Secondary Teachers' Meanings for Function Notation in the United States and South Korea [1]

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ABSTRACT

This study investigates what teachers in U.S. reveal about their meanings for function notation in their written responses to the *Mathematical Meanings for Teaching secondary mathematics* (MMTsm) items, with particular attention to how productive those meanings would be if conveyed to students in a classroom setting. We then report South Korean teachers' responses to see whether the meanings U.S. teachers demonstrated are shared with South Korean teachers. The results show that many U.S. teachers use function notation to name rules instead of to represent relationships. The data from South Korean teachers indicates that the problematic meanings in U.S. teachers' responses are shared with a minority of South Korean teachers. The results suggest a need for attention to ideas regarding function notation in teacher education for pre-service teachers and professional development programs for in-service teachers.

Keywords:
Mathematical meanings for teaching
Function notation
South Korea
United States

"Why do we say f of x when all we really mean is y?" (Algebra 2 teacher in a professional development workshop)

1. INTRODUCTION

Functions play a central role in mathematics, not just technically in terms of a definition, but conceptually in terms of ways of thinking. While the concept of function evolved in mathematics over centuries (Kleiner, 1989, 1993), discussions of their role in school mathematics began early in the 20th century (Hamley, 1934; Hedrick, 1921; Krüger, 2019). Discussions focused on representing quantitative relationships: "Functional thinking did not mean teaching the concept of function as we understand it today. Rather, it [focused] on a specific kinematic mental capability that can be described by investigating change, variability, and movement." (Krüger, 2019, p. 33). From this perspective, a focus on functions in secondary school mathematics entails ideas of variable, variation, and the co-variation of quantities. It is in more recent history that ideas of function as object, transformative process and relationship have emerged (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; DeMarois & Tall, 1996a; Dubinsky & Harel, 1992; Harel & Dubinsky, 1991; Sfard, 1992).

While research on students' understandings of function is abundant, there is little research on students' or teachers' understandings of notational conventions by which functions are represented. Studies too numerous to mention use function notation in tasks without probing what the notation itself means to students or teachers. For example, DeMarois and Tall (1996b, 1996c) mention function notation as an aspect of a function concept but focus on students' understandings of function as process or relationship and the extent to which they retained meanings across graphical, symbolic, and verbal registers. We do not learn what students understood function notation to *mean.* Two studies closest to probing students' meanings for function notation were by Carlson (1998) and Sajka (2003). We discuss Carlson's and Sajka's studies in more detail in the next section.

The issue of how to represent a functional relationship generally dogged mathematicians for centuries. Our reading of Cajori's history of function notation (Cajori, 1929, entries 642-646) is that a driving desire was to name three things simultaneously and economically: (1) a deterministic relationship between two quantities' values, (2) what today we would call the function's argument—a value of the independent quantity, and (3) the value related to the argument under the named relationship. To this end, Euler introduced today's convention in 1734 with the statement "If $f(\frac{x}{a}+c)$ denotes a function of $\frac{x}{a}+c$ …"[2] (Cajori, 1929, entry 643).

We unpack the conceptual sophistication and depth of Euler's convention in the next section as a backdrop for our aim of highlighting meanings teachers hold for function notation as a potential source of difficulties experienced by students. A natural question is whether undesirable meanings students hold are epistemologically necessary in coming to understand function notation well (similar to children first thinking of fractions like so many pieces of a pie) or are these meanings possibly conveyed unintentionally to students by teachers who also hold them?

[2] Si $f(\frac{x}{a}+c)$ denotet functionem quamcunque ipsius $\frac{x}{a}+c$ …

It is for this reason we investigated these questions: (1) *What meanings do high school teachers in the United States have for function notation*, and (2) *Are these meanings shared among teachers from the United States and South Korea*?

2. BACKGROUND

We first offer a conceptual analysis of Euler's proposal for representing the value of a function in relation to its argument before discussing past research on students' understandings of function notation. We do this to highlight the complexity of understanding function notation productively.

As we stated earlier, Euler's proposal satisfied three requirements addressed separately in prior notational proposals—the notation "$f(\frac{x}{a}+c)$" names the relationship between quantities' values "*f*", it specifies the function's argument (i.e., a value of the independent quantity)[3], and it represents the value of the dependent quantity in relation to the value of the argument. For this notation to be meaningful to a reader of it, she must hold three subsidiary meanings: a meaning for "relationship between two quantities values" that allows the relationship to be named, a distinction between what today we call "input" and "argument" of a relation, and a meaning for "the value of a dependent quantity in relation to a value of an independent quantity". We borrow Thompson and Milner's (2019) summary of this in the figures below.

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[3] There is some ambiguity regarding whether "x" or "$\frac{x}{a}+c$" in Euler's statement should be considered a value of *f*'s independent quantity. We settle on "$\frac{x}{a}+c$" because its value is what *f* maps to an element in its range. The ambiguity arises because "$f(\frac{x}{a}+c)$" implicitly denotes a composite function.

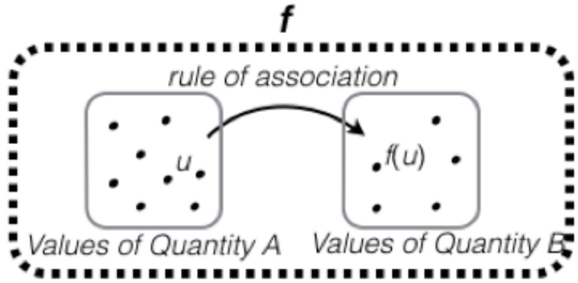

Figure 1. General meaning of function and function notation (Thompson & Milner, 2019, p. 56)

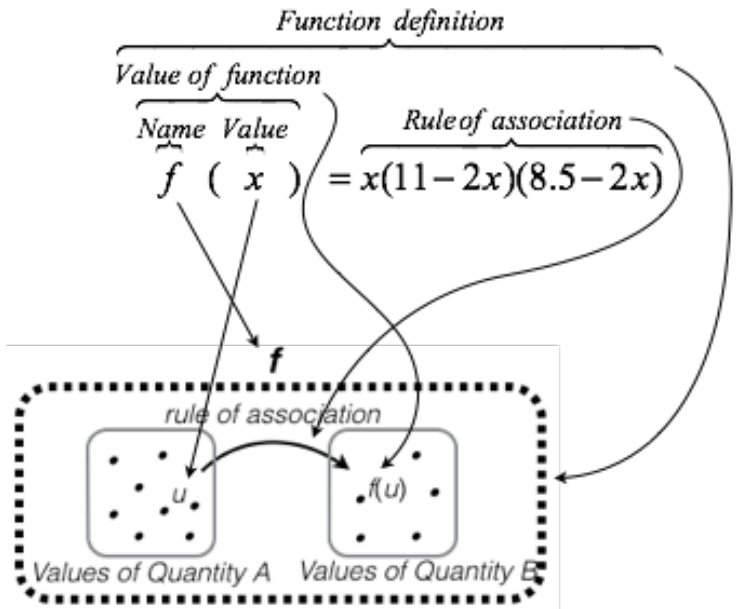

Figure 2. Relationship between function notation and general meaning of function (Thompson & Milner, 2019, p. 56)

Figure 1 is a common depiction of an image of a named relationship between values of two variables. Figure 2 is more subtle. It depicts how a function definition using function notation aligns with a general image of function while at the same time making it specific and re-usable.

The distinction between input and argument is an important aspect of Euler's notation. It affords the possibility of representing a composition of functions economically, and it supports an important flexibility of thought. For example, the cosine function is periodic with period 2π. But the period is with respect to cosine's *argument.* If $f(x) = \cos(3x + 2)$, then *f* has a period of 2π with respect to 3*x* + 2, not *x*. It is when $3x + 2$ varies by 2π that cosine repeats values.

Finally, if a person has a fully formed scheme for function notation as depicted in Figure 2, she can invoke the scheme even with partial information about it. She can understand a phrase like "The function *f* …" as referring to the entire scheme, albeit with parts of the scheme unspecified.

The ability to invoke one's function scheme in the presence of partial information supports using function notation representationally. By this we mean having a meaning for uses of function notation that do not involve a computational rule. We can say, for example, "Let *d*(*x*) be the distance between the center of Earth and center of Mars *x* years after 00:00 January 1, 1900." Even with this non-computational definition of *d* the meaning of *d*(52) is clear, as is the meaning of $(d(52.001) - d(52))/0.001$.

We began this section with a conceptual analysis of Euler's notation as backdrop for discussing research on students' and teachers' understandings of function. This research has focused predominantly on the sophistication with which they conceived calculating a function's values and less on the idea of function as a relationship between quantities' values or on representations of such relationships. Even (1993) reported a large number of pre-service teachers thought "functions are equations and can always be represented by formulas" (p. 104). Similarly, Vinner (1983) found that 10th and 11th grade students think a "function is a rule of correspondence", and that "function is an algebraic term, a formula, an equation, and arithmetical manipulations"(pp. 8-9). Hitt

(1998) reported that teachers were likely to expect continuous functions to be defined by a formula.

2.1. Research on understandings of function notation

Carlson (1998) and Sajka (2003) reported students' difficulties with function notation. Carlson included many tasks designed to reveal consequences of students' meanings for function notation. College algebra students in Carlson's study showed limited understanding of function notation. For instance, the students calculated $f(x+a)$ given $f(x) = 3x^2 + 2x - 4$ by simply adding an *a* to the expression on the right side (i.e. $f(x+a) = 3x^2 + 2x - 4 + a$). Carlson's analysis, however, focused on the extent to which students were successful on the tasks without reverse-engineering the meanings for function notation that might have led to their behavior. Sajka's (2003) case study involved a 45-minute interview with one high school student, Kasia, who had seen function notation for three years. Sajka's case study employed DeMarois's and Tall's (1996b, 1996c) focus on function as process or concept, but included a clear focus on the student's understandings of function notation itself. Sajka's (2003) penetrating analysis of Kasia's thinking showed Kasia moving from thinking "*f*" means "the beginning of a function formula" to thinking of "*f*(*x*)" as serving the same role as "*y*", to "*f*(*x*)" as part of a statement which itself will lead to a graph, to confusion over the meaning of "$f(x+y) = f(x) + f(y)$", to thinking that "$f(x+y) = f(x) + f(y)$" is an example of the distributive property. Sajka's report that Kasia understood "*f*(*x*)" as the start of a function formula is in line with Thompson's (1994) report of students thinking a function definition consists of the "*f*(*x*)" on the left side, the symbol "=", and an algebraic expression on the right side. Thompson (1994) also pointed to this way of thinking as behind the common mistake of mismatched letter on the left and letter on the right, such as $f(x) = n(n + 1)(n + 2)$.

Musgrave and Thompson (2014) and Thompson and Milner (2019) shifted focus from students' to teachers' understanding of function notation. Musgrave and Thompson (2014) and Thompson and Milner (2019) found that, for many teachers, function notation served as a label or a name for the defining formula rather than a representation of one quantity's values in relation to another quantity's values. While students meaning of function notation as a label might be a root of so many reports of students' difficulties with it (e.g., Carlson, 1998; Dreyfus & Eisenberg, 1982; Vinner & Dreyfus, 1989), it is important to investigate the possibility that teachers hold similar meanings.

In this study, we offer further support for findings of other studies related to students' or teachers' difficulties in understanding of function (Carlson, 1998; Dreyfus & Eisenberg, 1982; Even, 1993; Gray & Tall, 1994; Hitt, 1998; Musgrave & Thompson, 2014; Sajka, 2003; Sfard, 1991; Sierpinska, 1992; Vinner, 1983; Vinner & Dreyfus, 1989). We then extend the literature by exploring teachers' meaning for function notation.

Teachers' use of function notation can give information about sources of students' or teachers' difficulties in understanding of function; in particular, we hypothesize that many teachers name rules rather than construct representation of functions in terms of using function notation. Teachers need to develop a rich understanding of a particular mathematical idea to support students' conceptual learning of the idea (Silverman & Thompson, 2008). Tallman and Frank (2018) showed that a teacher's lack of awareness of conceptual affordances led to incoherent instructional actions. In exploring consequences of this study, we point to potential problems for these teachers in supporting students to develop fluency with function notation and an image of function notation as a representational tool.

3. THEORETICAL PERSPECTIVE

We use Thompson (2013a)'s construct of *meaning* to investigate what teachers are revealing about their thinking from their responses on the *MMTsm* items. We then present our perspective on a productive meaning that allows student to build a strong understanding of function notation.

3.1. Meanings

According to Piaget, to understand is to assimilate to a scheme (Skemp, 1962, 1971; Thompson, 2013; Thompson & Saldanha, 2003). Thus, the phrase "a person attached a meaning to a word, symbol, expression, or statement" means that the person assimilated the word, symbol, expression, or statement to a scheme. A scheme is an organization of actions, ways of thinking, images, or schemes (Thompson, Carlson, Byerley, & Hatfield, 2014). When we say *assimilate* we mean the ways in which an individual interprets and make sense of a text, utterance, or self-generated thought. According to Piaget, new schemes emerge through repeated assimilations, which early on require functional accommodations and eventually entail metamorphic accommodations (Steffe, 1991).

Thompson (2013a) said *meaning* is the space of implications of an understanding. For example, a teacher can understand function notation $f(x)$ as multiplication. This is her understanding of function notation in the moment. Then, she could think about $f(2)$ as "f times 2" when first looking at $f(2)$. This is an implication of her understanding in the moment. The teacher's meaning in the moment of an understanding is the space of implications of that understanding, or, as Dawkins (2018) rephrased it, the space of inferences made available once one holds an understanding.

While we cannot access the teachers' mathematical meanings directly, we can delimit categories of responses according to particular mathematical meanings that we discern from them. We categorize teachers' response based on meanings we believe might underlie the response based on the best available evidence of interviews and prior qualitative work. We assume that, for the most part, meanings that teachers used to construct their responses to an item are meanings that would guide their decisions in the classroom. Our focus on teachers' meanings as a root for their actions allows us to think of meanings students might construct based on meanings we attribute to teachers.

3.2. Productive meanings for teaching function notation

By productive meanings for teaching we mean meanings a teacher holds that would be productive for students' long-term learning were the teacher to convey them. Being productive for student learning is not the same as being mathematically correct. For example, a formal definition of function, "a subset of the cross product of two sets such that every member of one set appears as a first element in a pair and no member of the second set appears as the second element in two pairs" is mathematically correct. However, it is generally accepted that this meaning will not be productive for high school students. On the other hand, a meaning for continuous function that it has a graph which, in principle, can be sketched without taking pencil off paper is mathematically incorrect[4], but could be productive for high school students.

We presented our perspective on productive meanings for teaching function notation in Figure 2. Function notation $f(x)$ represents the function's values in relation to values of the independent variable x in the context of a relationship named "f". We may use "$f(u)$" or "$f(x)$" to represent the value of the function f regardless of whether we know a rule of assignment for it. We will say someone uses function notation representationally when he uses function notation to represent the value of a function in relation to values of its argument, without mentioning an actual rule of assignment.

4. METHOD

This study focuses on teachers' responses to two function notation items on the *MMTsm*. analyzing 252 U.S. and 366 Korean secondary mathematics teachers' responses. We then investigate relationships within each item as well as between the two items in order to see ways in which teachers thought of function notation. We include interviews with three teachers to illustrate the spectrum of responses.

4.1. Subjects

During the development phase of the *MMTsm*, several advisors and reviewers wondered if US teachers' problematic responses to items reflected unreasonable expectations for teachers' mathematical understandings. We wanted to see whether the difficulties we were seeing in US teachers' responses are shared with another country.

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[4] For example, $y = \frac{1}{x}, x \neq 0$ cannot be sketched, even in principle, without taking pencil off paper, yet is continuous on its domain.

We thought that the different results from another country would show whether it is reasonable to expect secondary teachers to give high level responses to the items.

The Project Aspire team administered a translated version of the *MMTsm* to 366 Korean secondary teachers in summer 2015 (264 high school[5], 102 middle school) and the English version to 252 US high school teachers in summers of 2013, 2014, and 2015 (see Table 1). The 366 SK (South Korea) teachers were taking a qualification program[6] at four locations: Seoul, a suburb of Seoul, Southwest South Korea, and Southeast South Korea. The 252 US high school teachers volunteered to participate in NSF Mathematics and Science Partnership summer professional development projects taking place in the Southwest and Midwest U.S. They took the *MMTsm* as part of their professional development program.

Table 1. US and SK teachers, school level by major

	Math Majors	MathEd Majors	Other Major	Total
Korea High School teachers	81	175	7	263
Korea Middle School teachers	33	49	19	101
U.S. ≥Calc teachers*	29	24	21	74
U.S. <Calc teachers, **	34	59	85	178
Total	177	307	132	616

* ≥Calc means U.S. high school teachers who taught calculus or higher at least once
** <Calc means U.S. high school teachers who never taught calculus
*** Two Korean teachers and one U.S. teacher did not report their degrees.

In Table 1, “Math Majors” means that a teacher reported having either a bachelor’s or master’s degree in mathematics whereas “MathEd Majors” mean that a teacher reported having either a bachelor’s or master’s degree in mathematics education. “Other Majors”

[5] In Korea, middle school teachers teach mathematics to grade 7th-9th students and high school teachers teach mathematics to grade 10th-12th students.
[6] In Korea, all teaches who have taught more than three years must take a qualification training program to earn “1st class” teacher certificates.

mean that a teacher reported degree(s) that were neither mathematics nor mathematics education. We separated U.S. teachers into teachers who taught calculus at least once and teachers who never taught calculus. U.S. teachers having taught calculus will have taught content similar to Korean high school teachers, and U.S. teachers who never taught calculus will have taught content similar to Korean middle school teachers. Korean teachers taught for an average of 4.3 years (some switched from middle to high or vice versa during this time). The 252 high school teachers in the U.S. taught at least one high school mathematics class (algebra and above). The U.S. high school teachers were asked to write how many times they had taught each subject and recorded the total number of high school classes taught in the assessment. On average the U.S. high school teachers had taught 26.2 year-long classes, which corresponds to approximately 5 or 6 years.

When we administered the *MMTsm*, we asked US and SK teachers to volunteer for an interview. Seventeen teachers (eight SK and nine US teachers) agreed to the interview. We conducted two task-based clinical interviews with the 17 teachers to see why they wrote the responses they did (Goldin, 1997). The difference between US and SK interviews was that SK teachers' interviews included looking at their original responses from a prior administration of the *MMTsm* whereas US teachers were asked to explain their responses after taking the *MMTsm* again. We will present item interviews with three teachers (two US and one SK teachers) because we think the three teachers' interviews illustrate the spectrum of responses from all 17 teachers after analyzing all interviews. In the following section, we present function notation items in the *MMTsm*.

4.2. Tasks

The first item, “Understanding Independent Variables” (see Figure 3) is designed to reveal how teachers thought of the left side of a function definition in relation to the right side in the context of a function definition.

Here are two function definitions.

$$w(t) = \sin(t-1) \text{ if } t \geq 1$$

$$q(s) = \sqrt{s^2 - s^3} \text{ if } 0 \leq s < 1$$

Here is a third function c, defined in two parts, whose definition refers to w and q. Place the correct letter in each blank so that the function c is properly defined.

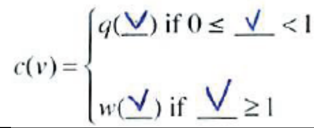
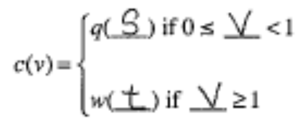
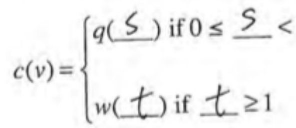
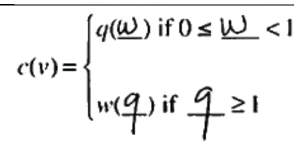
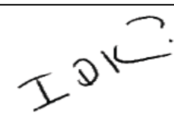
$$c(v) = \begin{cases} q(\_\_\_) \text{ if } 0 \leq \_\_\_ < 1 \\ \\ w(\_\_\_) \text{ if } \_\_\_ \geq 1 \end{cases}$$

Figure 3. The item, “Understanding Independent Variables” © 2015 Arizona Board of Regents. Used with permission.

We had observed that students and teachers often think of function notation idiomatically, “*c* (*v*)” is a four-character name. If a teacher thinks of function notation as a label for the formula that is on the right hand side of a function definition, we anticipate that he or she thinks *w* must always have *t* within parentheses and *q* must always have *s* within parentheses because *t* and *s* are part of the respective function name.

We categorize teachers’ responses in terms of what letters they inserted in each blank to complete the definition of *c*. The categorizations and sample responses are shown in Table 2. We categorize IDK (I don’t know) and NR (No response) separately. However, we will exclude IDK and NR responses from the data presentation in the next section because NR and IDK do not give us insight into teachers’ meanings for function notation.

Table 2. Categorizations and Responses to the item “Understanding Independent Variables”

Level	Categorization	Sample teacher’s response
3	All four spaces filled with the letter *v*	
2	Filled one, two or three spaces with *v*	
1	Filled blanks with *s* and *t*	
Others	- Filled something in blanks other than *s*, *t*, or *v* - The scorer cannot interpret the teacher’s response.	
IDK	I don’t know	
NR	No response	

The second item, “Hari’s Rock”, is shown in Figure 4. The purpose of “Hari’s Rock” is to see how teachers use function notation when prompted to use function notation to represent a dynamic situation. We added “at a non-constant rate” because, in earlier trials, many teachers assumed the radius increased at a constant rate and hence wrote a linear relationship such as $r = k \cdot t$.

Hari dropped a rock into a pond creating a circular ripple that spread outward. The ripple's radius increases at a non-constant speed with the number of seconds since Hari dropped the rock. Use function notation to express the area inside the ripple as a function of elapsed time.

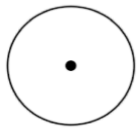


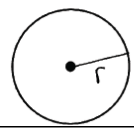
Figure 4. The item, "Hari's Rock" © 2015 Arizona Board of Regents. Used with permission.

The rubric for the second item focuses on two features of teachers' responses: where to use function notation, and how to use variables. We separate these features because they convey different information about teachers' function schemes. The two features are scored independently.

Our first focus in analyzing responses to the item "Hari's Rock" is to see whether a teacher uses function notation spontaneously to represent a quantity (namely, the radius) for which he or she does not have an explicit rule of assignment. For instance, teachers can use "$r(t)$" to represent the value of the radius regardless of whether they know a rule of assignment for it (see Figure 2). We anticipate many teachers would use function notation on the left side for the simple reason that this is "what one does" when using function notation, e.g. write "$A(t)$" instead of "A", and that many teachers would not think to use function notation to represent the radius' value.

Our second focus is on whether teachers used variables consistently on both sides of the equal sign (e.g., $A(t) = \pi\left(r(t)^2\right)$ vs. $A(t) = \pi r^2$). We categorize a response as "Used variables inconsistently" when this was clear in their responses. The new rubric is in Table 3. We categorize IDK (I don't know) and NR (No response) separately, but we exclude IDK and NR responses from the Table 3 because one dimension correlates automatically with IDK or NR on the other.

Table 3. Rubric for “Hari’s Rock” item. © 2015 Arizona Board of Regents. Used with permission.

	Used variables consistently	Used variables inconsistently
Level 3: Used function notation on both sides of the definition	$A(t)=\pi(r(t))^2$	$f(x)=\pi r^2(s)$
Level 2: Used function notation only in the defining rule	$\pi\left\{\int_0^t r'(t)\,dt\right\}^2$	$f(s)=$ non-constant function of t $A=\pi r^2=\pi(f(s))^2$
Level 1: Used function notation on left side only	$A=\pi r^2$ $A(t)=\pi t^2$	 $f(r)=r^t$
Others	$A=\pi r^2$ $=\pi(r^2)t$	$\pi r^2=\pi t^2$ ↑ seconds

Given our stance of categorizing teachers’ responses in terms of meanings rather than correctness, we are not concerned with the accuracy of the model a teacher generated. Rather, we focus only on the teachers’ use of function notation. For instance, we would consider $A(r)=2\pi r$ as a demonstration of using function notation on left side only even though the given formula does not accurately describe the area of the ripple. However, the statement $A(r)=2\pi r$ only uses function notation only on the left side of the function definition and does not use function notation to represent the circle’s radius in the defining rule (i.e. a function to account for the non-constant rate of change for the length of the radius with respect to the time elapsed). Thus, “$A(r)=2\pi r$” fits Level 1 in Table 3.

To us, using function notation in both sides of the function definition in Level 3 suggests that the teacher intentionally utilizes function notation representationally – to represent a relationship between quantities whose values vary, but for which there is no known rule for the relationship (see Figure 2). Level 2 responses contain function notation only in the defining rule. We see Level 2 responses being less productive for students' understanding of function notation than Level 3 responses because it fits a scheme wherein the "*A*" in "*A*(*t*)" means the same as *"A"* used as the name of a variable. We categorize Level 2 responses higher than Level 1 (function notation on left side only) because in writing "*r*(*t*)" the teacher is using function notation representationally. Level 1 responses fit a common meaning of "use function notation" as "write *A*(*t*) instead of *A"* (or "*f*(*x*) instead of *y*").

We feel the use of function notation in the defining rule is more significant than if teachers only used function notation on the left (Level 1 responses) because a teacher's definition such as $\pi\left\{\int_0^t r'(t)\,dt\right\}^2$ in Table 3 conveys to students that radius is a function of elapsed time without the need for a function rule.[7] We view the use of function notation in the defining rule as indicating teachers' spontaneous use of function notation representationally.

The difference between Level 2 and Level 1 lies in whether a teacher used function notation to represent the radius' length. In many Level 1 examples, the teacher tries to create a rule to represent the non-constant rate of change, such as $r(t)=t^2$, instead of simply using function notation to represent that variation. In constructing the rubric for

[7] We ignored the fact that this teacher used "*t*" both as upper limit of the integral and as the variable within $r'(t)dt$.

this item, we hypothesized that teachers' attempts to determine an explicit rule to describe the changing radius corresponds to a view of function as a rule and an inability to use function notation representationally. This aligns with previous research regarding meanings for function as a rule (Even, 1993; Sajka, 2003; Vinner, 1983).

The focus on whether teachers used variables consistently aligns with the first item in that we anticipate that teachers who used variables inconsistently thought of the left hand side ("*A*(*t*)") as the name of the function, imposing no constraint on variables used in the right side. For example, if a teacher wrote "*A*(*t*)" on the left side, but used "*r*" in the defining rule, we consider that he or she used variables inconsistently (e.g. $A(t) = \pi r^2$). The rubric for this item gives the benefit of the doubt regarding some usages of the letter "*r*". If a teacher defined the area as a function of time, but still included "r" in the rule such as $A(t) = \pi r^2 t$, we interpret that the teacher used "*r*" as a parameter and thus used variables consistently. However, we view "*t*" always as a variable to be consistent with conventional usage of "*t*" for varying time. Thus, the Level 1 example $f(r) = r^t$ from Table 3 is coded as using variables inconsistently. We used the Chi-square test to analyze the relationship between the first and the second items as well as the relationship between the two features of the second item,.

Inter-rater agreement scoring for SK responses was conducted by having the first author and Korean scorers score 30-response subsets. Inter-rater agreement scoring for US responses was conducted by having members of the project team score 30-response subsets. "Agree" meant a perfect match in scores. Inter-rater agreement for the first item was 93.3% for SK responses (0.845 Cohen's Kappa) and 88.0% for US responses (0.828

Cohen’s Kappa). Inter-rater agreement for the second item was 80.0% for SK responses (0.725 Cohen’s Kappa) and 96.0% for US responses (0.945 Cohen’s Kappa).

Two items shown in this study are among three function items discussed in Thompson and Milner (2019). Thompson and Milner (2019) showed the disparity between US and SK teachers’ responses in the two items, but did not deeply examine ways that teachers used function notation in the two items. We re-analyze the two items by (1) revising the scoring rubric for the second item, (2) investigating the relationship between the two items, and the relationship between the two dimensions of the second item, (3) presenting teachers’ interviews to validate the items and the rubrics. We substantially change the rubric for the second item by creating two dimensions: whether or not to uses function notation to represent varying quantities, and how to use variables. The change allowed us to better understand teachers’ use of function notation.

5. RESULTS

Our primary focus is to see meanings US teachers demonstrated, so we first present the distribution of U.S. teachers’ responses on the two items, the relationship between the items, the relationship between the two dimensions of the second item. We then report the results from South Korean teachers’ responses to see whether US teachers’ meanings are shared with SK teachers. We also discuss interviews with teachers who took the two items.

5.1. U.S. teachers’ responses

We gave the tasks in Figure 3 and Figure 4 to US high school mathematics teachers. All of 253 US teachers saw the first item “Understanding Independent Variables”. Only 241 US teachers saw the second item “Hari’s Rock”. We first present results from each item, and statistical analyses of the relationship between the items. We then display the relationship between the two features of US teachers' responses on the second item.

5.1.1. The first item "Understanding Independent Variables"

Table 4 presents results from the first item "Understanding Independent Variables". We exclude IDK (I don't know) and NR (No response) responses from the table. There are 16 IDK (3 US≥Calc and 13 US<Calc) and 14 NR (2 US≥Calc and 12 US<Calc) responses on the second item.

Table 4. U.S. Results for the first item "Understanding Independent Variables"

	Level 3 (*v* throughout)	Level 2 (Mix of *v*, *s*, and *t*)	Level 1 (*s* and *t*)	Others	Total
US ≥Calc	32	5	25	7	69
US <Calc	53	7	74	20	154
Total	85 (38.1%)	12 (5.4%)	99 (44.4%)	27 (12.1%)	223 (100.0%)

* Cells contain number of respondents total and percent of row total in the last row.

Approximately 43% of US ≥Calc teachers (32 of 74) filled the letter *v* in all four spaces. In addition, about 30% of US <Calc teachers (53 of 179) placed the letter *v* in all four blanks. Teachers in Level 3 seemed to think that "v" represents a value in the domain of "c", called the argument of the function "c"—the value at which to evaluate the function "c".

It seems that a large percentage of both levels of US teachers were insensitive to the role of independent variables (*s*, *t* or *v*). About 41% of US ≥Calc teachers (30 of 74) and 45% of US <Calc teachers (81 of 179) used *s* or *t* in at least one blank (Level 2 or

Level 1), which shows their tendency to keep the letter within parentheses in a function definition with the function name in which it occurred.

A large majority of US teachers seemed to think that "*v*" in *c*(*v*) is a part of the function name and they were therefore free to use other letters in the function's defining rule. Thompson (1994, 2013b) suggests a reason for the teachers who used *s* or *t* at least one blank in Level 2 and Level 1. Teachers thought of function notation as a four-character symbol that is used in place of the letter "y" (Thompson, 2013b). Teachers who filled the blanks with *s* or *t* might consider "*w*(*t*)" as one symbol because they thought they could replace "*w*(*t*)" with "*y*".

5.1.2. The second item "Hari's Rock"

Table 5 present results from the second item "Hari's Rock". We exclude IDK (I don't know) and NR (No response) responses from the table. There are 18 IDK (6 US≥Calc and 12 US<Calc) and 11 NR (1 US≥Calc and 10 US<Calc) responses on the second item.

Table 5. Results for the second item "Hari's Rock"

		Level 3 (FN both sides)	Level 2 (FN right side only)	Level 1 (FN left side only)	Others	Total
US ≥Calc	Use variable consistently	19	6	29	6	60
	Use variable inconsistently	2	0	5	0	7
US <Calc	Use variable consistently	20	11	64	23	118
	Use variable inconsistently	5	0	21	1	27
	Total	46 (21.7%)	17 (8.0%)	119 (56.1%)	30 (14.2%)	212 (100.0%)

* Cells contain number of respondents total and percent of row total in the last row.

Only 241 US teachers saw the "Hari's Rock" item. The first columns in Table 5 present teachers who used function notation in both sides (Level 3) to represent a relationship between quantities whose values vary. About 28% of US ≥Calc teachers (21 of 74) used function notation to represent the area and radius. Approximately 15% of US <Calc teachers (25 of 167) used function notation on both sides.

Responses were scored at Level 2 if a teacher used function notation to represent the radius increasing at a non-constant speed. Teachers in Level 3 or Level 2 used function notation to represent the value of a function in relation to values of its argument when they did not know a rule of assignment for the radius. To conduct statistical tests of the relationship between the two dimensions we combined Level 3 and Level 2 because teachers in both levels used function notation representationally. The results of the relationship between the two features will be presented in section 5.1.4.

Approximately 36% of US ≥Calc teachers (27 of 74) gave Level 3 or Level 2 responses to "Hari's Rock" item, using function notation to represent the radius that increases a non-constant rate. About 22% of US <Calc teachers (36 of 167) responded with function notation on the right side or both sides.

In addition, about 46% of US ≥Calc teachers (34 of 74) gave Level 1 responses, using function notation only to represent area. Approximately 51% of US <Calc teachers (85 of 167) responded with function notation only on the left side such as $A(t) = \pi t^2$, which indicates they used function notation to represent the area because of the prompt "use function notation" in the item.

Table 5 presents 34 responses that have clear evidence that the teachers used variables inconsistently. Of 34 US teachers who used variables inconsistently, 26 responses were in Level 1, where teachers used function notation only to represent the area. One example of these was $f(t) = \pi r^2$. This suggests that the 26 US teachers used function notation such as *f*(*t*) as a label to replace the word “Area”, since the function definition did not actually provide a rule dependent only on the varying quantity “t”.

We hypothesize that many teachers in Level 1 used function notation like “$A(t)$” as a four-character label, only because of the prompt “use function notation”, and therefore wrote $A(t)$ to represent the area. This would be consistent with students’ and teachers’ difficulties of writing $y = A(t) =$ a rule (Even, 1993; Hitt, 1998; Sajka, 2003).

There was no evidence that having mathematics or mathematics education degree was a statically significant predictor of using function notation to represent in the second item ($\chi^2(2, n = 212) = 2.03372, p = .36$).

5.1.3. The relationship between the two items

With regard to the relationship between the two items, we combined Level 2 and Level 1 in the first item “Understanding Independent Variables”, and combined Level 3 and Level 2 in the second item “Hari’s Rock”. Table 6 compares US teachers’ tendency to fill the blanks with letters to their use of function notation representationally. To conduct statistical tests we combined US ≥Calc and <Calc teachers.

Table 6. Responses to “Understanding Independent Variables” compared to responses to “Hari’s Rock” from US teachers

Count	Used function notation in the defining rule	Used function notation only to represent area	Others	Total
Filled the blanks with the letter *v*	35	35	6	76
Filled the blanks with the letter *s* or *t*	23	53	17	93
Others	5	31	7	43
Total	63	119	30	212

Table 6 shows a link between US teachers' use of consistent variables and idea of using function notation representationally. Approximately, 56% US teachers (35 of 63) who used function notation in the defining rule in the second item filled all blanks with the letter *v*. About 57% of US teachers (53 of 93) who typed "s" or "t" in the blanks used function notation only to represent the area.

The association between responses to *Understanding Independent Variable* and *Hari's Rock* from US teachers was statistically significant $\left(\chi^2(4, n = 212) = 19.037, p < 0.0008\right)$. These strong associations are consistent with the hypothesis that teachers who used function notation only to represent the area thought of "$f(t)$" as a label because they were likely to use variables inconsistently in *Understanding Independent Variables.*

5.1.4. The relationship between the two features on *Hari's Rock*

When analyzing the two dimensions' relationship in the second item, we first exclude IDK (I don't know) and NR (No response) responses from each table because IDK or NR on one dimension correlates automatically with IDK or NR on the other. Table 7 shows the relationship between the two features of US teachers' responses on *Hari's Rock*.

Table 7. US teachers' use of variables by their use of function notation on "Hari's Rock"

	Used function notation in the defining rule	Used function notation only to represent the area	Others	Total
Used variables consistently	56	93	29	178
Used variables inconsistently	7	26	1	34
Total	63	119	30	212

According to Table 7, the two features were associated. The first column of Table 7 shows that 89% of US teachers (56 of 63) who used function notation to represent the radius increasing at a non-constant rate used a consistent letter in the notation and the rule of the function. Similarly, the second row of Table 7 tells us that 76% of US teachers (26 of 34) who used variables inconsistently used function notation only to represent the area. The association between the two features of *Hari's Rock* from US teachers' responses was also statistically significant $\left(\chi^2(2, n=212)=7.716, p<0.0211\right)$. These statistical results are consistent with the hypothesis that teachers who used function notation only to represent the area (such as $A(t)$) think of t as a part of the function's name instead of an independent variable. We hypothesized that if they thought of $A(t)$ as one symbol, they might think "$A(t) =$" is just another way of writing "$y =$" (Thompson, 2013b).

5.2. South Korean teachers' responses

As mentioned earlier, we present SK teachers' responses on the two items to see whether meanings that US teachers demonstrated are shared with SK teachers. Table 8 shows results from the first item "Understanding Independent Variables". We exclude IDK (I don't know) and NR (No response) responses from the table. There are 3 IDK (2

HS and 1 MS teachers) and 16 NR (5 HS and 11 MS teachers) responses on the second item.

Table 8. Results for the first item "Understanding Independent Variables"

	Level 3 (*v* throughout)	Level 2 (Mix of *v*, *s*, and *t*)	Level 1 (*s* and *t*)	Others	Total
Korea HS	203	1	14	39	257
Korea MS	65	0	6	19	90
Total	268 (77.2%)	1 (0.2%)	20 (5.8%)	58 (16.7%)	347 (100.0%)

* Cells contain number of respondents total and percent of row total in the last row.

Approximately 77% of SK high school teachers (203 of 264) and about 64% of SK middle school teachers (65 of 102) filled the letter *v* in all four spaces. Approximately 6% of both SK high school teachers (15 of 264) and SK middle school teachers (6 of 102) filled *s* or *t* in the blanks (Level 2 or Level 1). It seems that both SK high school and middle school teachers were sensitive to the role of independent variables (*s*, *t* or *v*).

Table 9. Results for the second item "Hari's Rock" from SK teachers

		Level 3 (FN both sides)	Level 2 FN right side only	Level 1 (FN left side only)	Others	Total
Korea HS	Use variable consistently	86	77	21	48	232
	Use variable inconsistently	0	0	11	0	11
Korea MS	Use variable consistently	24	15	18	23	80
	Use variable inconsistently	0	2	5	0	7
	Total	110 (33.3%)	94 (28.5%)	55 (16.7%)	71 (21.5%)	330 (100.0%)

* Cells contain number of respondents total and percent of row total in the last row.

All 366 SK teachers saw the "Hari's Rock" item. We exclude IDK (I don't know) and NR (No response) responses from the table. There are 14 IDK (9 HS and 5 MS teachers) and 22 NR (12 HS and 10 MS teachers) responses on the second item.

Recall that teachers in Level 3 or Level 2 used function notation representationally. Approximately 62% of SK high school teachers (163 of 264) and 40% of SK middle school teachers (41 of 102) gave Level 3 or Level 2 responses to "Hari's Rock" item, using function notation to represent the radius that increases a non-constant rate.

We also conducted the statistical tests for SK teachers to see the relationship between the two items as well as the relationship between the two features of the second item.

Table 10. Responses to "Understanding Independent Variables" compared to responses to "Hari's Rock" from SK teachers

Count	Used function notation in the defining rule	Used function notation only to represent the area	Others	Total
Filled the blanks with the letter *v*	173	35	43	251
Filled the blanks with the letter *s* or *t*	4	5	5	14
Others	27	15	23	65
Total	204	55	71	330

Table 10 shows SK teachers' use of consistent variables and idea of function notation to represent varying quantities were also linked in SK responses. Approximately, 52% of SK teachers (173 of 330) used function notation in the defining rule on "Hari's Rock" *and* filled the blanks with the letter *v* on "Understanding Independent Variable". Looking through the first column of Table 10, we see that 85% of SK teachers (173 of 204) who used function notation in the defining rule in the second item filled all blanks with the letter *v*. Only 2% of SK teachers (4 of 204) who used function notation to represent the varying radius filled the blanks with *s* or *t*.

Scanning the first row of Table 10, we see that 69%of SK teachers (173 of 251) who filled the blanks with the letter *v* used function notation to represent the radius increasing at a non-constant rate. The association between responses to *Understanding Independent Variable* and *Hari's Rock* from SK teachers was statistically significant ($\chi^2(4, n = 330) = 24.010$, $p < 0.001$).

Table 11. SK teachers' use of variables by their use of function notation on "Hari's Rock"

	Used function notation in the defining rule	Used function notation only to represent the area	Others	Total
Used variables consistently	202	39	71	312
Used variables inconsistently	2	16	0	18
Total	204	55	71	330

According to Table 11, the two features of the second item were also associated in our sample of SK teachers. The first column of Table 11 shows that 99% of SK teachers (202 of 204) who used function notation to represent the radius increasing at a non-constant rate used a consistent letter in the notation and the rule of the function. In contrast, the second row of Table 11 tells us that 89% of SK teachers (16 of 18) who used variables inconsistently used function notation only to represent the area. The association between the two features of *Hari's Rock* from SK teachers' responses was statistically significant ($\chi^2(2, n = 330) = 71.598, p < .0001$).

SK teachers' responses show that the meanings US demonstrated in the tasks were shared with a minority of SK teachers. SK results also tell us it is not unreasonable to expect secondary mathematics teachers to give high level responses, contrary to the misgivings of our consultants and advisors. In addition, SK teachers' responses also allow us to think about the level of US teachers who taught calculus at least once (US ≥Calc teachers). U.S. teachers teaching calculus will have taught content similar to Korean high school teachers because calculus is in the high school curriculum in South Korea, but not in the U.S. SK middle school teachers' responses were more sensitive to the role of independent variable than were US ≥Calc teachers in our sample. SK middle school teachers' responses were also more likely to use function notation to represent varying quantities than were US ≥Calc teachers if we see teachers in Level 3 and Level 2 on the second item in our sample.

5.3. Interviews with three teachers

The first author interviewed 17 teachers (eight SK and nine US teachers) about their responses in the two items. We present item interviews with three teachers (two US and one SK teachers) to illustrate the spectrum of responses we found in all 17 teachers.

Figure 5 shows Teacher 1 (US)'s responses to the two items.

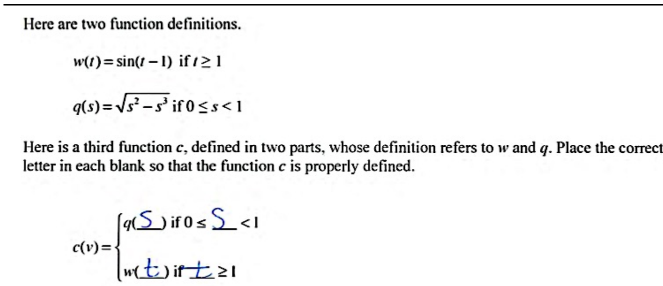
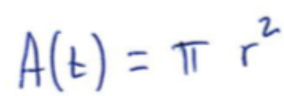
Response to *Understanding Independent Variables*	Response to *Expressing the Varying Area*
	

Figure 5. Teacher 1's responses to the two items.

Teacher 1 filled the blanks with *s* and *t* in *Understanding Independent Variables*. Her response to *Expressing the Varying Area* was $A(t) = \pi r^2$. She used function notation only to represent the area and used variables (t and r) inconsistently. We interviewed

Teacher 1 to better understand why someone might use variables inconsistently when defining a function.

Excerpt 1. Teacher 1's interview of *Understanding Independent Variable*

I: What's your interpretation of this item?

T1: Um so I am looking at two functions become a piecewise defined function. Um I am thinking and looking at the two functions w is a function of t and q is a function of s. Umm… anything that relates to q should have s in the parenthesis in the blanks and then anything that relates to w should have t. Since I am wondering if these two functions are coming from elsewhere with different inputs, so $c(v)$ would be $q(s)$ and $w(t)$ combined.

Teacher 1's response in Excerpt 1 tells us that she thought $q(s)$ was one inseparable symbol instead of thinking q is a function's name and s is an independent variable on *Understanding Independent Variables*. Although she said q is a function of s, she thought q always accompanies with s, and viewed $q(s)$ as one entity. Teacher 1 demonstrated her meanings for function notation when talking about her response to *Hari's Rock* (see Excerpt 2).

Excerpt 2. Teacher 1's interview of *Hari's Rock*

T1: (After writing $A = \pi r^2$, and then added (t) after A) I am not sure I am satisfied with my answer, but I think I'm done. But I am not quite sure… I am not totally sure how else I would express this.

I: What's your interpretation of this item?

T1 Yes, I was thinking about rock dropping and the area of the circle I am thinking is πr^2, the area of the circle. When I am using function notation I am thinking $A(t)$ because of the area with respect to time. The issue that I couldn't come up or couldn't figure out the rest of this was if the radius is increasing at a non-constant speed like due to time how can I make radius in terms of time? And I guess it made me like $A(t) = \pi r^2$ doesn't show the input of t, r is changing with respect to time. Would this be a situation where there would be multiple equations to represent the same situation or do I change the radius to be time, um but there are some relationships between time and radius, so I didn't feel satisfactory.

I: You feel like… there is the relationship between time and radius, but you have no idea how to represent the relationship. Am I right?

T1: Yes.

I: I saw you changed your answer *A* to *A*(*t*). Could you explain why you changed your answer?

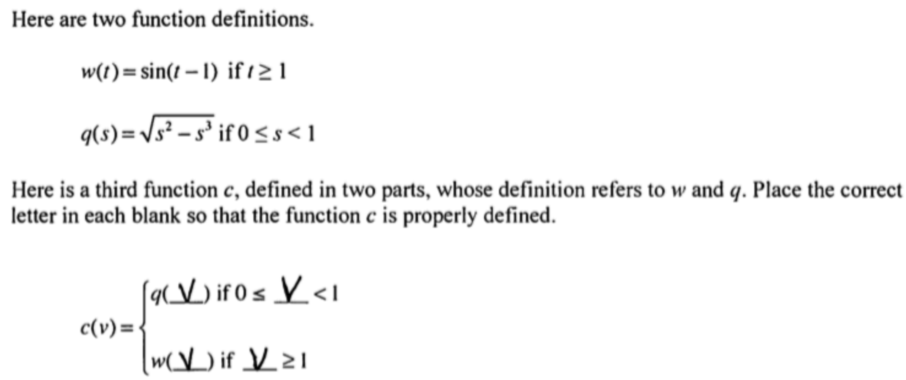
T1: Sure, first it (referring to the item) says use function notation, so I was trying to represent with function notation. It started with *A*. I was just noting the area of the circle which was what I was starting. Umm it says the area as a function of elapsed time, so using *t* to represent time. So I knew that the area was the function of time and I couldn't figure out what I wanted to do with *r*. I guess I could now change *r* to *t* although radius is not necessarily equal to time but there are some relationships between *r* and *t* that I want to represent. Since I don't know what's causing what relationship time and radius have. I don't exactly know how to factor time into the equation.

When taking *Hari's Rock*, Teacher 1 first wrote $A = \pi r^2$, and then added (t) after A. Her final answer was $A(t) = \pi r^2$. She used function notation only to represent the area on the left hand side and used *t* in the function notation and *r* in the defining rule. Her statement "when I am using function notation I am thinking *A*(*t*)" and "I was just noting the area of the circle which was what I was starting" is consistent with our hypothesis that teachers who used function notation only to represent the area think of *A*(*t*) as one symbol, and *A*(*t*) is a label for the formula on the defining rule.

It is worth noting that Teacher 1 expressed a need to relate radius and time and said she did not know how to do it. One reason might have been that to use function notation Teacher 1 needed a defining formula, and function notation did not serve as a representational tool for her without knowing a defining formula.

Teacher 2 (US)'s responses to the two items were in the highest levels according to the rubrics (see Figure 6).

Response to *Understanding Independent Variables*



Here are two function definitions.

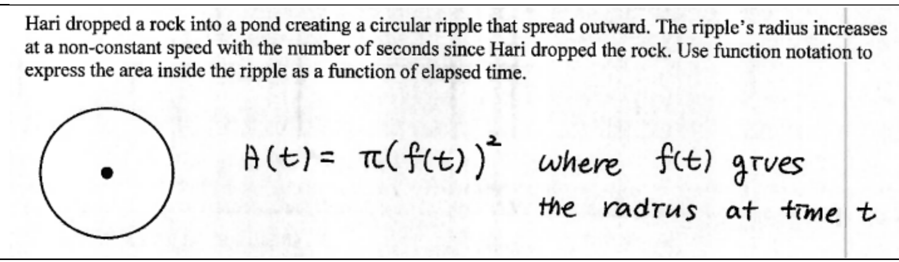
$w(t) = \sin(t-1)$ if $t \geq 1$

$q(s) = \sqrt{s^2 - s^3}$ if $0 \leq s < 1$

Here is a third function *c*, defined in two parts, whose definition refers to *w* and *q*. Place the correct letter in each blank so that the function *c* is properly defined.

$$c(v) = \begin{cases} q(\underline{V}) \text{ if } 0 \leq \underline{V} < 1 \\ w(\underline{V}) \text{ if } \underline{V} \geq 1 \end{cases}$$

Response to *Expressing the Varying Area*



Hari dropped a rock into a pond creating a circular ripple that spread outward. The ripple's radius increases at a non-constant speed with the number of seconds since Hari dropped the rock. Use function notation to express the area inside the ripple as a function of elapsed time.

$A(t) = \pi(f(t))^2$ where f(t) gives the radius at time t

Figure 6. Teacher 2's responses to the two items.

Teacher 2 filled the blanks with *v* on Understanding Independent Variables and wrote $A(t) = \pi\left(f(t)\right)^2$ where *f*(*t*) gives the radius of time *t*. We interviewed Teacher 2 to see whether teachers who wrote highest-level responses demonstrated coherent meanings.

Excerpt 3. Teacher 2's interview of *Understanding Independent Variable*

I: What's your interpretation of this item?

T2: Oh, I feel like I might be missing something. So the function *c* is properly defined. So, I mean this is *q*, so I would use *q* if my variable is between 0 and 1, and I would use *w* if my variable is greater and equal to 1. So, I mean I don't know I put *v*'s everywhere. (laughing) But since we are defining *c* of *v*, it seems the variable like it ought to be *v*. So the same domain restrictions would apply to *w* and *q*.

Excerpt 3 shows Teacher 2's awareness that (1) the letter inside of the parenthesis represents a variable, (2) the letter denoting the variable of a function should be consistent with the letter used in the defining rule, and (3) Teacher 3 used "*w*" and "*q*" as function names. Her interview on the second item also displays her coherent meanings for function notation (see Excerpt 4).

Excerpt 4. Teacher 2's interview of *Hari's Rock*

I: What's your interpretation of this item?

T2: Oh, well it seems… wanting me to show that I understand that the radius isn't constant so I have to do a function for the radius. And you know obviously area equals π radius squared, but the radius is…Given that there is no more information about how radius relates to time, I thought I just need to do a function then of time.

I: That's why you wrote $f(t)$?

T2: Yes.

I: To represent the varying radius?

T2: Yes.

I: You wrote also $A(t)$. Could you tell me know why you wrote $A(t)$?

T2: Because it wants the area as a function of elapsed time, so I wrote A for the area and t is for time as my variable.

Teacher 2's responses in Excerpt 4 indicate that Teacher 2 used function notation as a means to represent varying quantities in the case where an explicit rule is unknown. She also demonstrated that she was aware that t on both sides represents the independent variable for the function defined.

Teacher 3 (SK) filled the blanks with *x* in *Understanding Independent Variables*, and expressed his confusion because he did not know an explicit rule for the radius in *Hari's Rock* (see Figure 7).

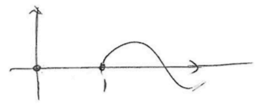
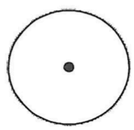
Response to *Understanding Independent Variables*	두 함수가 다음과 같이 정의되어 있다. $w(t)=\sin(t-1)$ if $t\geq 1$ $q(s)=\sqrt{s^2-s^3}$ if $0\leq s<1$  세 번째 함수 c 는 w 와 q 두 부분으로 정의되어 있다. 함수 c 가 알맞게 정의되도록 빈 칸을 채우시오. $c(v)=\begin{cases} q(\underline{x}) \text{ if } 0\leq \underline{x} <1 \\ w(\underline{x}) \text{ if } \underline{x} \geq 1 \end{cases}$ → 이것이 [0,1]에서 정의된 함수임에 문제가 있나...?? The teacher's scratch work: What is wrong with this for the function that is defined on [0,1]…??
Response to *Expressing the Varying Area*	메시가 연못에 돌을 던져서 생긴 원형 물결이 밖으로 퍼져나가고 있다. 원형 물결의 반지름은 메시가 돌을 던진 이후 지난 시간 (초)에 대해 일정하지 않은 속력으로 증가한다. 함수 기호를 사용하여 원형 물결 안의 넓이를 경과 시간 (초)에 대한 함수로 나타내시오.  ?? 원이 어떤 양상으로 커지는지 모르겠음. The teacher's response: ?? I don't know how the circle is increasing.

Figure 7. Teacher 3's responses to the two items.

Teacher 3's responses to the two items indicate his inclination of using *x* in function notation and finding out rules when thinking about functions. Unlike the two teachers (Teacher 1 and Teacher 2) we presented earlier, Teacher 3's interview included looking at his original responses from a prior administration of the *MMTsm*. Teacher 3's responses in Excerpt 5 show his reasoning on the first item.

Excerpt 5. Teacher 3's interview of *Understanding Independent Variables*

I: Could you explain your response to me?

T3: Doesn't this item want me to say it doesn't matter to use different letters?

I: What does that mean?

T3: This (referring to *w* function) is defined from 1, and this (referring to *q* function) is defined from 0 to 1, so it (referring to *c* function) is defined well, isn't it? But, c is defined in two parts, so I used the same letter. Ah, it is written *c*(*v*). Oh, there is *v*. Don't I have to fill the blanks with *v*?

I: Could you explain to me why you filled the blanks with *x*?

T3: At that time when I was taking this item? Was I in a hurry? Anyway I thought the intention of this item was…this (referring to *w* function) is in terms of *t*, this (referring to *q* function) is in terms of *s*. And the domains were connected, so I have to use one letter, my favorite letter. If I had seen *v* here (pointing to *v* in *c*(*v*)), I might have written *v*, but…Maybe I didn't see this (referring *v*) because I was in a hurry.

Teacher 3 expressed his tendency to use *x* as the letter inside of the parenthesis in Excerpt 5. He did recognize that the independent variable for a function is a place holder for a value and can be changed as long as a parallel change is made in the function's rule of association. At the time of taking the *MMTsm*, he did not recognize that the variable in a function definition should be consistent. It seemed my questions perturbed him during the interview because he eventually noticed that *v* was the argument to *c*.

Teacher 3 also said why he did not answer the second item in Excerpt 6.

Excerpt 6. Teacher 3's interview of Hari's Rock

I: Could you explain your response to me?

T3: I didn't understand this item at all. I asked teachers who took this item as soon as it was over. They said they chose any function in terms of time. But, I thought I needed a formula that represents how the circle is increasing. The radius is increasing at a non-constant rate, so I thought this item was asking me to find out how the radius is increasing. For example… (reading the item again) Does this (pointing to "the radius is increasing at a non-constant rate") mean that *t* and *r* are not directly proportional? For example, if I think of any function, I can take $r = t^2$ or $r = \sqrt{t}$ and apply the area formula

πr^2. Can this be one of correct answers? I feel comfortable with a specific function.

Teacher 3's response in Excerpt 6 indicates that Teacher 3 did not answer the second item because he could not come up with a formula that represent the radius increasing at a non-constant rate. His statement in this excerpt shows that his meaning for function is a formula such as $r = t^2$ or $r = \sqrt{t}$. It seemed that Teacher 3 first tried to come up with a rule when asked to use function notation. This teacher was also very explicit about how his own comfort level played into his actions. He used "*x*" in *Understanding Independent Variables* not because it was relevant to the function definition but because "it was his favorite variable", and he wanted to find a specific rule for how radius and time varied in *Hari's Rock* because "he feels comfortable with a specific function". It seemed he had a concept image of function notation that always includes "$f(x)$" as a four-character name (Tall & Vinner, 1981). Though other teachers did not indicate the same self-awareness of their own motivations, their responses indicated that many other teachers might have felt comfortable only when using specific function rules on the right-hand side of a function.

6. DISCUSSION

6.1. Discussion of the results

Prior research focused largely on students' and teachers' conception of calculating a function's values. In this study we shifted our focus to ways in which teachers used function notation. We were particularly interested in whether teachers used function notation to represent a relationship between varying quantities.

Teachers' responses to the two items in this study suggest that more than half of US teachers held unproductive, and sometimes incoherent meanings for function notation. The results show that attending to teachers’ meanings for function notation revealed useful information about teachers’ understanding of functions. From the first item, about 57% of US ≥Calc and 70% of US <Calc teachers’ responses suggest that they think of the left side of a function definition, e.g., $c(v)$, is a name for the rule on the right side (see Table 4). This is contrary to thinking of function notation as a method to encapsulate a number of entailed meanings (see Figures 1 and 2). U.S. teachers’ responses to the second item, in which 64% of US ≥Calc, and 78% of US <Calc teachers did not use function notation representationally, are consistent with responses to the first item. It seems they used function notation as a label to replace the word “Area”

The association between responses to *Understanding Independent Variable* and *Hari’s Rock* from US teachers’ responses was statistically significant. The association between the two features of *Hari’s Rock* from US teachers’ responses was also statistically significant. Those results and the interviews with teachers strongly suggest that (1) many U.S. teachers held a meaning for function notation as a label to replace a word such, as $A(t)$ to replace “area”; (2) teachers treated function notation “$A(t)$” as a variable name because they used different letters inside of function notation and in the defining rule, and (3) many U.S. teachers thought every function has to be defined by a rule.

We focused on meanings that teachers have for function notation. Our focus on meanings allowed us to explain what might lead to teachers’ common mistakes of function notation. For example, teachers' use of mismatched letter on both sides such as $A(t) = \pi r^2$ might come from thinking of function notation as a four-character symbol.

They might consider "$A(t)$" as one symbol because they thought they could replace "$A(t)$" with "y".

Our perspective on productive meanings for teaching function notation helped us think about potential productivity for students' long-term learning. Suppose a teacher's meaning for function is a formula. The teacher's use of function notation always accompanies with a computational rule. His students might think if there is no rule, there is no function. We think using "$f(x)$" representationally (i.e. using "$f(x)$" to represent the value of the function f in relation to a value of x regardless of whether we know a rule of assignment for it) would be productive for high school students because it supports students in developing a meaning for function beyond that of a rule. Additionally, we believe that the representational power of function notation could resolve thinking of $f(x)$ as a label because students can understand that $f(x)$ represents a value of one quantity with respect to the value of a second quantity according to a relationship between them named f. This suggests a need for attention to ideas of function notation as a means to represent one quantity's values in relation to another quantity's values in teacher education for pre-service teachers and professional development programs for in-service teachers.

SK teachers' responses showed it is reasonable to expect secondary mathematics teachers to give high level responses to our tasks, contrary to claims by our consultants and advisory board, and suggests the state of US high school teachers' meanings for function notation is problematic. Approximately 80% of SK high school teachers, 64% of SK middle school teachers, 43% of US≥Calc teachers, and 30% of US<Calc teachers filled the letter in the function notation that is consistent with the letter used in the

defining rule of the function. In addition, 62% of SK high school teachers, 40% of SK middle school teachers, 36% of US≥Calc teachers, and 22% of US<Calc teachers used function notation to represent a varying quantity that increases at a non-constant rate. It demonstrates that SK middle school teachers were more likely to understand the role of independent variables and to use function notation representationally than were US high school teachers who taught calculus at least once in our sample. SK data shows that US teachers' difficulties when reasoning with function notation are not due to epistemological obstacles to understanding meanings of function notation. The SK data suggests that US teachers' problematic responses are a systemic aspect of mathematics education in the U.S. Thus, it is plausible that US teachers convey problematic meanings to students unintentionally.

6.2. Limitations

Teachers in this study were not selected randomly and so cannot be taken as representative of all U.S. and South Korean teachers. Thus, generalization of the results is not possible. However, the sample is large for an educational study. We therefore think it is possible that many U.S. teachers are likely to convey to their students' meanings for function and function notation that are useful only in limited situations or are counterproductive for future mathematical learning.

6.3. Our attention to teachers' meanings and future research

Thompson (2013a) outlined ways to address the systematic inattention to meaning in U.S. mathematics education, explaining that addressing this problem will require a long-term effort. Attending to the representational power of function notation would be one way to help teachers develop and teach productive meanings for function (see Figure 1 and Figure 2).

We acknowledge that a teacher's meanings do not determine his or her actions. However, we cannot expect a teacher to help students develop a productive mathematical meaning when the student's teacher does not have that meaning. Holding productive meanings for ideas teachers teach is a necessary condition for them to convey those meanings to students, but it is not a sufficient condition.

We do not claim all teachers would express their meanings demonstrated in the *MMTsm* in their classrooms. However, our data supports our claim that teachers' responses in the *MMTsm* reflect meanings that can support or constrain what they might express during their lesson, and that these meanings can be characterized by responses to our items. We are continuing to investigate this claim by observing teachers' classrooms to see the relationship between what teachers write on the *MMTsm* items, what they say and do in teaching, and meanings students construct in trying to understand what teachers intend.

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