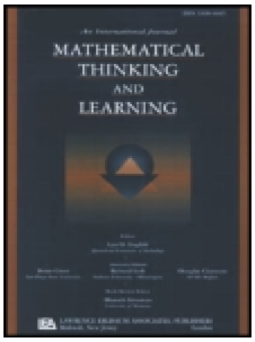

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An Exponential Growth Learning Trajectory: Students' Emerging Understanding of Exponential Growth Through Covariation

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ABSTRACT

This article presents an Exponential Growth Learning Trajectory (EGLT), a trajectory identifying and characterizing middle grade students' initial and developing understanding of exponential growth as a result of an instructional emphasis on covariation. The EGLT explicates students' thinking and learning over time in relation to a set of tasks and activities developed to engender a view of exponential growth as a relation between two continuously covarying quantities. Developed out of two teaching experiments with early adolescents, the EGLT identifies three major stages of students' conceptual development: prefunctional reasoning, the covariation view, and the correspondence view. The learning trajectory is presented along with three individual students' progressions through the trajectory as a way to illustrate the variation present in how the participants made sense of ideas about exponential growth.

Introduction

Exponential growth is an important topic beginning in the middle grades and is encountered in increasingly complex variations in subsequent grades. A focus on the conceptual underpinnings of exponential growth is highlighted in many national standards documents (e.g., Australian Ministerial Council on Education, Employment, Training and Youth Affairs, 2006; National Governor's Association Center for Best Practices, 2010; Finnish National Board of Education, 2004; Singapore Ministry of Education, 2006), and a deep understanding of exponential functions plays a critical role in university mathematics courses such as calculus, differential equations, and complex analysis (Weber, 2002). Recent years have seen an increased emphasis on the ideas related to exponential growth in early adolescence (e.g., Haese, Haese, & Humphries, 2013; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006).

Despite the importance of a robust understanding of exponential growth, instruction proves challenging given students' documented difficulties in understanding this topic. Weber (2002) found that university students struggle to understand the rules of exponentiation and to connect them to rules for logarithms. Research on secondary students reveals challenges in making the transition from linear representations to exponential representations and in identifying what makes data exponential (Alagic & Palenz, 2006). Further, studies investigating teachers identify struggles in both the understanding and instruction of exponential growth. Teachers appear to have a strong understanding of exponential growth as repeated multiplication, but struggle to connect this understanding to the closed-form equation, to recognize growth as exponential in nature, and to appropriately generalize rules such as the multiplication and power properties of exponents (Davis, 2009; Presmeg & Nenduardu, 2005; Strom, 2006, 2008). These limitations may contribute to teachers' difficulty in anticipating what supports students may require in learning about exponential properties (Davis, 2009).

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The challenges in supporting students' learning suggest a need to better understand how to foster student understanding of exponential growth. In order to develop this line of research, we constructed an Exponential Growth Learning Trajectory (EGLT), one possible trajectory tracing students' initial and evolving understanding of exponential growth throughout a series of lessons with an instructional emphasis on covariation. By using the term "learning trajectory," we follow Clements and Sarama's (2004) definition of a learning trajectory as:

> Descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (p. 83)

As Simon and Tzur (2004) noted, the most important aspect of a learning trajectory is for teaching concepts whose learning is problematic. Better understanding how to support student learning is a critical aspect of addressing challenging topics. This article reports on the results of two teaching experiments investigating students' understanding of exponential growth within the context of covarying quantities. We present a learning trajectory specifying students' evolving understanding, identifying connections between students' conceptions and the tasks and teaching actions promoting the development of those conceptions. Our findings suggest that situating an exploration of exponential growth in a model of covarying quantities can support both students' understanding of what it means for data to grow exponentially and how to algebraically express exponential relationships.

The need for learning trajectories research

Learning trajectories research has the potential to support a better understanding of student learning, enable more effective teaching strategies, and guide better curriculum and standards design. Steffe (2004) noted that the construction of learning trajectories of children's mathematical thinking constitutes "one of the most daunting but urgent problems facing mathematics education today" (p. 130). To date, the majority of learning trajectories address ideas in early mathematics, such as those for equipartitioning (Confrey, 2012); length measurement (Clements & Sarama, 2007; Sarama & Clements, 2002); children's composition of geometric figures (Clements, Wilson, & Sarama, 2004); relational thinking about addition, subtraction, and division (Stephens & Armanto, 2010); and integer addition and subtraction (Stephan & Akyuz, 2012). Although substantial work has occurred in understanding students' conceptual development in the secondary topics of ratio and proportion, algebra, and functions, few studies have been devoted to the development of learning trajectories on those topics. Given students' difficulty in developing rich ideas about algebraic concepts, learning trajectories research could support a better understanding of students' thinking and learning in those content areas.

Learning trajectories research can also provide opportunities to translate research-based findings into improved teacher education and professional development, enabling teachers to diagnose students' understanding and provide appropriate feedback (Amador & Lamberg, 2013; Wilson, Mojica, & Confrey, 2013). Clements and Sarama (2004) found that superior teachers used learning trajectories to support student learning by focusing on their students' reasoning and accordingly making adjustments to their tasks and teaching actions. In general, researchers suggest that learning trajectories can guide teachers in making sense of their students' understanding and in choosing the most appropriate instructional activities (Clements & Sarama, 2012; Szilagyi, Clements, & Sarama, 2013), an avenue for supporting curriculum design and the development of standards documents (e.g., Baroody, Cibulskis, Lai, & Li, 2004; Battista, 2004; Clements et al., 2004; Confrey & Maloney, in press).

Background and theoretical framework

Hypothetical learning trajectories: Definitions and characteristics

The notion of a hypothetical learning trajectory has different meanings among mathematics education researchers. Simon's (1995) original description of a hypothetical learning trajectory consists of "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Clements and Sarama (2004) elaborated on this definition by emphasizing three parts of a learning trajectory: A mathematical goal, a model of cognition they called developmental progressions, and instructional tasks providing experiences for students to progress through the developmental levels. Many researchers' working definitions of learning trajectories emphasize progression in sophistication. For example, Battista (2004) relied on the notion of levels of sophistication, strata through which a student progresses from one cognitive level to the next until reaching formal mathematical concepts. Similarly, Wilson and colleagues (2013) defined learning trajectories as research-based descriptions of how students' thinking evolves over time from informal ideas to increasingly complex ones.

Some researchers also highlighted the aspect of learning trajectories emphasizing a conjectured model of learning. For example, Steffe (2004) described a learning trajectory of children's thinking as including "A model of their initial concepts and operations, an account of the observable changes in those concepts and operations as a result of the children's interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes" (p. 131). Steffe's description is compatible with our model, in which we aim to develop conjectures about both a possible learning route and a specific means that can be used to support and organize learning along this route. We also attend to Confrey and associates' (2009) depiction of a learning trajectory, which emphasized individuals' refinement of their own understanding while acknowledging that conceptual growth is influenced by instruction. The work of building learning trajectories includes developing a cognitive model of students' learning that is sufficiently explicit to describe students' operations involved in constructing increasingly sophisticated mathematical ideas; thus, "the creation of learning trajectories always implies conceptual analysis" (Clements & Sarama, 2004, p. 85). Clements and Sarama also pointed out that a learning trajectory does not describe the only path of learning, or even the best of many paths. Instead, it represents one possible characterization of student learning over time. Within that perspective, we offer a learning trajectory that is an empirically based model of students' initial understandings about exponential growth and an account of how those understandings changed as the students interacted with mathematical tasks, tools and representations, and deliberate teaching actions.

Alternate models and critiques of learning trajectories

Daro, Mosher, and Corcoran (2011) described a learning trajectory as an "empirically supported hypotheses about the levels or waypoints of thinking, knowledge, and skill in using knowledge, that students are likely to go through as they learn mathematics" (p. 12). An example of this type of skill-based trajectory can be found in the Ongoing Assessment Project multiplicative reasoning framework (Vermont Mathematics Partnership Ongoing Assessment Project, 2011), which identified strategies important to multiplicative reasoning such as repeated addition, skip counting, doubling, and halving. Similarly, Brendefur, Bunning, and Secada (2014) identified a group of secondary teachers' collective development of a learning trajectory for exponential functions, which is a characterization of students' strategies for a doubling bacteria problem, such as a *doubling the time* strategy, a *divide the number of bacteria by two* strategy, and a *change the base from two to 1.5* strategy. In contrast, we focus on students' conceptual development, rather than emphasizing strategies or skills.

Sikorski and Hammer (2010) argued that couching learning trajectories as progressively more sophisticated ideas typically means progressively more correct. They critiqued learning progressions in science for only including correct ideas, particularly in cases in which the formation of an incorrect idea was historically generative for later progress. Because our learning trajectory is based on a conceptual analysis of empirical data, the trajectory includes the important ideas students developed about exponential growth, including those of limited generalizability or correctness. We do not offer a learning trajectory in which the selection of topics and pathways was based on a logical task analysis of content domains (e.g., American Association for the Advancement of Science, 2001; Harlen, 2010).

Lesh and Yoon (2004) discussed a form of learning trajectories in which researchers structure levels of knowledge development into ladder-like sequences, with each rung of the ladder representing a more sophisticated understanding of the construct in question. They noted that although such sequences can be useful for curriculum development, it can be dangerous to rely on such models to make inferences about how learning occurs. Ladder-like sequences imply a linear view of development, whereas there is ample evidence that mathematical ideas develop along a variety of dimensions. We take these cautions to heart and therefore are careful to characterize our learning trajectory as a reflection of the development of groups of students as we analyzed them in situ. Further, evidence from our corpus of data suggests that it does not make sense to identify an individual student's thinking as occurring at a particular developmental level, when that claim would imply that the student then functions at that same level across a variety of tasks and settings. Students' thinking is more complex, fluid, and dynamic than what can be easily represented by a static model of learning. Our learning trajectory is an attempt to characterize the nature of the evolution of students' thinking in a particular instructional setting. Therefore we present not only the learning trajectory itself, but also the individual trajectories of three students as a way to illustrate the differences and commonalities present in the ways that the students made sense of ideas about exponential growth.

Covariation and the rate of change perspective

A typical approach to exponential growth emphasizes the operation of repeated multiplication. For instance, in the middle grades (ages 11–14 years) curriculum *Connected Mathematics Project* (Lappan et al., 2006), students encounter a problem in which coins are placed on a chessboard in a doubling pattern. Students must examine the relationship between the number of squares and the number of coins, perform repeated multiplication, and then connect this operation to exponential notation. This approach follows the recommendation of researchers suggesting the introduction of exponential growth as repeated multiplication with natural numbers (e.g., Goldin & Herscovics, 1991). However, generalizing to non-natural exponents may then pose challenging for students, as it can be difficult to imagine a constant b multiplied by itself a fractional (or irrational) number of times (Davis, 2009).

In an alternate approach, Confrey and Smith (1994, 1995) introduced the operation of splitting; a splitting structure is a multiplicative structure in which multiplication and division are inverse operations, such as repeated doubling and repeated halving. Confrey and Smith posited that splitting can be an operational basis for multiplication and division, whereby students treat the product of a splitting action as the basis for its reapplication; thus, a split can be conceived of as a multiplicative unit. Basing multiplication on repeated addition, rather than splitting, neglects the development of ideas such as equal sharing, magnification, and repeated copies (Confrey & Smith, 1994). As the basis of a rate of change approach to exponential growth, splitting can enable students to calculate ratios between successive y-values for constant changes in x-values. We highlight this conception as an important foundational idea for a rate of change approach to exponential growth.

An emphasis on rate of change (Smith, 2003; Smith & Confrey, 1994) and covariation (Saldanha & Thompson, 1998; Thompson & Carlson, in press) fosters an examination of a function in terms of coordinated changes of x- and y-values. In this case a student may coordinate change between y_m to y_{m+1} with change between x_m to x_{m+1}. This approach differs from the more typical correspondence view, in which a function is seen as the fixed relationship between the members of two sets. From the correspondence perspective, $y = f(x)$ represents y as a function of x, in which each value of x is associated with a single value of y (Farenga & Ness, 2005). This static view underlies the typical treatment of functions in school mathematics, but research on middle grades students' emerging understanding of functional relationships suggests that beginning with a covariation approach can support a flexible, connected understanding of rate of change that supports eventual formalization and transition to the correspondence view (Ellis, 2007, 2011; Smith & Thompson, 2007; Thompson, 1994; Thompson & Thompson, 1992).

Saldanha and Thompson (1998) addressed the idea of covariation in terms of the images that can support one's ability to think covariationally, describing covariational thinking as the ability to mentally hold a sustained image of two quantities' values simultaneously. Castillo-Garsow (2013) similarly addressed covariation as the mental act of imagining two quantities changing together simultaneously. Immersing students in situations with quantities that they can visualize, manipulate, and imagine could foster an ability to reason flexibly about dynamically changing events (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). This approach may be particularly helpful in supporting students' understanding of exponential growth as they grapple with coordinating additive change in x with multiplicative change in y.

A defining characteristic of exponential growth is the notion that the rate at which the function changes with respect to x is proportional to the value of the function at x (Thompson, 2008). An emphasis on covariation could foster an early understanding that the value of y_2/y_1 is dependent on $x_2—x_1$. This goal could be further supported by developing an exponential growth situation in which two quantities covary continuously, rather than in discrete chunks. Students who can conceive of continuous variation can imagine smooth changes as composed of smaller chunks with numerical values, with every smaller chunk within that change in progress itself imagined as being covered by a smooth change in progress (Castillo-Garsow, 2012; Castillo-Garsow, Johnson, & Moore, 2013; Thompson, 2011). Strom (2008) noted that the ability to reason multiplicatively must include considering multiplicative comparisons for input intervals other than one unit. To develop a mature understanding of exponential function, she argued, one must imagine a function such as $y = 2^x$ more flexibly than just a function that grows by a multiplicative factor of two for every one unit added to x. One must also be able to imagine some other constant factor relating two output values for intervals smaller than a unit, such as a factor of $\sqrt{2}$ corresponding to an interval size of 0.5 units. This type of thinking promotes what Strom referred to as partial factors, the idea that fractional exponents can represent a smaller part of a factor (such as conceiving of $2^{½}$ as 1/2 factors of 2). A context in which two quantities covary continuously could potentially support these ways of thinking because, unlike a scenario with coins on a chessboard, an input unit less than one can have quantitative meaning for students.[1]

Methods

Overview: Structure of the study

The data presented in this article are from two teaching experiment studies conducted in the United States over the course of two years. The first study was an exploratory teaching experiment conducted in order to (a) understand students' emerging conceptions of exponential growth; (b) identify shifts in students' conceptions over time; and (c) hypothesize potential mechanisms responsible for promoting the identified shifts. The results of the first study produced an initial learning trajectory (Ellis, Ozgur, Kulow, Williams, & Amidon, 2013), which supported the design of an instructional sequence for a second, larger-scale teaching experiment. The results of the second

[1]We acknowledge that this approach does not address how to reason with irrational inputs, such as 2^{π}.

teaching experiment led to the revision and refinement of the learning trajectory reported here. The following sections detail the participants, the hypothetical learning trajectory and task development, and the process of data analysis.

Participants and the teaching experiments

Teaching experiment #1

The first teaching experiment (TE1) included five eighth-grade students (aged 13–14 years), all female. Our recruitment efforts yielded five students who agreed to participate, and all five students were accepted into the teaching experiment. Two of the students participated intermittently. Thus, we restricted our analysis to the three students who participated regularly, Uditi, Jill, and Laura (all pseudonyms). Jill and Laura were enrolled in an eighth-grade general mathematics course, and Uditi was enrolled in an eighth-grade pre-algebra course. None of the students had encountered a formal exponential functions unit in their mathematics class at the time of the study. On the first day of the teaching experiment, the students' engagement with initial exploratory activities suggested that their understanding of exponentiation was from a repeated multiplication model; they understood an expression such as 2^3 to represent $2 \times 2 \times 2$. Regarding exponential growth, all three students described the growth in qualitative ways, such as "the graph increases very quickly," but did not quantify the nature of the growth they encountered. The students participated in a 12-day teaching experiment (Cobb & Steffe, 1983; Steffe & Thompson, 2000) over the course of three weeks, in which the first author was the teacher-researcher. Two project members observed and videotaped each teaching session, which lasted approximately one hour. The project team met daily to debrief and discuss the events that occurred during each session, adjusting and modifying the planned activities for the following session based on what had transpired.

Teaching experiment #2

The second teaching experiment (TE2) included eight participants who had just completed eighth grade (all aged 14 years). The participants were all members of a university-sponsored program, which partners with local schools to provide support for students of color and first-generation college students. The program hosts a five-week on-campus summer session, which served as the setting for the second teaching experiment. There were five male participants and three female participants. The students took part in a five-week teaching experiment that addressed linear and exponential growth, in that order. The exponential growth portion of the teaching experiment occurred over a period of 9 days, with each session lasting 90 minutes. The second and third authors were the teacher-researchers.

Roughly half of the TE2 students had encountered an exponential growth unit in their classrooms prior to the teaching experiment. However, the students' ideas about exponentiation and exponential growth were strikingly similar to those of the students from the first teaching experiment. On the first day, all of the students described an expression such as 2^3 as representing repeated multiplication. Those students who had heard about exponential growth could only describe it as a type of growth that increased very quickly, but like the TE1 students, they could not quantify the nature of the growth. Due to the shorter number of days and the greater number of students, the students in the second teaching experiment did not encounter as many of the tasks as those in the first teaching experiment. One or more project members observed and videotaped each teaching session. The project team met daily to debrief and discuss the events that transpired during each session. Pseudonyms were assigned to all of the participants.

The general instructional approach employed in both teaching experiments was an inquiry-oriented approach. The teacher-researchers provided students with tasks and allowed the students to explore the tasks together while discussing ideas with one another. Students worked together in groups on a daily basis. The teacher-researchers followed the students' lead by probing their thinking, asking them to explain and justify their strategies, and encouraging the free exchange of ideas. Justification was a strong norm in both teaching experiments; the teacher-researchers encouraged and expected the students to justify their reasoning on a regular basis.

The hypothetical learning trajectory and task development

We constructed an initial hypothetical learning trajectory before the first teaching experiment based on extant research, a review of the relevant literature, and pilot interviews with middle school students. This supported the development of an initial set of learning goals in the context of a specific set of tasks (Clements & Sarama, 2004; Simon & Tzur, 2004). The task context relied on a scenario in which a plant called the Jactus grew by doubling its initial height every week. Students could explore the growing Jactus by comparing its height over time with a specially designed Geogebra script (Figure 1). Students could manipulate the Jactus by dragging its base with the mouse, observing as the plant continually increased or decreased its height as it moved along the x-axis. Later in the teaching experiment, we changed the growth factor to values other than 2 and the initial height to values other than 1 inch. Our aim was to develop a context in which students could explore two continuously covarying quantities.

A central goal for developing the instructional context was that it be experientially real and meaningful to the students (Gravemeijer, 1994). The students and the teacher-researcher discussed the feasibility of a plant growing exponentially, and we considered whether the unrealistic nature of the scenario would interfere with the students' sense making. Although the Jactus context is not realistic, we found the tradeoff worthwhile in that it represents a situation with continuously varying quantities that younger students could understand, visualize, and mathematize (following Webb, Van Der Kooij, & Geist, 2011). Webb and his colleagues argued that the realistic aspect of Realistic Mathematics Education is not just about using real world contexts, but rather about relying on situations that are imaginable; contexts can be idealized to motivate powerful mathematical strategies.

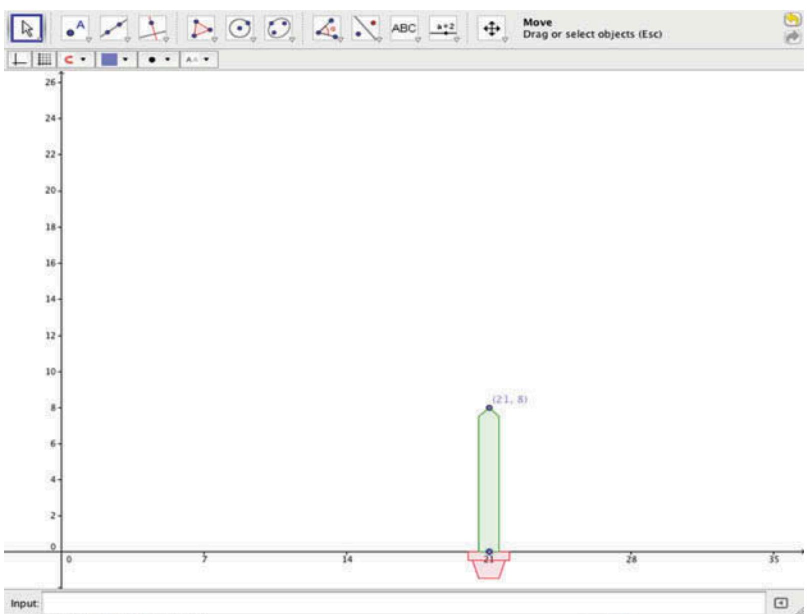

Figure 1. The Jactus Geogebra script.

Although relying on continuous contexts does not guarantee that continuous reasoning will occur, the Jactus context afforded the possibility of continuous reasoning in a way that a discrete situation might not. It is possible to visualize a plant that is somewhere between 1-inch tall when it starts growing and 2 inches tall after a week, whereas it may be more difficult to imagine that type of growth in more traditional scenarios (e.g., a cell in the midst of dividing into two cells). However, we anticipated that continuous reasoning would be difficult to achieve, particularly with younger adolescents. We thus developed tasks and strategies to encourage this way of thinking, such as (a) emphasizing scaling images of growth in the plant's height as it varied with time, (b) designing tasks highlighting the nature of growth within unit intervals, and (c) introducing problems requiring students to shift from calculation to anticipation. We discuss these instructional decisions in more detail in the Results section.

Building on the students' conception of exponential growth as repeated multiplication, our learning goals emphasized fostering students' understanding of the following five ideas for an exponential function $y = ab^x$:

1. The period of time x for the y-value to double (or increase by the growth factor b) is constant, regardless of the value of a or b.
2. There is a constant ratio change in y-values for each constant additive change in corresponding x-values.
3. The ratio of the growth in y is always the same for any same Δx; for example, a Jactus plant will grow the same amount multiplicatively (relative to its own height) from Week 102 to Week 104 as it grows from Week 2 to Week 4.
4. The value of y_2/y_1 is dependent on $x_2—x_1$.
5. The constant ratio change in y-values is dependent on both the growth factor b and on Δx in the following manner: $\frac{y_2}{y_1} = b^{x_2 - x_1}$. This relationship will hold even when $\Delta x < 1$.

The teaching experiment model demands flexibility in instruction; thus, any initial set of activities would operate only as a starting point for instruction (Simon et al., 2010). We implemented iterative cycles of teaching actions, in which we build models of students' thinking and consequently revised current and future tasks on an ongoing basis. The preliminary design enacted in the first teaching experiment was then refined and revised for the purpose of the second teaching experiment.

Data analysis

Clements and Sarama (2004) noted that a learning trajectory must be emergent, with teacher-researchers constructing new models of students' mathematics as they interact with students over time. We employed ongoing and retrospective analysis techniques (Simon et al., 2010; Steffe & Thompson, 2000) in order to characterize students' changing conceptions throughout the course of the teaching experiments. All sessions were transcribed and enhanced into documents including not only verbal utterances but also all images of student work, descriptions of relevant gestures, and other nonverbal actions (Arzarello, Paola, Robutti, & Sabena, 2009). A hypothetical learning trajectory developed before the first teaching experiment served as a source of preliminary codes as the research team coded the enhanced transcripts for evidence of one or more of the component understandings. *Component understandings* are the individual concepts, ideas, and particular ways of reasoning and representing we identified in the students' thinking about specific aspects of exponential growth. For example, one component understanding is the idea that the magnitude of the growth factor, b, determines how the plant's height grows. Another component understanding is the idea that one can express repeated multiplication for a growth factor b algebraically as $y = b^x$, where x represents time in weeks and y represents height in centimeters or inches.

For each student response to a task, we determined as a group whether it provided evidence of one of the component understandings in the hypothetical learning trajectory. This initial round of coding produced new emergent codes (Strauss & Corbin, 1990), which were not part of the initial hypothetical learning trajectory. Those codes supported a first round revised learning trajectory, which we call the first emergent learning trajectory, accounting for the events from the first teaching experiment. As an example, consider the component understanding that one can coordinate the ratio of two y-values, y_2/y_1, for any additive increase in x-values. For example, students can determine the ratio of y_2/y_1 for Δx and recognize this ratio as a representation of how many times taller the plant would grow in a given interval of time, regardless of the size of the interval. Early transcript excerpts of the students' thinking in the first teaching experiment indicated that they appeared to understand this idea. For instance, given a Jactus with two heights at Weeks 24 and 29, Uditi could divide the two height values and interpret the quotient, 32, as meaning that the plant grew 32 times as tall in a five-week span. However, later transcript excerpts revealed that Uditi did not take the quotient of height values for all intervals. In one instance, when the interval was a quarter of a week, Uditi could divide the two height values but could not interpret the quotient as a representation of how many times taller the plant grew in 0.25 weeks. The students' task responses and utterances suggested that determining growth for intervals smaller than one week was a conceptually different activity than determining growth for a span of multiple weeks. Therefore, the component understanding on the initial hypothetical learning trajectory that one can coordinate the ratio of two y-values for any additive increase in x-values was refined into three separate component understandings in the first emergent learning trajectory, which distinguished between coordinating multiplicative change in y for intervals of (a) one unit, (b) multiple units, and (c) any unit, including intervals less than one week.

The first emergent learning trajectory then supported the revision of the task sequence for the second teaching experiment. After the conclusion of the second teaching experiment, all episodes were transcribed and enhanced, and the analysis process continued with the first emergent learning trajectory now serving as the starting point for coding. Through multiple rounds of analysis, the trajectory underwent another series of revisions, with the process continuing until the trajectory was stabilized into the second emergent learning trajectory. The research team them returned to the data from the first teaching experiment and re-analyzed that data corpus with the second emergent learning trajectory, making a final round of minor revisions until the trajectory had stabilized into the final emergent learning trajectory, the EGLT. Given the challenge in clarifying the nature of the students' mental images, particularly when the students struggled to articulate their thinking, we relied on a process of triangulating data from multiple sources in order to infer students' thinking. We relied on students' drawings, written work, nonverbal gestures, and trends and shifts in their problem-solving approaches in order to draw on multiple sources of data, where possible, for inferring student thinking.

During all three rounds of coding, two members of the research team coded the entire data corpus independently, meeting weekly with the project team in order to discuss boundary cases and clarify and refine uncertain codes. Once this phase was complete, the research team met to code every transcript together, comparing each code and discussing any differences until reaching agreement. A subset of two researchers then re-coded the entire data set, again meeting weekly with the research team to discuss any final refinements. This iterative process of coding, refining, and recoding continued until no new codes emerged and no more refinement was necessary. We then chose 20% of the data corpus, which included a 118 instances of codes, encompassing the entire set of existing codes. A new project member who had not participated in the prior coding activities independently coded this portion of the data. We calculated Cohen's kappa to determine the inter-rater agreement between the coders. The resulting Cohen's kappa coefficient, $\kappa = 0.830, p < 0.001$, indicates an almost perfect agreement between the coders (Viera & Garrett, 2005).

Results: An EGLT and three students' progressions

Overview of the learning trajectory

The progression of the students' conceptual development occurred in three major stages of reasoning, which we call prefunctional reasoning, the covariation view, and the correspondence view. We borrow the term stage from Steffe (2012) and Glasersfeld and Kelly (1981), who described a stage as a period through which a characteristic is present and remains constant throughout the period. Although prefunctional reasoning preceded the development of both the covariation and the correspondence views, the latter two ways of thinking did not occur in a sequential nature. Rather, students constructed an early covariation understanding of exponential growth, and then began to develop both a more sophisticated covariation perspective and a correspondence understanding in tandem. The students' covariation thinking influenced the ways in which they constructed and interpreted correspondence rules, and they were eventually able to leverage both perspectives as needed when approaching novel tasks. Figure 2 presents a visual representation of the three stages and the component understandings in the learning trajectory. Arrows identify the ways in which the students demonstrated different types of movement between each of the individual component understandings over time.

The following four sections address prefunctional reasoning, early covariation reasoning, correspondence reasoning, and sophisticated covariational reasoning. In each case we present the relevant portion of the learning trajectory and discuss students' understandings and the tasks that supported the development of those understandings.

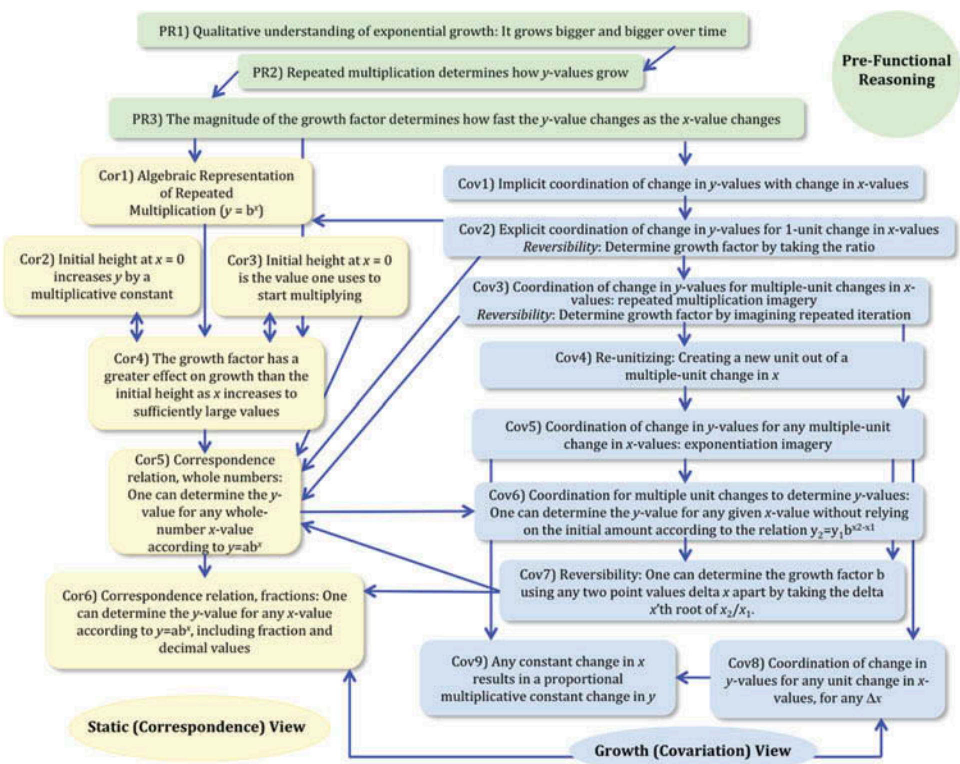

Figure 2. The stages and component understandings in the exponential growth learning trajectory.

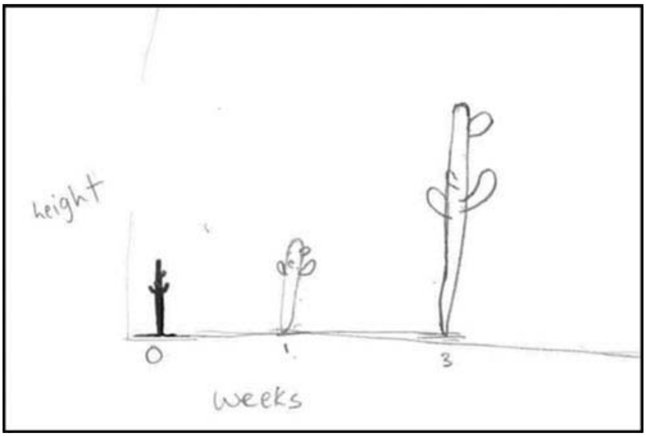

Figure 3. Uditi's drawing of the Jactus' growth over time.

Prefunctional reasoning

In both teaching experiments the students entered the sessions with a qualitative understanding of exponential growth, describing such growth as beginning slowly and becoming faster over time. Many students accompanied these explanations with gestures indicating a curve that grew sharply steeper over time. The students could not, however, quantify the manner in which the plants grew. Exploration with the Geogebra program enabled the students to begin to solidify and articulate a repeated-multiplication understanding of exponential growth. Table 1 details each of the three component understandings in the prefunctional reasoning section of the learning trajectory, along with associated data examples and tasks. We discuss each in turn.

With a qualitative understanding of growth, students understand that the y-value, in this case the plant's height, grows larger at an increasing rate over time. However, they are unable to quantify the manner of increase. Students first explored the Jactus situation within the Geogebra program, moving the plant along the x-axis representing time in order to explore how the height changed. They then encountered a task prompting them to draw a picture of how the Jactus grows over time. Troy, a student from Teaching Experiment 2 (TE2), drew an upward curve and explained, "It starts moving a little bit faster … in between Week 1 it started going up a little bit faster and then a little bit faster and then a little bit faster and then a little bit faster." Uditi, a student from Teaching Experiment 1 (TE1), drew a picture with cactus plants (Figure 3) and explained, "The difference in these two [pointing to the plant at Week 0 and Week 1] are like really little, but then it grows to Week 3 and then it's a lot more." In both cases, the students used qualitative terms such as "a little bit faster" or "a lot more" but could not quantify either the height of the plant or the amount of growth.

Prompts to measure and describe the plant's height at different points in time encouraged students to explicitly attend to the height in inches and the magnitude of growth over time. This fostered a repeated multiplication understanding of growth, the notion that a process such as doubling or tripling determined how the plant grew. However, at this point, the students' focus was on multiplication of the height values without explicit attention to changes in time. For example, for a task requiring students to calculate the plant's height each week, Carter (TE2) explained how he determined the height at Week 5:

Carter: Times it by 3.
Instructor 1: Times it by 3. Why were you timesing it by 3?
Carter: Because that's the pattern.
Instructor 1: How do you know?
Carter: Because 1 times 3 is 3, 3 times 3 is 9, 9 times 3 is 27, 27 times 3 is 81.

Table 1. Prefunctional reasoning.

Component Understanding	Definition	Data Examples	Tasks
PR1) Qualitative Understanding	Students understand that *y*-values grow larger at an increasing rate over time, but the manner of increase is unquantified.	TE2, Troy: "It starts moving a little bit faster … in between Week 1 it started going up a little bit faster and then a little bit faster and then a little bit faster and then a little bit faster."	Exploration with the Geogebra Program via sliders; tasks requiring students to draw, graph, and describe the nature of the plant's growth.
PR2) Repeated Multiplication	Students understand that repeated multiplication determines how height (*y*) grows without attending to time (*x*).	TE1, Uditi: "They're all going up by like times 4, like 16 times 4 is 64 and then 64 times 4 … that's 1024."	Tasks requiring students to measure the plant's height, record the height in tables, find missing values in a provided table, and complete far-prediction problems for large *x*-values.
PR3) Growth Factor Magnitude	Students understand that the magnitude of the growth factor determines how growth occurs.	TE1, Laura: "The tropical [is fastest], because it's growing 4 times and it's more than 2 times and 3 times."	Compare plants with different growth factors in order to determine which grows the fastest over time.

Absent from Carter's language is attention to how much time it took for the plant's height to triple.

Through comparing plants with different growth rates, students began to realize that the growth factor determined the nature of the growth, identifying plants with larger growth factors as those that would grow taller over time. For example, given a comparison problem across three plants with growth factors of 2, 3, and 4, Laura (TE1) could determine that the quadrupling plant grew faster than the others, "because it's growing 4 times and it's more than 2 times and 3 times." Evan (TE2) provided a similar description, explaining that the plant with the growth factor of 4 was the fastest growing plant because "it multiplies its height more than the others." At this stage the students did not attend to the connection between the growth of the y-values and the growth of the x-values; for this reason, we coded their understanding as prefunctional.

The tasks the students encountered early in the teaching experiments were designed to encourage quantification of both height and time. Because the students' initial understanding of exponential growth was qualitative, the research team constructed activities in which the students had to measure, record, calculate, and predict height values at different time points. The teacher-researchers' teaching actions also encouraged students to think about quantification. For example, when students referred to growth as "fast growth," the teacher-researcher responded, "How do you know that this is fast growth? What about this shows that it grows extremely fast?" When a student described a plant as getting bigger, the teacher-researcher pressed, "How much bigger?" The focus on quantification encouraged attention to the growth factor and the repeated multiplication action. However, at each component understanding in the pre-functional reasoning stage of the learning trajectory, the students did not make explicit connections between time and height; they did not consider height as a function of time. For that reason, it became important to develop tasks and scenarios requiring explicit attention to the time values as well as the height values.

Early covariation reasoning

In an attempt to encourage coordination of the plant's height with the number of weeks it had been growing, we introduced a task in which students had to compare three student-produced drawings of a Jactus that doubled in height every week, and determine which drawing accurately represented the plant's growth (Figure 4). Two students from TE2, Troy and Benito, discussed why they thought

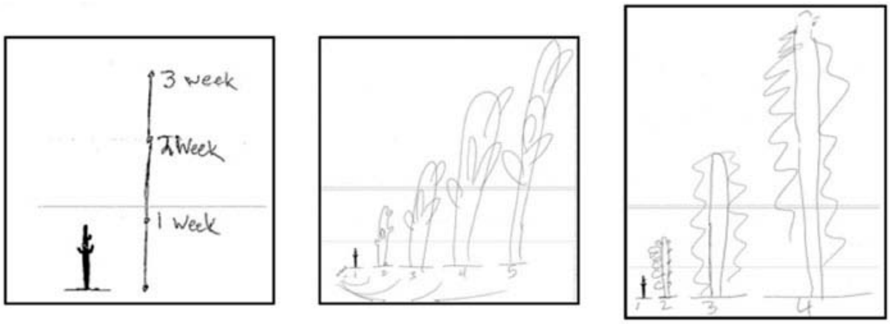

Figure 4. Three student drawings of growth over time.

the third drawing was correct. Benito suggested they write the heights above each plant, and Troy remarked, "So if you're doubling it by height each time, this is what it should be for Week 2, about that long." Troy and Benito assigned a numerical value to the plant, considering it to be 1 inch at Week 1, and then they understood that the height of the plant should double "each time." They could compare the height of the line representing the plant at Week 3 to the height of the line at Week 2 and declare it to be twice as tall. Although the students attended to the weeks, they did not quantify the growth in weeks. Attention to coordination of growth in height with growth in weeks was implicit, with the phrase "each time" a common one to describe students' multiplying process. The following quotes are examples of students' descriptions of their actions across both teaching experiments:

Paj (TE2): You double each time.
Uditi (TE1): It's starting out half the size of that and then it's doubling each time and that one's doubling too.
Jose (TE2): The plant doubles in size from the week before each time. So 2 it goes 4, 4 it goes 8.
Evan (TE2): It quadruples each time for growth.

In some cases, such as with Troy and Benito's conversation above, the students explicitly referenced the weeks. However, these references appeared to function only as a way to keep track of how many times one had doubled (or multiplied by the growth factor). The students viewed each week as a marker or a counter, rather than as a quantity of time (Table 2), as evidenced by both their language and their gestures in tapping each week.

Tasks requiring students to draw pictures of the Jactus encouraged more explicit coordination between time as quantified as a number of weeks and height in inches. For example, Carter (TE2) encountered a task with a drawing of a Jactus that was 1/2 inches tall at Week 0, and he had to draw the Jactus at Weeks 1–4. After producing his drawings, he explained:

Carter: This is Week 0, this is the starting height and then, so I took this height and I doubled it, and I doubled it and this is Week 1. So I doubled … so I doubled Week 0. Wait … wait, no, hold on. I doubled Week 1 and then I got Week 2. I doubled Week 2 and I got Week 3.
You get it?

The hesitation in Carter's explanation suggested a struggle he experienced in beginning to articulate the explicit connection between coordinating doubling in height with additive increases of 1 week. Carter also placed each plant's height on the same vertical line, which could have also accounted for his hesitation in explaining his thinking. Additional drawing tasks that included drawing heights for missing weeks continued to support this coordination.

Tables in which the inputs increased by nonuniform amounts also encouraged more explicit attention to time and the need to coordinate growth in height with growth in time. For example, the students from TE1 worked with a table of values requiring the determination of the plant's height at Week 10 (Figure 5). Laura doubled the inches for each successive week by filling in the gaps in the

Table 2. Early covariational reasoning.

Component Understanding	Definition	Data Examples	Tasks
Cov1) Implicit Coordination	Students understand that the *y*-value grows by a constant multiplicative factor "each time," but the time values are not explicitly quantified.	TE1, Jill: "It's going up by 3 each time." TE2, Jose: "The plant doubles in size from the week before each time. So 2 it goes 4, 4 it goes 8."	This was not a targeted conception, but could be encouraged through tasks requiring students to reflect verbally and in writing on relationships between height and time; tasks requiring students to draw pictures of growing plants for different week values and evaluate others' pictures; tasks requiring students to make sense of nonuniform tables of data.
Cov2) Explicit Coordination for 1-Unit Changes	Students can coordinate multiplicative growth in *y* with change in *x* for $\Delta x = 1$. If students achieve reversibility at this stage, it means they can take the ratio of two consecutive *y*-values (for $\Delta x = 1$) in order to determine the growth factor.	TE1, Laura: Filled in missing gaps in a table of data by doubling the inches for each successive week: "For 4 (weeks) I got 48 (inches)."	Tasks of drawing height values at specific week values; tasks with nonuniform tables of data and missing entries.
Cov3) Explicit Coordination for Multiple-Unit Changes (Repeated Multiplication)	Students can coordinate the change in *y*-values for multiple-unit changes in *x*-values, but their mental imagery is grounded in the actions of repeated multiplication. Achieving reversibility means students can determine growth factors by imagining the *y*-values repeatedly multiplying Δx times.	TE2, Paj: "I took this number (the height at Week 16) divided by this number (the height at Week 14) and get 9, and I tried to do this number (the height at Week 14) times 3 … 3 times 3 equals 9 and then got this number (the height at Week 16)."	Tasks with two or three data points between 2 and 5 weeks apart, requiring students to either (a) determine missing height values, or (b) determine unknown growth factors.

table, and she explained, "For 4 (weeks) I got 48 (inches)." Laura's language reflected an explicit attention to both weeks and inches, but she could only double the previous week's height to find the next week's height. Laura did not yet understand that she could, for instance, find the height at Week 10 by taking the height at Week 8 and multiplying it by 4, or 2^2 for a jump of 2 weeks. This component understanding is characterized by the students' need to include every missing value in the table in order to accommodate weeks, rather than being able to coordinate growth in height with growth in time for multiple-week jumps.

Introducing tasks in which students had to think about how to move directly from one time (e.g., Week 0) to another time (e.g., Week 5) encouraged the coordination of multiplicative growth in height with additive growth in weeks for multiple-week values. Tasks requiring the determination of an unknown growth factor given two week and height values also helped students begin to connect the repeated multiplication action with the increase in the number of weeks. For example, Evan's work (TE2) exemplifies this coordination for multiple-unit intervals when he encountered a table with three data points, (0, 1); (5, 1024); and (18, 68719476736). He drew a line between the 1 at 0 Weeks and the 1024 at 5 weeks and wrote, "$\times 4 \times 4 \times 4 \times 4 \times 4$," coordinating repeated multiplication by 4 five times with the 5-week increase in time.

The teacher-researchers' actions also encouraged the students to coordinate repeated multiplication with an additive increase in weeks. For instance, in TE1 the teacher-researcher drew a picture of the Jactus at Week 0 and then asked the students to predict what they would have to calculate in order to jump directly from (a) Week 0 to Week 2, and (b) Week 0 to Week 3. Uditi explained that to find the height at Week 2, one would have to multiply by 4 twice: "From this (gestures to Week 0) to this (gestures to Week 1) it's times 4, so 4 times 4 is 16." Her explanation for part (b) is similar:

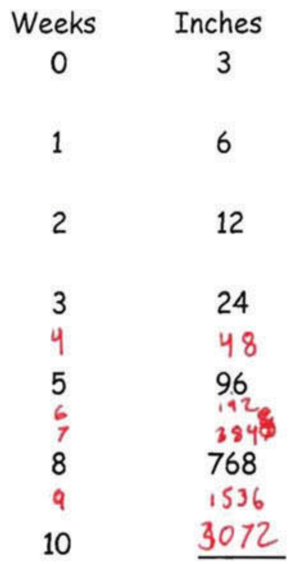

Figure 5. Laura's table of values for the doubling Jactus.

"Times 4 times 4 times 4." The teacher-researcher asked the students to reflect on what they would have to do in order to jump directly from the height at Week 0 to the height at Week 7, and after thinking, Uditi replied, "Four times 4 is 16, 16 times 4 is 64, 64 times 4 is 256 then … 1024 times 4, 4096, times 4 and then it's 16,384."

A commonality in Uditi's and Evan's actions is that both needed to account for each step between the multiweek intervals, coordinating an increase in one week with the action of multiplying by the growth factor. The teacher-researchers eventually introduced the notation b^m as a way to express $b \times b \times b \ldots m$ times, and although the students quickly made use of this notation, for them it was a representation of the action of repeatedly multiplying by the growth factor b. Students also developed the reverse understanding, determining unknown growth factors by dividing two height values and imagining what number they could repeatedly multiply the correct number of times. For example, given two data points (14, 9565938) and (16, 86093442), Paj (TE2) divided the two height values to get 9. She then took the first height value and multiplied it by 3 twice to result in the second height value. At this point, however, the students' reliance on mental images of repeated multiplication limited their coordination to small-week intervals, intervals for which they could imagine repeatedly multiplying by the growth factor. It was difficult to generalize this thinking to an interval that was arbitrarily large or small.

Correspondence reasoning

Many of the students' ideas about exponential growth as a correspondence relation emerged simultaneously with their covariation reasoning. We refer to these ideas as correspondence reasoning because they reflect the students' ideas about the direct relation between x and y, rather than students' ideas about the rate of change of y for corresponding changes in x (Table 3). For example, for a function $y = 2^x$, students may think about y as the result of taking 2 to the x power, relating the plant's height of 8 inches to the number of weeks it had been growing, 3, by thinking of 8 as 2^3.

When the students understood that the growth factor b represents the multiplicative change in height per week, they began to algebraically express this relationship. Tasks encouraging students to describe generally how to determine the plant's height on any given week fostered the correspondence view. For example, Carter (TE2) responded to this question for a doubling Jactus with an initial height of 1 inch. He explained the equation "$y = 2^x$" by stating, "y is the number of weeks and then if you times it by 2 to the power of x then it's like, that'll gives you the height that it goes for each week." When the teacher-researcher followed up by asking the students to think about how that would help the students determine the plant's height at Week 6, Benito explained, "It'd be 2 to the 6th power," elaborating, "It's doubling [for] 6 weeks so you power it to the sixth power."

Both Benito and Carter's language revealed a common phenomenon in that the equation $y = \mathrm{b}^x$ was a representation of both a direct correspondence relation ("you times it by 2 to the power of x," or "you power it to the sixty power") *and* an expression of covariation ("the height that it goes for each week"). When Benito said that the plant had been multiplying for six weeks, he appeared to be thinking about the plant's growth covariationally. The equations the students wrote often reflected this orientation. This was likely a result of the instructional emphasis enacted in the teaching experiments, as the teacher-researchers designed the task activities to encourage the coordination of co-varying quantities early in the sessions.

The next two rows of Table 3 address the students' component understandings of the parameter "a" in $y = \mathrm{a} \bullet \mathrm{b}^x$. The students across both teaching experiments conceived of the initial height "a" as either the "starting value," meaning the initial height at Week 0 one can use to begin the repeated multiplication process, or as a multiplicative constant, the value that multiplicatively changes the height at any given week by the constant a. An example of the first component understanding occurs in Uditi's language from TE1. She explained why a doubling Jactus plant needed to be multiplied by 0.2 inches, the initial height: "Because that's the starting number and you start multiplying from that point." Uditi considered the 0.2 inches to represent the value at which one begins the multiplication process. In contrast, Evan (TE2) compared two doubling Jactus plants, one with an initial height of 1 inch and the other with an initial height of 3 inches. Evan wrote two equations, "height = $1 \bullet 2^{\#\text{ of weeks}}$" and "height = $3 \bullet 2^{\#\text{ of weeks}}$," and explained, "So this part of the equation (pointing to the initial height 3) means how many times the original value is multiplied." For Evan, the 3 in "height = $3 \bullet 2^{\#\text{ of weeks}}$" acted as a multiplicative constant.

The fourth row of Table 3 addresses students' understanding of the growth factor b in $y = \mathrm{a} \bullet \mathrm{b}^x$. We introduced tasks encouraging students to compare plants with different initial heights and different growth factors in order to determine which plant would grow the fastest in the short term and in the long term. Uditi, Laura, and Jill's conversation from TE1 demonstrates the nature of this understanding. They encountered a task in which the blooming Jactus was 10 inches tall when it began growing, and it doubled each week. The evergreen Jactus, in contrast, tripled each week but was only one inch tall when it began growing. The students all agreed that the evergreen would be taller in the long run, with Jill explaining, "Because it triples so like when you keep going in the weeks, it's going to be bigger than the one that doubles." The teacher-researcher introduced other scenarios in which the blooming Jactus began with a height of 100 inches and 1000 inches. The students maintained their belief that after enough time, the evergreen would always pass the blooming Jactus, because "someday going to catch up and it's going to get bigger than that." In subsequent conversations, the teacher-researcher varied the initial height values and the growth factors, and the students maintained the idea that as long as one growth factor was larger than the other growth factor, the plant with the larger growth factor would eventually overtake the plant with the larger initial height but smaller growth factor. Calculating heights for different weeks and creating and comparing graphs supported this notion.

The students eventually encountered tasks in which they had to determine the height of a plant for a large interval, such as 20 weeks, given a known initial height and growth factor. For instance, given a doubling Jactus with an initial height of 3 inches, the students in TE2 had to determine the plant's height at 20 weeks. All of the students acknowledged that they could begin with a height of 3 inches and repeatedly double the height until they reached 20 weeks, but this process was sufficiently cumbersome that they preferred the direct correspondence relation. Carter explained that he would determine the plant's height by calculating 3×2^{20}. Generalizing his thinking, he then explained that he could just as easily find the plant's height at 50 weeks or 1000 weeks by calculating

Table 3. Correspondence reasoning.

Component Understanding	Definition	Data Examples	Tasks
Cor1) Algebraic Representation of Repeated Multiplication	Students express the repeated multiplication pattern for a growth factor b algebraically as $y = b^x$.	TE2, Carter: "*y* is the number of weeks and then if you times it by 2 to the power of *x* then it's like, that gives you the height that it goes for each week."	This was not a targeted conception, but could be encouraged through tasks requiring the determination of the plant's height for any given week; tasks requiring the determination of the plant's height for Week *x*.
Cor2) Initial Height is a Multiplicative Constant	Students view the initial height value (or the "a") in $y = ab^x$) as the value magnifying the height at any given week by the constant "a." Thus the height value for any week *k* is transformed to $a \cdot k$.	TE2, Evan: "So this part of the equation (pointing to the initial height) means how many times the original value is multiplied."	Contexts with initial heights other than 1 inch; tasks comparing plants with different initial heights and same growth rates.
Cor3) Initial Height is the Starting Value	Students view the initial height value "a" as the value at which the multiplying process begins.	TE1, Uditi: "Because that's the starting height and you start multiplying from that point."	This was not a targeted conception, but could be encouraged through tasks requiring students to reflect how many times larger a plant at *n* weeks is compared to the initial height.
Cor4) Effect of Growth Factor	Students understand that the growth factor has a greater effect on the plant's ultimate height than the initial height, thus for sufficiently large *x*-values, the value of *y* depends more on the growth factor than on the initial height.	TE1, Jill: "The evergreen … because it triples, so like when you keep going in the weeks, it's going to be bigger than the one that doubles."	Tasks comparing plants with different initial heights and growth factors; comparing and creating graphs for plants with different initial heights and growth factors.
Cor5) Correspondence Relation, Whole Numbers	Students understand that one can determine an unknown *y*-value for any given whole-number *x*-value according to the relation $y = ab^x$.	TE2, Kathy: "For the 3-inch one, *y* equals 3 times 2 [to the] *x*."	Tasks requiring the determination of *y*-values for very large *x*-values; tasks requiring the determination of *y*-values for any general *x*-value, given an initial height other than 1 inch.
Cor6) Correspondence Relation, Fractions	Students understand that one can determine an unknown *y*-value for any given *x*-value according to the relation $y = ab^x$, including fractions and decimals.	TE1, Uditi: Writes $y = 1 \times 2^{o.5}$ and explains, "1 is the starting number and … I'm trying to put that number 0.5 because that's the week."	This was not a targeted conception, but could be encouraged through tasks requiring the determination of *y*-values for nonwhole-number *x*-values, such as determining the plant's height at 0.5 weeks.

3×2^{50} and 3×2^{1000}, respectively. Kathy then shared a general formula, $y = 3 \bullet 2^x$, explaining the growth factor 2 as representing the plant's height doubling and the x as representing the number of weeks. The students were able to write equations for many different Jactus scenarios, varying both the initial height and the growth factor. Evan (TE2) explained the utility of this approach, writing, "You can do it the long way, which is doubling the starting point for each week, or you could use the equation $y = \text{b(starting point)} \bullet 2^x$." Evan's inclusion of "starting point" was his way of explaining what the parameter "b" represented.

The last row of Table 3 addresses a similar component understanding, that students are able to determine unknown y-values given an x-value. The only difference between this component understanding and the prior one is that now students can determine y-values for non-whole-number x-values. For example, Uditi (TE1) wanted to check the correctness of the height of a doubling Jactus plant at 0.5 weeks when the plant's initial height was 1 inch. She wrote "$1 \times 2^{o.5}$" and explained, "1 is the starting number and … I'm trying to put that number 0.5 because that's the week." The achievement of this understanding came later than the prior understanding, and we decided that the challenge the students appeared to experience in achieving it merited categorizing this component understanding separately.

Covariation reasoning

As discussed, the students' correspondence reasoning and covariation reasoning emerged in tandem, rather than one occurring before the other. More sophisticated covariation reasoning included the ability to explicitly coordinate the ratio of y-values for multiple-unit changes in x, including, eventually, scenarios in which the interval was less than one. Students' abilities to make this transition required a shift from mental images of repeated multiplication to imagery that did not rely on a repeated multiplication model. This transition began with the students' process of re-unitizing, which is described in the first row in Table 4.

Re-unitizing emerged as a way of thinking as the students began to grapple with imagining ratio change in y for larger intervals; students would take a new chunk of time, such as three weeks, as a unit, and then operate on that unit as the basis for coordinating multiplicative growth. Carter (TE2) wrote a table to organize the growth of a tripling Jactus, but he ordered the weeks in units of 2 rather than 1. Writing the ordered pairs (10, 59049), (12, 531441), (14, 4782969), and (16, 43046721), Carter wrote "×9" between the successive height values in the table. Re-unitizing appeared to be the first step in fostering a transition to imagining exponential growth for larger intervals and non-standard intervals. Re-unitizing thinking coincided with the initial shifts in students' language away from repeated multiplication imagery. This shift may have been motivated in part because one must conceive of growth embedded within growth in order to repeat a process of repeated multiplication for multiweek intervals of time. For example, repeated multiplication imagery involves thinking of a plant growing twice as tall for one week, repeating that process three times for a new interval of three weeks, and then repeating the entire embedded process for multiple three-week intervals. It may have made more sense to instead shift to thinking about the plant growing eight times as tall for an interval without imagining that process as a repetition of $\times 2 \times 2 \times 2$.

Prior to this point, the students' reliance on images of repeated multiplication constrained their ability to make sense of nonwhole number intervals in time as a measure of growth. Although students could use their correspondence reasoning to write equations such as $b^x = y$, for large values of x they could only conceive of this equation as a relation between a static height value, y, and a static point in time, x. An equation such as $b^{\Delta x} = R$ as a representation of a measure of growth (with R = ratio of two y-values, y_2 and y_1 and Δx = the change in corresponding x values, x_2—x_1) would not have meaning to students relying on images of repeated multiplication if Δx represented an interval that was, for instance, a very large number of weeks—perhaps too large to mentally coordinate an image of repeated multiplication—or a value less than one week.

The second row in Table 4 describes students' abilities to coordinate growth in height for larger intervals, shifting to images such as scaling and smooth change. This shift was supported by activities with the Geogebra script that modeled continuous scaling motions, as well as by prompts to draw graphs of exponential functions without specific values. The teacher-researchers explicitly attempted to foster a different type of imagery for growth among the students by re-directing their attention to the Jactus plant growing continuously over time in the Geogebra script. They asked students to describe verbally and in gestures the nature of the plant's growth between intervals, and introduced tasks in which the students were encouraged to draw and graph general exponential functions (rather than specific functions for which they could plot points). In addition, students encountered tasks in which they had to (a) determine the growth factor of a plant with only two data points with a large-week interval in between, or (b) use the growth factor to determine a new height value over a large-week interval. We hypothesized that providing only two height values for a large interval in time would encourage thinking that was not reliant on repeated multiplication, because it would be

Table 4. Covariation reasoning.

Component Understanding	Definition	Data Examples	Tasks
Cov4) Re-unitizing	Students create a new unit out of a multiple-unit change in x in order to operate on the new unit multiplicatively.	TE1, Uditi: "It's 8 for 3 weeks … 8 times 8, I mean 8 times 8, yeah, equals 64 for the 6 weeks."	Tasks requiring students to determine the growth factor for different periods of time for the same data; e.g., "How does this plant grow each week, every three weeks, and every half a week?"
Cov5) Explicit Coordination for Multiple-Unit Changes (Exponentiation Imagery)	Coordination of the ratio of y-values for any $\Delta x > 1$; students no longer rely on repeated multiplication imagery.	TE2, Carter: "This (height of 6144 inches at 12 weeks) times 2 to the power of 7 (yields the height of a doubling Jactus at 19 weeks)."	Prompts to draw non-specific exponential functions without plotting points; tasks with only two data points with large x-value intervals.
Cov6) Coordination for Multiple-Unit Changes to Determine New y-Values	Students determine new height values through coordinating multiplicative change in y with additive change in x; in particular, they no longer need to rely on the initial value but can instead determine y_2 from any y_1 by multiplying y_1 by $b^{(x2\text{-}x1)}$ for a growth factor b.	TE1, Uditi: (Multiplying 0.6×3^{13}): "0.6 is the week number 1, and then 3 is how much it goes by and 13 is the difference between these two [Pointing to Week 1 and Week 14]."	Tasks requiring the determination of y-values for large x-values; Prompts to determine whether a given point in a table of values is correct or incorrect.
Cov7) Reversibility (Determining Growth Factor)	Students can determine an unknown growth factor b by coordinating the ratio of two y-values, y_2 and y_1, with the corresponding difference in x-values, Δx, taking the Δx'th root of y_2/y_1.	TE2, Benito: [Given two points, (8, 25.6) and (18, 26,214.4), Benito took the tenth root of 26,214.4/25.6 = 1024]: "If it's 26,214.4 divided by 25.6, equals 1,024. And then, 1,024, you do that to the number of weeks."	Tasks or tables with provided data points requiring students to determine an unknown growth factor. Data points with large intervals between the x-values particularly target the development of this idea.
Cov8) Coordinating Multiplicative Change in y with Additive Change in x, any Δx	Coordination of the ratio of y-values for any change in x, including when Δx is < 1.	TE1, Uditi: [Given a plant that triples each week, how much does it grow in a day?] "For one week there are 7 days, and I divided 1 by 7 … and I got 0.14, so I did 3 over 0.14." [Writes $3^{0.14} = 1.17$].	Tasks prompting students to determine the nature of growth for Δx values < 1.
Cov9) Constant Change in x Yields Proportional Multiplicative Constant Change in y	Students understand that for any Δx, the ratio of the two corresponding heights y_2 to y_1 will be $b^{(x2—x1)}$ and does not depend on the individual x_1 or x_2 values.	TE1, Jill: [How many times taller would a plant with a growth factor of 2 grow from Week 100 to Week 103?] "It's going to always go up by the same amount. It's always going to double every week and go up by 8 every 3 weeks."	Tasks requiring students to determine the growth factor for different time periods; tasks requiring students to predict the growth factor for time periods with values too large to calculate.

too difficult to mentally keep track of the multiplication process a great many times (such as for 10 or 15 weeks). For instance, Troy (TE2) was able to determine that a doubling Jactus grew 1024 times as large every 10 weeks without writing an equation, explaining, "I did 2 to the power of 10." The expression 2^{10} represented, for Troy, how many times larger the plant would grow over a 10-week interval.

It was not always straightforward to clarify the nature of the students' mental images, because they did not easily articulate their thinking. Instead, we made inferences based on the students' gestures and drawings, clues from their language as they solved problems, and noticeable shifts in students' strategies. For example, one strategy shift was evidenced by students becoming able to easily determine a new height value by multiplying an existing height value by $b^{\Delta x}$, when he or she could previously not do so for large intervals or for nonstandard intervals. An example of a shift in students' language is evident from Uditi (TE1) as she determined the growth factor of a plant that was 256 inches at 4 weeks and 1,073,741,824 inches at 15 weeks. In prior days, the students struggled to make sense of this problem because the 11-week interval was too large to coordinate imagined repeated multiplication. Now, however, both Uditi and Jill could take the ratio of the two height values (4,194,304) and write the equation "$\_\_\_^{11}$ = 4,194,304" as a way to express the relationship between additive growth for 11 weeks and the quotient of the two height values. When asked to describe her thinking that led to this equation, Uditi stated, "It's growing in an exponential growth," with an accompanying hand gesture indicating a smooth process of increasing height over time. Uditi's shift in language when describing the growth provided additional evidence that her images of growth had changed.

Students' shifts to coordinating multiplicative growth in y for large intervals fostered the use of this coordination in order to determine new y-values, as described in the third row of Table 4. As an example, Carter (TE2) examined a plant with a height of 1024 inches at 5 weeks and 68,719,476,736 inches at 18 weeks. Guessing that the growth factor was 4, Carter multiplied 1024 inches by 4^{13} to confirm the new height. Carter explained that he multiplied the height at 5 weeks by 4^{13} because "five is 13 weeks away from 18," and "Four to the power of 13, that way, you don't have to keep timesing it by 4 every time." The students' use of exponential notation truncated the repeated multiplication process and further supported a shift in thinking to multiplicative growth that could occur for large or nonstandard intervals.

The students' growing comfort with the exponential notation $b^{\Delta x}$ as a representation of multiplicative growth was evidenced in their eventual use of the root operation to determine unknown growth factors. In particular, once the students had interiorized multiplying a height value by $b^{\Delta x}$ as way to multiplicatively scale that value by b for Δx weeks, they could then think about the reverse operation, taking the Δx'th root of y_2/y_1 in order to determine b. This reasoning can be seen in Benito's (TE2) work in determining the growth factor b given two points, (8, 25.6) and (18, 26,214.4) (Figure 6).

Benito explained, "See this, if it's 26,214.4 divided by 25.6, equals 1024. And then, 1024, you do that to the number of weeks." Benito was not familiar with formal language such as taking the tenth root, but his written work indicates that he did take the tenth root of 1024. He struggled to explain what operation the tenth root accomplished: "If you divide that 10 times, if you divide it … wait. If you *do it down*, and then by the number of weeks that went by." It is reasonable that Benito first tried to describe the operation as repeated division, since he conceived of it as the opposite operation as exponentiation, but Benito then adjusted his language. Later, he explained the operation as "We're seeing what times what equals that number, but it has to be the same number squared. So you square root it." Similarly, Evan (TE2) could describe this strategy in general terms: "I divide the height at more weeks by the height at less weeks. And then I find the difference in the number of weeks, and then I use that to root the quotient of the height." Benito and Evan understood the quotient, 1024, as how many times taller the plant became in 10 weeks;

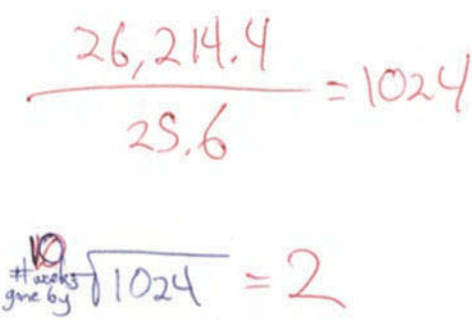

Figure 6. Benito's method for determining the growth factor from two height values.

1024 was the growth factor for an interval of 10 weeks. The students could take the tenth root to determine the growth factor for 1 week, 2, because they understood that if 2 was the growth factor for an interval of 1 week, 2^{10} must be the growth factor for an interval of 10 weeks. However, we do not have evidence that Benito or Evan would have understood this relationship for partial intervals, such as an interval of half a week or one day.

It was necessary for students to be able to coordinate multiplicative growth in y with additive growth in x for large intervals in a manner that no longer relied on images of repeated multiplication in order to then make sense of cases in which the time intervals were less than one. This occurred with Uditi's (TE1) thinking about a problem with only two data points. The plant had a height of 12.513 inches at 2.3 weeks and 13.967 inches at 2.4 weeks, and the students had to determine how the plant grew every 0.1 weeks and every week. All of the students took the ratio of the two height values, which was approximately 1.116, and stated that the plant would grow 1.116 times as tall for each tenth of a week. While Jill and Laura then struggled to use this information to determine how much the plant would grow in one week, Uditi wrote "$\_\_^{0.1} = 1.12$", indicating that the blank represented the weekly growth factor. At this point Uditi could now write an expression with a decimal exponent representing a measure of growth rather than a static value, explaining that the exponent, 0.1, represented "the difference between the height [between Week 2.3 and Week 2.4]" and the quotient, 1.12, as how much the plant "goes 1.12," which appeared to mean how many times taller the plant grew in 0.1 week. Uditi could then determine that the weekly growth factor was 3, explaining 3 as "the difference between the height for like every week, like a change of 3."

In a separate task on a different day, the students in TE1 again determined the growth factor to be approximately 1.116 for an increase of 0.1 weeks. Uditi was able to relate the growth for 0.1 weeks to the growth for 1 week by raising the growth factor 1.116 to the tenth power, concluding that the weekly growth factor was 3. The teacher-researcher asked Uditi, "How come it's to the tenth power instead of times 10?" and Uditi replied, "Because it grows, like, up for this one [indicates movement from 0 weeks to 0.1 weeks], and then you'll have to do that again to get this one, it's, like, two times to get this one, [indicates from movement from 0.1 weeks to 0.2 weeks] so, and then it goes by 10." This suggests that Uditi may have constructed an image of the plant's height stretching in a scaling-up process such that at each time point (i.e., at 0.1 weeks, at 0.2 weeks, etc.), the height was 1.116 times as tall as it was at the prior time point.

We designed and implemented two types of tasks to support students' thinking of exponents as measures of growth rather than as static values for intervals less than one. First, we encouraged students to imagine what occurred with the plant's growth between intervals. For example, if a plant triples each week, what happens after 1 day, or 1 hour? We encouraged discussion about how the plant grew in between a given interval. Second, we developed tasks that required a shift from calculation to anticipation. The students typically solved problems that provided specific height and time values, and they took ratios in order to determine growth factors. A task we introduced at the end of the teaching experiment provided the growth factor, or a way to easily determine it, and then asked students to predict what the ratio of y-values would be for different time intervals. These tasks required students to work with a known interval ($x_2—x_1$) and a known growth factor b and then anticipate how to determine, rather than calculate, a corresponding ratio y_2/y_1. Relying on numbers that were too large to directly calculate may have encouraged students to think directly about the relationship between the growth factor, b, and the interval, ($x_2—x_1$).

The final component understanding in Table 4 addresses the idea that a constant additive change for x yields a constant ratio for y, regardless of the individual x-values; for instance, for a function with a growth factor of 3, y will grow 9 times as large for any 2-week interval, regardless of whether that interval occurs between $x = 1$ and $x = 3$ or between $x = 101$ and $x = 103$. Troy (TE2) demonstrated an awareness of this idea when he sought to confirm that a plant with a growth factor of 4 would grow 64 times as tall for 3 weeks. Working with a uniform table of data with x-values ranging from 3 to 8, Troy took the height value at Week 6, which was 4,096 inches, and divided it by the height value at Week 3, which was 64 inches. Finding the quotient to be 64, Troy then suggested that a way to check his work would be to take the quotient of two height values for a different 3-week interval, between Week 4 and Week 7: "Try 256 [The plant's height at Week 4] divided by Week 7."

Students' progressions through the learning trajectory

The existence of a learning trajectory does not imply that all students will progress through the component understandings in an identical or similar path. In tracing the trajectories of individual students through both teaching experiments, we identified multiple paths through the trajectory. However, despite this variation, there also appeared a general progression of ideas that manifested in different ways for different students. In order to discuss these similarities and differences, we present three students' progressions through the learning trajectory. All three progressions identify a parallel development of covariation and correspondence reasoning, which we discuss in more detail in the next section.

As a reminder, the final version of the EGLT was not in place before the development of the tasks used in both teaching experiments, but rather emerged after the enactment of both teaching experiments. It is the result not only of the retrospective analysis process, but was also influenced by the initial hypothetical learning trajectory and by the understanding the students demonstrated at the beginning of the teaching experiments. Thus the initial task sequences and the final learning trajectory are related, but the component understandings in the trajectory emerged after the task development process.

We present the task sequences enacted in the two teaching experiments in Figure 7. The graphs in Figure 7 depict the progression of tasks and the component understandings targeted by each task; this was determined after the completion of the final EGLT. Each graph is divided into days of the teaching experiments, with TE1 consisting of 12 days and 54 tasks and TE2 consisting of 9 days and 37 tasks. The prefunctional stage is depicted in orange, early covariation stage in blue, correspondence stage in purple, and the more sophisticated covariation ideas are shown in green.

Because the final learning trajectory was developed after the task sequences were developed, the graphs depict the component understandings targeted by each task in relation to this final version,

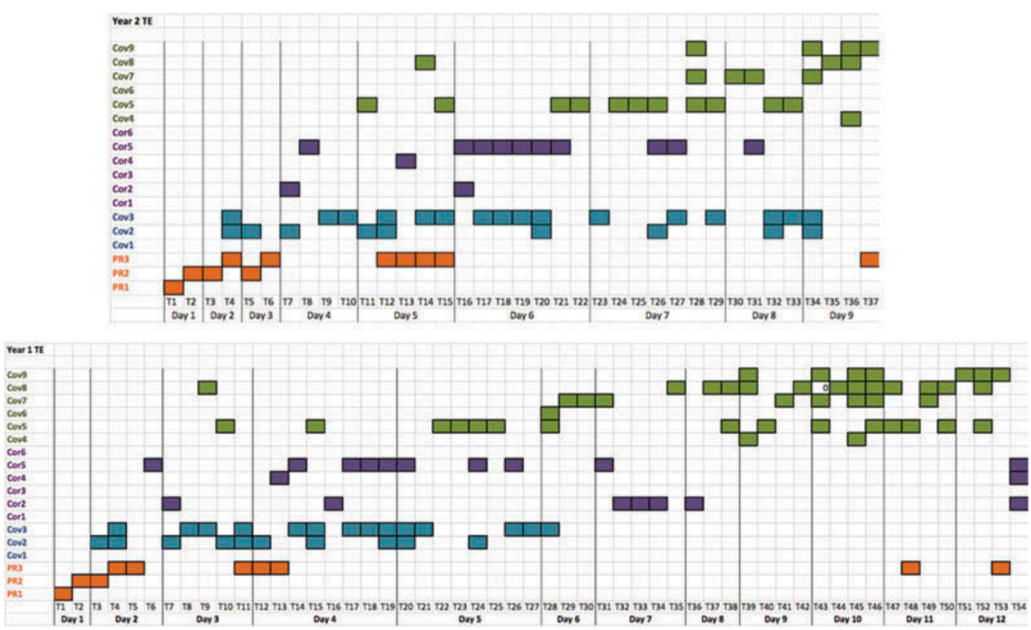

Figure 7. Target component understandings for each task in the teaching experiments.

rather than to the hypothetical version in place before the enactment of the teaching experiments. Thus some component understandings on the trajectory are not targeted by any of the tasks. For example, this is the case with Cov1, the implicit coordination of the change in y-values with the change in x-values. This is a component understanding not targeted by any task because our intention was to foster explicit coordination (Cov2). Cov1 emerged from our retrospective analysis, and thus is part of the learning trajectory even though it was not an idea deliberately fostered by any of the tasks. The same is true for the component understandings Cor1, Cor3, and Cor6, which were all emergent understandings that were not targeted by the task sequence.

An important distinction between the two teaching experiments was that the second teaching experiment only implemented about two thirds of the tasks used in the first teaching experiment. This occurred due to the limited time frame of the second teaching experiment, which lasted only 9 days instead of 12 days. Moreover, the tasks that were not implemented were the later tasks, which were specifically aimed at sophisticated covariation reasoning. The lack of inclusion of these tasks is evident in the individual trajectories of the majority of the students in the second teaching experiment.

Individual students' trajectories

All of the students began with prefunctional reasoning. The students in TE1 then began to engage with early covariational reasoning and then with correspondence reasoning, whereas the students in TE2 more predominantly engaged with early covariation and correspondence reasoning concurrently. This may be due to differences in the students' experiences with functions; the students in TE1 were eighth-grade students with fewer classroom experiences with function families than the students in TE2, who were rising ninth graders and may have been more comfortable with correspondence reasoning in general.

All of the students roughly progressed in similar manners throughout the major stages of the learning trajectory; individual differences appeared in (a) the degree to which students could engage in sophisticated covariation reasoning, and (b) the prevalence and timing of the development of correspondence reasoning. In this section we present the learning trajectories for three students: Uditi, Jill, and Evan. In each case, the student's trajectory is depicted along with the task trajectory from Figure 7 superimposed on the same representation. The colored portions represent the student's actual trajectory, and the black outlines represent the superimposed task trajectory. This enables a comparison between the intended component understandings targeted by each task and the component understandings for which each student demonstrated evidence on that task. In some cases, the evidence for a particular understanding was considered weak evidence by the research team; these are depicted with lighter-colored boxes. Weak evidence occurred in cases in which the research team thought it likely that a code applied to a student's thinking, but the data were open to multiple interpretations. For example, in one case Uditi determined that a Jactus grew 4,194,304 times as tall from Week 4 to Week 15. When asked whether the plant would always grow 4,194,304 as tall for any 11-week interval in time, and she wrote, "Yes, because it is the rate in which it is growing for every 11 weeks and the equation shows that too." Although her response appeared to indicate an understanding that a constant interval in x yields a constant multiplicative growth in y (Cov8), her written explanation was not conclusive and the transcript did now show any additional discussion of this idea. Thus, we considered it plausible that Uditi's understanding of Cov8 was emerging.

Uditi (TE1). Uditi's progression through the learning trajectory is depicted in Figure 8. The n/a for Task 22 indicates that Uditi did not complete that task, and the 0 depicted in Task 43 indicates that although Cov8 appears to be a targeted component understanding for that task, because it is outlined in bold, it was not targeted; instead, the targeted component understandings for Task 43 were Cov5, Cov7, and Cov9.

Uditi is one of the students who developed some of the most sophisticated and flexible ideas about exponential growth, with her progression through the learning trajectory demonstrating

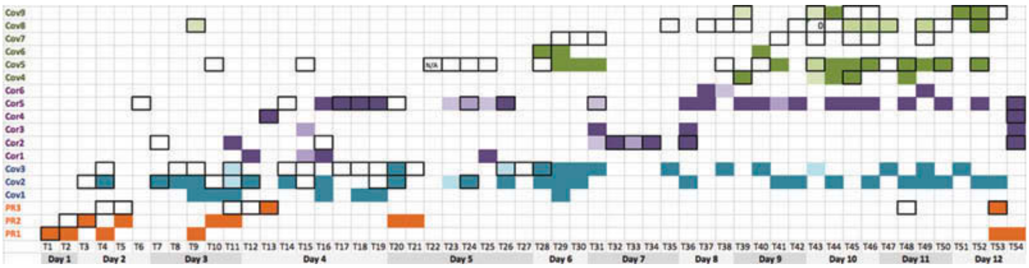

Figure 8. Uditi's progression through the learning trajectory.

multiple instances of the higher covariation understandings. Initially Uditi engaged in early covariation reasoning, and beginning on the third day, she demonstrated evidence of correspondence reasoning, which continued throughout the rest of the teaching experiment. Early covariation reasoning and correspondence reasoning co-occurred throughout the first half of the teaching experiment. During the second half of the teaching experiment, Uditi developed sophisticated covariation reasoning, which became a strong presence in her thinking overall while simultaneously not eliminating earlier forms of reasoning. Uditi's progress shows that early covariation ideas, correspondence reasoning, and sophisticated covariation ideas all mutually occurred. For example, Uditi's strong covariational reasoning enabled her to develop a more sophisticated correspondence strategy in which she did not have to rely on "the starting value" to find the height value at a given time and thus build a correspondence equation. Instead, she could take a height value at any point in time and multiply it by the growth factor raised to the power of the relevant time interval. For example, when encountering a tripling Jactus with only two height values at Week 14 and Week 16, Uditi had to determine whether a height of 0.6 inches at Week 1 was correct. In order to check, Uditi multiplied 0.6 by 3^{13} to see whether the new height was the same as the table's value at Week 14. Uditi could conceive of the expression $0.6(3^{13}) = 956{,}593.8$ as a representation of how many times taller the plant grew in a 13-week interval.

Jill (TE1). Jill's progression can be seen in Figure 9. Jill was absent on Days 3 and 8, as shown by the greyed out columns. Although Jill's progression through the trajectory has many similarities to Uditi's, one can observe more gaps in her early covariation understanding. Jill engaged in covariation reasoning less than Uditi did throughout the teaching experiment, and she relied heavily on her correspondence thinking. During the last three days of the teaching experiment Jill began to develop some sophisticated covariation component understandings, but she did not engage with Cov6 or Cov7 at all, and demonstrated only one instance with weak evidence of Cov8. The third day of the teaching experiment reflected a strong emphasis on building an understanding of covariation, and Jill's absence on that day may have hampered her development moving forward. Because Jill lacked a

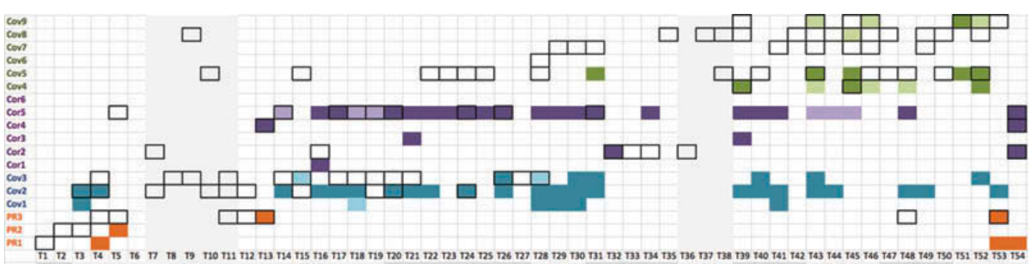

Figure 9. Jill's progression through the learning trajectory.

robust understanding of covariation, she often relied on finding correspondence rules, even when the tasks did not require it. Jill's strong reliance on correspondence reasoning may have also been responsible for the relatively late emergence of more sophisticated covariation component understandings.

Evan (TE2). Evan's progression through the learning trajectory (Figure 10) depicts some interesting contrasts to Uditi and Jill's. (The n/a depicted for Tasks 24 and 25 indicate that Evan did not complete those tasks.) During the first three days of the teaching experiment, Evan frequently demonstrated evidence of component understandings further along the trajectory than what was intended in the tasks. For example, the first task on Day 2 was intended to elicit PR2, the idea that repeated multiplication determines how the y-values grow over time. Evan demonstrated the component understandings Cov2 and Cor1, showing an explicit coordination of the change in y-values for a 1-unit change in x, and representing that coordination algebraically as $y = b^x$. Beginning with Day 4, however, Evan demonstrated evidence of the target component understandings, or understandings below those targeted, similar to Uditi and Jill.

Evan was able to engage in both covariation reasoning and correspondence reasoning, but his progression does not convey a robust grasp of the sophisticated covariation component understandings. In part this occurred because the structure of the second teaching experiment did not always provide Evan with sufficient time to engage with the tasks he found particularly challenging. However, it is notable that Evan demonstrated evidence of one of the most sophisticated covariation component understandings, Cov7, the idea that one can determine an unknown growth factor by taking the Δx'th root of x_2/x_1; this was a component understanding that neither Uditi nor Jill demonstrated. It is also worth noting that Evan only participated in the teaching experiment for 9 days and 37 tasks in contrast to 12 days and 54 tasks. Given more time, we would expect to see a stronger prevalence of sophisticated covariation reasoning in Evan's progression.

The task sequence depicted in Figure 7 is not identical to any one student's progression through the learning trajectory. The students' progressions did not mirror the progression of component understandings the tasks were supposed to elicit, but across all three students one can observe the covariation and correspondence perspectives developing in tandem. A strong emphasis on developing a covariation understanding of exponential growth was present in the tasks we developed and enacted, which reflected our goals as researchers in privileging covariational reasoning. Our belief was that an emphasis on covariation would support the development of meaningful correspondence rules, which was borne out by the data. In general, the individual differences in students' progressions emphasize the point that a given task sequence is not determinative of student conceptual development. One cannot ignore the role of instruction, students' existing ways of thinking and

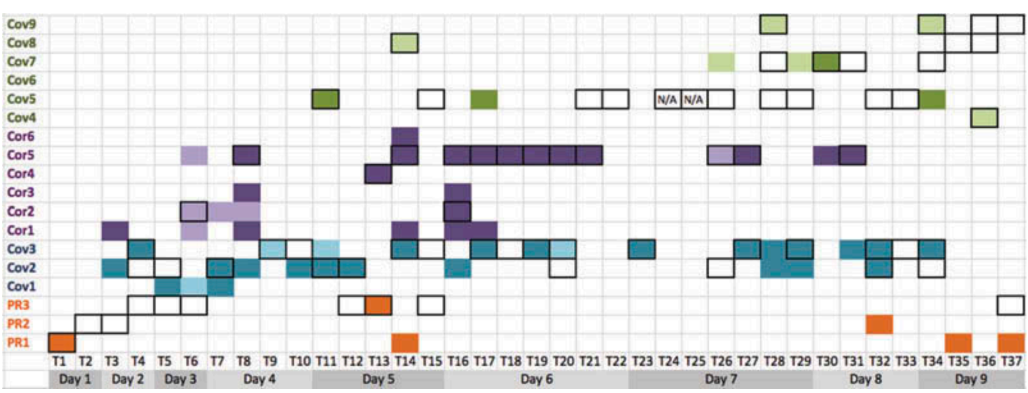

Figure 10. Evan's progression through the learning trajectory.

operating, students' dispositions, the interactions students engage in with one another and with the tools in their environment, and other factors in influencing student learning. However, we found the exercise of identifying individual student progressions helpful in supporting our goals of understanding why the students developed the ideas that they did in relation to their mathematical activity when engaging in the provided tasks.

Discussion

The construction of learning trajectories is a critical task facing mathematics education researchers (Steffe, 2004), particularly for challenging or poorly understood conceptual areas. By offering an empirically based learning trajectory articulating the development of students' initial understanding of exponential growth, our aim has been to contribute to the field's knowledge of students' learning processes and how they can be supported. Our findings suggest that situating an exploration of exponential growth in a scenario in which students can manipulate continuously covarying quantities in a dynamic environment can foster their ability to coordinate multiplicative growth in y with additive growth in x, a key element in understanding the nature of exponential growth. As a caveat, we caution that the EGLT is grounded in data from two groups of students whose initial understanding of exponentiation was limited to models of repeated multiplication. Secondary students with more in-depth exposure to exponential growth, particularly those who understand exponential growth as a function, will likely follow a trajectory with different entry points.

In addition, the students' abilities to coordinate the ratio of height values for corresponding time intervals played a significant role in their development of algebraic representations. In general, the students' early covariational thinking preceded their ability to develop correspondence rules of the form $y = f(x)$. This was particularly true for the students in the first teaching experiment, who had not yet had any formal exposure to nonlinear functions in school. This finding lends credence to Smith and Confrey's (1994) assertion that students typically approach functional relationships from a covariational perspective first, although the instructional emphasis on covariation as enacted through the task progression and the teacher-researchers' mathematical emphases certainly played a role as well.

EGLT offers a proof of concept that even with a relative lack of algebraic sophistication, middle school students can generalize their understanding of exponential growth to view b^a as a factors of b, even for non-natural values of a, a finding suggested in theory by Weber (2002), elaborated in Strom's (2008) discussion of partial factors, and borne out in our data for one student in particular, Uditi. Our findings suggest that reasoning with covarying quantities is a critical aspect of building this particular understanding of exponential growth. Although the Jactus context introduced some constraints, particularly in terms of limiting the growth factor to four or less in order to enable numbers small enough for the students' calculators to accommodate, we considered the constraints acceptable because the Jactus scenario offered a context in which students could make meaningful sense of non-natural exponents by imagining the height of the plant growing over time.

The students' abilities to view an image of a plant growing smoothly over time, however, does not mean that they achieved smooth continuous variation thinking (Thompson & Carlson, in press). While it is efficacious to enable students to reason about quantities that, from the researcher's perspective, vary continuously rather than in a discrete manner, our data also indicate that placing students in continuous contexts does not guarantee that students will then reason continuously. Smooth continuous variation would require the ability to imagine variation in the height's value as its magnitude increases in bits while simultaneously anticipating that within each bit, the value varies smoothly. This way of thinking is challenging for middle school students, in part because exponential growth is defined geometrically, with the function values forming a geometric progression. How the students thought about growth covariationally involved considering a new value as the product of a prior value and a growth factor, a manner of thinking that is inherently chunky.

The students had to shift away from repeated-multiplication imagery in order to make sense of larger intervals or partial intervals. Scenarios that afford these images, such as a growing plant, may support this shift better than scenarios that are discrete, such as coins placed on a chessboard in a doubling manner. Some students, such as Uditi and Benito, could ultimately think about variation in the height's value as increasing by intervals of a fixed size. Uditi in particular could consider time varying for different intervals and knew that she could determine the plant's height for any value in between two time values. For example, she knew that if the plant grew for a tenth of a week, there is a constant growth factor for a 1/10th time period, and that growth factor raised to the tenth power yields the growth factor for a week.

Confrey and colleagues (2009) reminded us that a learning trajectory, while emphasizing students' refinement of their own understanding, is also influenced by instruction. The EGLT is a reflection of the research team's emphasis on the coordination of covarying quantities, which is seen in the co-evolution of the covariation and correspondence views both in the progression of tasks and in the progressions of individual students. School mathematics instruction typically emphasizes the correspondence perspective at the expense of covariational reasoning, which may foster a restricted concept image of function, as students are not encouraged to think about change between variables (Thompson & Carlson, in press). Students may therefore miss out on important opportunities to meaningfully engage in correspondence rules, creating expressions and equations that represent a flexible understanding of a constant ratio change in y for each constant additive change in corresponding x-values according to the relation b^{x2-x1}. A focus on correspondence rules alone may run the risk of supporting a shallow understanding of exponential equations that is more procedural than conceptual in nature (Ozgur et al., 2013). In contrast, researchers (e.g., Carlson et al., 2002; Smith, 2003) have suggested that the covariation approach can support more powerful generalizations that can be later expressed algebraically as a correspondence relationship; our findings indicate that this can be the case for exponential growth.

Lesh and Yoon (2004) cautioned against assuming that a student identified as thinking at a particular level in a trajectory will then function at that level across all other tasks. The individual students' progressions through the EGLT lend credence to this warning; in particular, evidence of a student's functioning at a particular component understanding or stage did not result in any student remaining at that stage of thinking in a stable manner throughout the teaching experiment. Instead, our findings indicate that students transition back and forth between different component understandings, continuing to demonstrate some of the foundational early ideas concurrently with more sophisticated ones. Thus, it is important to provide repeated opportunities to reason through tasks targeting the same ideas in order to allow students time to reflect on and solidify their thinking (Ellis, Ozgur, Kulow, Williams, & Amidon, 2015).

Findings from the teaching experiments and the subsequent development of the EGLT suggest a number of instructional implications to support algebra students' emerging understanding of exponential growth. First, students may enter an exponential functions unit without strong functional reasoning in place; this may be particularly true for younger adolescents. It is therefore important to begin addressing ideas of exponential growth by encouraging students to identify the relevant quantities in a situation and to explore how those quantities co-vary. Students in the later grades may be better poised to think covariationally if they enter a unit already reasoning with the Cor1/Cov1 component understandings in place. Providing students with situations in which they can observe, visualize, and manipulate the relevant quantities, such as the Jactus scenario, can support these goals. Further, students' abilities to coordinate growth between quantities may likely be implicit at first, and they may require deliberate support in order to explicitly attend to how both the x-values and the y-values change together.

The EGLT depicts one possible set of understandings students can develop when reasoning with ideas about exponential growth. While it is not prescriptive, the learning trajectory offers a greater understanding of students' learning of a challenging topic along with insights into how particular tasks and instructional moves can support such learning. These findings enabled us to identify a set of instructional and curricular recommendations for supporting students' function understanding. While it will be important to study the effects of scaling up such recommendations to whole-class implementations, a task in which we are currently engaged, the learning trajectory itself serves as a valuable tool for supporting curriculum design and framing pedagogical recommendations. These findings contribute to a body of work aimed at explicating, understanding, and supporting students' learning and development as they engage in mathematically challenging and meaningful ideas.

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