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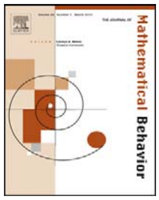

An operational definition of learning

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ABSTRACT

An operational definition offered in this paper posits learning as a multi-dimensional and multi-phase phenomenon occurring when individuals attempt to solve what they view as a problem. To model someone's learning accordingly to the definition, it suffices to characterize a particular sequence of that person's disequilibrium–equilibrium phases in terms of products of a particular mental act, the characteristics of the mental act inferred from the products, and intellectual and psychological needs that instigate or result from these phases. The definition is illustrated by analysis of change occurring in three thinking-aloud interviews with one middle-school teacher. The interviews were about the same task: "Make up a word problem whose solution may be found by computing 4/5 divided by 2/3."

> An *operational definition* is a showing of something—such as a variable, term, or object—in terms of the specific process or set of validation tests used to determine its presence and quantity. Properties described in this manner must be publicly accessible so that persons other than the definer can independently measure or test for them at will. An operational definition is generally designed to model a *conceptual definition* (Wikipedia)

1. Introduction

Conceptual definitions of *learning* successfully convey the messages about different theoretical perspectives on learning, but suffer from the lack of operability. For instance, many frequently used in mathematics education discourse conceptualizations of learning, such as *learning as acquisition*, *learning as participation*, *learning as problem solving* or *learning as assimilation and accommodation*, refer to the key processes involved, but are insufficient in order to operationally capture the essence of the intended change (Sfard, 1998; Skinner, 1950; Von Glasersfeld, 1995).

The need to build research programs on *operational* definitions of learning is recognized by many scholars (e.g., diSessa & Cobb, 2004; Siegler, 1996; Simon et al., 2010; Steffe, 2003). Simon et al. (2010) distinguish between two main operational approaches to studying learning. The first one focuses on fostering perturbations and characterizing the students' learning trajectories by specifying a resulting series of understandings or conceptual steps through which students pass. This approach is consonant with Von Glasersfeld's (1995) description of Piaget's theory of learning, and is exemplified in Simon's et al. (2010) paper by detailed discussion of several studies utilizing micro-generic analysis of students' reasoning while solving challenging for them arithmetical tasks (Steffe & Thompson, 2000; Steffe, 2003). The second approach – presented as recently emerging from Martin Simon's and his colleagues own research – focuses on examination of the *process* by which students

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progress from one of these understandings or conceptual steps to a subsequent one. Fostering perturbation is a possible but not necessary part of this approach. Simon et al. (2010) describe the main differences between the approaches as follows:

> In an approach in which the emphasis is on promoting perturbation, the data generally reveal the *result*, the successful solution to the new problem. Because previously the student was unable to produce a solution of this type and now is able to, there may be little or no data on the learning (change) process. In our approach, the researchers use a sequence of tasks to engender particular activity on the part of the students that foster the intended learning. If the task sequence is successful, the researchers are able to observe the students' activity over the course of the entire task sequence. This provides data on the *process* by which the new learning comes about, that is, evidence of modifications in the students' *thinking* [italics added] as a function of their mathematical activity as they participate in the sequence of tasks (Simon et al., 2010, pp. 107-108).

The scholars note that the approaches are rather complementary than antagonistic, and may highlight different aspects of learning as modeled through different types of situations. Our goal is to take this idea a step further and offer an operational definition of learning which would balance the two approaches. Note: we do not suggest in this paper a *conceptually* new definition of learning, but offer instead an *operational* approach to modeling change in individual's knowledge structures over time in a way accounting for both processes and products of the change. The suggested approach is rooted in Piaget's theory of equilibration and is asserted in terms of a particular conceptual framework, called *DNR-based instruction in mathematics* (Harel, 2008a, 2008b, 2008c). The selected elements from DNR are outlined in Section 2. In Section 3, we illustrate the approach by discussing three episodes with one middle-school teacher. The episodes are about the teacher's attempts to solve the same task: "Make up a word problem whose solution may be found by computing 4/5 divided by 2/3." We conclude, in Section 4, with a discussion of the operational nature of the presented definition of learning and its possible contribution to the existing literature on the topic.

2. DNR-oriented definition of learning

"Learning" in DNR is operationally defined as a continuum of disequilibrium–equilibrium phases manifested by (a) *intellectual needs* and *psychological needs* that instigate or result from these phases and (b) *ways of understanding* or *ways of thinking* that are utilized and newly constructed during these phases. The italicized two pairs of terms in this definition are concepts from DNR; they will be discussed below—the first pair in Section 2.3, and the second pair in Section 2.2. These concepts are oriented within a particular set of premises; they will be briefly discussed in Section 2.1. Prior to all this, however, we will describe in few words what DNR is.

DNR is a conceptual framework that stipulates conditions for achieving critical goals such as provoking students' *intellectual need* to learn mathematics, helping them to construct mathematical *ways of understanding* and *ways of thinking*, and assuring that they internalize and retain the mathematics they learn. The framework can be thought of as a system consisting of three categories of constructs: *premises*—explicit assumptions underlying the *DNR* concepts and claims; *concepts* oriented within these premises; and *instructional principles*—claims about the potential effect of teaching actions on student learning justifiable in terms of these premises and empirical observations. The system states three foundational principles: the *duality principle*, the *necessity principle*, and the *repeated-reasoning principle*; hence, the acronym *DNR*. The principles are presented in Section 2.4. Admittedly, here we only discuss those *DNR* elements that are needed for our definition of learning. For an extended discussion of *DNR* see Harel (2008a, 2008b, 2008c).

2.1. Three DNR premises

DNR is based on a set of eight premises, seven of which are taken from or based on known theories. For the concerns of this paper, we only need three premises that concern knowledge and knowing:

Knowledge (of Mathematics): Knowledge of mathematics consists of all *ways of understanding* and *ways of thinking* that have been institutionalized throughout history (Harel, 2008a).
Knowing: Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium (Piaget, 1985).
Knowing-knowledge linkage: Any piece of knowledge humans know is an outcome of their resolution of a problematic situation (Piaget, 1985).

The Knowledge (of Mathematics) premise concerns the nature of the mathematics knowledge by stipulating that *ways of understanding* and *ways of thinking*—the terms to be defined below—are the constituent elements of this discipline, and therefore instructional objectives must be formulated in terms of both these elements, not only in terms of the former, as currently is largely the case (Harel, 2008a). The next two premises are Piagetian: The Knowing premise is about the mechanism of knowing: that the means—the only means—of knowing is a process of assimilation and accommodation. A failure to assimilate results in a disequilibrium, which, in turn, leads the mental system to seek equilibrium, that is, to reach a balance between the structure of the mind and the environment. The Knowledge-Knowing Linkage premise, too, is inferable from Piaget, and, in addition, is complementary to Brousseau's idea that "for every piece of knowledge there exists a family of situations to give it an appropriate meaning" (Brousseau, 1997, p. 42).

2.2. Way of understanding and ways of thinking

Antecedent to the concepts of *way of understanding* and *way of thinking* is the primary concept of *mental act.* Examples of mental acts include the acts of interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving. When one carries out a mental act, one produces a particular outcome. For example, when reading a string of symbols, statement or problem, one of the mental acts an individual carries out is the interpreting act, which results in a particular meaning for it. Similarly, upon encountering an assertion, one may carry out the justification act and produces, accordingly, a particular justification. Such a product of a mental act is called a *way of understanding* associated with that mental act (cf. Simon et al., 2010, for the compatible use of the word 'understandings'). Different individuals are likely to produce different ways of understanding associated with the same mental act. For example, students engaged in a Geometer's Sketch Pad activity, may carry out the conjecturing act and, accordingly, produce different conjectures. And they may produce different justifications for their conjectures. Each of the conjecture and justification is a way of understanding—a product of the conjecturing act and justification act, respectively.

An example that is germane to the interview task discussed in this paper involves the phrase, "4/5 divided by 2/3." In the process of responding to the task mentioned earlier, different teachers interpreted this phrase differently, whereby producing different ways of understanding associated with the interpreting act. The following are some of these ways of understanding: "4/5 divided by 2/3" is the result of multiplying 4/5 by 3/2; the solution to the equation $(2/3)x = 4/5$; the number of times by which 2/3 goes into 4/5; and the quantity that results from dividing 4/5 into 2 equal parts and multiplying it by 3.

A cognitive characteristic of a person's ways of understanding associated with a particular mental act is referred to that person's way of thinking associated with that act. For example, a teacher or researcher may infer, from a multitude of observations, one or more of the following characteristics: that a student's interpretations of arithmetic operations are characteristically inflexible, devoid of quantitative referents, or—alternatively—flexible and connected to other concepts; that a student's justifications of mathematical assertions are typically based on empirical evidence or—alternatively—based on rules of deduction (Harel & Sowder, 1998). Each of these characteristics is a way of thinking (cf. the use of the word 'thinking' in the above quotation from Simon et al., 2010). Furthermore, problem-posing approaches are ways of thinking associated with the problem-posing act. While an actual problem posed in response to a problem-posing task is a way of understanding—because it is a cognitive product of the person's problem-posing act—a problem-posing approach is a cognitive characteristic of that act and hence is a way of thinking. For example, a problem-posing approaches *context in the center* (i.e., choosing the context for the problem and then considering how to incorporate the prescribed operation in it), or, alternatively, *solution in the center* (i.e., analyzing the specificity of the intended solution and then choosing a context that would fit it) characterize the problem-posing act; hence, they are instances of ways of thinking.

Another particularly relevant to this paper example is a ways of thinking that we tagged as *quantitative coordination* and its opposite, *the lack of quantitative coordination.* Analysis of these ways of thinking—including the methodology used to extract them from observable ways of understanding—are described in detail elsewhere (Harel, Koichu, & Manaster, 2006; Koichu, Harel, & Manaster, submitted for publication). Here they are briefly described in the restricted context of the task mentioned earlier, for the purpose of using them to illustrate our definition of learning.

Quantitative coordination, in the context of the task at hand, is when the process of constructing the problem whose solution can be found by dividing 4/5 by 2/3 involves the coordination of two mental operations: (a) assigning 4/5 and 2/3 as inputs for the division operation, and (b) assigning 2/3 as an operator acting on 4/5. Absence of one of these activities would render the problem posing as *the lack of quantitative coordination*. The quantitative coordination way of thinking was observed as one of the teachers constructed the problem: "*What is the length of a rectangle with the area 4/5 and the width 2/3?*" On the other hand, the problem "*4/5 of an $80 bill is divided among 2/3 part of a group of 6 persons. How much should each person pay?*", also produced by one of the teachers, manifests lack of coordination since, in the construction process of this problem, it was evident that the teacher did not conceive of 2/3 as an operand of an arithmetic operation.

2.3. Intellectual need and psychological need[2]

In this section we discuss the second pair of terms in DNR definition of learning: *intellectual need* and *psychological need.*

Disequilibrium, or perturbation, is a state that results when one encounters an obstacle. Its cognitive effect is that it "forces the subject to go beyond his current state and strike out in new directions" (Piaget, 1985, p. 10). Equilibrium is a state in which one perceives success in removing such an obstacle. In Piaget's terms, it occurs when one modifies her or his viewpoint (accommodation) and is able, as a result, to integrate new ideas toward the solution of the problem (assimilation). But what constitutes perturbation, and more relevant to this paper, what constitutes perturbation in mathematical practice? *DNR* defines perturbation in terms of two types of human needs: *intellectual need* and *psychological need*.

[2] This section is an abridged and slightly modified version of a section in Harel (2008c).

If *K* is a piece of knowledge possessed by an individual or community, then, by the Knowing-Knowledge Linkage Premise, there exists a problematic situation *S* out of which *K* arose. Such a problematic situation *S*, prior to the construction of *K*, is referred to as an individual's *intellectual need*: *S* is the need to reach equilibrium by learning a new piece of knowledge. Intellectual need has to do with disciplinary knowledge being created out of people's current knowledge through engagement in problematic situations conceived as such by them. Motivation, on the other hand, has to do with people's desire, volition, interest, self-determination, and the like. These characteristics are manifestations of *psychological needs*: motivational drives to initially engage in a problem and to pursue its solution. Psychological needs, thus, belong to the field of motivation, which addresses conditions that activate and boost—or, alternatively, halt and inhibit—learning in general. In contrast, intellectual needs refer to the epistemology of a particular discipline with an individual or community from the knowledge they currently hold.

Two notes of methodological nature should be made here, for the sake of the forthcoming analysis. First, an intellectual need cannot be determined independently of what satisfies it. Second, all intellectual needs are inextricably linked—a feature that makes it difficult to discuss them in separation from each other and from the context in which they are manifested.

2.4. The DNR foundational principles

According to the Knowledge (of Mathematics) Premise, mathematics consists of ways of understanding and ways of thinking. Of particular pedagogical importance is the question concerning the developmental interdependency between these two categories of knowledge. The Duality Principle asserts:

> Students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they possess.

The reciprocity between ways of understanding and ways of thinking claimed in the duality principle is of mutual effect: a change in ways of thinking brings about a change in ways of understanding, and vice versa. Moreover, not only do these two categories of knowledge affect each other but also a change in one cannot occur without a corresponding change in the other.

The Necessity Principle deals with students' intellectual needs for learning new knowledge. It asserts:

> For students to learn the mathematics we intend to teach them, they must have a need for it, where 'need' here refers to intellectual need.

The principle points out the crucial importance for learners to construct new ways of understanding and ways of thinking in response to an intellectually perturbing task that otherwise cannot be solved. For example, one result of the study from which the illustration described in the next section was taken, was that all teachers had been fluent in the use of invert-and-multiply rule for calculating an exercise "4/5 divided by 2/3," yet they lacked the knowledge of situations in which such an operation should be carried out (Harel et al., 2006). For many of them, division of fractions was a memorized technique devoid of connections to other arithmetic operations or of contexts that would necessitate the use of the technique.

Implied from the Duality Principle and the Necessity Principle is that preaching ways of thinking to students would have no effect on the quality of the ways of understanding they would produce. For example, just advising students to use particular heuristics would have minimal or no effect on the quality of the problems and solutions they would produce. Only by producing desirable ways of understanding—by way of carrying out mental acts of, for example, solving mathematical problems—can students construct desirable ways of thinking. This seems obvious until one observes, for example, teachers teaching problem-solving heuristics explicitly and students trying to follow them as if they were general rules rather than rules of thumb (Koichu, Berman, & Moore, 2007; Schoenfeld, 1992).

The Repeated Reasoning Principle deals with internalization, organization, and retention of knowledge. It asserts:

> Students must practice reasoning in order to internalize desirable ways of understanding and ways of thinking.

The emphasis here is on repeated opportunities to reason on perturbing tasks that may help in constructing and reinforcing particular ways of understanding and ways of thinking. Repeated reasoning, not mere drill of routine problems, is essential to the process of learning.

2.5. Summary

DNR's definition of learning incorporates two kinds of needs: intellectual need and psychological need, and attends the knowledge currently held and newly produced during the equilibration–disequilibrium phases. Further, since our interest is restricted to mathematics learning, this knowledge is defined in terms of ways of understanding and ways of thinking, by the Knowledge (of Mathematics) Premise. Thus, again, *DNR*'s definition of *learning* is:

> *Learning* is a continuum of disequilibrium–equilibrium phases manifested by (a) intellectual and psychological needs that instigate or result from these phases and (b) ways of understanding or ways of thinking that are utilized and newly constructed during these phases.

3. Definition's operational nature

In this section, we illustrate the operational nature of the *DNR* definition of learning by analyzing three interview protocols with one mathematics teacher, Burt. The second interview took part a year after the first, and the third one eight months after the second. In all the interviews, Burt was instructed to think aloud about the interview task, and the interviewer refrained from revealing anything about the quality of his responses. If the subject kept silent for more than 20–25 s while working on the task, the interviewer prompted him to think aloud in a neutral manner, by saying "Keep talking" or asking "What are you doing right now?" (see Koichu & Harel, 2007, for the detailed analysis of the interview design).

The interviews all deal with the task "make up a word problem whose solution may be found by computing 4/5 divided by 2/3." This task was chosen for two reasons. First, Ma (1999) has shown that US mathematics teachers, in contrast to Chinese teachers, experience major obstacles making up problems involving division of fractions. In her study, only 1 out of 23 US teachers was able to compose a conceptually correct problem. Hence, we anticipated that the subjects in our study, 24 US mathematics teachers who took part in the first year of DNR-oriented professional development institute, were likely to experience detectable perturbations—intellectual as well as psychological—with this task, especially in the first round of interviews. However, we also expected at least some of the teachers to show progress in the second and third rounds of the interviews, if only as a result of the repeated reasoning about the task. The second reason for choosing the task was that posing word problems was not a part of the middle-school mathematics curricula taught by the participants of our study, and the professional development institute never addressed this or similar tasks. Thus, we expected that even though the interviews were separated by long periods of time, the teachers would think of posing problems in the context of division of fractions only during the interviews, and, consequently, their ways of dealing with the task could be fully observed and properly recorded.

Burt was chosen to be presented in this paper for the following reasons. First, he was one of 11 teachers who also took part in the second year of the project and who were interviewed three times. Second, he was one of four teachers who did not know how to solve the task during the first interview, and then showed considerable progress from one interview to the next. Third, Burt was capable of thinking aloud with relatively little attention to the interviewer and the video-camera. Note that such reasons for the choice of a particular case for demonstration of our approach are fully compatible with Simon's et al. (2010) criteria for choosing what they called "useful data sets" (p. 89) for purposes similar to ours. Note also that though only one case is presented in this paper, its analysis relies on the categories developed and validated in the analysis of all 46 interviews (Harel et al., 2006; Koichu et al., submitted for publication); some of these categories are outlined in Section 2.

At this point, let us recall that this paper is *not* a research report aimed at contributing to the knowledge on the teachers' problem posing in the context of division of fractions. Instead, our goal is to demonstrate how learning can be modeled in terms of the above definition. Consequently, two types of analyses are illustrated below. First, we show how aspects of the definition are reflected in the aspects of the data. Second, we follow the changes in Burt's ways of understanding and ways of thinking that occurred through the repeated reasoning about the same task.

The last methodological note is that ways of understanding, by definition, are extractions from observations made with particular attention to a particular mental act, whereas ways of thinking, by definition, are second-order inferences from the observations (or first-order inferences from the ways of understanding), and as such these are more speculative: they are conjectured characteristics of one's mental act identified by means of inductive analysis (e.g., Dey, 1999; Johnson & Christensen, 2004). Methodologically, this distinction means that researchers should be worried about *reliability* of the extracted ways of understanding, and about *plausibility* of the inferred ways of thinking (Clement, 2000).

3.1. First interview

During the first interview, Burt undertook three attempts to make up a requested problem, and then asked for the interviewer's assistance. Burt's first attempt was:

Burt: The image that comes to mind is. . .4/5 of a group of people, uh. . .I have to find the answer first actually [Pause, Burt writes: 4/5÷2/3=6/5]. OK. The fact that the answer is greater than 1 would eliminate certain kinds of problems. . . If you have 4/5 of a group, how. . .when are we going to divide by 2/3? Well, when you're taking 2/3 of a group of people, you're multiplying. . .So the question wouldn't involve 2/3 of a group. When indeed do you divide by 2/3? [Laughs, stops talking]

We consider this excerpt an *observation*—an observer-made re-presentation of the evident interviewee's actions undertaken while solving the interview task. One way to summarize this observation is this:

WoU1.1: Burt considers 4/5 as "4/5 of a group of people," computes 4/5÷2/3=6/5, observes that the result of the division is greater than 1, notes that "2/3 of a group of people" leads to multiplication by 2/3, and realizes that he does not have a context that would involve division by 2/3.

This summary represents a way of understanding (WoU) extracted from Burt's first attempt to solve the given task. He made two more attempts, from which two additional ways of understanding were extracted:

WoU1.2: Burt says: "The task is carefully selected. . . It's not occurring to me immediately how I can use fractions on both ends of the problem." He then considers 4/5 as "4/5 of a piece of wood" and notes that it is easy to make up a problem about dividing that piece of wood into whole number of parts, but it is meaningless to divide it into 2/3 parts.

WoU1.3: Burt says that he is "greatly compelled to come back to 4/5 of a group of people or of other objects," and, whatever the objects are, it would be the beginning of the problem. Then Burt says that he can easily ask a question that leads to multiplication by 2/3 or to division by 2 but not to division by 2/3, and he stops.

These ways of understanding are extracted with particular attention to the problem-posing mental act carried out by Burt. They also indicate some products of the mental act of interpreting division as it continuously complements problem-posing in Burt's performance. Accordingly, two kinds of ways of understanding are discussed below: problems or fragments of problems that Burt made up in response to the interview task and his ways of understanding division.

The major product associated with the problem-posing act was an assertion that the requested problem must include "4/5 of a group of people or other objects." It is a rather modest result of Burt's problem posing, and he was unable to develop it further. Indeed, he started and finished with the same idea, and thus, demonstrated *circular* problem-posing behavior (Koichu, Berman, & Moore, 2006). One product of the interpreting mental act is that Burt evidently perceived division as *even partitioning* (see e.g., Greer, 1992).

A common characteristics of WoU1–WoU3 is that Burt could not attend to the two given numbers, 4/5 and 2/3, as inputs of one operation. In his words, "*It's not occurring to me. . .how I can use fractions on both ends of the problem.*" Consequently, it seems that Burt's problem-posing is characterized by *the lack of quantitative coordination*. In our study, *the lack of quantitative coordination* was indicated in the interview with Burt as well as in 13 additional interviews conducted at the beginning of the professional development institute; *quantitative coordination* was indicated in 10 interviews (Harel et al., 2006). Burt's performance is governed by an additional way of thinking mentioned in Section 2.2, namely, *context in the center*. Indeed, each WoU starts with considering a particular context, and finishes with its evaluation as the inappropriate one for the incorporation of the division of fractions.

We now discuss the disequilibrium–equilibrium aspect of the above excerpts. The fact that Burt was given the interview task did not necessarily mean that he was perturbed. For example, Burt could have refused to solve the task or, given that he accepted a social role of an interviewee, he could stop thinking about the task after realizing that he had no readily available way to solve it. Three teacher-participants out of 24 indeed stopped thinking about the task after finding no ready solution, but Burt persisted. As we have seen, Burt was struggling: he started solving the task (WoU1.1), evaluated the attempt as unsatisfactory, approached the task once more (WoU1.2), reflected on what he had done (WoU1.3), and gave up when he had no additional ideas. Hence, Burt was perturbed by the interview task or, in other words, Burt encountered disequilibrium.

We now consider intellectual and psychological needs that instigated the perturbation. It is reasonable to assert that certain psychological needs were inherent in the interview setting. Indeed, Burt had agreed to cooperate with the researchers and to think aloud about unknown questions, so he was ready to immerse himself in a problem. Thus, *the need for relatedness*, i.e., the need to be involved in relationships with significant others (Alderfer, 1972), can be conjectured as a stimulus that instigated Burt's initial engagement in the interview task. This conjecture is further supported by the fact that Burt was very cooperative during the study. Second, Burt was evidently discomfited by the fact that he, a professional mathematics teacher, was unable to respond quickly to the "simple" task concerning division of fractions. His speech was emotionally loaded, indicating he was embarrassed, which was particularly noticeable when he laughed and uttered the question "When indeed do you divide by 2/3?" (WoU1.1). Thus, *the need for competence* (e.g., Ryan & Deci, 2000) can be inferred as another stimulus. To indicate an additional stimulus, consider the continuation of the interview.

After the episode summarized in WoU1.3, the following dialogue took place:

Int.: I'm going to go onto another question if that's OK.
Burt: Yeah, let's. . . Well can you [inaudible].
Int.: Can I help you? Do you want me to help?
Burt: Yeah, no, I mean with uh, with this particular question.
Int.: Well, let's say [Pause 5 sec.]
Burt: I know that, that's not what we're doing but I'm [inaudible]
Int.: Well let's just say. . . because I see you're a little bit frustrated. . . let's say that you were walking at the rate of 2/3 of a mile an hour. So every hour you cover 2/3 of a mile [Pause 10 sec.]
Burt: Yes, thank you. . .How many hours would I need to walk. . . how many hours would I be walking. . . no, excuse me, if I were walking at 2/3 of a mile per hour and traveled a distance of 4/5 of a mile. Yes, then I would be dividing, and I would end up getting 12/10 of an hour [pause, Burt shows that he is ready for the next question, but looks unsatisfied].

The interviewer provided the requested assistance (which was a violation of the interview procedure) because he felt that otherwise the rest of the interview would be jeopardized. Indeed, even knowing that the interviewer was not allowed to assist interviewees with problem-solving—this was explained at the beginning of the interview—Burt asked for help immediately when it appeared to him that the interviewer was going to move on to another question. This can be seen as a sign of another stimulus that governed Burt's behavior during the interview, namely, *the need to know* (Maslow & Lowery, 1998) or, in Harel's (2008c) terms, *the need for certainty*. After the interviewer's intervention, Burt seemed to stop thinking about the first interview task. However, Burt's perturbation had not been completely eliminated, which would become evident during the second interview.

3.2. The second interview

The second interview was conducted a year after the first one, and the format of the first interview was implemented by a new interviewer. Burt quickly recognized the interview task: "*It's the same problem. I remember struggling with finding a way, but I shouldn't struggle again*." The summary of his first approach to the same problem-posing task during the second interview follows:

WoU2.1: Burt said: "We have a group of something that needs total to be four fifths of this [group], and we need to divide it into groups of 2/3. So, I believe that I want to use measurements." Then he considered a carpenter cutting a board of 4/5 feet into pieces of 2/3 feet each and made up a question: "How many pieces of 2/3 feet each can be cut from a board of 4/5 feet?"

Interestingly, Burt remembered that a year ago he had thought about a piece of wood 4/5 feet long (see WoU1.2). He also remembered that the first interviewer had provided him with a useful hint at the end of the first interview, but he did not remember what the hint was. This became evident when the interviewer asked Burt to clarify what he remembered from the first interview and what he invented during the second one. In Burt's words:

> [I remember that] at the end [of the first interview] I asked the interviewer [for help] and he created some analogies that made perfect sense at that time, but I could not remember [them today]. So I went back to the way I tried to solve the problem last year.

When Burt articulated the problem about cutting a board (see WoU2.1), the following dialogue occurred:

Burt: OK, do you want me to try again? . . . I don't like this problem. If my desire were to get my students to learn about dividing fractions, that's kind of a bad question.
Int.: Why so?
Burt: Oh, no, everything is OK, except for the answer. To solve it, I need to divide 4/5 by 2/3, and it would be 12/10. But the actual answer is something other than 12/10. The actual answer is. . .1 piece. . .The number of pieces cannot be fractional. . .

At this point, let us note that Burt composed a problem, which was very similar to that reported in Ma (1999) as the only "conceptually correct" problem about division of fractions composed by a US teacher. That teacher composed a problem with the answer "3(1/2) students" (p. 67), and had felt as uncomfortable about it as Burt was.

Burt then attacked the task again:

WoU2.2: Burt mentioned two possible problem-posing contexts: time ("something like 4/5 of a month needing to be divided into 2/3") and money ("4/5 of a dollar. . ."). He did not develop these ideas since he noted that it was unclear how to make sense of the answer 6/5 in these contexts.

We now analyze the presented data in accordance with the scheme developed in the previous subsection. As a way of understanding division, Burt used a quotitive (measurement) interpretation of division (e.g., Greer, 1992), which led him to composing the problem about cutting a board. Noticeably, Burt was first driven by the *solution in the center* way of thinking (see WoU2.1), and then came back to the *context in the center* one (see WoU2.2). Furthermore, the *quantitative coordination* way of thinking can be seen in WoU2.1, and its lack is indicated in WoU2.2. Indeed, WoU2.1 includes an attempt to create a joint structure involving the given numbers prior to interpreting them separately. WoU2.2 seems to be governed by the *lack of quantitative coordination* since Burt planned how to interpret 4/5 without having an idea of how to incorporate 2/3 into the chosen contexts.

Next, we attempt to characterize the disequilibrium–equilibrium phases that Burt encountered during this interview. For reasons similar to those presented in the analysis of the first interview, we conjecture that *the need for relatedness and the need for competence* were among the stimuli that instigated disequilibrium. It was particularly instructive that, when Burt recognized the task, he said in a self-persuasive manner: ". . .I shouldn't struggle again with this problem!" It was also evident that Burt wanted to continue thinking on the task, even when the interviewer was ready to move on to the next task (namely, "Do you want me to try again?" and WoU2.2).

The need for competence, indicated in the first interview mainly as the need to be a competent problem solver, was enriched in the second interview by a new aspect. One can see in the above data a manifestation of *the need for pedagogical* *applicability*. Burt produced the problem about cutting a piece of wood, which, technically speaking, correctly addressed the interview question as it involved division of 4/5 by 2/3. Burt knew this fact (see WoU2.1 and Burt's follow-up comments), but he still was perturbed because "If my desire were to get my students to learn about dividing fractions, that's kind of a bad question" (see WoU2.2).

WoU2.2 also helped us to better understand what constituted a state of equilibrium for Burt. Indeed, from the first interview we inferred that knowledge of a mathematically acceptable solution was a necessary component of equilibrium. From the second interview, we inferred that the equilibrium was not achieved even when Burt reached a solution independently because the achieved solution seemed to lack pedagogical applicability.

3.3. Third interview

The third interview with Burt was conducted eight months after the second one by the same interviewer. This time Burt not only recognized the interview task, but, as he admitted, figured out what would be asked and found an answer during a 5-min walk from a parking lot to the building in which the interview was scheduled. Hence, it was not surprising that, immediately after reading the interview task, Burt articulated the following word problem and its solution:

WoU3.1: "What is the rate of a car in miles per hour if the car has driven 4/5 of a mile in 2/3 of an hour? And the rate would be determined by dividing 4/5 by 2/3."

Burt was completely satisfied with this problem. He did not show any signs of dissatisfaction and just wanted to share with the interviewer how he had invented it. For this reason, the third interview turned to be retrospective rather than concurrent in nature (Van Someren, Barnard, & Sandberg, 1994). Burt explained that, after the first and especially after the second interview, he had been thinking a lot about why this seemingly simple task was so difficult for him, and at some point realized what the obstacle was:

Burt: I suppose I got stuck on things that we commonly do with fractions like 4/5 and 2/3. . .I was stuck with measurements and when would they be divided. . .And as I thought about the problem [after the second interview], I ended up concluding that what was the most important is the division aspect, and rate comes to mind, as rate is the ratio, and ratio means division. . .And the reason that this came quickly to mind is, in my classes we were covering word problems involving rate a lot recently. So it made it much easier for me to focus on a situation where division would be required, as opposed to [a situation where] fractions would be required.

Greer (1992) classified a problem very similar to the Burt's last problem as *a rate problem involving division by multiplier* and argued that producing such a problem requests quotitive rather than partitive understanding division; thus, Burt exhibits the former way of understanding. Regarding ways of thinking, it is very clear from the above excerpt, and especially from the last sentence of the excerpt, that Burt's problem posing was governed by the quantitative coordination: he deliberately thought of a joint structure that would involve the two given fractions. In addition, the first part of the excerpt points to that he was driven, for considerably long time, by *solution in the center* way of thinking.

The third interview does not provide insight regarding stimuli and needs associated with Burt's disequilibrium–equilibrium phases. Indeed, the equilibration occurred when Burt was not being observed, and, apparently, a new perturbation did not emerge from the achieved equilibrium state. We can only suggest that, for the reasons indicated above, it was important to Burt to succeed in the interview task. However, the third interview enriched our understanding of what constituted the state of equilibrium for Burt. To us, it was prominent that Burt was eventually satisfied with the word problem's formulation involving a *car* which "has driven 4/5 of a mile in 2/3 of an hour." To us, a tired pedestrian or at least a car stuck in heavy traffic might have been a more appropriate object that could illustrate the rate "6/5 miles per hour." However, Burt was not bothered at all by such considerations. From Burt's perspective, it was important that the produced formulation clearly met the requirements of the interview task and was pedagogically applicable in the sense that it would sound familiar to his students. Evidently, *practical applicability of a word problem* (e.g., Greer, Verschaffel, & de Corte, 2002) was not a component of Burt's state of equilibrium manifested during the third interview.

3.4. Recapitulation of Burt's learning

Within the DNR conceptual framework, to model someone's learning, it suffices to characterize a particular sequence of that person's disequilibrium–equilibrium phases in terms of ways of understanding, ways of thinking, and intellectual and psychological needs that instigate or result from these phases. We have described these phases for Burt based on three interviews. Fig. 1 summarizes the main findings with a map of Burt's learning across the three interviews.

4. Discussion

The definition of learning discussed in this paper posits learning as a multi-dimensional and multi-phase change occurring when individuals attempt to resolve what they view as a problematic situation. The definition is rooted in the Piagetian

		Interview 1	*Interview 2*	*Interview 3*
Ways of understanding	*with respect to problem posing act:*	Considers *"4/5 of a group other objects"* and does not know how to continue.	*"How many pieces of 2/3 feet each can be cut from a board of 4/5 feet?"*	*"What is the rate of a car in miles per hour if the car has driven 4/5 of a mile in 2/3 of an hour?"*
	With respect to the act of interpreting division:	*Division as even partitioning*	*Quotitive division*	
Ways of thinking:		*Lack of quantitative coordination*	*Approaching quantitative coordination*	*Quantitative coordination*
		Context in the center	*Solution in the center* and then *context in the center*	*Solution in the center*
Psychological and intellectual needs:		*The need for relatedness, the need for competence and...*		
		the need to know or the need for certainty	*the need for pedagogical applicability*	
Ways of eliminating perturbations:		*Approaching the task and...*		
		requesting assistance	*independently solving the task*	
Missing components of the state of equilibrium:		*Solving independently*	*pedagogical applicability of the solution*	-

Fig. 1. Map of Burt's learning.

conceptualization of learning, and is operational in the sense that it specifies which types of (interrelated) changes should be accounted for. Our claim about the usefulness of the definition relies on the claim that both *products* and *processes* of learning should be adequately considered in contemporary mathematics education research programs (Simon's et al., 2010). Specifically, the suggested operational definition accounts for the products of one's learning as it includes identification of one's ways of understanding, i.e., products of a particular mental act, at different stages. It also is sensitive to a process of change because it deals, though in a suggestive manner, with gradually evolving ways of thinking, i.e., cognitive characteristics of the mental act, which, by the Duality Principle, govern the construction of ways of understanding.

The definition also accesses the question of why the changes occurred, by the attention to the reciprocity between ways of understanding and ways of thinking, as well as by the attention to intellectual and psychological needs instigating the initial involvement in a problematic situation and pursing it in a particular way. For instance, the entire analysis undertaken above explains how and why Burt was eventually capable of independently solving the interview task through the repeated reasoning on it. Generally speaking, the identified combination of psychological and intellectual need plausibly (though partially) explained how the interview task necessitated for Burt scrutinizing his experience as a mathematics teacher and caused gradual, over a long period of time, shifting from naïvely adapting the available word problem contexts to making subtle observations about the essence of the division of fractions; his learning is reflected in interrelated change in his ways of understanding and ways of thinking. Note that to us the definition was equally instrumental for understanding why some other teachers who took part in three interviews under similar conditions, had not eventually solved the interview task. The failures, which presentation is beyond the scope of this already lengthy paper, could be, again, explained in terms of psychological and intellectual needs of the teachers and the absence of some particularly useful ways of understanding and ways of thinking across the interviews.

In summary, we contend that the DNR-oriented definition of learning aids both educational theory and methodology in that it exemplifies how a particular conceptual definition of learning can be operationalized. We also believe that the definition has a clear pedagogical implication: instructions should be aimed at achieving desirable changes in all the dimensions of learning.

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