The Development of The Concept of Speed and its Relationship to Concepts of Rate*

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Running Head: Concepts of Speed and Rate

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Can we measure the speed of a car in miles per century? *No, because you would die or the car would rust away before a century.* Suppose we travelled from here to Peoria at 30 miles per hour and returned at 70 miles per hour. What would be our average speed for this trip? *50 miles per hour—because 100 divided by 2 is 50.* What does it mean when we say we averaged 45 miles per hour on a trip? *That we mostly went 45 miles per hour.*

INTRODUCTION

Hidden in these questions and several fifth-grader's responses are a myriad of issues regarding not only a concept of speed, but also regarding measurement, proportionality, ratio, rate, fraction, and function. But, do we see primitive conceptions of speed and primitive conceptions of average in these children's responses, or do we simply see the effects of schooling? I propose that it is all three, but predominantly the latter. Also, I will demonstrate that by removing certain insidious practices of schooling, such as insisting that students be able to calculate any expression they encounter in solving a problem, combined with a concerted effort to focus students' attention on their conceptions of situations, it is possible for students to advance far beyond what we typically expect of them.

The setting for the interviews from which the above excerpts were taken is a project to investigate the development of students' quantitative reasoning and the relationship of quantitative reasoning to reasoning algebraically in applied situations. The premise behind this investigation is that when people reason mathematically about situations, they are reasoning about things and relationships. The "things" reasoned about are not objects of direct experience and they are not abstract mathematical entities. They are objects derived from experience—objects that have been constituted conceptually to have qualities that we call mathematical.

Abstract mathematical objects are the constructions of a relatively few people who have reflectively abstracted qualities and relationships constructed through experience into formal systems of relationships and operations. If the person doing the reasoning about a situation is also a person who has reflectively abstracted systems of relationships and operations having to do with conceiving it, reconstituting them as *mathematical* objects (Harel & Kaput, in press), then, from that person's point of view, the situation is composed of mathematical objects. This project aims to capture the multiple re-constitutions that take place in individual people as they progress toward the construction of mathematical objects such as ratio and rate. Vergnaud (1983, 1988, this volume) and Greer (1989, this volume) have given valuable explications of broad characteristics of situations that lend themselves to being conceptualized in terms of ratios and rates. It is also valuable to have an idea of how individual people come to comprehend such situations, and to have an idea of how such comprehensions become possible.

One crucial part of sound mathematical development is students' construction of powerful and generative concepts of rate. What that means is a source of controversy. I will discuss controversies surrounding the concept of rate in the next section as part of my explication of quantitative reasoning. In the third section of this paper I will discuss a teaching experiment done to investigate individual children's construction of the concept of speed as a rate and to investigate the components of a conception of average rate.

Quantitative Reasoning

To characterize quantitative reasoning I must first make clear what I mean by quantity and what I mean by a quantitative operation. I will draw heavily on Piaget's constructs of internalization, interiorization, mental operation, and scheme.

The notion of goal-oriented action (praxis) is foundational to understanding Piaget's notions of internalization, interiorization, mental operation, and scheme. It may prove helpful if I point out that, in Piaget's usage, an action is not the same as an observable behavior. An action is an activity of the mind which might be expressed in behavior, but it need not be expressed in anything detectable by an observer (Powers, 1973, 1978). It is important to keep this distinction in mind when giving meaning to terms like action, image, and operation.

Constructs from Piaget's Theory

Internalization and Interiorization

Actions are tied more or less to experience. At the lowest level are reflexes, at the highest level is intelligent thought. Internalization is the process of reconstructing actions so as to enable mental imagery of situations involving them (Lewin, 1991). Piaget distinguished among three types of images:

(1) An "internalized act of imitation … the motor response required to bring action to bear on an object … a *schema* of action."

(2) "In place of merely representing the object itself, independently of its transformations, this image expresses a phase or an outcome of the action

performed on the object. … [but] the image cannot keep pace with the actions because, unlike operations, such actions are not coordinated one with the other."

(3) "[An image] that is dynamic and mobile in character … entirely concerned with the transformations of the object. … [The image] is no longer a necessary aid to thought, for the actions which it represents are henceforth independent of their physical realization and consist only of transformations grouped in free, transitive and reversible combination." (Piaget, 1967, pp. 294-296).

Internalization refers to an assimilation, an initial "image having" (Kieren & Pirie, 1991). Interiorization refers to the progressive reconstruction and organization of actions so as to enable them to be carried out in thought, as mental operations. Later in this chapter I will say that a child "internalized" the text of a situation. By that I will mean that she read the text and constructed an image—probably unarticulated—of the situation and its elements, with the meaning that her image contains actions of moving, filling, comparing, and so on.

Mental operation

As intimated in Piaget's quotations concerning images, a mental operation is a system of coordinated actions that can be implemented symbolically, independently of images in which the operation's actions originated. Mental operations are always implemented in an image, but the image need not be one tied historically to the origins of the operation (von Glasersfeld, 1991). Mathematicians speak frequently of such things as, say, partitions with the understanding that the actions taken to make one have definite requirements, characteristics, and products, yet if they conjure an image it has no special status relative to the generality of the operation of partitioning.

Scheme

What makes an operation repeatable? This was the problem Piaget addressed with his notion of scheme. His characterization of a scheme, "whatever is repeatable and generalizable in an action" (Piaget, 1971, p. 42) hardly seems helpful. Cobb & von Glasersfeld (1983) provide a useful elaboration. A scheme is an organization of actions that has three characteristics: an internal state which is necessary for the activation of actions composing it, the actions themselves, and an imagistic anticipation of the result of acting. The imagistic anticipation need not be iconic. It could just as well be symbolic, kinesthetic, or any other re-presentation of experience (von Glasersfeld, 1980)—although the specific characters of its actions and anticipations may affect the generality with which a scheme is applied (Steffe, this volume).

Piaget stated his view on the source of schemes quite clearly: Schemes emerge from assimilations of experience to ways of knowing.

> Assimilation thus understood is a very general function presenting itself in three nondissociable forms: (1) functional or reproductive assimilation, consisting of repeating an action and of consolidating it by this repetition; (2) recognitive assimilation, consisting of discriminating the assimilable objects in a given scheme; and (3) generalizing assimilation, consisting of extending the field of this scheme .... It is therefore assimilation which is the source of schemes … assimilation is the operation of integration of which the scheme is the result. Moreover it is worth stating that in any action the driving force or energy is naturally of an affective nature (need and satisfaction) whereas the structure is of a cognitive nature. To

> assimilate an object to a scheme is therefore simultaneously to tend to satisfy a need and to confer on the action a cognitive structure (Piaget, 1977, pp. 70-71).

The third form of assimilation, “generalizing assimilation,” provides one way of understanding the important notion of transfer, and it provides a foundation for understanding how people meet occasions requiring reflection or distinction. We recognize situations by the fact we have assimilated them to a scheme. When features of that situation emerge in our understanding that do not fit what we would normally predict, we introduce a distinction, and the original scheme is accommodated by differentiating between conditions and subsequent implications of assimilation.[1] One way to understand the idea of generalizing assimilation is to consider that all situations as constituted in someone’s comprehension are the products of actions and operations of thought, and that when there is a large commonality between the operations of thought activated on different occasions, the person constituting the situations will experience a feeling that the later comprehension is somehow similar to the earlier one.[2] Later on in this paper I will discuss an attempt to orient a student so that she would construct a scheme for speed that is powerful enough that she recognize (what we take as) more general rate situations as being largely the same as situations involving speed. My attempts were to get this child to engage in generalizing assimilations.

[1] von Glasersfeld (1989) provides a marvelous example of generalizing assimilation and subsequent accommodations. He discusses a child’s assimilation of (what we would see as) a spoon to her “rattle scheme,” and subsequent distinctions (no rattle sound), and accommodations (produces a wonderful bang when hit on the table).

[2] Judgements of similarity may range from figurative (situations “resemble” one another) to operative (the situations are, *in principle*, identical).

Reflective abstraction

Reflective abstraction is perhaps the most subtle, important, and least understood of Piaget's constructs (Steffe, 1991; von Glasersfeld, 1991). It is the motor of interiorization—the process whereby actions become organized, coordinated, and symbolized (Bickhard, 1991; Thompson, 1985a, 1991). As von Humbolt put it, "In order to reflect, the mind must stand still for a moment in its progressive activity, must grasp as a unit what was just presented, and thus posit it as object against itself" (1907, Vol. 7, part 2, p. 581, as quoted in von Glasersfeld, 1991).

Reflective abstraction has two aspects in Piaget's theory. The first is the re-construction of actions so as their activation is progressively less dependent on immediate experience. The second is an assimilation to higher levels of thought—to schemes of operations. The first is closely allied with what we normally think of as learning, while the second is closely allied with what we normally think of as comprehending (Piaget, 1980).

Quantity

Quantities are conceptual entities. They exist in people's conceptions of situations. A person is thinking of a quantity when he or she conceives a quality of an object in such a way that this conception entails the quality's measurability. A quantity is schematic: It is composed of an object, a quality of the object, an appropriate unit or dimension[3], and a process by which to assign a numerical value to the quality. Variations in people's conceptions of a quantity occur in correspondence with variations in level of development of components within their schemes. Also, I want to make clear that objects are constructions that a person takes as given, and that qualities of an object are imbued by the subject conceiving it—as Piaget pointed out repeatedly. To a young child watching a passing car, the car probably is an object, and it may have the quality *motion*. However, for this young child the car probably does not have the quality *distance moved in an amount of time*.

My characterization of quantity differs somewhat from others. Schwartz (1988), Shalin (1987), and Nesher (1988) characterize quantities as ordered pairs of the form (*number*, *unit*). To characterize quantities as ordered pairs may be useful formally, but it does not provide insight into what people understand when they reason quantitatively about situations, and it severely confounds notions of number and notions of quantity (Thompson, 1989, in press). Steffe (1991b), in extending the work of Piaget (1965), characterizes quantity as the outcome of unitizing or segmenting operations. I take the operations of unitizing and segmenting as foundational to a person's creation of quantities, but I have found it productive to use "quantity" more broadly than Steffe does. By characterizing the idea of quantity schematically we are able to capture important structural characteristics of people's

[3] By "dimension" I mean a person's understanding of a quality as being measured potentially by any of a number of appropriate units. When a person's understanding of a quality includes the awareness of the possibility of using any of a multitude of mutually-convertible units (i.e., an equivalence class of units),

reasoning when they reason about complex situations that involve a myriad of multiplie-related quantities (Thompson, 1989, in press).

Quantification

Quantification is a process by which one assigns numerical values to qualities. That is, quantification is a process of direct or indirect measurement. One does not need to actually carry out a quantification process for a quantity in order to conceive it. Rather, the only prerequisite for a conception of a quantity is to have a process in mind. Of course, a person's grasp of a process may change as he or she re-conceives the quality and the process in relation to one another.

Piaget (1964) made a valuable distinction between two dramatically different kinds of quantification: gross quantification and extensive quantification. Gross quantification refers to a conception of a quality in ways that objects having it can be ordered by some experiential criteria (e.g., "appears bigger than"). Extensive quantification refers to a conception of a quality as being composed of numerical elements which arise by operations of unitizing or segmenting (see also Steffe, 1991; Steffe, von Glasersfeld, Richards, & Cobb, 1983). Both kinds of quantification are essential for conceiving situations quantitatively; persons limited to gross quantification are blocked from conceiving situations mathematically.

Quantitative Operation

A quantitative operation is a *mental operation* by which one conceives a new quantity in relation to one or more already-conceived quantities. Examples of quantitative operations are: combine two quantities additively, compare two

then this person possesses an "adult" conception of dimension.

quantities additively, combine two quantities multiplicatively, compare two quantities multiplicatively, instantiate a rate, generalize a ratio, combine two rates additively, and compose two rates or two ratios. It is important to distinguish between constituting a quantity by way of a quantitative operation and evaluating the constituted quantity. One can conceive of the difference between your height and a friend's height without giving the slightest consideration to evaluating it.[4]

A quantitative operation creates a quantity in relation to the quantities operated upon to make it. For instance, comparing two quantities additively creates a difference; comparing two quantities multiplicatively creates a ratio. A quantitative operation creates a structure—the created quantity in relation to the quantities operated upon to make it.

Quantitative operations originate in actions: The quantitative operation of combining two quantities additively originates in the actions of putting together to make a whole and separating a whole to make parts; the quantitative operation of comparing two quantities additively originates in the action of matching two quantities with the goal of determining excess or deficit; the quantitative operation of comparing two quantities multiplicatively originates in matching and subdividing with the goal of sharing. As a person interiorizes actions, making mental operations, these operations-in-the-making imbue him or her with the ability to comprehend situations representationally, and enable him or her to draw inferences

[4] While you may not give any consideration to evaluating the difference between your height and your friend's height, you probably compare the two with full confidence that the difference could be evaluated by any of a number of methods.

about numerical relationships that are not present in the situation itself.

Not every action is interiorized as a mental operation. The problem-classification scheme developed by Carpenter and Moser (1983), for instance, can be thought of as corresponding to classes of action schemata that become internalized as mental images of action and then interiorized as the mental operations of additive combination and additive comparison.

Quantitative Operations vs. Numerical Operations

To understand the notion of quantitative reasoning, it is crucial to understand the distinction between quantitative operations that create quantities and numerical operations used to evaluate quantities. The following problem was used in a teaching experiment on complexity and additive structures with six fifth-graders (Thompson, in press).

> Team 1 played a basketball game against Opponent 1. Team 2 played a basketball game against Opponent 2. The captains of Team 1 and Team 2 argued about which team won by more. The captain of Team 2 won the argument by 8 points. Team 1 scored 79 points. Opponent 1 scored 48 points. Team 2 scored 73 points. How many points did Opponent 2 score?

Five children, after varying degrees of effort and intervention, subtracted 48 from 79 (getting 31), then added 8 to 31 (getting 39). To a child, these five could not say what “39” stood for (i.e., the difference between Team 2’s score and Opponent 2’s score). It was evident in the interviews that these children understood that the situation was about comparing two differences, and it was evident that to them, having *added* to get 39, 39 could not be the value of a difference. These children had not distinguished among the numerical operation actually used to evaluate a quantity (in this case addition), the quantitative operation used to create the quantity (in this case additive comparison), the kind of quantity being evaluated (in this case a difference), and the operation used to evaluate the quantity under stereotypical conditions (in this case subtraction). They had over-identified the quantitative operation of additive comparison with the numerical operation of subtraction.

The distinction between quantitative and numerical operations is largely tacit in mathematics education, and the two are often confounded, as in the following quote.

> A partitive model for 12 ÷ 3 would be 12 miles divided by 3 hours equals 4 miles per hour. A quotitive interpretation would be 12 miles divided by 3 miles per hour equals 4 hours. Finally, as the inverse of cross product multiplication, a model could be 12 outfits divided by 3 skirts equals 4 blouses. (Hiebert & Behr, 1988, p. 5)

Here is a confusion: division is a numerical operation, it is not a quantitative operation. We do not "divide" one object by another. If 12 ÷ 3 interpreted as quotitive division means "how many composite units of 3 are contained in the composite unit of 12" (Hiebert & Behr, 1989, p. 5), then "12 miles divided by 4 miles per hour" means we are asking how many units of 4 miles/hour are contained in 12 miles? A decision to divide need not be based always on considerations of partition or quotition. It could also be made relationally—as when one conceives of a situation multiplicatively and the information being sought pertains to an initial condition, such as "How long must one travel at 4 miles/hour to go 12 miles?" or "How many blouses does Sally have if she has 3 skirts and 12 skirt-blouse pairs?"

We may, in the final analysis, short-change our students by holding numerical operations on high, proposing models for them so that they become meaningful. I have difficulty seeing how the "models" approach differs from some ways of teaching formal, axiomatic systems: Instantiate the axioms, thereby creating a model of the formal system. One proposes a model in order to make the formal system more tangible, but the ultimate aim is to teach a formal system that has no special relationship with the proposed model. In some ways this turns the idea of learning on its head.

A quantitative operation is non-numerical; it has to do with the *comprehension* of a situation. Numerical operations are used to evaluate a quantity. It is understandable, however, that the two are easily confounded. First, in simple situations being conceived by an adult who has both, the two are so highly related that they appear indistinguishable. It is in more complex situations that distinctions must be made, and often people make them so rapidly that they are made unconsciously. Second, we do not have a conventional notation for quantitative operations independent of the arithmetic operations of evaluation. Thus, arithmetic notation has come to serve a double function. It serves as a formulaic notation for prescribing evaluation, and it reminds the person using it of the conceptual operations that led to his or her inferences of appropriate arithmetic. The double-use of arithmetic operations provides power and efficiency for persons who can make these subtle distinctions while using the notation. It is a source of confusion to many students and teachers who have not constructed distinctions between quantitative relationship and numerical operations. In (Thompson, 1989) I describe an extension of a notational system for quantitative operations and quantitative relationships originally devised by Shalin (Greeno, 1987; Shalin, 1987) that can be used to represent non-numerical comprehensions of situations.

Comprehension of Quantitative Situations

A person comprehends a situation quantitatively by conceiving of it in terms of quantities and quantitative operations. Each quantitative operation creates a relationship: The quantities operated upon with the quantitative operation in relation to the result of operating. For example, the quantities "girls in this class" and "boys in this class" being compared multiplicatively produces the quantity "ratio of girls to boys." Those three quantities in relation to one another constitute a quantitative relationship. Comprehensions of complex situations are built by constructing *networks* of quantitative relationships.

The process of constructing a comprehension of a specific situation is a dialectic among reflectively abstracting features of the situation to schemes of operations, expressing mental operations in action schemata, and reflecting results of activated schemata back to schemes of operations. Variations among people's comprehensions of a specific situation correspond to variations in the operational constitution of their schemes and to the classes of actions represented by their mental operations.

Ratio vs. Rate

The fact that we have two terms "ratio" and "rate" would suggest that we have two ideas different enough to warrant different names. Yet, there is not a conventional distinction between the two, and there is widespread confusion about such distinctions. The confusion is not limited to school classrooms; confusion is evident even in the mathematics education research literature. Ohlsson (1988) wrote the following as an analysis of ratio:

> … If I get 8 miles per gallon out of my car during the first leg of a journey but for some reason get only 4 miles per gallon during the second leg, then the correct description of my car's performance over the entire trip is *not* (4+8)=12 miles per gallon. Similarly, if one classroom has a ratio of 2 girls per 3 boys and another classroom has 4 girls per 3 boys, the combined class does not have 2/3 + 4/3 = 6/3, or 6 girls per 3 boys....The correct analysis of the examples is, I believe, the following. The fuel consumption during the first leg of the journey in the first example was 8 miles per gallon. We represent this with the vector (8,1). The fuel consumption during the second leg of the journey was (4,1). Adding these vectors … gives (12,2), which is equivalent to (i.e., has the same slope as) the vector (6,1). Vector theory predicts that the fuel consumption during the entire trip was 6 miles per gallon, which is correct. (Ohlsson, 1988, p. 81; emphasis in original).

Claims of correctness notwithstanding, Ohlsson's analysis of his car's mileage is correct only if he traveled ($8x$) miles in the first leg of his journey and ($4x$) miles in the second leg. Otherwise, it is incorrect. He evidently took a *rate* of consumption (8 miles per gallon) for actual consumption (went 8 miles and used 1 gallon). Without information about the relative number of miles travelled or relative number of gallons used in each leg of his journey, we can say nothing about his car's *rate* of fuel consumption.[5]

Perhaps the lack of conventional distinction between ratio and rate is the reason that the two terms are used often without definition. Lesh, Post, and Behr noted that "… there is disagreement about the essential characteristics that distinguish, for example rates from ratios … In fact, it is common to find a given author changing terminology form one publication to another" (1988, p. 108). The most frequent distinctions given between ratio and rate are:

1) A ratio is a comparison between quantities of like nature (e.g., pounds vs. pounds) and a rate is a comparison of quantities of unlike nature (e.g., distance vs. time; Vergnaud, 1983, 1988).

2) A ratio is a numerical expression of how much there is of one quantity in relation to another quantity; a rate is a ratio between a quantity and a period of time (Ohlsson, 1988).

3) A ratio is a binary relation which involves ordered pairs of quantities. A rate is an intensive quantity—a relationship between one quantity and one unit of another quantity (Kaput, Luke, Poholsky, & Sayer, 1986; Lesh, Post, & Behr, 1988; Schwartz, 1988).

While there is an evident controversy about distinctions between ratio and rate, each of these distinctions seems to have at least some validity. My explanation for this controversy is that these distinctions have been based largely upon situations *per se* instead of being based on the mental operations by which people constitute situations. When we shift our focus to the operations by which people constitute "rate" and "ratio" situations, it becomes clear that situations are neither one nor the other. Instead, how one might classify a situation depends upon the operations by which one comprehends it. In Thompson (1989) I illustrate how an "objective" situation can be conceived in fundamentally different ways depending on the quantitative operations available to and used by the person conceiving it. When we take the perspective that ratios and rates are the products of mental operations,

[5] Suppose, for instance, that the first leg of the journey was 99.99 miles, and the second leg of the

classification schemes for separating situations into "rate" and "ratio" categories are no longer of great importance.

A ratio is the result of comparing two quantities multiplicatively.

This definition is in accord with most given in the literature. One slight difference is that it does not specify how the result of a multiplicative comparison is denoted or expressed. For example, a collection of 3 objects can be compared multiplicatively against a collection of 2 objects in either of two ways: a comparison of the two collections *per se*, or a comparison of one as measured by the other (Figure 1). The first comparison is of the two collections as wholes. The second comparison is of one quantity measured in units of the other. Both are expressions of a multiplicative comparison of the two quantities. The second is propitious for concepts of fraction, and may be more sophisticated than the first. It needs to be noted about the second comparison that even though the result is expressed in the same way as what is often called a "unit rate," the comparison described is between two specific, non-varying quantities, and hence is a ratio comparison.

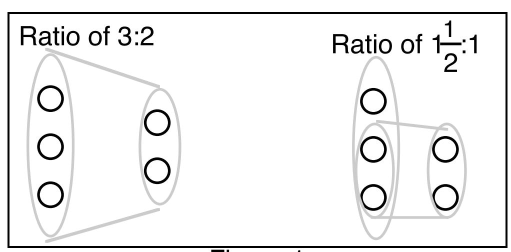

Figure 1

One conception of ratio that is propitious for constructing rates is the conception of a ratio with a fixed value being the result of comparing two quantities having indeterminate values. I use the term "ratio" even in this last case, since it is two specific quantities (e.g., a car's distance and that same car's time of travel, thought of as fixed but unknown) that are present in thought when the person conceives the comparison.

journey was 0.01 miles. It is evident that the car's rate of fuel consumption is very close to 8 mi/gal.

When we focus on the mental operation of multiplicative comparison, it is evident that it makes no difference if the quantities are of the same dimension or not. What matters is that the two quantities are being compared multiplicatively. If the quantities being compared are measured in the same unit, then the comparison happens to be a direct comparison of qualities. If the quantities are measured in different units, then it is segmentations (measures) of their qualities that are being compared. In either case, the salient mental operation is multiplicative comparison of two specific quantities, and the result of the comparison is a ratio.[6]

A rate is a reflectively abstracted constant ratio.

A rate is a reflectively-abstracted constant ratio, in the same sense that an integer is a reflectively abstracted constant numerical difference (Thompson, 1985a, 1985b; Thompson & Dreyfus, 1988). A specific numerical difference, as a mental structure, involves a minuend, a subtrahend, and the result of subtracting. An integer, as a reflectively abstracted numerical difference, symbolizes that structure as a whole, but gives prominence to the constancy of the result—leaving minuend and subtrahend variable under the constraint that they differ by a given amount.

Similarly, a specific ratio in relation to the quantities compared to make it is a mental structure. A rate, as a reflectively abstracted constant ratio, symbolizes that structure as a whole, but gives prominence to the constancy of the result of the

[6] Values of compared quantities need not be known in order to *conceive* them as being compared multiplicatively. The process by which one actually achieves a numerical value for the ratio is the quantification of the ratio. This is one more way in which quantitative reasoning differs from numerical reasoning.

multiplicative comparison.[7]

Once a specific situation is conceived in a way that involves a rate, it is implicit in the way the concept of rate is constructed that the values of the compared quantities vary in constant ratio. Hence, a specific conceived rate is (from our point of view) a linear function that can be instantiated with the value of an appropriately conceived structure. To say that an object travels at 50 miles/hour quantifies the object's motion, but it says nothing about a distance traveled nor about a duration traveled at that speed (Schwartz, 1988). However, conceiving speed of travel in relation to an amount of time travelled produces a specific value for the distance traveled.[8]

When one conceives of two quantities in multiplicative comparison, and conceives of the compared quantities as being compared in their, *independent*, *static* states, one has made a ratio. As soon as one re-conceives the situation as being that the ratio applies generally outside of the phenomenal bounds in which it was originally conceived, then one has generalized that ratio to a rate (i.e., reflected it to the level of mental operations). The wording "as soon as one re-conceives …" is important. It is possible, perhaps likely, that people first conceive a multiplicative comparison in terms of a ratio and re-conceive that ratio as a rate. One occasion for

[7] This is one way to explain why a rate has the feel of being a single quantity.

"re-conceiving" a ratio as a rate in schools is when students are asked to "assume (something) continues at the same rate" when the initial situation is described in a way that there are two quantities to be compared multiplicatively.[9]

It is in responding to the question of how people come to conceive of rates that I rely on Piaget's notion of reflective abstraction. The first sense of reflected abstraction is that a class of actions are re-constructed and symbolized at the level of mental operations. The second sense of reflected abstraction is that as a situational conception is constructed, the figurative aspects of the conception are "reflected" to the level of mental operations. In the first sense, we would say a person has learned (e.g., learned speed as a rate). In the second sense we would say a person has comprehended (e.g., conceived an object's motion as a rate).

One activity that provides occasions for students to think in ways propitious for constructing rate as a reflection of constant ratio is the use of "building up" strategies in the solution of proportional reasoning tasks (Hart, 1978; Kaput & West, this volume). If a child is trying to find, say, how many apples there are in a basket where the ratio of apples to pears is 3:4 and there are 24 pears, and the child thinks "3 apples to 4 pears, 6 apples to 8 pears, …, 18 apples to 24 pears" (a

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[8] Some might say that "50 mi/hr" can be instantiated also by a value for distance, thereby producing a value for time. I would argue that a person making this claim bases it on either of two reasoning processes: (1) an inference based on some form of the numerical formula "d = rt", whereby one reasons that in, say, $125=50t$, t must be 2.5, or (2) that since the object goes 50 miles per hour, it must also go 1/50 hours per mile. The first inference, while valid, is not a quantitative inference. It is valid, formal, numerical reasoning. The second inference employs a sophisticated mental operation: inversion of a rate. Even in the second case, however, "50 mi/hr" is not instantiated with a value for a distance to produce a time. Instead, its *inversion* (measured in hours per mile) is instantiated with a value of a distance.

[9] Unfortunately, this occasion is too often taught by teachers and recognized by students as the time to perform certain numerical rituals.

succession of ratios), then this provides an occasion for the child to abstract the relationship "3 apples for every 4 pears" (an iterable ratio relating collections of apples and pears as the amounts of either might vary), and eventually "there will be $\frac{3}{4}$ of an apple or part thereof for every pear or proportional part thereof" (an accumulation of apples and pears, which carries the image that the values of both can vary, but only in constant ratio to the other). The former conception—accumulations made by iterating a ratio—I call an *internalized* ratio, whereas the latter conception—total accumulations in constant ratio—I call an *interiorized* ratio, or a *rate*.[10]

Motion vs. Speed

Piaget (1970), in investigating the development of children's concepts of movement and speed, made a clear distinction between motion as a phenomenon—the experience or observation of movement—and speed as a quantity. Objects move; speed is a quantification of motion. Thus, the quantity normally identified as speed is more accurately thought of as motion together with a quantification of it.

Piaget explained the development of children's concept of speed in terms of the emergence of the general mental operations entailed in proportional reasoning, and used the language of centration (focus of attention), decentration (coordinations of

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10 Kaput and West (this volume) raise an interesting extension of this way of thinking about rates. They speak about conceptions of "homogeneity" as characteristic of rate-like reasoning. Harel (1991) has used this concept of rate to explain students' difficulties with traditional "mixture" problems, where students must conceive of samples from a mixture as *always* being composed of constituent elements in constant ratio to one another, *regardless of the amount of mixture sampled*. In this conception, the notion of variability is not of accumulation, but instead is of random value. Accumulation still may be part of a person's conception of a given random value, but accumulation is not the predominant conception of

centrations), and regulations (construction of mental operations that balance conflicting results of centrations, such as a conflict between noticing that one object moved farther than another, which suggests it went faster, and noticing that it took more time than the other, which suggests it went slower). He concluded that the concept of speed is constructed as a proportional correspondence between distance moved and time of movement—"the elaboration first of concrete and later of formal metrical operations" (p. 259).

Piaget described the emergence of a concept of speed (quantified motion) as a process wherein children first re-conceive motion as entailing changes in position simultaneously with changes in time, then coordinate the two dimensions of distance and time as changing in proportion to one another (Piaget, 1970, pp. 279-280). He intimated a distinction between ratio and rate, but he did not elaborate on it (Piaget, 1970, p. 280).

The emergence of conceptions of speed investigated by Piaget will inform our discussions of ratio and rate, but only to a point. Conceptions of ratio and rate were not issues to Piaget, so we must not lean too heavily on his analyses. Moreover, in his investigation of speed, as in most investigations he directed, Piaget was not interested in children's operations as used by them in the solution of specific problems, which is of considerable interest to researchers today. Rather, he was interested in the broad characteristics of mental operations held by the "epistemic subject"—a knower in general. He had no great interest in the "psychological subject"—an individual knower (Piaget, 1971a). Today we realize the importance of

sample.

understanding individual children in order to understand children in general. Also, we know now that we can see wider variations in operativity of a concept when it is required for comprehension of a situation that is more complex than ones dominated by the concept itself. That is, Piaget's investigation may not have captured concepts of speed richly enough to suggest how students must understand speed so that they may understand, for example, the concept of average speed.

A TEACHING EXPERIMENT

As a preliminary step to investigating students construction of the quantities speed, average speed, and the construction of rates in general, I simply spoke with children and adults about what they meant when, in specific situations, they concluded that something had an average speed of so much. The explanations of average speed given by adults, and given by school students who could solve the problems posed, were quite consistent: An average speed is the constant speed at which a different trip would be taken so that it would entail traveling the same distance in the same amount of time as the original trip. The explanations centered around the same person repeating the trip at a constant speed, or centered around a different person following the same path as the original. In either case, there always was a second trip involved.

I wrote a computer microworld (described later) that captured the sense of two trips happening sequentially or simultaneously, and which presented components of speed: distance traveled, time spent traveling, and rate of travel. This microworld then served as a metaphor within which instruction on speed took place. The microworld-as-metaphor was later used canonically in that instruction on rates *per se* always referred back to the metaphor established by the microworld.

The logic of a teaching experiment is to use instruction as the primary site for probing students' comprehensions, and as the primary site for gaining insights into their constructions (Cobb & Steffe, 1983; Hunting, 1983; Steffe, in press; Thompson, 1982). Instruction at the outset of a teaching experiment is highly bounded by conceptual analyses done prior to instruction. However, instruction is modified continually according to limitations in students' thinking as evidenced in their struggles and according to avenues that arise which promise greater insight into powerful ways of thinking. The hallmark of a teaching experiment is that it is opportunistic; one must continually rely on serendipity.

Subject

One 10 year-old fifth-grader, JJ, took part in the teaching experiment. JJ was a quiet, reflective student; she was not especially strong mathematically according to conventional indicators—Iowa Test of Basic Skills percentile scores for JJ were Concepts-84, Problem Solving-87 , and Computation-83. JJ participated in two earlier teaching experiments, one on complexity and additive structures (Thompson, in press), the other on area and volume.

Computer Microworld

The computer microworld (called OVER & BACK) presents two animals, a turtle and a rabbit, who run along a number line (Figure 2). Both can be assigned speeds at which to run. The turtle's speed can be assigned two values: one for the turtle's speed while running "over", the other for the speed at which it will run "back." The rabbit's speed can be assigned only one value, which applies to both its run over and its run back. Each animal can be made to run separately from the other, or they can be made to run simultaneously (as in a race). A timer shows elapsed time as either of the animals runs. One can press the "Pause" button to interrupt a race; when "paused," the distances travelled by either or both animals is displayed on the screen (Figure 3). Any assigned value can be changed during a pause; the animals will renew their race with speeds having the re-assigned values.

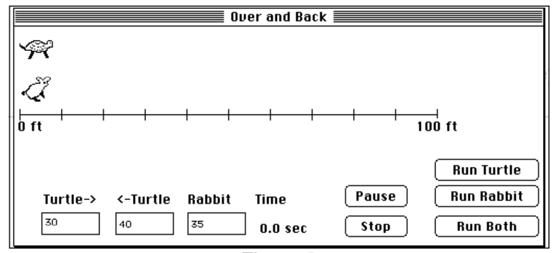

Figure 2

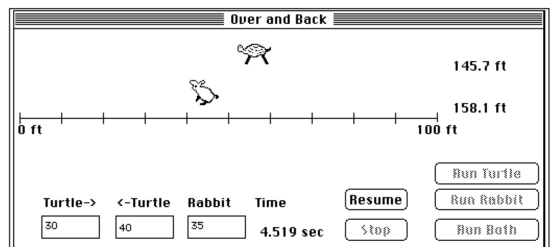

Figure 3

Overview of Instruction

Instruction occurred over eight sessions, each lasting approximately 55 minutes, between the dates of April 25 and May 14, 1990. The instructional format was that of a prolonged clinical interview. My remarks were given usually as questions asking for clarification or given for purposes of orienting JJ's attention. Direct instruction was given only to establish conventions, to explain how the program worked, and to present directions for the performance of tasks.

Tasks for a session were based on JJ's progress in that and previous sessions. Sometimes tasks were modified during a session when opportunities arose for generalization or clarification. JJ had a computer at home on which to run OVER & BACK , so homework often involved her use of the program.

Development of Concepts of Speed, Average Speed, and Rate

My discussion of the teaching experiment will be given in three parts, which correspond to a division of the teaching experiment's activities from my perspective. I saw the teaching experiment pass through three phases. Phase I focused on probing and extending JJ's concept of speed. Phase II focused on extending JJ's concept of speed so as to include what we normally take as an understanding of average speed. Phase III focused on giving JJ the opportunity to extend her concept of speed to a more general concept of rate by the process of generalizing assimilations and subsequent reflection.

Concept of Speed

I began the first session by demonstrating OVER & BACK to JJ and by trying to determine her understanding of speed. She explained that 40 ft/sec means that "every second he runs 40 feet." JJ's explanation suggested a good comprehension of speed; it soon became evident that I had given it greater significance than was appropriate.

My intention was to get as quickly as possible to problems involving average speed. With this in mind, I asked JJ to determine in advance how much time it would take for the rabbit to go over and back at speeds of 50 ft/sec and 40 ft/sec, and how long it would take the turtle to go over and back when it would go over at 20 ft/sec and back at 40 ft/sec. She answered each question successfully. I then asked

JJ to give the rabbit a speed so that it and the turtle would tie, where the turtle went over at 20 ft/sec and back at 40 ft/sec. JJ said, "30 ft/sec for the rabbit. [Why?] Because its 20 and 40 and in between is 30." It was at this moment that the teaching experiment began in earnest.

JJ's method for determining an amount of time needed to go a given distance at a given speed emanated from a primary notion of speed as one speed-length in one unit of time (e.g., one length of 40 feet in one second). She conceived the total distance as being measured in units of speed-length. JJ could even reason proportionally in making a determination of time: the rabbit would go 100 feet at 30 ft/sec in $3\frac{1}{3}$ seconds, "because there are three 30's in 100, and 10 is $\frac{1}{3}$ of 30, so $3\frac{1}{3}$ seconds" (Session 1). However, in JJ's conception, an amount of time was *made* by traveling, and its determination was *produced* by comparing a total distance with a speed-length. In other words, for JJ it was the case that *speed was a distance* (how far in one second) and *time was a ratio* (how many speed-lengths in some distance).

JJ's early conception of speed and time constrained the ways in which she could constitute speed, distance, and time in relation to one another. On four occasions in Session 1, when asked to give the rabbit a speed that would make it and the turtle tie, JJ resorted to a guess-and-check strategy of "pick a speed, figure a time, adjust up or down" even when she knew that the rabbit and turtle needed to use the same amount of time, and she knew the amount of time in which the rabbit needed to complete its trip. JJ's actions were strongly suggestive of trying to produce a desired measure of a given length by adjusting and re-adjusting the length of the unit according to by how much she missed the desired measure.

The notion that speed is a distance was predominant in JJ's thinking. For example, she claimed that the turtle would always be 10 feet behind the rabbit when the turtle moved at 20 ft/sec and the rabbit moved at 30 ft/sec. A change in her conception began to appear after she played with the "Pause" button (see Figure 3) while running the turtle and rabbit. She eventually reconciled her prediction and her observation that the turtle was rarely 10 feet behind the rabbit by re-constituting the comparison as "the distance [between them] will increase by 10 [each second] as they are going." JJ's conception of *distance* was changing, but not her conception of speed. Distance could accumulate, but the accumulation was still measured in units of speed-length.

JJ's homework after Session 1 had two parts. In the first part she was to formulate predictions for how much time the turtle or rabbit would take to go over and back at various speeds, and then test her predictions using OVER & BACK. The second part was to formulate predictions of who would win when the two raced with various speeds, again testing her predictions using OVER & BACK. JJ formulated her time predictions with ease, using her measuring technique. On one task, which asked JJ to formulate a time prediction for when the turtle was to go over at 20 ft/sec and back at 50 ft/sec, JJ understood the task as "make them tie." She said that she tried to "figure it out, but got confused," and resorted to a guess-and-test method. I took this as our starting point for Session 2.

In Session 2 I attempted to orient JJ's reasoning toward re-thinking her conception of time. She had previously conceived time-of-travel as being produced by moving in increments of speed-length, so I designed situations which, from my perspective, she might assimilate by that scheme, but in which she needed to work from the basis of a given time. The situations all presented a fixed distance (200 feet total) and times that were integral divisors of 200. JJ seemed to assimilate them as I predicted, looking for rate-lengths that when multiplied by the given number of seconds would produce 200 feet, or when multiplied by half the given time would produce 100 feet. My intention was that JJ eventually form an image of two segments, one for distance and one for time, where the segments were partitioned proportionally according to units of time.

Late in Session 2 JJ began to envision a segmented, total amount of time in relation to a to-be-traversed distance, and to envision that a segmentation of time imposed a segmentation on distance (Excerpt 1, ¶'s 1-5). That is, JJ began to conceive of distance and time as measured attributes of completed motion, but she was still more aware of distance than of time.

Excerpt 1 (Session 2)

1. PT: Okay, how about if we give a little tricky one. Make it [rabbit] go over and back in 6 seconds. *JJ sits up and looks at screen. Pause.* What are you saying to yourself?
2. JJ: Well, if it were 6 seconds there and back, it would have to go 3 seconds there and 3 seconds back. And, well, I was trying to figure out…*pause.*
3. PT: Excuse me just a second. *Gets up and shuts the door.* It's too noisy out there. Okay. You were saying, 3 seconds over and 3 seconds back?
4. JJ: Yeah, and, well, you could, if you could figure out how many times 3 went into 100, that would, well…
5. PT: Okay, if we figured out how many times 3 would go into 100?
6. JJ: Well…*pause.*
7. PT: … You know what you want to do?
8. JJ: Yeah.
9. PT: So what is it we need to do to see how many times 3 goes into 100.
10. JJ: Divide 100 by 3. (*Long* pause.)
11. PT: What is it that you've got in your hand?

12. JJ: *Has calculator in hand. Looks at it and laughs. Calculates 100÷3.* 33.3.

JJ's comment ("How many times 3 goes into 100", Excerpt 1, ¶ 4) was ambiguous. It is not evident whether she was thinking of "how many 3's in 100" or of cutting 100 feet into 3 pieces. This was clarified in Session 3 (she meant the latter). Also, JJ's application of division in the previous excerpt was not generalizable to non-integral values of time. Excerpt 2 presents the discussion following immediately after that in Excerpt 1.

Excerpt 2 (Session 2)

1. PT: (PT and JJ have been discussing what all the 3's after a decimal place might stand for, and whether JJ is confident of her answer. She is, "mostly.") Mostly. Well let's give it a shot. *PT enters 33.333 in the "Rabbit's Speed" box.* 6 seconds on the nose. You're right! How about making it go over and back in 7 seconds? *Pause.* What do you need to do?
2. JJ: Well, it has to go across one way in 3 and one-half seconds. And if it went 30 feet per second it would go 3 and one-third of a second. So, it would have to be more than 30.
3. PT: Well, last time you didn't do that. You didn't … estimate "has to be more than 30 or less than 50," or something like that. Last time you said you needed to know how many times 3 goes into 100. Why is this different, or is it different?
4. JJ: It…well…*pause.* Um…
5. PT: Is this different from the last time?
6. JJ: Not really. It's just that I'm not, I'm not dividing it. But I'm sort of guessing it and then checking it.
7. PT: Um, would it make sense to ask the same kind of question you asked last time? *Pause.* It's going to go from the beginning to the end in 3 and one-half seconds, right?
8. JJ: Uh-huh.
9. PT: And so what question would you ask if you were to ask the same kind of question as you did last time?
10. JJ: How many times does 3 and one-half go into 100, or how many times does 7 go into 200? *JJ has a quizzical look on her face. Bell rings, ending Session 2.*

JJ had formed a strong association between the operation of cutting-up a segment and the numerical operation of division, which would explain her remark that the situations involving 3 seconds and $3\frac{1}{2}$ seconds were not really different, but in the latter case she was not dividing (Excerpt 2, ¶'s 5-6). I suspected then, and still do, that JJ's discomfort in the latter case about using division to evaluate speed stemmed from this association. My hunch was that her discomfort would cease if she came to understand the operation of "cutting up" as one of making a proportional correspondence between two quantities. With this image of "cutting up," division is no less appropriate when cutting up segments into $3\frac{1}{2}$ parts than when cutting segments into 3 parts.

JJ's homework for that day was to determine speeds to give the rabbit so that it would make a complete trip in given amounts of time (5, 10, 8, 6, 7, 6.5, 7.5, and 8.3 seconds). During the next session JJ said that, in doing her homework, she figured some out with her calculator and did some by "guess and check," using OVER & BACK. She used "guess and check" for 6.5 seconds and 7.5 seconds, but actually calculated a speed for 8.3 seconds. Excerpt 3 presents JJ's justification for over and back in 6 seconds. In it I raised the question of what she was finding when she divided 100 by 3.

Excerpt 3 (Session 3)

1. PT: In 8 seconds, 25; 6 seconds, 33.3. Can you tell me how you got 33.3? You don't have to tell me blow by blow details.
2. JJ: Well, I… *pause*
3. PT: What was your method?
4. JJ: Well, I, I, um, I, well, half of 6 is 3. And so I, um, took 3 into 100.
5. PT: Okay, and what were you finding out when you divided the 100 by 3.
6. JJ: Um, how many, how many, well, how many threes were in 100.
7. PT: How many threes were in 100? Is that because the rabbit is jumping in threes? *Points to the number line on screen.*
8. JJ: Um, well…*pause*…the…okay, um, well…if the rabbit has to go over and back in 6 seconds and so, and that would be 200, he has to go 200 feet in 6 seconds. So, if I made it so he can go 100 feet in 3 seconds. And I divided 100 by 3…to figure out…how many, well I can't remember what I did.
9. PT: Can you, can you sort of explain, perhaps using … pointing to the number line, what it was that you were finding?
10. JJ: Well…
11. PT: Would that help?
12. JJ: Okay, I wanted…well to see if you could divide this, *pointing to number line*, into three different parts that are equal.
13. PT: Uh, I see. Now I get it, okay. So you're trying to divide this into three parts that are the same size, *points to number line indicating three equal sections*, that would mean that each part he goes … takes how long to go in each part?

14. JJ: 3.
15. PT: 3 seconds for each part? *Pointing to three sections on the number line.*
16. JJ: Oh, 1 second for each part.

Excerpt 3 shows that distance still predominated in JJ's reasoning, but that she now took both the total distance *and total time* as given, and that she constituted completed motion as segmented distance in relation to a segmentation of time (3 units of time, so 3 segments of distance). This is further suggested in Excerpt 4, where, for the first time, JJ argued for the "sensibleness" of segmenting distance so that it (the segmentation) corresponded to a non-integral value for amount of time travelled.

Excerpt 4 (Session 3)

JJ and PT have discussed how to determine half of 8.3 seconds. JJ figured it would be "4.1 and one-half" and guessed that she should use "4.15" with her calculator.

1. PT: Alright. Now, if you didn't already have this written down on the paper, then how would you figure out how fast it needs to go to go from 0 to 100 in 4.15 seconds? *Points to 0 feet and 100 feet on the number line.*
2. JJ: Well, you could divide 100 by 4.15.
3. PT: Now, I followed you when you said you would divide 100 by 3 because you're thinking of the rabbit going in 3 … 3 pieces, one second for each piece, but what do you have in mind when you say you would divide by 4.15?
4. JJ: Well, you want to figure out how many feet the rabbit can go in, per second, and…and it would have, it would have to be equal up to 4.15 seconds when you get to the end of the 100 feet.

I interpreted JJ's remark, "it would have to be equal up to 4.15 seconds when you get to the end of 100 feet," as an expression of an image that time accumulated (in seconds) as the rabbit moved. This, together with her frequently-expressed conception that distance accumulated as the rabbit moved, suggested that she had come to conceive of completed motion as being constituted by a segmented, total distance in relation to a segmented, total amount of time. As such, it was important to see if JJ was actually thinking primarily in terms of segments of time, so I focused upon the significance she gave to 0.15 seconds in relation to distance travelled by the rabbit.

Excerpt 5 (Session 3)

JJ has evaluated 100 ÷ 4.15 on her calculator.

1. PT: Okay, 24.096385. … What, what did that number stand for?
2. JJ: How many, how many feet the rabbit would go per second.
3. PT: That's how far he's going to go each second. Okay? So now can you tell me precisely how far he will go in 4 seconds?
4. JJ: 96 feet.
5. PT: 96 feet? Exactly?
6. JJ: Um.
7. PT: You nodded your head yes. Is he going exactly 24 feet per second?
8. JJ: No.
9. PT: So then how would you tell me exactly where he's going to be?
10. JJ: Um, you would multiply this number by 4.
11. PT: Okay, why don't you do that.
12. JJ: *Uses calculator.* Ninety-six point three eight five five four (96.38554).
13. PT: And that number is a number of what?
14. JJ: Of how far he will be after 4 seconds.
15. PT: In miles, inches?
16. JJ: Feet.
17. *PT runs rabbit, pausing it at 4.1 seconds. Rabbit is at 98.8 feet.*
18. PT: Oops, missed it. At the end of 4 seconds it will be near the end. What about that little bit left over, that extra 4 feet … a little less than 4 feet? How long will it take him to go the rest of the way?
19. JJ: Point one five (.15).

After the discussion presented in Excerpt 5, I asked JJ to give the rabbit a speed to tie the turtle when the turtle goes over at 20 ft/sec and returns at 40 ft/sec. JJ calculated the total time for the turtle (7.5 sec), and then divided 200 by 7.5. This was the first occasion where JJ understood that it was the turtle's time that would determine the rabbit's speed if they were to tie. JJ had evidently abstracted time from her intuitions of the co-variation of distance and time. Amount of time was no longer tantamount to a number of speed-lengths.

To further verify that JJ had re-constituted speed by way of quantitative operations, rather than merely having abstracted a pattern in the numerical operations she used to answer questions, I asked JJ to consider a major variation in the kinds of problems she had been solving: The turtle goes over at some speed, comes back at 70 ft/sec, and the rabbit goes over and back at 30 ft/sec. Give the turtle an “over-speed” that will make it and the rabbit tie (Excerpt 6).

Excerpt 6 (Session 3)

1. PT: This one is a little bit different. It says the turtle is supposed to go over at some speed that you find out, come back at 70 feet per second, and the rabbit is going to go at 30 feet per second. The distance is 100 feet, and they’re supposed to tie, but we don’t know the turtle’s over speed. So how, how could you figure that out?
2. JJ: Well, first you could divide 30 into 200 to figure out how many seconds it would take for the rabbit to go…
3. PT: Alright.
4. JJ: …there and back.
5. PT: Okay.
6. JJ: And then, um, you would divide 70 into 100 to see how many seconds or how many times it would take, how many times the turtle could go in, how many seconds the turtle could travel per feet at 70 feet per second.
7. PT: Alright.
8. JJ: So you would divide…so you would divide 70 into 100. And then you would subtract, could subtract that number by the, how many seconds the rabbit took to go there and back, and then figure out how many seconds the turtle would have to go, and how many feet the turtle would have to go.
9. PT: How many feet?
10. JJ: Or, how many, well, how many feet per second.

The significance of JJ’s reasoning as presented in Excerpt 6 is that she presented a plan for finding the values of various quantities. Every calculation identified by JJ was selected in order to evaluate a quantity (e.g., divide 70 into 100 *to see how many seconds it would take*; subtract turtle’s “back time” from rabbit’s total time *to see how many seconds it would take for the turtle to go over only*). I take JJ’s solution to this problem as the first solid indication that she had constituted “turtle and rabbit” situations through a scheme of mental operations, for it is only through mental operations that she could have supplied quantities which were unmentioned in my presentation but which were related structurally to those that were mentioned. JJ evidently had an operative image of the situation. That is, her quantitative operations were part of her image of the situation, which means that she constituted the situation with them. This gave her both something to reason *about* and something to reason *with* in determining appropriate numerical operations for evaluating the various quantities that were present in her image.

JJ's homework after Session 3 was to solve more "turtle and rabbit tieing" problems. In the next session, to further test the conclusion that JJ was using arithmetical operations representationally rather than constitutionally, I made up a situation similar to that presented in Excerpt 6. Excerpt 7 presents the first part of the discussion.

Excerpt 7 (Session 4)

Text of problem reads, "Tomorrow you are going to be given the turtle's "over" and "back" speeds and the length of the race track. Tell what arithmetic you would do to determine a speed to give the rabbit so that it and the turtle tie."

1. PT: Okay, so let me write "Turtle over" for the number that the turtle goes over. "Turtle back" for the number the turtle goes back, okay? *Writes "TO" for turtle over and "TB" for turtle back.* (See Figure 4).

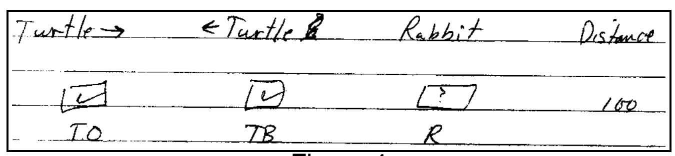

Figure 4

2. JJ: And…
3. PT: We'll call them "TO" and "TB."
4. JJ: Okay. Well…okay, you divide the number that the turtle goes over…
5. PT: Okay.
6. JJ: …in, into 100.
7. PT: And that number stands for what? "TO" stands for what?
8. JJ: How many feet per second he went over.
9. PT: Okay. So 100, and then you divide that by the speed that the turtle goes over by.
10. JJ: And, um, and then you, and you should get how many times, you should get how many times that goes into 100 and so you could change that into seconds.
11. PT: And what is, what does this stand for?
12. JJ: Uh, that stands for how many…
13. PT: So, I'll write that down, how many… *Writes "How many."*
14. JJ: Um, how many feet, well, how many feet per second the turtle went over in 100 feet.
15. PT: Okay, what do you get when you do that division?
16. JJ: A number.
17. PT: And what does that number stand for?
18. JJ: It stands for how many times that number goes into 100.
19. PT: Alright.
20. JJ: But then, you can … you can convert it into seconds.
21. PT: Convert it into seconds? How do you convert it into seconds?
22. JJ: Well change it to seconds.

23. PT: You just say it's the number of seconds?
24. JJ: Yeah. Well, like if you take 25 into 100, it goes 4 times, so you can say that's 4 seconds.
25. PT: Oh, I see. Okay. Alright. So that's how many, how many of those numbers are in 100 and change that to a number of seconds. *Writes an expression.* I'm going to write division like this…like it's a fraction. *Records what JJ said. See Figure 5.* Is that right?
26. JJ: Um-hum (yes).
27. PT: Okay.
28. JJ: And then, you do the same thing for the turtle to go back.
29. PT: Well, what, what do you mean the same thing?
30. JJ: Well…
31. PT: Divide 100 by "TO"?
32. JJ: No.

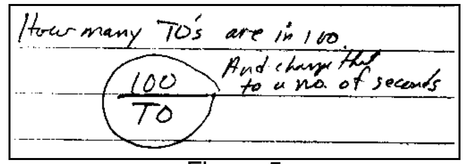

Figure 5

33. PT: By the turtle's over speed?
34. JJ: The turtle back speed.
35. PT: Oh, okay. Divide 100 by the turtle back speed and what does that stand for?
36. JJ: How many, um, of the thing, how many whatever, well how many of the numbers you have in…
37. PT: Okay, so that's…
38. JJ: How many…
39. PT: What number?
40. JJ: How many "TB's" you have in 100.
41. PT: So how many "TB's" there are in 100. *See Figure 6.* Okay.
42. JJ: And then you change that to seconds, too.
43. PT: Change that to seconds. *See Figure 6.* Alright.
44. JJ: And then, you add those two things of seconds, those two. *Pointing to the paper.*

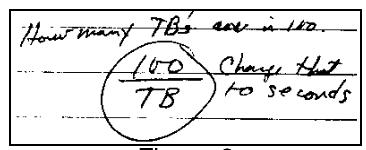

Figure 6

45. *PT and JJ discuss how she might write the addition of the two seconds. JJ writes the two "seconds" expressions in vertical format. She labels the expression with "The number you get will be how many seconds the turtle travels over and back."*
46. PT: Okay. So, now you've got the number of seconds the turtle travels over and back, and … what was it that we were trying to do? I forgot.
47. JJ: To figure out what the number is of the rabbit going over and back.
48. PT: To do what?
49. JJ: So they tie.
50. PT: … Now, what would you do to find out the rabbit's speed over and back so they tie?
51. JJ: Uh … you would divide the number you get into 100, or … well … you could divide that into … well … you could either divide that into 200 with that number you get, or divide that by 2 and then divide it into 100.

Two things are significant about the conversation in Excerpt 7. First, JJ did not need actual values in order to reason about what arithmetic was appropriate. She inferred appropriate arithmetic according to what quantities she wished to evaluate and according to relationships of those quantities to others, and did all of this with indeterminate values.

Second, JJ referred to the result of dividing the value of distance travelled by the value of speed as something that needed to be *converted into* a number of seconds (Excerpt 9, ¶'s 15-24). That JJ needed to "convert it into seconds" provides insight into her emerging comprehension of speed, distance, and time taken together as a quantitative structure. In Session 1 JJ needed to actually imagine measuring a given distance with a given speed-length to determine an amount of time. In Excerpt 7 we see that she had interiorized this notion to a level of thought, so that she understood that *any* rate-length would determine a *partition* of the total distance into units of rate-length and parts thereof, and that this partition would correspond to a partition of the total time into units of seconds and proportional parts thereof (Figure 7).

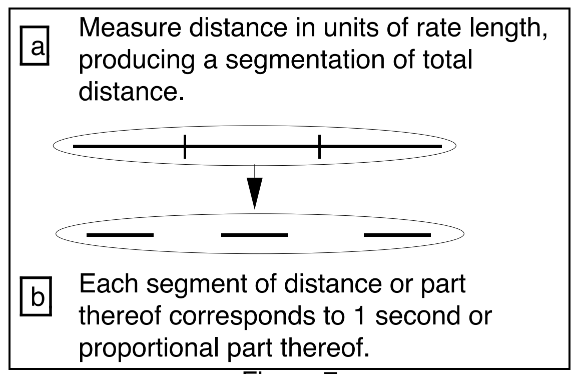

Figure 7

I asked one further question, following the discussion in Excerpt 9, as a final assessment of the degree to which JJ had established speed as a proportional correspondence between between distance and time: What would happen if you changed the distance from 100 to 200 feet and kept the turtle's speeds the same? JJ responded that "It would be pretty much the same … the time would just double." I then asked JJ to suppose she wanted to keep the time the same, to which she responded "You would make it so it would be more feet per second. [How much more?] Doubled." JJ's explanation was very imagistic: "If you just divide it in half (pointing to the OVER & BACK screen) it would be just like it used to be." The evident obviousness to JJ that the time would double if the speeds were to remain the same, and that all the speeds would have to double if the time were to remain the same, suggests strongly that she had interiorized "turtle and rabbit" situations as a quantitative structure—that she had developed an image of turtle and rabbit situations that she constituted by way of quantitative operations.

Average Speed

To further investigate the generality of JJ's mental operations I introduced "average speed," characterizing it as one person making a trip at (possibly) varying speeds, a second person travelling the same itinerary at a constant speed, and they tie. The second person's speed is the first person's average speed. (I also recast it as the first person repeating his trip at a constant speed so that he takes the same amount of time.) My hypothesis was that, if JJ had actually constructed a speed as a scheme of operations, then the idea of average speed should be easily generalized to other situations of average rate (e.g., average price).[11]

Excerpt 8 (Session 5)

1. JJ: *(Reads text.)* John traveled 35 miles per hour for 100 miles and 45 miles per hour for 50 miles, what was John's average speed on this trip?
2. PT: Okay, now don't, don't think that, uh, [I'm asking you to give] an answer right now. Tell me what is…tell me what this told you. Explain to me what that told you.
3. JJ: That he traveled 35 miles per hour for 100 miles and then 45 miles per hour for 50 miles.
4. PT: Now, how is that like the turtle and the rabbit?
5. JJ: Um…well…um, he traveled a certain amount of miles per hour for a certain amount of miles and that traveled a certain amount of feet per second for a certain amount of feet.
6. PT: Okay. So it's, it's in that way, it's not very different. Is there anything that, um, that is different, that's about the situation described here and the turtle and the rabbit?

[11] It turned out that JJ had already studied "average" as arithmetical mean, which caused much confusion at first, but we eventually straightened things out by agreeing that anytime I wanted her to think of "add up and divide" I would refer to it by the name of her teacher—"Mrs. T's average."

7. JJ: Um, well, the first time he went for 100 miles and the second he went for 50.
8. PT: Okay. Now is that, is that like the, the turtle and the rabbit or different from the turtle and the rabbit?
9. JJ: Well the turtle and the rabbit go, um, 100 feet both ways, or 200.
10. PT: Okay, and, and this doesn't go 100 both ways. *(Referring to "mileage" problem.)*
11. JJ: Huh-uh. *(No.)*
12. PT: Alright. What would you need to know, oh…I'm sorry, I'm getting ahead of myself. In this question, *(pointing to "mileage" problem)*, it asks what was John's average speed on this trip. What does that mean?
13. JJ: Um…well…um…*pause*. It means that…you want to figure out his average speed. *(Giggles.)*
14. PT: Alright…I'm asking now what does average speed mean in this context?
15. JJ: How…for going one…how many miles per hour for…and just for 150 miles but just…and stay that same speed.
16. PT: Okay. And what's special about that speed?
17. JJ: It should come out the same as this. *(Pointing to paper in front of her.)*
18. PT: The same in what way?
19. JJ: Um, it should go the same amount of time or the same amount of miles.
20. PT: Just one of them or both of them?
21. JJ: Both.

JJ went on to calculate John's average speed, which was not surprising. However, a better question for assessing how well she had interiorized speed as a quantitative structure would have been, "John traveled at an *average speed* of 35 miles per hour for 100 miles and at an *average speed* of 45 miles per hour for 50 miles." Had JJ responded as straightforwardly to that question as she did to the one posed it would have been clear that she was constituting the situation by way of a scheme of mental operations, for it is only through mental operations that she could have supplied quantities which were unmentioned in the text but which were related structurally to those mentioned.

After the discussion in Excerpt 8 I asked JJ to think about a situation that was quite different from any she had seen.

Excerpt 9 (Session 5)

1. JJ: *(JJ reads text.)* Sue paid nine dollars and forty-six cents for Yummy candy bars at forty-three cents per bar and paid six dollars and eight cents for Zingy candy bars at thirty-eight cents per bar. What was her average cost of a candy bar?
2. PT: Now before even talking about how to answer the question, explain…can you tell how this

one, *(points to "candy bar" text)* is like the one you just did about John traveling?

3. JJ: Well, you want to figure out the average of the cost … of a candy bar and …
4. PT: Aside from the, okay, leave the question out, how is, how is this situation, *(points to "candy bar" text)*, similar to the situation of John traveling at different speeds?
5. JJ: Because, um, it's, it's like, you could change that…it's…okay, it's, it's exactly just the same but with a different…it's with candy bars and it's, it's like forty-three cents per bar is like 35 miles per hour, and the nine dollars and forty-six cents for a Yummy candy bars and, uh, for 100 miles … So it's…and then that would be the same thing down, for this. *(Points to "candy bar" text.)*
6. PT: This…this meaning the Zingy candy bars? So how…how then would you, uh, find the average cost of a candy bar?
7. JJ: Um, you would divide forty-three cents into nine dollars and forty-six cents.
8. PT: And what would you find when you did that?
9. JJ: The…how many candy bars she got. And then you would do the same for these *(Zingy)*.
10. PT: And how is that similar to what you did up here with John?
11. JJ: Because um …
12. PT: "That" meaning…being, um, finding the number of candy bars?
13. JJ: Because you want to find out how many hours it went *(pointing to "mileage" text)* and this, you want to figure out how many candy bars you could get. *(Pointing to "candy bar" text.)*

I take JJ's analysis of similarity between the two problems (average speed and average cost) as evidence that she assimilated both to a scheme of mental operations. One other possibility is that JJ's explanation of the similarity between the two could have been based on a textual correspondence of parts of one text and parts of the other (matching "per" phrases, for example). This turned out not to be the case, as a follow-up comparison between two problems had non-corresponding quantities mentioned explicitly, and yet JJ said they were "pretty much the same," and explained why.

Concept of Rate

To investigate further the nature of JJ's scheme, I presented her with additional correspondence tasks. The discussion is presented in Excerpt 10.

Excerpt 10 (Session 8)[12]

1. JJ: Carol has a swimming pool that holds forty-two thousands five hundred gallons of water. She fills it with two pumps, one pump put into the … one pump put water into the … water into the pool at forty-five gallons per minutes, the other pump put water into the pool at seventy gallons per minute. How long will it take Carol to fill the swimming pool while using both pumps at the same time if it is now empty?
2. PT: Okay, and how is that like the rabbit and the turtle? The rabbit or the turtle?
3. JJ: Well, that's like the turtle would go across at forty-five … um, feet per second or whatever it is and then come back at seventy feet per second or minute, and it's like how long will it takes the turtle to … *Pause*.
4. PT: Okay, so the rabbit doesn't even enter into to it. … Could you write the arithmetic that you would do to answer that question. In another words, just write the division sentence, don't … don't actually do the division.
5. JJ: *Sighs. Long pause.*
6. PT: Did you run into a problem?
7. JJ: Um … Yes.
8. PT: Okay, what's that?
9. JJ: Well, I'm trying to figure out … *Pause*. … Well, … *Pause.*
10. PT: Do you still think this is like the turtle going over and back?
11. JJ: *Pause.* It's like the turtle going across at forty-five feet per second or whatever it is, and the rabbit going at seventy feet per second, that's it. *Pause.*
12. PT: How was the forty-three hundred thousand gallons like the distance?
13. JJ: It's like how … it's like how long … what the distance is between two points, but that's how much *(long pause)* the turtle and rabbit go together?

JJ evidently internalized the text as being similar to turtle situations because of the presence of two rates and one amount. When she attempted to interiorize her internalized situation (i.e., assimilate her initial comprehension of the text to her scheme of operations) for the purpose of deciding upon appropriate arithmetic, it did not fit. Most importantly, JJ apparently could identify the reason why it did not fit—it would be as if the turtle and rabbit were running together and both their distances were being accumulated as one quantity. I asked JJ to change the "Carol's Pool" problem so that it would be exactly like the turtle and/or rabbit, but also so that it would still be about the pool.

[12] Sessions 6 and 7 were devoted to dealing with confusions JJ had about "average." Her sister decided to help JJ with her homework, reiterating to JJ that "average" means "add up and divide." It took two full sessions to deal with this confusion; those sessions could be the subject of another paper.

Excerpt 11 (Session 8)

1. PT: (Asks JJ to change "Carol's Pool" problem so that it is just like the turtle and the rabbit.)
2. JJ: That one pump would go on, for maybe a minute and then the next pump, and then other pump would, and then they would switch on and off.
3. PT: And then they would switch on and off. When we had the turtle going, it went over at one speed and back at another, right? You're right, we could have them switch on and off, and it would be more like the turtle. But the turtle didn't go seventy feet per second for a little bit, and then forty feet per second. It went over at one speed and back at the other.
4. JJ: Um. *Pause.* You could change it that, if it went for a minute for forty-five feet or whatever, and then another minute for seventy … feet.
5. PT: You said that the forty-two thousands five hundred gallons is like the distance, is that right? … Is that like the distance one way, or the distance over and back?
6. JJ: Over and back.
7. PT: Okay, when the turtle went over and back at two different speeds, how far did it go at one speed, and how did it go at the second speed.
8. JJ: A hundred feet.
9. PT: Okay, what part of the turtle trip was that?
10. JJ: Half.
11. PT: Half, because sometimes you'd have it go two hundred feet, didn't you? So you could have it go different distances, but it always went over at one speed and back at the others, and it always went half way at one speed, and half way at the other. Does that give you any ideas about now how to change this so it just like the turtle?
12. JJ: *Long pause.* You could … you could divide the, if you … *Pause.* If you divide that in half. *Points to text of problem.*
13. PT: That being the forty-two thousands and five hundred gallons?
14. JJ: So it would be like the distance from one place to another and then back.
15. PT: Alright.
16. JJ: But the half would be just one place to another and then the second half would be back. … And then the forty-five gallons would be … the speed … the speed going one way and that it would go forty-five or whatever per minute and another way back it would be the seventy gallons, so it'd be like seventy feet … feet back.

JJ saw that the turtle needed to go at one speed and then another, but evidently accommodated to that need by focusing on different pump-rates for different times, "switching on and off." When I oriented her attention to the particulars of the turtle's situation, she then created explicit relationships among the capacity of the pool, the distance travelled by the turtle, the flows of water from the pumps, and the speeds at which the turtle ran.

The next problem differed from the turtle's situation on yet another characteristic: One knows the pool capacity, the pumps' rates, and the amount of time required to fill the pool.

Excerpt 12 (Session 8)

1. JJ: Janna has a swimming pool that holds ninety-three thousand gallons of water. She began using one pump to fill her pool, but it broke down, and she had to replace it. Her first pump put water into the pool at seventy-three gallons per minute. The second pump put water into the pool at sixty-eight gallons per minute. It took one thousand three hundred and twenty-one minutes of pumping time to fill the pool. How long did each pump work?
2. PT: Okay, how is that one like the turtle and the rabbit? Is this situation like the turtle and the rabbit? Or just the turtle or just the rabbit?
3. JJ: It's like one of them … the turtle not the rabbit … just one.
4. PT: Is it … is it like or different from the story you just wrote? *(Referring to JJ's modification of "Carol's Pool".)*
5. JJ: It's like because … she's used two different pumps and … they were both different, they both put waters into the pool at different gallons, different …
6. PT: The word is rate. At different rates.[13]
7. JJ: At different rates per minute, but … in this one you want to figure out how long each pump works?
8. PT: Okay, now, so how is that unlike the rabbit? *[PT misspoke—he meant "turtle."]*
9. JJ: Well … *Long pause.*
10. PT: Can you tell, how it is unlike the story that you wrote? *Puts the two papers side-by-side.* Is there any real big difference?
11. JJ: No. *She studies the papers.* But, in this you know how long that one pump … You know, because in this one it was filled half full and then they switched. But in this one you don't know.
12. PT: You don't know how far up the pool was filled?
13. JJ: That's what you want to figure out.
14. PT: Oh, if you find out how much the pool was filled then you can answer this question?
15. JJ: Well … *long pause* … that would be the answer if you knew.
16. PT: So, if we knew that the pool was one-fourth filled when she switched, then that would be the answer?
17. JJ: That would be, umm, you could get the answer very easily from that because then you would just divide 73 into, well, how much … well you divide … then you could … find what's one-fourth of 93,000, and then divide that by 73.

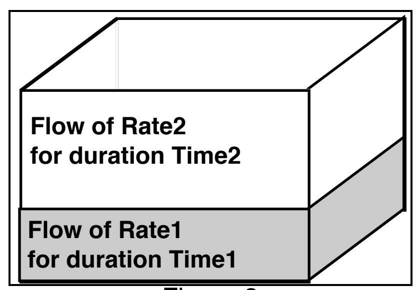

Figure 8

JJ's remarks in Excerpt 12 again provides evidence that she possessed a scheme of mental operations into which she assimilated the situation. Not only did JJ recognize the similarities and differences between this situation and those she encountered with the turtle and rabbit, she knew what she needed to know in order to proceed toward answering the question of for how long each pump worked. This

[13] It turned out that the word for which JJ searched was "amount," as in "different amounts per minute."

suggests that she came to conceive the filled pool as being composed of two parts: the part filled by Pump1 running at Rate1 for duration Time1, and the part filled by Pump2 running at Rate2 for duration Time2 (see Figure 8).

Excerpt 10 showed JJ internalizing a text inappropriately, and experiencing disequilibrium when attempting to assimilate the product of that internalization into her scheme of mental operations. In the next excerpt, JJ again internalizes a text inappropriately. I could have promoted disequilibrium by asking her to decide upon appropriate arithmetic operations. Instead, I took a different approach, relating this problem to the "Carol's Pool" problem so that we would have an occasion to discuss the combination of two rates.

Excerpt 13 (Session 8)

1. JJ: Jim has a swimming pool that holds 35,000 gallons of water. He has two pumps. One pump can fill his pool in 912 minutes. The other pump can fill his pool in 532 minutes. How long will it take Jim to fill the swimming pool using both pumps if it's now empty? *Long pause.* Well, you could do the same thing as you could do in the first one and you could divide that in half and … well, and then see how long it would take if it could fill his pool in 912 minutes and see how long it could fill half of his pool. And then you could switch and see how long you could fill half of his pool, and what, regularly, you could fill it in 532 minutes.
   PT turns page to "Carol's Pool" text: Carol has a swimming pool that holds forty-two thousands five hundred gallons of water. She fills it with two pumps. One pump put water into the pool at forty-five gallons per minutes. The other pump put water into the pool at seventy gallons per minute. How long will it take Carol to fill the swimming pool while using both pumps at the same time if it is now empty.
2. PT: Let's go back to this one. She's got both pumps running. Is that correct? *JJ nods*. Suppose that this was just about one pump filling the swimming pool, 42,500 gallons, one pump is running at 45 gallons per minute, could you answer the question of how long it would pump to fill the swimming pool?
3. JJ: Yeah.
4. PT: And how would you do that?
5. JJ: You would divide 45 into 42,500.
6. PT: Okay. Suppose that I had a pump that pumped water at 100 gallons per minute. Could you answer the question how long would it take to …
7. JJ: Yeah.
8. PT: Same way?
9. JJ: Uh huh.
10. PT: And what would that be?
11. JJ: You would divide 100 into 42,500.
12. PT: Okay. So if he's running both pumps at the same time, then that's like one great big pump isn't it?
13. JJ: So you could keep adding, you could add 45 and 70 and you would get a number and …

14. PT: And what does that number refer to?
15. JJ: How many gallons for, well, per minute.
16. PT: Okay.
17. JJ: And then you could keep on adding those numbers until you get to 42,500.
18. PT: Okay. Or, could you do it an efficient way?
19. JJ: You could multiply. You could guess a number and multiply it.
20. PT: Now, let's go back to what I was asking you before. Suppose I told you that I had one pump that pumped water at 115 gallons per minute. How long would it take that pump to fill the pool? How would you answer that question?
21. JJ: You would divide that into 42,500. So you could add those and divide that into 42,500!

Each of the problems previous to "Jim's Pool" gave prominence to rates of flow, and JJ at first interiorized the text according to that image (Excerpt 13, ¶ 1). Instead of attempting to promote a state of disequilibrium in JJ so that she would re-conceive the setting, I returned to the "Carol's Pool" problem (see Excerpt 10) to discuss the notion of a combination of flows (Excerpt 13, ¶'s 2-15).

JJ's remark, "And then you could keep on adding those numbers until you get to 42,500" (¶ 17) suggests two conclusions about her emerging comprehension of the "Carol's Pool" situation: (1) she constructed a total rate of flow as a combination of the two rates of flow mentioned in the text, and (2) that she came to conceive of a filled pool as being made by a (measured) flow of water happening over a (measured) duration of time. In regard to the latter, it is as if JJ thought, "Add 45 and 70, then iterate that sum until you fill the pool (get 42,500 gallons)." Her later remark that she could use the strategy of guess and check with multiplication (¶ 19) strengthens this interpretation.

JJ's approach here resembles her initial method of determining how many seconds it would take the rabbit to go a given distance at a given speed. She segmented distance in units of "speed-length," thereby constructing a corresponding duration (number of time units). In the present case, JJ evidently thought of segmenting the capacity of the pool by way of iterating the combined "flow amount," thereby constructing a duration over which the flow rate will fill the pool. It appears that with respect to combining two flow rates to produce a new flow rate, JJ recapitulated her construction of speed as a rate: segment the total amount of the extensive quantity being created in units of the "rate amount," thereby inducing a duration amount. In previous cases, the "rate amount" was a single quantity; in the present case, it was a *combination* of rates that she needed to construct as a rate. The fact that JJ evidently needed to recapitulate the construction of speed in the case of a combination of flow rates suggests that this was her general method of constructing rates of any kind. The fact that JJ so quickly constructed a combination of flow rates as a rate (relative to her construction of speed as a rate) suggests that recapitulation is a process by which reflective abstraction of the second kind happens—JJ became "skilled" at constructing rates, and the *process* of construction became suggestive of the kind of thing the process produces (Cellerier, 1972).

There remains the question of how structural was JJ's construction of combined flow rate. Was "combination of rates" a quantitative operation for JJ? Was the result of combining two flow rates of the same stature as an uncombined flow rate? That is, could she infer a combination in the context of complex relationships and then use it relationally as a single quantity? The latter portion of Excerpt 14 (below) suggests that, yes, combining flow rates was a quantitative operation for JJ; the result of combining was indeed a single quantity.

Excerpt 14 (Session 8)

PT turns page to "Jim's Pool" text: Jim has a swimming pool that holds 35,000 gallons of water. He has two pumps. One pump can help fill his pool in 912 minutes. The other pump can fill his pool in 532 minutes. How long will it take Jim to fill the swimming pool using both pumps if it's now empty?

1. JJ: Well…you could do the same and add those two together and divide that into 35,000.
2. PT: Okay. Now, how are these numbers different from the one's in the first problem (Carol's Pool)? *He puts the two papers side-by-side.* Look closer.
3. JJ: The 912 and the 532 is how long it would take to fill this pool, and this is how many gallons per minute it would take.
4. PT: Okay. Can you do the same thing here (Jim's Pool) that you did over there (Carol's Pool) and just add these two numbers? Well, I mean of course we can add these two numbers. You can always add numbers. If we do add these numbers, 912 and 532, then we would get 1444. Okay, we would get 1444 minutes. If we were to say that's how long it takes something to happen, what is it that would happen?
5. JJ: How long…it would take to fill…two pools.
6. PT: Very good. That's right. That's how long it would take to fill two pools. This one would fill it in 912 minutes and this one would fill a second one in 532 minutes. Okay. Now, can you do something with these numbers so that you have exactly that situation (points to "Carol's Pool" text)?
7. JJ: *Long pause.* You could change that into how many gallons per minute and change that into how many gallons per minute.
8. PT: And how would you do that?
9. JJ: You would…you could divide…if one pump fills a pool in 912 minutes, then you could divide that into 35,000 because that's how much the pool holds. And then see how many gallons per minute it , the pump, can fill this pool. And then, that would be, you could, that would be that number and you would do the same for 532.
10. PT: Alright, then you get the gallons per minute for each pump. Then what would you do?
11. JJ: *Long pause*. And then…you could…*Pause*…you could…add those two numbers so you get together…so then you'll get, so it will seem like one pump.

I dismiss JJ's inappropriate internalization of the text (¶ 1) as being largely unimportant.[14] The remainder of the excerpt contains three important episodes: The first is where JJ distinguished between the numbers in the two texts as being values of two different kinds of quantities (¶ 3). The second is where JJ understood that adding the two numbers of minutes would evaluate the quantity "time to fill two pools" (¶'s 4-5). The third is where JJ re-conceives the "Jim's Pool" situation as involving two flow rates each of which can be evaluated by division, and that those flow rates can be combined to produce a total flow rate (¶'s 7-11).

The third episode (Excerpt 14, ¶'s 7-11) is important for understanding the extent to which JJ had interiorized rate as constant ratio. Her comment, "You could change that into how many gallons per minute …" (¶ 7), her hesitation about how to make the change (¶ 9), and her comment that "… see how many gallons per minute the pump can fill the pool" (¶ 9) taken together suggests that she conceived of the situation in terms of rates. Her conception evidently involved an image of the pool being filled, as opposed to an image of breaking up of a filled pool into 912 (or 532) portions, one for each minute. Had I been a better interviewer, I would have probed this further by asking JJ if she thought first of the pool being filled or first of the pool as already filled. Also, at the end of Excerpt 14, JJ again expressed a conception that the result of combining the two flow rates produces one flow rate: "…so it will seem like one pump" (¶ 16).

DISCUSSION

My discussion of issues surrounding JJ's emerging conceptions of speed and rate is in two parts. My first remarks have to do specifically with JJ and with questions of conception. Second are my remarks relating events of this teaching experiment to issues that are more broadly associated with the field of multiplicative structures and mathematical development in general.

Construction of Speed as a Rate

The only situations JJ could fully conceive early on had to do with determining

[14] I suspect these inappropriate internalizations were more a function of schooling than of conception. Students (including JJ, according to her) come to expect new problems to be just like the ones they have recently solved.

the time required to go a given distance at a given speed. She could not conceive situations having to do with determining a speed at which to go a given distance in a given amount of time—unless the amount of time corresponded quite specifically with a whole-number segmentation of the distance. This had nothing to do with a preferred strategy; it was not the case that JJ preferred to reason "within" one measure space (specifically distance) in order to draw conclusions about another (specifically time). Rather, JJ was constrained to reason primarily about distance because time was not a quantity of the same stature as distance; time of travel, for JJ, was implicit in how many "rate-lengths" (or parts thereof) were required to make a distance (Figure 9). With this interpretation of JJ's conception of speed we are able to explain her early inability to fully conceive situations in which she was required to determine a speed with which the rabbit would go a given distance in a given amount of time (Figure 10).

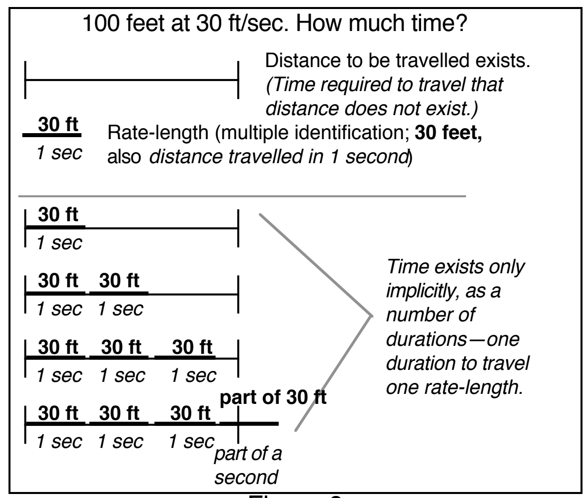

Figure 9

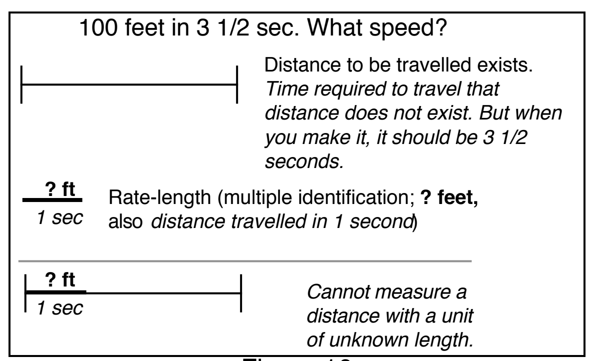

Figure 10

It was only later (Session 3 on) that JJ began to conceive of turtle and rabbit situations as involving co-varying *accumulations* of quantities. Evidently, in working on the homework assignment between Session 2 and Session 3, JJ abstracted time from her intuitions of speed so that it was a quantity in and of itself—one that could be measured in correspondence to distance, and one that varied in constant proportion to distance (Figure 11). This new conception of speed not only allowed JJ to answer questions that before were inaccessible to her, it also provided her with occasions to create situationally-bounded mental structures that she would eventually interiorize as a constant ratio structure. JJ's abstraction of time from her intuitions of speed enabled JJ to conceive of distance and time in relation to one another, which in turn enabled her to conceive of speed as the result of such a comparison.[15]

I would like to make an observation here lest it be missed. JJ's initial scheme for speed did not involve conceiving quantified motion as a quotient—it was not a ratio between distance and time. Rather, her initial conception of speed was that it was a *distance*, and her initial conception of time was that it was a *ratio*. JJ first had to construct speed as a ratio (i.e., as the result of a multiplicative comparison between distance travelled and duration of travel),

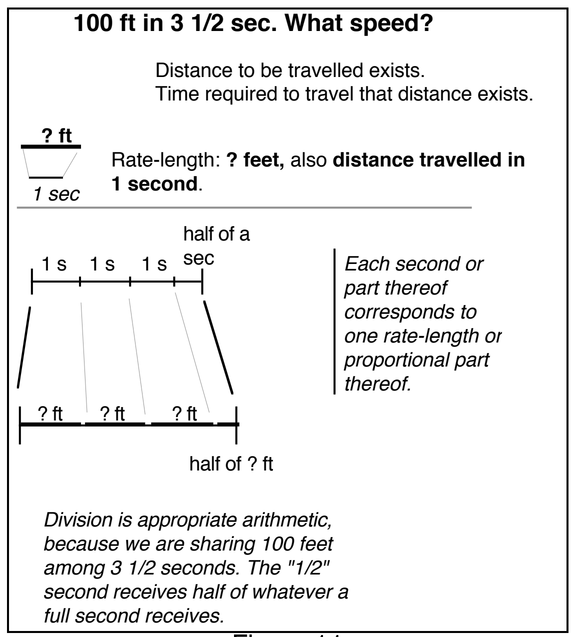

Figure 11

15 This insight, which amazed me at the time of gaining it, turns out to have been anticipated by Piaget. He stated, "The relation $v = d/t$ makes speed a relation and makes d as well as t two simple intuitions. The truth is that certain intuitions of speed, like those of outdistancing, preced those of time. Psychologically, time itself appears as a relation (between space traveled and speed …), that is, a coordination of speeds, and it is only when this qualitative coordination is completed that time and speed can be transforemed simultaneously into measurable quantities (Piaget, 1977, p. 111).

and to do that she had to create time as an extensive quantity and abstract it and distance from her intuitions of motion. This, together with conceiving travelling as if it were completed, enabled JJ to construct speed as a ratio. JJ's ability to construct speed as a ratio had two ramifications: it enabled her to reason flexibly about determining an appropriate arithmetic operation to evaluate some quantity in a relationship, and it led to occasions wherein JJ reflected upon speed as constant ratio, thereby interiorizing speed as a rate.

There is an important lesson to be learned from this teaching experiment. The standard method for introducing speed in schools is "distance divided by time." While we cannot know the course of events had I introduced speed as "distance divided by time," I say this with confidence: (1) "Distance divided by time" would have had little, if any, relevance to JJ's initial understanding of speed, (2) JJ would not have developed a concept of speed as a ratio, and therefore she would not have constructed speed as a rate, and (3) JJ would not have progressed as far as she did.

To tell students that speed is "distance divided by time" with the expectation that they comprehend this locution as having something to do with motion, assumes two things: (1) they already have conceived of motion as involving two distinct quantities—distance and time, and (2) that they will not take us at our word, but instead will understand our utterance as meaning that we move a given distance in a given amount of time and that any segment of the total distance will require a proportional segment of the total time. In short, to assume students will have any understanding of "distance divided by time" we must assume that they already possess a mature conception of speed as quantified motion. This places us in an odd position of teaching to students something which we must assume they already fully understand if they are to make sense of our instruction.[16]

General Issues

Arithmetic of Units

Schwartz (1988) has made a lengthy analysis of what he termed "referent-transforming operations." In his characterization, one "transforms" two quantities by an arithmetic operation to produce a resulting quantity. It is evident that JJ did not do this. The operands of her arithmetic operations were *values* of quantities, they were not quantities. The operations she performed on quantities were *quantitative* operations, and her inferences of appropriate arithmetic were based on relationships among quantities that, in turn, were created by her quantitative operations. When JJ "spoke arithmetic," it was evident that she was using it representationally, as she did not have a vocabulary to express her conceptions of situations.

While it is not recommended by the science community, it is still common to see references to what is called "dimensional analysis," or the arithmetic of units (e.g., miles ÷ miles/hour = hour) both to determine an arithmetical operation's resulting unit and to suggest appropriate arithmetic to do (e.g., Musser & Burger, 1991). JJ did not do arithmetic of units. Rather, she knew the quantities in the relationships, their units, and inferred a numerical evaluation for a quantity's

[16] Actually, we are telling them something new: Forget what you already understand; instead, do this

value, knowing ahead of time the unit of a quantity's value. We should condemn dimensional analysis, at least when proposed as "arithmetic of units," and hope that it is banned from mathematics education. Its aim is to help students "get more answers," and it amounts to a formalistic substitute for comprehension.

Problem Typologies

The problems given JJ fell into the "between measure spaces" category (e.g., distance vs. time; cost vs. number of candy bars; capacity vs. time), which made them "rate problems" according to several authors (Karplus et al., 1983; Nesher, 1988; Vergnaud, 1983, 1988, this volume). This view, however, emphasizes characteristics of texts. When we consider JJ's emerging conceptions of the situations depicted in texts we get a different perspective on the matter. JJ at first conceived problems concerning relationships between distance and time as what these authors would call "within measure space"—distance vs. distance. It was only after JJ constructed time as an extensive quantity that she came to conceive of these problems as what these authors would call "between measure spaces." It is my experience that any problem typology suffers this same deficiency, namely that any given situation can be conceived in a multitude of ways that cut across the boundaries of the typology.

Practice and the development of schemes

In a quotation given earlier Piaget expressed the strong position that assimilation is the source of schemes. One might interpret Piaget's position as a sophisticated form of Thorndike's Law of Effect or of the commonism "Practice

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calculation anytime I ask you something about speed.

makes perfect," but it would be misdirected to do so. Cooper (1991) has made the extremely important distinction between the notion of practice as it is normally used and what Piaget had in mind, which was the notion of repeated experience. "Practice" normally refers to an activity conceived by a designer of the practice; it is the repetition of observable behavior. We have known since the days of Brownell that what a child "practices" in a mathematics classroom often has little to do with developing habits of reasoning. "Repetitive experience," however, focuses our attention on the fact that what we want repeated is the constitution of situations in ways that are propitious for generalizing assimilations, accommodation, and reflection.

Quantity vs Number

While concepts of quantity and concepts of number are most assuredly highly related, both historically and ontogenetically, concepts of number quickly become confounded with matters of notation and language whereas concepts of quantity remain largely non-linguistic and largely unarticulated. I cannot say whether the inattention to matters of quantity is due to development or culture, but I suspect it is a result of unexamined cultural and institutional assumptions about mathematical learning and reasoning.

One aspect of the teaching experiment that is not evident from the excerpts in this chapter is JJ's unexamined assumption that she was obliged to use paper-and-pencil algorithms to perform numerical calculations. Moreover, JJ felt that she could not proceed in her reasoning *until* she had calculated any intermediate numerical result. If I contributed anything to JJ's progress, it was to relieve her of these constraints. For example, on one occasion JJ felt that it would be appropriate to divide by 100 by 3 but had no idea how to do it (see Excerpt 3). Had I not pointed out to her that she had a calculator, either I would have had to suggest we leave it at that or we would have had to digress from our discussions of relationships among speed, time, and distance. This was but one of many occasions where JJ would have been stymied had I demanded that all issues of calculation and numerical representation be settled before moving on in our discussions. This is not to say that matters of notation and numerical evaluation are unimportant. Rather, it is only to say that there are times and places for attention to issues of notation and calculation, but those times and places should not interfere with the development of foundational concepts (Thompson, 1992). In fact, the case of JJ would suggest that remaining oriented primarily to matters of conception has a salutary effect on the development of calculation skills, for it sets a context in which calculations normally considered "ugly" are seen as natural and necessary.

On this last matter, namely "ugly" calculations, I must agree with Confrey (this volume) that our interpretations of students' calculational difficulties and "misconceptions" concerning numerical operations are highly dependent upon customary foci of school instruction. Confrey observes that students' inclinations to associate certain operations with certain representations of number have more to do with their conceptual constitution of numbers and notations than it does with anything innate about numerical comprehension per se. Her observation is quite consistent with our observations of JJ. We saw JJ became "comfortable" with ugly numbers and figural distractions simply because her attention was oriented away from the activation of notationally-based computational procedures. As Greer (this volume) and Harel, Post, Lesh, and Behr (this volume) observe, we need to exercise great care to distinguish issues of notational convention and notationally-based procedures from issues of meaning and application.

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