Advances in Research on Quantitative Reasoning

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It is indeed a pleasure to comment on the chapters in this section. They each provide an advance in thinking about the importance of students' quantitative reasoning for learning mathematics. While the chapters report work on a variety of topics, there is one theme that is common to each—the central role of quantification in students' development of productive mathematical thinking.

As I said in (Thompson, 2011), to quantify an object's attribute goes hand in hand with conceiving the object and attribute so that the attribute is quantifiable. Put another way, quantifying an attribute entails a dialectic among conceiving the object, conceiving an attribute of it, and conceiving a method to measure that attribute. In all cases that I've seen, the "object and its attribute" that is being quantified entails a system of relationships and interactions, and the motive for quantification is to solve a problem. The problem is to answer the question of "how much" of something there is in a way that is coherent with the quantity's conceptual milieu and which produces "good" measurements. A good measurement is one that produces measures that are consistent with the nature of what is being measured. Many students think that an angle measure measures the area "trapped" by the sides of an angle, where "sides" are segments, not lines. This is not a good quantification of angle measure. One problem with it is that extending the segments changes the "trapped" area, and hence the angle's measure, contradicting the fact that it is still the same angle and should have the same measure.

The dialectic among object, attribute and quantification is captured with remarkable clarity in Klein's *The World of Measurements* (Klein, 1974). Klein could have titled his book *The World of Quantification* without loss of meaning. For example, in Klein's telling, the concept of specific gravity arose from King William II's edict to tax alcoholic beverages according to their alcohol content (Klein, 1974, pp. 544-555). Initial methods to measure alcohol content were for the tax assessor to pour a small amount of a beverage on a wooden bench and gauge the amount of alcohol in it by the beverage's stickiness. The assessor determined "stickiness" by sitting on the bench with leather breeches and then standing up (Klein, 1974, p. 547). Stakeholders' desire for more accurate and objective measures of alcohol content is understandable. Addressing this need led to refinements in the concept of alcohol content, which led eventually to concepts of mixture, concentration, density, and specific gravity.

Another example from Klein is that of James Watt's attempt to convince mill and mine owners to replace their horses (which turned a turnstile to power their operations) with his steam engine. For this argument Watt needed a way to predict the cost of using horses in comparison with the cost of using his machine to produce the same amount of milled grain or mined ore. In this effort Watt created the concept of accomplished work, and its quantification as force times the distance over which the force is applied, to argue that his machine would be more cost effective than horses (Klein, 1974, pp. 237-238).

A third example of the quantification process is one that spans centuries. Newton's concept of mass was that it is a measure of an amount of matter. This way of thinking about mass presumed that one has a concept of matter, the "stuff" of which there is an amount. In Mashood's (2010) account, the concept of matter arose originally in Aristotle from his commitment to the idea that in growth and decay there is an invariance of something. His concept of matter, together with his distinction between form and substance, resolved the question, "What is the 'something' that is conserved within the processes of growth and decay?" According to Mashood, Aristotle's way of thinking about matter became central to theological explanations of transubstantiation—how bread became the flesh of Christ. Scientific treatments of mass (“amounts of matter”) arose first in Galileo’s observation that objects resisted change in motion (they have the property of inertia) and that inertia is proportional to the object’s amount of matter. Newton clarified the concept of mass by stating that an object’s mass is the number of elementary particles composing it, and used a pendulum to measure an object’s inertia, which he presumed to be proportional to the object’s mass (Brown, 1976, p. 375). Thus, the concept of mass evolved from Aristotle’s positing the existence of matter to resolve his need to distinguish form from substance in accounting for growth and decay, through theological needs to account for transubstantiation, to Newton’s concept of atomic nature of matter and quantification of its amount.

Before commenting on this section’s chapters I must highlight an important difference between quantification in the physical sciences and quantification in mathematics. With regard to physical quantities, scientific ideas and mathematical ideas have essentially the same origins and the same development. However, the role of quantity in mathematical thinking departs from its role in science thinking once one has mental operations that constitute the quantity and its quantification. In science, once you have a model of a physical system that entails its quantification, you use the model to delve further into the physical system in order to understand it better. In mathematics, once you have a model of a physical system that entails its quantification, you begin to explore properties of the model itself, its structure, ways in which the model can be generalized, and implications for its mathematical properties of loosening or tightening its assumptions. The physical system that gave rise to the model fades over time. In this sense, the systems addressed in this section are in some instances foundational to understanding physical quantities and are in some instances beyond physical quantities.

I offered the three examples above to emphasize the point that quantification is nontrivial and requires as much attention to what is being quantified as to how to quantify it. Each chapter in this section unpacks in unique ways the dialectic among conceiving an object, conceiving its attribute, and conceiving the attribute's quantification. Each chapter contributes to our understanding of quantification and its implications for learning in important ways. In the following pages I discuss aspects of the chapters that struck me as supporting a coherent narrative about research on quantitative reasoning. Because of limited space (and time) I am forced to choose issues raised and covered by these chapters eclectically. Whether the result is coherent is for you to judge.

Ulrich addresses students' construction of directed change in a quantity's value and the reconstruction of arithmetic to include operations on changes. "Directed change" becomes the object, magnitude in a direction its attribute, and signed difference its quantification. Arithmetic on those quantities must be reconceived to account for their nature – directed changes are no longer cardinal amounts; they are changes in amounts. Combining directed changes means to combine transformations, and a sum is a net transformation instead of a total amount. The integer $^{-}27$, as a number in a number system, must eventually be conceived as an equivalence class of changes, and arithmetic of integers will be conceived as arithmetic of equivalence classes in exactly the same way that the arithmetic of vectors is conceptualized as an arithmetic of equivalence classes. Ulrich points out with great insight that the statements (a) "I added" and (b) "There is a change of 2 from A to B" have different entailments. In (a), the student is an actor on numbers. In (b) the student is an observer of quantities and relationships among them. She also makes clear that the difference between "take away 20" and "add negative 20" is that "20" has different meanings. In the former it is a cardinality. In the latter it is a transformation.

Tillema addresses students' construction of an object that we might call an "outcome space" and their quantification of it is by structured counting. In a sense, structured counting is at the root of conceiving an outcome space at the same time that it provides a quantification of it. Tillema offers a theoretical framework for the emergence of structured counting that extends Steffe's theory of nested sequences. His framework accounts for the creation of two-dimensional objects from one-dimensional objects as well as for the quantification of the two-dimensional object. He explains that creating one ordered pair as a unit involves disembedding one unit from each of two collections and combining them to make one unit from two. When students anticipate applying this process iteratively, by disembedding one unit from one collection and distributing it combinatorially across the units of another collection, they anticipate producing one "row" of an $m \times n$ matrix, where the unit of each row is an ordered pair. Tillema uses his framework to demonstrate the effort required of middle school students to develop and coordinate the disembedding operations and combining operations to create a conceptual matrix. I am reminded that Piaget considered logical "and" to be the core of multiplicative reasoning—that a person take two things simultaneously so they become one thing (Bringuier, 1980; Inhelder & Piaget, 1969). Tillema's framework for explaining students' quantification of a Cartesian product of finite sets seems ready-made for research on students' probabilistic reasoning, and also seems promising for insights into student's understandings of coordinate planes and graphs in them.

Moore argues that students' construction of angles and angle measures, if addressed as a problem of quantification, can provide coherence to their thinking in all aspects of trigonometric reasoning. He argues convincingly that coherence as a property of a student's mathematical understandings emerges from a fit amongst mathematical meanings. In the case of trigonometry, Moore argues that if a students' meaning for angle measure is a quantification of an angle's "open-ness", and the quantification is the relative arc length that the angle subtends on a circle, then both degree and radian measures of an angle fit as seamlessly in meaning as does metric and English measures of a table's width. The same attribute of the same object is simply measured in different units. Moore also demonstrates how this meaning of angle measure supports both triangle and unit-circle trigonometry and supports extending triangle trigonometry to the concept of trigonometric function. The coherence of trigonometry rests completely on a particular quantification of an angle's open-ness. Moore demonstrates the power of thinking this way in one student, and demonstrates the difficulties encountered by another student that are due to her meanings of angle measure and trigonometric ratio being incompatible with ideas of trigonometric functions.

The Johnson, Castillo-Garsow, and Ellis *et al.* papers, taken together, form an interesting study of variation and covariation of quantities' values. Castillo-Garsow's and Ellis *et al.*'s studies take variation and covariation as foundational concepts to further examine students' thinking in regard to geometric and exponential growth.

The studies by Ellis *et al.* and Castillo-Garsow form an interesting contrast. Both studies investigate students' emerging conceptions of exponential growth, but the underlying contexts seem to support different conceptual developments of smooth continuous covariation. Castillo-Garsow focused on the value of bank accounts over time according to different banks' compounding policies. Ellis *et al.* focused on the height of a Jactus plant that grew exponentially over time. In the bank context, it was evident to students that an account had a value at every moment in time and that within a compounding period the account's value grew linearly. Thus, it was within students' capability to evaluate account value as a function of time at any particular value of elapsed time since the account opened. Their understanding of the value at endpoints of the compounding period was also computable because the number of completed compounding periods was a whole number. Their major realization was that if annual interest accumulates at a rate of rP per year, then the interest rate is $\frac{r}{n}P$ per $1/n$ year (i.e., that in $1/n$ year, you will accumulate $1/n$ the interest normally accumulated in a year). In other words, students had the opportunity to think smoothly about a quantified account value with respect to time. In the Jactus context, students anticipated that the Jactus would have a height at every moment in elapsed time, but I suspect they felt no opportunity to think smoothly about its *quantified* height in relation to elapsed time. By quantified height I mean a height that grows exponentially as time varies smoothly.

When Ellis *et al.* asked for heights between two successive integral time periods, the more advanced students thought in terms of partial factors (Strom, 2008), while one student thought in terms of linear interpolation—which is precisely the kind of reasoning that Castillo-Garsow intended to support in his bank contexts. The partial-factor way of thinking allows one to think about *specific* values of the Jactus' height at specific times, but it does not support the idea of smooth variation. It seems that "smoothness" entered students' thinking about the Jactus plant's height when they *stopped* thinking about constant ratios of successive values and created a process view of b^t . That is, constant ratio and repeated factors provided a conceptualization of exponential function, but seemed to hinder students' thinking about time varying smoothly. When students thought in terms of constant ratio, it was always in the context of finding specific values that would satisfy the geometric sequence created by introducing partial factors to account for Jactus' heights at evenly-spaced intervals between two specific times. This is similar to thinking of x in $3x+7=12$ as standing for a single number. The "single number" meaning of x in $3x+7=12$ is different from imagining that the value of x varies through its domain and having a bell ring at the moment x takes on a value that makes the statement $3x+7=12$ true. The parallel with the Jactus context is for students to have confidence that b^t is a meaningful number for every value of t as it varies. Students seemed to gain this process view of b^t after they had become confident that they understood what b^t meant for arbitrary, non-integral values of t even if they didn't actually calculate it.

On the other hand, Castillo-Garsow's approach emphasized smooth thinking from the start (though Tiffany thought chunkily throughout) and the idea that rate of change in account value with respect to time at any moment is proportional to the account's value at that moment. This approach was at the expense of highlighting the aspect of geometric growth and repeated growth factors over time that Ellis' approach entailed. Repeated factors and growth factor were "there" in Castillo-Garsow's approach, but they were not the focus. To me, it is an open question as to the tradeoffs of each approach with regard to the ultimate goal that students conceptualize $f(x) = b^x$, $b > 0$, $b \neq 1$, as a continuous function with all its entailments. I look forward to more on this from Ellis *et al.* and Castillo-Garsow.

Johnson's study of covariation of quantities sheds some light on the issues of variation that were central to Ellis *et al.*'s and Castillo-Garsow's studies. Johnson found, as had others (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998), that students conceptualized the joint accumulation of volume and height by first coordinating their images of small changes in one quantity with associated changes in the other. Her account of students' thinking while trying to imagine a graph of the water height in the bottle in relation to volume of water in a bottle as the bottle filled reminded me of Newton's approach to variation and rate of change. He imagined quantities as "flowing" in relation to conceptual time. Not time on a clock, but an imagined, smoothly changing, quantified time—a measured duration that grows in extent. In modern notation, we would write $x = x(t)$. Since quantities flowed with respect to conceptual time, their values' rate of change, which he denoted $\dot{x}$, was with respect to conceptual time. In modern notation, we would write $\dot{x} = \frac{d}{dt}x(t)$. The rate of change of volume V with respect to height h is, in Newton's notation, $\frac{\dot{y}}{\dot{x}}$. Thus, students in Johnson's study seemed to be constructing the joint accumulation of water's volume and height in a bottle as $(V,h) = \big(V(t),h(t)\big)$, where t is time.

Notice that I did not say that t in $(V,h) = \big(V(t),h(t)\big)$ was *conceptual* time for Johnson's students. Rather, it seemed that they began by placing the variation in experiential time—the felt time that passed as they imagined the bottle filling. It seems that they made progress in conceptualizing covariation as they made time more conceptual, which then allowed them to suppress it because they no longer, without awareness, incorporated time into the joint accumulation of height and volume. Perhaps in becoming more aware of time as a conceptualized, measured quantity they could consciously separate time from their considerations of the relative changes in volume and height.

The connection between Johnson's study and the studies by Ellis *et al.* and Castillo-Garsow is that perhaps the students struggling with chunky thinking were not operating in conceptual time. As Castillo-Garsow suggested, to operate with change happening in conceptual time, one must extract time from change, so that change happens in relation to time as opposed to happening because of the passing of time. It could be that for someone to imagine change happening smoothly, they must have conceptualized time as passing smoothly and changes happening in relation to smooth-changing, measured, conceptual time. I am the first to admit that this hypothesis is, at this moment, entirely speculative. But it seems like a worthwhile area of research on the development of quantitative reasoning.

In closing, I commend the authors of this section's chapters on pushing forward with important, interesting, and promising areas of research. They show clearly the promise of research on quantitative reasoning as a foundational component of students' learning of important mathematical ideas and contributing to their capacity to participate in important mathematical practices.

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