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Collaboration between Mathematics and Mathematics Education

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Abstract

Our chapter is in four sections. Michèle Artigue tells the story of her transition from mathematical logic to mathematics education and of collaborations at a wide variety of institutional levels. Günter Törner gives a history of collaboration between mathematics and mathematics education in Germany along with a list of recommendations to foster collaboration. Ehud de Shalit shares lessons learned from personal experiences collaborating in the production of a math fair and in the design of a mathematics education major. Pat Thompson tells of several collaborative efforts at his home institution and examines ways that mathematics education contributed mathematically to them. A concluding section provides a reflection on our charge – structural and cultural issues involved in collaborations between mathematics and mathematics education.

Keywords: collaboration, constraints, affordances, mathematics, mathematics education

Introduction

The editors of this book asked our group to address the matter of collaboration between mathematics and mathematics education. For some time we debated whether to change our charge so that it referred to people rather than to disciplines—collaboration between *mathematicians* and mathematics *educators*. We finally decided there was much wisdom in the organizers' original charge. We therefore attempted to focus on aspects of the disciplines and their organizations that might lend to collaboration among the people populating them.

Section 1: Collaboration between mathematics and mathematics education

Michèle Artigue, Université Paris Diderot – Paris 7, France

Collaboration between mathematics and mathematics education is first collaboration between individuals who belong to the corresponding communities or navigate at their interface. Preparing my contribution on this theme has given me an opportunity for reflecting on my personal experience and for trying to draw some lessons from it. In this contribution, I summarize this reflection and its outcomes.

Such a reflection is necessarily subjective. For that reason, it is important that I start by pointing out some characteristics of my professional life that necessarily influence my perception. I was trained as a mathematician at the Ecole Normale Supérieure in Paris, and logic was my first research area. My Ph.D. was on recursivity issues and then I got a position at the mathematics department of the University Paris 7 and entered a research group working on non-standard models of arithmetics and bicommutability between theories. One of my professors at the Ecole Normale Supérieure, André Revuz, had been recently recruited there and he was in charge of a new and original institution, called IREM (Institute of Research on Mathematics Teaching). IREMs are specific structures attached to universities with close links to mathematics departments. The first three IREMs were created in 1969, but there are now 28 that form a network covering the whole country, and there are even some IREMs abroad (http://www.univ-irem.fr). Their mission is to contribute to teacher professional development, to develop innovation and research, and to produce resources both for teaching and for teacher education. For fulfilling these missions, the IREMs create mixed thematic groups including university mathematicians, teachers and teacher educators working part time collaboratively. For instance, at the creation of the IREM of Paris at the University Paris 7 in 1969, the mathematics department was allocated six specific positions by the Ministry of Education, and mathematicians from the department were invited to spend part of their academic duty contributing to IREM activities, in collaboration with the 20 secondary teachers delegated half time to the IREM by the academic authorities[1].

I was soon invited by André Revuz to join the IREM team, and this was the origin of my engagement in educational issues. In the mid 1970s, Revuz proposed that two colleagues and I take charge of mathematics teaching in an experimental elementary school that had been recently attached to the IREM . We had a lot of freedom for organizing mathematics teaching and learning in that school, and in this capacity I collaborated with Guy Brousseau. Brousseau had obtained the creation of a similar school attached to the IREM of Bordeaux, a laboratory where the main constructs of the theory of didactic situations were being developed and put to the test (Brousseau, 1997).

I thus entered the emerging community that would be later known as the French didactic community, and had the possibility to contribute to its development while pursuing my research in logic. This context also led to the fact that, even when years later I stopped doing research in mathematics, I could continue to collaborate with mathematicians. I taught with them at the undergraduate level or in teacher education programs; I worked with them at the IREM as well as in different academic institutions and commissions, such as the CNU (National Council of Universities) which is in charge of the qualification and promotion of university academics[2] and, later on, worked with mathematicians in the CREM (Commission of Reflection on Mathematics Teaching) presided by the mathematician Jean-Pierre Kahane and in ICMI, the International Commission on Mathematical Instruction. These characteristics of my professional life

[1] The current means of the IREM are far from this idyllic state, for instance the many secondary teachers contributing to its activities only receive some extra salary.

[2] In France, most didacticians are attached to the section of CNU in charge of applied mathematics and applications of mathematics.

made me move regularly at the interface between communities, and they certainly influence my vision of collaboration between mathematicians and didacticians.

Collaboration between mathematicians and didacticians is necessary, and I am personally convinced that no substantial and sustainable improvement of mathematics education can be obtained without building on the complementarity of their respective expertise, without their common engagement and coordinated efforts. However, I am perfectly aware that productive collaboration is not easy to create and that maintaining it, once established, requires continued effort.

The situation was certainly different in the sixties and even the seventies, a time when didactic research was just emerging. The proceedings of the first ICME congresses, for instance, attest to the existence of such collaborations, as well as to the existence of many individuals combining research activity both in mathematics and mathematics education. But, as I explained in the closing lecture of the symposium organized for celebrating the centennial of ICMI in Rome in 2008 (Artigue, 2009), the development and professionalization of research in mathematics education and the increasing pressure put on researchers—whatever their field of expertise—inexorably increases the distance between the communities and their respective agendas. This distance makes individuals who can maintain a substantial and recognized research activity both in mathematics and in mathematics education more and more an exception. Of course, there are still some people who span the boundary between mathematics and mathematics education, and they play a particularly important role in maintaining and even strengthening the connections between the communities. However, the quality of relationships between mathematicians and didacticians depends increasingly on the establishment of productive collaboration between individuals or groups who do not have full expertise in both domains, but who think that collaboration is needed for the improvement of mathematics education and are ready to invest part of their time and energy for making this possible and productive. As I also pointed out in my Rome lecture, the development of mathematics education as a genuine field of research has led to the building of theoretical frameworks, constructs, of technical terms that make communication more problematic. Mathematics education has progressively built a specific form of discourse in which research articles are written and results expressed. Making these results and ideas accessible outside the research community needs appropriate transposition of discourse. I am not sure that the didactic community does this so well. The difficulty of the work is often under-estimated and the efforts of those who invest in it are not valued enough by academic institutions.

Committed individuals are certainly essential for initiating and developing fruitful collaboration , but, without appropriate structures and institutional support, any impact remains necessarily limited and its sustainability is impossible to ensure. Looking for institutional support and creating adequate structures is thus crucial. Moreover, priority should certainly be given to actions where collaboration can make a visible difference while being accessible at a reasonable cost. I would say that teacher education and professional development, teaching and learning at the university, popularization and enriching activities are in some sense natural candidates as many mathematicians already are engaged in them. Also, collaboration is possible at a policy level in curricular commissions and in joint reactions to policy decisions that we think damaging for mathematics education or teacher education.

In looking at my personal experience, many positive examples come to mind that reflect different forms of collaboration . I will briefly evoke six of these and give an extended seventh example:

1. Collaboration at a personal level: collaboration with the mathematician Adrien Douady, and then with Marc Rogalski for instance was essential for my research and engineering work on the teaching and learning of differential and integral processes, and of differential equations (Artigue & al., 1988; Artigue, 1992). Conversely they benefited from our collaboration for their teaching at university. There is no doubt that my research interests in the area of Calculus and Analysis favored such a form of collaboration .
2. Collaboration in innovative university programs, teacher education and professional development. As I explained above, this form of collaboration was something normal for me as I have spent more than three decades working both in a mathematics department and in an IREM . For instance, in the early eighties, together with mathematicians and physicists, I was involved in the development of a very innovative and successful experimental mathematics and physics program for first year university students. I also worked with Jean-Luc Verley, an historian of mathematics leading the corresponding group at the IREM , in the creation of an experimental course combining history of mathematics and didactics in the master's program of mathematics. More recently, I worked with François Sauvageot, a mathematician colleague, in the creation of a course on modeling in a master's program devoted to the education of mathematics teacher educators. There is no doubt that the existence of the IREM structure was essential for initiating these innovations and making them successful.
3. Collaboration in national groups and commissions, such as the CREM (Commission of Reflection on Mathematics Teaching), asked in 1999 by the Ministry of Education to reflect on what should be taught in mathematics, why and how, and whose main reports published in 2001 still constitute a reference in France (Kahane, 2001).
4. Collaboration in the supervision of doctorate students. René Cori, who leads the IREM group on logic, and I are jointly supervising a thesis on the teaching of logic at senior high school (some elements of logic were recently reintroduced from grade 10). Even if logic was my initial research area, this was a long time ago and I feel my collaboration with René is necessary.
5. Collaboration in popularization and dissemination activities. This form of collaboration that I personally experienced in the conception of the UNESCO travelling exhibition "Experiencing Mathematics!" is increasingly developing.
6. Collaboration in the organization and management of actions, but also at a reflective level that is required for more systematic study of the functioning and effects of such activities. Doctoral theses such as the recent thesis by Nicolas Pelay in France, which was co-supervised by the mathematical historian Jean-Pierre Crépel and the didactician Viviane Durand-Guerrier (Pelay, 2011) are very promising from this perspective.

My seventh example involves collaboration at the ICMI level. One of the main ambitions of ICMI is to organize the collaboration of all those who can contribute to the improvement of mathematics education worldwide, to guide and support their efforts, and to disseminate their outcomes. As I explained in the Rome lecture, my election as ICMI vice-president in 1998 occurred at a moment of tension between ICMI and its mother institution, the International Mathematical Union. We had to reflect on what we wanted to achieve and how it could be possible. At that time, voices were being raised in the community of mathematics education asking ICMI to allow it to become independent. The ICMI Executive Committee resisted these voices while acknowledging that the *status quo* was not acceptable. Thanks to mutual efforts among mathematicians and mathematics educators, the situation progressively improved to a quality of relationships and collaboration today that were difficult to imagine in 1998. Along the years, I experienced the decisive role that can be played by influential and respected individuals (members of the two executive committees, especially their presidents and secretaries, such as Hyman Bass, ICMI President from 1998 to 2006, and Bernard Hodgson, ICMI General-Secretary from 1998 to 2009, Jacob Palis, John Ball and László Lovász, the successive IMU Presidents during that period).

I also witnessed how collaborative work on common projects is essential for overcoming mistrust and bad experiences. I could mention many different experiences. I will limit myself to two of them. The first one is the Felix Klein project, whose aim is to make the mathematics developed since the Klein era accessible and source of inspiration for teachers (focusing on senior high school teacher in the first phase of the project). This project was inspired by the work of Felix Klein himself, who a century ago gave a series of lectures for German teachers that led to the famous series *Elementary mathematics from an advanced standpoint.* The Klein project is now a joint project of ICMI and IMU, led by a team made of mathematicians and mathematics educators (http://blog.kleinproject.org).

The realization of the Klein project cannot be envisaged without a collaboration between mathematics and mathematics education, and we observe that the interest such collaboration raises can reinforce further collaboration between communities and even provoke it. A paradigmatic case is that of Brazil where a Klein project in the Portuguese language has been launched at the initiative of the Brazilian Mathematical Society with the participation of all societies in charge of mathematics and mathematics education, and a strong support from the government (http://klein.sbm.org.br).

I must also mention the recent CANP project (CApacity and Networking Programme)[3] jointly launched by ICMI and IMU with the support of UNESCO and ICIAM (International Council for Industrial and Applied Mathematics) for the development of teacher educators in developing countries. One realization of this project is a conference held annually in a different part of the world. The first conference took place in September 2011 in Bamako (Mali) for Francophone sub-Saharan Africa, the second held in August 2012 in Costa Rica for Central America. The next conferences are planned in Cambodia (2013) and Tanzania (2014). As explained in the description of CANP, this project is jointly led by mathematicians and mathematics educators, and addresses all those involved in teacher education: expert teachers, didacticians, mathematicians, inspectors and advisors. The conferences aim to foster the collaboration among all communities engaged in teacher education in a given country as well as

[3] http://www.mathunion.org/icmi/other-activities/outreach-to-developing-countries/canp-project/

regional collaboration . Conferences take the form of a two-week workshop in which didactic and mathematics themes are combined together with study of questions of specific interest for the region. Once again, the first conferences look very promising, showing that collaboration between mathematicians and didacticians, even if it is unusual in many countries, is nevertheless possible and can be rewarding when carefully organized and planned in a spirit of mutual respect. Such projects also show how the ICMI spirit of collaboration among communities can disseminate and impact local situations.

This contribution to the reflection undertaken in our panel will perhaps appear too optimistic to some whose experience is quite different from mine. This vision is nevertheless realistic. Fruitful collaborations indeed exist at different levels and in many different contexts. Even in difficult contexts, actions are possible which in the long term can move positions and visions, by relying on the individual forces that always exist, and by patiently cultivating these. However, these positive descriptions and outcomes should not hide that collaboration is always costly. Whatever are its conditions, collaboration needs personal efforts of decentration, the building of intermediate languages, the building of an appropriate semiosphere where communication is made possible between members of different communities. Mathematicians, historians, teachers and didacticians have all to learn a lot for productively collaborating; they have to accept the limitation of their knowledge and expertise but also to be convinced of its value and importance for others. There is no alternative because, as already stressed above, each community alone will not produce sustainable and large-scale improvement of mathematics education.

Establishing a collaborative culture requires patience and determination. It is easier to destroy than to build. But when collaboration works, it is so rewarding!

Section 2: Collaboration between mathematics and mathematics education: Some personal experiences and remarks

Günter Törner, University of Duisburg-Essen, Germany

To paraphrase Euclid, *There is no royal road to collaboration between mathematics and mathematics education*. I am fully aware that each country has its own mathematical tradition and culture, which has been lived for many centuries by mathematicians. The TED-symposium confirmed that situations differ among different countries.

Nevertheless, there might be a chance that each reader reflects on the variables that I identify here for Germany. Eventually in the reader's country there is some latitude, which might be filled and framed. Also, I am aware that my viewpoint is a personal one.

From 1997 I was a member of the Executive Board of the German Mathematical Society (DMV). In the late 1990's the alarming results of TIMSS were published, which showed German students performing much lower than we anticipated, and soon the question arose: Who is "guilty" – the teachers, the society of mathematics educators, or the DMV through teacher education at universities?

Soon we realized that our Society is not very large and thus our influence is limited, since at that time we counted 4,000 members. Thus it was not straightforward for the Society to be invited to public hearings to present our views. Our Society therefore joined efforts with two other learned societies covering the field of mathematics education.

After that we started to produce jointly-authored declarations on various occasions when mathematics at school was discussed in the press. Through this effort, we spoke in the name of more than 10,000 members. Next we argued against the claim that schoolteachers were responsible for the poor outcomes of German mathematics teaching and for the claim that the problems were systemic.

We also realized that there is no such thing as *the* mathematician, *the* mathematics educator, or *the* mathematics teacher. Rather, there is great variability in values and perspectives within each group. Further, we had to confess that few people in the mathematics community were intimate with the processes in the ordinary mathematics classroom. Upon getting in contact with mathematics educators and coming to truly appreciate their research, we understood that we do have a deficit in research on teaching and learning, but rather an implementation problem. However, contrary to the expectations of administrators and politicians, carrying out sustainable and effective research in mathematics education about implementation is much more difficult than just publishing some interesting new results in mathematics.

But mathematics education is not the only area facing an implementation problem. Mathematics itself is facing many implementation problems – cooperation of pure mathematics with applied is just one – and these are often ignored in the daily practice of mathematics departments! To summarize: To change mathematics education, so as to develop collaboration between mathematics and mathematics education, is not solely a personal problem or endeavor, but should be classified as a communal challenge. Collaboration between mathematics and mathematics education should be regarded as a task of cooperation between societies.

Lest I be misunderstood, I must say that in there are fruitful local collaborations between mathematicians and schools or teachers in numerous places in Germany, including mathematics departments. These efforts are important, and should be emphasized publicly. However, though such projects are necessary, they are not sufficient. *Bottom-up approaches* must be complemented by *top-down initiatives.*

Looking back to earlier times at the beginning of the last century, in the glorious time of Felix Klein, mathematics education at school was in accordance with the insights of mathematics education at university. It might sound incredible today, but Felix Klein worked to strengthen school education in the lower secondary grades and in kindergarten. In a seminar (circa 1910) he invited famous educational researchers to share their thoughts about the implications of Pestalozzi's research for mathematics education at school. Pestalozzi was a famous (primarily non-mathematics orientated) educationalist. This was Klein's idea—to learn from an educationalist. I am sorry to say that, today, I do not know any mathematician, internationally recognized as a leading researcher in his or her field, who is so convincingly and simultaneously engaged in mathematics and school mathematics education.

The New Math movement of the 1960's and 1970's provoked a separation of teachers and mathematics educators in the DMV. Mathematicians, sometimes with hubris, ignored teachers' needs, their proposals, and their contributions. Thus there was a "divorce" within the DMV of mathematics and mathematics education. So far mathematics education was a session of some few mathematicians at their annual meetings, now 'Gesellschaft für Didaktik der Mathematik' (GDM) was established as an independent learned society, also attracting primary school teachers. Some members of

DMV left the mathematical society and joined GDM. GDM organized its own annual meetings and there was no correspondence between these societies, no discussion on common topics, nearly hostile neighborhood.

Forty years later, both sides recognized that this splitting was a mistake. Unfortunately, there is now so much accumulated divergence that it makes little sense to propose a reunification. We have to accept that, for example, primary school teachers are very far from DMV intellectually and probably would not join DMV, but there are at least more than one hundred persons which are members in both societies. Also, some of the DMV members, in parallel with mathematics education, are aware that foundations for mathematics are laid in the primary grades and thus feel also responsible for that group of teachers. Meanwhile, about 2000, in Germany we started some gentle cooperation between mathematicians and mathematics educators in various projects financed by foundations (see Hoechsmann & Törner, 2004). It is a partnership on a level playing field.

Nevertheless, we struggled (and continue to struggle) with closely held beliefs among mathematicians and among mathematics educators. One problematic belief held by many mathematicians is that the problem of learning and teaching is trivial or unimportant. Mathematicians do not entertain elaborate models for learning. Rather, they like to generalize their private opinions to the general case—even though they would never extend knowledge from one mathematical example to all related examples.

Finally, there is too much hubris among mathematicians. Mathematicians often speak as if mathematical objects are real, available for inspection. On the other hand, it is too often the case that mathematics educators do not have a command of the larger body of mathematics that mathematicians see as arising from the ideas under discussion. Mathematicians then form the impression that educators have underdeveloped worldviews of mathematics—on the role of systems, the role of axiomatics, and on formal aspects of mathematics. It is also true that mathematicians overemphasize these same aspects, without appreciating the nature and role of meaning and understanding in students' mathematical learning.

Over the years I have become aware of multiple Do's and Don'ts. Here I offer an incomplete list:

- STEP 1: Avoid philosophical discussions. Although some or our problems are rooting in the philosophy of mathematics, be careful to start discussing them. Philosophies are often deeply anchored and hard to change, hence such discussions might not affect our daily practice.
- STEP 2: Cooperation must first be grounded in communication.
- STEP 3: You may solve the "problem" under discussion in your department, but this will - if successful at all – lead only to a local solution.
- STEP 4: Be sensitive in communication processes. Don't act as a missionary. Don't try to convince. Your collaborators have many experiences and insights.
- STEP 5: Be modest in your expectations. Expect to invest years of effort.

While it is easy to call for win-win collaborations, they are not easy to accomplish in our context. Thus, the DMV tried to define win-win situations on a large scale where groups and societies are involved. Certainly it is a win-win situation for a country when mathematicians and mathematics educators are willing to cooperate. Math wars create losers: teachers, students and finally mathematics itself.

Successes in Germany can also be attributed to additional factors.

- The International Congress of Mathematicians (ICM 1998) at Berlin provided an opportunity for an inventory in the field, and the DMV made use of it. Magnus-Enzensberger, an internationally respected essayist, gave a famous talk (Enzensberger, 1999): *Draw-Bridge Up, the Ivory Tower*, portraying misconceptions of mathematics.
- The presidency of Martin Grötschel, who is now serving as the Secretary of IMU, changed the self-view of the DMV by bringing many applied mathematicians into the society and into offices of the DMV. Groetschel also began an initiative in 1993 that granted a seat for mathematics education on the Executive Committee of the DMV. Today it is no longer disputable that there should be a mathematics education representative in the DMV's internal discussions.
- The 2007 Joint annual conference of mathematicians and mathematics educators in Berlin was a success. However, the 2010 Joint annual conference in Munich was not as successful. We learnt from this conference that success is highly dependent upon the local organizers. As a consequence, at this moment there is no plan for a further joint conference.

We aimed to establish a culture of a reciprocal appreciation among mathematicians and mathematics educators, and we are practicing it. We came to understand that blaming the other side does not improve the situation. We accepted that poor textbooks do exist (in school and in university mathematics) and that poor teaching exists (at school as well as at university).

We are also convinced that transparency and openness generate confidence. We try to abolish envy and jealousy. We are practicing graciousness: Invite math education representatives to all EC-meetings of our mathematical society. It is also important to note that we invite our mathematics education colleagues into our "private homes" and "temples" like The Mathematical Research Institute Oberwolfach, the Fields Institute, and the Banff International Research Station. We know this is not easy, since we have to reject a mathematically oriented conference topic to host mathematics educators for a week; but it is paying off.

Meanwhile there are well-established projects:

- A joint commission on issues of teacher education with delegates from the German Mathematical Society (DMV) and two more societies representing the mathematics educators and teachers of mathematics.
- A joint commission dealing with the transition problems of students starting to study mathematics after leaving school, and who have a high drop rate—which must be lowered.

We are widening our views and are eager to gain more friends in mathematics education. We are convinced they do exist. Together with Celia Hoyles (NCETM, London) and the Deutsche Telekom Foundation[4] (DTS) we are inviting charity foundations, NGOs and institutions to the FOME-conference (Friends of Mathematics Education) in Berlin (March 2013)—organizations that are parallel to mathematics and which are sponsoring projects for mathematics classrooms. The International Mathematical Union (IMU), the International Commission on Mathematical Instructions

[4] http://www.telekom-stiftung.de/dtag/cms/content/Telekom-Stiftung/en/396336

(ICMI) and the European Mathematical Society (EMS) will support us. All this will serve to improve mathematics education, not least by the help of mathematicians. Better school education improves the success of students at university.

Section 3: Collaboration between Mathematics and Mathematics Education: Two personal examples

Ehud de Shalit, Einstein Institute of Mathematics, The Hebrew University, Israel

The session on collaboration between mathematics and mathematics education at the TED conference has given me the opportunity to reflect upon two very rewarding experiences I have had in recent years, and to share my thoughts about them with the other members of the panel. I would like to precede my description of these two enterprises, though, with a confession. Coming from the side of mathematics, I often feel unsure in the company of math educators. Math education is by now a mature field that has its own paradigms and methodology, and its own language that I do not speak. In a paradoxical way, serving mathematics for over 30 years blinded me to some very basic truths about math education. I often find the observations made by math educators eye-opening and awe inspiring. I can only regret the fact that despite a somewhat growing trend toward collaboration in recent years, in Israel at least, the two communities still remain largely disjoint.

The Meet Math exhibition

Some 8 or 9 years ago I was recruited by Prof. Hanoch Gutfreund to participate in a full-scale, 400 square-meters math exhibition. The *Meet Math* exhibition, an Italian-Israeli-Palestinian co-production, opened two years later for 3 months in the Città della Scienza in Napoli, before moving for 8 more months to the Bloomfield Science Museum in Jerusalem, finally settling in its permanent residence at Al-Quds university in Abu-Dis.

I will not say anything here about the very interesting experience of working with people from other nationalities to enhance peace in the region. I will only focus on the scientific experience per se. Our team was incredibly large. It included mathematicians, curators, designers, educators, carpenters, as well as financiers. Everything had to be done from scratch – defining the goals, the target audience and the concepts, and of course, building the exhibits and writing the texts. Focus shifted rather early from History of Mathematics (with emphasis on Arab contributions in the Middle Ages) to the subject matter itself, with hands-on exhibits. To the surprise of the Italians, it was the Palestinian members of the team who preferred an exhibition that would benefit their school children directly, over a learned historical exhibition that would pay tribute to their heritage but attract fewer viewers, mostly adult. Perhaps at my insistence, the target audience was set at junior-high and high school children. I felt that too often science museums catered either to the very young, or to adults and professionals, leaving out the formative years in which the child chooses his or her future direction.

Some of the messages that we wanted to communicate were obvious – the usefulness of mathematics, its role as a language for other sciences, that doing math can be fun, etc. Other messages were subtler and not always easy to explain. Does the mathematician discover or invent the mathematical world? What is the difference between an illustration of a mathematical fact and its proof? We used a rather standard exhibit of Pythagoras' theorem, in which liquid flows from one square to fill up the other two, to address this point. Next to it, we also had a tangram-based proof of the theorem, and the activity around the exhibit focused on which of the two was more convincing and why.

Is there room for ugly mathematics (to paraphrase G. H. Hardy)? What is an algorithm and what is algorithmic complexity? We used the Tower of Hanoi to illustrate this last point. What is an open problem? We presented a computer game in which the visitor chose a number x, and then successively applied to it the transformation $3x+1$ if what they had at hand was odd, or $x/2$ if it were even. It is an open problem (called the Collatz conjecture) to show that this game always ends with 1. The statistics can be quite amazing – some very high numbers are reached, and the game lasts for quite a long time, before it finally ends with 1.

Some exhibits dealt with fundamental notions encountered in school. Against a background of Leonardo's Vitruvian man, children measured their heights and arm-spans, and a computer recorded their measurements and calculated their ratio, showing that it was almost constant. A toy car moved on a rail by one's hand, produced on a screen a graph of distance versus time, allowing the visitor to "feel" what constant-speed motion or acceleration meant, and relate it to the graph. Other exhibits dealt with more advanced subjects – tiling the floor with "darts" and "kites" to produce a non-periodic Penrose tiling (fun and aesthetic), classifying knots (learning about chirality), or following Euler's path across the Koenigsberg bridges with a rope.

The organization of the exhibition was basically thematic. Its core was arranged in four halls, called Number, Shape, Pattern and Computing respectively. Nevertheless, the unity of mathematics and relations between the various areas were constantly emphasized. Balance between computer-based exhibits and mechanical ones was another issue. Whenever possible, we had a preference for the latter.

As a mathematician, I had to set aside my preconceptions and listen to the experience of curators and educators. Nevertheless, I believe that some of the messages, and the ways in which they were presented, would not have come across, if not for the involvement of the mathematicians. As much as it is important to present science in a friendly, appealing and accessible way, it is also important to adhere to its true nature and meaning, as perceived by the scientist. Resolving the potential conflict between these two goals is possible when Scientist and Educator work together in harmony.

Fundamental issues of math education – building a new teacher education course at the Hebrew University

High school math teachers in Israel are required to hold both a B.Sc. in math, or in a related area, and a teaching certificate. Unfortunately, the two programs at the Hebrew University (and to the best of my knowledge at most other universities in Israel) are not coordinated. The prospective teacher takes the same math classes, from logic to topology and differential equations, which any other undergraduate in mathematics would take. In fact, nobody at the department of mathematics takes notice of which of the 100 students in each cohort intends to become a teacher.

He or she then start, in their third year, certificate studies at the School of Education, where they focus mostly on pedagogy and general education courses. Very little is done to address didactical issues pertaining to mathematics. The practicum is conducted in the fourth year in selected participating schools, but it is often left to the older teachers in those schools to guide the would-be teachers in their first field experience. The old practices of teaching-to-the-test and emphasizing technique at the expense of understanding are then instilled from day one, and whatever spirit of reform the new teacher brings is washed away. Mathematicians, or researchers from the science teaching unit at the university have not been involved with the School of Education's teacher education program.

To add insult to injury, the mathematics department and the school of education at the Hebrew University are located in different campuses, separated by a 30-minute bus ride.

Changing this unfortunate scenario was the ultimate goal of Prof. Baruch Schwarz from the school of education, Prof. Abraham Arcavi from the Weizmann Institute Department of Science Teaching, and myself, when we met 2 years ago with the idea of upgrading the teacher education program, and in particular, forming collaboration between educators, science teaching experts, and scientists.

As a pilot for such a program, we devised and ran a year-long seminar on *Fundamental Issues of Math Education*, which met every Sunday for 2 hours at the school of education. All three of us were present in every class, as well as some 14 students of variable background and age. This was a unique experience. I am not aware of a similar joint effort in Israel, although courses dealing with didactical issues are probably well established worldwide. Every week we met for several hours to discuss between ourselves the coming weeks and the division of labor. The issues were discussed in depth, each of the three organizers contributing his particular angle. I have been exposed to articles and examples that enriched my understanding of mathematical teaching, and I hope my colleagues have profited here and there from my perspective as a mathematician.

The course was structured in such a way as to facilitate the collaboration . It was divided into 7 sections, and 4 weeks were devoted to each of them. Five of the sections were thematic: they dealt with the teaching of (1) arithmetic, (2) algebra and functions, (3) geometry and trigonometry (4) probability and data analysis and (5) calculus. Two sections were "horizontal" – dealing with (6) mathematical modeling and (7) problem solving.

Within each section, each of the organizers gave one 2-hour lecture. Naturally, I would speak about the mathematics of the concept, often in historical perspective, and my colleagues would discuss studies related to didactical questions, or the cognitive and psychological development of mathematical thinking. The last week within each section was a workshop conducted by one or two of the students, who were assigned tasks related to the material discussed in class. Typically, they had to prepare an activity or a school lesson, followed by a discussion among all of us.

The material did not necessarily overlap school curriculum, and no attempt was made to cover every aspect. Some of the students, having years of teaching experience behind them, contributed important insights. At other times we were surprised to see them miss what seemed to us obvious didactical points.

Examples of issues that were discussed included:

- Components of good teaching: understanding math, skill-building, developing mathematical sense and intuition,
- How to avoid compartmentalization: the unity of mathematics,
- Revisiting ideas and making connections among ideas in teaching,
- Procedural vs. conceptual learning,
- Order, pace and age adaptation,
- Application of advanced technologies in teaching.

All were discussed in-context within the sections, and not abstractly. As an example, the section on Functions and Algebra Teaching included:

Lecture 1: Ehud de Shalit: *Evolution of the function concept* (following Kleiner, 1989)
Lecture 2: Baruch Schwarz: *Different presentation of functions – a didactic analysis*
Lecture 3: Abraham Arcavi: *Dynamical software and its use in teaching functions*
Lecture 4: Student workshop: *Four schemes for grade adjustment* (a class presentation).

We believe that courses of this sort can serve as a model for collaboration between mathematicians and mathematics educators in teacher education programs in the future.

Section 4. What Can Mathematics Education Bring to Mathematics?

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The editors asked our group to address the matter of collaboration between mathematics and mathematics education. Collaboration between the two often is viewed from the perspective that mathematical content is within the purview of mathematicians and pedagogy is within the purview of mathematics education. A consequence of this view is that discussions of collaboration assume that each field brings to a collaboration what is in its purview. I would like to pursue a different perspective – one in which mathematics education actually can contribute to mathematics in regard to the mathematical preparation of mathematics majors as well as to the mathematical preparation of future mathematics teachers.

I will develop this thesis through two examples at Arizona State University. The first is the development of a calculus course; the second is the development of a B.Sc. Mathematics degree program with a concentration in mathematics education.

Calculus

Calculus courses, in the U.S. at least, are plagued by an orientation that calculus is nothing but procedures and facts. Students' understandings of calculus are often incoherent when viewed as a body of ideas; any coherence in their understandings is too often about just connections among procedures.

At ASU we have designed an experimental introductory calculus course (differential and integral calculus of one variable) that aims from the beginning to have mathematics and non-engineering science majors learn the calculus as a coherent body of ideas that are necessitated intellectually (Harel, 1998, 2008a, 2008b). The course is necessitated by two fundamental problems: (1) You know how fast a quantity is changing and you want to know how much of it there is. This problem leads to the idea of accumulation functions and of an indefinite integral as an accumulation function. (2) You know how much of a quantity there is and you want to know how fast it is changing. This problem leads to the idea of instantaneous rate of change and properties of a function's behavior that can be discerned from its rate of change. A detailed description of the course appears in (Thompson, Byerley, & Hatfield, 2013).

The course's curriculum emerged from a combination of mathematics education research on students' understandings of accumulation and rate of change (Carlson, Persson, & Smith, 2003; Schnepp & Nemirovsky, 2001; Thompson, 1994; Thompson & Silverman, 2008; Yerushalmy & Swidan, 2012), insights into the ways that curricula and instruction can be designed to motivate students' mathematical interest (Harel, 2008a; Harel & Sowder, 2005), and research on students' quantitative reasoning and uses of notation to represent it (Carlson, Larsen, & Lesh, 2003; Ellis, Ozgur, Kulow, Williams, & Amidon, in press; Gravemeijer & Doorman, 1999; Johnson, 2012; Kaput, Blanton, & Moreno, 2007; Schoenfeld, 2007; Selter, Verschaffel, Greer, & De Corte, 2000; Smith & Thompson, 2007; Thompson, 1993, 1995). This body of research did not dictate to us what should constitute a calculus curriculum. Rather, it provided a way to think about meanings that belong evidently to the calculus as emerging from a mosaic of understandings that students typically build in school.

On one hand the design and experimentation of this course could be seen solely as a mathematics education effort. On the other hand, however, it is a true example of collaboration between mathematics and mathematics education. Our effort could not have happened without the support and trust of the Department's director and of the mathematics faculty. Several members of the first-year mathematics faculty are trying this new approach. More are participating with us in planning a grant proposal to investigate what students learn from in-principle different curricular and instructional approaches to major ideas in the calculus.

B.Sc. Mathematics with Mathematics Education Concentration

The Bachelor of Science degree in mathematics is the primary undergraduate degree in ASU's School of Mathematical and Statistical Sciences ("the School"). The School had already created concentrations within the B.Sc. in computational mathematics and statistics. Most recently it created a concentration in mathematics education. Moreover, with the support of ASU's Mary Lou Fulton Teachers College (MLFTC), Arizona's Department of Education (AZDoE) granted graduates of the Math Education concentration what is called *institutional recommendation*, meaning that graduates of the B.Sc. Mathematics/Mathematics Education will automatically receive a license to teach secondary mathematics. The School's B.Sc. Math/Math Education is Arizona's first program not housed in a college of education whose graduates receive an institutional recommendation for licensure.

ASU's Bachelor of Science in Math/Math Education resulted from a long collaboration among the School's mathematicians and mathematics educators (Luis Saldanha, Pat Thompson). In particular Fabio Milner (applied mathematics) was instrumental in obtaining university approval for the program. Bruno Welfert (mathematics) and Matthias Kawski (applied mathematics) supported our effort in their successive terms as Director of Undergraduate Studies. In addition, ASU's MLFTC was instrumental in assisting us to prepare proper documentation to support an application to the AZDoE for institutional recommendation of our program's graduates, and for including our application as part of theirs. MLFTC's support was essential, as it is the only body within ASU from which AZDoE will accept such proposals.

We designed the B.Sc. Math/Math Education degree so that it focuses deeply on its graduates' *Mathematical Knowledge for Teaching secondary mathematics* (MKTsm). Specifically:

1) Students in our Math/Math Ed program take the School's standard program in mathematics for its B.Sc. Mathematics degree.
2) They take a subset of the MLFTC program for secondary education majors. Students are not required to take MLFTC's general education courses that overlap with the specialized math education courses described below, in (3).
3) Students take five courses that the School designed specifically to draw connections between mathematics and mathematics education. The five courses are:
   a) *Algebra and Geometry in the High School* (Year 1). This is a conceptual overview of the secondary mathematics curriculum. At the same time that students take this course, they enroll in a field experience course called *Mentored Tutoring*. Mentored Tutoring has students in the review course work with students in remedial mathematics courses under the guidance of the review course's instructor.
   b) *Technology and Mathematical Visualization* (Year 2). The TMV course is designed to have students re-conceive the mathematics they know so that symbolic representations have imagistic content. This is not a programming class. We use software, primarily Geometer's Sketchpad (GSP) for geometry and Graphing Calculator (GC; Avitzur, 2011) for everything else. The idea of the course is that students need to engage in mathematical thinking to create visualizations that might help them convey a particular mathematical idea to students. As a simple example, we set the problem of how to define a function that takes two points (of dimension 2 or 3) as input and produces a graph of the segment that connects them.[5] They must not only define the function, they must explain why the function produces what was requested. A second example is that students must define a function g whose graph will be a plane that is tangent to the graph of an arbitrary function of two variables f at an arbitrary point on f's surface. A user of a student's project must have control over the definition of f and the location of the arbitrary point. The plane must adjust dynamically as the user moves the point of tangency.
   c) *Curriculum and Assessment in Grades 7-12.* In this course we introduce students to curricula from various countries and to principles of assessing school students'

[5] One solution: $f(X,Y)=(1-t)X+tY$, $0\le t\le 1$. GC produces a graph that is a segment in 2- or 3-space, depending upon the dimension of vectors X and Y. An explanation of why this works necessarily involves two things: imagining the value of t varying in small increments and describing the role of proportionality in traversing the hypotenuse of a right triangle.

understandings of the mathematics in them. We have not yet offered this course, but we anticipate that, for U.S. students, it will be an eye-opening experience for them to see the mathematics that other countries expect their students to learn in high school.

d) *The Development of Mathematical Thinking.* In essence, this course will introduce students to research on the development of additive and multiplicative reasoning. This is another course we have yet to offer, but our intent is for students to become consciously aware of different ways that school students' might understand mathematical ideas that teachers often take as unproblematic. We also see this course as helping our students conceptualize the school mathematics curriculum as entailing their students' development of systems of ideas over time.
e) *Research Project in Mathematics Education.* This is a seminar in which students will design, conduct, and interpret a teaching experiment with one or two high school students. We see the Project course as a culminating experience through which our students will draw from what they learned in the courses described above.

We counsel students enrolling in the B.Sc. Math/Math Ed to enroll from the start in the experimental, conceptually oriented calculus that we designed. We feel that moving future teachers from a procedure-oriented mathematics to an idea-oriented mathematics is a long process, and that it is unlikely to happen if their university mathematics continues their school practice of mathematics as memorization.

Mathematics educators in the School are also engaged in the design of curriculum for students who are not in education. Kyeong Hah Roh, who has a Ph.D. in mathematics education and a Ph.D. in mathematics (differential geometry), worked with mathematicians on our faculty to redesign *Mathematical Structures*, a course required of all students in any mathematics concentration. The course gives an introduction to proof and higher mathematics. Dr. Roh also redesigned our undergraduate advanced calculus and real analysis courses based on research on students' learning of proof, functions, and limits. Marilyn Carlson led a 10-year research and development project to transform the School's precalculus course, which is a remedial course for students who are unprepared to take calculus. The redesign is rooted firmly in developmental research on students' difficulties in learning mathematical ideas that are essential for students to succeed when they reach calculus, such as deep understandings of linearity, rate of change, and the concept of function.

Comments on collaboration and its outcomes

It might be useful to discuss the nature of the collaborations I've described and about places of friction where things did not go smoothly. The calculus redesign was an outgrowth of my and Marilyn Carlson's research. The School's contribution was to allow the redesign on an experimental level. Actual collaboration began with the attempt to increase the number of course sections using the redesigned curriculum. Jay Abramson and Mark Ashbrook have been instrumental in that effort. However, other instructors have been reluctant to adopt this new curriculum and approach.

The redesign of *Mathematical Structures*, advanced calculus, and real analysis were an outgrowth of Kyeong Hah Roh's research on teaching and learning mathematics. Her redesign was successful in terms of outcome measures, but other instructors of these courses have been slow to pick up Roh's changes.

Marilyn Carlson's redesign of precalculus, also an outgrowth of her research, has been the most successful of the innovations. After several years of resistance among perennial instructors of precalculus, all sections at ASU are now using her curriculum. Fabio Milner and several first-year mathematics instructors were integrally involved in the redesign, giving substantive input regarding the mathematical treatment of ideas, and were important supporters in the politics of curriculum change.

Regarding the B.Sc. Mathematics/Mathematics Education, the School faculty voted to approve this concentration. So the general acceptance among mathematicians that mathematics has an important stake in mathematics education is evident.

The friction in all these moving parts comes from the fact that few mathematicians understand aims, methods, and results of mathematics education as a discipline. The comment, "So, you train teachers how to teach math, right?" is not uncommon. It is a revelation to many who spend time working with us that we take mathematics seriously—in some ways more seriously than they do. Conceptual coherence in the mathematics that is actually conveyed through discourse is of central importance in mathematics education, and we find that it is less important in mathematics. By "less important in mathematics" I mean that language and actions in a mathematician's classroom often have little chance of being interpreted by students as anything remotely resembling what the instructor intended. When we address this problem (intended meaning is the meaning actually conveyed) in curriculum, mathematicians are often puzzled by what we are trying to teach. They are accustomed to discourse in which their personal mathematical language is the language in which ideas are offered to students. They fail to realize that the courses we designed often are more conceptually rigorous than the versions they teach, because our courses are designed with the goal (and expectation) that students actually understand the mathematical ideas taught. As I say to my mathematics colleagues, "Mathematics education is easy—until you take student learning seriously."

Though mathematics education as a discipline is sometimes understood poorly, good things happened nevertheless. There is enough trust, little enough mistrust, and enough shared commitment to address problems in our students' learning to let innovation blossom.

Conclusion

The four discussions of collaboration between mathematics education and mathematics highlight many ways that collaboration can happen and many levels of social organization at which it can happen. Sometimes collaboration is between professional societies; sometimes collaboration is between individuals engaged in a shared task.

Running through the authors' examples is the theme laid by Artigue when she said, "… no substantial and sustainable improvement of mathematics education can be obtained without building on the complementarity of [math and math ed] expertise, without their common engagement and coordinated efforts." Törner illustrated this in his discussion of the separation of mathematics and mathematics education in Germany decades ago and the subsequent realization that, to have an influence at a national level, the two disciplines needed each other to address the problem of systemic sources of unmet expectations about students' mathematical learning.

This chapter's examples also illustrate that, at all levels of collaboration , individuals matter and institutions matter—simultaneously. At a level of collaboration between societies, it is important that individual players have vision and commitment to address problems of mathematics education—and a standing within their respective fields that allows them to exert influence with others in their societies. At a level of personal collaboration , collaborators' efforts happen within institutions whose structures either enhance or obstruct their efforts. The physical separation of education and mathematics at Hebrew University constrained collaborative efforts to improve the University's teacher education program. The inclusion of mathematics education within ASU's School of Mathematical and Statistical Sciences afforded collaboration in the design of a program in which mathematics and mathematics education are often addressed simultaneously within individual courses. The location of mathematics education within the School, and the School's support of it, was also a major factor in the University's approval of the program.

The chapters' examples also point to a shared a trait noted by Törner: successful collaboration requires mutual trust and respect among collaborators in the context of a shared commitment to solving a problem. This is not to say that there cannot be misunderstanding of each other's values, commitments, or competence regarding the nuances of the problem. Rather, the nature of trust is that collaborators carry a commitment to listen respectfully to each other and be open to modifying their positions. The examples by de Shalit of the Meet Math Exhibit and the design of a teacher education program at Hebrew University illustrate this point well. In Thompson's example of calculus redesign there were deep and prolonged discussions of the meanings of rate of change and of differential that would prove foundational for students' future learning, which led to sustained conversations of how the course might be shaped to support students development of those meanings and how it might be shaped to build upon those meanings.

We end by emphasizing a comment by Törner and illustrated by the other three authors. It is that successful collaboration between mathematics and mathematics education is most probable when collaborators have a shared commitment to a problem and believe that others in the effort have something to contribute to its solution.

References

Alibert, D., Artigue, M., Courdille, J.M., Grenier, D., Hallez, M., Legrand, M., Menigaux, J., Richard, F., Viennot, L. (1988). Le thème "différentielles" - un exemple de coopération maths-physique dans la recherche. In G. Vergnaud, G. Brousseau, M. Hulin (Eds.), *Didactique des acquisition des connaissances scientifiques. Actes du Colloque de Sèvres*. Mai 1987. (pp. 7-45). Grenoble: La Pensée Sauvage éditions.

Artigue M. (1992). Functions from an algebraic and graphic point of view : cognitive difficulties and teaching practices. In, E. Dubinski & G. Harel (eds), *The concept of function – aspects of epistemology and pedagogy*, p. 109-132. MAA Notes n°25. Mathematical Association of America.

Artigue M. (2009). ICMI: A century at the interface between mathematics and mathematics education. In, M. Menghini, F. Furinghetti, L. Giacardi, F. Arzarello (Eds.), *The first century of the International Commission on Mathematical Instruction (1908-2008). Reflecting and shaping the world of mathematics education*, pp. 185-198. Istitituto della enciclopedia Italiana. Roma.

Avitzur, R. (2011). Graphing Calculator (Version 4.0). Berkeley, CA: Pacific Tech.

Carlson, M. P., Larsen, S., & Lesh, R. A. (2003). Integrating a models and modeling perspective with existing research and practice. In R. A. Lesh & H. Doerr (Eds.), *Beyond constructivism in mathematics teaching and learning: A models and modeling perspective* (pp. 465-478). Hillsdale, NJ: Erlbaum.

Carlson, M. P., Persson, J., & Smith, N. (2003, July). Developing and connecting calculus students' notions of rate-of-change and accumulation: The fundamental theorem of calculus. In N. Patemen (Ed.), *Proceedings of the 2003 Meeting of the International Group for the Psychology of Mathematics Education - North America*, (Vol 2, pp. 165-172). Honolulu, HI: University of Hawaii.

Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (in press). Quantifying exponential growth: The case of the Jactus. In R. Mayes, R. Bonillia, L. L. Hatfield & S. Belbase (Eds.), *Quantitative reasoning: Current state of understanding*, WISDOMe Monographs (Vol. 2). Laramie, WY: University of Wyoming.

Enzensberger, H. M. (1999). Drawbridge Up - Mathematics - A Cultural Anathema. Nattick, MA: A K Peters, Ltd.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*(1), 111-129.

Harel, G. (1998, June-July). Two dual assertions: The first on learning and the second on teaching (or vice versa). *Mathematical Monthly, 105*(6), 497-507.

Harel, G. (2008a). DNR perspective on mathematics curriculum and instruction, Part I: focus on proving. *ZDM Mathematics Education, 40*, 487-500.

Harel, G. (2008b). What is mathematics? A pedagogical answer to a philosophical question. In R. B. Gold & R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics From the Perspective of Mathematicians* (pp. 265-290). Washington, DC: Mathematical Association of America.

Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and development. *Mathematical Thinking and Learning, 7*(1), 27-50.

Hoechsmann. K., & Törner, G. (2004). Schisms, breaks, and islands - seeking bridges over troubled waters: a subjective view from Germany. In McDougall, D. (Ed.) (2004). Proceedings of the 26th Conference for the International Group for the Psychology of Mathematics Education (PME) - North America Chapter (PME-NA) at Toronto 3, 993-1001.

Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *Journal of Mathematical Behavior, 31*(3), 313-330.

Kahane J. P. (2001). *L'enseignement des sciences mathématiques*. Paris : Odile Jacob.

Kaput, J. J., Blanton, M. L., & Moreno, L. (2007). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19-56). New York: Erlbaum.

Kleiner, I. (1989). Evolution of the function concept: A brief survey. *College Mathematics Journal, 20*, 282–300.

Pelay N. (2011). Jeux et apprentissage mathématique. Elaboration du contrat didactique et ludique en contexte d'animation scientifique. Doctoral Thesis. Université de Lyon 1. http://bit.ly/XaywDi

Schnepp, M., & Nemirovsky, R. (2001). Constucting a foundation for the fundamental theorem of calculus. In A. Cuoco & F. Curcio (Eds.), *The roles of representations in school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A. H. (2007). Early algebra as mathematical sense making. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 479-510). New York: Erlbaum.

Selter, C., Verschaffel, L., Greer, B., & De Corte, E. (2000). Making sense of word problems. *Educational Studies in Mathematics, 42*(2), 211-213.

Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). New York: Erlbaum.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165–208.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*(2-3), 229–274.

Thompson, P. W. (1995). Notation, convention, and quantity in elementary mathematics. In J. Sowder & B. Schapelle (Eds.), *Providing a foundation for teaching middle school mathematics* (pp. 199–221). Albany, NY: SUNY Press.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics*, MAA Notes (Vol. 73, pp. 43-52). Washington, DC: Mathematical Association of America.

Thompson, P. W., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools, 30*, 124-147.

Yerushalmy, M., & Swidan, O. (2012). Signifying the accumulation graph in a dynamic and multi-representational environment. *Educational Studies in Mathematics, 80*(3), 287-306.