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UNDERSTANDINGS OF MARGIN OF ERROR

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The goal of this study was to develop a theoretical framework for understandings of margin of error and to explore eight high school mathematics teachers' understandings of margin of error as they engage in a two-week professional development seminar. Conceptual analysis of the concept of margin of error highlighted that understanding margin of error entails 1) an imagery of repeated sampling, 2) knowing that margin of error has nothing to do with the particular sample statistics. It is, rather, about the sampling method, and 3) understanding the concept of confidence level and how it relates to margin of error. Analysis of teachers' interpretations of margin of error suggested that while teachers made significant progress in developing a scheme of repeated sampling, the idea of margin of error as a characteristic of sampling method, and the idea of confidence level were particularly hard for teachers to understand.

Research Topic

Margin of error is the signature index of sampling variability in poll results that appear in non-technical publications such as newspapers and magazines. Yet it is also one of the least understood statistical concepts by the public. There is abundant confusion in both the lay and technical literature about margin of error (Saldanha, 2003). For example, the writings of ASA (1998) and Public Agenda (2003) misinterpreted margin of error as "95% of the time the entire population is surveyed the population parameter will be within the confidence interval calculated from the original sample". Against this background, the goal of our study was to address a series of interconnected questions: What does it mean to understand margin of error? How do people understand it, and how might we support people's development of a more coherent understanding of margin of error? We view the answers to these questions crucial not only for creating models of understanding margin of error, but also for supporting instructional design intended to promote the learning of margin of error. To tackle these questions, we examined a set of data collected from a professional development seminar that we had conducted with a group of eight high school teachers that aimed to investigate their understanding of probability and statistical inference (Liu & Thompson, 2004).

Background Theories & Methodology

Our study was guided by a radical constructivist perspective on human knowledge and human learning. Radical constructivism entails the stance that any cognizing organism builds its own reality out of the items that register against its experiential interface (Glasersfeld, 1995). As such, in our study that aimed to understand others' mathematical understanding, it is necessary to attribute mathematical realities to subjects that are independent of the researchers' mathematical realities. This is what Steffe meant when he described the researcher' activity in a constructivist teaching experiment as that of performing the act of de-centering by trying to understand the *mathematics of [others]* (Steffe, 1991).

To construct models of others'/teachers' understanding, we adopted an analytical method that Glasersfeld called conceptual analysis (Glasersfeld, 1995), the aim of which is "to describe conceptual operations that, were people to have them, might result in them thinking the way they evidently do." Engaging in conceptual analysis of a person's understanding means trying to think as the person does, to construct a conceptual structure that is isomorphic to that of the person. In conducting conceptual analysis, a researcher builds models of a person' understanding by observing the person' actions in natural or designed contexts and asking himself, "What can this person be thinking so that his actions make sense from his perspective?" In other words, the researcher/observer puts himself into the position of the observed and attempt to examine the operations that he (the observer) would need or the constraints he would have to operate under in order to (logically) behave as the observed did (Thompson, 1982).

Research Design & Data analysis

The seminar, which lasted two weeks, was conducted in the summer of 2001. Table 1 presents demographic information on the eight selected teachers. None of the teachers had extensive coursework in statistics. All had at least a BA in mathematics or mathematics education. Statistics backgrounds varied between self-study (statistics and probability through regression analysis) to an undergraduate sequence in mathematical statistics.

Table 1: Demographic information on seminar participants

Teacher	Years Teaching	Degree	Stat Background	Taught
John	3	MS Applied Math	2 courses math stat	AP Calc, AP Stat
Nicole	24	MAT Math	Regression anal (self study)	AP Calc, Units in stat
Sarah	28	BA Math Ed	Ed research, test & measure	Pre-calc, Units in stat
Betty	9	BA Math Ed	Ed research, FAMS training	Alg 2, Prob & Stat
Lucy	2	BA Math, BA Ed	Intro stat, AP stat training	Alg 2, Units in stat
Linda	9	MS Math	2 courses math stat	Calc, Units in stat
Henry	7	BS Math Ed, M.Ed.	1 course stat, AP stat training	AP Calc, AP Stat
Alice	21	BA Math	1 sem math stat, bus stat	Calc hon, Units in stat

Each session began at 9:00a and ended at 3:00p, with 60 minutes for lunch. All seminar sessions were led by a high school AP statistics teacher (Terry) who had collaborated in the seminar design throughout the planning period. We interviewed each teacher three times: prior to the seminar about his or her understandings of sampling, variability, and the law of large numbers; at the end of the first week on statistical inference; and at the end of the second week on probability and stochastic reasoning. This paper will focus on day 3 & 4, in which we focused on parameter estimation.

Results

Part I: Theoretical framework for understandings of margin of error

Margin of error (for a population with known standard deviation), when centered around a population parameter, yields an interval that captures a certain percentage of sample statistics collected from repeatedly taking samples of a given size. Expressed symbolically, this interpretation is:

$$\textit{The interval } p\pm r \textit{ captures } x\% \textit{ of } s_i, \quad x \in [0,100]. \tag{1}$$

Reciprocally, when margin of error is centered around the sample statistics, it yields confidence intervals *x%* of which contain the population parameter.

$$x\% \textit{ of intervals } s_i \pm r \textit{ contain } p. \tag{2}$$

Although typically, report of margin of error follows a sample estimate of an unknown population, margin of error in fact does not communicate to us how far off that sample statistic is from the population parameter. Rather it tells us that if we were to repeat the same sampling method, a certain percentage of all sample statistics will be within a given range of the population parameter. Therefore, with respect to one particular confidence interval, the best we can say is:

We don't know whether the interval p±r captures s, and (3)
we don't know whether the interval s±r contains p
(but we do know that x% of intervals s_i*±r contain p).* (4)

Understanding of margin of error is not complete until one also understands that

x% is the statistic's confidence level. (5)

In other words, the percentage of sample statistics captured by *p±r* is the confidence level of a sampling method. The combination of interpretations 1&3&5 conveys the definition/ways of thinking about margin of error. The combination 2&4&5 conveys a conventional interpretation/understanding of confidence interval.

Analysis of literature as well as data from the teachers seminar and prior teaching experiments found interpretations or ways of thinking that are incompatible with understanding margin of error. A classic misunderstanding of margin of error is:

The interval s±r contains p. (6)

This interpretation is completely devoid of the idea of confidence level and a distribution of sample statistics. It exhibits a perspective that focuses on the accuracy of one individual sample statistic, and takes the margin of error as a measure of the distance between the sample statistic and the population parameter. Note that (6) is the direct opposite of the idea expressed in (4).

There are three other interpretations that indicate either a lack of or an erroneous understanding of margin of error. One interpretation is:

There is an x% probability that the interval p±r will contain s. (7)

This interpretation is not wrong in itself, but it is vague. "*x%* probability" could mean *x%* of sample statistics, in which case (7) is the same as (1). It could also denote a subjective belief, which means it does not convey a distribution of sample statistics. In this paper, we will remove the ambiguity by assigning a subjective meaning to the word, "probability". That is, if a teacher says (7) but we have evidence that she is thinking (1), and we would assign (1) to her thinking.

The second interpretation is

The interval s±r captures x% of s_i*,* (8)

The interpretation conveys a distribution of sample statistics. However, it says that *x%* of the sample statistics would be captured by the confidence interval constructed from the sample statistics, instead of the confidence interval centered on the population parameter. The difference between (8) and (1) is the center of confidence interval constructed from the margin of error.

The third interpretation is

The interval p±r contains x% of the intervals s_i *±r.* (9)

This interpretation is incoherent because all confidence intervals are of the same width (*2r*). It does not make sense to think that one interval will contain other intervals. Note that the interpretations 1, 2, 8, and 9 are all interpretations of margin of error that contain an image of distribution of sample statistics.

The above interpretations, taken together, constitute a theoretical framework/coding scheme (Figure 1) for understanding teachers' conceptions and interpretations of margin of error.

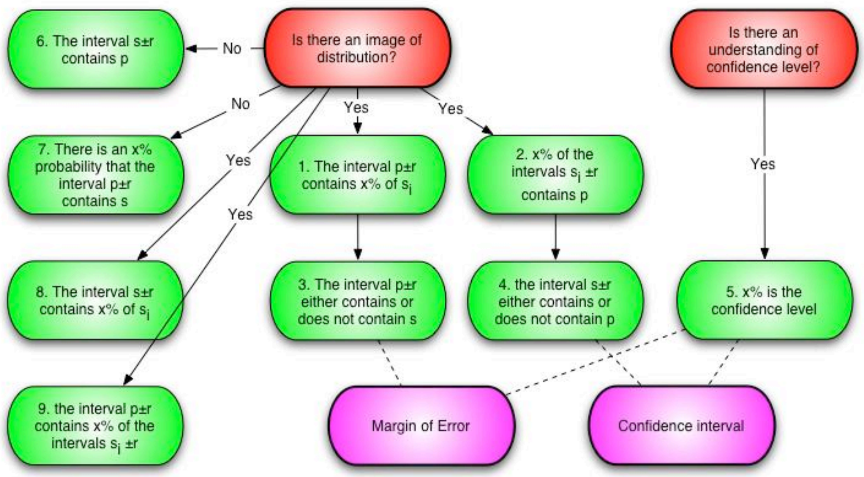

Figure 1: Theoretical framework for understandings of margin of error

Part II: Teachers' understandings of margin of error

In our attempt to explore teachers' understandings of margin of error, we provided the following table of results obtained by resampling samples of size 500 from various populations with known parameters.

Percent of Yes in Population	Number of People in a Sample	Number of Samples Drawn	% of Sample Percents within 1 Percentage Point of Population %	% of Sample Percents within 2 Percentage Points of Population %	% of Sample Percents within 3 Percentage Points of Population %	% of Sample Percents within 4 Percentage Points of Population %
65%	500	2500	36.7%	64.5%	84.8%	91.5%
32%	500	2000	37.1%	65.8%	83.9%	91.1%
57%	500	6800	36.2%	64.9%	84.2%	91.3%
60%	500	5500	36.1%	65.2%	84.3%	91.4%

Afterward, we asked teachers this question:

Stan's statistics class was discussing a Gallup poll of 500 TN voters' opinions regarding the creation of a state income tax. The poll stated, "... the survey showed that 36% of Tennessee voters think a state income tax is necessary to overcome future budget problems. The poll had a margin of error of ±4%." Stan said that the margin of error being 4% means that between 32% and 40% of TN voters believe an income tax is necessary. Is Stan's interpretation a good one? If so, explain. If not, what should it be?

This question queried teachers' understandings of margin of error by having them comment on a particular interpretation of the reported margin of error for a public opinion poll of 500 people. We coined the scenario so that the information on the table could determine the confidence level associated with the scenario's sampling method and the reported margin of error. A "conventional" interpretation of the reported margin of error is: *The margin of error ±4% means that if we were to repeatedly sample 500 TN voters, around 91% of the sample statistics will be within ±4% of the true population proportion. We don't know if 36% is within that range.* The same interpretation expressed with the idea of confidence interval is: *We don't know if the interval 36%±4% will contain the true population proportion, but we do know that if we were to repeatedly sample 500 TN voters, around 91% of the intervals constructed like this will contain the true population proportion.* This question was given as a homework on day 3 of the seminar. Teachers were asked to give a written answer. After a 2-hour discussion on day 4, we asked the teachers to give a second answer to the question.

Teachers' first written answers (Table 2) showed that none of the teachers agreed with Stan's interpretation. Three teachers, John, Betty, and Alice, interpreted the margin of error ±4% as meaning "95% of sample statistics fall within ±4% of the unknown population parameter". Henry believed that the margin of error ±4% meant, "95% of the confidence intervals constructed from this margin of error will contain the unknown population parameter". These two interpretations of margin of error, conveyed by codes 1 and 2, are two coherent interpretations of margin of error, both of which build on an image of a distribution of sample statistics. Nicole had the misconception that the *interval s±4% contains x% of the sample statistics* (code 8). Three teachers, Linda, Lucy, and Sarah, used the word "probability" to relate the sample statistic and the population parameter (code 7). These interpretations of margin of error were not built on an image of a distribution of sample statistics. Although none of the teachers agreed with Stan's interpretation, only one teacher, Henry, explicitly stated the idea that countered Stan's interpretation, that *the interval s±4% does not necessarily contain p* (code 4). With respect to the idea of confidence level, all teachers used the number 95% where they hoped to convey their subjective level of confidence. Only three teachers, John, Sarah, and Linda, stated that the 95% was the "confidence level". None of the teachers utilized the table to infer that the confidence level (standard sense: number of sample statistics that are within the interval $p \pm r$) was 91%.

Table 2: Teachers' initial interpretations of the ±4% margin of error

	1	2	3	4	5[i]	6	7	8	9	1or2or8or9	1&3&5	2&4&5
John	√				*					√		
Nicole								√		√		
Sarah					*		√					
Lucy							√					
Betty	√									√		
Linda					*		√					
Henry		√		√						√		
Alice	√									√		
Counts	3	1	0	1	0	0	3	1	0	5	0	0

Table 2 shows that five teachers' interpretations of margin of error were built on an image of a distribution of sample statistics (code 1or2or8or9). Codes 1&3&5 or 2&4&5 are used to denote two different ways of understanding margin of error that are both coherent and complete[ii]. As we can see from the table, none of the teachers understood margin of error as indicated by either combination.

Teachers' second answers were summarized in Table 3.

Table 3: Teachers' second interpretations of the ±4% margin of error

	1	2	3	4	5	6	7	8	9	1or2or8or9	1&3&5	2&4&5
John	√	√			*					√		
Nicole	√					√				√		
Sarah			√		*	√						
Lucy	√	√								√		
Betty	√	√				√				√		
Linda	√	√		√						√		
Henry	√	√		√						√		
Alice	√									√		
Counts	7	5	1	2	0	3	0	0	0	7	0	0

Table 3 shows that all the teachers, except Sarah, understood the margin of error ±4% to mean "95% of sample statistics fall within ±4% of the unknown population parameter". Five teachers also interpreted the margin of error ±4% as "95% of the confidence intervals constructed from this margin of error will contain the unknown population parameter". None of the teachers used the word "probability" to relate the sample statistic and the population parameter, or had the misconception that *the interval s±4% contains x% of the sample statistics*. All teachers except Sarah had an image of distribution of sample statistics in their understandings of margin of error. Compared to their first written answers, 3 additional teachers, Nicole, Lucy, and Linda, had a coherent image of the distribution of sample statistics and understanding of how it relates to margin of error.

Three teachers, Sarah, Linda, and Henry, stated explicitly that *the interval s±4% does not necessarily contain p*, or *the interval p±4% does not necessarily contain s,* as opposed to only one teacher (Henry) in prior answers. However, a conflicting result was while no teacher agreed with Stan's interpretation in prior answers, three teachers, Nicole, Sarah, and Betty, held the same interpretation as Stan's interpretation this time around.

With respect to confidence level, only John and Sarah mentioned the phrase. Like in the prior answers, all teachers used the number 95% where they needed to convey their confidence level. None of them utilized the table to infer that the confidence level was 91%. As a result, once again none of the teachers had a complete understanding of margin of error.

In the post-interview, we asked the teachers the following question: A Harris poll of 535 people, held prior to Timothy McVeigh's execution, reported that 73% of U.S. citizens supported the death penalty. Harris reported that this poll had a margin of error of ±5%. Please interpret "±5%. How might they have determined this? How could they test their claim of "±5%"? Table 4 summarized the teachers' answers to this question.

Table 4: Teachers' interpretations of the ±5% margin of error

	1	2	3	4	5[iii]	6	7	8	9	1or2or8or9	1&3&5	2&4&5
John	√	√								√		
Nicole								√		√		
Sarah			√									
Lucy	√		√		√					√	√	
Betty	√	√								√		
Linda	√		√							√		
Henry	√	√		√						√		
Alice	√									√		
Counts	6	3	3	1	1	0	0	1	0	7	1	0

As we can see from Table 4, all but two teachers, Nicole and Sarah, understood the margin of error ±5% to mean "95% of sample statistics fall within ±5% of the unknown population parameter". Three teachers, John, Betty, and Henry, also understood the margin of error ±5% as "95% of the confidence intervals constructed from this margin of error will contain the unknown population parameter". Nicole took up again her understanding that *the interval s±r contains x% of the sample statistics*. All teachers except Sarah built their interpretations of margin of error on an image of a distribution of sample statistics. Four teachers, Sarah, Lucy, Linda, and Henry, stated explicitly that *the interval s±4% does not necessarily contain p*, or *the interval p±4% does not necessarily contain s*.

With respect to confidence level, all teachers used the number 95% where they needed to convey their confidence level (Note that the question did not specify a confidence level). Only Lucy explicitly assumed a confidence level of 95% before using it to refer to the percent of samples what are within the interval *p±r*.

Table 5 compared the teachers' interpretations of margin of error in both questions.

Table 5: Comparison of teachers' interpretations of margin of error

Counts	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1or2or8or9**	**1&3&5**	**2&4&5**

Table 2	3	1	0	1	0	0	3	1	0	5	0	0
Table 3	7	5	1	2	0	3	0	0	0	7	0	0
Table 4	6	3	3	1	1	0	0	1	0	7	1	0

Table 5 shows that there was a significant increase in the number of teachers who interpret margin of error coherently (captured by codes 1 and 2). However, there were no significant changes in teachers' understanding of confidence level, and of the idea that *the interval s±4% does not necessarily contain p.*

Conclusion

1) Understanding margin of error entails an image of repeated sampling, and knowing that margin of error, when centered around the population proportion, captures a portion of all sample statistics. Analysis of teachers' interpretations of margin of error showed that more teachers understood this idea towards the end. However, we also found both inconsistencies and instability in teachers' images. 2) Understanding margin of error entails knowing that margin of error has nothing to do with the particular sample statistics. It is, rather, about the sampling method. We found that only a few teachers understood this idea. 3) Understanding margin of error entails knowing that the proportion of sample statistics is the confidence level of the sampling method. It tells us the percent of times we obtain a sample result that is within a certain range of the true population proportion. Results showed that this idea was particularly hard for the teachers to understand.

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[i] We assign √ when an answer indicates that the confidence level is 91%, and assign * when a teacher uses the phrase "confidence level" to refer to the percentage of samples that are within the interval *p±r*.

[ii] By "coherent", we mean understanding margin of error as "95% of sample statistics are within the interval [population parameter ± margin of error]". By "complete", we mean understanding of margin of error that also include an understanding of confidence level, and an understanding that "a particular sample statistic might be one of those sample statistics that are not within the interval [population parameter ± margin of error]".

[iii] In this particular situation, we assign √ only when a teacher explicitly assumes a confidence level when talking about a percentage of samples that are within the interval *p±r*.

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