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Talking About Rates Conceptually, Part II: Mathematical Knowledge for Teaching†

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Part I of this report (Thompson & Thompson, 1994) described one teacher's, Bill's, attempts to teach concepts of speed to one child, Ann. We analyzed sources of their eventual dysfunctional communication, concluding that Bill had encapsulated his deep understandings of rate and proportionality within his language for numbers and operations. Part II presents Pat's session with Ann, analyzing his instructional actions in light of his agenda for having Ann build a scheme of operations by which to understand distance, time, and speed. We contrast Pat's agenda with Bill's, focusing on the mathematical knowledge that guided their instructional decisions and actions. We briefly discuss implications for the content knowledge preparation of teachers.

Most people agree that teachers' knowledge of mathematics is essential to their ability to teach effectively. Yet, historically, researchers have had great difficulty elucidating the roles that mathematical knowledge plays in effective mathematics teaching. Early investigations consisted of correlational studies whose results, by and large, failed to show a clear link between what the teacher knows and what the students learn (Begle, 1972; Begle, 1979; Eisenberg, 1977). The most plausible explanation for these puzzling results is that the two variables, teacher subject matter knowledge and student learning, were inadequately conceptualized (Byrne, 1983). These early attempts suggest that teacher and student knowledge and the relationship between the two are complex theoretical constructs that defy simplistic definitions and conceptualizations.

Recent studies have focused on the form, nature, organization, and content

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of teachers' mathematical knowledge (Ball, 1990; Lampert, 1991; Leinhardt & Smith, 1985; Marks, 1987; Steinberg, Marks, & Haymore, 1985; Thompson, 1984; Thompson, Philipp, Thompson, & Boyd, 1994). More attuned with our intuitions, this work has highlighted the critical influence of teachers' mathematical understanding on their pedagogical orientations and decisions — on their capacity to pose questions, select tasks, assess students' understanding, and make curricular choices.

Joining these perspectives on important aspects of mathematical knowledge with careful conceptual analyses of students' understanding of rate, we get a synergy between a person's knowledge of mathematics and knowledge of learning that we refer to as "knowledge for conceptual teaching." This article, which is the sequel to Part I (Thompson & Thompson, 1994), examines what it means for teachers to have clear images of understanding a mathematical idea conceptually, how those images might be expressed in discourse, and what benefits might accrue to students by addressing the conceptual sources of their difficulties.

The Context

In Part I we described the first two days of a teacher's, Bill's, attempt to teach conceptually during a four-day teaching experiment with a sixth-grade student, Ann. The teaching experiment focused on the ideas of speed and average speed. The objective of focusing on speed and average speed was that Ann come to understand speed as a rate. The image of speed we intended Ann to construct through this unit is composed of these items, which themselves are constructions:

1. Speed is a quantification of motion.
2. Completed motion involves two completed quantities — distance traveled and amount of time required to travel that distance (this must be available to students both in retrospect and in anticipation).
3. Speed as a quantification of completed motion is made by multiplicatively comparing distance traveled and amount of time required to go that distance.
4. There is a direct proportional relationship between distance traveled and amount of time required to travel that distance. That is, if you go m distance units in s time units at a constant speed, then at this speed you will go $(a/b)\cdot m$ distance units in $(a/b)\cdot s$ time units. Described imagistically, to say that there is a direct proportional relationship between distance and time means that one "sees" that partitioning a traveled total distance implies a proportional partition of total time required to travel that distance, and partitioning the total time required to travel a distance implies a proportional partition of the distance traveled.

In previous research that builds upon Piaget's (1970) theory of the epigenesis of speed we found that this image develops through a progressive internalization of measuring total distances in units of speed-lengths — distance traveled in one unit of time (Thompson, 1994; Thompson & Thompson, 1992). Children first internalize the process of measuring a total-distance-traveled in units of speed-length (the measurement producing an amount of time required to travel that distance). When children have internalized the measurement of a total distance in units of speed-length they can anticipate that traveling a distance at some constant speed will produce an amount of time. This implies that children first conceive speed as a distance and time as a ratio ($\frac{\text{total length}}{\text{speed length}}$). With this anticipation they can reason about their image of completed motion, thinking about corresponding segmentations of accumulated distance and accumulated time. Their internalization of the dual measurement process provides a foundation for their conceptualizing constant speed as a rate. Prior to internalizing this measurement process, children are unable to conceptualize a situation where one is to determine a speed at which one will travel a given distance in a given amount of time. This is not to say that they are unable to answer questions about at what speed one must travel to go a given distance in a given amount of time. Rather, the only questions they can answer are those where they can use the equivalent of guess-and-test, looking for a distance (speed-length) which will produce a given travel time when they measure the given total distance in units of the speed-length.

The materials for the teaching experiment were a set of questions having to do with a computer program called Over & Back (Thompson, 1990). Over & Back presents two animals, Turtle and Rabbit, who run along a number line (Figure 1). Both can be assigned speeds at which to run. Turtle's speed can be assigned two values: one for its speed while running "over", the other for its speed "back." Rabbit's speed can be assigned only one value, which applies to both its trip over and its trip back. Each animal can be made to run separately from the other, or they can be made to run simultaneously (as in a race). A timer shows elapsed time as either of the animals runs.

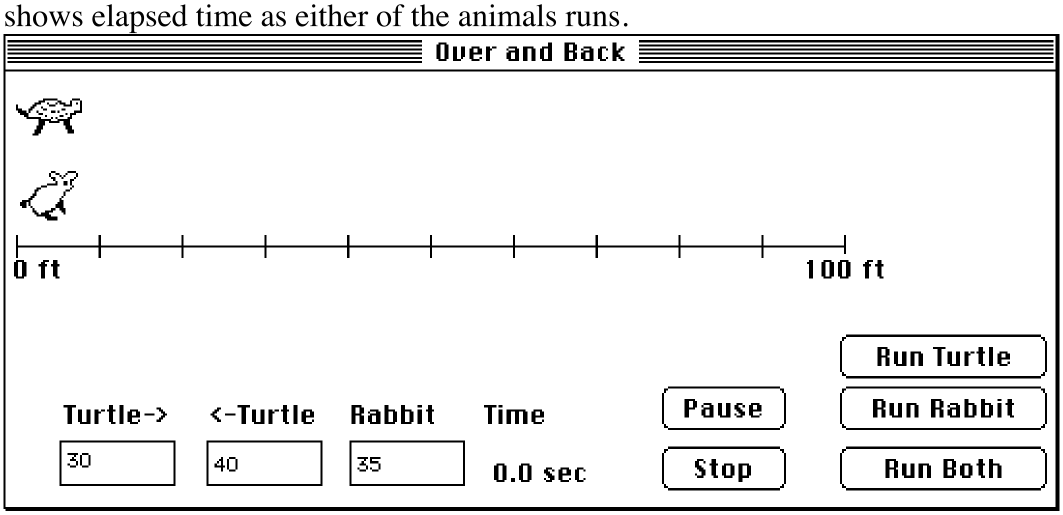

Figure 1. Over & Back's startup screen.

A Recap of Part I

In Part I we reported that at the end of the second day of the four-day teaching experiment, Bill and Ann reached an impasse that caused Bill frustration and brought Ann to the verge of tears. Our analyses, in Part I, focused on the source of breakdowns in communication between Bill and Ann. We speculated about two primary sources: (a) Bill's "packed" understanding of division and proportionality, which expressed itself in his orientation to speak in ways that were mostly calculational (Thompson & Thompson, 1994) and which were insensitive to conceptual subtleties in the situations, and (b) Ann's fundamental image of speed as a distance.

At the end of the second day's session Bill placed an "emergency call" asking Pat (co-author of this paper and co-researcher in the study) to take over Ann's instruction for the third day. Part II starts with Pat's session with Ann on the third day of the teaching experiment. It examines the teacher's, Pat's, understanding of rate in relation to matters of cognition, communication, and pedagogy. The analysis focuses on Pat's subject-matter knowledge for teaching rates conceptually — what comprises it, how it is expressed, and how it might be fostered in others.

We urge the reader to read Part I for a detailed description and analysis of the events that led to Day 3. Space restrictions preclude us from repeating them here.

CONTINUATION OF THE TEACHING EXPERIMENT

Pat prepared for Day 3 by watching the videotape of Bill and Ann's session for Day 2. While viewing the videotape it occurred to him that three essential features were missing from Ann's images of the situations she and Bill had been discussing: (a) an image of Rabbit's motion as entailing the *simultaneous* accumulation of distance from its starting point and time since it began; (b) an image of completed motion as entailing both a completed distance and a completed amount of time required to move that distance; and (c) the *anticipation* of Rabbit's completed motion so that she could construct a proportional correspondence between accumulated distance and accumulated time in relation to anticipated total distance and anticipated total time. Accordingly, Pat's agenda for his session with Ann was to help her think of: (a) distance and time as covarying quantities; (b) the idea that going all the distance will take all the time, and conversely, that going all the time will cover all the distance; and (c) the proportional correspondence between accumulated distance and accumulated time. Pat also knew he would need to be somewhat direct in his instructional approach because he had only 20 minutes to spend with Ann.

Pat's Session with Ann

Pat and Ann met in the same setting in which she and Bill had met the previous two days. Bill was present as an observer. The session began by Ann telling Pat, "You should see how fast this rabbit can go." Pat pretended not to know. Ann gave Rabbit a speed of 9,999,999 ft/sec and made him run. Rabbit's motion was barely perceptible. Ann claimed it took Rabbit no time to run and pointed to the timer to support her claim (the timer showed 0 seconds). Pat adjusted the timer to show several decimal places (it read 0.000) and asked Ann if indeed, it took Rabbit no time to run. Ann said that it took too little time for the computer to be able to record it. This exchange established the casual tone for the conversation that ensued in which Pat turned Ann's attention to the difficulties she had experienced the previous day.

Ann's spontaneous use of division during the previous two days had been to evaluate how many "speed lengths" were contained in some distance. Pat wanted to ensure that Ann also understood division as an appropriate operation to evaluate the length of a segment when it was made by a uniform partition of a longer segment. He also wanted Ann to think of distance, time, and speed in relation to one another and not to think merely about engaging in answer-getting activities. These two issues appear in an excerpt occurring early in the session.

Excerpt 1 — 01:58 to 03:41[1]

1. Pat: Did you have some confusion yesterday?
2. Ann: Yeah.
3. Pat: A little bit, huh? Well instead of talking about Turtle and Rabbit right now, let me ask you some questions about time, speed, and distance, okay?
4. Ann: Okay.
5. Pat: Let's try and clear things up [*draws a distance line with tick marks as end points*][2]. Now yesterday Mr. B drew something like this, right?
6. Ann: Yeah.
7. Pat: And gave that a distance. I'm going to give it a really messy distance, like five hundred and twenty-three feet [*writes "523" at the right end of the distance line*]. Okay? Now if we cut this up into [*puts four equally-spaced tick marks between the end points*] … one, two, three, four, five parts, what would you do to find out how long one of these parts was?
8. Ann: … I would …[3]
9. Pat: I'm not asking for an answer, I'm just asking what you would do.
10. Ann: Divide five into five hundred and twenty-three?
11. Pat: All right. So one of these parts [*writes "523 ÷ 5"; draws a line from it to the rightmost interval*] would be five hundred and twenty-three divided by five. Okay? [*Ann nods*] So that's distance [*writes "ft" next to the "523"*]. I'm calling that feet. Remember I'm not asking for an answer.
12. Ann: Uh huh [*meaning yes*].
13. Pat: That [*points to "523÷5"*] would tell you how long just this one is? [*points to the last interval again*]
14. Ann: Uh huh .
15. Pat: Or would it tell you how long each of them is? [*points to each interval individually*]
16. Ann: It would tell you how long one of them was.
17. Pat: And how would you find out how long this one was, right there … the second one? [*draws brackets under the second interval from right*]
18. Ann: You … if you wanted to find out just these two? [*points to the two rightmost intervals*]

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[1] These numbers indicate both elapsed time for the excerpt and when in the session the excerpt occured.

[2] Here and throughout we use "line" when we should use "line segment." We adopt this convention because Ann and Bill used it in earlier sessions and Pat continued that usage in his session with Ann.

[3] We use an ellipsis ("…") to indicate a pause. It does not indicate omitted text.

19. Pat: No, not how much they are together, how much they are individually [*again points to all the intervals from right to left*].
20. Ann: They would all be the same!

In (Excerpt 1, speech 7) Pat decided to use a distance of 523 feet so that Ann could not easily produce numerical answers to the questions he was about to ask. He anticipated that Ann would employ basic multiplication facts if he used a distance of 100 feet, as Bill had consistently used in his two earlier sessions with Ann. Had Ann worked with a distance of 100 feet and 5 sections, and had she answered on the basis of knowing that 5×20=100, it would have been more difficult for the discussion to focus upon division as an appropriate operation for evaluating the length of each section. Also, in (speeches 13-20), Pat's questions oriented Ann toward generalizing her understanding of division as evaluating only the length of one section to understanding division as evaluating the length of each section. Pat felt it was important for Ann's later reasoning that she understand that each segment in a uniform partition is the same length, and because division evaluates the length of one, it evaluates the length of each.

We should note that Pat was content with Ann's having associated division and segmentation. It was an association he hoped to build upon pedagogically; he was not concerned about ensuring that Ann had a deep understanding of it. To ensure her deep understanding of division in relation to segmentation might have required a prolonged instructional development, which might have confused her regarding the session's main point.

Pat then moved on to discuss time in relation to motion and distance and to initiate a conversation about distance and time as varying simultaneously in a linked manner — as covarying quantities (Excerpt 2).

Excerpt 2 — 04:39 to 06:01

1. Pat: Now down here … okay, let's forget distance for a little while. Suppose that we're running Rabbit just one way [*moves hand across the computer screen to indicate Rabbit's trip over*]. And I'm going to use a segment here to talk about how much time it takes [*draws a smaller line below the distance line, with tick marks as end points*]. Now here's the way I want you to think about this, because I'm not talking about distance with this line [*points to time line*]. I'm just talking about time. As Rabbit goes over … now watch my finger [*puts his left index finger at the beginning of the distance line*]. As Rabbit goes over here [*slowly moves index finger across the distance line*], he's going to … what's this timer doing? [*points to the computer's Time Counter*]
2. Ann: It's timing how long it's taking him.
3. Pat: Okay.
4. Ann: … to go from one place to another.
5. Pat: So the number of seconds … what's happening to the number of seconds as he goes? [*moves finger slowly across the distance line*]
6. Ann: They're increasing.
7. Pat: It's increasing. Okay. So as he goes we can also think, if this is seconds … [*writes "seconds" at the right end of the time line*] … As he goes along a distance line [*drags finger across the distance line*], we can think of the number of seconds increasing also [*moves index finger on the right hand across the time line*]. We can think of them doing it together [*moves fingers across both lines simultaneously*].

8. Ann: … Yeah.

In Excerpt 2 (speech 1) Pat started to speak of a line segment representing Rabbit's travel time and to make explicit that it did not represent a distance, but interrupted himself. He digressed to ensure that Ann could in fact envision time as a variable quantity within the phenomenon itself — time's measure increasing as Rabbit traveled (speeches 1-4). Pat had observed in Days 1 and 2 that Ann already imagined motion as entailing distance traveled, but not as entailing an amount of time. He presumed that her understanding was that an amount of time is something to be determined after the fact, that is, after having moved an amount of distance (see Figure 2).

Figure 2. Ann's initial image: Motion entails moving a distance, but moving some distance did not automatically entail an amount of time. Rather, an amount of time was something needing to be determined after having moved an amount of distance.

In (Excerpt 2, speeches 5-7) Pat moved through a subtle development aimed at having Ann envision distance and time covarying as Rabbit moved (see Figure 3). He focused her attention on the distance line as he moved his finger across it while at the same time asking Ann about the number of seconds that would show on the timer as Rabbit moved. It is worth noting that, whereas Ann said "*They're* increasing" (speech 6), Pat said "*It's* increasing." This suggests that Ann was seeing an amount of time as a plurality of seconds, whereas Pat saw an amount of time as a magnitude (Steffe, von Glasersfeld, Richards, & Cobb, 1983; von Glasersfeld, 1981). Evidently, Pat did not notice this discrepancy, and it is not clear that he would have acted differently had he noticed.

Figure 3. Ann's subsequent image: Motion entails moving a distance, and moving a distance entails using some time to do so.

In (Excerpt 2, speech 7) Pat aimed to have Ann see increases in Rabbit's amount of distance and amount of time as happening simultaneously. His intention is depicted in Figure 4. The bi-directional relationship which constitutes covariation entails thinking of both quantities varying simultaneously *without* a necessary dependency. Rather, if one focuses first on distance, one can determine time; if one focuses first on time, one can determine distance. It was Pat's intention to have Ann understand motion as covariation of distance and time in this way — as a bi-directional, reversible relationship. His action in (Excerpt 2, speech 7) was his opening attempt at moving Ann's understanding toward that state.

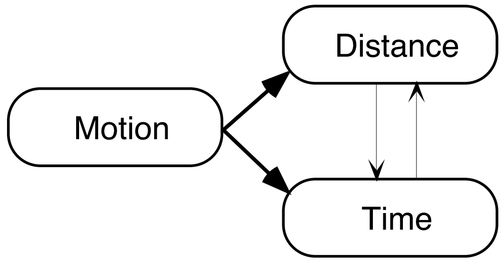

Figure 4. Pat's intention for Ann's understanding: Motion entails moving a distance, and it also entails some amount of time to move that distance. Distance and time become covarying quantities, as opposed to one being dependent upon the other.

After the exchange in Excerpt 2 Pat continued by pointing out to Ann that the time-line represented all of the time Rabbit took to travel the entire distance and that they could talk about "all the time" without knowing what it actually was. He then pretended that they had frozen Rabbit at the moment it had traveled one fifth of the distance and asked Ann where the time marker would be frozen. Ann said that Rabbit would be frozen at one fifth of the total time. Pat then told Ann to suppose that one fifth of the time was two seconds, and asked how long it would take Rabbit to make the whole trip. Ann responded by saying "Since there are five parts, you would just times five by two." Pat then asked how long it would take Rabbit to go two fifths of the way, three fifths of the way, the whole way. In each case, Ann responded correctly. Pat suggested that they drop the two seconds and asked how much time it would take Rabbit to go three fifths of the distance. Ann responded: "It'll be three fifths of the . . . total number of seconds that it would take him." Pat reiterated the idea that we can speak of fractional parts of a total amount even when we do not know the actual amount.

Following this Pat told Ann to pretend that Rabbit traveled at 60 ft/sec and asked how far he would go in two and one-half seconds. Ann responded immediately, "He would go … a hundred and fifty feet"—explaining that she had mentally calculated $60 + 60 + \frac{1}{2}(60)$. They then proceeded to mark the distance and time lines accordingly, with Pat emphasizing the covariation of distance and time.

In Excerpts 1 and 2 the discussion was about fixing an accumulation of distance and thinking subsequently about a corresponding accumulation of time. In Excerpt 3 Pat turned their discussion to the reverse relationship — fixing an accumulation of time and thinking subsequently about a corresponding accumulation of distance.

Excerpt 3 — 11:02 to 12:10

1. Pat: Now. Suppose I turn that around. We've still got distance and we've still got time, okay? … [*On a new piece of paper, draws two lines like that on the previous page, the bottom one shorter than the other*] We're going to do … another one. Now you understand, still, that even though I've got a length here [*points to the bottom line*], I'm talking about time.
2. Ann: Yeah.
3. Pat: All right [*writes "Time" next to the bottom line*]. And that's just all the time it's

going to take him.
4. Ann: Uh huh [*yes*].
5. Pat: And this is all the distance [*writes "Distance" beside the top line*] that he's going to go. And let's suppose that this is one hundred feet that he's going to go [*writes "100" at the right end of this new distance line*]. Suppose that we say, "All right, let's cut up the time that he's going to go." … [*Interrupting himself.*] When he uses up all the time [*drags the pen across the time line*] where is he going to be?
6. Ann: Over and back?
7. Pat: Let's just talk about him going one way [*holds up one finger*].
8. Ann: Okay. He'd be all the way … a hundred feet.
9. Pat: A hundred feet. So when he uses up all that time [*gestures across the time line*], do we have to know how much time there is [*again gestures across the time line*] to say . . .
10. Ann: No [*shakes head*].
11. Pat: So you just know that when he uses up all that time [*gestures across the time line*] he's going to go a hundred feet [*gestures across the distance line*].
12. Ann: Uh huh [*nods*]

In (speech 1) Pat made his opening move toward raising a relationship between time and distance where Ann thinks first of an amount of time. He digressed momentarily (speeches 1-4) to ensure that Ann was thinking about time and distance and not about the segments representing them, as she had been prone to do in earlier sessions. In (speeches 5-12) he also ensured that Ann understood that the amount of time she thought of was not completely arbitrary — it was however much time it would take Rabbit to go the entire distance. Pat felt that in order for them to be able to speak meaningfully about proportional correspondences between an accumulated amount of time and an accumulated amount of distance, it was essential that Ann have firmly in mind that "all the time" is however long it takes Rabbit to travel "all the distance."

Excerpt 4 — 12:10 to 13:37

1. Pat: Okay? Suppose that I tell you that when he uses up half his time [*puts pen at the halfway point on the time line*; *looks up; pauses*] … No, I'm getting confused on this... When he uses up half his time … where will he be when he uses up half his time?
2. Ann: Fifty feet.
3. Pat: Fifty feet. Where will he be when he uses up one fourth of his time [*places pen about a fourth of the way along the time line*]?
4. Ann: ... Umm … twenty feet?
5. Pat: Okay. And how are you coming up with twenty?
6. Ann: If half of a hundred is fifty then, … umm, like a fourth of a hundred is like twenty or thirty.
7. Pat: ... I'll tell you twenty is wrong. But if you had told me that it's a fourth of a hundred, that's right. So if you're not sure, like, what a fourth of a hundred is, just say a fourth of a hundred [*shrugs shoulders. Ann nods*]. Because we can always use a calculator to get a number, right?
8. Ann: Yeah.
9. Pat: Okay. So if he goes a fourth of the way in the time [*points about one fourth the way across the time line*], he's going to go how far in the distance?
10. Ann: A fourth of the way.

11. Pat: [*Nods*] A fourth of the way. So a fourth of one hundred. Okay. And what would you do with the calculator to find out what a fourth of one hundred is?
12. Ann: Umm, you divide … four into a hundred?
13. Pat: … Yes [*nods*]. That's right … A seventh of the way [*indicates time line]*, he would be how far on the distance?
14. Ann: A seventh of the way.
15. Pat: A seventh of … ?
16. Ann: He'd be a seventh of a hundred.
17. Pat: A seventh of a hundred. All right.

Pat caught himself as he began to misspeak (Excerpt 4, speech 1). He began to say, "When he uses up half his time, how *far* will he be on the distance line" (speech 1), but saw that, were he to ask that question, Ann might think of amounts of distance instead of fractional parts of the total distance. Even so, his choice of 100 feet as the total distance and his choice of one half as the fractional part of time still allowed Ann to concentrate upon "how far" instead of upon "what part of the distance" (speeches 2-4). Ann's answer of 20 feet when asked where Rabbit would be after using one-fourth of the time (speeches 3 & 4) was fortuitous for Pat. It provided a natural context for him to request that Ann express a distance as a formula for determining it rather than as an actual number of feet (speeches 5-10).

In Excerpt 4, Ann said that in one seventh of the total time Rabbit would travel one seventh of the total distance (speeches 13-17). Pat then changed the context slightly, so that Rabbit traveled for a specific number of seconds (Excerpt 5).

Excerpt 5 — 3:37 to15:12

1. Pat: Suppose I tell you that it takes him seven seconds [*writes "7" at the end of the time line*] to go a hundred feet [*gestures across the distance line*]. Now, let's not worry about speed, okay? Let's just talk about [*inaudible*]. So, if it takes him seven seconds [*gestures across the time line*] to go across the time, how many parts [*counts imaginary intervals on the time line*] have we cut up the time into?
2. Ann: … Seven.
3. Pat: Okay. So [*marks 7 intervals on the time line; draws a squiggle under the first interval*] … And then each second is what part of the total time?
4. Ann: One seventh.
5. Pat: One seventh. So that's one seventh of the time [*writes "1/7 of the time"*]. How much would one seventh of the time go with up here [*points to the distance line*]?
6. Ann: One seventh of the distance.
7. Pat: One seventh of the distance. Okay. So one second, this is one second [*writes "1 sec" above the first interval on the time line*], is one seventh of the time, and it would go with … I'm going to put a dotted line up here like this [*connects the right-end of the first time interval to a point about 1/7th of the way along the distance line; draws a squiggle under the distance interval; turns paper so that Ann can see it*]. That [*time interval*] would go with one seventh of the distance [*writes "1/7 of the distance" beneath the squiggle*]. Okay?

Excerpt 5 reveals a subtle transition. Pat and Ann had spoken previously only of a fractional part of the total time corresponding to the same fractional part of the total distance. He moved the conversation from speaking only of "one seventh of a total amount of time" to speaking of "one second out of a total of seven seconds as being one seventh of the total time" (speeches 1-4). He then asked Ann to repeat her earlier inference that one seventh of the time goes with one seventh of the distance (speeches 5-6), and summarized both the context they had built to that moment and the inferences they had drawn from that context (speech 7). Excerpt 6 begins with Ann's response to Pat's summary.

Excerpt 6 — 15:12 to16:31

1. Ann: ... So we're just … so to find out the answer, you just do what you did before like on the other one [*gestures toward the computer and an earlier-completed work on scratch paper*]?
2. Pat: I don't know what you did before. Why don't you explain that.
3. Ann: Like this [*points to scratch paper with work from opening situation — "Cut up a distance into 7 parts, how would you find the length of each part?"*]. You just divide a hundred by seven.
4. Pat: Yeah!
5. Ann: And you come up with the answer.
6. Pat: So, if it takes him … if one second is one seventh of the time [*drags pen over the first interval on the time line*], then in that one second he's going to go one seventh of the distance [*drags pen over first interval on the distance line*]. And if the whole distance is one hundred feet [*gestures across the distance line; points to "100 ft"*], then what's this part [*points to the first interval on the distance line*] that he went in one second?
7. Ann: One seventh … of a hundred.
8. Pat: Try that … I mean how would you calculate one seventh of one hundred [*gestures to the calculator*]?
9. Ann: [*Looks down at the calculator*] One hundred divided into seven, or seven divided into a hundred? [*Ann uses calculator to calculate 100÷7, getting 14.286.*]
   [*24 seconds of transcript omitted as Ann and Pat discuss how to read the calculator's display.*]
10. Pat: [*Writes "14.286 feet" at the top of the page.*]

Ann seemed to have drawn a connection between dividing to evaluate the length of one segment in a uniform partition of total distance and inferring corresponding fractional parts of time and distance (Excerpt 6, speeches 1-5). On the other hand, it was a major mistake that Pat did not ask Ann what question her answer was answering (speech 5). He assumed that she was answering the question, "How far will Rabbit go in one second if it goes 100 feet in seven seconds?" but the next excerpt (Excerpt 7, speeches 1-10) suggests that this question may not have been precisely what she had in mind.

Excerpt 7 — 16:31 to19:01

1. Pat: Okay. So … these [*pointing to "14.286"*] are what? That's fourteen point two eight six what?
2. Ann: Umm, … seconds … Distance? Distance.
3. Pat: Where are we finding it? In time [*points to the time line*] or distance [*points to the distance line*]?

4. Ann: No [*points to the distance line*], in distance.
5. Pat: And we didn't say what this is [*points to "100" on the distance line*]. That's one hundred feet [writes "ft" after 100]. So, what's this segment here [*holds up the sheet so that Ann gets a good view; points to the first interval on the distance line*]?
6. Ann: It's fourteen and … it's fourteen and two hundred and eighty-six thousands feet.
7. Pat: Two hundred eighty-six thousandths of a foot [*writes "ft" after 14.286]*; ... So, if he's going to go a hundred feet [*drags pen across the distance line*] in seven seconds [*drags pen across the time line*], what do we end up saying that he goes each second [*points to the first interval on the distance line*]?
8. Ann: Hmmm?
9. Pat: Remember. This [*points to the first tick interval on the time line*] was … no. What's your question [*looks closely at Ann*]?
10. Ann: I didn't get what you just said … I didn't, I didn't hear it.
11. Pat: Remember [*moves pen across the distance line*] I started out by saying "Suppose that it takes him seven seconds to go the whole way" [*drags pen across the distance line*]. Then you said that one second [*points to "1 sec" over the time interval*] is one seventh of the time [*points to "1/7 of the time" below the time line*]. So the distance he goes in one second [*drags pen across first tick interval on the distance line*] is one seventh of the distance [*points to "1/7 of the distance" below the distance line*].
12. Ann: Uh huh.
13. Pat: So how many feet does he go each second [*drags pen across the first tick interval on the time line*]?
14. Ann: He goes ... fourteen and two hundred and eight-six thousandths of a feet, of a foot each second.
15. Pat: And in the next second?
16. Ann: He would go the same amount of time as in the second before.
17. Pat: Yeah. He's going the same amount of time which is one second.
18. Ann: And the same amount of distance.
19. Pat: Which is?
20. Ann: Fourteen and . . .
21. Pat: You can just say 14 point 2, 8, 6 for now.
22. Ann: 14 point 2, 8, 6.
23. Pat: Okay. And in the next second he'll go … [*draws a line from the third tick mark on the time line to a point about halfway along the distance line, making a third distance interval*]
24. Ann: 14 point 2, 8, 6.
25. Pat: [Writes "14.286" above the third interval on the distance line] And in the fourth second [*draws a line to the distance line from the fourth time interval's endpoint*]?
26. Ann: 14 point 2, 8, 6.
27. Pat: Point 2, 8, 6. And tell me what that number is. It's a number of what?
28. Ann: Feet.
29. Pat: Okay. So it's [*writes "ft" after each 14.286*] so each second how far does he go?
30. Ann: 14 point 2, 8, 6.
31. Pat: Okay [*nods*].
32. Ann: So, like, if you divided that number [*points to "100" on the distance line*] by seven you could come up with the answer too, right?

Ann understood that she was finding an amount of distance (speeches 1-

6), but it is not clear that, at that moment, she understood the correspondence between segments of time and segments of distance as a quantification of Rabbit's motion (speeches 7-10). Pat interpreted Ann's hesitation as indicating that she had lost a coherent image of the total context. Rather than simply repeat his question he instead elaborated the context and summarized Ann's reasoning up to that moment — and then repeated his question (speeches 11-14). Pat's follow-up (Excerpt 7, speeches 15-31) was intended to have Ann elaborate her image of Rabbit's motion so that she understood *each* second as one seventh of the total time, being one of seven seconds, and hence that in *each* second Rabbit would travel one seventh of the total distance.

Pat ended his session with Ann by asking her how far Rabbit would go in one twelfth of the total time; she responded, "He would go one twelfth of a hundred feet." After this Pat summarized the context he and Ann had created — the context of knowing a total time for Rabbit's trip and knowing the distance Rabbit would travel in that amount of time, and that when Rabbit moved for a fractional part of its total time it would move a corresponding fractional part of its total distance. Pat congratulated Ann for her "good reasoning" and then spoke with Bill about when they could next meet. While Pat and Bill talked, Ann set Rabbit's speed at 14.286 ft/sec and ran Rabbit . Bill took over the session.

Bill's Follow-up

Bill congratulated Ann on having done so well in the previous 20 minutes, and suggested that they pick up where they had stopped at the end of Day 2, performing the worksheet task of giving Rabbit a speed that would make it go over and back in six seconds.

During the remainder of Day 3 and during their session on Day 4, Ann expressed valid reasoning (with some fragility), often drawing distance and time lines and partitioning both appropriately, while answering questions of what speed to give some animal. The tasks ranged from "Give Rabbit a speed that will make it go over and back in __ seconds" to "Turtle will go over at some speed and will go back at 70 ft/sec. Rabbit will go over and back at 50 ft/sec. Give Turtle a speed going over so that it and Rabbit tie." By the time she solved the latter problem she had ceased drawing line segments to represent distance and time, instead simply saying in prose what she had earlier acted out. We will not analyze these sessions, as Ann performed appropriately and Bill's instructional actions and orientations resembled closely those he expressed during Days 1 and 2.

Bill ended their last session (Day 4) by asking Ann if she had any more questions. Ann asked what had given Pat the idea for making up Over & Back. Bill's response reveals some of his basic imagery regarding rate and pedagogy.

Excerpt 8 — 38:53 to 41:32

1. Bill: Any other questions?
1. Ann: Yeah. How did Dr. Thompson make it up?
2. Bill: How did he write the program?
3. Ann: Yeah.

4. Bill: Well you mean as far as actually writing the program code in the computer, I don't know how he did that because, I haven't actually seen the program. But do you mean the actual information that's in here or how to come up with the idea of doing something like this?
5. Ann: How did he come up with the idea of just doing it?
6. Bill: Okay. The concept of speed, which is distance divided by time that you learned in science,[4]
7. Ann: Uh huh.
8. Bill: is a difficult concept for a lot of people to understand. Sixth, seventh graders especially, because we're talking about a ratio of two numbers that are independent of each other. And by that I mean on one thing you have speed, and on the bottom side of this fraction, some people call it a fraction although it is a ratio [*puts one hand above the other to indicate numerator and denominator*], you have, umm distance … I'm sorry, distance and time. And those are two entirely different units. And trying to work with them, as you found out the first or second day we were working here, "Uh-oh something is kind of different here, it doesn't seem to work right." So the reason for the program is to help teach what that relationship is. That we're dealing with distance and time [*makes an imaginary distance line with his hand*]. If we're going to divide a distance of 200 feet up [*uses his other hand to make pretend tick intervals in his left hand*] into 5.2 seconds of time, that going up proportionally and dividing up the distance into 5.2 segments will give me the speed. That's the part that's hard to understand.
9. Ann: Okay.
10. Bill: But you're a past expert on that now, see? [*Chuckles*] Any other ones?
11. Ann: Yeah. Do we have to do algebra?

It is instructive that after Pat's and Bill's attempts to have Ann develop the idea of speed conceptually, Bill characterized it for her as "distance divided by time" (speech 6), and spoke of speed as if it were a ratio of two numbers (speech 8). Bill also spoke of "dividing a distance of 200 feet up into 5.2 seconds of time," a phrase that was at cross purposes with the central image of Rabbit's *motion* as being what made gave sense to the idea of thinking about some segment of distance corresponding with some segment of time. We will return to these observations in our general discussion of mathematical knowledge for teaching. For now, we point out that Bills remarks in Excerpt 8 are consistent with our observations in Part I (Thompson & Thompson, 1994) that Bill's deep understandings of rate were often encapsulated within the language of numbers and operations. He injected meaning into the image of a ratio of two numbers (a fraction), and that the injection happened by his understanding of each number as being the value of some quantity and the ratio of those two numbers as coming from a multiplicative comparison of those two quantities. But to a listener who could not likewise inject meaning into numbers and into operations on them, he was just speaking about numbers and operations.

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[4] Unknown to us, Ann had studied "distance-rate-time" in her science class two weeks prior to this teaching experiment. This became known to us during Day 4 when Ann interrupted herself, saying, "Wait a minute! This is science. Are you trying to trick me here?"

BILL'S REFLECTIONS

Bill and Alba (first author) held their regularly scheduled weekly meeting on the afternoon of Day 3, approximately 5 hours after Bill and Pat had finished their session with Ann. Alba had not yet seen the videotape of that session, so she asked Bill to describe his impressions of it.

Excerpt 9: Afternoon of Day 3

1. Alba: Tell me about your impressions of today's session.
2. Bill: Well, the thing that was really helpful today was having Pat come over, because he came over and actually taught her the first fifteen or twenty minutes before he had to leave. To me, that was really helpful, because where I got hung up yesterday was that I didn't know how far to demonstrate or do diagrams, etc. I didn't know how far to go with the demonstrations of trying to make the connections between the distance versus the time. I kept trying to have her come up with that, and she got to the point where she just became very frustrated yesterday. It didn't show up on camera or anything because her face was not to the camera, but she was almost in tears.
3. Alba: Is that right?
4. Bill: Yes, a couple of times. She gets so tense inside.
5. Alba: I think (another teacher) said something about her being that way. I don't recall exactly the nature of (the other teacher's) comments, but I do remember her saying that she closes up in herself.
6. Bill: Well, when she got to that point … then I tried a couple of ways to do the connection, to show the connection without telling her what the connection was that I was looking for. And by that time she was so frustrated that she …
7. Alba: What connection did you have in mind?
8. Bill: I was trying to get her to see that if we take — just like Pat was doing today with a distance line and a time line — if we take the distance line and divide it into five equal sections, we've got to do the same thing with the time line to see that that connection exists between them. Essentially, what he did today was that very thing. But he went very slowly, step by step, and broke it down for her. And I was trying to get her to do it instead of showing her how, and I'm glad he did that. Number one, because she immediately picked up on it and realized what it was, and after that she was fine. She could do every one of the problems and did them all very well and very quickly. She continued to draw the lines for the next three problems that we did after Pat left.
9. Alba: I see.
10. Bill: And she drew out the distance line and the time line, and she labeled them. And she showed that if it's 8.3 seconds, that I have to take a hundred and divide it by 8.3. And then she'd do that on the calculator and write it in there. And I had her explaining and she went step by step, "Okay you've got this answer, I6.6. What does that answer represent?"
11. Alba: Did she understand what each one of those segments was … when she divided 100 by 8.3?
12. Bill: That was the distance it would have to travel in each second.
13. Alba: And that was clear to her?
14. Bill: Yes, very clear. So we just started to work on the next worksheet when the bell rang.

Bill's impression of Pat's session with Ann was unarticulated and focused on Pat's actions, not on what effect they might have had on Ann. Bill saw that Pat had done "essentially … the very thing" he had tried to do when he and Ann reached their roadblock (speech 8). This was to get Ann to see that both time and distance lines must be cut up into "the same number of segments." He noted that Pat "went very slowly, step by step, and broke it down for her," while Bill "was trying to get her to do it, instead of showing her how."

Two things are noteworthy in (Excerpt 9, speech 8). One is that Bill referred only to the need for the *number* of segments in both lines to be the same. He did not say anything about the proportionality of corresponding segments — an understanding of which Pat considered essential to Ann's ability to solve the problems, and to which he devoted much of his session with Ann.

The second is Bill's perception that Pat showed Ann how to do the tasks and that Pat's success with Ann was due to his going through the tasks "step by step." This suggests to us that, while observing Pat work with Ann, Bill understood Pat to be helping Ann learn to solve these particular problems instead of as Pat helping Ann construct a more powerful, coherent image of distance, time, and speed.

Later in her conversation with Bill, Alba brought up the idea of proportionality *per se*. She asked Bill whether he saw any connection between the tasks he used with Ann and some standard textbook tasks involving ratios and proportions that he had used earlier in the year with his eighth-grade class. In particular, they discussed a task involving the ratio of some number of red marbles to some number of black marbles.

Excerpt 10: Afternoon of Day 3

1. Alba: Now did you see any connection between the kinds of things that you have been doing with this, with the notion of speed as a rate, and the kinds of discussions that we had earlier? Did you see any connection between this stuff and ratio and proportion?
2. Bill: Yes, definitely they're connected because we're talking about proportions here, both ratios and proportions. The only part I see a difficulty with, let me go back a step. [*A portion of the transcript is omitted here. Bill digresses to discuss managerial difficulties he encountered when he first used Over and Back with "an unruly eighth grade class."*]
3. Bill: One reason I'm interested in doing this with Ann is to go through the process with her to find out how to use this, if I can, with the seventh grade class. My concern is whether or not to start off with something like this, dealing with rates, as opposed to dealing with other general definitions of ratios and proportions like the ones we were talking about before, "oranginess" [*referring to Noelting's* (1980a; 1980b) *task in which one is to decide which of two mixtures tastes more "orangy" based on how many units of water and orange pulp each contains]* and/or other kinds of comparisons.
4. Alba: Or when you were talking about, for example, the red marbles and the black marbles.
   [*Portion of transcript omitted.*].
5. Alba: ... How does this connect with that kind of lesson, where you were trying to get the kids to realize that for every three red marbles there would be, say,

five black marbles?

6. Bill: Well, only in the realization of the distance and time. That's the same kind of a ratio or proportion.
7. Alba: Right.
8. Bill: But this is, in one way, more complex. Marbles are fixed. They don't change. But here we have a speed that can change, a distance that can change, and a time that can change. Whereas if I'm dealing with a ratio of, say, two marbles out of seven are red, how many out of two hundred fifty are red? I don't have the variation of the distance and time thing. In effect, I should say, I'm bringing in a third parameter when we deal with this. The distance and time are two individual things.
9. Alba: Yes.
10. Bill: Then we have to compare. The ratio of those together is what? So this one seems to be more complex to the student.
11. Alba: I don't follow you. Can't I think of red marbles as distance, black marbles as time, then the ratio of red to black being the speed?
12. Bill: I see what you're saying. Yes, they do compare directly that way. For example, changing the number of one of those two sets of marbles —
13. Alba: I thought you were going to say that in the case of the marbles, they are discrete objects that can actually be set aside. Whereas, when you talk about distance and when you talk about time, these are continuous. Their unit is not readily visible or perceptual. In the case of the marbles, you can count them out. When you break up that line segment representing that distance into several parts, each little segment over here stands for a certain distance, say twenty miles. And for every one of these twenty miles there is a small segment down here.
14. Bill: And a time that goes along with it.
15. Alba: Right, and in this case the comparison, the multiplicative comparison, for every twenty of this I have one second down here... for every piece of this line segment that represents distance I have a piece of this line segment down here that represents time, and I must have the same number of line segments up here that I have down here, because I have to make this correspondence. The student has to realize that where this line segment ends the next one immediately begins so that there is this continuity with distance, that there is this continuity with time. Whereas with marbles, you can actually pile them apart… but as far as the multiplicative comparison...
16. Bill: It's virtually the same.
17. Alba: The same, yes.
18. Bill: And the part I was talking about earlier I didn't explain well. It really doesn't follow, either. What I was talking about here … we have the speed, let's say forty feet per second. And we have seconds going along with those down on that bottom line. The third thing that goes into this is this distance of a hundred feet, or two hundred feet. But that compares directly to saying, "If I have two out of five marbles that are red, how many out of two hundred fifty," which is the total distance.
19. Alba: Right.
20. Bill: So it's . . . really it's pretty much the same thing.
21. Alba: Right.

Despite the fact that Bill evidently reasoned proportionally when solving Over and Back problems himself, he may not have been aware that he was indeed reasoning proportionally. Proportionality was not an explicit topic of his pedagogical agenda when he worked with Ann. The above excerpt indicates that, although he possessed all the ingredients to make the connection between Over and Back tasks and typical ratio and proportion tasks, he had not made the connection explicit until his conversation with Alba. It seems that, for Bill, ideas of proportionality lived within the equation "$a/b = c/d$." They did not live within the formula "$d = rt$." We should point out that, despite his insight during his conversation with Alba, Bill did not raise the issue of proportionality during his next session (Day 4) with Ann.

MATHEMATICAL KNOWLEDGE FOR TEACHING CONCEPTUALLY

In many regards we have given additional substance to the claims by Grossman and others (Grossman, 1992; Grossman, Wilson, & Shulman, 1989) and by Thompson (1984; 1992) that how one teaches a subject is influenced greatly by the many ways one understands it. But there is another aspect of this study that cuts across the types of knowledge typically embraced by phrases such as "content knowledge" or "pedagogical content knowledge" (Shulman, 1986). It is that teachers' images — the loose ensemble of actions, operations, and ways of thinking that come to mind unawarely — of what they wish students to learn, and the language in which they have captured those images, play important roles in what teachers do, what they teach, and how they influence students' understandings.

Pat's actions were highly image-oriented and his language was deliberately chosen to help Ann in two ways: to become oriented likewise and to form, in fact, those images. He felt that one of his most important moves was to sweep his fingers simultaneously along the two line segments that Ann understood to represent Rabbit's distance and Rabbit's time. By sweeping both index fingers along the distance and time lines simultaneously, Pat focused Ann's attention on the idea that distance and time were varying simultaneously. Once Pat was assured that Ann was indeed imagining *motion* as an essential aspect of the situation, and that her idea of motion entailed simultaneous changes in distance traveled and changes in the duration of travel, he felt comfortable moving on to relationships between uniform covariation and proportional correspondence.

Pat's suggestions to Ann that she use line segments to represent distance and time and that she think about Rabbit's motion within these representations, and his questions to her about fractional amounts of one in relation to fractional amounts of the other led Ann to think about corresponding segmentations of accumulated distance and accumulated times. Ann's thinking of corresponding segmentations led her to examine how partitioning the total distance traveled implied a proportional partition of total time required to travel that distance, and vice versa. Ann thus became able to "see" (form an image of) the proportional relationship between distance traveled at a constant speed and the amount of time required to travel that distance. In turn, this led to Ann resolving for herself her original difficulty finding a constant speed that would enable Rabbit to travel a given distance in a specified amount of time.

Pat's actions as he interacted with Ann were constantly informed by his knowledge of three complementary ideas, and his goal that Ann understand them implicitly:

1) Division is an appropriate calculation to evaluate the size of a whole piece when any quantity is partitioned into a number of equal-sized pieces,
2) Constant speed implies a bi-directional, proportional correspondence between segments of accumulated distance and accumulated time;
3) Total time as a number of seconds can be imagined also as a *partition* of total time into a number of equal-sized pieces.

An important aspect of Pat's interactions with Ann, as illustrated especially in (Excerpt 7, speeches 7 ff.), is that after Ann had answered a question correctly, he attempted to keep her aware of the entire context — including inferences that she had made — as the context built up. It was important to him that, at each moment, Ann be aware of the entire set of conditions upon which she based an inference. He expected that, in the future, she would need a coherent image of where she began, where she was at the moment of making an inference, and how she moved between the two if she was to be able to reconstruct such reasoning on her own. Pat's intent, as reflected in his goal that Ann develop the three complementary ideas listed above, was that Ann build a speed *scheme* (Cobb & von Glasersfeld, 1983; Johnson, 1987; Steffe, 1994; Thompson, 1994) and that she be able to answer questions about determining any one of speed, distance or time when given the other two by assimilating such questions to this scheme. That is, Pat's goal was that Ann come to understand motion in relation to speed, distance, and time sufficiently well that her ability to solve problems posed within the teaching experiment be a *consequence* of that understanding — as distinct from having the goal that she learn how to solve such problems.

We doubt that teachers need to be able to articulate their schemes explicitly or in principle, as Pat did for this report. However, we think teachers will need to *possess* these schemes — whether by having built them in school or in a teacher preparation program. Moreover, their schemes should not support an understanding of the mathematics curriculum as being composed of prescribed activities, and they should not support an orientation to expressing oneself in the language of procedures, numbers, and operations — what elsewhere we have called a *calculational orientation* (Thompson et al., 1994). Instead, teachers' schemes should support a *conceptual* orientation to teaching mathematics.

> A teacher with a conceptual orientation is one whose actions are driven by:
> - an image of a *system of ideas* and *ways of thinking* that she intends the students to develop,
> - an image of *how these ideas and ways of thinking can develop*,
> - ideas about *features of materials, activities, expositions, and students' engagement with them* that can orient students'

> attention in productive ways (a productive way of thinking is one that is generative of a "method" that generalizes to other situations),
>
> - an *expectation and insistence that students be intellectually engaged* in tasks and activities.
>
> Conceptually-oriented teachers often express the images described above in ways that focus students' attention away from thoughtless application of procedures and toward a rich conception of situations, ideas and relationships among ideas. These teachers strive for conceptual coherence, both in their pedagogical actions and in students' conceptions. As a result, conceptually-oriented teachers tend to focus on aspects of situations that, when well understood, give meaning to numerical values and which are suggestive of numerical operations. (1994, p. 86)

In (Thompson et al., 1994) we discussed a conceptual orientation as expressed in instruction surrounding individual problems. The present article describes a conceptual orientation expressed in instruction surrounding a significant and foundational concept in the middle-school curriculum — rate of change. In both cases the defining characteristic is an intertwining of images and expectations regarding something other than numbers and operations. This resembles Lampert's notion of "intertwining content and discourse"(Lampert, 1990), with the exception that within "content" we include ways of reasoning and imagining that lend coherence to the ideas being developed (Thompson, in press).

Pat's instructional actions were guided by what he understood to be Ann's reasoning in relation to how he imagined she understood the tasks. In this sense Pat's instruction was cognitively-guided. His instruction was guided by his understanding of the image-based reasoning he hoped Ann would develop.

Recent research on cognitively-guided instruction has focused on primary grades, where the content, coming largely from the additive conceptual field (Carpenter & Moser, 1983; Thompson, 1993; Vergnaud, 1982), is largely intuitive for an adult. We suspect researchers will need to address the matter of imagery and image-informed schemes when they consider cognitively-guided instruction in multiplicative conceptual fields (Harel & Confrey, 1994; Vergnaud, 1983; Vergnaud, 1988; Vergnaud, 1994) — areas in which elementary- and middle-school teachers commonly do not possess the schemes we hope students will build (Post, Harel, Behr, & Lesh, 1991; Simon & Blume, 1994).

IMPLICATIONS FOR TEACHER EDUCATION

It seems evident to us that successful conceptual teaching requires, at minimum, that teachers possess the schemes we hope children will build. However, there is little research on teachers' schemes and on teachers' abilities to reason conceptually. The little there is tends to emphasize the mathematics teachers cannot do or cannot explain and not the actual schemes by which they reason. It is essential that there be more research on the variety of teachers' actual schemes in regard to central mathematical ideas if the long-term problem of preparing conceptually-oriented mathematics teachers is to be addressed productively. The few studies that do attempt to account for the schemes behind poor teaching (Boyd, 1992; Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; Norman, 1992; Stimpson, 1992; Thompson, 1991; Thompson & Thompson, 1994) and poor understanding in prospective teachers (Ball, 1990; Even, 1993; Harel & Dubinsky, 1991; Simon, 1993; Simon & Blume, 1994; Zaskis & Khoury, 1994) suggest that making conceptually-oriented teaching occur in even a significant minority of classrooms will be an enormous task.

It is unclear to us what success might be expected reasonably of efforts to transform current teachers' and entering prospective teachers' schemes so that they could support conceptual teaching. Our experience in teacher education, and reports of other programs cited earlier, suggest to us that we must not only ask what level of success might be expected, but also to examine at what cost success might be had.

How can prospective or practicing teachers come to understand a mathematical idea so that they may teach it conceptually? We see teachers gaining this knowledge through sustained and reflective work with students and with mathematical ideas — comparing their attempts to influence students' thinking with disinterested analyses of what those students actually learn — and reflecting on both what they intended and what (they understand) they achieved. In a sense, we are recommending that teacher education turn Cobb and Steffe's (1983) idea of "Researcher as teacher and model builder" into "Teacher as researcher and model builder." We should add that, in our experience, the level of support teachers need to do this far exceeds what most teacher enhancement and teacher education programs provide.

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