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INTEGERS AS TRANSFORMATIONS

PATRICK W. THOMPSON, *Illinois State University*
TOMMY DREYFUS, *Center for Technological Education, Israel*

To investigate whether elementary school students can construct operations of thought for integers and integer addition that are crucial for understanding elementary algebra, 2 sixth graders were taught for 6 weeks in eleven 40-minute sessions using a computerized microworld that proposed integers as transformations of position, integer addition as composition of transformations, negation as an operator upon integers or integer expressions, and representations of expressions as defined words. By the final session, both students had constructed mental operations for negating arbitrary integers and determining the sign and magnitude of a sum and had constructed a rule of substitution that allowed them to negate integer expressions. One student could negate represented expressions.

Arithmetic in the elementary grades is important because of its applications, but it is important also because of the mental operations that students are to develop in preparation for their study of algebra. Arithmetic as a basis for algebra (or algebra as generalized arithmetic) has both historical and pedagogical foundations (Eves, 1969; Herscovics & Chalouh, 1984; Kieran, 1984). The question investigated in this article is whether it is possible to organize instruction on the arithmetic of signed numbers so as to facilitate students' development of mental operations that directly parallel features of algebraic thinking commonly accepted as important.

Our investigation emphasized the conception of integers as transformations of quantities and the conception of negation as an operation upon integers. The concept of an integer as a transformation of quantity has inspired textbooks' use of arrow diagrams to illustrate integers and integer addition (Weaver, 1982). But such presentations typically do not emphasize the operation connotation of the arrows. Rather, they are used to point at the position on the number line that names the result.

A theoretical foundation for proposing integers as transformations is found in Vergnaud's (1982) framework for distinguishing among the operations of thought required by children to solve problems in K-8 mathematics having an additive structure. A significant feature of Vergnaud's framework is his construct of a relational calculus, a system of mental operations for analyzing the quantitative relationships present in a situation. The problems that proved most difficult in his experiments (Fisher, 1979;

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Marthe, 1982) correspond closely to problems normally identified as involving addition of integers. All of the problems could be described by the arithmetic of whole numbers, but according to Vergnaud's analysis, to comprehend the more difficult problems children needed to conceive of them in terms of addition of transformations of quantities instead of addition of cardinal quantities. Janvier (1983) found similar results.

Vergnaud's analysis did not address the issue of how one must conceive of the negation of integers for it to be consistent with a relational calculus, nor did his analysis address the concept of integer expressions. Research on concepts of expressions is typically found in studies of algebraic concepts (Davis, 1975; Davis, Jockusch, & McKnight, 1978; Lewis, 1981; Matz, 1982; Sleeman, 1982; Sleeman & Smith, 1981). This body of research has established that, by and large, students conceive of algebra not as generalized arithmetic but instead as a set of rules for manipulating letters and operation signs. Experience suggests that the case is similar with regard to integer expressions. Typical answers given by eighth graders in simplifying, for example, $-(5 + -7)$ are $-5 + -7$ and $-5 - 7$. A rule for distributing the negative sign across an expression is a generalization. Students should be able to make the generalization correctly from their experiences in arithmetic.

SAMPLE

The study involved 2 sixth graders, Kim and Lucy. They were chosen by their teacher to fulfill our request for two students of middle ability. Neither Kim nor Lucy was exceptional in mathematics. Both scored at grade level on the mathematics part of the California Test of Basic Skills. Both seemed well adjusted socially. Kim was more reflective and more assertive than Lucy, though Lucy was by no means reserved. Neither student had used a computer in instruction before her involvement with the study. Sixth graders were chosen so that the subjects would not have received extensive instruction on integers—especially instruction on rules for operating on integers.

METHOD

The method of the investigation was the constructivist teaching experiment (Cobb & Steffe, 1983; Steffe & Richards, 1983; Thompson, 1982). In a constructivist teaching experiment, one teaches the concepts being investigated to students who do not have them. After each instructional session, students' performance is analyzed to establish the boundaries of their understanding, and instruction for the next session is planned in light of this analysis. In this study, we attempted to teach Kim and Lucy the following conceptualizations of integers and operations upon integers:

1. Integers as transformations
2. Addition of integers as the composition of transformations
3. Negation as a unary operation upon integers and integer expressions

Instruction took place over a 6-week period in the spring of 1984. The interviewer (a research assistant) met with the students together twice weekly for approximately 40 minutes per meeting, with the exception of the first week in which they met only once. Every session was audiotaped.

THE MICROWORLD FOR INTEGERS

The microworld used in this study, called INTEGERS, has been designed to present integers as unary translation operators acting on positions on a number line (Dreyfus & Thompson, 1985; Thompson 1985a, 1985b). The model for this representation contains a turtle that walks right and left. The concept of an integer as a transformation is realized by displacing the turtle according to the value of the integer: Entering an integer or integer expression causes the turtle to walk according to specific rules, shown in Table 1.

Table 1
Integers as Commands to Move the Turtle

number	The turtle walks *number* steps in its current direction.
–*number*	The turtle turns around, walks *number*, and then turns back around.
START.AT *number*	Puts the turtle at the *position* named *number*. This is where the turtle will begin its next itinerary.

INTEGERS also incorporates a grammar for integers, which stipulates what are allowed as values of ***number*** (described in Table 1). The grammar for integers is as follows:

- A whole number is an integer.
- The negative of an integer is an integer.
- The composition of two integers is an integer.
- The representation of an integer is equivalent to an integer.
- An operation defined on representations of integers is equivalent to an integer.

Figure 1 illustrates the effect of entering **50** when the turtle is at the position 70. (Numbers and words in boldface type signify commands to INTEGERS and hence have its special semantics.) The turtle moved 50 steps from its starting position when **50** was executed. The vertical lines mark the turtle's beginning and ending positions. The displacement is shown by a heavy arrow. The arrow showing the turtle's displacement is the graphic correspondent of the integer that caused the displacement. Figure 2 shows the effect of entering **–90** after having entered **50**. The turtle turned around, moved 90 steps, and then turned back around.

The fact that the turtle begins its movements from its current position emphasizes the distinction between a state (position) and an integer (change of position). The effect of entering a number, in terms of change of position, is the same regardless of where the turtle starts.

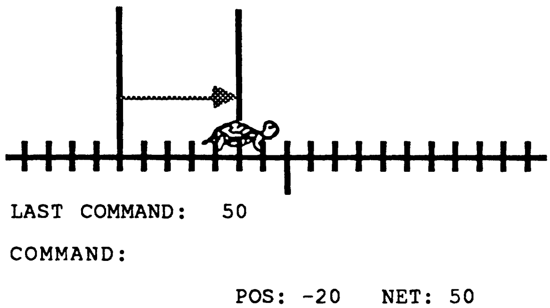

Figure 1. Entering 50 caused the turtle to walk 50 steps in its current direction.

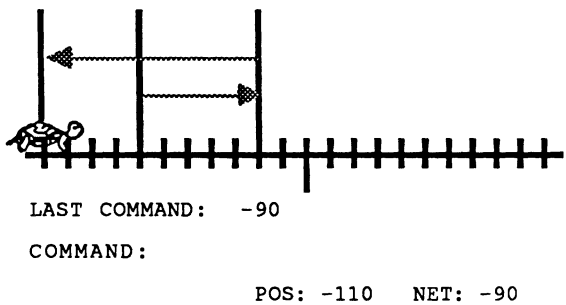

Figure 2. Entering −90 caused the turtle to turn around, walk 90 steps, and turn back around.

The reason for modeling the effect of negation by having the turtle turn around rather than back up is twofold. First, if backing up were used to express negation, then the change in the turtle's default motion of moving in the direction it faces would not be manifested in an observable action on the screen. Second, were the turtle to back up instead of turn around, there would be nothing concrete to correspond to the end of the scope of a negation sign.

Two integers can be composed in INTEGERS by entering them on the same line. Figure 3 shows the effect of entering −60 90. INTEGERS executed −60 and 90 as if they had been entered separately but consecutively. The heavy arrows show the effects of −60 and 90. The light arrow shows the net effect (composition) of the expression −60 90. The information line tells that the net effect of −60 90 is 30.

The example in Figure 3 illustrates the necessity of having the turtle turn back around after executing *number* in the command −*number*. In −60 90, were the turtle to not turn back after executing the 60 of −60, then the net effect of −60 90 would be −150. This example illustrates our aim of having an observable action reflect the end of the scope of a negation.

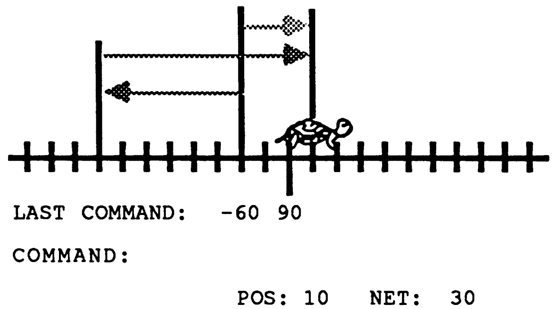

Figure 3. The effect of entering −60 90: The turtle did −60 (lower heavy arrow) and then 90 (upper heavy arrow), which resulted in a net effect of 30 (top arrow).

Because integer expressions denote integers, they can themselves be negated. Figure 4 shows the effect of entering −[−60 90] after having entered −60 90. The use of brackets in −[−60 90] is essential because they denote the beginning and end of the negation's operand. Brackets may optionally be used for nonnegated expressions: [−60 90] means exactly the same as −60 90.

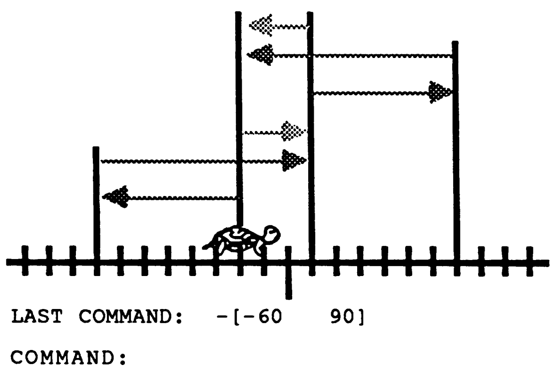

Figure 4. The effect of entering −[−60 90] after having entered −60 90; the arrows should be read from bottom to top.

Integers and expressions can be explicitly represented by letters and words, and letters and words that have been given values can in turn be used in expressions. For example, the command DEF "JOE[30 −50 −40] says to make the word JOE stand for the expression [30 −50 −40]. The word JOE can then be used in place of what it has been defined to represent. Figure 5 shows the effect of entering − −JOE after JOE has been defined as above. In doing − −JOE, the turtle first turned around to do −JOE. In doing −JOE, it first turned around to do JOE. After doing JOE, the turtle turned around to finish −JOE, then turned around again to finish − −JOE.

The effect of − −JOE is the same as that of JOE, but in executing − −JOE the turtle makes two turns at both the beginning and the end.

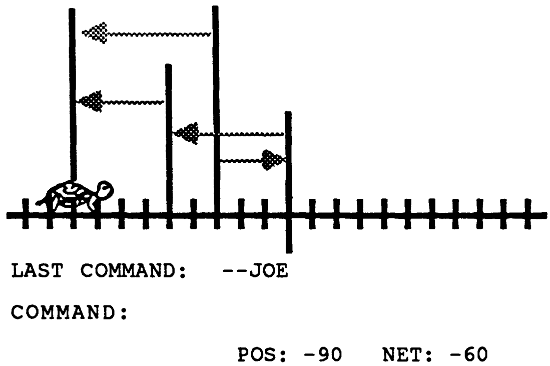

Figure 5. The effect of entering − −JOE where JOE has been defined as representing the integer expression [30 −50 −40].

The DEF command can also be used to define general operations upon integers and integer expressions, but this capability was not used in the present study.

INSTRUCTION

The instruction typically consisted of asking Kim and Lucy to predict the result of a command before it was executed, executing the command, and then discussing their prediction. Discussions of their predictions commonly led to questions about generalizations that might follow from specific examples and to questions about misconceptions that might have been evident in their predictions.

Particular attention was given to evidence of the quality of Kim's and Lucy's understanding. Opportunities for our gaining a better picture of their understanding were pursued in depth. We viewed the teaching episodes as occasions for testing and expanding the boundaries of Kim's and Lucy's understanding of integers and operations upon integers. Our perspective was that an informal measure of a student's understanding is the ease with which he or she can generalize a point of discussion.

The agendas for the first two sessions were fixed in advance. All sessions were audiotaped and transcribed; agendas for each of the 3rd through 11th sessions were determined after an evaluation of the preceding lesson. The aim was to take Kim and Lucy as far as possible toward exploring integer operations while at the same time trying not to exceed their capacities to accommodate to instruction. Whenever the interviewer judged that either Kim or Lucy was lost or confused, she would return to concepts with which they had shown some facility.

The first session was used to introduce Kim and Lucy to the conventions of the microworld (position, effect of number, clearing the screen, etc.). The next two sessions focused upon the distinction between position and change of position. The distinction was drawn through three types of problems. The three types are exemplified by the following:

1. The turtle began at 50. Sally entered −60. Where did the turtle end?
2. The turtle began at −20. George entered a number. The turtle ended at 50. What number did George enter?
3. Herman entered −40. The turtle ended at −10. Where did the turtle begin?

Sessions 4 to 8 concentrated on two themes: the net effect of *−number* (including the case of *− −number*) and the net effect of composing two integers. Though Lucy and Kim had just spent two sessions on composite integers such as **60 −90** and **−10 70**, in the eighth session they asked the interviewer to postpone the planned agenda so that they could devise an explicit rule by which they could judge whether the "net effect arrow" would point to the left or to the right. The negation of an integer expression (e.g., **−[50 40]**) was introduced in the latter part of the eighth session.

Sessions 9 to 11 focused on the idea of the equivalence between an expression and its net effect and on the idea of an additive inverse. Represented expressions (see Figure 5) were introduced in Session 9. The concept of additive inverse was introduced in Session 11 with questions like "What could you enter after the turtle does *expression* that would make it go back to where it started?"

KIM'S AND LUCY'S CONCEPTUALIZATIONS

Integers as Transformations

Both Kim and Lucy had been introduced in their mathematics class to the negative side of the number line and could locate positions to the left and to the right of 0 under the convention that slashes on INTEGERS' number line were 10 "turtle steps" apart. Verification was obtained by having Kim and Lucy predict where the turtle would appear upon entering **START.AT *x*** for various positive and negative values of *x*.

Kim soon distinguished between the ideas of position and of change of position. Lucy consistently confused the two throughout the first three sessions. Excerpt 1 illustrates the nature of Lucy's confusion.

Excerpt 1 (Session 2)

Interviewer: How would you put the turtle on the place called thirty?

Lucy: Start at thirty.

Interviewer: Okay. Go ahead and do it.

Lucy: [Enters **S.A 30.**; **S.A** is an abbreviation for **START.AT.**]

Interviewer: What would happen if you just put in thirty?

Lucy: Would it just stay there?

Interviewer: Try it.

Lucy: [Enters 30.] Oh yeah, it does that line.

Interviewer: And what did the turtle do?

Lucy: It—then it goes thirty.

Interviewer: Okay, try it again. Put in thirty, but don't press RETURN.

Lucy: [Types **30**.]

Interviewer: Where is the turtle going to end up when you press RETURN?

Lucy: He'll end up over there [pointing to the position 30].

Interviewer: Go ahead and press RETURN.

Lucy: [Presses RETURN.] Oh, it'll go another thirty spaces.

After a few sessions, Lucy distinguished between position and transformation, but she made the distinction reliably only when her attention was drawn to it. Her distinction between position and displacement frequently vanished when she attended to new ideas. Excerpt 2, which reports an exchange that took place during Session 5 when Lucy was trying to attend to composition of transformations, illustrates the fragility of her distinction between displacement and position.

Excerpt 2 (Session 5)
[Turtle is at 40.]

Interviewer: What would be the net effect of forty twenty? [No response.] Put in forty twenty.

Lucy: [Enters **40 20**; the turtle moves accordingly, ending at position 100.] The net was 100.

Both Kim and Lucy conceived initially of a negative transformation in two distinct parts: a displacement and a direction. We distinguish between direction as a qualifier of displacement and direction-displacement as a multiplicative structure. The distinction is completely analogous with preconceptions of torque. To children, distance and weight are mutual qualifiers prior to the emergence of distance-weight as a multiplicative structure called "torque."

When predicting the effect of entering a negative integer, Kim and Lucy would often give the turtle's displacement but not the direction in which it moved. When they included the direction in their responses, the direction was commonly given as a *qualifier* of the displacement: "It will go thirty spaces, and it will go this way (left)" (Kim, Session 3). During the third session, Kim began to consolidate an integer as a direction-displacement pair.

Excerpt 3 (Session 3)

Interviewer: Okay, tell me what will happen when I put in negative fifty.

Kim: It will make a line, and then turn around, and go the space. Fif-, negative fifty, and then turn around and make the arrow.

Kim's change of mind as to how far the turtle would go after it had turned around ("Fif-, negative fifty") indicates her attempt to describe the product of the turtle's movement as a multiplicative structure, namely direction-displacement. We interpret Kim's confusion in Excerpt 3 as an expression of the more general phenomenon of creating a conceptual structure that links a process and the product of that process (Dijkstra, 1976; Mandelbrot, 1983; Thompson, 1985c).

Both students quickly apprehended the process of negating numbers. They understood that " – " denoted both a turn and a turn back, and they reliably expressed that knowledge in their predictions of the turtle's movements. We will see in the next section that determining *where* in an expression the turn back occurred was a problem for both girls; for single numbers this problem never arose.

It took four sessions before either Kim or Lucy would agree that, for instance, $--20$ was the same as 20. They insisted that even though $--20$ and 20 ended up making the same arrow, they were different because the turtle does "a bunch of turnarounds" for $--20$ and doesn't do them for 20 (Session 3). This again shows that Kim's and Lucy's attention was on the processes of the turtle's movements rather than on its movement's effects. Indeed, the processes are different in $--20$ and 20.

Excerpt 4 shows Kim's and Lucy's eventual acceptance, and indeed insistence, that $--$*number* is the same as *number*. We note that Kim and Lucy insisted in pointing out that *their* definition of "the same" had changed to mean "same net effect." The question of what entities they would allow as numbers is discussed in a following section.

Excerpt 4 (Session 5)

Interviewer: Last Tuesday we talked about putting the negative of a negative number in. And you said . . .

Lucy: You could just put in that number.

Interviewer: Well, I told my boss, Pat—he's the one who listens to these tapes—and he didn't believe that you could always just put in the number. He thinks that he can find a number so that when you put the negative of the negative of it, it just won't do the same thing as the number. I think what you say is true, but I couldn't explain it to him. Can you explain it?

Lucy: Like if . . . like you mean when you put negative negative?

Interviewer: Yeah.

Lucy: It's true!

. . . .

Interviewer: But how do we know there isn't some number that it doesn't work for?

Kim: Because it . . . it turns twice, and then it does . . . it's facing the way it started out facing [and then it does the number].

Addition of Integers as a Composition of Transformations

Both Kim and Lucy quickly understood that when they entered two integers together, the turtle executed each in succession. They had no difficulty predicting the net effect when both integers were positive, and only minor difficulties at the outset when first introduced to the composition of two negative integers. They added the numbers to predict the net effect of two positive integers; they typically acted out the effect of the composition when it included one or more negative numbers. In these cases their difficulty was not with the process associated with an expression but with determining the *result* of the process. The dialogue in Excerpt 5 was typical of their difficulties.

Excerpt 5 (Session 6)
[Turtle is at 30.]

Interviewer: Put in negative thirty and twenty.

Lucy: [Types **−30 20**; does not press RETURN.] Negative thirty twenty.

Interviewer: Okay, what's it going to be?

Kim: Fifty . . . negative fifty . . . no.

Lucy: [Presses RETURN.] Ten . . . twenty . . .

Interviewer: What was it?

Kim: Negative fifty.

Interviewer: Look at the dotted arrow up there.

Kim: [To herself] No . . . it's not negative.

Interviewer: What's the net?

Kim: Ten?

Lucy: No . . . ummm . . .

Kim: Fifty . . . negative fifty?

Interviewer: Are you looking at the dotted arrow?

Kim: Yeah, right there.

Interviewer: Boy, negative fifty got real small all of a sudden.

. . . .

Lucy: That's fifty.

Kim: No! It's ten!

Excerpt 5 illustrates the complexity of Kim's and Lucy's attempts to relate position, transformation, negation, and composition ("net effect"). Both students tended to give "total turtle steps" as their evaluation of the composition of two integers. When both integers were positive, this yielded a correct answer. When both integers were negative, both girls knew the answer again was total displacement, but qualified by "it went left." When the integers were of different signs, their answers were unreliable. It was not until the eighth session that both girls could act out the process of an expression and reliably identify the expression's net effect. The dialogue in which they appeared to conceptualize a rule for the sign of a composition is given in Excerpt 6.

Excerpt 6 (Session 8)

Lucy: I didn't understand which . . . how you would know the, which way the net arrow would be facing.

Interviewer: You decided that when you said negative thirty twenty, you knew that it would be ten, but you didn't know if it would be negative or positive ten.

Lucy: Right.

Kim: Yeah.

. . . .

Interviewer: Suppose you started off somewhere.

Lucy: Okay.

Interviewer: And you moved a real long ways to the right, and then you moved a little ways to the left, and you look from where you are from where you started. Are you to the right or left of where you started?

Kim: What do you mean?

Interviewer: If you move a long way to the right and then a little bit to the left, would you be right or left of where you started [demonstrates with arms]?

Kim: Right.

Lucy: Right.

Interviewer: Suppose you move a whole bunch to the left and a little bit to the right. Where are you going to be?

Lucy: Left. . . .

Interviewer: In INTEGERS, suppose the turtle moves sixty and then negative thirty.

Lucy: Oh wait! I see! It'll be . . . will the umm . . . arrow be pointing to the positive side, because it's more . . .

Kim: Wait. I have to . . .

Lucy: Like the negative . . . the net effect arrow will be pointing to the number that moved more. Like, if you have umm . . . let's say seventy and negative fifty.

Interviewer: Hm hmm.

Lucy: It'll be pointing to the positive side, because it moved seventy spaces, and seventy moved more than fifty.

Interviewer: So, can you predict the net effect of sixty negative thirty with your new idea?

Lucy: Thirty! [Enters 60 −30; turtle moves accordingly.]

Kim: Thirty.

Negation of Integer Expressions

In determining the net effect of −[−30 70], for instance, Kim and Lucy had the opportunity to reason in terms of equivalencies and make their reasoning apparent. We looked for instances of the following kind of reasoning as an indicator that they equated an expression and its net effect: −[−30 70] is −40, since −30 70 is 40.

As noted in the previous section, both Kim and Lucy quickly apprehended negation of integers. In particular, they understood that each negative sign entailed a turn at the beginning and a turn back at the end. When negation was applied to expressions, however, they typically applied their knowledge as "theorems in action" (Vergnaud, 1982) without formalizing the net result of the action.

Kim and Lucy each experienced difficulty in placing the "turn back around" in their predictions of negated expressions. When first introduced to negated expressions (Session 8), both students negated only the first term (e.g., the net of −[20 70] is 50). That Kim and Lucy eventually became skilled at maintaining the scope of a negation, and at maintaining a hierarchy of negations, is illustrated in Excerpt 7.

Excerpt 7 (Session 9)

Interviewer: See if you can tell what will happen with this one. [Types −[−40 −30]; does not press RETURN.]

Kim: Oh.

Interviewer: Negative negative forty negative thirty.

Kim: The same thing as the other one [−[−50 −20], which they had just finished discussing and which they described correctly].

Interviewer: Okay.

Kim: It's going to turn around.

Interviewer: Hm hmm.

Kim: It's going to turn around again . . .

Lucy: And move the number.

Kim: And then it's going to move forty, and then it's going to . . .

Both: Turn around.

Kim: And then turn around again, and then move thirty, and then turn around, and turn around.

Excerpt 7 illustrates clearly Kim's and Lucy's understanding of the semantics of a negated expression. However, it does not tell us that they were either capable or incapable of reasoning in terms of equivalencies. The interviewer went on to investigate this issue.

Excerpt 8 (Session 9)

Interviewer: . . . Okay, let's talk about an easy way to do these [negated expressions]. What's the net effect of this number? [Types 40 50; does not press RETURN.]

Both: Ninety.

Interviewer: Ninety, okay. [Presses RETURN; turtle moves accordingly.] What's going to be the net effect of negative forty fifty? [Types −[40 50]; does not press RETURN.]

Lucy: Oh . . . let's see. Hmm . . . ten.

Interviewer: Why would you say it's ten?

Lucy: Because it would turn around . . . huh. No . . . negative ten?

Interviewer: What is it going to be, Kim?

Lucy: It turns around . . .

Kim: It's going to be ten. Because it's going to turn around, and then go forty, and then turn around, and then go fifty . . . no.

Lucy: And then turn around again.

Kim: Yeah.

Lucy: To finish the first one.

[Students proceeded to correct their predictions of −[40 50] in terms of turtle actions.]

Interviewer: Let's watch. [Presses RETURN; turtle moves accordingly.] Was the net effect negative ten?

Kim: No.

Interviewer: What was it?

Lucy: Ten.

Kim: Ninety . . . ninety . . . negative ninety!

Interviewer: Negative ninety.

Lucy: Negative ninety! Negative ninety!

Kim: Because it moved the whole way, and it didn't turn around enough.

Interviewer: Now, if I ask you what's the net effect of forty fifty?

Both: Ninety.

Interviewer: You tell me it's ninety. If I ask you what's the net effect of forty fifty, but that number with a negative in front of it—negative forty fifty . . .

Lucy: Negative ninety . . . Oh yeah, like if it'd be fifty thirty, it'd be eighty. And if it'd be negative fifty thirty, it'd be negative eighty.

Interviewer: Okay. Now, what is the net effect of negative fifty thirty? [Types −50 30; does not press RETURN.]

Both: Negative twenty.

Interviewer: Okay, now what's going to be the net effect of negative negative fifty thirty? [Deletes −50 30; types −[−50 30]; does not press RETURN.]

Lucy: Negative negative twenty.

Interviewer: Do we know what negative negative twenty is?

Lucy: Twenty.

Excerpt 8 illustrates two points. The initial part of the excerpt illustrates that the reliability of Kim's and Lucy's schemes for negating expressions was not high (see Excerpt 7). The lack of reliability may be traced to the fact that they usually reasoned in procedural terms: What is the turtle going to do? The degree of complexity in this kind of reasoning is high, and confusion is likely to result as soon as there are two or more negative signs in the expressions. The latter part of Excerpt 8 suggests that both Kim and Lucy could reason in terms of equivalencies of expression and composition (net effect). The question remains as to the degree of spontaneity of that reasoning. The dialogue in Excerpt 9 addressed this question.

Excerpt 9 (Session 9)

[Before this dialogue, the interviewer introduced the DEF command (Figure 5) and illustrated it with [20 20], [30 50], and [60 −30]

Interviewer: What's going to happen with this one? [Enters DEF "H[−70 40]; computer responds H DEFINED.]

Lucy: Well . . . it'll be negative thirty.

Kim: Yeah.

Interviewer: Let's put in H just to make sure. [Enters H; turtle moves accordingly.]

Kim: Uh huh.

Interviewer: Okay, what if I put in negative H? [Types −H; does not press RETURN.] You remember what H is?

Lucy: Thirty.

Interviewer: What is it? We just did it up here [pointing to screen].

Kim: Negative thirty.

Lucy: Negative thirty.

Interviewer: Negative thirty. So, we're going to put in negative H. What's it going to be?

Lucy: Negative thirty.

Kim: Thirty.

Interviewer: Negative H is equal to what?

Lucy: Negative thirty.

Kim: Thirty.

Interviewer: [To Lucy] No, what is H equal to? We want to do a negative, and what's H equal to?

Lucy: Negative thirty.

Interviewer: We want to do negative H. . . . What's negative negative thirty?

Lucy: Sixty. No, no, no, no! . . . It'll be sixty . . . negative sixty.

Apparently, Kim had formalized the equivalence of net effect and expression to a greater degree than Lucy had. Kim had a knowledge of equivalence to use spontaneously in solving a more complex problem. Lucy did not. Lucy most typically equated composition and expression when that was stated explicitly as the goal of the problem.

We took a second approach to assess Kim's and Lucy's ability to identify composition with expression. The interviewer typed an expression (without pressing RETURN) and asked what number would take the turtle back to where it started were the expression executed. The reasoning we looked for was this: If z is the net effect of *expression*, then $-z$ will undo the effect of *expression*. Both students correctly predicted the negative of an expression in some cases and got confused in others. The dialogues in Excerpt 10 were typical.

Excerpt 10 (Session 11)

Interviewer: Try negative thirty sixty. [Types −30 60; does not press RETURN.] Can you tell me what number I can put in to make it come back to where it started?

Lucy: Thirty.

Interviewer: Thirty?

Kim: Ninety.

[The interviewer then asked them to press RETURN, whereupon they understood that the correct answer would have been negative thirty. A few minutes later, the dialogue continued.]

Interviewer: Suppose I did this . . . suppose I said the negative of thirty forty? [Types −[30 40]; does not press RETURN.] What number could I put in to make it come back?

Kim: Thirty forty.

Interviewer: Thirty forty?

Kim: Or . . . seventy.

Interviewer: Can you explain why we put in thirty forty?

Lucy: Well, the negative is in front of the whole number. [Lucy then starts acting out the expression in terms of turtle movements.]

Much of the dialogue in Session 11 showed a considerable amount of confusion. Immediately following the dialogue in Excerpt 10, Kim was asked to predict what number would undo the effect of 60 −30. She insisted that 30 would suffice. Further probing made it apparent that Kim anticipated that the turtle faced to the left after executing 60 −30 (an atypical error for her) and hence that 30 would take it back to its original position.

On the other hand, in the last part of Excerpt 10, Kim identified the composite transformation [30 40] with its net effect, 70, and both students demonstrated an understanding that an easy way to bring the turtle back to its starting position was to find the expression's net effect and then enter its negation.

From the dialogues in Session 11 we concluded that the girls' confusion was not attributable solely to the difficulty of the task of negating an expression. Rather, their difficulty originated with the thought operations involved in equating an expression and its net effect. These difficulties were then accentuated by the added cognitive load of negating the expression. However, the fact that both students did employ a primitive form of a rule of substitution suggests that they were at least beginning to formalize a relationship between their concepts of expression and net effect.

CONCLUSION

We have shown that sixth graders can conceive of integers as transformations in nontrivial ways. They can address problems that parallel algebraic structures, such as finding the negation of an expression. Further research is required to determine whether or not such conceptual development in integers in fact makes a difference in the way students learn algebra.

It became apparent in our analysis of the transcripts that the road to a formal understanding of integers and integer operations is long and bumpy. The difficulty with which Kim and Lucy constructed concepts of transformation, composition, and inversion—in the semantically rich environment of INTEGERS and in the course of 11 highly individualized lessons—calls into question the depth of understanding we may expect from students studying integers under more traditional instruction. Moreover, research on students' understanding of integers and integer operations through traditional instruction needs to focus upon operations of thought that generalize to elementary algebra as well as upon skill with standard addition and subtraction algorithms.

The instability of Kim's and Lucy's conceptualizations was remarkable. We have no explanation for it, nor are we aware of a theory of learning that speaks specifically to their difficulties. Theories that espouse learning as knowledge compilation (e.g., Neves & Anderson, 1981) address the general phenomenon of unreliable knowledge, but knowledge in these theories is depicted as declarative (i.e., linguistic) structures that bear little resemblance to the nonlinguistic, action-based knowledge expressed in Kim's and Lucy's explanations and predictions. An extension of diSessa's (1983) approach, in which nonlinguistic, phenomenological primitives provide the foundation of mathematical mental representations, would appear to be more appropriate (Janvier, 1985).

It seems warranted to conjecture that an exposure to INTEGERS over a longer period than in this study, such as half a year, would result in more stable and reliable knowledge. A study of longer duration than reported here, during which the conceptualizations of composite and inverse transformations become stabilized, could serve as a basis for examining the effects of such instruction on the acquisition of parallel thought processes in elementary algebra. Similar studies of the developing conceptions of functions and operations and of abelian group structures would also be feasible but would have to be preceded by a careful theoretical analysis of the thought processes involved and the transfer effects that might be expected.

The effect of specific models on the quality of students' learning also must be investigated. The INTEGERS microworld is not the only model of integers one might use. One could use a "charged particle" model (Battista, 1983; Bennett & Nelson, 1979) or a balloon model (Janvier, 1983), either of which might or might not be implemented on a computer. Two shortcomings of these models, however, are that neither model proposes negation as an operation upon integers and that there is no way in which to represent an expression within either model.

The model entailed by INTEGERS need not be restricted to a computer. One could have students act out definitions by walking on a number line. We would expect it to be more difficult for students to assimilate the model when used in other than a computerized environment, for they would not have objective feedback on the efficacy of their thinking. Nevertheless, the relative influence of various models of integers and integer operations is an issue that warrants further investigation.

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AUTHORS

PATRICK W. THOMPSON, Associate Professor, Department of Mathematics, Illinois State University, Normal, IL 61761

TOMMY DREYFUS, Senior Lecturer, Center for Technological Education, P.O. Box 305, Holon 58368, Israel

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