Model Response to Sample Exam Question

"The image of speed we intended students construct through this unit is composed of these items, which themselves are constructions".

a) Describe what each of these items means.

1. Speed is a quantification of motion;

An adequate response to this question should entail an explication of the ideas of motion and of quantification, both of which should be viewed from the frame of Thompson's theory of quantitative reasoning. In Thompson's theory, a quantity is a conceptual entity—a (conceived) quality of an object (also conceived) that one can envision ascertaining how much of it there is, and imagine expressing in some standard unit of measure. The process of assigning a numerical measure (in some agreed unit) to the quality *in order to express how much of it "exists"* is called an act of quantification. Thus, quantification involves assigning numbers to a quality (or imagining doing so) *with an intention* of measuring how much of the quality the conceived object possesses. [Incidentally, without such an intention, an act of assign numbers need not be an act of quantification, it may merely be associating numerical values to an object without a clear sense of the rationale for doing so.] In the context of motion, we need to define a sufficiently rudimentary concept of motion that is reasonable for a young child to have in mind as a quality. One such quality of an object (say a car) might be that it is perceived to be at different positions relative to a reference position at different times, even if time is only implicit in the mind of the child (as it often is). So if a child has the experience of noticing and tracking a car's various positions over time, relative to some original position, that feature of the car—namely, it being at different positions at different points in time—constitutes the quality called the car's *motion*. Quantifying the car's motion, then, would be an act of describing the extent of that quality of the car in some unit of measure. A child might begin to quantify this motion by, say, imagining the measured length of the straightline displacements of the car from one position to the next, while also imagining the time elapsed during each of these displacements. They might begin to make correspondences such as "it moved 1 foot in 2 seconds", and "then it moved 2 feet in 3 seconds", etc. This might extend to "every 2 seconds it moves 3 feet" or "it moves 1.5 feet per second". Such comparisons would be a rudimentary form of conceiving speed as the quantification of motion. Notice that this last anticipates a description of component 2 of the scheme.

Notice, also, how this response tries to explain both the ideas of motion and quantification, rather than assume that readers already understand what is meant by them—this is the level of explication you should aim for in exam responses.

2. Completed motion involves two completed quantities—distance traveled and amount of time required to travel that distance (this must be available to students both in retrospect and in anticipation);

An explanation of the meaning of this component was presaged in the one to component 1 (above), but this is more explicit and elaborated. There are a couple of key ideas involved in this component: one is that speed is conceived as a quantity that is made by coordinating two other directly measurable quantities—namely, the distance traveled by an object and the time it takes the object to travel that distance. Further, to think of this as "completed" motion *in retrospect* means that one can imagine *having* traveled some amount of distance and there *having* simultaneously elapsed an amount of time taken to do so. To think of this *in anticipation* means being able to imagine such completed motion as a future event--that if an object *were* to travel some distance, there *would* be a simultaneous elapsed amount of time taken to do so. The significance of having these two (in retrospect and in anticipation) is that it suggests an interiorization of completed motion—this means that a person having these is at a developmental level whereby she can *imagine* completed motion without need for the perceptual elements from which the idea was initially abstracted (something that children at an early developmental level may be unable to do).

3. Speed as a quantification of completed motion is made by multiplicatively comparing distance traveled and amount of time required to go that distance;

This component builds on the preceding ones by adding the condition that the comparison of the two quantities (distance traveled and elapsed time) be *multiplicative*. This was presaged in the description of the first component, but it can be elaborated further. The comparison being multiplicative entails images/conceptions such as "for every unit of time elapsed, the object traveled such and such units of distance" or "for every n-units of distance traveled, so many units of time elapsed" (ratio-like thinking), or more overt multiplicative comparisons such as "for every unit of time elapsed, the object traveled *n* times as many units of distance". This idea is extended and elaborated by the fourth component.

4. There is a direct proportional relationship between distance traveled and amount of time required to travel that distance. That is, <u>if you go *m* distance units in *s* time units at a constant speed, then at this speed you will go *a/b x m* distance units in *a/b x s*."</u>

The meaning of this is almost self-explanatory, since an operationalization of the way of thinking it encapsulates is given by the underlined statement. Another way to describe this is to say that understanding constant speed means understanding that the co-accumulation of distance traveled and time elapsed are in a direct proportional correspondence, such that one can think of that correspondence reciprocally: *any* fraction the distance traveled corresponds to exactly that same fraction of the time it would take to travel that distance, and *any* fraction of the time it would take to travel some distance will correspond to traveling that same fraction of the distance.

b) Describe two arguably distinct ways in which the authors used the above scheme of ideas in their research study.

Two distinct, yet interrelated, uses of the above scheme of ideas—which are essentially models of understandings the authors intended for the student (Ann) to develop—are as a data interpretation framework and as an instructional design framework. The former use is seen in the conceptual analysis of speed as a rate that was developed in the two Thompson and Thompson articles (1992, 1994), and in Thompson's 1994 chapter. This use is also evident in the authors' use of that analysis as a basis for interpreting Ann and JJ's thinking that emerged as they engaged with instruction designed to foster the development of those understandings. The authors used these concepts/constructs as a lens through which to make sense of and describe the thinking of the students (including ways of thinking they had no yet developed) at certain points in time in the progression of the teaching experiments. For instance, we saw many instances of the authors give analyses of the transcripts that framed the students' thinking explicitly in terms of components/ways of thinking that were either consistent with, or in contrast to, those specified in the speed scheme elaborated above. Regarding the latter, the authors introduced the construct of speed-lengths as a way to capture an aspect of Ann's and JJ's thinking that was qualitatively different from the conceptions that formed the above speed scheme. Although the speed-length is not part of this scheme, its development as an explanatory construct was implicitly shaped by the author's development of the speed scheme. We saw that the speed scheme was also used to guide the authors in designing a sequence of tasks intended to foster the development of the various components/ways of thinking in both Ann and JJ. In addition to the design of an a priori sequence of tasks seen in the Thompson and Thompson articles, we saw PT's more extemporaneous and responsive use of the double number line as a representing and thinking tool to help Ann develop the proportional correspondence of distance traveled and time elapsed specified in component 4 of the speed scheme. This is a clear example of reverse engineering an instructional tool and activity from a model of a targeted way of thinking.