative.

4.1.4. Terri's Adjustments (Lesson 6)

Terri adjusted her instructional actions as she planned. What she presented in Lesson 6 after watching students' video clips where the two students wrote $f'(x) = \lim_{h \to 0} \frac{f(x+h) + f(x)}{h}$ (Amy) or $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + f(x)}{\Delta x}$ (Alex) focused on what to do and how to do it with the definition of derivative. Terri's instructional actions in the excerpt below show that her goal was to help students remember how to use the formula, and her comment "that might help you to remember the formula" shows that her purpose for the connection to the slope formula was to help students memorize the definition of derivative correctly.

Excerpt 12. Terri's Adjustments in Lesson 6

Terri: Just a quick reminder (*writing the image below on the board)*

$$\boxed{\frac{f(x+h)-f(x)}{(x+h)-x}}$$

This is what it looks like before you simplify.
Remember isn't that just a version of this, isn't it? Isn't it? Yes?
If you kind of lost that in the shuffle. It's possible to have lost that in the shuffle. Um what we did is what? Hack and slashed (*See the image below*)

$$\boxed{\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(b)-f(a)}{b-a}}$$

Yes? Through the limit front make my h approaching zero. It is no longer the slope between two points (*erasing* $\frac{f(b)-f(a)}{b-a}$ *and clapping*), but it's using the concept of slope.
But the points are what? Infinitesimally close to each other as the h is approaching zero. That formula…**That might help you remember how to DO the formula cause you can write actually it that way** (*pointing to the image below).*

$$\boxed{\lim_{h \to 0} \frac{f(x+h)-f(x)}{(x+h)-x}}$$

But my h is that little teeny tiny change in x. This is new and this is calculus. But when you do $\frac{f(b)-f(a)}{b-a}$ you are doing pre-calculus and you're doing average rate of change, which leaves me to do the last part. The last part is most expansive.
What we've been doing lately? (*asked students & got responses*) Yes, we found slope at a point then we went though step 1, 2, 3 the equation of the tangent line.
What is the other meaning of average rate of change? I heard it. Slope! (*Writing slope on the top of Average ROC: see the image below*)

pages

Slope

Part VII Average ROC

Had Terri wanted Amy to think about the difference quotient as changes she would have emphasized the meaning of the different parts of the difference quotient in Lesson 6. There are two possible reasons why Terri's adjustment was about actions again. The first hypothesis is that Terri thought Amy knew the numerator represents change in y, but Amy accidentally used a plus, because Terri put her own meaning into her model of Amy's action. The second hypothesis is that Terri only focused on Amy's action and wanted Amy to remember to use a minus instead of wanting Amy to have a meaning for the difference quotient. Regardless, Terri's behavior in Lesson 6 confirms that she did not identify the difference between a student's ability to use a formula, and the student's meaning for that formula.

4.1.5. Amy's Understanding of Terri's Adjustments (Post-Lesson 6 Interview)

I conducted the Post-Lesson 6 Interview with Amy after she experienced Terri's adjustments during Lesson 6. Amy's understanding of Terri's adjustment in Lesson 6 was that the derivative formula is connected with the slope formula, so the difference quotient should be rise over run. Then she tried to understand why Terri said the numerator was change in y. She ended up with the conclusion that the notation $f(x)$ represents y, so the numerator is about y. Amy's meaning for slope (rise over run) led her to understand her self-produced expression $\frac{f(x+\Delta x)+f(x)}{\Delta x}$ as rise over run because rise over run was y values over x values and, to Amy, $f(x+\Delta x)+f(x)$ represented a y-value. As a result, Amy concluded $f(x+\Delta x)+f(x)$ was change in y, $h+h$ was a change in x, and $\frac{f(x+\Delta x)+f(x)}{h+h}$ was change in y over change in x (rise over run).

Amy's meaning for the definition of derivative was useful for her to get another formula by substituting a function (an inscription) into the derivative formula. This is why Amy called the definition of derivative the derivative formula, which was consistent with what Terri said in lessons. In Lesson 5, Terri actually said learning the derivative was about learning how to determine formulas for derivatives. Terri also said "when you have the formula $g'(x)$, it's a derivative formula." It seems that Amy paid attention to Terri's instruction and tried to remember what Terri said. Terri also explained lessons regarding the concept of derivative in terms of actions such as "plugging in" or "getting rid of *h*", and Amy constructed her meanings for derivative that were about actions to perform on inscriptions. It seems Amy recalled static images of what Terri wrote and put them together as best she could, but with little underlying meaning to guide her. However, at no point did Terri use a meaning for change that was anything like "by how much final exceeds initial". Because of this, Amy continued to think of changes in the numerator as being represented by adding the final and initial values. Indeed, she then tried to mirror this sum in the numerator. Additionally, she used both Δx and h, which suggests that these might not represent the same quantity to her. Amy's final expression for the different quotient was $\frac{f(x+\Delta x)+f(x)}{h+h}$, and she was satisfied with it because she felt she had incorporated what Terri taught in Lesson 6.

4.1.6. Terri's Thinking about Amy's Understanding of Terri's Adjustments (Post-Lesson 7 Interview)

Terri watched Amy's Post Lesson 6 interview, in which Amy continued to use a plus in the numerator of the difference quotient, saying that $f(x+\Delta x)+f(x)$ was a change, even after Terri's adjustments in Lesson 6. Terri said she did not know why Amy kept using a plus, but she thought Amy did understand it was a change because she said "change" and she solved problems correctly on her quiz. Terri finally wanted to call Amy's usage of a plus "a habit" that is hard to undo, but she believed Amy would write the difference quotient correctly in the future.

Terri focused on Amy's actions, such as using a plus where Terri wanted her to use a minus. Terri did not attend to the source of Amy's difficulty nor what Amy might have understood about the difference quotient or the idea of change. Terri noticed that Amy mentioned "change", but Terri did not think about Amy's meanings for change. For example, it was not significant to Terri that Amy thought a change from one value to another did not have to involve the difference between them. Moreover, Terri did not consider Amy's way of thinking as the *cause* of her difficulty. This interview suggests Terri put her own meaning into her model of Amy's statement, so she thought Amy understood that the numerator represents change in y as well as what a change is, but she accidentally used a plus.

4.2. Slope (or Rate of Change) Between Two Points Versus Slope (or Rate of Change) at One Point (Lessons 1-7)

Overview

Amy and Alex both needed two points to think about slope and rate of change. They therefore had difficulties understanding slope and rate of change at one point. Before Terri's lessons, both Amy and Alex had "rise over run" as their meaning of slope between two points. However, their meanings for " rise over run" differed. Amy's meaning was going up and over because she drew first the vertical line and the horizontal line (see *Figure **20***). Alex said "slope is how much the y value has risen over how much x

value has changed" while pointing to the two changes. Alex's meaning for slope was the relationship between changes in *x* and associated changes in *y* while Amy's meaning for slope was more like the actions of plotting a point.

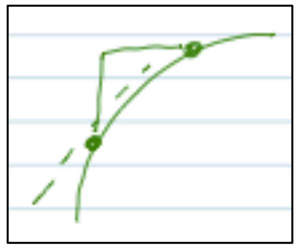


Figure 20. Amy's Meaning for Slope: Going up and Over

When asked about rate of change, both Amy and Alex said they needed two points to think about rate of change because they needed changes. However, their meanings for changes were also different. Amy meant connecting two different points with an up-and-over motion, whereas Alex thought about a comparison of two quantities (viz., changes in *x* and *y*).

Amy and Alex both tried to make sense of Terri's lessons as they experienced Terri's lessons. After watching student interviews, Terri decided to adjust the next lesson but her adjustments were not relevant to Alex and Amy's difficulties. In all lessons, Terri switched back and forth between talking about slope (or rate of change) at one point and two points. It seems that Terri envisioned two points getting closer and closer together until they collapsed into one point.

4.2.1. Interviews with Terri and Terri's Lessons (Pre-Lesson 1 Interview and Lesson 1)

Terri's image of derivative (tracking slopes of tangent lines as she imagined moving along the graph) and her primary meaning for slope (visual slantiness of lines and index of slantiness) prior to Lesson 1 led her to explain that finding a derivative is finding the slope of a tangent line, which is slope at one point. In the Pre-Lesson 1 interview, she said that she wanted her students to understand (1) the idea of derivative is to find the slope of a tangent line, which meant finding slope at one point by finding the equation of tangent lines, (2) the slope of a tangent line is the value of the derivative at one value of *x*. In Lesson 1 Terri said two points are infinitesimally close to each other and then said that this defined the idea of slope at one point.

Terri's instructional actions in Lesson 1 suggested that the ideas of derivative and instantaneous rate of change only make sense with two points, not one point, but she also repeatedly said that bringing two points together changed the scenario to involve only one point. Terri explained the idea of slope at one point by referring to "the limit" because she said the limit allowed them to talk about slope at a point. She expressed two thoughts about slope or rate of change at one point during interviews with me: 1) limiting process—*h* approaching zero, 2) slope of secant line approaching slope of tangent line. Terri mentioned "rate of change at a given value" in Lessons 1-3, but she did not attempt to convey a meaning for rate of change that works with only one point.

Terri used "slope" and "rate of change" interchangeably in her lessons. She always spoke of slope as a visual aspect of a line. I hypothesize that Terri always mentioned slope because people can see slopes when they have graphs, but rate of change (being a relative size of changes) cannot be perceived. After she introduced a line that is tangent to a graph at one point, she only referred to visual "slantiness" to give meaning to the derivative. It is noteworthy that none of Terri's examples showed a "tangent" at a point of inflection, which students often insist is a case where there is no tangent line to the function's graph because any line passing through this point necessarily crosses the graph.

4.2.2. Amy's and Alex's Understandings of Slope and Rate of Change at One Point (Post-Lesson 3 Interview)

At the beginning of Post-Lesson 3 Interview Amy said slope and rate of change were different from derivative. Slope and rate of change needed two points (going up and over) whereas she needed only one point to talk about derivative. Although Terri said derivative was slope at a point, Amy maintained that slope always involves two points. However, after she substituted a value for x into the derivative definition, she was perturbed by my request for the meaning of her result and determined that the number represented the slope at that point because of the letter m in $m_{\tan}$. Amy connected derivative with slope at a point because she associated the letter "m" with "slope". She learned from Terri's lesson that after substituting a value she got something that was labeled $m_{\tan}$ because that was what Terri wrote on the board, and Terri used the same word "slope" and the letter "m" together during the lessons. Amy resolved her conflict between meanings of "slope with two points" and "slope at one point" by associating them both with the letter "m", but did not use the phrase "rate of change" when she dealt with one point.

During Post-Lesson 3 Interview Amy expressed her confusion by saying there were two different ways to evaluate slope. The two different ways were (1) rise over run (slope of a secant line) and (2) getting a number after "plugging a number into the derivative formula". The first meaning for slope, rise over run, is Amy's original meaning, and she added a new meaning for slope at one point because she was not able to use her old meaning for slope to make sense of “slope at one point”. She developed a new meaning for slope at one point that was the numerical result after she went through the algebraic procedure. Amy accommodated her meaning for slope by building an additional, separate meaning for slope at a point. She chose to use only slope for one point even after watching Terri’s video clip where Terri talked about rate of change at one point because slope was a visual aspect from a line and its slantiness. Amy’s accommodation for rate of change at one point was to give up trying to use rate of change when only one point was involved.

Alex understood slope as the relationship between changes in x and y prior to Terri’s lessons. In Post-Lesson 3 Interview Alex said he had never thought about slope at a point and developed the idea from Terri’s lessons. He explained slope at one point by moving two hands closer to show two points were getting closer and closer, becoming extremely close. Alex’s understanding of slope at one point was the value of a derivative, or a limit of a quotient as h approaches zero. He used the definition of derivative to understand slope at one point via the concept of limit. Alex was making a connection between slope of two points and slope at one point by thinking about Δx (or an interval) getting smaller and two points getting closer. However, he expressed his confusion when talking about the connection between slope and rate of change at one point. It seemed that he could imagine slope of tangent line at a point visually, but he could not think of rate of change at one point. Recall that Terri explained derivative as slope of tangent line at a point, and she mentioned rate of change at one point without elaborating the meaning for it. Alex was confused because while he could envision slope at a point and derivative being the same, and relate them to slope between two points, he could not reconcile rate of change at one point and slope at one point.

Alex finally accepted slope at one point as a visual aspect of a line—the slantiness of the line. His meaning of slope was changed when he talked about slope at one point. He thought slope was rise over run, which means a comparison of two changes, but his new image of slope at one point was not compatible with what he was thinking about slope between two points. Slope at a point no longer consisted of changes. As Amy did, Alex added another, alternative meaning for slope so that there were two kinds of slope—between two points and at one point. His new meaning for slope of a point was slantiness of tangent line at a point by applying a visual meaning for limit. Alex also made an accommodation to add a new meaning for slope at one point and had two different meanings for slope in different contexts.

4.2.3. Terri's Thinking about Amy's and Alex's Understandings of Slope and Rate of Change at One Point (Post-Lesson 4 Interview)

Terri and I watched video clips of Amy and Alex sharing their understandings of slope at one point. She told me the two students' understandings were very different but both of them seemed to have trouble understanding slope or rate of change at one point. She first watched Amy's video clip. In the excerpt below Terri kept saying Amy did not understand slope at one point when I asked her about Amy's understanding. Amy was saying the number that she got after substituting values for x in the definition of derivative was labeled m and realizing m of m_{tan} represents slope at one point. Terri only talked about what Amy did not understand instead of what Amy did understand.

Excerpt 13. Post-Lesson 4 Interview with Terri

1 T: (*While watching Amy's video clip where Amy was confused between slope of two points and the value of the derivative at one point*) She has a misunderstanding.

2 I: What do you think about her understanding?

3 T: She is still stuck on slope means between two points. She is saying. Do you understand what she is saying?

4 I: No. Please let me know.

5 T: Okay. Well she is saying on this curve, right? (*drawing the image below*)

She is saying this. The slope between these two points ((0,0) and (1,1)) is 1. She doesn't understand that the slope at that point is 3. She has not made the jump from average rate of change to instantaneous rate of change

6 I: But she said the slope at the point.

7 T: I know. But she is not still understanding. She is confused between the slope here (*slope between (0,0) and (1,1)*) and the slope there (*slope at (1,1)*).

8 I: Yes, she said there are two things.

9 T: Uh huh. She has still not understanding that slope is actually at a point not between two points.

10 I: Do you think to her she needs two points to get a slope?

11 T: In her head. Yeah. And here is what I feel is in her head. She has not made the jump from average rate of change to instantaneous rate of change. This is the jump when I tell them, when I teach them. This is the jump between what they knew before and what they establish in calculus. Because this is foundational for calculus, but she hasn't made the jump.

12 I: But she said the slope at the point.

13 T: Yes, she finally said that.

14 I: What about her understanding of slope, derivative and rate of change?

15 T: I still think it's lacking.

16 I: She said derivative is slope, but she needs two points to get a slope.

17 T: Right. But I think she hasn't made this jump. I don't know. I'm trying to think about how I can do that. Um.

18 I: Do you remember? She did today find the equation of the tangent line.

19 T: Yeah. And hopefully she is the person who did it. But then I think my strategy for showing with the calculator that actually that tangent line is unique should have enhanced her understanding that it's at that

> point and only that point that I am talking about. I'm not talking about between two points.

Terri began to talk about how she might adjust her lesson to help Amy understand slope at one point. She said she would teach how to find the equations of tangent lines at two different points and show that there are two different slopes of tangent lines at different points x=-1 and x=2 by using the graphing calculator. Terri said this adjustment might help students' understanding of slope at one point because they can see at every point there is a slope and at every point there is also a tangent line.

Terri focused on neither the source of Amy's difficulty with the idea of slope at one point, nor what Amy meaning for slope at one point was. Rather, she mentioned what Amy did not understand compared to Terri's own way of thinking. Terri acknowledged that Amy was thinking about slope as between two points, but Terri did not think of Amy's meaning as the *cause* of her difficulty. Moreover, Terri did not see any significance to Amy's meaning for slope as "going up and over" as an inadequate foundation for thinking of slope and rate of change as *being the same thing*. Her decisions to adjust her lessons were also unrelated to what Amy understood, so her adjustment did not address the sources of Amy's difficulties.

Terri continued to watch Amy's video clip where she said rate of change needed two points. The excerpt below shows that Terri thought understanding of the numerator might be the source of Amy's problem that rate of change has to have two points. Terri thought Amy interpreted the numerator in the difference quotient as dealing with two points because it has $x+h$ and x, suggesting that to Terri the numerator does not have two points after the limit is taken. Terri seemed to keep focusing on Amy's understanding of the difference quotient because she just watched Amy's video clip where Amy used "+" in the numerator of the difference quotient. It seemed that Terri tried to connect Amy's confusion about rate of change at one point to Amy's inability to write the difference quotient. However, Amy did not express confusion about the difference quotient when she talked about rate of change at one point.

Excerpt 14. Terri Watched Amy's Video Clip in Post-Lesson 4 Interview

1	T:	(*While watching Amy's video clip where she said rate of change at one point does not make sense to her because there is no change at one point*) She didn't get what the numerator means (*By the numerator Terri meant the numerator of the difference quotient of the definition of derivative*).
2	I:	(*After watching Amy's video clip where she said rate of change or slope needs two points*) What do you think about her understanding?
3	T:	She hasn't, she hasn't grasped the h approaching zero.
4	I:	Why do you think so?
5	T:	I don't know. I just think that she is really kind of stuck in her preconception. It's her preconception that rate of change has to have two points. She is kind of stuck there. Because that's all she has done prior to this year. And the big leap in calculus is to know I wanna find rate of change at one point. I think that conceptually is very much more abstract than something that they are very comfortable with and I probably have not made it more clear. I think I have not made it clear at least for her. I don't know if I made it clear but not for her because we did exactly what I showed you and we talked about well here is the secant how can I get a better result? If I make my change in x very small and I get as close as I want to that point. It's much more abstract concept and she still does not get that. Maybe she will through it this week. I don't know.
6	I:	If you teach this again that part?
7	T:	I will spend more time.
8	I:	Is there anything that you want to change?
9	T:	Yes, I think in my textbook in my old notes I actually did not do this. I have um making the change .1 making the change .01 making the change .001. I actually have some problems like that I could've done with them. And I did not. I chose not to. Well like take a function any function that you could do with. And do that difference quotient with an actual h being .1, .01, .001 and I do see the advantage of that. In BC I didn't do that cause I have so much

> curriculum it's so hard, but for AB kids probably I should've done that. I think it would help her. I don't know everyone needs that, but I think it would help her.

Terri mentioned the possibility of making the value of h have small values in her examples. It is noteworthy, however, that Terri never acknowledged "$h \rightarrow 0$" is usually accompanied by "$h \neq 0$", meaning that the difference quotient in the limiting process is *always* computed over an interval of non-zero length. Put another way, the difference quotient always computes the slope of a secant, even when the secant's interval has infinitesimal length. By focusing on limit instead of convergence, Terri's meaning for derivative *excluded* changes over intervals.

In the excerpt above, Terri developed a negative hypothesis of Amy's understanding of rate of change at one point based on what Amy did not understand. She thought Amy did not grasp h approaching zero, so making h smaller would help Amy understand rate of change at one point. It seems it did occur to Terri that Amy could not think of "infinitesimally small" in a way that two points are "essentially" one, but Terri seemed to focus on "getter smaller" rather than on "infinitesimally small" (Line 9). Terri's hypothesis of Amy's difficulty was built upon "she is not thinking correctly". The genesis of Terri's decision to help students' difficulties was based again on what they did not understand.

Terri also watched Alex's video clip where he talked about the idea of slope at one point. When I asked Terri in the excerpt below about Alex's understanding of rate of change at a point, she said repeatedly he misunderstood instead of saying what he understood. In the video clip, Alex used the idea of the limit and made Δx (or an interval) get smaller to think about slope at one point. Finally, Alex ended saying slope at one point was the slope of tangent line, which was a visual attribute.

Terri then talked about her planned lesson adjustment to help Alex's struggle. She said having students find different tangent lines at different points would help their difficulties because she thought showing there is a unique tangent line at every point would help them imagine a slope anywhere on a curve. However, Terri did not say anything about Alex's understanding of slope at a point. Terri said she knew Alex was struggling, but she had no idea why Alex did not understand the idea of slope at one point. Terri's statement "they have been trained to do slope between two points for so long" indicates that she saw students' difficulty as a matter of procedural habit. She did not see it as *part of their fundamental meaning of slope*. Therefore, Terri's adjustment for Alex also did not address the origin of his struggle.

Excerpt 15. Terri Watched Alex's Video Clip in Post-Lesson 4 Interview

1 I: (*After watching Alex's video clip where Alex talked about slope between two points and slope at one point and saying rate of change at one point did not make sense because he needed to compares two points to take about rate of change)* what do you think about his understanding?

2 T: I think that he is still even though he is saying the words "infinitesimally close" he is still stuck with slope between two points. Not to the extent that she (Amy) is, but he is still kind of stuck.

3 I: What do you think about his understanding of slope? He literally said slope at a point.

4 T: I know he did. But then he said I can't do this with rate of change at a point. So he is having trouble. I think maybe it's the definitions and the things he learned before. Again you make an assumption that they have done average rate of change between two points. He gets I think intellectually he gets the idea you're getting closer and closer to the point, but he is still misunderstanding that the derivative is an instantaneous rate of change. And that to me okay. That is a higher level of thinking or understanding. He is also struggling with. Maybe not to the extent she (referring to Amy) is.

5 I: What do you think he is struggling with?
6 T: I don't, I don't know. I don't know.
7 I: You concluded he has misunderstanding of slope because…
8 T: Oh well I think he has had slope like he said he learned slope for so many years at a certain context only with linear things. So when you get to calculus you make a leap. It's a leap. It's making sense to me now, but kids they have a certain level of understanding, but they are just not quite there yet. They are just not. I don't know the reason for that other than they have been trained to do slope between two points for so long.

The genesis of Terri's decision to adjust her lessons after watching Amy's and Alex's video clips was not based on their understandings of slope or rate of change at one point. Amy's understanding of slope at one point was metonymic—identifying a label *m* from $m_{\tan}$ as representing slope – and Alex had already developed a new meaning for slope at one point as slantiness of the tangent line. Terri's plan, which was showing that there are different tangent lines at different points, was relevant to neither Amy's understanding nor Alex's understanding. Rather, Terri's adjustment was to emphasize a correct way of thinking from her perspective regardless of students' understanding exemplified in the video clips.

4.2.4. Terri's Adjustments in Lesson 5

After watching Amy and Alex's Post-Lesson 4 video clips, Terri said she would clarify slope and rate of change at a point, not between two points, in Lesson 5. She started to say to the class she wanted to clarify something and asked a student to explain what he wrote on the board (he had written the equation of tangent line of $f(x) = x^3$ at x=-3 correctly). She said she was finding a slope at a specific point on the curve and said "we are finding the unique tangent line, the only line that touches the curve at that point" and used the graphing calculator to show the tangent line.

Terri continued to say “at that point, the tangent line mimics the curve” and said she did not make this clear yesterday (Lesson 4). She said “the tangent line really gives an accurate value near the point (x=2) but I did not talk about yesterday, so I want to clarify this today” while pointing to $g(x)=\frac{1}{x}$ and the table in the graphing calculator. Then, she explained the two tangent lines to $g(x)=\frac{1}{x}$ at (2, $\frac{1}{2}$) and (-2, $-\frac{1}{2}$) were not the same line although they have the same slope of $-\frac{1}{4}$ (see *Figure 21*).

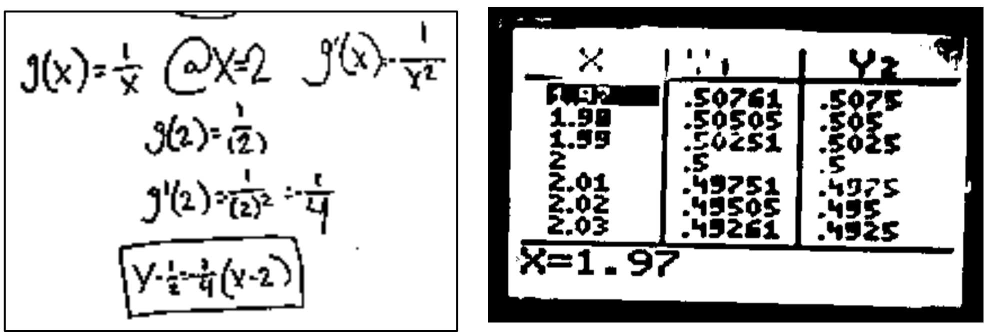


Figure 21. Terri Found the Equation of Tangent Line at 2 and Showed the Slope of the Line at 2 in Lesson 5

After showing different tangent lines at different points, Terri emphasized slope (or rate of change) at one point, not between two points in Lesson 5 as she planned. In Lesson 5 she said,

> So, the derivative is the rate of change of my original function. But it’s the instantaneous rate of change. Okay. I’m not sure everyone in this class still gets we are not doing slope between two points anymore aren’t we. Not really. We are doing the slope at a point by using the concept of the limit. Okay. I am finding the rate of change of a quadratic. And it’ll always be linear. Okay? (Terri, Lesson 5)

Terri used slope and rate of change interchangeably and talked about slope or rate of change at one point by using the concept of the limit. Although she mentioned the concept of the limit she had not explained how two points became one. It seems Terri understood Amy's and Alex's concern about slope involving two points as being about tangent lines at two points.

4.2.5. Amy's and Alex's Understandings of Terri's Adjustment in Post-Lesson 6 Interview

I played the video clip from Terri's Lesson 5 in which she emphasized slope or rate of change at one point during Amy's Post-Lesson 6 Interview. Amy said,

> Basically she said that instead of like you have to change your meaning of slope. So like before the meaning of slope is like you have two points. It is rise over run. And then, you have to change from that to like finding the slope like at one point. It's not two points. It's one point. And you have to if you have a quadratic you have to have a linear line with that to find where they intersect at a point. (Amy, Post-Lesson 6 Interview)

Amy said she adjusted her meaning of slope after listening to what Terri said by creating a new meaning for slope at one point, which was when she substituted a number into the derivative formula. Additionally, Amy only said "slope at a point" although Terri used "slope", "tangent line", and "rate of change" interchangeably in regard to any of them at one point. Amy tried to understand what slope or rate of change at one point might mean and decided upon the visual attribute of slantiness of a line. Slantiness of a line was a new meaning of slope for Amy because, slope at a point was still slope even though it did not mean going up and over.

It seems Amy only accepted slope at one point among slope and rate of change at a point because she could imagine slope of tangent line at a point visually, because she kept saying rate of change needed two points whereas slope at one point made sense to her. It is plausible that she accommodated her difficulties with rate of change at one point by simply giving up the phrase “rate of change at one point” because Terri did not explain what she meant by slope or rate of change at one point. Rather, Terri just used the terms slope and rate of change at one point interchangeably.

Alex also thought rate of change needed two points because it involved changes. Thus, he only accepted the idea of slope at one point, but not rate of change at one point. I showed Alex Terri’s Lesson 5 video clip, in which Terri said

> … the derivative is the rate of change of my original function, but it’s the instantaneous rate of change. Okay. I am not sure everyone in this class understands we are not doing slope between two points anymore aren’t we. We are doing slope at one point using the concept of the limit (Terri, in Lesson 5).

Alex expressed surprise by saying “she *did* say ‘instantaneous rate of change’.” Alex originally thought rate of change was different from slope because slope at one point was imaginable whereas rate of change at one point was not. After I had him re-watch the lesson clip where Terri said rate of change at one point was called instantaneous rate of change, he seemed to develop a new meaning for rate of change at one point because of the word “instantaneous”. He developed a new idea that slope at a point and rate of change at a point are different names, but represent the same thing. He ended up connecting slope at a point and rate of change at a point by the idea of limit “approaching”.

Alex had previously connected slope between two points and rate of change between two points, but had not been able to find a way to connect slope at one point and rate of change at one point. To connect them, he added a new meaning of rate of change so that one could have *instantaneous* rate of change at one point that was slope of a line at that point. His revised meaning for rate of change had two sub-meanings—

instantaneous rate of change at one point (slope of tangent line) and rate of change between two points (average rate of change). Depending on the number of points involved, sometimes rate of change is about quantities changing and sometimes it is about a visual attribute. The word "instantaneous" allowed him to think that instantaneous rate of change and slope at a point are the same.

4.2.6. Terri's Thinking about Amy's and Alex's Understandings of Rate of Change at One Point in Post-Lesson 7 Interview

After watching Amy's Post-Lesson 6 Interview where Amy expressed her understanding of Terri's adjustment in Lesson 5, Terri started to differentiate getting a correct answer from conceptual understanding.

> She (Amy) focused on only how. And I don't think she has conceptual understanding. She can solve problems on the quiz correctly, but that always doesn't mean she understands what she is finding, which is too bad. You are getting at their understanding. It is very hard sometimes to write questions on quizzes that really at their understanding. They need to know how to this but they … it doesn't always mean they understand why they are doing it and what it's gonna do from. (Terri, Post-Lesson 7 Interview)

Terri had thought Amy understood what she (Terri) said because Amy solved problems on the quiz correctly. After watching Amy's video clips Terri began to reflect on her image of Amy's understanding. She began to notice the difference between a student's ability to solve a problem, and the student's underlying understanding of the mathematical ideas. Terri also started to reflect on her assessment questions and how they did not necessarily reveal her student's understandings in the ways she had intended.

Terri also watched Alex's video clip where he tried to make sense of rate of change at one point. In the excerpt below, Terri focused on what Alex did not understand. Terri expressed her difficulty to think about what students did not grasp because the idea of rate of change at one point was really obvious to her (Line 6). It seems that she did not think about the possibility that students can struggle to think about rate of change at one point because she had not attempted to adopt a way of thinking that would be consistent with what the students expressed. Thus, what Terri could think about as the source of students' difficulty was "I did not make it clear" or "I jumped too quickly" (Line 12), which was not about students' understanding.

Excerpt 16. Terri Watched Alex's Video Clip in Post-Lesson 7 Interview

1 T: (*Watching the video clip where Alex was confused because he thought slope and rate of change were the same, but he could not think rate of change at one point whereas slope at one point is acceptable and I asked him "are slope and rate of change the same?")* Yes, they are. He didn't get the difference between average rate of change and instantaneous rate of change.

2 I: (*After watching Alex's clip*) what do you think about his understanding?

3 T: I think that he is still even though he is saying the words "infinitesimally close" he is still stuck with slope between two points. Not to the extent that she (Amy) is, but he is still kind of stuck.

4 I: What do you think about his understanding of slope? He literally said slope at a point.

5 T: I know he did. But then he said I can't do this with rate of change at a point. So he is having trouble… I think maybe it's the definitions and the things he learned before. Could you? Again, you make an assumption that they have done average rate of change between two points. He gets I think intellectually he gets the idea you're getting closer and closer to the point, but he is still misunderstanding that the derivative is an instantaneous rate of change. And that to me … okay. That is a higher level of thinking or understanding he is also struggling with. Maybe not to the extent she (Amy) is.

6 T: The idea of having a rate of change at a point seems really obvious to me now, but it doesn't seem obvious to them.
Because rate of change they are assuming there are two it's from a point to a point.

7 I: Why do you think so?

8 T: I don't know. Because they say rate of change they are assuming there are two. It's from a point to a point.

9 I: He said slope is rate of change, slope is derivative, slope at a

point, but he never said rate of change at a point.

10 T: No, he has not connected them, those words were all to be the same. So I think it's some part of this is the vocabulary and some part of this is they are so conditioned. Average rate of change… So conditioned. We say slope way more.

11 I: What do you think about his understanding of rate of change?

12 T: I think he has a limited, a limited view of rate of change. Because here is a thing. If I talk to him tomorrow, if you talk to him about physics he will understand at a certain moment in time there is a velocity of this position. He will understand that. But he might not understand what he is doing is rate of change. But see when you think about it he would understand it's a rate. That's like a 50 miles per hour. He will understand that. He might not understand that the 50 miles per an hour is a rate of change at that point. I'm trying to think about why he is confused. Because he put it in the context of something moving I think he would he would get that. And again,
I am feeling like I didn't make a clear enough. I jumped too quickly between average rate of change to instantaneous rate of change.

It is noteworthy that in (6-8), Terri acknowledges students' apparent need to think about changes when speaking of rate of change. But this point is not significant to her. Instead of dwelling on why they might think this way, she expresses dismay and instead speaks about correct ways to understand slope at a point and instantaneous rate of change.

Terri watched Alex accommodated his meaning for rate of change by building a new, separate meaning for rate of change at one point (instead of two points). Terri was satisfied with Alex's understanding. Terri said Alex's understanding was good because he started to use rate of change at one point. It seems that Terri thought Alex had a strong understanding because she compared his answers to what she would answer to the same question.

Excerpt 17. Terri Watched Alex's Accommodation in Post-Lesson 7 Interview

1 T: (*While watching Alex's video clip where he said rate of change cannot be applied to one point because rate of change needed an interval or two points*) He hasn't really differentiated between average rate of change and instantaneous rate of change. He is stuck with his prior knowledge. He is willing to accept derivative, he is willing to accept how to do the limit. He is not really made that there is one little thing. He hasn't made the leap to this is a slope at a point, it is an instantaneous rate of change. Cause I think in his mind rate of change has to involve two points. Still not clear. It's interesting. Okay.

2 I: (*Watching Alex's video clip where he said slope and rate of change were not interchangeable, and rate of change at point is the limit of rate of change as the interval gets smaller and smaller*) what do you think about his understanding?

3 T: His understanding is good. He just said "I am not comfortable. I am not gonna use them interchangeably". That's to me he gets it, he gets what we were doing, but he is like there is a little wall.

4 I: At this moment, I showed the lesson clip when you explained rate of change at one point. After watching the clip let's see how he changed. (*Watching Alex's video clip where he made a connection between rate of change at one point and slope at one point by using the word "instantaneous" after watching Terri's lesson clip*)

5 T: Oh that's really good though. Talking to you helps with that.

6 I: Is that consistent with what you intended?

7 T: Yes.

Terri put her own meaning into her model of Alex's statement "instantaneous rate of change" and was pleased with his understanding. Regardless of what he thought about rate of change at one point or how he connected rate of change at one point and slope at one point, Terri focused only on whether he said what she wanted to hear. Terri did not notice that Alex still felt a conflict between his meanings for slope at a point, rate of change at a point, and his fundamental meanings of slope and rate of change.

5. DISCUSSION OF RESULTS

My research questions were

1. The meanings Terri conveyed to students
   a. What are Terri's meanings for the ideas she is teaching?
   b. How do Terri's meanings influence her instructional actions?
   c. What meanings do students already possess in relation to the ideas Terri is teaching?
   d. How do those meanings influence their interpretations of Terri's utterances and actions?
2. Terri's development of Key Pedagogical Understanding (KPU)
   a. To what extent does Terri think about ways her students understand her?
   b. How does Terri's image of student thinking influence her instructional decisions?

To answer Question 1 I need to explain Terri's meanings for the derivative and slope (or rate of change). Terri's primary meaning of "the definition of derivative" was "Taking a derivative gives a formula, then you can substitute numbers in. You get values out and the values mean slopes". She expressed her meaning during her lessons by putting the most emphasis on algebraic procedures such as eliminating h to obtain a closed form derivative.

Terri's students, Amy and Alex, experienced her lessons at the same time and in the same place, but their different schemes led them to arrive at different meanings after the lessons. Amy's meaning for slope was rise over run, which meant "going up and over". Her meaning for slope was connecting two different points with an up-and-over motion, entailing nothing about changes; she tried to restate what Terri said about changes but ended making problematic statements such as that "$f(x+\Delta x)+f(x)$ is change in y". It seems that Amy tried to accept what Terri said in lessons by trying to force a square peg into a round hole.

Alex's story was different from Amy's. Alex's original meaning for slope was a comparison of two quantities (changes in x and the associated changes in y), which was about changes. While Alex tried to remember what Terri said and did in lessons, he made an error that was writing $f(x+\Delta x)+f(x)$ in the difference quotient. However, his scheme for slope led him to catch the error, so the Post-Lesson 3 Interview with him showed his realization that what he had represented was not what he had intended, and he was able to correctly represent the definition of derivative by referring to his meaning for changes.

Terri's primary meaning for slope (visual slantiness of lines) and her image of derivative (tracking different slopes of tangent lines as she moves along (see *Figure **16***) led her to explain that finding a derivative is finding the slope of a tangent line, which is slope at a point during her lessons. In addition, when she talked about a tangent line at one point, she only referred to visual "slantiness" by saying "high" or "low" to give meaning to the derivative. Terri did mention "rate of change at one point", but she did not explain meaning for rate of change that works with one point during Lessons 1-3.

Recall that Amy's meaning for slope and rate of change was going up and over (connecting two points), whereas Alex's meaning for slope and rate of change was about two changes. They said slope and rate of change needed two points before Terri's lesson. After Amy experienced Terri's lesson, she developed a new meaning for slope at one point, which was getting a number after substituting a number into the derivative formula. She ended up with two different meanings for slope between two points (her original meaning for slope) and for slope at one point (a new meaning). Alex also developed a new meaning for slope at one point, which as slantiness of lines. It seemed that he accepted slope at one point among slope and rate of change at one point because slantiness of lines is a visual attribute.

After Terri adjusted her lesson by saying "the derivative is the rate of change at one point, which is the instantaneous rate of change" to help students understand rate of change at one point, Alex developed a new meaning for rate of change at one point because of the word "instantaneous". He said "instantaneous" meant "one point" in Post-Lesson 6 Interview. However, Amy still said rate of change at one point did not make sense. She only accepted slope at one point among slope and rate of change at one point.

My answer to Question 2 is that Terri slowly moved from KPU Phase 1 to Phase 2 and then Phase 3 as she watched students' understandings in the video clips with me over the observations, but did not enter Phase 4 or beyond. At the beginning of the observation it seems that she was in KPU Phase 1 because student thinking was not an issue. In Pre-Lesson 1 Interview she said "The majority of them will get my intention. I don't know." when I asked her "What do you think how your students might understand about the idea that is different than what you intend?"

Terri slowly changed her KPU phases as she watched students' understandings in the video clips with me over the observations. After watching two students' video clips she began to say "Different students think differently", but she did not talk about how the two students think differently during Post-Lesson 4 Interview. As Terri found that her students' understandings were not consistent with her intention, her anticipation about students' difficulties seemed to become concrete. In Pre-Lesson 8 Interview, she said "I anticipate there will be some students who have difficulty understanding when this function is going up the derivative is going down because it is increasing at a decreasing rate (*drawing an increasing and concave down curve*)." However, Terri still did not imagine different ways of thinking that Amy and Alex construct. A teacher in KPU Phase 4 tries to understand student thinking by projecting her own ways of thinking, and a teacher in KPU Phase 5 tries to understand student thinking by constructing models of ways students may understand the ideas; Terri did not enter either of these phases during my time with her.

It seems that Terri arrived at Phase 3 at the end of my observations. Terri was aware that Amy and Alex thought differently than she anticipated they would after watching their video clips, but she had not attempted to adopt a way of thinking that would be consistent with what the students expressed. Rather, Terri's adjustment was to emphasize a correct way of thinking from her perspective and to repeat her previous actions, regardless of students' understanding exemplified in the video clips.

I describe evidence that shows Terri's KPU phases.

1. Terri taught the ideas in ways that they are obvious or clear to her (KPU Phase 2).

Terri said "the idea having rate of change at a point seems really obvious to me, but it doesn't seem obvious to them" in Post-Lesson 4 Interview after watching two students video clips where they said rate of change at one point did not make sense to them, because they needed two points to talk about rate of change.

2. Terri focus on what Amy and Alex did not understand instead of what they understood (Not KPU phase 4).

Terri kept saying her students did not understand when she found that their understanding was inconsistent with her intention. It seems that she did not think about the possibility that students can struggle to think about rate of change at one point because she did not struggle with it herself. After hearing them express their struggles several times, she still did not attempt to hypothesize a way of thinking that would be consistent with what the students expressed, but instead always characterized their thinking only by how it differed from her own.

3. Terri focused on neither the source of students' difficulty nor what they thought about the idea (Not KPU 4).

In Post-Lesson 3 Interview Amy said the number that she got after substituting values for x in the definition of derivative was labeled m and realizing m of m_{tan} represents slope at one point. Amy was making a connection between the words (m and slope), not between the meanings. Terri acknowledged that Amy was thinking about slope as between two points, but Terri did not think of Amy's way of thinking as the *cause* of her difficulty. Moreover, Terri did not see any significance to Amy's meaning for slope as "going up and over" (connecting two points) as an inadequate foundation for thinking of slope and rate of change *being the same thing*.

4. Terri frequently ascribed student understanding inconsistent with her own to student habits or mistakes or to their prior teachers. She occasionally blamed her teaching in terms of what she did not do (not go slowly, not explain something

clearly, etc.) but did not reeflect on whether what she did do in lessons helped them build these understandings (Not KPU phase 4).

Terri did not think about her own contribution to students' understandings. In other words, it seems that Terri did not think about her instructional actions and their relationship to students' understandings when she found discrepancies between her intention and students' understanding.

In Post-Lesson 4 Interview, when I asked Terri about Alex's understanding of rate of change at a point, she said repeatedly he misunderstood instead of describing what he did understand or what he was thinking. In the video clip, Alex used the idea of the limit and made Δx (or an interval) get smaller to think about slope at one point. Finally, he ended saying slope at one point was the slope of tangent line, which was a visual attribute. However, Terri did not say anything about his understanding of slope at a point. Rather, Terri said she had no idea why Alex did not understand the idea of slope at one point, but she knew he was struggling. Terri then said "they have been trained to do slope between two points for so long", which suggests she saw students' difficulties as a matter of procedural habit. She did not see *their fundamental meaning of slope* as the source of their confusion and difficulties. Terri did not reach KPU Phase 4 because she only developed negative hypotheses for both her own actions and her student understandings; she always talked about what she had failed to do or what her students had failed to understand, but did not thought about her students' image of what she said and did in class. Terri always assumed that if students paid attention and absorbed everything she said and did, they must inevitably build the same meanings as she had.

5. Terri's lesson adjustment was not about student's understanding (Not KPU 4).

Her decisions to adjust her lessons were unrelated to what Amy and Alex understood, so her adjustment did not address the sources of their difficulties.

Terri always focused on what Amy and Alex did not understand. She expressed her frustrations with how her students did not grasp the idea of rate of change at one point, because it was really obvious to her. What she could think about the source of students' difficulty was "I did not make it clear" or "I jumped too quickly", which was always about what she had not done and not about her students' interpretations of what she had done actions.

The process of Terri's lesson adjustment was as follows: 1) She detected what students said in the video clips was not consistent with what she intended. 2) She concluded they did not understand "the idea" according to her own understanding. 3) She thought about the way that they should think about the idea. 4) She thought she did not make it clear. 5) She planned to change her lesson a little bit in ways that it was clear to her such as putting in more algebraic steps or showing more visual attributes. Terri's lesson adjustment was always based in her own ways of thinking because she did not envision students' understanding.

Terri's lack of orientation to her students' mathematics led her to teach the ideas in ways that were clear to her, which led to miscommunication between Terri and the two students. When Terri detected miscommunication she focused on what Amy and Alex did not understand instead of what they understood. Moreover, Terri did not think of students' way of thinking as the *cause* of their difficulties. For example, Terri did not see

Amy's meaning for slope as "going up and over" as an inadequate foundation for thinking of the difference quotient of the definition of derivative, but instead focused on any parts of her own (Terri's) understanding that Amy lacked. In addition, Terri's decisions to adjust her lessons were unrelated to what Amy and Alex understood, thus her adjustment did not address the sources of their difficulties and did not help Amy or Alex with their struggles. For instance, Amy just tried to remember what Terri said and concluded $\frac{f(x+\Delta x)+f(x)}{h+h}$ was change in *y* over change in *x* (rise over run) after experiencing Terri's adjustment where Terri taught the difference quotient in the definition of derivative was change in *y* over change in *x* again.

6. LIMINTATION AND IMPLICATIONS

6.1.Limitations

An obvious limitation of this study is no one teacher can be taken as representative of teachers in the U.S., nor can two students (Amy and Alex) be taken as representative of students in the U.S. Thus, generalization of the results is not possible. The method could be a limitation because I was also a participant in this study. I perturbed students by asking about Terri's lesson and provided Terri with opportunities to think about students' thinking and her lessons. Thus, what I observed might not be the same as what actually happens in the classrooms. For example, teachers rarely have opportunities to listen in-depth to students' understandings as described to an outside interviewer.

6.2. Implications for Teacher Education

The findings from this study suggest that students can develop various meanings through interactions with their teachers, and teachers need to think ahead to how their students might understand their statements or actions. Thinking about teaching in this way means that we must also think about the most productive meanings that teachers can have for mathematical ideas, as well as teachers' image of student thinking.

By using a conceptually coherent framework to explain what happens in classrooms, this study demonstrated that to maximize the potential of their classroom teachers must a) think about productive mathematical meanings for student learning, b) have clear intentions on how to convey those meanings to their students, c) remain cognizant of the multiple ways in which students might interpret their statements and actions, and d) develop hypotheses about their students' understandings of the teacher's instruction based on expressions of their understandings in written work and classroom conversations, test those hypotheses with further tasks, and adjust their lessons accordingly. My research could therefore contribute to improving teacher preparation, professional development programs.

7. CONCLUSION

Teachers constantly experience discrepancies between what they intended and what their students understood, and wonder why students did not understand what they tried to convey. The results described here point to a breakdown in the conveyance of meaning from teacher to student when the teacher has no image of how students might understand his or her statements and actions. This suggests that supporting in-service teachers in deliberating on student thinking so that they can convey their meanings and what they intend to students in classroom is critical. This study also indicates that it is crucial to help pre-service teachers improve what they are capable of conveying to students and their images of what they hope to convey to future students.

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APPENDIX A

HUMAN SUBJECTS APPROVAL LETTER FOR STUDY ONE

EXEMPTION GRANTED

Patrick Thompson
Mathematics and Statistical Sciences, School of
480/965-2891
Pat.Thompson@asu.edu

Dear Patrick Thompson:

On 4/7/2015 the ASU IRB reviewed the following protocol:

Type of Review:	Modification
Title:	Project Aspire: Defining and Assessing Mathematical Knowledge for Teaching Secondary Mathematics
Investigator:	Patrick Thompson
IRB ID:	1201007353
Funding:	Name: NSF: National Science Foundation, Funding Source ID: NSF-National Science Foundation
Grant Title:	None
Grant ID:	None
Documents Reviewed:	• HRP-503a protocol , Category: IRB Protocol; • oral recruitment, Category: Recruitment Materials; • Aspire Consent (Tchr) (Korean)_2015summer.pdf, Category: Translations; • support letter_SNU.pdf, Category: Other (to reflect anything not captured above); • back translation form, Category: Translations; • participant consent form, Category: Consent Form; • diagnostic instrument (Korean), Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Bio form (Korean), Category: Other (to reflect anything not captured above);

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (1) Educational settings on 4/7/2015.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

Sincerely,

IRB Administrator

cc:

APPENDIX B

HUMAN SUBJECTS APPROVAL LETTER FOR STUDY TWO AND STUDY THREE

EXEMPTION GRANTED

Patrick Thompson
Mathematics and Statistical Sciences, School of
480/965-2891
Pat.Thompson@asu.edu

Dear Patrick Thompson:

On 3/8/2016 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	Investigating relationships between teachers' mathematical meanings, their instructional actions, and meanings that their students construct in the United States and Korea
Investigator:	Patrick Thompson
IRB ID:	STUDY00004002
Funding:	None
Grant Title:	None
Grant ID:	None
Documents Reviewed:	• Student Assent_Korean_20160229.pdf, Category: Consent Form; • backtranslation form_2016.pdf, Category: Translations; • Parent Consent _20160229_Korean.pdf, Category: Consent Form; • Teacher Consent_2016_Korean.pdf, Category: Consent Form; • HRP-503a_20160227.docx, Category: IRB Protocol; • Student Interview Questions.pdf, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Student Assent_20160229.pdf, Category: Consent Form; • Teacher Consent_2016.pdf, Category: Consent Form; • Parent Consent_201620160229.pdf, Category:

	Consent Form; • Teacher Interview Questions.pdf, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions);

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (1) Educational settings on 3/8/2016.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

Sincerely,

IRB Administrator

cc: Hyunkyoung Yoon