Shape thinking and students' graphing activity

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We describe a construct called shape thinking that characterizes individuals' ways of thinking about graphs. We introduce shape thinking in two forms–static and emergent–that have materialized in our work with students and teachers over the past two decades. Static shape thinking entails thinking of a graph as an object in and of itself, and as having properties that the student associates with learned facts. Emergent shape thinking entails envisioning a graph in terms of what is made (a trace) and how it is made (covarying quantities). We provide illustrations of the shape thinking forms using examples from data that we have gathered with secondary students, teachers, and undergraduate students. We close with future research and teaching directions with respect to students' shape thinking.

Key words: Graphing; Function; Covariational reasoning; Quantitative reasoning

Students' and teachers' graphing activity remains a critical focal area in mathematics education, as their difficulties with graphs have short- and long-term consequences for their success in mathematics and other STEM fields (Oehrtman, Carlson, & Thompson, 2008). Despite their difficulties, students do construct stable and organized ways of thinking over the course of their schooling. Numerous researchers (including ourselves) have claimed that students develop ways of thinking about functions and their graphs that often lack a basis in reasoning about generalized relationships or processes between quantities' values (Dubinsky & Wilson, 2013; Lobato & Siebert, 2002; Oehrtman et al., 2008; Thompson, 1994b, 1994c). If graphs are intended to be representations of related quantities under a coordinate system (with a coordinate system itself being an organization of quantities), then we must ask:

1. *If students do not see a graph representing a relationship between quantities, then what do they think it represents?*
2. *What do we intend students to understand that a graph represents?*
3. *What ways of thinking are involved in understanding a graph as representing a relationship between quantities' values?*

We elaborate on a construct, called *shape thinking,* that we and others (Weber, 2012) have found useful in addressing each of these questions, both clarifying different ways of thinking students hold for graphs and characterizing a productive way of thinking about graphs as emergent relationships between quantities. We discuss shape thinking in two forms–*static* and *emergent*–that clarify important differences among students' understandings of graphs. In detailing the two forms of shape thinking, we draw illustrations of shape thinking from prior studies over the past two decades. For this reason, our purpose is not to report a single study, nor to report on the development or progress of a particular set of individuals. Instead, our purpose is to address the important questions above by describing distinguishable ways of thinking that students and teachers have for graphs.

Two Vignettes

We introduce the forms of shape thinking with two vignettes from clinical interviews (Goldin, 2000) of undergraduate students. Each vignette is a response to the prompt: *A middle-school student graphed the relation defined by* $y = 3x$ *as shown in Figure 1. How might he/she have been thinking when producing the graph?*

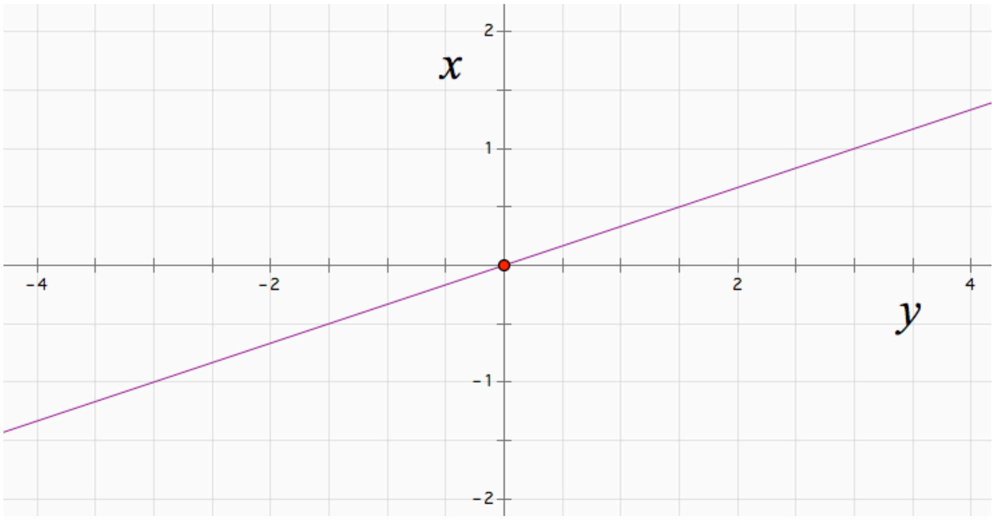

Figure 1. The middle-school student's graph.

Vignette 1

Student 1: He's thinking like this [*turning paper 90 degrees counter clockwise*]. But that's still not right because this is now a negative slope [*tracing the line*].

Int.: What if the student said, "Here's [*rotating back to Figure 1*] how I'm thinking"?

Student 1: The only way I can think of it is like this [*turning paper 90 degrees counter clockwise*] and it's still wrong because after I turn it, this [*laying the marker on the line sloping downward left to right*] is now a negative slope. When I was in middle school we learned a trick to remember positive, negative, no slope, and zero [*making hand motions to indicate directions*]. It's stuck with me so it's important to know which direction the slopes are going, where the slopes are.

Vignette 2

Student 2: He's thinking the rate of change is three, so the change in y is three times the change in x. He put the horizontal axis as y, so whatever he increased by in x, he increased by three times that in y [*tracing her pen to the right three and up one from the origin, then right three and up one from that point*]. He's right.

Int: So what if I do this [*turning paper 90 degrees counter clockwise*]?

Student 2: Well, the relationship is the same because the graph rotated with the axes, so it's still y equals three x. Change in x, change in y [*indicating an arbitrary change in x and a corresponding change in y*]. Change in y is three times change in x.

Student 1 and Student 2 each understood the graph, but their understandings were quite different–they assimilated the graph to different schemes. Student 1 drew on ways of thinking for slope (or rate of change) that were based in perceptual cues, such as thinking a line falling downward left-to-right unquestionably means negative slope. Because Student 1 associated properties of slope with a line's direction and location (e.g., "where the slopes were"), he concluded that rotating the graph changed the slope and, in his understanding, changed the represented relationship. Student 2 examined the graph in terms of how quantities varied in tandem within the axes orientation *as given*. By examining the graph in terms of covarying quantities, she understood that changing the axes' orientation or rotating the sheet of paper does not change the represented relationship. We consider Student 1's actions, which focused on perceptual cues and global properties of shape, to be indicative of *static* shape thinking. We consider Student 2's actions, which focused on the graph as an emergent object constituted by images of covarying quantities, to be indicative of *emergent* shape thinking.

Theoretical Framing

Our interest is to characterize persons' meanings and ways of thinking. To do so, we draw on Thompson and Harel's (Thompson, Carlson, Byerley, & Hatfield, 2014) description of understanding, meaning, and ways of thinking, which has roots in Piaget's (2001) notions of action, operation, scheme, and image. *Understanding* is an in-the-moment state of equilibrium, which may occur from assimilation to a scheme or from a functional accommodation specific to that moment in time. A *Meaning* is the space of implications that the moment of understanding brings forth—actions that the current understanding implies. *Ways of thinking* are "when a person has developed a pattern for utilizing specific meanings…in reasoning about particular ideas" (Thompson et al., 2014, p. 12). Returning to Student 1 and 2, each student's activity suggests that they had constructed specific meanings and ways of thinking about graphs in terms of slope or rate of change.

Data Sources and Motivation

The shape thinking construct originated during the second author's work with middle grade to post-secondary students and teachers. This work included clinical interviews (Goldin, 2000), teaching experiments (Steffe & Thompson, 2000), and professional development, during which he noted students and teachers holding particular dispositions toward understanding graphs. As our research moved forward, differences in students' and teachers' dispositions toward graphing became more apparent during studies in which we explored supporting students' and teachers' covariational and quantitative reasoning (Moore, 2014; Moore, Paoletti, & Musgrave, 2013; Thompson, 1994b, 2013). We often found students' and teachers' ways of thinking about graphs to have little connection to images of covariation, leading us to elaborate on the shape thinking construct. For instance, Weber (2012) introduced notions of *expert* and *novice shape thinking* when characterizing students' reasoning about rate of change in the context of multi-variable functions. Most relevant to the work here, the term static shape thinking emerged as a way to reference a student interpreting a graph statically and giving it meaning by way of association with learned facts, and emergent shape thinking emerged as a way to reference a student interpreting a graph through schemes and images of quantitative and covariational reasoning.[1]

Our motivation for introducing shape thinking and its forms also stems from our asking what a graph represents *to a student*. As mathematics teachers, our instruction on functions and relationships became more productive for student learning once we developed an ear for whether students were thinking about a graph as a static image and object in and of itself or as a trace of quantities having covaried. In what follows, we describe and illustrate both forms of shape thinking. Our illustrations originate from ongoing and retrospective conceptual analyses (Thompson, 2008; von Glasersfeld, 1995) of these clinical interviews, teaching experiments, and professional development.

Static Shape Thinking

Static shape thinking involves operating on a graph as an object in and of itself, essentially treating a graph as a piece of wire (graph-as-wire). Static shape thinking entails assimilations and actions based on perceptual cues and the perceptual shape of a graph. Because static shape thinking entails thinking of the graph as an object, meanings associated with static shape thinking treat mathematical attributes as properties of the graph-as-wire. For instance, in his moment of understanding, Student 1 (Vignette 1) assimilated slope (or rate of

[1] See (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Smith III & Thompson, 2008; Thompson, 1994a, 2011) for more extensive treatments of quantitative and covariational reasoning.

change) more as a property of the graph-as-wire as he perceived it (e.g., the wire rising or falling left to right) than as a measure of how one quantity changes with respect to another.

Another example of static shape thinking is when students treat equations, names, or analytic rules as facts of shape regardless of its coordinate system or orientation. For instance, Excerpts 1 and 2 show responses by two mathematics education undergraduate students to the prompt that a secondary student named Ralph thought that the graph in *Figure 2* displays the inverse sine function because, "*…we are graphing the inverse of the sine function, we just think about x as the output and y as the input*". We designed the task so that Ralph's claim captured the understanding that $y = \sin(x)$ implies $x = \arcsin(y)$ with the appropriate restriction on x. More generally, Ralph's claim rests on the fact that a graphical representation of a single-valued function can be thought of as simultaneously representing the function and its inverse relation (or its inverse function if the original function is strictly monotonic).

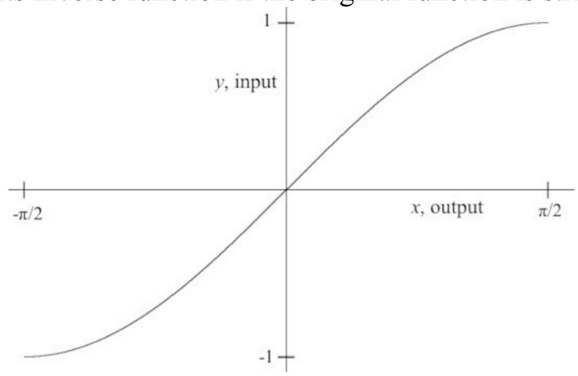

Figure 2. Graph and prompt posed to the PSTs.

Excerpt 1: *Sansa's response to Ralph's statement.*

Sansa: You can't just label it like that…I feel like he's missing the whole concept of a graph…I know you can call whatever axis you know if you are doing time and weight or volume or whatever. You can flip-flop those and be OK. But not necessarily with the sine graph. A sine graph's…a graph everyone knows about.

Excerpt 2: *Brienne's response to Ralph's statement.*

Brienne: I'm thinking this just kind of looks like…the plain sine graph (*laughs*). Which is going to be different. So, no… I guess what I'm like thinking, like struggling with thinking is that like, like I don't know if, or like an inverse function, like the graph of an inverse function, like, can't be the same as the original graph.

To Sansa, the given graph was "a sine graph…everyone knows about." We understood her to mean that the graph was a shape all mathematics students should know exclusively as "the sine graph". Brienne anticipated that a graph of the inverse sine function should appear, in shape, different than that of the given shape because a different function is being graphed. Both students' ways of thinking involved associating a shape with a named function such that no other function could name that shape; the students understood a function's name as a fact of shape (e.g., graph-as-wire).

Emergent Shape Thinking

Emergent shape thinking involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation). As opposed to assimilating a graph as a static object, emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities. Because we are limited to a static medium in this paper and cannot convey a trace in progress, we convey this way of thinking through snapshots of an emergent trace (*Figure 3*). Emergent shape thinking is more complex than depicted because it entails imagining what happened between snapshots.

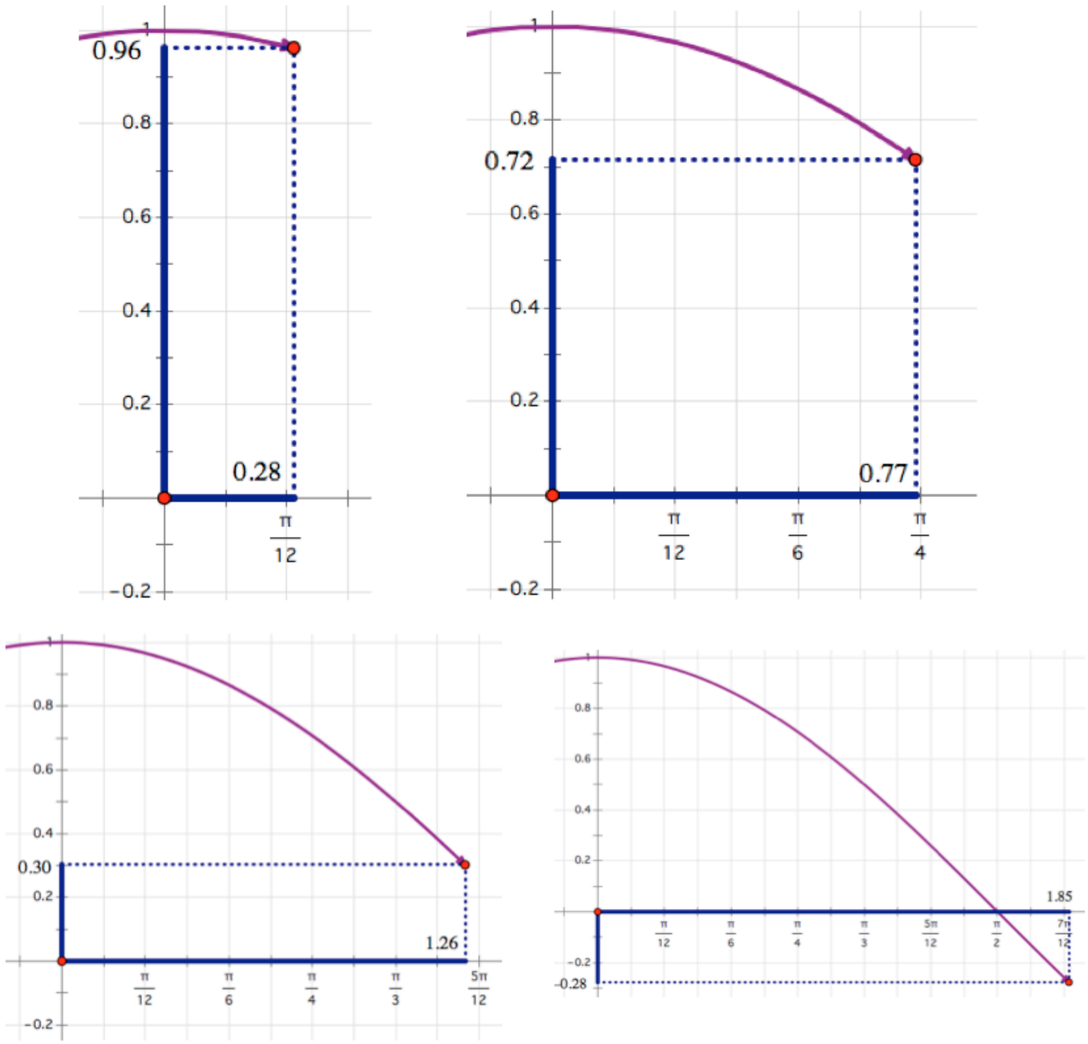

Figure 3. Instantiations of emergent shape thinking.

Because emergent shape thinking entails understanding a graph as an emergent trace of covarying quantities, meanings associated with emergent shape thinking treat mathematical attributes as properties of covariation. To Student 2 (Vignette 2), rate of change was a property of how x and y change together regardless of the graph's orientation. As another example, Shae, a mathematics education undergraduate, said that the graph in Figure 2 represents both $y = \sin(x)$ and $\sin^{-1}(x) = y$. The interviewer then asked Shae how the given graph related to a conventional graph of the arcsine function (e.g., input on the horizontal axis). He asked this question to determine whether presenting Shae with a different shape but the same stated function could perturb her enough that she evoke static shape thinking. Shae went on to describe that both graphs represent "the same thing" (Excerpt 3).

Excerpt 3: *Shae compares nonstandard and standard graphs of arcsine.*

Shae: You could just like disregard the y and x for a minute, and just look at, like, angle measures. So it's like here [*referring to graph of* $\sin^{-1}(x) = y$], with equal changes of angle measures [*denoting equal changes along the vertical axis*] my vertical distance is increasing at a decreasing rate [*tracing graph*]. And then show them here [*referring to graph of* $\sin^{-1}(y) = x$] it's doing the exact same thing. With equal changes of angle measures [*denoting equal changes along the horizontal axis*] my vertical distance is increasing at a decreasing rate [*tracing graph*]. So even though the curves, like, this one looks like it's concave up [*referring to graph of* $\sin^{-1}(x) =$

y from $0 < x < 1$] and this one concave down [*referring to graph of* $\sin^{-1}(y) = x$ from $0 < x < \pi/2$], it's still showing the same thing. [*Shae denotes equivalent changes on each graph as shown in Figure 4*]

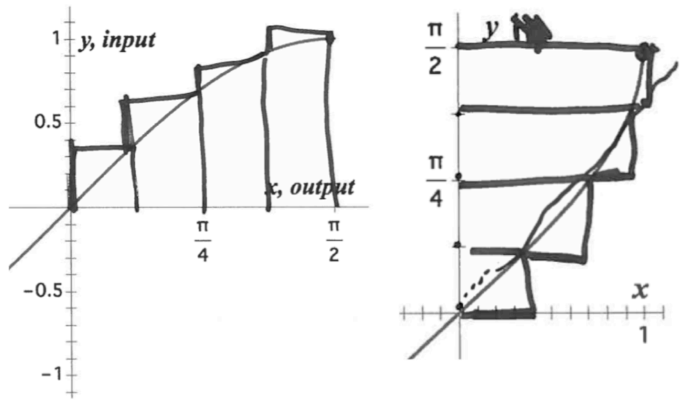

Figure 4. Two graphs that represent one relationship.

Shae reasoned that both graphs convey that some quantity increases at a decreasing rate as another quantity increases in successive equal amounts, while at the same time representing the respective quantities on her graphs (*Figure 4*). By conceiving each graph as an equivalent emergent relationship, Shae understood that a "concave up" trace conveys the same information as a "concave down" trace if the axes are switched. In contrast to Sansa and Brienne's understandings, Shae understood the traces she perceived as representing both the (restricted) sine and inverse sine functions. The function names were not names of a shape; they named a covariational relationship.

Reasoning about or with Quantities

A notable feature of students' static shape thinking is that images of covariation and points as projected quantities represented along the axes are absent from their in-the-moment thinking. In fact, coordinate axes and their representation of quantities are not critical to static shape thinking except in their labeling and orientation–static shape thinking's basis in observables and shape necessitates the labeling and orientations in which the students' abstracted their ways of thinking. In making the claim that images of covariation are absent from their in-the-moment thinking, we do not mean that students engaging in static shape thinking cannot or will not think about quantities and their relationships. Static shape thinking can entail drawing inferences about quantities and their relationships, where these inferences are drawn from a graph's appearance or shape. Hence, static shape thinking does not exclude reasoning about quantities, but the type of reasoning is indexical: a particular shape or property of shape implies something about quantities. Said another way, information about quantities and their relationships are implications of assimilation. To illustrate, students exhibiting actions like Student 1 (Vignette 1) can describe slope in terms of covarying quantities, but this is done after drawing inferences about the slope of the line from perceptual cues (e.g., using that a line falls left to right line to infer a negative slope and conclude y decreases in some manner as x increases).

Students who are limited to thinking about graphs in terms of static shapes can learn to associate information about quantities and their relationships with particular shapes. Students thinking statically are limited to making empirical abstractions from their activity. Students who are capable of thinking about graphs emergently gain insight into relationships between quantities that are more organic to the quantities and relationships. Also, students thinking about graphs emergently are positioned to reflect on their reasoning to form abstractions and generalizations from their reasoning. Hence, unlike associations abstracted through static shape thinking, relationships abstracted through emergent shape thinking are not constrained to a particular labeling and orientation. When changing coordinate conventions and systems, the shape associated with a particular relationship changes in a visual sense. But, the mental operations involved in emergent shape thinking enable constituting a trace in any coordinate system as representative of the same covariational relationship given that the student understands the coordinate system's quantitative structure.

Future Directions for Shape Thinking

Students' shape thinking raises a number of new directions for mathematics education research and teaching. First, researching students' capabilities and constraints when limited to static shape thinking or capable of emergent shape thinking will form a productive line of inquiry. We consider this line of inquiry to offer a new perspective on multiple representations by enabling researchers to be clearer about what a graph represents *to a student*, and thus what students understand multiple representations to be representations of. We find that emergent shape thinking enables students to move among representations while maintaining a subjective sense of invariance in the form of covarying quantities (see Excerpt 3 and Moore et al. (2013)), thus supporting them in conceiving the 'something' that multiple representations are to represent. Second, because static shape thinking can entail inferences about quantities and their relationships, meanings associated with static shape thinking might stem from abstractions partly involving emergent shape thinking. A student who is capable and prone to think about graphs emergently might come to know a family of graphs so well that that thinking statically about a graph implies what they know about its emergence. Understanding these relationships between students' static and emergent thinking will entail exploring the ways that students' static and emergent thinking do or do not complement each other. Such explorations will provide insights into how students' ways of thinking about graphs develop and will contribute additional clarity to the notions that Weber (2012) identified as expert and novice shape thinking. Lastly, we see our work providing a lens to organize and frame curricular approaches more carefully with respect to intended student ways of thinking for graphs and their functions. For instance, we hypothesize that supporting students in thinking emergently will better position them to envision functions as mappings once they can envision completed covariation as having produced all possible pairs of values associated with a mapping. On the other hand, we hypothesize that an approach to function that introduces a parent shape and treats other functions as translations of the parent shape is more likely to promote students thinking of graphs as objects in and of themselves. We find it difficult to envision students making sense of translations as more than moving a graph-as-wire to different locations in the plane according to learned rules without a robust understanding of graphs as having emerged through covariation.

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