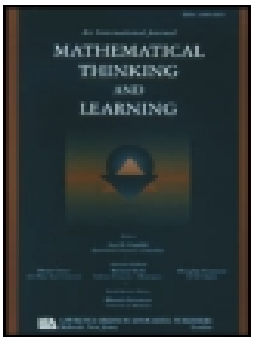

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Advanced Mathematical-Thinking at Any Age: Its Nature and Its Development

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Advanced Mathematical-Thinking at Any Age: Its Nature and Its Development

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This article argues that *advanced mathematical thinking*, usually conceived as thinking in advanced mathematics, might profitably be viewed as advanced thinking in mathematics (*advanced mathematical-thinking*). Hence, advanced mathematical-thinking can properly be viewed as potentially starting in elementary school. The definition of mathematical thinking entails considering the epistemological and didactical obstacles to a particular way of thinking. The interplay between ways of thinking and ways of understanding gives a contrast between the two, to make clearer the broader view of mathematical thinking and to suggest implications for instructional practices. The latter are summarized with a description of the DNR system (Duality, Necessity, and Repeated Reasoning). Certain common assumptions about instruction are criticized (in an effort to be provocative) by suggesting that they can interfere with growth in mathematical thinking.

The reader may have noticed the unusual location of the hyphen in the title of this article. We relocated the hyphen in "advanced-mathematical thinking" (i.e., thinking in advanced mathematics) so that the phrase reads, "advanced mathematical-thinking" (i.e., mathematical thinking of an advanced nature). This change in emphasis is to argue that a student's growth in mathematical thinking is an evolving process, and that the nature of mathematical thinking should be studied so as to

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lead to coherent instruction aimed toward advanced mathematical-thinking. These arguments are embodied in our responses to four questions:

1. What is meant by "mathematical thinking"?[1]
2. What are the characteristics of advanced mathematical-thinking?
3. What are concrete reasoning practices by which advanced mathematical-thinking can be enhanced?
4. What are concrete reasoning practices by which advanced mathematical-thinking can be hindered?

We address these questions, in turn, in the four sections comprising this article.

Our earlier research necessitated these questions in the context of mathematical proof (cf. Harel, 2001; Harel & Sowder, 1998). In this article, however, we do not restrict our discussion to the process of proving. Rather, we demonstrate our claims in a range of mathematical contexts across the grade-level spectrum, to demonstrate that advanced mathematical-thinking is not bound by advanced-mathematical thinking.

ARTICLE'S ORGANIZATION

This article is organized in three sections (followed by a brief conclusion section):

1. Our definition of advanced mathematical thinking is based on an important distinction between two categories of knowledge: *ways of understanding* and *ways of thinking*. In the first section, we define these notions and discuss several responses and solutions by students to illustrate them. In particular, we show how "proof schemes" (what constitutes truth for an individual), "problem-solving approaches," and "beliefs about mathematics" are instances of one's ways of thinking.

2. Our definition of advanced mathematical thinking also utilizes Brousseau's (1997) notion of *epistemological obstacle*. Therefore, in the second section, we discuss this important notion to argue for our relativistic view of the property "advanced" and to discuss examples of epistemological obstacles involved in the development of advanced mathematical thinking (relative to our definition).

3. Finally, in the third section, we point to general reasoning practices by which advanced mathematical thinking can be advanced or hindered.

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[1]We use the terms *mathematical thinking, a way of mathematical thinking,* or just *a way of thinking* interchangeably, although we are always referring to a mathematical context.

WHAT IS MEANT BY "MATHEMATICAL THINKING"?

Underlying the analysis presented in this article is the fundamental premise that humans' mental actions, observable or inferred, are induced and governed by their general views of the world, and, conversely, humans' general views of the world are formed by these actions. Our probe into the above four questions through the lenses of this duality led to a distinction between two categories of knowledge: ways of thinking and ways of understanding.

The particular meaning students give to a term, sentence, or text, the solution they provide to a problem, or the justification they use to validate or refute an assertion—are ways of understanding, whereas students' general theories—implicit or explicit—underlying such actions are ways of thinking. This distinction, to be elaborated upon shortly, has been both essential and valuable for our research and for its instructional implications. We have observed that teachers often form, at least implicitly, cognitive objectives in terms of ways of thinking, but their efforts to these teach ways of thinking are often counterproductive because their efforts do not build on ways of understanding. Conversely, teachers often focus on ways of understanding but overlook the goal of helping students construct effective ways of thinking from these ways of understanding. This observation is the basis for the Duality Principle, one of the fundamental principles that underlie the instructional treatment employed in our teaching experiments (see Harel, 1998, 2001). We return to the Duality Principle in the third section of this article.

WAYS OF THINKING VERSUS WAYS OF UNDERSTANDING

We describe the distinction between ways of thinking and ways of understanding in the context in which it initially arose. Consider the following three central, often interrelated, mathematical activities:

1. Comprehension of mathematical content, as when reading texts or listening to others.
2. Carrying out an investigation, as when solving a problem.
3. Establishing truth, as when justifying or refuting.

Although it is pedagogically useful to distinguish among the three activities, cognitively they can easily be subsumed under item (2), problem solving; for comprehension and communication, as well as justifying or proving, are all problem-solving processes.

Corresponding to these three types of mathematical activities, the phrase, ways of understanding, refers to

1. The particular meaning/interpretation a person gives to a concept, relationships between concepts, assertions, or problems.
2. The particular solution a person provides to a problem.
3. The particular evidence a person offers to establish or refute a mathematical assertion.

Examples of ways of understanding for (1) include the following: A student may read or say the words, "derivative of a function," understanding the phrase as meaning the slope of a line tangent to the graph of a function, as the best linear approximation to a function near a point, as a rate of change, etc. On the other hand, a student may understand this concept superficially (e.g., "the derivative is nx^{n-1} for x^n") or incorrectly (e.g., "the derivative is the quotient $(f(x + h) - f(x)) / h$"). Similarly, a student may understand the concept of a fraction in different ways. For example, the student may understand the symbol a / b in terms of *unit fraction* (a / b is a $1 / b$ units); in terms of *part-whole* (a / b is a units out of b units); in terms of *partitive division* (a / b is the quantity that results from a units being divided equally into b parts); in terms of *quotitive division* (a / b is the measure of a in terms of b-units). All of these would be ways of understanding derivatives or ways of understanding fractions.

Examples of ways of understanding for (2)—particular methods of solving a problem—can be seen in the following. A ninth-grade class was assigned the following problem:

> *Problem 1*: A pool is connected to 2 pipes. One pipe can fill the pool in 20 hours, and the other in 30 hours. Assuming the water is flowing at a constant rate, how long will it take the 2 pipes together to fill the pool?

Among the different solutions provided by the students, there were the following four—each represents a different way of understanding.

> *Solution 1.1:* In 12 hours the first pipe would fill 3/5 of the pool and the second pipe the remaining 2/5. (The student who provided this solution accompanied it with a sketch similar to Figure 1. We return to this solution later in the article).

> *Solution 1.2*: It will take the 2 pipes 50 hours to fill the pool.

> *Solution 1.3*: It will take the 2 pipes 10 hours to fill the pool.

> *Solution 1.4*: It would take x hours. In one hour the first pipe will fill 1/20 of the pool, whereas the second will fill 1/30. In x hours the first pipe would fill

FIGURE 1 Student's sketch for Solution 1.1.

> $x/20$ and the second, $x/30$. Thus, $x/20 + x/30 = 1$. (The student then solved this equation to obtain $x = 12$.)

Examples of ways of understanding for (3)—justifying or refuting—include the following justifications by prospective secondary teachers to the problem:

> *Problem 2*: Prove that $\log(a_1 \cdot a_2 \ldots a_n) = \log a_1 + \log a_2 + \ldots + \log a_n$ for all positive integers n.

> *Solution 2.1*:
> $\log(4 \cdot 3 \cdot 7) = \log 84 = 1.924$
> $\log 4 + \log 3 + \log 7 = 1.924$
> $\log(4 \cdot 3 \cdot 6) = \log 72 = 1.857$
> $\log 4 + \log 3 + \log 6 = 1.857$
> Because these work, then $\log(a_1 \cdot a_2 \ldots a_n) = \log a_1 + \log a_2 + \ldots + \log a_n$.

> *Solution 2.2*:
> i. $\log (a_1a_2) = \log a_1 + \log a_2$ by definition
> $\log (a_1a_2a_3) = \log a_1 + \log a_2a_3$. Similar to $\log (ax)$ as in step (i), where this time $x = a_2a_3$.
>
> Then
>
> ii. $\log (a_1a_2a_3) = \log a_1 + \log a_2 + \log a_3$
> We can see from step (ii) any $\log (a_1a_2a_3 \ldots a_n)$ can be repeatedly broken down to $\log a_1 + \log a_2 + \log a_n$.

In our usage, the phrase way of understanding, conveys the reasoning one applies in a local, particular mathematical situation. The phrase way of thinking, on the other hand, refers to what governs one's ways of understanding, and thus expresses reasoning that is not specific to one particular situation but to a multitude of situations. A person's ways of thinking involve at least three interrelated categories: beliefs, problem-solving approaches, and proof schemes.

BELIEFS—VIEWS OF MATHEMATICS

"Formal mathematics has little or nothing to do with real thinking or problem solving," and "The solution of a problem should not take more than five minutes" are detrimental common beliefs among students (Schoenfeld, 1985, p. 43). On the other hand, in our work with undergraduate mathematics students, we found that enabling beliefs such as "A concept can have multiple interpretations" and "It is advantageous to possess multiple interpretations of a concept," although essential in courses such as linear algebra, are often absent from the students' repertoires of reasoning. The development of these ways of thinking should not wait until students take advanced-mathematics courses, such as linear algebra. Elementary school mathematics and secondary school mathematics are rich with opportunities for students to develop these ways of thinking. For example, the different ways of understanding fractions we just presented should provide such an opportunity to develop the above ways of thinking for all elementary-grade students; likewise the (correct) different ways of understanding derivatives should provide such an opportunity for secondary-school students who take calculus.

PROBLEM-SOLVING APPROACHES[2]

"Look for a simpler problem," "Consider alternative possibilities while attempting to solve a problem," "Look for a key word in the problem statement" are examples of problem-solving approaches. The latter way of thinking might have governed the way of understanding expressed in Solution 1.2. Which ways of thinking might have governed the other three solutions to Problem 1? Of particular interest is Solution 1.1. Only one student, G, provided this solution, and she was briefly interviewed. G indicated that she drew a diagram—a rectangle to represent the pool (Figure 1)—and divided it into 5 equal parts. Then she noticed that 3(20/5) is the same as 2(30/5). G was unwilling (or unable) to answer the question of how she thought to divide the rectangle into 5 equal parts, so we can only conjecture that a juxtaposition of ways of thinking had driven G's solution. These may have included "Draw a diagram," "Guess and check," and "Look for relevant relationships among the given quantities." It was shocking to learn that G's score on this problem, as well as on three other problems she solved in a similar manner (i.e., without any "algebraic representation") was zero. Her teacher's justification for this score was something to the effect that G did not solve the problems algebraically, with unknowns and equations, as she was expected to do.

[2]We chose not to use the term "heuristics" here because, although every heuristic is a general approach to solving problems, the converse is not true. Heuristics are defined as "rules of thumb for effective problem solving" (Schoenfeld, 1985, p. 23); students' approaches to solving mathematical problems—needless to say—are not always heuristics in this sense.

PROOF SCHEMES

Proving is defined in Harel and Sowder (1998) as the process employed by a person to remove or create doubts about the truth of an observation, and a distinction is made there between ascertaining for oneself and persuading others. A person's *proof scheme* consists of what constitutes ascertaining and persuading for that person. Thus, proof schemes include one's methods of justification. In this sense, "proving" and "justification" are used interchangeably in this article. One of the most ubiquitous proof schemes held by students is the *inductive proof scheme*, where students ascertain for themselves and persuade others about the truth of a conjecture by direct measurements of quantities, numerical computations, substitutions of specific numbers in algebraic expressions, etc. (Harel & Sowder, 1998). We found that this way of thinking governed the way of understanding expressed in Solution 2.1 (Harel, 2001). The way of understanding expressed in Solution 2.2, on the other hand, was found to be a manifestation of a different way of thinking, called *transformational proof scheme*. In Harel (2001) it is shown why Solution 2.1 contains the three essential elements that characterize the transformational proof scheme: (a) consideration of the generality aspects of the conjecture, (b) application of mental operations that are goal oriented and anticipatory—an attempt to predict outcomes on the basis of general principles—and (c) transformations of images that govern the deduction in the evidencing process.[3]

WHAT ARE THE CHARACTERISTICS OF ADVANCED MATHEMATICAL-THINKING?

It is clear that some ways of thinking are flawed (e.g., relying solely on empirical observations to justify mathematical arguments, as we have seen in Solution 2.1; over-generalizing mathematical ideas, as in the common inference students make: "since $2(a + b) = 2a + 2b$ is valid, then $(a + b)^2 = a^2 + b^2$ must also be valid" [Matz, 1980]), although others are sound (e.g., looking for elegant solutions to problems; generalizing mathematical ideas). But in what sense is "mathematical thinking" advanced? Does "advanced" imply "effective," "efficient," or "elegant"? Is nonadvanced mathematical thinking necessarily lacking or faulty? "Advanced" implies there is also an "elementary." If so, in what sense is "mathematical thinking" elementary? It is extremely difficult to characterize these properties, even if we share an intuitive understanding of their meaning, and it is even more difficult to build a taxonomy that differentiates among properties of mathematical thinking. Yet it is of paramount importance to characterize qualities of mathematical thinking to

[3]For the full taxonomy of proof schemes, see Harel and Sowder (1998) and Harel (in press).

> translate them into essential cognitive objectives—objectives that would position elementary mathematics content for the successful subsequent learning of advanced mathematical content. But what is the complete set of such ways of thinking? Is the set a mere list, or does it have an underlying structure and is it guided by a small number of principles? Advanced mathematical-thinking research can and should take the lead in answering these critical questions. (Heid, Harel, Ferrini-Mundy, & Graham, 2000, p. 35)

One goal of this article is to contribute to our understanding of these issues.

The term "advanced" implies that a developmental process is involved. "Advanced" is, therefore, not an absolute but a relative term, both in relation to a single way of thinking and in relation to different ways of thinking. The attainment of a certain way of thinking is not all or nothing but gradual, and likewise, one might demonstrate a high level of mastery of one way of thinking and little or none of another.

In addition to this relativistic view of the property "advanced," we consider the kind of obstacles one encounters in developing a way of thinking. We adopt Brousseau's distinction between didactical obstacles and epistemological obstacles. The former are the result of narrow or faulty instruction, whereas the latter are unavoidable due to the nature of the development of human knowledge (Brousseau, 1997). But what are the criteria for determining whether the development of a particular type of mathematical thinking necessarily involves epistemological obstacles? Although this question itself requires serious research—cognitive, historical, and epistemological—there already exist some criteria with which to begin a debate on this question. Duroux (1982, cited in Brousseau, 1997) lists necessary conditions for a piece of knowledge to be considered an epistemological obstacle. The first of Duroux's conditions is that epistemological obstacles have traces in the history of mathematics. The second condition is that an epistemological obstacle is not a missing conception, or a lack of knowledge; rather, it is a piece of knowledge or a conception that produces responses that are valid within a particular context, and it generates invalid responses outside this context. To overcome the epistemological obstacle, one must construct a notably different point of view. The third and last condition is that an epistemological obstacle "withstands both occasional contradictions and the establishment of a better piece of knowledge. Possession of a better piece of knowledge is not sufficient for the preceding one to disappear" (Brousseau, 1997, pp. 99–100).

These considerations—the relativistic view of the property "advanced," and the obstacles involved in the developmental process—led us to the following definition, which suggests a research agenda for determining ways of thinking that are advanced, as well as the level of their development:

> Mathematical thinking is advanced, if its development involves at least one of the above three conditions for an obstacle to be epistemological. The level

of acquisition of a way of thinking by an individual is determined by the extent to which the individual has overcome these obstacles.

It should be noted that the first condition—that an epistemological obstacle must have traces in the history of mathematics—is particularly problematic. It is difficult, and in many cases it may not be possible, to establish whether an obstacle has manifested itself in the history of mathematics. Many obstacles have likely occurred in the historical development of mathematics but have never been observed by historians.

A ready example of an obstacle that satisfies one of Duroux's conditions is the transition from solely additive reasoning to proportional reasoning, a commonly observed difficulty (see discussion following). Also, the notion of epistemological obstacle applies to the construction of both ways of understanding and ways of thinking. For example, the understanding of negative integers and imaginary numbers meets some, if not all three, criteria. The mathematical community of the time (17th century) had to reconstruct—even revolutionize—its ways of thinking about the concepts of number and quantity to accept these new constructs (Klein, 1968; see also Kline, 1972, p. 252).

We conclude this section with two episodes to illustrate the above considerations—not the definition per se.

THE RELATIVISTIC VIEW OF THE PROPERTY "ADVANCED"

This consideration is discussed in the context of Problems 3–5 below, about a student who can do proportional reasoning but is not yet able to reason in terms of functional representation, and hence does not work in a mathematically efficient fashion.

In a secondary mathematics lesson on exponential decay, the homework included the problem:

Problem 3: The annual rate of inflation in a certain year is 8%. How much will the dollar lose of its purchasing power during this year?

Student H's solution was the following

Solution 3:

H: What costs \$1 at the beginning of the year will cost \$1.08 at the end of the year. If a product costs \$1 at the beginning of the year, that product would cost \$1.08 at the end of the year. We want to know how much of the product we can buy for \$1 at the end of the year. We are not going to be able to buy the whole product for one dollar, only a portion of it. Let's say we can buy x of it for \$1. Then [reasoning proportionally] $1/1.08 = x/1$. $x = 1/1.08 = 1/(1$

+ 8/100) = 100/108. We can buy only 100/108 = 92.6% of the product. The dollar lost about 7.4% of its purchasing power.

Following H's presentation of her solution, the teacher introduced the following generalization (without labeling it so):

Problem 4: The annual rate of inflation in a certain year is a%. How much will the dollar lose of its purchasing power during this year?

The teacher went on to present the following solution:

Solution 4:

Teacher: As H said, a product that costs \$1 at the beginning of the year would cost \$(1 = a/100) at the end of the year. Our goal is to find out how much of the product we can buy for \$1 at the end of the year. If x is the fraction of the product we can buy for \$1, then, as H did, x can be obtained from the equation: $1/(1 + a/100) = x/1$. Solving for x, we get: $x = 1/(1 + a/100) = 100/(100 + a)$, or $100(100/(100 + a))$%. Thus, if the annual inflation rate is a%, then the dollar loses $(100 – 10000/(100 + a))$% of its purchasing power.

Following this work, the teacher discussed with the students the graph of the function $f(a) = 100 – 10000/(100 + a)$, and its physical (economic) meaning. Specifically, he discussed these questions: What are the roots of the function? Where is it defined? What is the behavior of the graph of the function, and what is the economic meaning of these behaviors (e.g., the economic meaning of $a = –100$, or an annual rate of 100% deflation)?

The next set of homework included the following problem.

Problem 5: During one year, the dollar lost 12.7% of its value. What was the annual rate of inflation during that year?

H applied a similar reasoning to that which she used to solve Problem 3:

Solution 5: At the end of the year, with \$1 I can buy only (100 – 12.7)% = 87.3/100 of the product. The whole product would cost \$$y$. $y/1 = 1/(87.3/100)$. $y = 100/87.3 \approx 1.145$. The annual rate of inflation is about 14.5%.

H's solution involves an application of proportional reasoning—a sophisticated way of thinking that warrants the label "advanced," by our definition. First, additive reasoning—an antecedent to proportional reasoning—produces responses that are valid within a particular context but generates invalid responses outside this context. Indeed, the transition from additive reasoning to proportional reasoning requires one to construct a different way of understanding relationships between quantities. Second, research has shown that additive reasoning withstands "occasional contradictions," in that students continue to reason additively after they are shown its inapplicability in certain situations. Finally, the "establishment of a better piece of knowledge," that of proportional reasoning, does not completely remove its application in multiplicative situations—students continue to use it after they have been exposed to the concept of proportionality.

Going back to Solution 5, note that H did not realize that she could obtain the solution by substituting 12.7 for $f(a)$ and solve the equation, $12.7 = 100–10000/(100 + a)$ to obtain the annual rate of inflation a. When she was shown this approach, she had difficulty comprehending it. The latter approach exemplifies a way of mathematical thinking that manifests, among other things, economy of thought. "Economy of thought," in this case, has to do with one's ability to reify Solution 4 into a "solution method." It has been shown that reification is one of the most complex processes in the conceptual development of mathematics—with the individual (e.g., Dubinsky, 1991; Greeno, 1983; Harel & Kaput, 1991) and in the history of mathematics (Sfard, 1992).

It is critical to emphasize that one cannot and would not appreciate the efficiency of the latter solution if he or she has not gone, in various problematic situations, through an elaborated solution, such as that offered by H. Hence, although we desire to label the functional solution as more advanced than the elaborated solution, it may be unlikely that the former could be constructed without the latter. Of course, the student's background plays a critical role. For example, if a student understood functions before studying inflation, the function solution would likely be easier for her. This raises a question that is important to curriculum development and instruction: What possible instructional treatments can help H construct this and other ways of advanced mathematical-thinking?

"Proportional reasoning" and "reification of a solution into a solution method"—the two ways of mathematical thinking that emerged in the analysis of this last problem—are examples of what we, as mathematics educators, feel should be labeled "advanced." This is so because we recognize that these develop during a long period of intellectual effort and have proved essential and effective in doing and creating mathematics. Proportional reasoning, for example, is indispensable in many areas of mathematics, and it demands a reconceptualization of mathematical reality—from a world that is organized solely according to additive principles to a world that is organized according to a differentiation of additive phenomena from multiplicative ones. Noelting (1980a, 1980b) found that even among students who had had the usual instruction dealing with proportions, it was quite common for the students to instead use a unit-rate thinking in working proportion problems, a practice also observed among practicing teachers in the intermediate grades (Harel & Behr, 1995). Lamon (1999) has identified several steps in a possible development of proportional thinking, and Cai and Sun (2002) have described the carefully planned development of proportion in a Chinese curriculum. Arriving at a level of thinking that might be called genuine proportional thinking is not just a matter of telling students about cross-multiplication.

OBSTACLES INVOLVED IN THE DEVELOPMENTAL PROCESS

This consideration is discussed in the context of Problem 6 below. It shows a major obstacle—to our knowledge little discussed in the literature—that students encounter in building the way of thinking of representing word problems algebraically. The obstacle is not in forming a propositional representation of the problem; rather, its roots seem to lie in the subtle distinction between "variable" and "unknown"—a difficulty that might be appreciated through historical considerations.

Problem 6. Find a point on the number line whose distance from 1 is half its distance from –4.

Solution 6. L, a prospective elementary school teacher, drew a number line and marked on it the points, 1 and –4. After a long pause, L indicated that he did not know what to do next. His teacher proceeded by asking him to describe the problem. In the process of doing so, L indicated—erroneously—that the unknown point couldn't be to the left of 1. It was clear from his description that he understood the problem. L's argument about the location of the unknown point—despite being erroneous—supports this claim.

Teacher: Very good. What is the distance between x and –4?
L: x plus 4
Teacher: Write that down, please.

L writes $x + 4$.

Teacher: And what is the distance between x and 1?
L: Half of $x + 4$

L writes $(x + 4)/2$.

Teacher: How else can you express the distance between x and 1?

L reads the problem again.

L: It says it is half the distance from –4.

At this point L was unable to express the distance between x and 1 in a different way from $(x + 4)/2$.

In our experience, the difficulty of forming equations, as in this case, is common among students. A possible conceptual basis for this difficulty is the following. For an expert, a value x representing an unknown in a word problem would involve two ways of understanding. One is expressed in the condition of the problem; the other in the variability of the quantities involved. In our case the condition is "The distance of the unknown point x from 1 is half its distance from –4," and the variability is that of the functional expressions $x + 4$ and $x – 1$. These two ways of understanding are independent of each other. In the former x is an unknown whereas in the latter it is a variable. There might be different explanations for L's difficulty. L may not have constructed these two ways of understanding, may have had difficulty coordinating them, or once he constructed one way of understanding had difficulty attending to the other.

The distinction between "variable" and "unknown" is likely to be more epistemological than didactical—a claim that can be supported by the historical development of the notion of "variable" in the 17th century. As we have discussed earlier, for an obstacle to be epistemological it is necessary that it has occurred in the historical development of mathematics.

WHAT ARE REASONING PRACTICES BY WHICH ADVANCED MATHEMATICAL-THINKING CAN BE ENHANCED?

Our answer to this question is an instructional treatment guided by a system of learning-teaching principles, called the DNR system. The three chief principles of the system are Duality, Necessity, and Repeated Reasoning. In this section we briefly describe the first and last; the middle will be mentioned in the next section. (For the complete description of the system, see Harel, 1998, 2001.)

The Duality Principle.

This principle asserts that

Students' ways of thinking impact their ways of understanding mathematical concepts. Conversely, how students come to understand mathematical content influences their ways of thinking. (Harel, 1998, in press)

Clearly, one's ways of thinking, both good and bad, influence one's further ways of understanding. A student whose way of thinking involves believing that a mathematics story problem should be solved quickly by looking for a key word and then waiting for a teacher's reaction to the answer will certainly derive a different way of understanding for story problems (i.e., will solve them differently) than a student willing to spend several minutes making a drawing, looking for relationships, and then striving for some sort of self-verification. The Duality Principle asserts that the converse is also true, and so teachers and curriculum developers in all grade levels should structure their instruction in a way that provides students with opportunities to construct advanced mathematical-thinking from ways of understanding.

There are powerful examples of the relationship of advanced mathematical-thinking in school mathematics to advanced-mathematical thinking. Consider again the "multiple ways of understanding" we mentioned earlier.

Most students' repertoires of reasoning do not include the way of thinking that "A concept can be understood in different ways," and that "It is often advantageous to change ways of understanding of a concept when attempting to solve a problem." The learning of linear algebra, an advanced-mathematical thinking topic, requires multiple ways of understanding, for one must realize, for example, that problems about systems of linear equations are equivalent to problems about matrices, which, in turn, are equivalent to problems about linear transformations. Students who are not equipped with these ways of thinking are doomed to encounter difficulties. At the precollege level, there are various opportunities to help students think in these ways. The list of ways of understanding fractions mentioned earlier provides one such opportunity. Students should learn, for example, that the fraction 3/4 can be understood in different ways: 3 individual objects, each of quantity 1/4; the result when 3 objects of the same size are shared among 4 individuals; the portion of the quantity 4 that equals the quantity 3; and 3/4 as a mathematical object, a conceptual entity, a number. Similarly, students should become comfortable with the different ways in which many functions can be represented—table, graph, equation, for example—and translations among these representations. Students should also learn that depending on the nature of the problem, some interpretations or representations are more advantageous than others. We believe that it is from these kinds of ways of understanding that students construct the aforementioned ways of thinking.

THE REPEATED REASONING PRINCIPLE

Research has shown that repeated experience, or practice, is a critical factor in enhancing, organizing, and abstracting knowledge (Cooper, 1991). The question is not whether students need to remember facts and master procedures but how they should come to know facts and procedures and how they should practice them. This is the basis for the Repeated Reasoning Principle: "Students must practice reasoning internalize and interiorize specific ways of thinking and ways of understanding" (Harel, 2001).

Consider again two important ways of thinking we mentioned earlier: "mathematical efficiency" and "transformational proof scheme."

Two elementary school children, S and T, were taught division of fractions. S was taught in a typical method, where he was presented with the rule $(a/b) \div (c/d) = (a/b) \cdot (d/c)$. The rule was introduced to him in a meaningful context and with a mathematically correct justification that he understood, but was asked to repeat. T, on the other hand, was presented with no rule but consistent with the duality principle and the repeated reasoning principle, she was always encouraged to justify her mathematical actions. Each time she encountered a division of fractions problem, she explained its meaning using her understanding of division of whole numbers as the rationale for her solution. S and T were assigned homework problems to compute divisions of fractions. S solved all the problems correctly, and gained, as a result, a good mastery of the division rule. It took T a much longer time to do her homework. Here is what T—a real third-grader—said when she worked on (4/5) ÷ (2/3):

> How many 2/3s in 4/5? I need to find what goes into both [meaning: a unit-fraction that divides 4/5 and 2/3 with no remainders]. 1/15 goes into both. It goes 3 times into 1/5 and 5 times into 1/3, so it would go 12 times into 4/5 and 10 times into 2/3. [She writes: 4/5 = 12/15; 2/3 = 10/15; (4/5) ÷ (2/3) = (12/15) ÷ (10/15)]. How many times does 10/15 go into 12/15? How many times do 10 things go into 12 things? One time and 2/10 of a time, which is 1 and 1/5.

T had opportunities for reasoning of which S was deprived. T practiced reasoning and computation, S practiced only computation. Further, T eventually discovered the division rule and learned an important lesson about mathematical efficiency—a way of thinking S had little chance to acquire.

In Harel and Sowder (1998) we argued that a key to the concept of mathematical proof is the transformational proof scheme—a scheme characterized by consideration of the aspects of the conjecture, application of mental operations that are goal oriented and anticipatory, and transformations of images as part of a deduction process. The education of students toward transformational reasoning must not start in college. Otherwise, years of instruction that focus on the results in mathematics, rather than the reasons behind those results, can leave the impression that only the results are important in mathematics, an opinion sometimes voiced even by university mathematics majors. We argued that instructional activities that educate students to reason transformationally about situations are crucial to students' mathematical development, and that these activities must begin at an early age.

The building of environments in which students regard the giving of reasons as a natural part of mathematics is one of the more exciting aspects of some studies with children in the primary grades (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Fuson et al., 2000; Maher & Martino, 1996; Yackel, Cobb, Wood, Wheatley, & Merkel, 1990). Having discussions about which of 2.12 and 2.113 is larger can reveal something important about the children's ways of understanding, and hence, have implications for their ways of thinking. Some may rely erroneously on the number of digits, a way of understanding that naturally develops with whole number work. Or, in comparing 4.21 and 4.238, it may come out that some students focus on the right-most place value and decide that 4.21 is larger because hundredths are larger than thousandths (Resnick et al., 1989). Such discussions would seem to be more valuable in the long run than practicing a teacher-given rule about annexing zeros until each number has the same number of decimal places, especially if the discussions led naturally to the rule.

Similarly, ready-made theorems, formulas, and algorithms, even when motivated and completely proved, are often hastily introduced in undergraduate mathematics courses. An interesting phenomenon was observed in our teaching experiments (Harel, 2001; Harel & Sowder, 1998). It illustrates the importance of practicing mathematical reasoning. Until a mathematical relationship was declared a theorem, the students continued—either voluntarily when they needed to use the relationship or upon request—to justify it. Once the relationship was stated as a theorem, there seemed to be a reduced effort, willingness, and even the ability of some of the students to justify it. This phenomenon was explained in terms of the students' authoritarian view of mathematics (another example of an undesirable, yet common, way of thinking): For them, the label "theorem" renders the relationship into something to obey rather than to reason about. Or, possibly, in the teaching experiment context these students had not practiced enough the reasoning behind the theorem.

WHAT ARE REASONING PRACTICES BY WHICH ADVANCED MATHEMATICAL-THINKING CAN BE HINDERED?

Epistemological obstacles are perhaps more fascinating as objects of scholarly study than didactical obstacles, but we must attend to the latter, for if narrow or faulty instruction leads to problems in thinking or understanding, it should be easier to correct such instruction than it may be to overcome an epistemological obstacle.

Certain teaching practices are still in existence, and even widely used, despite the consensus among mathematics education researchers that they lead to didactical obstacles that are difficult to eradicate. The emphasis on "key" words in instructing students on how to decide what operation to do in solving story problems is an example. Students learn that the phrase "all together" in a problem statement should signal addition; "left" should signal subtraction; "per" should summon multiplication or division, etc. Such instruction, although well intentioned, will give at best short-lived success, and will fail completely if problems are not always written to follow such guides (as in "Thirty rows, with 42 seats in each row, will seat how many, all together?"). More important, these ways of understanding would reinforce faulty ways of thinking—that in doing mathematics what counts is the result, not the reasoning process.

It is fair to say that most instructional planning is a mix of art and science, with art playing the major role. In an effort to be provocative, we challenge some of the usual principles—in our view they are myths—that might guide one's instruction. Like the "key words" approach above, these principles may be helpful in the short run, but may prove to be unhelpful or even counterproductive in the long run. Each of them certainly merits research attention.

Myth 1: In sequencing instruction, start with what is easy. For example, it is common to introduce methods of solving equations with examples like $x + 2 = 7$ and $3x = 15$. Because these can be solved virtually by inspection, the students may see no need for the usual canons for solving equations, and thus the Necessity Principle (Harel, 1998)—students are more likely to learn when they see a genuine need (intellectual, not necessarily social or economic)—is violated. Much better first examples might be $x + 75.6 = 211.3$ and $1.7x = 27.2$ or even $2.4x + 9.6 = 17.28$, examples not likely to be easily solved by inspection or guessing. In the same vein, perhaps a treatment of congruent figures should start with complicated figures rather than the usual congruence of segments, angles, and triangles. Dienes and Golding long ago suggested that such a "deep-end" approach might be appropriate in many cases:

> At first it is not always wise or useful to present a new mathematical concept in its simplest form .… It has been found that, at least in some cases, it is far better to introduce the new structure at a more difficult level, relying upon the child to discover the less complex sections within the whole structure. (1971, p. 57)

Hence, a building-blocks metaphor in designing curriculum may not be the most useful one, especially if the learner has no idea of the building that will eventually be finished. A more apt metaphor for designing curriculum might be based on some sort of deep-end metaphor, perhaps starting with a picture of the building and the question, "How would you build this?"

Myth 2. The best mental model is a simple one, preferably one quite familiar to the students. For example, instruction in linear algebra often uses coordinate 2-D and 3-D geometry as the first examples of a vector space. Harel

(1999) argued that these examples constrain students' understanding, so that they think vector space ideas are just ideas about geometry: Linear algebra "=" geometry. Consequently they have difficulty dealing with nongeometric vector spaces. He suggests that using systems of linear equations as a first way of understanding vectors at least keeps the students' thinking in an algebraic domain.

Here is another instance in which starting with the simplest situations may create a didactical obstacle. Multiplication is always introduced as repeated addition; this natural but confining approach seems to lead almost inexorably to the erroneous "multiplication makes bigger" idea (e.g., Fischbein, Deri, Nello, & Marino, 1985; Greer, 1987). Perhaps introducing multiplication as meaning "copies of" would serve the students better (Thompson & Saldanha, 2003): 2 × 4 tells you how many are in 2 copies of 4, and 2/3 × 6 tells you how many are in 2/3 of a copy of 6—thus enveloping repeated-addition and fractional-part-of-an-amount interpretations into one way of thinking about multiplication. We do recognize that other ways of understanding multiplication should also, and usually do, come up in the mathematics curriculum. Such an instructional approach is needed to advance the ways of thinking "a concept can have multiple interpretations" and "it is advantageous to have multiple ways of understanding."

In general, instruction that uses examples limited in some irrelevant or confining way runs the risk of over-generalization, with the irrelevant characteristic perhaps becoming a part of the concept—everyone knows what to draw when asked to draw an "upside-down" trapezoid (cf. Sowder, 1980). The first choices of examples may be crucial, as Marshall's work (1995) with schemas for story problems suggests.

Myth 3. In advanced undergraduate mathematics, begin with the axioms. Starting with the basic rules of the game might seem sensible, but we feel that the typical undergraduate student is not yet ready to play the game that way. Our argument builds on our notion of "proof scheme" mentioned earlier—a proof scheme guides what one does to convince oneself and to convince others (Harel & Sowder, 1998). Our studies of the proof schemes of undergraduate mathematics majors suggest that extensive earlier work entailing deductions by the student, putting two or more results together to get a new result (deductive proof schemes) must precede any meaningful work with axiomatic developments (axiomatic proof schemes). Otherwise the student may just go through the motions, often rotely, without any genuine appreciation of the development from axioms.

Myth 4. In school practice, use mathematical proofs to convince the students that a mathematical result is certain. We know that an argument of "but how can you be sure, without a proof" is often used, and that of course mathematicians do look for arguments to assure themselves (and their referees) that the result is indeed established. But mathematicians often look for more than certainty in their proofs—What is the key to the result? Or, does a slight modification in the proof suggest another result? Rav (1999) even claimed that mathematical knowledge is embedded in the proofs, with the theorem only a "headline" (p. 20). But, to repeat an earlier point, we have noticed that a proof for many students is either something to ignore in favor of studying the result, or something only to be dutifully memorized for purposes of repetition on an examination. Indeed, labeling a result with "theorem"—and that labeling alone—often means that the result is certain and requires nothing more, as we noted earlier.

We hypothesize that it is better to emphasize the reasoning, perhaps in several examples, that a proof generalizes. The earlier example in which the child continually utilized a meaning-based argument for calculating divisions by fractions illustrates our point. Brownell (1956) emphasized that the quality of practice, rather than just practice itself, was most important. Carefully planned practice could guide the student's thinking to a higher level. For example, the exercises in Figure 2 could precede, indeed could generate, the result about the relation between the measures of vertical angles, at the same time they are providing practice with the angle sum for a linear pair.

Here is another example of practice paving the way to a result. Suppose the target is one version of the fundamental theorem of calculus: Under certain conditions on f, with F an antiderivative of F, $\int_a^b f(x)dx = F(b) - F(a)$. A common starting point for this version is another version of the fundamental theorem: $\frac{d}{dx}\int_a^x f(t)dt = f(x)$, again with conditions on f. Paraphrasing the latter gives that the integral is an antiderivative of $f(x)$. Hence, for example, $\int_2^x (\cos t)dt$ is the antiderivative of cos x, or cin x + C. (Then the practice begins.) Therefore, $\int_2^3 (\cos t)dt = \sin 3 + C$, but $\int_2^2 (\cos t)dt = \sin 2 + C = 0$, so C = –sin2, and $\int_2^3 (\cos t)dt = \sin 3 + C = \sin 3 - \sin 2$. Repetitions of the argument with other integrals sets the stage for the general argument (= proof) that $\int_a^b f(x)dx = F(b) - F(a)$, with $F(x)$ an antiderivative of $f(x)$.

Find the measurements x and y.

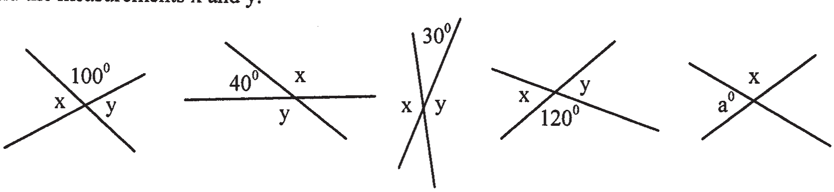

FIGURE 2 Practice leading to a general result.

Opportunity to Learn

The most serious didactical obstacle is a lack of opportunity to learn. In particular, we have in mind the (good) ways of thinking and understanding mentioned earlier, and the "habits of mind" of Cuoco, Goldenberg, and Marks (1996). Instruction (or a curriculum) that ignores sense-making, for example, can scarcely be expected to produce sense-making students. Computational shortcuts like "move the decimal point" or "cross-multiply" or "invert and multiply" given as rules without any attention as to why these work turns elementary school mathematics into what is deservedly called a bag of tricks. Also, students who never have a chance to make conjectures cannot become more skilled at conjecturing—and it may be hypothesized that students who have never conjectured do not see any need for mathematical proof. And so on.

Not an Exclusive Or

Whether a particular obstacle to learning is didactical or epistemological, in an exclusive-or sense, is, we believe, too limiting. Harel (in press) offers the view that an obstacle may be partly didactical and partly epistemological. Consider, for example, "multiplication makes bigger" (MMB), the well-documented misconception mentioned earlier that is an obstacle for many students (through college) in choosing an operation for solving a story problem (Greer, 1987). MMB clearly meets Duroux's partially-valid and obstinacy criteria, and one might argue that it also has historical roots, with multiplication probably first formalized with whole numbers. Yet, MMB could perhaps have its influence allayed, if not nullified, by some instructional modification like some more-inclusive view, say the "copies of" interpretation mentioned earlier, or perhaps by exploring "what it would be" via a calculator calculation of something like 0.2×15 or $(1/2)x24$ at an age before extensive experience with whole numbers leads to MMB. Hence, MMB might be positioned on a didactical versus epistemological set of axes as in Figure 3.

In a similar way, one can conjecture difficulties with proportional reasoning, with understanding $(-1)(-1) = +1$, with linear independence, or with some notational conventions like $\sin^{-1} x$, as being both didactical and epistemological in nature, as we have speculated in Figure 4.

SUMMARY

Our view is that the roots of mathematical thinking for advanced mathematics must be fostered during the study of elementary mathematics. General ways of thinking, built on rich ways of understanding in elementary mathematics, can then symbiotically support further ways of understanding and of thinking in advanced

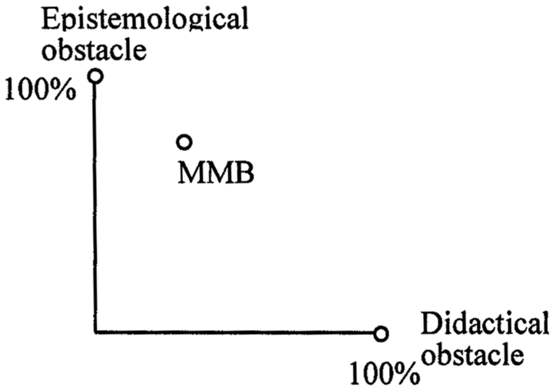

FIGURE 3 Multiplication makes bigger (MMB) as a mix of obstacles.

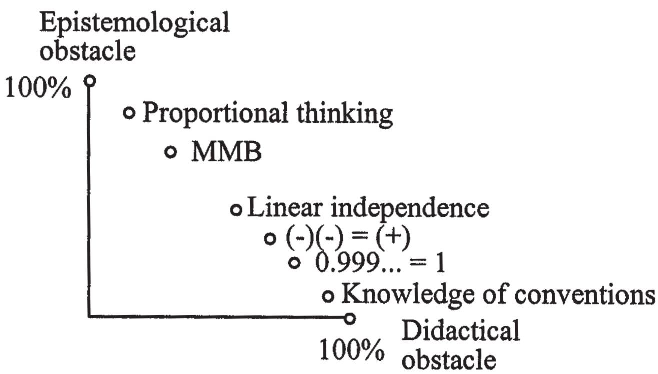

FIGURE 4 Hypothesized mixes of types of obstacles.

mathematics. Obstacles to ways of thinking and ways of understanding may be epistemological and/or didactical, with didactical obstacles more easily identified and perhaps more easily overcome than epistemological obstacles. We propose that a way of mathematical thinking be called "advanced" if its development necessarily involves at least one of the three necessary conditions for epistemological obstacles identified by Duroux (1982, cited in Brousseau, 1997). An important next step will be to identify ways of thinking that meet this criterion.

We endorse the DNR-based instruction for furthering ways of thinking and ways of understanding: (Duality Principle) make the dually supportive roles of ways of thinking and ways of understanding a conscious, carefully planned part of the cognitive objectives for coursework in mathematics; (Necessity Principle) build instruction via problems that contain intellectual appeal to the students; and (Repeated Reasoning Principle) involve repeated reasoning to give a firm foundation for ways of thinking and ways of understanding. Finally, by labeling them "myths," we offer a critique of some teaching "axioms" that have face validity but might actually hinder the development of fruitful ways of thinking and ways of understanding.

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