TEACHERS' UNDERSTANDINGS OF PROBABILITY AND THEIR IMPLICATIONS FOR TEACHER PROFESSIONAL DEVELOPMENT*

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Abstract

Probability is an important idea with a remarkably wide range of applications. However, psychological and instructional studies conducted in the last two decades have consistently documented poor understanding of these ideas among different population across different settings. The purposes of this study are to understand teachers' understandings of probability; and to develop theoretical frameworks for understanding teachers' understandings. To this end, we conducted an eight-day seminar with eight high school statistics teachers in the summer of 2001. The data we collected include videotaped sessions and interviews, teachers' written work, and researchers' field notes. Our analysis of the data revealed that there was a complex mix of conceptions and understandings of probability, both within individual teachers and among the group of teachers, that are often situationally triggered, which are often incoherent when the teachers try to reflect on them, and which do not support their attempts to develop coherent pedagogical strategies regarding probability and statistical inference. The implications of these results include: 1) Understanding probability and teaching effectively entails a substantial departure from teachers' prior experience and their tacit beliefs; and 2) the goal of teachers professional development should be helping the teachers develop understandings of probability and statistical inference as a scheme of interrelated ideas by exerting a great amount of coerced effort in helping teachers develop the capacity and orientation in thinking of a distribution of sample statistics.

Teachers' Understandings of Probability and their Implications for Teacher Professional Development

Probability is one of the most important ideas that we expect students to understand in high school. Since its origin in the mid-seventeen century, probability theory has had an enormous impact on the scientific and cultural development in the human society. The range of its applications spread from gambling problems to jurisprudence, data analysis, inductive inference, and insurance in eighteen century, to sociology, physics, biology and psychology in nineteenth, and on to agronomy, polling, medical testing, baseball and innumerable other practical matters in twentieth century (Gigerenzer, Swijtink et al. 1989). Along with this expansion of applications probability has shaped modern science, transformed our ideas of nature, mind, and society, and altered our values and assumptions about matters as diverse as legal fairness to human intelligence. Given the extraordinary range and significance of these transformations and their influence on the structure of knowledge and power, and on issues of opportunity and equity in our society, the question of how to support people's development of coherent understanding of probability takes on increased importance.

However, the historical development of the theories of probability has been riddled with controversy, which in turn, has shaped the way the curriculum and pedagogy of probability are designed and implemented today in school. The concept of probability is often used to refer to two kinds of knowledge: *frequency-type probability* "concerning itself with stochastic laws of chance processes," and *belief-type probability* "dedicated to assessing reasonable degrees of belief in propositions quite devoid of statistical background" (Hacking 1975 p. 12; Hacking 2001 pp. 132-133). Since 1654, there was an explosion of conceptions in the mathematical community that were compatible with this dual concept of probability, for example, frequentist probability, subjective probability, axiomatic probability, and, probability as propensity (cf. Von Plato 1994; cf. Gillies 2000). Yet, until today, mathematicians and scientists continue to debate and negotiate meanings of probability both for its theoretical implication, and for its application in scientific research. There are subjectivists, e.g., de Finetti, who have said that frequentist or objective probability can be made sense of only through personal probability. There are frequentists, e.g. von Mises, who contend that frequentist concepts are the only ones that are viable. According to Hacking (1975), although most people who use probability do not pay attention to such distinctions, extremists of these schools of theories "argue vigorously that the distinction is a sham, for there is only one kind of probability" (*ibid*, p. 15).

The controversy surrounding the theories of probability presents a difficult question to educators: "What do we teach?" (2003). Current instructional practices have sidestepped this question by simultaneously introducing multiple definitions of probability while ignoring the differences and conflicts among the theoretical underpinnings of these definitions (e.g., name a few stat textbooks here). These practices have resulted in shortsighted design that does not take into account of the consequence of students' learning over the long run, and does not support efforts in making connections between the idea of probability to that of statistical inference.

Since 1960s, there have been abundant research studies conducted to investigate ways people understand probability. Psychological and instructional studies consistently documented poor understanding or misconceptions of these ideas among different population across different settings (Kahneman and Tversky 1973; Nisbett, Krantz et al. 1983; Konold 1989; 1991; Konold, Pollatsek et al. 1993a; Fischbein and Schnarch 1997). The well-known research of Kahneman and Tversky (Kahneman, Slovic, & Tversky, 1982, Kahneman & Tversky, 1972, 1973) suggested that many people used non-standard strategies to reason probabilistically. Later research, such as by Konold and his colleagues (Konold, 1989, Konold et al, 1993) and by Goldstein and Hogarth (1997), suggested that Kahneman's and Tversky might have over-interpreted their data. Kahneman and Tversky seemed to assume that their subjects had a relative frequency meaning for "probability" when it may have been that many of them had a deterministic understanding of events, and that numerical probability simply reflected their degree of belief in the outcome. Konold also discovered that many students were not only reasoning deterministically, they were reasoning about the events portrayed probabilistically as if they were never to be repeated again. As such, subjects assigned probabilities between zero and one to events whose probabilities were, as conceived, either zero or one. Later studies supported their suspicion that this is a widespread conception among people asked to reason about events probabilistically and may have been common among Kahneman's and Tversky's subjects and among subjects in other studies that found seeming incoherence between abilities to calculate probabilities and abilities to reason about events probabilistically (e.g., Fischbein and Gazit 1984).

Despite the rich insights generated from these studies, our understandings of peoples' probabilistic reasoning remain fragmented and incomplete. Particularly, we do not have a theoretical framework for modeling how people think and why they think the way they do. Against this background, we designed and conducted this study. Our goal was to understand *how people understand probability*. Specifically, we hope to

1) construct a description of how people understand probability; and
2) develop a theoretical framework for modeling people' understandings of probability.

To explicate our research purposes, let us first explain what we mean by "understanding" and the method we use in developing descriptions of an understanding. By "understanding" we mean that which "results from a person's interpreting signs, symbols, interchanges, or conversation—assigning meanings according to a web of connections the person builds over time through interactions with his or her own interpretations of settings and through interactions with other people as they attempt to do the same." (Thompson and Saldanha 2002) Building on earlier definitions of understanding based on Piaget's notion of assimilation, e.g. "assimilating to an appropriate scheme" (Skemp 1979), Thompson & Saldanha (*ibid.*) extend its meaning to "assimilation to a scheme", which allows for addressing understanding people do have even though it could be judged as inappropriate or wrong. As a result, a description of understanding require "addressing two sides of the assimilation—what we see as the thing a person is attempting to understanding and the scheme of operations that constitutes the person's actual understanding." (*ibid.*, p. 11)

To construct a description of a person's understanding, we adopt an analytical method that Glasersfeld called conceptual analysis (Glasersfeld 1995), the aim of which is "to describe conceptual operations that, were people to have them, might result in them thinking the way they evidently do." Engaging in conceptual analysis of a person's understanding means trying to think as the person does, to construct a conceptual structure that is isomorphic to that of the person. In conducting conceptual analysis, a researcher builds models of a person' understanding by observing the person' actions in natural or designed contexts and asking himself, "What can this person be thinking so that his actions make sense from his perspective?" (Thompson 1982) In other words, the researcher/observer puts himself into the position of the observed and attempt to examine the operations that he (the observer) would need or the constraints he would have to operate under in order to (logically) behave as the observed did (Thompson 1982).

As a researcher engage in the activity of constructing description of his subjects' understanding, he should in the mean time subject his very activity to examination, i.e., to reflectively abstract (Piaget 1977/2000) the concepts and operations that he applies in constructing descriptions. When the researcher becomes aware of these concepts and operations, and can relate one with another, he has a theoretical framework, which usually opens new possibilities for the researcher who turns to using it for new purposes (Steffe and Thompson 2000). There is a dialectic relationship between these two kinds of analyses—constructing models of a person' understanding and creating a theoretical framework for constructing such models. The theoretical framework and the explanations exert a reciprocal influence upon each other as they are simultaneously constructed. Theoretical framework is used in constructing explanations of understandings. As one refines the understandings, the appearance of the framework changes, as one refines the framework, the understandings may be modified (Thompson 1982).

It is important to note that a theoretical framework does not emerge entirely from the empirical work of trying to understand a person's actions and thinking. It could draw upon theoretical constructs established in an earlier conceptual analysis, or informed by others' work in the existing literature. And most often it is heavily constrained/enabled by the epistemology or background theories that the researcher embraces in his work (e.g. Thompson 1982).

Methodology

In conducting this study, we adopted a modification of constructivist teaching experiment methodology (Thompson 1979; Steffe and Richards 1980; Cobb and Steffe 1983; Hunting 1983;

Steffe 1991; Steffe and Thompson 2000). The constructivist teaching experiment methodology was adapted from the Soviet-style teaching experiment (Kantowski 1977) to serve the purpose of developing conceptual models of students' mathematical knowledge in the context of mathematics instruction.

Radical constructivism entails the stance that any cognizing organism builds its own reality out of the items that register against its experiential interface. As such, it is necessary, in a teaching experiment, to attribute mathematical realities to subjects that are independent of the researchers' mathematical realities. While acknowledging the inaccessibility of the subjects environment as seen from their points of view, constructivists also believe that the roots of mathematical knowledge can be found in general coordination of the actor's actions (Piaget 1971). These assumptions then frame the specific research goals of a teaching experiment as being constructing models of subjects' mathematical realities while inducing changes to their understanding and probe their conceptual adaptability and endurance.

A distinguishing characteristic of the constructivist teaching experiment methodology is that the researchers induce changes in the subjects whose knowledge are to be investigated, whether the teaching episodes are realized with the researcher acting as teacher (Cobb and Steffe 1983; Steffe 1991), or in collaboration with teachers (Cobb 2000). The primary purpose of doing so is for the researchers to experience, firsthand, subjects' mathematical learning and reasoning. Without such experiences, there would be no basis for coming to understand the mathematical concepts and operations they construct or even for suspecting that these concepts and operations may be distinctly different from those of researchers/teachers (Steffe and Thompson 2000). In a teaching episode, the teacher-researcher has the initial tasks of 1) interpreting what he or she sees the subjects doing; 2) attempting to perform the act of de-centering by trying to understand the *mathematics of the children [other]* (Steffe 1991). After gaining experiential acquaintance with and making essential distinctions in subjects' ways and means of operating in domains of mathematical concepts and operations that are of interest, in the next phase, the major goal moves to that of generating and testing research hypotheses. The primary purpose of doing so is to formulate the boundaries of subjects' ways and means of operating by teaching them with the goal of promoting the greatest progress possible in all participating subjects. In the actual instruction, however, the teacher-researcher may be too immersed in interaction to be able to step out of it, reflect on it and take action on that basis. It is, therefore, critical to appeal to an observer of the teaching episode for an alternative interpretation of events, which may help the teacher-researcher both to understand the subjects and to posit further actions (Steffe and Richards 1980; Steffe and Thompson 2000).

As a matter of course, the interactive mathematical communication in a teaching experiment are audio- or video- recorded. Retrospective analysis of the records is a critical part of the methodology. Careful analysis of the audio- or video- tapes offers the researchers the opportunity to activate the records of their past experience with the subjects and to bring them into conscious awareness. It also provides a chance for the researchers to make a novel or an alternative interpretation in terms of their evolving concept of the subjects' mathematics. Through retrospective analysis, the activity of model building is brought to the fore. This effort is equivalent to that of proposing answer to the question "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?"(von Glasersfeld 1995 p. 78) Steffe and Thompson (2000) illustrated the nature of modeling subjects' understanding by comparing it to modeling in science: "As scientists, we want to provide explanations for the phenomena we observe. That is we want to propose conceptual or concrete systems that can be deemed intentionally isomorphic to the systems that generate the observed phenomena"(Maturana 1978).

Design and Implementation

The teachers' seminar was conducted in June 2001. Eight high school mathematics teachers—six female and two male teachers— participated in the seminar. The following table presents demographic information on the eight selected teachers. None of the teachers had extensive coursework in statistics. All had at least a BA in mathematics or mathematics education. Statistics backgrounds varied between self-study (statistics and probability through regression analysis) to an undergraduate sequence in mathematical statistics. Two teachers (Linda and Betty) had experience in statistics applications. Linda taught operations research at a Navy Nuclear Power school and Betty was trained in and taught the Ford Motor Company FAMS statistical quality control high school curriculum.

[Insert Table 1]

We designed the seminar activities and artifacts during the year prior to the seminar in collaboration with a school math teacher (Terry), who subsequently hosted most of the seminar activities and conversations. The seminar discussion progressed over eight sessions in two weeks. Table 2 listed the themes and topics of conversation as it transpired in the seminar.

[Insert Table 2]

The seminar began at 9am each day and concluded at 3pm each day, with a 30- minute lunch break. At the end of each day, the research team met briefly to discuss our observations and suggestions on modification of next days' activities. At the end of the seminar sessions, we also made photocopies of teachers' notes. Each teacher was interviewed 3 times for about 45 to 60 minutes each time. Interviews were conducted once before the seminar and once at the end of each week. We video recorded all interviews and kept record of teachers' work during the interviews.

Data Analysis

The data for analysis include video recordings of all seminar sessions made with two cameras and individual interviews, the teachers' written work, and documents made during the planning of the seminar. The analytical approach we employed in generating descriptions and explanations was consistent with Cobb and Whitenack's (1996) method for conducting longitudinal analyses of qualitative data and Glaser and Strauss' (1967) grounded theory, which highlights an iterative process of generating and modifying hypotheses in light of the data. Analyses generated by iterating this process were aimed to develop increasingly stable and viable hypotheses and models of teachers' understanding.

We began by first reviewing the entire collection of videotaped seminar sessions and interviews. Our primary goal in this process was to develop a rough description of what had transpired in the seminar, and a sense of ways of organizing the data. In reviewing the tapes, we partitioned the sessions into video segments: a segment of video is defined by a) a chronological beginning and end; b) a task/artifact; c) the rationale of task design, or the issues that the task was designed to raise; d) discussion/verbal exchange around the task and the issues that arise. For each session, we wrote a summary of all video segments chronologically. Descriptions of video segments consist of the above four components. For interview data, we organized the summary around the questions being asked, i.e., for each interview question, we juxtaposed teachers' answers and explanations so that it facilitated easy comparison as well as developing an overall sense of teachers' understanding. In general, our interaction with data in this phase can be characterized by the openness in documenting the data. In other words, instead of looking for answers to specific questions from the data, we consistently documented what happened without differentiating certain segments from the others in terms of how well they might reveal teachers' understanding. Our purpose was to develop an overall sense of what had transpired in the seminar sessions and interviews, and to create an organizational structure and brief summaries that would facilitate further analysis of the data.

In the next phase, we review the videotaped seminar sessions and interviews again, this time with the purpose of identifying video segments that seem potentially useful for gaining insight into one or more teachers' understandings of probability and statistical inference. Segments containing direct evidence of teachers' thinking, miscommunication in discussions of problems or ideas, or controversy about mathematical meanings or pedagogical practices were especially significant. We then enriched the first level summary by providing thick descriptions of these segments. Along with these descriptions, we also made observations and developed initial hypotheses of what these segments seemed to have revealed about teachers' understanding.

Differentiated video segments in the past phase are then transcribed. A transcriber working with us produced transcripts of the conversations, and we completed the transcripts by including in them the contextual information employed during the discussion, e.g. activities handouts, video screen shots, sketches, etc.

The unit of transcript analysis was conversation around activity. For each activity, we captured its global structure by parsing the conversation into hierarchies of episodes. The first level episode was defined as conversations around the organizing questions in the activity, and thus can be conveniently named as the organizing question. The second level episodes depict the significant themes of the conversation in the first level episodes. We then took each of these second level episodes as primary unit of annotation—local interpretation. Our primary goal in the annotation was to clarify the meanings of teachers' utterances, i.e., discern from their utterances what they had in mind. For any utterance *x*, the questions we ask include:

- *What motivated a teacher to say x? What was the point the teacher tried to make?*
- *How does it build on this teachers' interpretation of conversation preceding x?*
- *How did the other teachers interpret x?*

The guiding question for making sense of a teacher' utterance is: *What might he have been thinking (or seeing the situation, or interpreting the previous conversation) so that what he said made sense to himself?* An interpretation of teachers' thinking is a conjecture that subjects to further confirmation, refutation, or modification in light of further evidences. Thus, the viability of the interpretations is achieved by triangulation of evidences across the data.

After annotating an episode, we synthesized the conjectures developed during the annotation. These conjectures were then subject to constant comparison and modification with those of the rest of the data. More substantiated conjectures would emerge from this process and serve as potential answers to the research questions. In this phase, the hierarchical structure of conjectures made during transcript analyses became data that were further analyzed (Cobb and Whitenack 1996) and re-organized along three categories of themes: 1) teachers' conceptions of probability; 2) conceptions of hypothesis testing; 3) understanding of variability and margin of error. Within each of these categories, there are sub categories of theoretical constructs that describe/capture the teachers' different conceptions or understanding. As the collection of these constructs emerges and their relationship among one another becomes clearer, they form a theoretical framework that will serve as a basis for constructing narratives of teachers' understanding of probability and statistical inference.

Results

In this paper we focused our analysis on teachers' understandings of probability. In this section, we report the results in three parts. First, we elaborate a powerful conception of probability that underlies the ideas of statistical inference. We place this discussion in the result section because while we embedded the stochastic conception as an instructional goal in our design of the seminar activities, we found the intricacy of the conception and the difficulties teachers have in developing this conception only during, and as a result of our post-analysis of, the seminar. Second, we present the theoretical framework that we have developed as a result of analyzing teachers' conceptions of probability. Growing from yet extending beyond the teacher seminar, this theoretical framework provided a way to model the different possible ways people understand probability. Finally, we describe the teachers' understandings of probability in the context of the seminar activities. Since teachers' emerging ideas and their responses to tasks were coupled tightly with the instruction/conversation that provoked them, we present descriptions of seminar activities along with the results instead of describing instruction separately from presenting the results.

Stochastic Conception of Probability

Statistical inference builds on a stochastic conception of probability. Statistical inference is about inferring a population parameter by taking one sample. In hypothesis testing, an inference is made on the basis of a probabilistic statement about *the relative frequency* of the observed sample over *a large number of repetitions*. In parameter estimation, the probability of a confidence interval containing the true population parameter is the *relative proportion* of all confidence intervals that contains the true population parameter. Hence, understanding statistical inference entails a stochastic conception of probability.

In a stochastic conception, an outcome A's probability being *x* means "an expectation that the long run repetition of the process that produces the outcome A will end with an outcome like A *x* percent of the time." To say an outcome has a probability of *.015* is to say that we *expect* the outcome to occur 1.5 percent of the time as we perform some process repeatedly a large number of times.

A person having a stochastic conception of an event conceives of an observed outcome as but one expression of an underlying repeatable process, which over the long run will produce a stable distribution of outcomes. The conceptual operations entailed in this conception are:

1. Conceiving of a probability situation as the expression of a stochastic process;
2. Taking for granted that the process could be repeated under essentially similar conditions;
3. Taking for granted that the conditions and implementation of the process would differ among repetitions in small, yet perhaps important, ways;
4. Anticipating that repeating the process would produce a collection of outcomes;
5. Anticipating that the relative frequency of outcomes will have a stable distribution in the long run. (Thompson and Liu 2002)

For example,

What is the probability that 18 out of 30 people favor Pepsi over Coca Cola?

To conceive of the underlying situation stochastically entails

1. Conceiving of a random sampling process: selecting a number of people from a population, and asking each person whether he or she favors Pepsi or Coca;
2. Imagining repeatedly taking samples of size 30, and recording the number of people in each sample that favor Pepsi;
3. Understanding that this repeatable process will produce a collection of outcomes;
4. Understanding that because of the *random* selection process there exists variability in the collection of outcomes, but over the long run, the distribution of outcomes will become stable.

Probability of "18 out of 30 people favor Pepsi" is the relative frequency of this outcome within the distribution of outcomes.

The Theoretical Framework

A stochastic conception is a coherent and powerful conception of probability that supports understanding of statistical inference. It is what one might take as an instructional objective when designing teaching of probability. Using the stochastic conception of probability as an overarching organizing construct, we developed a theoretical framework (Table 2 and Figure 1) that, when applied to teachers, will result in descriptions of teachers' actual understandings of probability.

[Insert Table 2 and Figure 1]

[Insert Table 3]

Interpretation 4 (outcome approach) is a subjective conception of probability. One who holds an outcome approach thinks that probability is a subjective judgment based on personal experiences.

Interpretations 5 and 7 both adopt an approach that reduces the sample space, for probability of outcome A, to [A, not A]. Interpretation 7 further applies the principle of indifference—the probability of each outcome is 1/n where n is the number of outcomes. Note that people who hold such interpretations of probability will not be able to make sense of common probabilistic statements, such as, the chance of rain tonight is 40%. They will think instead that the chance of rain is either 1 or 0, or the chance of rain is always 50%.

Interpretation 6 employs what I call proportionality heuristic—evaluating the likelihood of a sample statistic by comparing it against the population proportion, or statistic of a larger sample. For instance, In Example II (Table 4) *what is the probability that 18 out of 30 people favor Pepsi?* A person employing proportionality heuristic could say, "If you were to sample a larger number of people, would you get the same result?" It meant that in this scenario, 18 out of 30 people favor Pepsi; if we take a larger sample, say 90 people, and around 54 out of 90 favor Pepsi, then we would deem the result of "18 out of 30 favor Pepsi" as highly likely. The logic behind this reasoning could be that: "A larger sample is more representative of the population. So if you get the same result in a larger sample, that confirms that the result of the small sample is very likely." The problem of proportionality heuristic lies in the question: What does it mean for two results to be *the same*? Would the two results be the same only when the larger sample has precisely 60% of people preferring Pepsi? What if there is 61% preferring Pepsi? Would that count as the same? If 61% could be counted as same/similar, where does one draw the line? In other words, the difference between proportionality heuristic and a stochastic reasoning is that a stochastic conception of likelihood is quantifiable, but likelihood in a proportionality heuristic is not. The answers to the question "how likely is this result" have to be expressed in qualitative terms, such as, "very likely" or "not likely".

[Insert Table 4]

Interpretations 8 and 9 reveal a relative proportion conception of probability. Within this conception, the probability of an event is the relative proportion of the outcomes making that event compared to all possible outcomes. As we can see from the Example I of Table 4, the results of the interpretations 8 and 9 conflict with each other. This is because interpretation 8 confounded the values of the random variable (sum of the dots on the uppermost faces) with the process' sample space (the set of all possible states in which the process can terminate). Notice that interpretation 9 came from a standard method of computing probability. The limit of this method is that it can be applied only to experiments that have a sample space that can be defined so that all outcomes are equally likely. From this framework, we can see that a person who thinks probability is relative proportion of results might or might not think stochastically, depending on whether or not he/she has the set of conceptual operations listed in 1 to 3.

Teachers' Understandings of Probability

Activity 1: Chance and likelihood

> What does "a 45% chance of rain" mean?
>
> A pollster asked 30 people about which they liked better, Pepsi or Coca-Cola.
> 18 said Pepsi
> How likely is this result? (What does this mean?)

The key to interpret chance and likelihood stochastically is to conceive of a stochastic process that generates a collection of outcomes of which the particular phenomenon in question is but one of those outcomes. For example, "a 45% chance of rain" means "of all those days having the similar conditions like today, 45% of them rain." The stochastic process is to examine the weather conditions of all the past days having the similar conditions as today. Interpreting the Pepsi situation stochastically means conceiving of 1) an underlying population having a relatively stable proportion of people who favor Pepsi and 2) a process of taking random samples of 30 people out of this population. Conceiving of the situation as such allows one to think of the result "18 people out of 30 favor Pepsi" as one of the possible outcomes of the sampling process. The likelihood of this outcome can then be quantified as the relative frequency of outcomes like this one against the total number of times the sampling process is repeated.

Discussion around the first situation shows that three out of eight teachers interpreted the situation non-stochastically. They were concerned, essentially, about *today's* chance of rain.

Betty	[It] means you'll probably wanna take your umbrella.
Linda	I would do it based on what you were going to do that day. If you had an outdoor wedding planned, I'd say there's a good chance… you need to plan something else. But if I was planning for some outdoor sports activity, then I'd probably go ahead and do it. It is a relative—I don't know, It is not a number that really means 45 of anything. It is just…forty-five percent of anything, like if it were a hundred percent chance of rain then you know it is going to rain for sure, if it is eighty-five percent then you're almost sure it is going to rain, ten percent chance, well it probably won't rain so go ahead and do whatever you want that day.
John	There's always a 50% chance of rain, it just so happens that on that day there's a little bit less than a fifty percent chance. It is either it will rain or it will not.

For Linda, a 45% probability of rain conveys the strength of belief about the likelihood of rain that lies somewhere between "definitely going to rain" to "no rain". John's conception of probability closely resembled the "principle of indifference"-"conversion of either complete ignorance or partial symmetric knowledge concerning which of a set of alternatives is true, into a uniform probability distribution over the alternatives" (Fine 1973 p. 167). In John's conception, the probability of rain on any given day is 50% if we remain completely ignorant of the possible conditions or information that might provide evidence for or against any one of the two outcomes [rain] and [no rain].

Only two teachers, Henry and Nicole, Interpreted this situation stochastically:

Nicole If you had 100 days just like today, that in forty-five days it is going to rain in the immediate area.

Henry I think today they run a lot of models, meteorological models…if you had a hundred days just like today, the model says it is going to rain forty-five of those a hundred days.

They understood that the probability of rain on one particular day was calculated on the basis of a collection of days having similar conditions.

The discussion around the second scenario revealed that none of the teachers interpreted the situation stochastically.

John Yes, it is highly likely that it is 18. I would expect this number to be between 10 and 20, I doubt that it would be 0 and I doubt that it would be 29. But, I'm not trying to be funny, I'm being serious, it is either, either you like Pepsi or you don't like Pepsi…. So the other choice would be "not Pepsi", which would be Coca Cola. So it is either Pepsi or it is Coca Cola. And uh…how-- what does that tell us? Eighteen out of thirty, what does that tell me? Well, I can't say that 60% of the nation likes Pepsi because that's not enough data. That's just one sample of 30. But I would expect to see one sample of 30, say, 10 people liking Pepsi, or 20 people liking Pepsi, I would expect to see a lot of things, but I wouldn't expect to see 0 and I wouldn't expect to see 30.

Linda It is about half way, so I would say it is very likely. Because 15 is half, so … I mean Pepsi and Coca Cola are, are always competitive with each other.

Betty As a Coke drinker I would think more people would choose Coke, because I like Coke better and I can't see how anybody can drink Pepsi.

Alice Is that what you would have expected? ... that 18 would have preferred Pepsi.

Lucy Is that a valid result? …like if you were to sample a larger number of people, would you get the same result?

Sarah Would we consistently get that result over and over?

John, Betty, and Linda evaluated the likelihood of "18 out of 30 people favor Pepsi" based on their beliefs of how the population was distributed with respect to their preference. Betty, a Coke drinker, believed more of the population favored Coke, while Linda and John assumed that the population was evenly split. John's justification was, "either you like Pepsi or you don't like Pepsi… it is either Pepsi or it is Coca Cola." The logic of this reasoning is: "Because any particular person either likes Pepsi or doesn't like Pepsi, about half of the population like Pepsi and half don't." The absurdity of this logic can be easily seen by an analogy: "Because you either have AIDS or you don't, about half of the population have AIDS." The fault of this reasoning lies in the fact that one is thinking about individual cases while making statement about a collection. It is a coerced application of the "principle of indifference", always acting in a state of complete ignorance. That is, for any event that has *n* multiple outcomes, the chance of any particular outcome's occurrence in a number of repetitions of this event is always a fixed number *1/n*.

The defining characteristic of these three teachers' reasoning is that they tried to answer the question "how likely is this result?" on the basis of 1) the *particular* result, and 2) their beliefs of how the population was distributed—"I believe the population is distributed in this way. Given what I believe, the likelihood that 18 out of 30 people favor Pepsi is…" This way of reasoning is what I called a proportionality heuristic—evaluating the likelihood of a sample statistic by comparing it against the population proportion, or statistic of a larger sample, as we will see below in Lucy's thinking.

Lucy also applied this heuristic. By saying "If you were to sample a larger number of people, would you get the same result?" what she meant was something like this—in this scenario, 18 out of 30 people favor Pepsi. If we take a larger sample, say 90 people, and around 54 out of 90 favor Pepsi, then we would deem the result of "18 out of 30 favor Pepsi" as highly likely. The logic behind this reasoning could be that: "A larger sample is more representative of the population. So if you get the same result in a larger sample, that confirms that the result of the small sample is very likely." What distinguished Lucy's heuristic from John, Betty, and Linda's was that Lucy conceived of a repeatable sampling process.

The problem of proportionality heuristic lies in the question: What does it mean for two results to be *the same*? In Lucy's case, for example, would the two results be the same only when the larger sample has precisely 60% of people preferring Pepsi? What if there is 61% preferring Pepsi? Would that count as the same? If 61% could be counted as same/similar, where does one draw the line? In other words, the difference between proportionality heuristic and a stochastic reasoning is that a stochastic conception of likelihood is quantifiable, but likelihood in the proportionality heuristic is not. The answers to the question "how likely is this result" have to be expressed in qualitative terms, such as, "very likely" or "not likely".

Sarah's answer "would you get the result over and over" implies that she had conceived of a repeatable process and hinted on the idea of likelihood as long run expectation. However, since she did not elaborate on her thinking, we could not know what she had in mind about the process and how it related to the likelihood of the result.

A discussion occurred later revealed that the teachers' interpretations of the word "likelihood" might have contributed to their interpretation of the situation. They argued that the word *likelihood* did not call for a quantitative interpretation.

Nicole	But "likely" feels like a weaker word than "what percent". "What percent" means to me that I need to do some floundering around to figure out a "mathie"-type answer and um, that's what "what percent" means. "How likely" just was (shrugs)..I mean in that case I wasn't making the assumption that it was 50-50 so it seemed quite likely 'cause I'd been basing it on my um…on a general knowledge… that's, you know, they're sort of equally distra-dist-distributing in Kroger, but I've never stood there that long and counted them up. So, sure!
Sarah	"Likely" is less definitive than percent=
Nicole	=That's right! That's how I felt and so=
Sarah	=and I don't disagree with that, but I'm like Henry, I couldn't come up with a better phrase! So..
Terry	When you think of the word "likely" , what question do you think, when you talk about likelihood, just in general=
Nicole	=what could've happened!?

Terry	And how would you measure "Could've happened"?
Nicole	Well, I wasn't measuring it because I thought that that's what the slide, you know=
Terry	=Ok=
Nicole	=I mean if, if you would ask me, you know, "three percent favor Pepsi, how likely is that result?" Just sort of based on my common knowledge I would've said, "I don't think it is likely, no."
Terry	OK.
Lucy	"Likely" sounds more like it is asking for a question like, for an answer like "fairly-likely" or "not very likely" or something.

This discussion revealed that teachers were wrapped around in the colloquial meaning of the word *likelihood*. They did not equate likelihood to the technical meaning of probability. Likelihood to probability is like, for example, *warmth* to *temperature*. When we ask a question, "How warm is the water?" we expect answers such as, "Yes, very warm", "Not very", or "It is not." When we ask, "What is the temperature of the water?" we expect a measurement of the temperature expressed in quantitative terms, such as "70 degrees". By the same token, teachers thought that the question of "how likely" called for answers such as "fairly likely" or "not very likely". This non-quantitative conception of likelihood is consistent with, and perhaps could account for, the teachers' non-stochastic interpretations of the Pepsi situation.

Table 5 summarized the teachers' interpretations and ways of thinking about these two situations.

[Insert Table 5]

In summary, most teachers had a non-stochastic conception of chance and likelihood. Their interpretations to chance and likelihood situations were subjective, expressed either as purely personal beliefs or as results of a coerced application of the "principle of indifference". They held a non-quantitative meaning of likelihood, in which case likelihood expresses strengths of beliefs about the possibility of chance occurrence, as opposed a mathematical expectations.

Activity 2: Movie theatre scenario

Ephram works at a theater, taking tickets for one movie per night at a theater that holds 250 people. The town has 30 000 people. He estimates that he knows 300 of them by name. Ephram noticed that he often saw at least two people he knew. Is it in fact unusual that at least two people Ephram knows attend the movie he shows, or could people be coming because he is there? (The theater holds 250 people.)

This activity was adapted from Konold (1994). It centered on investigating the question: Is it *unusual* that at least two people Ephram knows attend the movie he shows? Ways of thinking about this question that indicates a stochastic conception of usualness would be to conceive of a scheme of repeated sampling: seeing "Ephram sees *x* people he knows" as a random event and evaluating the likelihood of outcomes "Ephram sees at least two people he knows" against the distribution of a large number of possible outcomes. That is:

1. Assuming that people go to the theatre randomly, i.e., people do not go to the theatre because Ephram is there;
2. Thinking of a collection of nights, when random samples of 250 people from the population of 30000 go to the theatre;
3. Recording the number of people Ephram knows each night;
4. Plot a distribution of these numbers, and calculate the density of "at least 2": the chance of at least two people Ephram knows attend the movie he shows;
5. If the proportion is smaller than 5% (a conventional significance level), then conclude that it would be unusual that at least two people Ephram knows attend the movie.

The discussion around this activity lasted about 106 minutes. Initially, the teachers gave their gut-level answers. All teachers said it was <u>not unusual</u> that Ephram saw at least two people he knew. There was no discussion about what unusual meant. Nor did any teacher give a quantitative measure to unusualness. They seemed to use unusualness to express a subjective feeling about chance occurrences. This scale of unusualness expressed in terms of "somewhat unusual" and "non unusual" paralleled with teachers' non-quantitative conception of likelihood: A question of "how likely is … " has answers such as "not likely", "highly likely", etc.

The teachers set out to investigate whether the event "Ephram sees at least two people he knows" was in fact not unusual, as their gut instincts told them. John proposed the first method—comparing two relative proportions: 1) the proportion of people Ephram knows by name out of the entire population in town, and 2) the proportion of people Ephram knows out of all the people coming to the theatre, i.e. 2/250.

Terry	How would you figure that out?
Henry	Develop a proportion.
Terry	Okay, and how would you develop a proportion?
John	Well, he knows 300 and there's 30,000 people so there he developed a proportion from that. Out of the proportion of 30,000, how many does he know? So basically he knows 1 out of every 100 people.
Terry	Okay
John	So, that's what I did.
John	I set up a proportion and basically got that he knows 1 out of every 100 people=
Terry	Okay and then what did you do with that?
John	So therefore I said he'd know approximately 2.5 people at the movie theater because if 250 come, out of the first 100, he should know 1 person out the next 100 he should know one and of the last 50 he should know half=

John's method was: *Since Ephram knows 300 people out of 30,000 people in his town, it means for every 100 people, he knows 1 person. On any given night he should know 2.5 people out of 250 people who come to the theatre, given that this 250 people is a random sample of 30,000 in his town. Therefore, it is not unusual that he saw in the theatre at least 2 people he knows.* This method employed the proportionality heuristic. John's method suggested his conception of unusualness was about *one particular event*: how likely is it that on one particular night Ephram sees at least two people he knows? This way of conceptualizing unusualness does not quantify unusualness. Rather, it leads to a subjective judgment on the likelihood of the outcome (of Ephram seeing at least two people he knows) in non-quantitative terms—"Yes, it is unusual," or "No, it is not".

The following excerpt revealed Sarah's conception of unusualness.

Sarah	I thought if he had a night where he saw 50 people that he knew, that would be unusual.

…

Terry	… I'm hearing people saying 'one night, just by proportions, 2-3 people that he knows are going to be there and what I'm saying is that that, to me, is not talking about whether that was unusual or not.
Sarah	I think that would be your expectation. Something unusual would have to be something different than that, like a whole herd of people comes in that he knows.
John	We'd have to go back to the assumption=
Terry	Okay, okay stop! Let's not even talk about this situation. Just tell me what your definition of unusual is.
Sarah	Something that does not meet the expected.

Sarah claimed that the event "Ephram saw *a whole herd of* people he knew" was unusual. This was another example of teachers not conceptualizing "Ephram sees x people he knows" as a random repeatable event. While John was fixating at one particular night when x=2; Sarah was fixating at one particular night when x=*a big number*. Neither one of them had conceived of a collection of nights with varying outcomes. Sarah's conception of unusualness was entirely subjective—something is unusual if it is unexpected, and expectations are made on the basis of personal experience.

The following excerpts showed that Linda and Betty's conceptions of unusualness were also subjective.

Linda	Can I draw a different example? …Okay, in a college you expect that for every college algebra class, 30 people are going to sign up. But if one instruction has a 60 person waiting list, that's unusual. Okay? Because you don't see that. There's something going on there. That's unusual.

Terry	What is my criterion for me saying that [something] is unusual?
Betty	It is out of the ordinary. It is out of normal circumstances.

These excerpts suggested that they both had a subjective conception of unusualness. More specifically, Betty exhibited a typical case of someone employing the outcome approach, i.e. an event/outcome is unusual if it is unanticipated to occur.

Only two teachers, Alice and Henry, had a quantitative conception of unusualness.

Terry	Is it unusual that people get struck by lightning?
Alice	[Yes, because] It happens to very few people.
Terry	Okay when you say it doesn't happen very often, what do you mean?

Alice Think about the whole population. Compared to the number of times it happens within the whole population.

Henry There may be a lot of people at the table who have a mathematical definition of unusual, if you want to talk about numbers and problems and so forth, and then you may have a personal definition of unusual. For instance, myself, I tend to think 'well, something's unusual if I'm doing it less that 50% of the time'. That's just sort of my rule of thumb for me as a person. If I'm not doing it 50% of the time or more, then it is unusual. It doesn't occur for the majority. It is less than the majority. It is the minority.

Both Alice and Henry's interpretations of unusualness involved some underlying repeatable process. To Alice, the unusualness of "people get struck by lightning" was measured by the low frequency of its occurrence over a large number of times. To Henry, an event was usual if it was repeated less than 50% of the time.

Table 6 showed that, of the six teachers whose conceptions of unusualness were revealed in the discussion of the movie theatre scenario, all but one teacher had non-stochastic conceptions of unusualness. John applied a proportionality heuristic to "measure" the unusualness, while Sarah, Betty, and Linda had an outcome approach to unusualness. Only two teachers, Alice and Henry, conceived of unusualness as a statistical value.

[Insert Table 6]

Activity 3: PowerPoint presentation

#	Title	Slides	
1	Risky Rides	Your risk of being killed on an amusement park ride? One in 250 million.	What does that mean? How do you suppose they determined these values?
		Your risk of being killed driving home from an amusement park ride? One in 7,000.	
2	Vitamin Use	In a study of over 88,000 women, total folate (vitamin Bc) intake was not associated with the overall risk of breast cancer. However, higher folate intake (or multivitamin use) was associated with a lower risk of breast cancer among women who regularly consumed alcohol.	What does this mean to you? How do you suppose they determined this?
3	Gustav's Bad Luck	Gustav read in Newsweek magazine that drivers with three or more speeding tickets are twice as likely to be in a fatal accident as are drivers with	What does this mean?

		fewer than three tickets.	
		The very next day he received his third speeding ticket.	What does this imply?
4	Car and weather situations	What is the probability that my car is red? What's the probability that the temperature tonight is below 40? What does that mean?	
5		Probability is what we anticipate will happen in the long run.	
6	Rishad's situation	Rishad's sister, Betty, rolled ten sixes in a row while playing a board game.	Rishad: "That is impossible!
			Rishad: "If a billion people rolled a die 10 times, what fraction of them would roll all sixes?"
7		The situation per se is not probabilistic. It is how you conceive the situation that makes it probabilistic.	
8	Rephrase the situations probabilis-tically	What's the probability that the next U.S. Attorney General is a woman? What's the probability that it will snow tomorrow? …the BHS gymnasium ceiling is 30' high? …you are off by no more than 2" when you measure the BHS gymnasium's height? …the Titans go to Super Bowl XXXXV? …you are dealt one pair in a 5-card hand from a standard deck?	
9		Two ways of think about probability: empirical and theoretical. You can determine that either empirically or theoretically. Experiment is the process that's being repeated. Outcome is a state an experiment could end. Event is a collection of outcomes.	

Discussion around the slides 1-3 suggested that Linda conceived of risk/probability as relative proportion of two quantities.

Terry	All right, the risk of being killed on an amusement park ride is one in 250 million. What does this mean to you?
Linda	If you took, it took 250 million park rides for one person to get killed.
Terry	Your risk of being killed driving home from an amusement park ride was 1 in 7000. Ok? How do you suppose they figured that statistic out?
Linda	The actual road deaths and the number of cars that they would estimate going from point A to point B.

Careful examination of Linda's responses suggested that there existed inconsistencies in the way she conceived of both situations. For the first situation, Linda conceived of a repeatable process of people taking park rides. However, the conditions of this process were not specified. Because while the risk in the situation intended to mean, "For every 250 million *people who take park rides*, one person dies", Linda's interpretation was, "For every 250 million *park rides*, one person dies." The same was true for the second situation. Linda conceived of two incompatible collections: actual road deaths and the number of cars. She did not conceive of a stochastic process of sampling from a collection of *people* (who drive home from an amusement park ride) and recording the number of those who were killed in auto accident.

Discussion around the slides 2 and 3 again revealed Linda's conception of probability as relative proportion.

Terry — But apart from the design of the study. Apart from whether there was a control group and all this stuff, just talking about when they talk about overall risk or lower risk, what does that mean?
Linda — Less occurrence of breast cancer out of the total number.

Terry — 'Gustav reads in the Newsweek magazine that drivers with three or more speeding tickets are twice as likely to be in a fatal accident…Ok. What does that mean? What does that statement mean, that people being killed in wrecks have lots of speeding tickets?
Linda — The same number of drivers that had 3 or more speeding tickets, compared to the same number of drivers that had fewer than 3, there was a greater number of those that were killed, in the first group.
Terry — Ok, when you say 'greater number'?
Linda — Out of the same proportion, there's a greater proportion, but if you look at the same number, like 10,000 drivers of each group, there would be more that were killed in the top group than the bottom group.

The highlighted utterances suggested that Linda conceptualized a collection of people (having *x* number of tickets) and a portion of that collection (of people who were in fatal accidents), and she interpreted likelihood (of expected outcome) as the relative proportion of outcomes like this one in a group of all possible outcomes.

With respect to the car situation in Slide 4, Linda believed that the probability is either 1 or 0.

Linda — It is either 0 or 1. Because there's only two outcomes. It either is or it isn't.

In her thinking, there was one particular event (color of my car) with two possible outcomes (red and not red). Thus, the probability that "my car is red" is 1 when the outcome is red, and 0 when the outcome is not red.

Sarah Wouldn't the probability of buying a red car have to do with how many red cars out of how many cars were on the lot?

Sarah's comments suggested that she attempted to interpret the situation in terms of relative proportion. That is, the probability of buying a red car is the relative proportion of the red cars out of all cars [that are for purchase].

Terry, the seminar host, dismissed this interpretation on the ground that the situation under discussion was about *my car*. She emphasized the phrase *my car* and insisted that the car situation was about a single event/object, and argued that it did not make sense to talk about probability in this situation.

Terry Does it even many sense to talk about that as a probability?
John No. It doesn't make sense.
Terry No, it just doesn't even really make sense to talk about it because think about probability as repeating a process, and one of the characteristics being that we don't necessarily know what's going to happen, but once you go out there and see what color my car is, you know. It is not until I buy, you know, if I don't buy a new car, it is going to be the same color. There's no unknown there.

The second question, "What's the probability that the low temperature tonight will be below 40 degrees?" was very similar to the probabilistic statement the teachers had discussed earlier, "Today there is a 40% chance of rain." Nicole, who had interpreted that statement stochastically, had a similar interpretation to the weather situation. However, as we will see in the following excerpt, when Sarah proposed a different stochastic interpretation (different in the sense that Sarah specified a particular collection where Nicole vaguely expressed as "over a long period of time"), Nicole questioned her own interpretation.

Terry What's the probability that the low temperature tonight will be below 40 degrees? What does that mean?
Nicole Well, don't you think that it means that if we had conditions just like today, and the weather conditions generally around here over a long period of time, what's the probability that we'll get a temperature below 40 degrees?
Sarah Or does it mean that on June the 18th, of all the June 18ths how many have we had that were below 40 degrees?
Nicole That's right, is it tonight or is it all of these nights?
…

Nicole   Anyway, it is zero for tonight, folks.

Sarah's mention of "June the 18th"(the day of the discussion), despite the fact that she was thinking of many years of June 18th, made Nicole realize that there could be two ways of interpreting this question. One is non-stochastic, i.e. thinking of the situation as being about one particular event: "The temperature *tonight* is below 40". The probability in this case is, as she said, "It is zero for tonight." The other way of interpreting the question is stochastic, i.e., imaging looking at the night temperature over an extended number of June 18ths.

Henry's observed that the weather situation was interpreted in two ways. One way was to say: tonight temperature is or isn't going to be below 40 degrees; another way was to look at historical data about temperatures of days like today. But the only acceptable interpretation for the car situation was "it is either red or isn't red." Henry raised the question, i.e., essentially, "If there were two ways of interpreting the weather situation (stochastically and non-stochastically), why couldn't the car situation be interpreted stochastically?"

Linda   One has a large number of nights to base your---your statement about tonight, you can base it on a large number of previous nights, but if you talk about your car—if I said something about my car, I wouldn't base it on anything that I've ever owned before. I mean…
Henry   I've got a friend who's owned 50 cars. What's the probability=
John   I see. You've made a good point, Henry. Those are exactly the same question. Because, if the weatherman had to make a forecast about what kind of color your car would be, the weatherman would have the model go and look at the previous cars, just like he goes and looks at previous weather situations, so predict what tonight is going to be like.
Linda   But there's a question about tonight. There's no question about what kind of car she has.
John   I think those are the exact same question=
Linda   There is no, no question. She either has a red car, or she doesn't=

Linda   Maybe the question should be is it determined now. Whether we know it or not is really not the issue, it is is it determined?
Terry   Right. Is it determined what the low temp—that's a good way to put it. It is determined what color my car is. There's no chance there. My car is a certain color. It is not going to change color when I walk out there. Whereas, it is not predetermined, at this moment, what the low temperature's going to be. So there is some chance involved there. So there's an element of—I can't predict what the low temperature is going to be.

Discussion around Henry's question revealed that, while one group of teachers (Henry, John, and Nicole) realized that a probability situation could be interpreted in both ways, another group (Terry and Linda) insisted that the two situations in slide 4 must be interpreted differently. The difference in the two groups' opinions hinged on the question: What was the outcome? To Terry and Linda, the outcomes for the weather situation were the possible temperatures for either tonight, or a collection of nights, but the outcomes for the car situation were "red" or "not red". The epistemological difference, it seems, is that in one case they anticipate variability (temperature) and in the other case they do not (color of car). In the former, they would be asking the same question about *different nights (or different time tonight).* In the latter, they would be asking the same question about *the same car.* The other group, Henry and John, had a different point. The outcomes of weather situation could be either "below 40" or "above 40" when the situation was seen non-stochastically, which was parallel to the non-stochastic interpretation of car situation. By the same token, the car situation could be seen from a stochastic perspective, in which case, the outcomes were colors of a collection of cars (previously owned by a person).

Table 7 summarize teachers' conceptions of probability revealed in this activity.

[Insert Table 7 here]

Interview 3-1: Five probability situations

The following Post-Interview question investigated 1) teachers' interpretations of probability situation, and 2) the extent to which teachers were aware of multiple interpretations of probability situations.

Consider this list of statements:

1. What are the chances that it will snow in Billings, Montana on April 23, 2002?
2. What's the probability that the length of Dean Benbow's driveway is between 29' and 30'?
3. Your risk of being struck by lightning is 1 in 400,000.

4. How likely is it to be dealt one pair in a 5-card hand from a standard deck?
5. What's the probability that you are off by no more than 2" when you measure the length of your driveway?

Tell me about the underlying situation that is being referred to in each statement.

Table 8 summarized the teachers' interpretations of these probability situations.

[Insert Table 8 here]

Looking across the interpretations of each question, we find that the teachers consistently interpreted Statement 4, the situation of "dealing 5 card" stochastically. Majority of the teachers also interpreted Statement 1 and 3 stochastically. Most of the teachers conceived of statement 2 non-stochastically. Only two teachers, Lucy and Henry, interpreted several situations in both ways (highlighted in red, Lucy 2 out of 5, Henry 3 out of 4).

Looking at interpretations across situations given by each teacher: Nicole, Linda, and Alice seemed to have an orientation in conceiving of probability situations stochastically. John, Sarah, Lucy, Betty, each conceived of two situations non-stochastically. Most teachers made statements about whether the situation was or was not probabilistic, as opposed to whether they conceived of the situation as probabilistic. Henry was the only one who consistently tried to interpret all situations in two ways. He acknowledged that he had difficulty seeing the third statement (risk of being struck by lightning) from a non-stochastic perspective. Perhaps it was because had he done so, then the probability would be either 0 or 1, which was in conflict with the given measure of probability1/400000.

Interview 3-4: Gambling

You must make a choice between:

a. Definitely receiving $225,
b. A 25 percent chance of winning $1,000 and a 75 percent chance of winning nothing.

1. Suppose this is a one-time choice. That is, you are presented with these options once and you will never be presented with these options again. Would you choose (a) or (b)? Why?

2. Suppose you are a gambler who will be presented with these same options many, many times. Would you choose (a) or (b)? Why?

This interview question resembles the slide 6 *Rishad's situation* in the PowerPoint presentation in that the distinction between non-stochastic and stochastic interpretations was made clear in the question. Teachers' answers to both questions (Table 9) revealed that they understood the distinction between making a one-time choice and making repeated choices, and they understood that consistently choosing *b* would yield more benefit over the long run. Four teachers made their choices based on this understanding. The other four teachers, while understanding the distinction, nevertheless made their choices based on their personal preference of avoiding risk. It is worthwhile noting that Henry chose to accept risk even in the one-time situation. Perhaps he thought of this one-time situation as one of many one-time situations.

[Insert Table 9]

Summative analysis

Finally we look at each teacher's interpretations of probability across all the problem scenarios.

[Insert Table 10 to Table 17]

These summative tables revealed that five out of all eight teachers (John, Linda, Sarah, Betty, Lucy) had a situational conception of probability. That is, their interpretations of probability vary according to the particularity of the situation or context in which the probability is embedded. Nicole, Henry, and Alice had a stochastic conception of probability. Henry also had a non-situational conception of probability. He was able to distinguish a situation from conceptions of the situation, and offer multiple interpretations to one situation.

Conclusions

Summary

This paper explored the teachers' conceptualizations and interpretations of probability situations as they engaged in designed activities that purported to revealed their understandings of probability. We structured the activities and interview questions around two important ideas: 1) A stochastic conception of probability is one that supports thinking about statistical inference. Thus, we want to know the extent to which the teachers reasoned stochastically, and what difficulties the teachers might have experienced in reasoning stochastically; 2) We wanted the teachers to understand that a situation is what you conceive it to be. That is, a probability situation can be interpreted either non-stochastically or stochastically, depending on how one conceives of the underlying process behind the stated situation. Moreover, when a situation is conceived of non-stochastically, there are multiple ways to interpret what probability is in that situation. So is stochastic interpretation. The multiplicity of these ways of thinking and interpretations were presented in the theoretical framework for probabilistic reasoning. We believe that as teachers of probability, they must be aware of, and be able to control, different interpretations of probability situations so that they will be equipped to understand students' various conceptions of probability.

With respect to the first question - To what extent the teacher reasoned stochastically? - we found that in the beginning of the seminar, most of the teachers had a non-stochastic conception of probability. Specifically, only two teachers interpreted one (out of two) probability situations stochastically. In the beginning of the second week when we began to focus on probability, we saw a turning point during the teachers' discussion on the PowerPoint slide 4: *Rain & Temperature*. This slide presented two probability situations. While one group of teacher consisting of Henry, Nicole, and John argued that the two situations could be interpreted both non-stochastically and stochastically, another group consisting of Terry and Linda insisted that one had to be non-stochastic and the other stochastic. At the end of this debate, there appeared to have a shared understanding among the teachers that a situation is not stochastic (probabilistic) in and of itself, and that it is how one conceives of it that makes it stochastic.

In the post-interview, teachers' interpretations to the five probability situations revealed that most of the teachers only interpreted the situations in one way or the other. Only two teachers, Henry and Lucy, gave both non-stochastic and stochastic interpretations in a number of occasions (Henry 3 out of 4, Lucy 2 out of 5).

Summative analysis of teachers' interpretations across all situations revealed that three teachers, Nicole, Alice, and Henry, had predominantly stochastic conceptions. Sarah had a non-stochastic conception. The remaining teachers' conceptions of probability were situational: Their interpretations of particular probability situations were contingent upon how these situations were stated.

Contributions and Implications

The most salient findings of this study were the theoretical framework that emerged from the analyses of teachers' understandings of probability. This theoretical framework, in comparison to prior relevant research studies, opens up the "black box" of probabilistic reasoning: It explicates what constitutes a coherent and powerful understanding of probability, what are the non-conventional ways of understandings that people might have, and how these various levels of understandings relate to, or develop into, a coherent understanding. This framework provides a tool for understanding and supporting people's learning of probability by allowing us to model their understandings of probability and generate insights about possible instructional interventions to support their learning.

This study also provides a rich description of the kinds of difficulties the teachers experienced in developing coherent and powerful understandings of probability. It also develops, whenever possible, conjectures about what it is that might have hindered the teachers' attempts in doing so. The set of descriptions and conjectures provide an insight into the complexity in understanding probability, and what we should reasonably expect of the understandings of the content knowledge of high school teachers who teach, or are going to teach, probability. In the general population, we should expect a complicated mix of understandings of probability that are often incoherent and highly compartmentalized, which do not support teachers' attempts to develop coherent pedagogical strategies regarding probability.

These theoretical frameworks and our knowledge of the teachers' understandings, together, provide many insights as to how instructions of probability and statistical inference should be designed in future professional development in order to support teachers' learning of probability and statistical inference. For example, we learned that teachers' understandings of probability and statistical inference were highly compartmentalized: Their conceptions of probability were not grounded in the conception of distribution, and thus did not support thinking about statistical inference. The implication of this result is that instructions of probability and statistical inference must be designed with the principal purpose as that of helping the teachers develop understanding of probability and statistical inference that cut across their existing compartments. A powerful conception of probability that supports reasoning in statistical inference built heavily on the conception of distribution of outcomes. To develop a stochastic conception, one has to develop a series of ways of thinking that include 1) conceiving of an underlying repeatable process, 2) understanding the conditions and implementations of this process in such a way that it produces a collection of variable outcomes, and 3) imaging a distribution of outcomes that are developed from repeating this process. This suggests a strategy for instructional design for professional development for probability: Start by engaging teachers in activities that support their building an image of *distribution of outcomes* from a random experiment, and ask probability questions about these distributions. The purpose of these activities is to broach the concept of *probability* and *distribution of outcomes*, and to help teachers developing a [stochastic] conception of probability as long run expectations and probability "regions" as regions of a distribution. Then, engage teachers in actual process or simulation of repeated sampling, and in discussions of features of the resulting distributions of sample statistics. In doing so, we can help teachers developing a stochastic conception of probability in the context of repeated sampling, and building connections among concepts of *probability*, *distribution of sample statistics*, and *p-value,* the ideas essential to statistical inference. Finally, we move on to topics in statistical inference. This strategy, by exerting a great amount of coerced effort in helping teachers develop the capacity and orientation in thinking of a *distribution of sample statistics*, allows them to develop a stochastic/distributional conception of probability, and incorporating the image of distribution of sample statistics in their thinking of statistical inference.

Limitations

In hindsight, there are many limitations of this study, and many of these are unavoidable due to the very nature of the study. Due to the deficiency of any systemic attempt at unpacking teachers' probabilistic understanding in the field of statistics education, this study is highly exploratory. This means two things. First, we must work with a small group of teachers so that we give a fair amount of opportunity for each teacher to reveal their understandings in the seminar. As a result, this small sample size does not support making claims about the prevalence of this study's central findings to a broader population of teachers.

Second, when we designed and conducted the seminar, we did not have as much understanding, as we do now as depicted in the theoretical framework, about what it means to understand probability coherently and how such understandings develop. As a result, on occasions the activities and interviews were not carried out in its optimal sequence or manner. This is suggested by the non-chronological order with which I organized the description of activities and interviews. Some data seemed to be weak or inadequate in hindsight. For example, Post-Interview 3-1 provided an opportunity for us to explore the teachers' stochastic conception of probability. At the time when the interview was conducted, the only distinguishing element that the interviewers believed that separated a stochastic conception from a non-stochastic conception was whether one conceived of a repeatable process for a probability situation. Therefore the questioning stopped once that information was collected. As such, the data turned out to be insufficient in probing how well the teachers understood these repeatable processes and whether they had an image of distributions of outcomes generated from the processes, which we learned at the end of the analysis are important benchmarks for stochastic conception.

Next steps

This study is an early step of a larger research program, which aims to understand ways of supporting teachers learning and their transformations of teaching practices into ones that are propitious for students' learning in the context of probability and statistics. As a precursor, this study has developed initial frameworks for understanding teachers' (in general, learners') understandings of probability. In the mean time, it also opened many doors to many different directions of future research. The most immediate follow-up work would be to refine these frameworks, i.e. to test their viability through working with a broader audience and revising them accordingly. One of the results from this work would be to generate insights about the prevalence of people's particular conceptions and understandings. These results could then inform instructional design of probability.

Although I documented the chronological change of teachers' thinking, in most cases I do not know how these changes occurred. This was partially due to the fact that this study was not designed as a traditional design experiment that takes teacher change as its primary focus. Naturally, one of the follow up studies would be to design a teaching experiment that explicitly focuses on ways of supporting teachers' development of coherent probabilistic understanding, given what we learned in this study about what it means to for them to have such understanding and what difficulties they might encounter in develop this understanding.

Tables and Figures

Table 1. Demographic information on seminar participants.

Teacher	**Years Teaching**	**Degree**	**Stat Background**	**Taught**
John	3	MS Applied Math	2 courses math stat	AP Calc, AP Stat
Nicole	24	MAT Math	Regression anal (self study)	AP Calc, Units in stat
Sarah	28	BA Math Ed	Ed research, test & measure	Pre-calc, Units in stat
Betty	9	BA Math Ed	Ed research, FAMS training	Alg 2, Prob & Stat
Lucy	2	BA Math, BA Ed	Intro stat, AP stat training	Alg 2, Units in stat
Linda	9	MS Math	2 courses math stat	Calc, Units in stat
Henry	7	BS Math Ed, M.Ed.	1 course stat, AP stat training	AP Calc, AP Stat
Alice	21	BA Math	1 sem math stat, bus stat	Calc hon, Units in stat

Table 2. Theoretical constructs in probability framework

Ways of thinking	1	Q1	Is there an image of a repeatable process?
	2	Q2	Are the conditions of the process specified?
	3	Q3	Is there an image of a distribution of outcomes?
Interpretations of probability	4	OA	Outcome approach
	5	ANA	Outcome is A or non A, prob.=1 or 0
	6	PH	Proportionality heuristic
	7	ANA	Outcome is A or non A, prob.=50%
	8	APV	Probability as relative proportion: all possible values of random variables
	9	APO	Probability as relative proportion: all possible outcomes
	10	RF	Probability as relative frequency: distribution of all outcomes

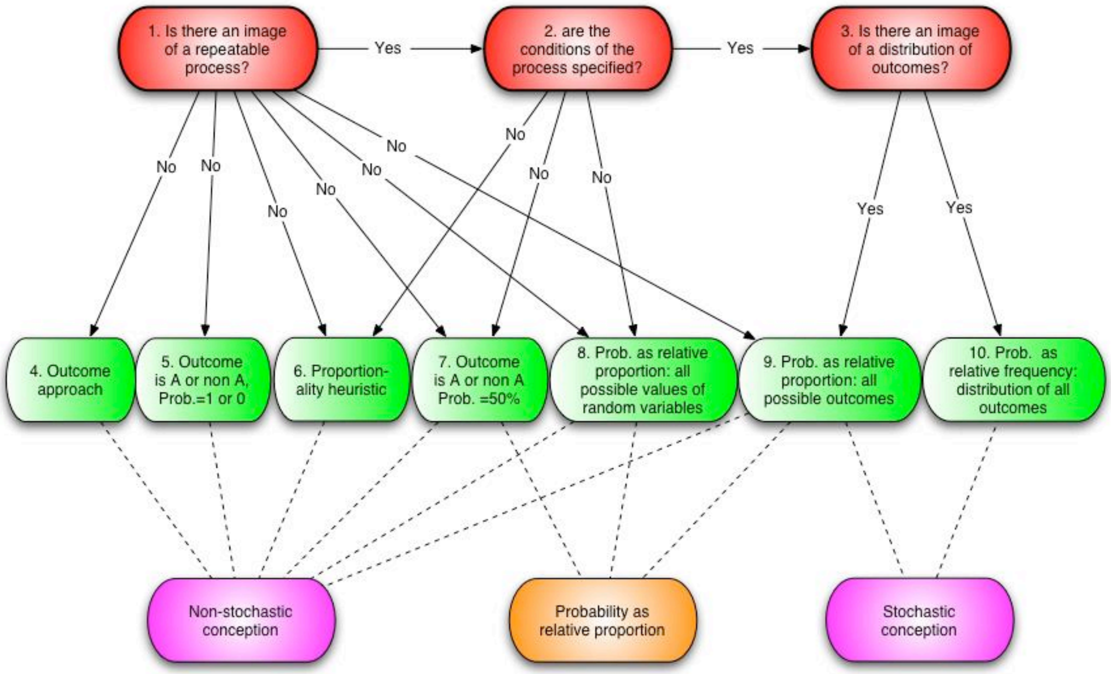

Figure 1: Theoretical framework for probabilistic understanding

Notes: The paths indicated conceptions of probability.
Non-stochastic conceptions: 1-4, 1-5, 1-6, 1-7, 1-8, 1-9, 1-2-6, 1-2-7, 1-2-8
Stochastic conceptions: 1-2-3-9, 1-2-3-10

Table 3. Explanation of Figure 3

Color	Meaning	Corresponding to
Red	Ways of thinking about a probability situation	Box 1, 2, 3.
Green	Interpretations of probability	Box 4 to 10.
Purple	Conceptions of probability	Path (See below)
Orange	A standard method of computing probability	Box 7, 8, 9.

Table 4. Examples of conceptions & interpretations of probability

Path	**Interpretation of Probability**	**Example I:** *What is the probability that I will get a sum of 11 when I throw two dice?*	**Example II:** *What is the probability that 18 out of 30 people favor Pepsi?*
1-4	Outcome approach	11 is my favorite number, so I believe the probability is going to be 80%.	I'm a coke drinker. I think it is unlikely that 18 out of 30 people favor Pepsi.
1-5	Outcome is A or non A, Prob.=1 or 0.	I will either get an 11 or not. The probability is 1 if I do get an 11, 0 if I don't.	The probability is 1 if 18 out of 30 favor Pepsi, is 0 if not.
1-2-6	Proportionality heuristic	N/A	The likelihood of 18/30 is high if a larger sample reflects similar or the same proportion.
1-6	Proportionality heuristic	N/A	A person either likes or doesn't like Pepsi (implying that population parameter is 50%), thus

			the likelihood of 18/30 is high.
1-7 or 1-2-7	Outcome is A or non A, Prob.=50%	I will either get an 11 or not. The probability of getting 11 is 50%.	The outcome is either 18 or not 18, so the probability is 50%
1-8 or 1-2-8	8 All possible values of random variables	There are 11 possible outcomes: 2, 3, … 11, and 12. The probability of getting 11 is 1/11.	N/A
1-9 or 1-2-3-9	9 All possible outcomes	There are 36 possible outcomes: (1, 1), (1, 2)…, and (6, 6). Two of these outcomes (5, 6) and (6, 5) will give the sum of 11. The probability of getting 11 is 2/36.	There are 31 possible outcomes. The probability is 1/31.
1-2-3-10	Probability as relative frequency	If I repeatedly throw two dice, what fraction of the time will I get a sum of 11?	If we repeatedly take samples of 30 people, what fraction of time will get results of 18 people favoring Pepsi?

Table 5. Teachers' conceptions of probability situations in Activity 1

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1Q1	John	N						Y			
	Nicole	Y	Y	Y							Y
	Betty	N									
	Linda	N			Y						
	Henry	Y	Y	Y							Y
D1A1Q2	John	N					Y	Y			
	Nicole	N			Y						
	Sarah	Y	N								
	Lucy	Y	N				Y				
	Betty	N					Y				
	Linda	N					Y	Y			

Instruction for reading the table—The first column "locator" tells where an interpretation is located. For example, "D1A1Q2" means "Day 1, Activity 1, Question 2". Columns numbered 1-10 are theoretical constructs of the probability framework. Detailed description is provided in Chapter 5. "Y" and "N" means "Yes" and "No". Examples: 1) the "Y" in column 1 and row 4 means "For Activity 1 Question 1, Nicole conceived of a repeatable process." 2) the "Y" in column 7 and row 3 means "John's interpretation of probability, as shown in Activity 1 Question 1, is "probability of an event A is 50% because there can be only two outcomes, A or non A." Codes across a row provide the information about a path. For example, row 3 denotes John's non-stochastic conception, path 1-5. Codes across a column tells the number of instances in which teachers exhibits a particular way of thinking. For example, column 1 tells the number of instances the teachers did or did not conceive of a repeatable process.

Table 6. Teachers' conceptions of probability situation in Activity 2

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D3A2	John	N					Y				

	Nicole										
	Sarah	N			Y						
	Lucy										
	Betty	N			Y						
	Linda	N			Y						
	Henry	Y	N								Y
	Alice	Y	Y	Y							Y

Table 7. Teachers' conceptions of probability situations in Activity 3

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D5A3S1	Linda	Y	N							Y	
D5A3S2	Linda	N								Y	
D5A3S3	Linda	N								Y	
D5A3S4Q1	John	Y	Y	Y							Y
	Nicole	Y	Y	Y							Y
	Linda	N				Y					
	Henry	Y	Y	Y							Y
	Terry	N				Y					
D5A3S4Q2	John	Y	Y	Y							Y
	Nicole	Y	Y	Y							Y
	Linda	Y	Y	Y							Y
	Henry	Y	Y	Y							Y
	Terry	Y	Y	Y							Y

Table 8. Teachers' conceptions of probability situations in Interview 3-1

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
I3-1Q1	John	N				Y					
	Nicole	Y	Y	Y							Y
	Sarah	Y									
	Lucy	N									
	Betty	Y	Y								
	Linda	Y	Y	Y							Y
	Henry	Y	Y	Y							
	Henry	N				Y					
	Alice	Y	Y	Y							
I3-1Q2	John	N				Y					
	Nicole	Y									
	Sarah	Y				Y					
	Lucy	N				Y					
	Betty	N				Y					
	Linda	N				Y					
	Henry	N				Y					
	Henry	Y	Y								
	Alice	Y	Y	Y							Y
I3-1Q3	John	Y	Y	Y							Y
	Nicole	N			Y						
	Sarah	N									
	Lucy	N									
	Lucy	Y									
	Betty	Y	Y	Y							Y
	Linda	Y	Y	Y							Y

	Henry	Y									
	Alice	Y	Y	Y							Y
I3-1Q4	John	Y									
	Nicole	Y	Y								
	Sarah	Y	Y	Y							Y
	Lucy	Y									
	Betty	Y	Y	Y							Y
	Linda	Y	Y	Y							Y
	Henry	Y	Y	Y							Y
	Henry	N				Y					
	Alice	Y	Y	Y							Y
I3-1Q5	John	Y									
	Nicole	Y	Y	N							
	Lucy	N									
	Lucy	Y									
	Betty	N				Y					
	Alice	Y	Y	Y							Y

Table 9. The choices teachers made in Interview 3-4

Name	1	2
John	*a*	*b*
Nicole	*a*	*b*
Sarah	*a*	*a*
Lucy	*a*	*a*
Betty	*a*	*a*
Linda	*a*	*b*
Henry	*b*	*b*
Alice	*a*	*a*

Table 10. John's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1-2Q1	John	N						Y			
D1A1-2Q2	John	N					Y	Y			
D3A1-6	John	N					Y				
I2-3	John	Y	Y	Y							Y
D5A2-2S4Q1	John	Y	Y	Y							Y
D5A2-2S4Q2	John	Y	Y	Y							Y
I3-1Q1	John	N				Y					
I3-1Q2	John	N				Y					
I3-1Q3	John	Y	Y	Y							Y
I3-1Q4	John	Y									
I3-1Q5	John	Y									
D6A2-4	John	N				Y					

Table 11. Nicole's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1-2Q1	Nicole	Y	Y	Y							Y
D1A1-2Q2	Nicole	N			Y						
I2-3	Nicole	Y									
D5A2-2S4Q1	Nicole	Y	Y	Y							Y
D5A2-2S4Q2	Nicole	Y	Y	Y							Y
I3-1Q1	Nicole	Y	Y	Y							Y
I3-1Q2	Nicole	Y									
I3-1Q3	Nicole	N			Y						
I3-1Q4	Nicole	Y	Y								
I3-1Q5	Nicole	Y	Y	N							
D6A2-4	Nicole	Y	Y	Y							Y

Table 12. Sarah's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1-2Q2	Sarah	Y	N								
D3A1-6	Sarah	N			Y						
I2-3	Sarah	Y	Y	N			Y				
I3-1Q1	Sarah	Y									
I3-1Q2	Sarah	Y				Y					
I3-1Q3	Sarah	N									
I3-1Q4	Sarah	Y	Y	Y							Y
D6A2-4	Sarah	N									

Table 13. Lucy's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1-2Q2	Lucy	Y	N				Y				
I2-3	Lucy	Y	Y	Y							Y
I3-1Q1	Lucy	N									
I3-1Q2	Lucy	N				Y					
I3-1Q3	Lucy	N									
I3-1Q3	Lucy	Y									
I3-1Q4	Lucy	Y									
I3-1Q5	Lucy	N									
I3-1Q5	Lucy	Y									
D6A2-4	Lucy	Y	Y	Y							Y

Table 14. Betty's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1-2Q1	Betty	N									

D1A1-2Q2	Betty	N					Y				
D3A1-6	Betty	N			Y						
I2-3	Betty	Y	Y	N			Y				
I3-1Q1	Betty	Y	Y								
I3-1Q2	Betty	N				Y					
I3-1Q3	Betty	Y	Y	Y							Y
I3-1Q4	Betty	Y	Y	Y							Y
I3-1Q5	Betty	N				Y					
D6A2-4	Betty	Y	Y	Y							Y

Table 15. Linda's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1-2Q1	Linda	N			Y						
D1A1-2Q2	Linda	N					Y	Y			
D3A1-6	Linda	N			Y						
I2-3	Linda	Y	Y	Y							Y
D5A1S1	Linda	Y	N							Y	
D5A1S2	Linda	N								Y	
D5A1S3	Linda	N								Y	
D5A2-2S4Q1	Linda	N				Y					
D5A2-2S4Q2	Linda	Y	Y	Y							Y
I3-1Q1	Linda	Y	Y	Y							Y
I3-1Q2	Linda	N				Y					
I3-1Q3	Linda	Y	Y	Y							Y
I3-1Q4	Linda	Y	Y	Y							Y
D6A2-4	Linda	Y	Y	Y							Y

Table 16. Henry's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1-2Q1	Henry	Y	Y	Y							Y
D3A1-6	Henry	Y	N								
I2-3	Henry	Y	Y	Y							Y
D5A2-2S4Q1	Henry	Y	Y	Y							Y
D5A2-2S4Q2	Henry	Y	Y	Y							Y
I3-1Q1	Henry	Y	Y	Y							
I3-1Q1	Henry	N				Y					
I3-1Q2	Henry	N				Y					
I3-1Q2	Henry	Y	Y								
I3-1Q3	Henry	Y									
I3-1Q4	Henry	Y	Y	Y							Y
I3-1Q4	Henry	N				Y					

Table 17. Alice's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D3A1-6	Alice	Y	Y	Y							Y
I2-3	Alice	Y	Y	Y							Y
I3-1Q1	Alice	Y	Y	Y							
I3-1Q2	Alice	Y	Y	Y							Y
I3-1Q3	Alice	Y	Y	Y							Y
I3-1Q4	Alice	Y	Y	Y							Y
I3-1Q5	Alice	Y	Y	Y							Y

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