CALCULUS STUDENTS' UNDERSTANDING OF RATE

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ABSTRACT

This study investigated first semester calculus students' understanding of rate of change and how their understandings were affected by instruction on the derivative. Ninety college students were given a pretest on the concept of rate of change at the beginning of a first semester calculus course and then given a posttest, the same test, soon after completing their study of the derivative. Six students who had taken both tests were interviewed on their understanding of the concepts presented on the tests.

There was little difference in performance between the results on the pretest and the results on the posttest. A study of responses to individual items on both tests and during the interviews enabled common errors and misconceptions to be identified. Implications are drawn for teaching rate of change and the derivative.

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CHAPTER I

INTRODUCTION

Current Failings of Calculus Instruction

Up through the 1950's calculus, at the undergraduate college level, was a course taken during a student's sophomore year, and the freshman year prepared for it (Douglas, 1986). However, the freshman year's course was not a "precalculus" course. The freshman course contained an overview of many topics, focusing on fundamental concepts and interesting ideas of mathematics. For most students it was a terminal course and all they needed for their majors. For those students who continued, it was the common base from which they started calculus. This freshman course disappeared in the 1960's when calculus became a freshman course.

Little changed in the freshman curriculum after calculus became a freshman course until 1979, when dissatisfaction with students' learning of calculus began to emerge. The failure rate among first semester calculus students had approached fifty percent. These factors contributed to a shortage of college students choosing mathematics, physics, or engineering as careers.

Conceptual Understanding

Calculus builds on many fundamental mathematical concepts and skills which must have been previously understood by students. Skemp (1978) distinguishes between two types of understanding, “relational understanding” and “instrumental understanding”. He wrote:

> “By the former is meant what I have always meant by understanding, and probably most readers of this article: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as ‘rules without reasons’ without realizing that for many pupils and their teachers the possession of such a rule, and ability to use it, was what they meant by understanding.”

There is a clear reference in this to the distinction between processes which can be carried out by using rules, such as taking the derivative of a polynomial function in first semester calculus, and the ideas or concepts which underlie the processes. These concepts may be extremely difficult to understand, particularly if insufficient time is spent on them. In calculus, understanding instantaneous rate of change, for example, depends on understanding functions, average rate of change, and on limits. Instructors from all college departments are less concerned that all the standard topics be covered than that students learn and understand these fundamental background concepts in order to become better problem solvers.

One reason that conceptual understanding of rate and ratio is so important is their usefulness in solving real world problems to acquire consumer skills, to develop proportional thinking as a problem-solving technique and to acquire skills for vocational application (NCTM, 1989). People in government, industry and business often communicate ideas using graphs; a knowledge of calculus helps to read and understand those graphs. Reading and understanding a graph includes the ability to understand relationships among change, average rate of change and instantaneous rate of change.

The concepts of rate, ratio and proportion are a major concern to science educators. Wollman and Lawson (1978, p 227) emphasize the importance of ratio and proportion as the most universal mathematical tools of any introductory science course. Douglas (1986) argued that almost all of science is concerned with the study of systems that change, and the study of change is the very heart of the calculus.

As argued earlier, the concept of rate of change is foundational to students' understanding of ideas in calculus, and most certainly is foundational to their understanding of the derivative as instantaneous rate of change. When do students come to understand the ideas of rate of change? Casual conversations with mathematics faculty suggest that they presume very little understanding of rate of change beyond "d = rt" and that any deeper understanding will come by studying the derivative in calculus. This leads to an empirical question: Do students enrich their understanding of rate of change by studying the derivative in a college calculus course?

Purpose

The purpose of this research is two-fold. It is intended to survey students' understanding of rate of change at the beginning of a first semester calculus course and again soon after completing their typical study of the derivative. By examining students' understandings of rate of change at the beginning of the first semester calculus course and soon after their typical study of the derivative, we can gain insight into the reasons why some calculus students have difficulty acquiring a conceptual understanding of the derivative.

CHAPTER II

REVIEW OF THE LITERATURE

This chapter will review the basis for the present study of calculus students' understanding of rate of change. A brief section will describe the current state of calculus education, its implications, and the role that the typical study of the derivative plays in students' understanding of the concept
of rate of change.

Current State of Calculus Education and the Understanding of Rate

The failure rate of first semester calculus students in the United States today is approaching fifty percent. Douglas (1986) stated that this failure rate among first semester calculus students "cannot be tolerated any longer". Also, regarding students who completed first semester calculus, he stated that "physicists and others complain that students don't understand the basics of calculus, such as motion, velocity or acceleration."

Many calculus students do not have a rich, meaningful, conceptual base from which to understand the fundamental concepts of the calculus. They do not know how to use the calculus, what can be done with it and do not have a feeling for what it is about. Yet calculus is one of the greatest technical advances in exact thinking in history. A reform of the calculus curriculum is underway, but as Douglas (1986) emphasized, "no real consensus has been reached on the problem of reducing the failure rate and the methods for teaching conceptual understanding of calculus."

Study of Rate at the Secondary Level

Indications are that many concepts within the domain of multiplicative structures are not well taught nor are they well learned. Numerous studies have shown that early adolescents and many adults have a great deal of difficulty with the basic concepts of fraction, rates, and proportion (Boyer, 1946; Confrey, 1992; Hart, 1978; Heller, Post, Behr & Lesh, 1990; Karplus, Pulos & Stage, 1983; Monk, 1992; Orton, 1984; Thompson, 1992; Tournaire & Pulos 1983). These studies indicate that students often use incorrect and inappropriate qualitative reasoning in solving problems involving basic multiplicative concepts, often using additive approaches where multiplicative approaches are required.

Heller, Post, Behr and Lesh (1990) stressed the importance of rate, ratio, and proportion in the interpretation of dynamic phenomena. They observed that one of the extreme deficiencies in the cognitive development of students at the secondary level is their failure to master reasoning involving ratios. Their poor performance on Piagetian tasks of ratio reasoning indicate that only about one-fifth of seventh graders and one-fourth of eighth graders have an understanding of simple rate and proportion. Lesh, Post, and Behr (1988) argue that proportional reasoning is both the capstone of the middle school mathematics program and the cornerstone of all that is to follow.

What is Rate of Change?

One indication of the complexity of the concept of rate of change is the lack of consensus in distinguishing between rate and ratio. Thompson (1994) illustrated the lack of consensus by noting:

> Perhaps the lack of conventional distinction between ratio and rate is the reason that the two terms are used often without definition. Lesh, Post, and Behr noted that "... there is disagreement about the essential characteristics that distinguish, for example rates from ratios ... In fact, it is common to find a given author changing terminology from one publication to another" (1988, p. 108). The most frequent distinctions given between ratio and rate are:
>
> 1) A ratio is a comparison between quantities of like nature (e.g., pounds vs. pounds) and a rate is a comparison of quantities of unlike nature (e.g., distance vs. time; Vergnaud, 1983, 1988).
>
> 2) A ratio is a numerical expression of how much there is of one quantity in relation to another quantity; a rate is a ratio between a quantity and a period of time (Ohlsson, 1988).
>
> 3) A ratio is a binary relation which involves ordered pairs of quantities. A rate is an intensive quantity - a relationship between one quantity and one unit of another quantity (Kaput, Luke, Poholsky, & Sayer, 1986; Lesh, Post, & Behr, 1988; Schwartz, 1988).

Thompson offered a unifying definition of rate by focusing on the ideas of rate and ratio from a developmental perspective.

Development of the Concept of Rate

Thompson stated that

> "the development of images of rate starts with children's image of change in some quantity (e.g., displacement of position, increase in volume), progresses to loosely coordinated images of two quantities (e.g., displacement of position and duration of displacement), which progresses to an image of the covariation of two quantities so that their measures remain in constant ratio."

Young children, according to Piaget (1970, p 226-325), understand movement as a change of location instead of as a distance traveled. That is, children's earliest conception of movement is concerned only with starting and stopping points, and is unconcerned with an amount of displacement. Speed, for young children, is simply the intuition of overtaking. That is, if two objects are in motion simultaneously, the one which is in front of the other has the greatest speed, regardless of where they started. The young child does not understand that speed is a relationship between distance and time but rather equates speed with distance or time. For example, some children think that taking a longer time to travel a certain distance implies a greater speed. Monk (1992) stated that perhaps students have only the most naive ideas about speed, distance, and time, regardless of how information about them is represented. Monk (1992) also stated that it is not uncommon for adults to be able to interpret a speedometer reading only in terms of a primitive intuition that might be called "fastness", having little idea of how to interpret this reading to changes in position in relation to changes in time.

Adolescents' ratio and proportional reasoning, according to Piaget (1970), develops from a single strategy, often additive in nature, to the mature scheme of constant rate of change which involves the understanding that increases are in the same proportion as that of the previously acquired amounts, and hence, in the same proportion as that of the total amounts acquired including the increases. He also stated that the scheme of rate and proportion is a late acquisition.

The ideas of average rate of change of one quantity with respect to changes in another, average rate of change of a function over an interval, secant to a graph and tangent to a graph are related. Average rate of change of one quantity with respect to changes in another is the constant rate of change at which one quantity would vary with respect to the other so that the same total changes occur as before. A function, f(x), may be used to describe the relationship between the real-life quantities. Then the average rate of change of one real-life quantity with respect to changes in the other, say from x to x+h, can be related to the average rate of change of the function f(x) over the interval [x, x+h] since they describe they same idea. In the latter case, the average rate of change of a function f as x ranges from x_0 to x_0+h would be the function which has a constant rate of change and which produces the same total change over [x_0,x_0+h] as did f. However, we may discuss the average rate of change of a function over an interval without referring to a real-life situation.

The secant to a graph can also be related to the average rate of change of one quantity with respect to another. If a function f(x), used to describe the relationship between the real-life quantities is graphed, the secant to the graph of the function is the graph of the function which has a constant rate of change and which produces the same total change over the interval [x, x+h] as did the original function. We think of the secant as a graphic illustration of the average rate of change of one quantity with respect to another and the average rate of change of a function over an interval which are numerical.

The tangent to the graph is the graph of the function which is the limit of the sequence of functions associated with the secants. The slope of the tangent to a graph is defined to be the instantaneous rate of change of a function.

The derivative, used to represent point properties of a curve or a function, can be thought of as the average rate of change "at a point", the velocity of a body in motion at any given time or the instantaneous velocity. Instantaneous velocity, the derivative, could be defined as the limit of the average velocity as the time interval approaches zero. Strang stated that, "the whole subject of calculus is built on the relation between velocity and distance." A conceptual understanding of the derivative of any function f differentiable at a point a, is thinking of f'(a) as the rate of change of f at a. Many students tested and interviewed do not have this conceptual understanding of the derivative. Also, it was seen that their informal image of rate of change is kept separate from related ideas of the derivative being studied formally.

An important aid in visualizing the relationship between average rate of change and the derivative is graphing. From the above definitions of (average) rate of change and derivative, if f is a function of t, it is possible to graph the relationship between f and t. The slope of the secant between two points on the graph, representing the average rate of change, and the slope of the tangent at a point, representing the derivative, are very closely related. On the graph of f(t) from a to a+h the distance up divided by the distance across gives the slope of the secant and represents the constant rate of change which produces the same total change over [a, a + h] as does the original function. As h approaches 0, the slope of the tangent at a, representing the derivative, can be calculated.

For grades 9 to 12, the Summary of Changes in Content and Emphasis in the Standards states that students be able to make "the connections among a problem situation, its model as a function in symbolic form, and the graph of that function". If a student were given a problem situation involving distance and time, he may be able to express the situation as a function f(t), where f(t) is the distance as a function of time, and then graph the function. He then may be able to relate the slope of a secant line to average velocity. Finding the slope of the tangent at a point on the graph of the function by defining the velocity at a particular point along the graph of the function would lead to the concept of the derivative. At any point along the graph of the function, the derivative or the slope of the tangent may be used to describe the rate at which a graph rises (or falls) and instantaneous velocity is the rate at which the distance changes with respect to time. We can apply the derivative concept to any quantity that can be represented by a function. Since quantities of this type occur in nearly every field of knowledge, applications of the derivative are numerous and varied, but each concerns a rate of change.

Students' Difficulties with the Derivative

Ferrini-Mundy and Graham (1994) studied connections students had made between the graphical and algebraic representations of a function and how these connections relate to their understanding of the limit, continuity, derivative and definite integral of a function. They stated that the students they interviewed had the most difficulty relating their graphical and formula-based understandings of a function to the derivative. Their ability to compute derivatives using algorithms was excellent, as well as their ability to test points, and determine positive and negative derivatives from curves. However, their connections between procedural and conceptual knowledge of the derivative was weak since they had no idea how the derivative of a function related to the function itself or how the tangent line related to the derivative. Also, they could not relate the idea of differentiability to the idea of the smoothness of a graph.

Orton (1984) investigated elementary calculus students' basic algebra and algebraic manipulation, their understanding of rate of change and its relationship with differentiation, their use of limits, and their understanding of basic calculus' symbols using a wide range of questions which occur when calculus is being introduced. Students found eight questions difficult; four concerned with understanding integration as the limit of a sum and four concerned with understanding differentiation based on rate of change and limits. He noted that students' routine performance on differentiation items was adequate, but that little intuitive or conceptual understanding of the derivative is present. He argued that calculus should be introduced only after background work on limits and rate of change had been included within the curriculum over a period of years.

Orton and Ferrini-Mundy had examined students' difficulties in understanding the derivative from a perspective that was highly symbolic. Another possibility was that students' difficulties with symbolic calculus was an expression of their inability to connect the symbolism with anything conceptual which would be naturally expressed by the notational methods they attempted to memorize. The present study investigates that possibility by examining students' conceptual understanding of rate of change before and after they study the derivative. If it turns out that their understanding of rate of change improves substantially, they probably connected their thinking, while studying the derivative, to intuitions they already had - thereby transforming both their conceptual understanding and its connections with the notational methods of calculus. If it turns out that their understanding of rate is poor both before and after studying the derivative, then it would be reasonable that they have difficulty using techniques of calculus skillfully and thoughtfully, for those techniques would have no basis in understanding.

CHAPTER III

METHODS

Subjects and Setting

Subjects for the study were students enrolled in Math 150, Single Variable Calculus, at San Diego State University during the Spring 1994 semester. All sections of Math 150 used (Swokowski, 1991) as their primary text.

Of the six Math 150 calculus sections offered during this semester, the three largest classes, with approximately 30 students each, participated in the investigation. The remaining three classes, with fewer than 10 students each, did not participate in the investigation. The three largest Math 150 classes were tested twice with one written examination administered on two occasions. The first administration was given to a total of 90 students. Two classes were given the pretest at the end of the first week of the course and the third class was given the pretest at the beginning of the second week of classes. The second administration was given soon after these same students had completed their study of the derivative, about six weeks after the start of the course. However, only 57 students of the original 90 students were present to participate in the posttest.

Initial Procedures

After developing the written examination, the investigator presented the examination to four people for review. Some of the questions were then modified for readability and others were removed or replaced.

Prior to the start of the study, the investigator obtained approval from the professors whose classes would be used for the written examinations. It was emphasized to the students and stated on the first page of the examinations that participation in the research was voluntary and their results on the examinations would not affect their grade in the course. It was assumed, however, that the students completed both written examinations to the best of their ability.

Written Examination Procedures

The pretest was administered during students' normal class time. The pretest was administered during the first or second week, prior to any discussion of the derivative. The posttest was administered after students had taken their last course exam on the derivative and before they began studying the integral. Times between the pretest and posttest ranged from 6.5 weeks to 8.5 weeks.

Three forms of the test (Form A, B, and C) were constructed by putting items of Form A in two random orders (Forms B and C). Each class was given approximately 10 copies of each form. By giving identical problems but in a different order each problem would be attempted by at least some of the students. The entire examination was designed to take approximately 50 minutes, the length of their normal class period. Some students finished earlier and others did not complete the pretest during this time.

The posttest was given just after the students had completed their study of the derivative, at about six weeks into the course. Each student took the same test form as on the pretest. Each question is presented and discussed separately in Chapter IV. The entire test is presented in Appendix A.

Interviews

Each pretest had a checkbox where students could indicate their willingness to be interviewed. Six students who took both pretest and posttest were selected to be interviewed. The selection of these six students was made as follows. First, the upper, middle, and lower range of the performance scores were determined. Then names of two students from each of the performance groups, who had responded with lengthy explanations of their answers on the written examinations and had indicated a willingness to be interviewed, were selected. Of the three classes, three students were from one class, two students were from another class and one student was from the third class. The investigator then contacted each selected student to schedule an interview.

Interviews were held in a faculty office with only the interviewee and investigator present. All interviews were videotaped, and each interview lasted approximately 2.5 hours. Students were asked to explain, in as much detail as possible, their understandings of and responses to each question on the examination. The investigator would interject prompts where appropriate to elicit clarifications and expansions (e.g., “What did you mean when you wrote ...?” and “I don’t quite understand what you mean when you say ...”). The investigator then watched the videotapes in order to gain a richer sense of students’ understanding of the concept of rate. Information from interviews was analyzed separately and was used to place students’ performance data in a larger context of how they connected the exam with what they already knew and with their work in introductory calculus.

Data Analysis

Tests were analyzed from two perspectives: performance and process. Performance data was gotten by scoring students' answers according to whether they were correct. Process data was gotten by analyzing students' solution methods and scratch work. Performance was scored 0 or 1; solution processes were categorized. In both cases, data from pretest and posttest were analyzed by constructing contingency tables to examine pre-post differences.

Students' performance on an item was scored 1 (correct), 0 (not correct), N (no response), or A (absent). Criteria for correctness varied from problem to problem, and is discussed in Chapter 4 in the context of each item's results. A score of "A" was given only on the posttest, and only for students who were present at the pretest but absent on the posttest. The score "N" (no response) was assigned only if an item was left completely blank. Thirty-three of 90 students who took the pretest were absent for the posttest. Data analyses were performed only on the 57 students who took both pretest and posttest.

Categories of students' processes were constructed from students' responses. For each item, the investigator sorted responses by initially "putting those together that go together." When finished with the initial sort, the investigator refined the categories by reassigning responses to groups that they seemed to fit better. When the sort was stable (i.e., the investigator saw no further refinements to be made), the investigator then examined the groupings and assigned descriptive names to each group of responses. The resulting categories are explained in Chapter 4 in the context of each item's results.

CHAPTER IV

RESULTS AND FINDINGS

Analyses of Individual Problems

Each problem revealed unique aspects of students reasoning, and students' performance varied considerably from problem to problem. As such, the investigator discusses data on each problem, and the discussion has this organization: Discussions of students' performance (on those problems where such analyses make sense), discussions of students' thought processes which the investigator inferred from their work, and a summary of the interviews held with the sample of six students.

In the discussions of students' performance, the investigator explains the intent of the problem, discusses criteria for assigning scores, presents pretest results, presents posttest results, and compares the two performance results. In the discussions of inferred thought processes, the investigator explains the categories into which students' work were put, presents the categories to which students were assigned on the pretest, the categories to which students were assigned on the posttest, and presents cross-tabulations of the two.

In the interview summaries, the investigator presents excerpts and illustrations of students' thinking which serve to illuminate the measurement data. At times the interviews will be consistent with the data, but at times it will show that students were thinking at a much less sophisticated level than their work might suggest.

Problem 1

A textbook stated: “A car traveled 5 miles across town in 15 minutes.” Is it reasonable for the teacher to ask how far the car went in the first 2 minutes?

(a) Yes. It went ____ miles.

(b) No. Because … (explain)

Intent of Problem 1

Problem 1 presents a situation which, in junior high school, is often intended to have students reason proportionally. What is often presumed, but unstated, is that in order to use proportional reasoning one must assume that the car traveled at a constant velocity. But nowhere in the text is such a condition stated. Alternative (a) was offered with the intention that students who assumed, consciously or unconsciously, that the car traveled at a constant velocity would select it. Alternative (b) was included so that students who recognized the inappropriateness of assuming constant velocity would be free to state as such.

Criteria for Assigning Scores

The nature of Problem 1 is such that students’ answers are not strictly correct or strictly incorrect. So performance data does not actually reflect correct performance. Rather, it reflects, to some extent, what students presumed about the situation.

The investigator assigned a “1” to any answer which suggested that the student thought that going 5 miles in 15 minutes says nothing about where the car was at any particular moment in time between when it started and when it ended. The investigator assigned a "0" to all other answers, whether or not their arithmetic was correct.

Table 1

Rows are levels of **1**
Columns are levels of **P 1**
No Selector

	0	**1**	**A**	**total**
0	33 36.7	7 7.78	21 23.3	61 67.8
1	5 5.56	12 13.3	12 13.3	29 32.2
total	38 42.2	19 21.1	33 36.7	90 100

table contents:
Count
Percent of Table Total

*Frequencies of students who assumed constant speed (**0**), who did not assume constant speed (**1**), or who were absent from the posttest (**A**). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

Table 1 shows that 61 students (67.8%), approximately two-thirds of the students, assumed that the car had traveled at a constant velocity. Twenty-nine students (32.2%) assumed the car had not traveled at a constant velocity. Of these 29 students, four students answered both that it was unreasonable to ask where the car was after two minutes, because you could not assume it traveled at a constant velocity, and that the car went 2/3 mile in the first two minutes (see Table C1).

Performance on Posttest

Of the 57 students who were present for the posttest, 38 students (66.7%), approximately two-thirds of the students, assumed the car traveled at a constant velocity.

This is consistent with the pretest results. Nineteen students (33.3%) assumed the car had traveled at an average velocity. Of these 19 students, three students answered both that it was unreasonable to ask where the car was after two minutes, because you could not assume it traveled at a constant velocity, and that the car went 2/3 mile in the first two minutes (see Table C2).

Comparison of Pretest and Posttest Performance

We can see how consistent were students pretest and posttest performance by examining the (0,0) and (1,1) cells of Table 1. These cells show that, of the 57 students who took both tests, 45 (78.9%) were consistent from pretest to posttest. The (0,1) cell shows that seven students (12.3%) changed from assuming constant velocity to not assuming it. Cell (1,0) shows that five students (8.8%) changed from not assuming constant velocity to assuming it.

Table C1

Rows are levels of **C1**
Columns are levels of **PC1**
No Selector

	averv	**both**	**cnstv**	**total**
averv	10 17.5	0 0	3 5.26	13 22.8
both	1 1.75	2 3.51	1 1.75	4 7.02
cnstv	5 8.77	1 1.75	34 59.6	40 70.2
total	16 28.1	3 5.26	38 66.7	57 100

table contents:
Count
Percent of Table Total

averv *Average velocity.* Realized this was an average speed not constant.

both *Both.* Answered both parts.

cnstv *Constant velocity*. Took 5 miles in 15 minutes as a constant speed, not an average speed.

Categories of Inferred Thought Processes

Table C1 shows the grouping of the inferred thought processes. The nature of Problem 1 was such that performance was essentially equivalent to inferred thought processes. Therefore, no further discussion will be given for Problem 1 regarding inferred thought processes.

Summary of Interviews

Of the six students interviewed, four assumed the car was traveling at a constant velocity during the car's trip across town. These students stated that the car traveled 1 mile in 3 minutes, hence 2/3 mile in 2 minutes. When these four students were asked to describe a trip through a typical town, they immediately realized that their velocity would not be constant. They changed from assuming that the velocity was constant to not assuming it, stating that it wouldn't be reasonable to ask how far the car traveled in the first 2 minutes. However, one of these students added that in the first 2 minutes the car "would be about 2/3 miles". From this last comment, it can be presumed that there is not much distinction between this student's concept of average velocity and constant velocity.

The remaining two students who were interviewed did not assume constant velocity for the car's trip through town. They stated that "the car did not travel at a constant speed due to stops, etc., so we cannot guess how far the car went in two minutes". However, one of these two students had assumed constant velocity for the car's trip through town during the pretest.

Problem 2

A car went from San Diego to El Centro, a distance of 93 miles, at 40 miles per hour. At what average speed would it need to return to San Diego if it were to have an average speed of 65 miles per hour over the round trip?

Intent of Problem 2

This problem explores whether a student can take into account the fact that an average speed is determined by traveling a total distance in some amount of time. Usually students simply assume that the arithmetic average of the one way speeds must equal the average speed for the entire round trip. For example, many students think that a car that traveled one way at an average speed of 50 mph and then back over the same distance at an average speed of 100 mph had an average speed for the entire round trip of 75 mph. The arithmetic mean is appropriate only when the amounts of time driven are the same; but it is inappropriate when only the same distances are driven. These same students do not have an understanding that, for the same distance, the car traveled twice as long at an average speed of 50 mph as 100 mph, making the average speed 66 2/3 mph.

Criteria for Assigning Scores

A “1” was assigned to any answer which suggested that the student had tried to solve Problem 2 by taking the difference in time between the entire round trip at an average velocity of 65 mph and the one way trip an average velocity of 40 mph. A “0” was assigned to those answers which were conceptually incorrect.

Table 2

Rows are levels of **2**
Columns are levels of **P 2**
No Selector

	0	**1**	**A**	**N**	**total**
0	49 54.4	1 1.11	28 31.1	2 2.22	80 88.9
1	2 2.22	1 1.11	2 2.22	1 1.11	6 6.67
N	1 1.11	0 0	3 3.33	0 0	4 4.44
total	52 57.8	2 2.22	33 36.7	3 3.33	90 100

table contents:
Count
Percent of Table Total

*Frequencies of students whose answers were conceptually incorrect (**0**), who considered the difference in times of travel between the entire round trip at an average of 65 mph and the one way trip at 40 mph (**1**), who were absent from the posttest (**A**), or who gave no response (**N**). Rows are levels of performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

Table 2 shows that 80 students (88.9%) solved this problem incorrectly. Only six students (6.67%), the third lowest number of correct responses overall, considered the times of travel for the two round trips in order to find the average velocity of the return trip. The remaining four students (4.44%) did not give a response to this question.

Performance on the Posttest

Of the 57 students present for the posttest, 52 students (91.2%) solved this problem incorrectly. Two students (3.5%) gave the correct response by considering the times of travel for the two round trips. The remaining three students (5.26%) did not give a response.

Comparison of Pretest and Posttest Performance

By examining the (0,0), (1,1) and (N,N) cells of Table 2, we see that of the 57 students who took both tests, 50 students (87.7%) were consistent. The (0,1) cell shows that one student changed from taking the arithmetic average of the one way speeds or some other incorrect response to considering the times of travel. Cell (0,N) shows that two students who had taken the arithmetic average of the one way speeds or had given some other incorrect response, gave no response on the posttest. Cell (1,0) shows that two students changed from considering the times of travel to taking the arithmetic average or giving some other incorrect response. Cell (1,N) shows that one student who had considered the times of travel on the pretest gave no response on the posttest. Cell (N,0) shows that one student changed from giving no answer to taking the arithmetic average of the one way speeds or giving an incorrect response. Cell (N,1) shows that of the students who gave no response on the pretest, none gave the correct response on the posttest.

Table C2

Rows are levels of C2
Columns are levels of PC2
No Selector

	N	aaver	cmprt	cni	nwk	onet	total
N	0 0	0 0	0 0	0 0	0 0	1 1.75	1 1.75
aaver	1 1.75	25 43.9	1 1.75	2 3.51	2 3.51	2 3.51	33 57.9
cmprt	2 3.51	2 3.51	1 1.75	0 0	0 0	0 0	5 8.77
cni	0 0	1 1.75	0 0	1 1.75	1 1.75	0 0	3 5.26
nwk	0 0	5 8.77	0 0	0 0	6 10.5	1 1.75	12 21.1
onet	0 0	1 1.75	0 0	0 0	2 3.51	0 0	3 5.26
total	3 5.26	34 59.6	2 3.51	3 5.26	11 19.3	4 7.02	57 100

table contents:
Count
Percent of Table Total

aaver *Arithmetic average* (got 90 mph by (40 + x)/2 = 65
cmprt *Compared time for the round trips*
cni *Could not interpret*. Tried one or more inappropriate methods but not (40+x)/2 = 65 (For example: 40mph/93mi + x mph/93mi = 65mph/186mi)
nwk *No work*
onet *One time*. Found the time it took from San Diego to El Centro, but didn't make any other observations or calculate the time for the round trip.

Categories of Inferred Thought Processes

The nature of Problem 2 was such that performance was essentially equivalent to inferred thought processes. However, Table C2 shows that three students (5.26%) on both tests wrote equations which the investigator could not interpret, for example, 40 mph/93 mi + x mph/93 mi = 65 mph/186 mi. Also, three students (5.26%) on the pretest and four students (7.02%) on the posttest calculated the one way time of travel from San Diego to

El Centro (from that trip's given distance and speed) but did not make any other observations or calculations.

Summary of Interviews

Of the six students interviewed, four students solved Problem 2 incorrectly by taking the arithmetic average of the one way speeds. These students concluded that the return trip needed be at an average speed of 90 mph for the entire round trip to have an average speed of 65 mph since the average one way speed was 40 mph. They were then asked to compare the times of travel for these two round trips. They did not perform any calculations but strongly believed that the times of travel were the same for these two equal distance round trips.

Four interview students solved Problem 2 by finding the arithmetic average of the one way speeds. Three of these four students did not use the equation $(40 + x)/2 = 65$. Instead, these three students discussed Problem 2 in terms of "balancing" the one-way speeds. They stated that "since 40 mph is 25 mph less than 65 mph, 25 mph would need to be added to the 65 mph on the return trip in order that the average speed for the round trip be 65 mph".

Another interviewed student stated that Problem 2 did not have an answer, since Problem 2 may be similar to Problem 1. That is, the car in Problem 2 may also be traveling through a town where the car's velocity varies along the way. It was presumed that this student had an inadequate conceptual image of the average velocities, not understanding that average velocity allows for variations during a trip.

The sixth interviewed student had received the highest score overall on both tests and was one of the two students who had given a correct response on the posttest to

Problem 2. He had focused on the arithmetic average of the speeds in the pretest but on the posttest had written “186 mi/65 mph = 93 mi/40 mph + 93 mi/x mph”. This last equation indicated to the investigator that he now had an understanding of the relationship between the times of travel in Problem 2. This equation indicated that he understood that the time of travel for the round trip at an average speed of 65 mph was the same as the time of travel for the round trip with a one way average speed of 40 mph and the other one way with an average speed of x mph. However, when asked to explain his understanding of Problem 2 and how this led to the equation which he had written, he replied, “I don’t know how I got this equation.” He explained that he had basically set up a proportion, being careful to place the same units in the numerator or denominator but was unconcerned about which units would be in the numerator as compared to the denominator. The order of placement, of course, is significant in order for the times of travel for the round trips to be equal. It was found through interviewing this student regarding his understanding of Problem 2, that he had selected the correct order of placement of the units in setting up his proportion purely by chance. The investigator had assumed, from the proportion that this student had written, that this student possessed the understanding in Problem 2 that the times of travel for the round trips had to be equal in order to solve this problem. This, however, was an incorrect assumption.

Problem 3

Fred and Frank are two fitness fanatics. On a run from A to B, Fred runs half the way and walks the other half. Frank runs for half the time and walks for the other half. They both run and walk at the same speed. Who finishes first?

Intent of Problem 3

The intent of Problem 3 was to see if students could reason imagistically and schematically about speed in relation to distance and time. For example, if we actually imagine two runners running, then they are side-by-side up to the moment one of them begins walking. After that, the one who continues running will move ahead, and will maintain his lead even after he begins to walk. Fred will begin to walk at the half-way point; Frank needs to continue running past the half-way point, since otherwise he would walk for a longer period of time than he runs (presuming he walks slower than he runs). Therefore, Frank will move ahead of Fred as soon as Fred starts to walk at the half-way point, and when Frank begins to walk he will maintain whatever lead he has at that moment. Therefore, Frank will win.

Criteria for Assigning Scores

A “1” was assigned to any answer which suggested Frank would finish first, even if the student did not give an explanation or show a representation of the situation. A “0” was assigned to all other answers.

Table 3

Rows are levels of **3**
Columns are levels of **P 3**
No Selector

	0	**1**	**A**	**N**	**total**
0	36	9	21	1	67
	40	10	23.3	1.11	74.4
1	2	8	6	0	16
	2.22	8.89	6.67	0	17.8
N	1	0	6	0	7
	1.11	0	6.67	0	7.78
total	39	17	33	1	90
	43.3	18.9	36.7	1.11	100

table contents:
Count
Percent of Table Total

*Frequencies of students who gave an answer other than Frank finished first (**0**), who answered that Frank finished first (**1**), who were absent from the posttest (**A**), or who gave no response (**N**). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

Table 3 shows that 67 students, about three-fourths (74.4%) of the students, gave an answer other than Frank finished first, such as, “Fred and Frank finished at the same time” or “Fred finished first”. Sixteen students (17.8%) stated that Frank would finish first, and seven students (7.78%) gave no response.

Performance on Posttest

Table 3 shows that, of the 57 students who took the posttest, 39 students (68.4%) gave an answer other than Frank finished first. Seventeen students (29.8%) stated that Frank would finish first and one student (1.8%) gave no response.

Comparison of Pretest and Posttest Performance

There was a substantial increase in the percentage of correct responses on the posttest as compared to the pretest. Also, the percentage of those who did not answer this question decreased from 7.78% on the pretest to 1.8% on the posttest. Cells (0,0), (1,1) and (N,N) of Table 3 indicate that, of the 57 students who took both tests, 44 students (77.2%) were consistent from pretest to posttest. The (0,1) cell indicates that nine students changed from saying that Frank did not finish first to thinking that Frank finished first. The (0,N) cell indicates that one student who thought that Frank did not finish first on the pretest gave no answer on the posttest. The (1,0) cell indicates that two students changed from thinking that Frank finished first to some other answer. The (1,N) cell indicates that, of the students who thought Frank finished first on the pretest, all of them gave a response on the posttest. Of the students who gave no response on the pretest, cell (N,0) indicates that one student gave an incorrect answer on the posttest and cell (N,1) indicates that there were no students who thought Frank finished first on the posttest.

Table C3

Rows are levels of **C3**
Columns are levels of **PC3**
No Selector

	N	**cni**	**deqt**	**nwk**	**tfrd**	**total**
N	0	0	1	0	0	1
	0	0	1.75	0	0	1.75
cni	0	2	2	1	2	7
	0	3.51	3.51	1.75	3.51	12.3
deqt	1	0	25	1	6	33
	1.75	0	43.9	1.75	10.5	57.9
nwk	0	2	1	1	0	4
	0	3.51	1.75	1.75	0	7.02
samsp	0	0	1	0	1	2
	0	0	1.75	0	1.75	3.51
tfrd	0	1	0	1	8	10
	0	1.75	0	1.75	14.0	17.5
total	1	5	30	4	17	57
	1.75	8.77	52.6	7.02	29.8	100

table contents:
Count
Percent of Table Total

N No answer.
cni Could not interpret.
deqt Distance equals time. They equated time with distance.
nwk No work. Just wrote "Fred."
samsp Same speed. Fred and Frank each walk and run at the same speed (they don't run any faster than they walk, e.g., they each run and walk at 5 mi/hr)
tfrd Time farther than distance. They understand that running half the time gets them further than running half the distance.

Categories of Inferred Thought Processes

Table C3 shows that, of the 57 students who took both tests, 10 students (17.5%) on the pretest, and 17 students (29.8%) on the posttest thought Frank, who ran for half the time, finished first. It was assumed that they understood that Frank ran a further distance than Fred, thus, Frank would finish first. However, giving credit for answering "Frank" with no explanation or an unclear explanation for this response was of some concern since the student's reasoning may not have been correct. For example, one student on his posttest had written "Time is faster than distance so that Frank finishes an obvious first".

Thirty-three students (57.9%) on the pretest, and 30 students (52.6%) on the posttest assumed that running half the time would be equivalent to running half the distance and concluded that Fred and Frank finished at the same time. Two of these students stated that "If they both run and walk at the same speed and the distance is the same, they would be side by side the whole way". And similarly, two other students wrote "If the run and walk speeds are the same, and they run and walk for the same amount of time, then they tie." This type of reasoning could be also be seen through many of their diagrams which showed midpoints on two line segments where one line segment represented time and the other line segment represented distance. They did not make a correct comparison between Fred's running time and Frank's running time nor did they make a correct comparison between Fred's running distance and Frank's running distance.

There were other responses to Problem 3. Seven students on the pretest and five students on the posttest gave answers which the investigator could not interpret, such as, "because the time it takes Frank to run half the time will be less than half the distance". Others stated that they needed the speed and time for each or the speed and distance for each in order to determine who would finish first, indicating that they could not relate time and distance. Four students on each test only wrote "Fred", an incorrect response, and gave no explanation for their answer. One student on each test did not answer this problem and two students, on the pretest only, stated that Fred and Frank's running speed was the same as their walking speed, i.e. running was not any faster than walking, hence, Fred and Frank finished at the same time.

Summary of Interviews

The six interviews support the fact that most students had difficulty relating time to distance in Problem 3. During one interview, a student stated that it would have been easier to solve this problem if the running distances for both Fred and Frank were given or if the running times for both Fred and Frank were given. During his interview he was never able to compare Fred's running distance to Frank's running distance or Fred's running time to Frank's running time from the given information.

All the students during their interviews were able to understand and represent Fred's situation, i.e. Fred ran half the distance and walked half the distance on a run from A to B. However, after representing Fred's situation using a line segment from A to B, they had tremendous difficulty representing Frank's situation which involved two half times from A to B and relating this to Fred's situation involving distances. One student during his interview tried to resolve this problem by stating that "time and distance are considered the same", hence, Fred and Frank finish at the same time.

During another student's interview, when given specific times that Frank ran and walked (Frank ran for 1/2 hr and walked for 1/2 hr from A to B), said, "Don't we need to know how far that is?" indicating that she was trying to relate these times to the distances running and walking but could not determine from these times that Frank ran for more than half the distance from A to B and walked less than half the distance. It is interesting to note, however, that when this same student was asked to focus instead on Fred's situation (Fred ran for half the distance and walked for half the distance), she was able to state that "running would take less time" than walking but she still could not determine who would finish first.

Another student stated that Fred finished first because she believed, "Running half the distance is greater than running half the time." Her statement indicated to the investigator that perhaps the concept of distance is dominant in her thinking as compared to the concept of time.

About half the students interviewed needed to use specific distances and times in order to determine an answer to Problem 3. This was also seen on their tests. One of these students, although he had represented Fred's running and walking distances and Frank's running and walking times correctly using specific values could not determine who would finish first.

The student with the best overall score on both tests, when interviewed, stated that "Fred (who runs half the way and walks the other half) reached the half way point sooner than half the time" but could not determine who would finish first. Not until he focused on Frank's situation (Frank runs for half the time and walks for the other half) and realized that Frank runs past the half way point, could he determine that Frank finished first.

Problem 4a

A car's speed from a standing start increases at the rate of 5 ft/sec/sec over a 25 second interval.

(a) What does 5 ft/sec/sec mean in this situation?

Intent of Problem 4a

The intent of Problem 4a was to investigate students' understanding of acceleration by having the students explain exactly what a constant acceleration of 5 ft/sec/sec means to them.

Criteria for Assigning Scores

A "1" was assigned to any answer similar to the following statement: "5 ft/sec/sec means that for every second the car's speed increased by 5 ft/sec, that is, 5 ft/sec/sec represents a rate of change of velocity of the car." If the student stated that for every second the speed doubled, this answer was also assigned a "1" since he may have recognized that "5 ft/sec/sec" implied a variable speed or thought only about the first two seconds. A "0" was assigned to the answer "acceleration" if the student gave no further explanation of what acceleration meant to him. The investigator felt that "acceleration" was just a word substitution for the statement "a car's speed from a standing start increases at the rate of ..." and was not an explanation of meaning. Also, a "0" was assigned to an answer which was a similar form of 5 ft/sec/sec, for example, 5 ft/sec^2. The investigator was seeking the particular interpretation of 5 ft/sec/sec stated above which showed that the student had a clear understanding of acceleration.

Table 4a

Rows are levels of **4a**
Columns are levels of **P4a**
No Selector

	0	**1**	**A**	**N**	**total**
0	24 26.7	4 4.44	20 22.2	6 6.67	54 60
1	4 4.44	8 8.89	4 4.44	1 1.11	17 18.9
N	5 5.56	1 1.11	9 10	4 4.44	19 21.1
total	33 36.7	13 14.4	33 36.7	11 12.2	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who did not explain what 5 ft/sec/sec means (0), who explained what 5 ft/sec/sec means (1), who were absent from the posttest (A), or who gave no response (N). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 4a shows that 54 students (60%) did not give a satisfactory meaning for 5 ft/sec/sec. Seventeen students (18.9%) gave a satisfactory meaning as stated above and 19 students (21.1%) did not give a response.

Performance on Posttest

Of the 57 students who took the posttest, 33 students (57.9%) did not give a satisfactory meaning for 5 ft/sec/sec. This is consistent with the pretest results. Thirteen students (22.8%) gave a satisfactory meaning and 11 students (19.3%) did not give a response.

Comparison of Pretest and Posttest Performance

Of the 57 people who took both tests, 36 students (63.2%) were consistent from pretest to posttest as seen from the (0,0), (1,1) and (N,N) cells of Table 4a. The (0,1) cell indicates that four students (7.02%) who had given an unacceptable meaning for 5 ft/sec/sec on the pretest gave an acceptable meaning for 5 ft/sec/sec on the posttest. Cell (0,N) shows that six students (10.5%) who had given an unacceptable meaning on the pretest for 5 ft/sec/sec gave no response on the posttest. Cell (1,0) shows that four students (7.02%) who had given an acceptable meaning on the pretest gave an unacceptable meaning on the posttest. This may be due to the fact that writing “acceleration” with no other explanation was an unacceptable meaning for 5 ft/sec/sec although the student may know that 5 ft/sec/sec means that for every second the car’s speed increased by 5 ft/sec. Cell (1,N) shows that one student who had given a correct meaning on the pretest gave no response to this question on the posttest. Of the students who gave no response to this question on the pretest, cell (N,0) indicates that five of these students gave an unacceptable meaning and cell (N,1) indicates that one of these students gave an acceptable meaning for 5 ft/sec/sec.

Table C4a

Rows are levels of **C4a**
Columns are levels of **PC4a**
No Selector

	N	**cni**	**cnstv**	**incrv**	**jstac**	**secsq**	**total**
N	4 7.02	1 1.75	1 1.75	1 1.75	3 5.26	0 0	10 17.5
cni	1 1.75	0 0	1 1.75	0 0	1 1.75	0 0	3 5.26
cnstv	1 1.75	1 1.75	1 1.75	1 1.75	1 1.75	0 0	5 8.77
incrv	3 5.26	1 1.75	1 1.75	9 15.8	2 3.51	0 0	16 28.1
jstac	0 0	1 1.75	1 1.75	1 1.75	8 14.0	1 1.75	12 21.1
jstd	0 0	0 0	1 1.75	0 0	0 0	0 0	1 1.75
secsq	2 3.51	1 1.75	1 1.75	0 0	2 3.51	4 7.02	10 17.5
total	11 19.3	5 8.77	7 12.3	12 21.1	17 29.8	5 8.77	57 100

table contents:
Count
Percent of Table Total

N No answer.
cni Could not interpret.
cnstv Constant velocity. The car traveled 5 ft for every second.
incrv Velocity increased by 5 ft/sec for every second.
jstac Just answered acceleration with no further explanation.
jstd 5 ft/sec/sec was a distance since the seconds cancelled.
secsq Rewrote 5 ft/sec/sec as 5 ft/sec^2.

Categories of Inferred Thought Processes

Since Problem 4a was asking for the meaning of 5 ft/sec/sec the inferred thought processes were essentially equivalent to the performance. However, Table C4a shows that five students on the pretest and seven students on the posttest wrote that this expression meant the “car goes 5 ft for every second”. This was either a careless error or these students equated the constant acceleration expression 5 ft/sec/sec with the constant speed expression 5 ft/sec with which they are probably more familiar. One student on the pretest stated that 5 ft/sec/sec represented a distance since he thought the seconds in the expression 5 ft/sec/sec canceled each other.

Summary of Interviews

The six interviews supported the fact that many students recognized 5 ft/sec/sec as an acceleration although they may not have explained what acceleration meant to them. Three of the students interviewed, who had written only “acceleration” on both tests, were able to explain, that the speed of the car increased by 5 ft/sec for every second. Their explanation went as follows: “at the end of first second, the speed would be 5 ft/sec, at the end of the second second, the speed would be 10 ft/sec, and at the end of the third second, the speed would be 15 ft/sec, etc.”

Another student, when asked what the car’s speed was at the end of the second second replied “15 feet” giving as his reason, “I always thought that accelerating, your distance increases.” His understanding of acceleration is apparently limited to the fact that during equal time intervals, the distance the car travels increases. During his interview, he was never able to give the car’s speed at the end of every second.

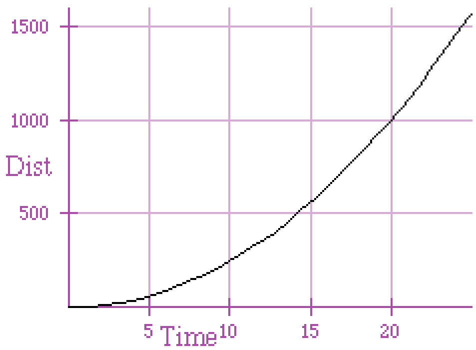
The remaining two students who were interviewed gave an acceptable meaning for 5 ft/sec/sec on both tests and during their interviews indicating that they appeared to have a clear understanding of acceleration.

Problem 4b

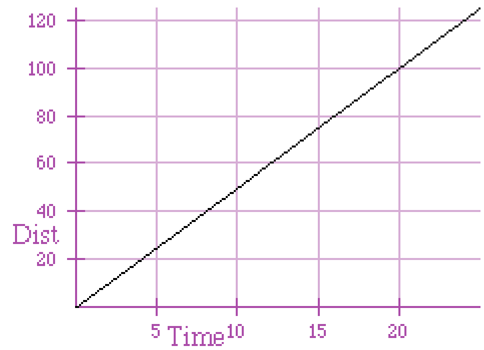
A car's speed from a standing start increases at the rate of 5 ft/sec/sec over a 25 second interval.

(b) Circle the graph that represents the car's distance over this 25 second interval.

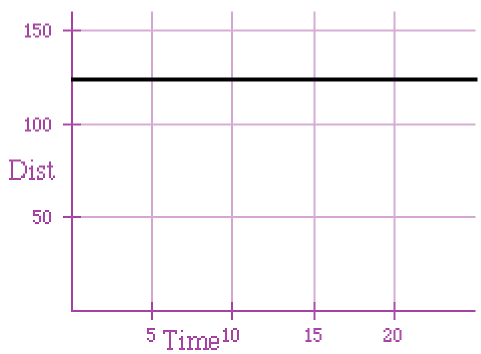
i



ii



iii



iv *None of these represents the car's distance.*

Intent of Problem 4b

The intent of Problem 4b was to determine if students could identify the graph which represented the distance traveled as a function of time of an accelerating object. The students were given a choice between three different graphs or the student could choose "none of these graphs". In addition to the correct graph, two other graphs depicted a constant rate (no acceleration) and a rate of zero (no displacement).

Criteria for Assigning Scores

A student received a "1" if he selected the quadratic graph. All other answers received a "0".

Table 4b

Rows are levels of **4b**
Columns are levels of **P4b**
No Selector

	0	**1**	**A**	**N**	**total**
0	16 17.8	5 5.56	16 17.8	1 1.11	38 42.2
1	7 7.78	21 23.3	11 12.2	2 2.22	41 45.6
N	2 2.22	2 2.22	6 6.67	1 1.11	11 12.2
total	25 27.8	28 31.1	33 36.7	4 4.44	90 100

table contents:
Count
Percent of Table Total

*Frequencies of students who did not select the quadratic graph (**0**), who did select graph (i) (**1**), who were absent from the posttest (**A**), or who gave no response (**N**). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

Table 4b shows that 38 students (42.2%) did not select the quadratic graph as their answer to Problem 4b. Nine of these 38 students were able to give a correct meaning for 5 ft/sec/sec in Problem 4a on the pretest. This appears to indicate that, although these nine students were able to relate acceleration to speed as a function of time in their description of 5 ft/sec/sec, they could not extend this understanding of an increase in speed during equal time intervals to determine which graph represented the distance the car traveled as a function of time. Table 4a also shows that 41 students (45.6%) chose the correct answer and 11 students (12.2%) gave no answer to Problem 4b on the pretest.

Performance on Posttest

Of the 57 students present for the posttest, 25 students (43.9%) did not choose the quadratic graph, 28 students (49.1%) did choose the quadratic graph, and four students (7.02%) did not give an answer.

Comparison of Pretest and Posttest Performance

By examining cells (0,0), (1,1) and (N,N) of Table 4b, of the 57 students who took both tests, we see that 38 students (66.6%) were consistent from pretest to posttest. The (0,1) cell shows that five students (8.77%) who did not select the correct graph, which represented distance as a function of time for an accelerating car, on the pretest chose the correct graph on the posttest. The (0,N) cell shows that one student who did not select the correct graph on the pretest gave no answer on the posttest. However, the (1,0) cell shows that seven students (12.3%) who selected the correct graph on the pretest selected an incorrect graph on the posttest. The (1,N) cell shows that two students who selected the correct graph on the pretest gave no answer on the posttest. The (N,0) and (N,1) cells show that of four students who did not give an answer on the pretest to Problem 4b, two students selected an incorrect graph and two students selected the correct graph on the posttest.

Table C4b

Rows are levels of **C4b**
Columns are levels of **PC4b**
No Selector

	N	**quad**	**lingr**	**none**	**nwk**	**total**
N	1 1.75	2 3.51	0 0	2 3.51	0 0	5 8.77
quad	2 3.51	22 38.6	4 7.02	1 1.75	1 1.75	30 52.6
lingr	0 0	3 5.26	8 14.0	2 3.51	0 0	13 22.8
none	1 1.75	3 5.26	3 5.26	2 3.51	0 0	9 15.8
total	4 7.02	30 52.6	15 26.3	7 12.3	1 1.75	57 100

table contents:
Count
Percent of Table Total

N No answer.
quad Quadratic graph represented the correct relationship between distance and time.
lingr Linear graph. Graph represented velocity as a function of time.
none None of the graphs represented the relationship between distance and time.

Categories of Inferred Thought Processes

Since the students were required to only select the graph which represented the distance as a function of time of an accelerating car in Problem 4b, it was difficult to determine their thought processes. However, a few students who selected the correct graph, graph (i), did give a reason for their selection which helped in understanding their thought processes. One student stated that “you are accelerating exponentially so (i) would be the curve” and another student stated that in the expression “5 ft/sec^2, the square causes the graph to increase exponentially not directly”. The first reason given is incorrect since the car’s acceleration is constant, not exponential, but more importantly, both of these students refer to an exponential graph representing distance as a function of time. One wonders that if the given graphs were not distance as a function of time, but velocity or acceleration as a function of time, if these students would still choose the graph that appeared curved, not linear. A curved graph seems to represent an exponential function to these students irrespective of what the axes may represent. Another example of students ignoring the axes and focusing on the shape of the graph can be seen when one student selected the linear graph as his answer because "constant acceleration would be a straight line."

Part of the confusion between distance, velocity and acceleration comes from students using incorrect units in describing these concepts. This was seen on both tests in statements such as, "It is the only one that works at 5 sec, it would be going 25 miles" and "Velocity increases 5 feet every second."

Summary of Interviews

During the six interviews, students usually mentioned that they had seen these graphs in classes other than their first semester calculus class, such as, in a physics class. However, they could not remember which graph represented distance as a function of time for an accelerating car.

One student stated that each graph looked like the derivative of the graph preceding it, adding that he had learned this in a "Math for Business and Economics Majors class". He explained that since the acceleration was constant, the constant-function graph was the acceleration, the linear graph was the velocity, and the quadratic graph was the distance as a function of time of the accelerating car. Another student had learned a similar technique but rather than stating that each graph looked like the derivative of the graph preceding it, he had learned to look at the increase in area under each graph, the integral, as the x values increased. He also explained that since acceleration was constant, the constant-function graph was the acceleration as a function of time. The increase in area under the constant-function graph as the x values increased led this student to conclude that the linear graph represented the velocity as a function of time and, similarly, the increase in area under the linear graph led to the quadratic graph. During the interview with these two students, they did not discuss nor were they able to understand the following relationship between acceleration, speed and distance as a function of time: If a car accelerates at a constant rate, its speed increases during each successive time interval and, hence, the distance traveled in each successive time interval increases also. Therefore, the quadratic graph represents the total distance traveled at any particular time t. Apparently, these two students had only learned a “trick”, using either the derivative or the integral, to help them decide which graph represented the distance as a function of time for a constantly accelerating object.

Two other students, during their interviews, stated that constant acceleration meant that the graph would be represented by a straight line. They chose the linear graph as their answer rather than the constant graph, since the horizontal line was at a y value of 125 which conflicted with the constant acceleration value of 5 ft/sec/sec. These students did not ignore the fact that Problem 4b was referring to the graph of the distance as a function of time and not constant acceleration as a function of time. In fact, one of these students gave the following reason for equating constant acceleration with the distance the car traveled: “The car’s speed is increasing at a constant speed so it should be increasing the same amount of distance too.” Also, note that this student’s phrase “increasing at a constant speed” is usually stated as “increasing at a constant rate” indicating that rate may only be synonymous to speed for this student.

Problem 5a

Joe dropped a ball from the top of a building. It took 9 seconds for the ball to hit the ground. The distance the ball fell in t seconds after it was released is given by the function d(t), where $d(t) = 16t^2$, $0 \le t \le 9$.

(a) What was the ball's average speed for the time between when it was released and when it hit the ground?

Intent of Problem 5a

The intent of Problem 5a was to offer a situation which clearly involved non-constant velocity and determine if students could see that the average velocity of an object is determined solely by how far it went and how long it took to go that far.

Criteria for Assigning Scores

I assigned a "1" to any answer similar to "average speed = (change in distance)/(change in time)" or "average speed = d(9)/9". Another answer which was also assigned a "1" and seen only on the posttest was "average velocity = d'(9)/2". This last answer, involving the derivative of d(t) at 9 seconds divided by 2, was given credit since it produced a correct answer when applied to the second degree polynomial d(t) = 16t^2, which involved a constant acceleration. However, this technique works only in the case of constant acceleration. All other answers received a "0".

Table 5a

Rows are levels of **5a**
Columns are levels of **P5a**
No Selector

	0	**1**	**A**	**N**	**total**
0	15	0	7	5	27
	16.7	0	7.78	5.56	30
1	9	9	9	3	30
	10	10	10	3.33	33.3
N	8	0	17	8	33
	8.89	0	18.9	8.89	36.7
total	32	9	33	16	90
	35.6	10	36.7	17.8	100

table contents:
Count
Percent of Table Total

*Frequencies of students who gave the incorrect average velocity (**0**), who gave the correct average velocity during the ball's travel (**1**), who were absent from the posttest (**A**), or who gave no response (**N**). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

Table 5a shows that 27 students (30%) did not give the correct average velocity for the ball's travel on the pretest. Four of these students wrote "v = (9.8 m/sec^2)(9) = 88.2" which, to the investigator meant, "velocity = (acceleration due to gravity) x (9 seconds). Perhaps these students were thinking about the formula "velocity = (acceleration)(time)" or "v = at". Thirty students (33.3%) gave a correct response, either "v = d/t = 16(9)^2/9" or "v = [d(9) d(0)]/[9-0]". Thirty-three students (36.7%) gave no response on the pretest.

Performance on Posttest

Of the 57 students who were present for the posttest, 32 (56.1%) did not give a satisfactory answer for the average velocity of the ball during its travel. Nine students (15.8%) gave a satisfactory answer and 16 students (28.1%) gave no response.

Comparison of Pretest and Posttest Performance

By examining cells (0,0), (1,1) and (N,N) of Table 5a, of the 57 students who took both tests, 32 students (56.1%) were consistent from pretest to posttest. The (0,1) and (0,N) cells show that, of those students who gave an unsatisfactory answer on the pretest, there were no students who gave a satisfactory answer on the posttest and five students who gave no answer on the posttest. Cells (1,0) and (1,N) indicate that, of those students who gave a satisfactory answer on the pretest, nine students gave an unsatisfactory answer on the posttest and three students did not give an answer on the posttest. Cells (N,0) and (N,1) indicate that, of those students who gave no answer on the pretest, eight students gave an unsatisfactory answer on the posttest and no student gave a satisfactory answer on the posttest. As stated previously, a correct response found only on one posttest was “average velocity from 0 to 9 seconds = d’(9)/2 = 32(9)/2”. These students used the derivative of the distance formula d(t) = 16t^2 to find the velocity v(t) = 32t. Since the velocity increased at a constant rate, dividing 32(9) by 2 would give the average velocity from 0 seconds to 9 seconds. However, many students, who used this last procedure wrote “v(t) = 32t = 32(9) = 288” and did not divide this answer by 2, in order to find the average velocity between 0 and 9 seconds. They did not realize that 288 is the velocity of the ball at 9 seconds and not the average velocity of the ball between 0 and 9 seconds. This misunderstanding accounted for the high increase in the percentage of incorrect responses on the posttest as compared to the pretest.

Table C5a

Rows are levels of **C5a**
Columns are levels of **PC5a**
No Selector

	N	**cd/ct**	**cni**	**grav**	**v=d/t**	**total**
N	8 14.0	1 1.75	7 12.3	0 0	0 0	16 28.1
cd/ct	0 0	1 1.75	6 10.5	1 1.75	0 0	8 14.0
cni	4 7.02	0 0	7 12.3	0 0	0 0	11 19.3
grav	0 0	0 0	4 7.02	0 0	0 0	4 7.02
v=d/t	4 7.02	0 0	4 7.02	1 1.75	9 15.8	18 31.6
total	16 28.1	2 3.51	28 49.1	2 3.51	9 15.8	57 100

table contents:
Count
Percent of Table Total

N No answer.
cd/ct Wrote “average velocity = (change in distance)/(change in time)”.
cni Could not interpret.
grav Wrote a formula involving the acceleration due to gravity, 9.8 m/sec/sec.
v = d/t Wrote “v = d/t = 16(9)^2/9”. This is not as sophisticated as v = (change in distance)/(change in time).

Categories of Inferred Thought Processes

Responses for Problem 5a fell into five categories based on students’ inferred thought processes (See Table C5a). Of the 57 students who took both tests, Table C5a shows that 16 students (28.1%) on both tests gave no response. Eleven students (19.3%) on the pretest and 28 students (49.1%), about half, on the posttest gave an answer which the investigator could not interpret, for example, “d(t) = 16(9)^2 = 1296” or “v(t) = 32t = 32(9) = 288”. As verified in the interviews, some students didn’t realize that v(9) = 32(9) = 288 is the speed at which the ball hit the ground at 9 seconds. These students did not appear to have a strong conceptual understanding of the fact that the derivative of the distance function at time t gives the velocity of the ball at that time t, not the average velocity. Table C5a also shows that 18 students (31.6%) on the pretest and nine students (15.8%) on the posttest found the average velocity of the ball's travel by using the equation v = d/t, substituting 9 for t in the denominator and 1296 for d in the numerator since d(9) = 16(9)^2 = 1296.

Eight students (14%) on the pretest and two students (3.51%) on the posttest used a more sophisticated formula for finding the "average" velocity of the ball. These students used the formula "average velocity = (change in distance)/(change in time)". It is particularly interesting that so few of the students used this formula since it is emphasized in first semester calculus in developing the concept of the derivative.

Four students (7.02%) on the pretest and two students (3.51%) on the posttest either wrote "gravity = 9.8 m/sec/sec" or "velocity = (acceleration)(time) = (9.8 m/sec/sec)(9 sec) = 88.2 m/sec", both of which involved the acceleration of the ball due to gravity. Although the ball accelerated due the gravitational pull on the ball, these equations did not give the average velocity of the ball during its travel.

Summary of Interviews

The investigator assumed that when students wrote "v = d/t = d(9)/9" in order to calculate the average velocity of the ball, they were not aware of the formula "average velocity = (change in distance)/(change in time) = [d(9) - d(0)]/[9 - 0]". Three of the six students interviewed seemed to support this assumption. These three students stated that they substituted 9 for t in d(t) = 16t^2, which gave them distance, then divided by 9 sec, the time, which would gave them the velocity. The student with the highest overall score on both tests gave the satisfactory response, "d(9)/9", on both tests. However, when asked to explain why this equation would give the average speed of the ball, he responded, "The speed at the beginning plus the speed at the end divided by the total time would give the average speed." He seems to have confused two methods for finding the average speed of the ball in this particular problem, which did not become clear until he was interviewed regarding his acceptable answers on the tests. One method would have been to add the speeds of the ball when it was released and when it hit the ground and divide by 2, and the other would have been to take the difference of the distances from where the ball was released and when it hit the ground and divide this difference by the difference of the times, in this case, the total time.

Another of these students was asked if this velocity would be the average velocity of the ball from the top of the building to the ground or the velocity of the ball when it hit the ground. She replied, "The rate that it hit the ground" and added that in order to calculate the ball's average velocity "take the value at the top of the building, 0, and add this to some other value (of which she wasn't sure) and then divide by 2". Since she could not determine this second value of the velocity, she would "take the average seconds between 0 and 9 seconds, 4.5 seconds, and try to find the ball's speed at this time" but she did not discuss how she would calculate the ball's speed at 4.5 seconds.

Another student, during her interview, used the more sophisticated formula "average speed = (change in distance)/(change in time)" in order to calculate the answer to this problem and stated that she had just learned this formula in her physics class.

Problem 5b

Joe dropped a ball from the top of a building. It took 9 seconds for the ball to hit the ground. The distance the ball fell in t seconds after it was released is given by the function d(t), where d(t) = 16t^2, 0 <= t <= 9.

(b) Write an expression that represents how far the ball fell during the period between t seconds and t+2 seconds after it was released.

Intent of Problem 5b

The intent of Problem 5b was to test students' ability to write an expression which represented the distance the ball fell in a certain interval of time given that the equation d(t) = 16t^2 described "the distance the ball fell in t seconds after it was released." Understanding what the equation represented was needed in order to determine that the solution to this problem was a difference between two distances. More importantly, this leads to understanding that rate is a change or difference in some quantity (distance) as compared to a change or difference in another (time).

Criteria for Assigning Scores

A "1" was assigned to the answer "d(t+2) - d(t)" or "16(t+2)^2 - 16t^2", which represents the difference of the distances at t+2 and t seconds, that is, how far the ball fell during the period between t and t+2 seconds. All other answers received a "0".

Table 5b
Rows are levels of **5b**
Columns are levels of **P5b**
No Selector

	0	**1**	**A**	**N**	**total**
0	12 13.3	3 3.33	13 14.4	3 3.33	31 34.4
1	4 4.44	4 4.44	0 0	2 2.22	10 11.1
N	7 7.78	2 2.22	20 22.2	20 22.2	49 54.4
total	23 25.6	9 10	33 36.7	25 27.8	90 100

table contents:
Count
Percent of Table Total

*Frequencies of students who incorrectly represented the distance the ball fell between t and t+2 seconds (**0**), who correctly represented the distance the ball fell between t and t+2 seconds (**1**), who were absent from the posttest (**A**), or who gave no response (**N**). Rows are levels of problem* *performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

Table 5b shows that 31 students (34.4%) gave an expression which incorrectly represented the distance the ball fell between t and t+2 seconds. Of these 31 students, five students wrote "[16(t+2)^2-16t^2]/[(t+2)-t]", a (change in distance)/(change in time), which represents the average velocity of the ball between t and t+2 seconds, not the change in distance between these times. Two students wrote "d(t)= (t+2) - t" which is the difference of the times, t and t+2. Also, of these 31 students, two other students wrote "16(t+2)^2" and "16t^2" but did not take their difference. Only 10 students (11.1%) gave a correct representation of the distance the ball fell between t and t+2 seconds and 49 students (54.4%) gave no response.

Performance on Posttest

Of the 57 students who took the posttest, 23 students (40.4%) gave an expression which did not represent the distance the ball fell between t and t+2 seconds, four students wrote "[16(t+2)^2-16t^2]/[(t+2)-t]", a (change in distance)/(change in time), the average velocity of the ball between t and t+2 seconds. Perhaps these four students were still thinking about Problem 5a, which asked for the average velocity of the ball from the time it was released until it hit the ground. One other student took the sum of 16(t+2)^2 and 16t^2, and two other students on the posttest wrote "16(t+2)^2" and "16t^2" but did not take their difference. Nine students (15.8%) gave a correct representation of the distance the ball fell between t and t+2 seconds and 25 students (43.9%) gave no response.

Comparison of Pretest and Posttest Performance

Of the 57 students that took both tests, cells (0,0), (1,1) and (N,N) of Table 5b indicate that 36 students (63.2%) were consistent from pretest to posttest. Cells (0,1) and (0,N) indicate that, of those who gave an unsatisfactory expression for the distance the ball traveled between t and t+2 seconds on the pretest, three students gave a satisfactory answer and three students gave no answer on the posttest. Cells (1,0) and (1,N) indicate that, of the students who correctly answered this problem on the pretest, four students gave an unsatisfactory answer on the posttest and two students gave no answer on the posttest. Cells (N,0) and (N,1) indicate that, of the students who gave no answer on the pretest, seven students gave an unsatisfactory answer on the posttest and two students gave a correct answer on the posttest. It was interesting to note that on the posttest only, two students had written the expression "[(32(t)+32(t+2))/2]t" to represent the distance the ball fell between t and t+2 seconds, however, the last t should have been a 2. The investigator presumed they used the equation "distance = velocity x time" and substituted t, not 2, for the time and "[32(t) + 32(t+2)]/2" for the velocity from t to t+2 seconds obtained from the derivative of d(t) = 16t^2.

Table C5b

Rows are levels of **C5b**
Columns are levels of **PC5b**
No Selector

	N	**cd/ct**	**cni**	**d(t)=**	**dif**	**sum**	**total**
N	20	1	5	0	3	0	29
	35.1	1.75	8.77	0	5.26	0	50.9
cd/ct	1	1	3	0	0	0	5
	1.75	1.75	5.26	0	0	0	8.77
cni	1	1	1	0	3	0	6
	1.75	1.75	1.75	0	5.26	0	10.5
d(t)=	0	0	2	0	2	0	4
	0	0	3.51	0	3.51	0	7.02
dif	2	1	1	2	4	1	11
	3.51	1.75	1.75	3.51	7.02	1.75	19.3
sum	1	0	1	0	0	0	2
	1.75	0	1.75	0	0	0	3.51
total	25	4	13	2	12	1	57
	43.9	7.02	22.8	3.51	21.1	1.75	100

table contents:
Count
Percent of Table Total

N No answer.
cd/ct (change in distance)/(change in time). Computed average velocity rather than the distance the ball fell between t and t+2 seconds.
cni Could not interpret.
d(t)= Wrote d(t)=16t^2 or d(t)=16(t+2)^2 but did not take their difference.
dif Took the difference between d(t+2) and d(t).
sum Took the sum of d(t+2) and d(t).

Categories of Inferred Thought Processes

For Problem 5b, the inferred thought processes were essentially equivalent to the performance. Therefore, no further discussion will be given for Problem 5b.

Summary of Interviews

From the responses on the pretest and posttest, it appeared that some students thought solely in terms of the completed trip of the ball from the time it was released until it hit the ground. This was verified during an interview with one student when she asked, “Doesn’t the t represent 1296 feet?” From this question, perhaps it can be concluded that this student believed that the t in the equation $d(t) = 16t^2$ was not a variable which had given the distance the ball fell at time t but could only equal 9 seconds, yielding a distance of 1296 feet.

Another student, during her interview, was asked what $d(t) = 16t^2$ meant. She gave a satisfactory answer which was the distance the ball fell in t seconds. However, when asked to substitute 2 seconds for t, she stated that the answer was 64 ft/sec, adding that the “64 was the rate it went for the distance”. When it was pointed out to her that her answer was incorrect, she stated that “the 64 was the seconds”. It was explained to her that 64 was the distance the ball had fallen in 2 seconds, and, if 1 second was substituted for t, the distance the ball would have fallen would be 16 feet. From this information, she was able find the distance the ball had fallen between 1 and 2 seconds by taking the difference between 64 and 16 feet. However, she was never able to find the distance the ball had fallen between t and t+2 seconds in order to answer this problem.

Another student, during his interview, computed the difference in time between t and t+2 seconds, which was 2 seconds, and, concluded that the distance the ball fell between t and t+2 seconds was “$d(2) = 16(2)^2 = 64$ feet”, not realizing that the distance the ball falls is not 64 feet for every 2 second interval of time.

Problem 5c

Joe dropped a ball from the top of a building. It took 9 seconds for the ball to hit the ground. The distance the ball fell in t seconds after it was released is given by the function $d(t)$, where $d(t) = 16t^2$, $0 <= t <= 9$.

(c) What was the ball’s average speed during the period between 1/2 second and 2 3/4 seconds after it was released?

Intent of Problem 5c

The intent of Problem 5c was similar to Problem 5a except this problem explores whether students would see that “velocity = distance/time” could not be used since the object is not at rest at 1/2 second. (The velocity at t = 1/2 second is not zero.) In order to solve this problem they would need to think in terms of the more sophisticated formula “average velocity = (change in distance)/(change in time)”, a generalized form for rate of change.

Criteria for Assigning Scores

I assigned a “1” to an answer similar to “average velocity between 1/2 and 2 3/4 second = (change in distance between 1/2 and 2 3/4 second)/(change in time between 1/2 and 2 3/4 second) = [d(2 3/4) - d(1/2)]/(2 3/4 - 1/2)”. Since the acceleration was constant, a “1” was also assigned to an answer involving the arithmetic average of the velocities at 1/2 and 2 3/4 seconds, that is, “average velocity between 1/2 and 2 3/4 second = [v(1/2) + v(2 3/4)]/2” where v(1/2) and v(2 3/4) were the values of the derivative of d(t) = 16t^2 at t = 1/2 and t = 2 3/4 seconds. A “0” was assigned to all other answers.

Table 5c

Rows are levels of 5c
Columns are levels of P5c
No Selector

	0	1	A	N	total
0	10 11.1	2 2.22	5 5.56	4 4.44	21 23.3
1	4 4.44	5 5.56	5 5.56	0 0	14 15.6
N	9 10	2 2.22	23 25.6	21 23.3	55 61.1
total	23 25.6	9 10	33 36.7	25 27.8	90 100

table contents:
Count
Percent of Table Total

*Frequencies of students who did not correctly calculate the average velocity between 1/2 and 2 3/4 seconds (**0**), who correctly calculated the average velocity between 1/2 and 2 3/4 seconds (**1**), who were absent from the posttest (**A**). or who gave no response (**N**). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

Table 5c shows that 21 students (23.3%) gave an unacceptable answer, such as, “32(2 3/4-1/2)”, which the investigator assumed was derived from “v(t) = d’(t) = 32t”. Another example of an unacceptable answer which some students gave was “16(2 3/4)^2 - 16(1/2)^2”, the difference of distances from 1/2 to 2 3/4 seconds. Fourteen students (15.6%) gave an acceptable answer as described above and 55 students (61.1%) gave no answer to this problem on the pretest.

Performance on Posttest

Of the 57 students who were present for the posttest, 23 students (40.4%) gave an unacceptable answer, nine students (15.8%) gave an acceptable answer as described above and 25 students (43.9%) gave no answer.

Comparison of Pretest and Posttest Performance

Cells (0,0), (1,1) and (N,N) of Table 5c show that 36 students (63.2%) of the 57 students who took both tests were consistent from pretest to posttest. Of those who gave an unsatisfactory answer on the pretest, cell (0,1) shows that two students (3.5%) gave a satisfactory response and four students (7.01%) gave no response on the posttest. Of those who gave a correct response on the pretest, cell (1,0) shows that four students (7.01%) gave an unsatisfactory response and cell (1,N) shows that all of these students gave a response on the posttest. Of those students who did not respond on the pretest, cell (N,0) shows that nine students (15.7%) gave an unsatisfactory answer and cell (N,1) shows that two students (3.5%) gave a correct response on the posttest.

Table C5c

Rows are levels of **C5c**
Columns are levels of **PC5c**
No Selector

	N	**cd/ct**	**cni**	**difd**	**difv**	**total**
N	21 36.8	4 7.02	6 10.5	0 0	1 1.75	32 56.1
cd/ct	2 3.51	8 14.0	3 5.26	1 1.75	0 0	14 24.6
cni	2 3.51	1 1.75	2 3.51	0 0	0 0	5 8.77
difd	0 0	1 1.75	1 1.75	0 0	0 0	2 3.51
difv	0 0	2 3.51	2 3.51	0 0	0 0	4 7.02
total	25 43.9	16 28.1	14 24.6	1 1.75	1 1.75	57 100

table contents:
Count
Percent of Table Total

N No answer.
cd/ct (Change in distance)/(change in time) = average velocity.

cni Could not interpret.
difd Difference of distances . They took the difference of distances from 1/2 to 2 3/4 seconds as the average velocity of the ball between 1/2 and 2 3/4 seconds.
difv Difference of velocities. They took the difference of velocities between 1/2 and 2 3/4 seconds as the average velocity of the ball between 1/2 and 2 3/4 seconds.

Categories of Inferred Thought Processes

Students' performance on Problem 5c was essentially equivalent to their thought processes. However, of the 16 students who used an acceptable approach in arriving at an answer to Problem 5c on the posttest, five students wrote "average velocity = [32(2 3/4) + 32(1/2)]/2", the arithmetic average of the velocities at 1/2 and 2 3/4 seconds, using the derivative of d(t) = 16t^2. This computation of the average velocity using the derivative of d(t) and the arithmetic average of the velocities at 1/2 and 2 3/4 seconds was only seen on the posttest and was a new way of thinking about this problem for several students. However, this new approach to solving Problem 5c lead to a great increase in the percentage of incorrect responses from pretest to posttest. Many students on the posttest incorrectly used the derivative of d(t) to calculate the average velocity of the ball between 1/2 and 2 3/4 seconds. For example, several students had written "average velocity = d'(t) = 32t = 32(2 3/4 - 1/2)" which is the change in velocity between 1/2 and 2 3/4 seconds, not the average velocity of the ball between these two given times.

Summary of Interviews

Prior to a student's interview regarding Problem 5c, it should be remembered that the student had discussed the concept of the average speed of the ball from start to finish in Problem 5a, and the distance the ball fell between t and t+2 seconds in Problem 5b and perhaps incorporated these concepts into their answer in Problem 5c.

One student, in answering Problem 5c, the average velocity of the ball between 1/2 and 2 3/4 seconds, calculated the change in distance between 1/2 and 2 3/4 seconds, recalling what he had calculated in Problem 5b. He did not proceed any further without explicit directions.

Another student, during his interview, tried to remember a formula that could be used to calculate the average velocity of the ball between 1/2 and 2 3/4 seconds. He ended his discussion of Problem 5c by stating that the only formula he knew which involved any kind of velocity was “velocity = distance/time” and this did not appear to help him solve this problem involving average velocity. He did not realize that this formula could lead to the correct answer for the average velocity of the ball from 1/2 to 2 3/4 seconds if he interpreted “distance/time” as “(change in distance from 1/2 to 2 3/4 second)/(change in time between 1/2 and 2 3/4 second)”.

Another student was able to find the average velocity of the ball between 1/2 and 2 3/4 seconds by stating that the velocity, v(t), was the derivative of d(t). He used the arithmetic average of the velocities at these times stating that “average velocity = [32(1/2) + 32(2 3/4)]/2”. However, when asked if he could think of any other way to solve this problem he was unable to do so (although he was asked to examine Problem 5b again which referred to a change in distance between two different times). The investigator was trying to determine if this student also knew that the average velocity = (change in distance)/(change in time).

One student was able to find the average speed of the ball between 1/2 and 2 3/4 seconds by using the formula “average velocity = (change in distance)/(change in time)” which she had learned in a physics class which she was attending. When asked to explain this formula, she referred to the slope of a line between the two points (1/2, d(1/2)) and (2 3/4, d(2 3/4)) if distance were graphed as a function of time and “instantaneous velocity was the tangent” (not mentioning the slope of the tangent at a particular point). She added, as did other students during their interview, that in first semester calculus they never covered real world problems such as these. However, it is the investigator's belief that this student did cover real world problems in first semester calculus but lacked the understanding that the slope of a line between two points on a graph represents the average rate of change between those points. Also, she may not understand that the slope of the tangent at a point is the instantaneous rate of change at that point. In this student's own words, "this method of finding the instantaneous velocity through graphing doesn't really help you see it."

Another student did say that they "discussed velocity, acceleration, average velocity, and instantaneous velocity in first semester calculus". When asked what instantaneous meant to him, he replied "when you put a limit in front of it or get closer and closer to our answer but it is an approximation to our answer." When asked again about the instantaneous velocity of the ball at, for example, 2 seconds, he talked about the average velocity from 0 to 2 seconds and ended by saying he was confused.

Problem 5d

Joe dropped a ball from the top of a building. It took 9 seconds for the ball to hit the ground. The distance the ball fell in t seconds after it was released is given by the function $d(t)$, where $d(t) = 16t^2$, $0 \le t \le 9$.

(d) Write an expression (a formula) for the ball's average speed during the period between u and u+h seconds after it was released, where $h>0$ and $u+h \le 9$.

Intent of Problem 5d

The intent of this problem was to determine whether the students could now write the general formula for the ball's average velocity during any time interval of the ball's travel after completing problems 5a and 5c which dealt with specific time intervals.

Again, students would need to have a clear understanding that average velocity is a change in displacement as compared to a change in time.

Criteria for Assigning Scores

A "1" was assigned to an answer similar to "average velocity = [d(u+h) - d(u)]/h" or "average velocity = [v(u) + v(u+h)]/2". A "0" was assigned to all other responses.

Table 5d
Rows are levels of **5d**
Columns are levels of **P5d**
No Selector

	0	**1**	**A**	**N**	**total**
0	6 6.67	2 2.22	4 4.44	3 3.33	15 16.7
1	1 1.11	5 5.56	3 3.33	1 1.11	10 11.1
N	8 8.89	2 2.22	26 28.9	29 32.2	65 72.2
total	15 16.7	9 10	33 36.7	33 36.7	90 100

table contents:
Count
Percent of Table Total

*Frequencies of students who gave an unsatisfactory formula for the ball's average velocity between u and u+h seconds (**0**), who gave a satisfactory formula for the ball's average velocity between u and u+h seconds (**1**), who were absent from the posttest (**A**), or who did not respond (**N**). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

Table 5d shows that 15 students (16.7%) gave an unsatisfactory formula for the ball's average velocity during the interval u and u+h seconds. Ten students (11.1%) gave a satisfactory formula, and almost three-fourths (72.2%) of the students on the pretest gave no response for the ball's average velocity between u and u+h seconds.

Performance on Posttest

Of the 57 students who took both tests, 15 students (26.3%) gave an unsatisfactory answer, nine students (15.8%) gave a satisfactory answer, and almost three-fifths (57.9%) of the students on the posttest, gave no response to Problem 5d which asked for a general formula for the average velocity of a ball between u and u+h seconds.

Comparison of Pretest and Posttest Performance

We can see how consistent were students pretest and posttest performance by examining cells (0,0), (1,1) and (N,N) of Table 5d. Of the 57 students who took both tests, 40 students (70.1%) were consistent. Of those students who gave an unsatisfactory response on the pretest, cell (0,1) indicates that two students (3.51%) gave a correct response on the posttest and cell (0,N) indicates that three students (5.26%) did not respond on the posttest. Of those students who gave a satisfactory answer on the pretest, cell (1,0) indicates that one student (1.75%) gave an unsatisfactory answer on the posttest and cell (1,N) shows that one student (1.75%) gave no response on the posttest. Of those who gave no response on the pretest, cell (N,0) indicates that eight students (14%) gave an incorrect response, and cell (N,1) indicates that two students (3.51%) gave a correct response on the posttest.

Table C5d

Rows are levels of **C5d**
Columns are levels of **PC5d**
No Selector

	N	**cd/ct**	**cni**	**slter**	**sum**	**total**
N	29 50.9	2 3.51	6 10.5	1 1.75	1 1.75	39 68.4
cd/ct	1 1.75	6 10.5	0 0	3 5.26	0 0	10 17.5
cni	2 3.51	0 0	3 5.26	1 1.75	0 0	6 10.5
sum	1 1.75	1 1.75	0 0	0 0	0 0	2 3.51
total	33 57.9	9 15.8	9 15.8	5 8.77	1 1.75	57 100

table contents:
Count
Percent of Table Total

N	No answer.
cd/ct	(Change in distance)/(change in time) = average velocity
cni	Could not interpret.
slter	Slight error from an acceptable answer. These were found on the posttest only.
sum	Summed two quantities, other than v(u) and v(u+h), and divided by 2.

Categories of Inferred Thought Processes

Responses for Problem 5d fell into five main categories based on students' inferred thought processes (See Table C5d). Table C5d shows that 39 students (68.4%) on the pretest, and 33 students (57.9%) on the posttest did not give a response.

Six students (10.5%) on the pretest and nine students (15.8%) on the posttest gave an answer which the investigator could not interpret. For example, five of the nine students on the posttest began their formula for the average velocity of the ball between u and u+h with "lim as h approaches 0", which perhaps indicated confusion between average velocity and instantaneous velocity of the ball.

Ten students (17.5%) on the pretest and nine students (15.8%) on the posttest thought about average velocity correctly, as the change in distance the ball traveled as compared to the change in time from u and u+h seconds. Two students (3.52%) on the pretest and one student (1.75%) on the posttest in writing an expression which represented the average velocity of the ball between u and u+h seconds summed two quantities, other than v(u) and v(u+h), and divided by 2. For example, these students wrote “[(u+h) + u]/2” or “[16u^2 + 16(u+h)^2]/2” perhaps not realizing that these arithmetic averages represent the average time and average distance the ball traveled between u and u+h seconds.

Only in the posttest were slight errors from an acceptable answer made but these may indicate that the student had some major misconceptions involving the average velocity. For example, one student wrote “[f(u+h) + f(u)]/h” which indicates a change in the time in the denominator but a sum of the distances in the numerator between u and u+h seconds. Another student, using the derivative v(t) = d’(t) = 32t, wrote “[32(u+h) - 32u]/2” which indicates this student confounded several different perspectives: the formula for average rate of change, the derivative of 16t^2, and the arithmetic mean of two numbers. The following three examples, “[(u+h) -u]/h”, “[d(u) -d(u+h)]/u”, and “[16(u+h)^2 -16u^2]/u”, indicate a change in some quantity over the change in another but either one or both of these changes were incorrect for determining the average velocity between u and u+h seconds. The investigator presumed that the students who had written one of the last five expressions for average velocity unsuccessfully memorized the formula for average velocity but had no conceptual understanding of what the formula for average velocity or what the formula they had written represented.

Summary of Interviews

Two of the six students interviewed, from what had been discussed in Problems 5a to 5c, appeared to understand that the average velocity of the ball during the time interval u to u+h was the change in distance during that interval as compared to the change in time during that interval. From their understanding of Problem 5b, these two students were able to write the expression for the change in distance during the time interval u to u+h. However, these students had difficulty writing the expression for the change in time during the interval u to u+h seconds which was needed for the denominator of their answer. One of these two students wrote 9 seconds in the denominator of their expression for the average velocity of the ball during the time interval u to u+h seconds.

Another student, using the derivative of the distance d'(t) = v(t) = 32t, took the arithmetic average of the velocities at times u and u+h stating that this was the average velocity of the ball between u and u+h seconds. Another student, during the interview, studied the answers she had given to this problem on the pretest and posttest, since they were incompatible, and was able to determine and explain why her answer on the pretest was correct. The remaining two students, who were repeating first semester calculus, had given a satisfactory answer to this problem on both tests and during their interviews.

Problem 5e

Joe dropped a ball from the top of a building. It took 9 seconds for the ball to hit the ground. The distance the ball fell in t seconds after it was released is given by the function d(t), where d(t) = 16t^2, 0 <= t <= 9.

(e) Suppose that t and w represent numbers of seconds and d is the function defined above. What does the expression [d(t+w) - d(t)]/w represent about the falling ball?

Intent of Problem 5e

The intent of problem 5e was to determine if students could interpret that a change in distance during the time interval from t to t+w seconds as compared to the change in time over the same interval represented the average velocity of the ball.

Students who have the limited view that (constant) velocity is simply distance/time might not be aware that (average) velocity actually involves a change in distance as compared to a change in time for a specific time interval. In other words, the distinction between constant velocity and average velocity would not have been made.

Criteria for Assigning Scores

A "1" was assigned to an answer which was similar to "This expression represents the average speed (or velocity) of the ball." A "1" was assigned this answer whether or not the time interval, t to t+w seconds, was specified. A "0" was assigned to all other responses including "speed" or "velocity" since these may have referred to instantaneous speed or instantaneous velocity which some students gave as their response.

Table 5e
Rows are levels of **5e**
Columns are levels of **P5e**
No Selector

	0	**1**	**A**	**N**	**total**
0	4 4.44	2 2.22	5 5.56	6 6.67	17 18.9
1	6 6.67	4 4.44	3 3.33	0 0	13 14.4
N	5 5.56	2 2.22	25 27.8	28 31.1	60 66.7
total	15 16.7	8 8.89	33 36.7	34 37.8	90 100

table contents:
Count
Percent of Table Total

*Frequencies of students who gave a response other than "average speed" or "average velocity" (**0**), who gave the response "average speed" or "average velocity" (**1**), who were absent from the posttest (**A**), or who gave no response (**N**). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

As Table 5e shows, 17 students (18.9%) gave a response other than “average speed” or “average velocity” and 13 students (14.4%) gave an acceptable answer. However, of the 13 students who gave the acceptable answer “average speed” or “average velocity”, only two students included the fact that this was during the time interval between t and t+w seconds. Two-thirds (66.7%) of the students gave no answer to this problem.

Performance on Posttest

Of the 57 students who took both tests, 15 students (26.3%) gave an unacceptable answer and eight students (14%) gave an acceptable answer as described above. Five of the eight students, who had given an acceptable answer, included the fact that this was during the time interval between t and t+w seconds. Thirty-four (59.6%) of the students gave no answer to this problem on the posttest.

Comparison of Pretest and Posttest Performance

From cells (0,0), (1,1) and (N,N) of Table 5e we can see that, of the 57 people who took both tests, 36 students (63.2%) were consistent from pretest to posttest. Of those who gave an unacceptable answer on the pretest, cell (0,1) shows that two students gave a satisfactory answer on the posttest and cell (0,N) shows that six students gave no response on the posttest. Of those who had given an acceptable response on the pretest, cell (1,0) shows that six students gave an unacceptable response on the posttest and cell (1,N) shows that all of these students gave a response on the posttest. Of those who gave no response on the pretest, cell (N,0) shows that five students gave an incorrect response on the posttest and cell (N,1) shows that two students gave a correct response on the posttest.

Table C5e

Rows are levels of **C5e**
Columns are levels of **PC5e**
No Selector

	N	**ac**	**averv**	**cni**	**instv**	**jstv**	**total**
N	28 49.1	1 1.75	2 3.51	3 5.26	1 1.75	0 0	35 61.4
ac	0 0	0 0	1 1.75	0 0	0 0	1 1.75	2 3.51
averv	0 0	1 1.75	3 5.26	1 1.75	0 0	2 3.51	7 12.3
cni	3 5.26	1 1.75	1 1.75	0 0	0 0	0 0	5 8.77
instv	2 3.51	0 0	0 0	0 0	0 0	0 0	2 3.51
jstv	1 1.75	0 0	0 0	3 5.26	1 1.75	1 1.75	6 10.5
total	34 59.6	3 5.26	7 12.3	7 12.3	2 3.51	4 7.02	57 100

table contents:
Count
Percent of Table Total

N	No answer.
ac	"Acceleration".
averv	"Average veleocity" or "average speed".
cni	Could not interpret.
instv	"Instantaneous velocity".
jstv	Answered just "velocity" or "speed".

Categories of Inferred Thought Processes

Responses to problem 5e fell into six main categories of inferred thought processes (See Table C5e). Table C5e shows that of the 57 students who took both tests, 35 students (61.4%) on the pretest and 34 students (59.6%) on the posttest gave no answer. Five students (8.77%) on the pretest and seven students (12.3%) on the posttest gave an answer which the investigator could not interpret. For example, one student wrote that the expression [d(t+w) - d(t)]/w represented "the slope". Perhaps this student was thinking about the graph of the function d(t) = 16t^2 and, hence, the expression "[d(t+w) - d(t)]/w" would be the slope of the line through the points (d(t), t) and (d(t+w), t+w). He did not state, however, what this slope represented about the falling ball, as the question had asked.

Seven students (12.3%) on both tests gave the acceptable answer "average speed" or "average velocity", but, one student on the pretest and two students on the posttest stated that the given expression was "the average velocity between t and w seconds", not between t and t+w seconds. Either this was a slight error or these students didn't realize that this expression referred to the time interval between t and t+w seconds.

Six students (10.5%) on the pretest and four students (7.02%) on the posttest just wrote the word "speed" or "velocity". These students were not given credit for this answer since the investigator assumed that this was an instinctive response to the given expression when these students realized that this expression represented a distance/time.

Other students who did not receive credit for this problem realized that the numerator involved distances but had difficulty understanding the denominator and how the denominator related to the numerator. For example, one student wrote "distanced traveled with respect to w" and another wrote "this says that distance is a function of time".

Two students (3.51%) on each test wrote that this formula represented the "instantaneous velocity", or "the derivative d'(t)". The formula given in Problem 5e does appear similar to the formula for the instantaneous velocity of the ball at time t, that is, lim as w approaches 0 of [d(t+w) -d(t)]/w, but perhaps these students did not understand the distinction between these two expressions. Perhaps these students have lost or never understood that the instantaneous velocity is the limit of the average velocity as the time interval approaches 0.

Two students (3.51%) on the pretest and three students on the posttest (5.26%) wrote that the expression in Problem 5e represented the acceleration of the falling ball. Perhaps these students thought about the ball accelerating downward due to gravity or perhaps the given formula was the rate of change of the average speed.

Answers to Problem 5e reveal that most first semester calculus students cannot identify the most common of rate of change, average speed, when presented with it. It is also interesting to note that these students had written comments regarding their confusion in answering the questions in Problem 5 more than they had any other problem. The following are some examples of the students comments: "I don't understand this concept very well", or "The concept of velocity is not well defined in my mind", or "The entire problem puzzles me" or "I've seen this type of problem in physics but I don't remember the right equations to use to solve these problems" or "This looks like a homework problem I just had and did not understand" or "Does the ball fall at the same speed constantly (till it hits the ground)?"

Summary of Interviews

All six students interviewed recognized that Problem, 5e was similar to Problem 5d which they had just discussed. Hence, five of the students stated that the given formula in Problem 5e represented the "average speed" while one student stated that the formula represented "speed since it was a distance over a time". This student did not appear to distinguish between constant velocity and average velocity. One of the students added, "or some kind of speed", also having difficulty distinguishing between constant velocity and average velocity, and another student thought the denominator should be w-t not w.

One student was then asked to refer to his answer on the posttest to the previous question, Problem 5d. He had used the derivative of d(t) to find the average speed of the ball between u and u+h seconds by taking the arithmetic average of the speeds at these times. When asked how he would find the average speed of the ball during the same time interval if the function for the distance were changed to d(t) = 16t^3 (a cubic equation) he replied that he would proceed in the same way. That is, he would take the derivative of d(t) = 16t^3 which would yield v(t) and calculate the arithmetic average of the velocities at the two times. This method is incorrect, yet the investigator believed that he never had or no longer thinks in terms of the rate of change expression for the average velocity, (average velocity = (change in distance)/(change in time)), after learning that the derivative of the distance function yields the velocity function. He appears to have developed a schema which he consistently uses to calculate the average velocity between two given times by taking the arithmetic average of the velocities at these times although this may be incorrect in some situations.

Problem 6a

A spherical storage tank stood empty one morning and then was filled to capacity with water. The water's volume increased as its height increased. A supervisor, who had a dip stick but no clock, measured the water's depth repeatedly as the tank filled. The graph at the right represents the water's volume, in cubic feet, as a function of its height above the tank's bottom. The tank is 8 feet high and holds 268 feet of water.

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(a) What would be the unit for "average rate of change of volume with respect to height"?

Intent of Problem 6a

This is the first problem that used the words "average rate of change" and, in addition, the rate in this problem did not involve speed or time. The intent of this problem was to determine if the students could see that the unit for average rate of change of volume with respect to height was found by focusing on the unit for volume and the unit for height and combining these units to form the composite unit of cubic feet per foot.

Criteria for Assigning Scores

A "1" was assigned to any answer similar to "(cubic feet)/foot" even if the student then changed this to "square feet". Later, in the discussions of the students' thought processes, these answers are not considered equivalent. A "0" was assigned to all other answers.

Table 6a

Rows are levels of **6a**
Columns are levels of **P6a**
No Selector

	0	**1**	**A**	**N**	**total**
0	8 8.89	10 11.1	12 13.3	4 4.44	34 37.8
1	2 2.22	9 10	4 4.44	3 3.33	18 20
N	8 8.89	3 3.33	17 18.9	10 11.1	38 42.2
total	18 20	22 24.4	33 36.7	17 18.9	90 100

table contents:
Count
Percent of Table Total

*Frequencies of students who gave an answer other than (cubic feet)/foot (**0**), who gave an answer similar to (cubic feet)/foot (**1**), who were absent from the posttest (**A**), or who gave no response (**N**). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

Table 6a shows that 34 students (37.8%) did not give an answer similar to "(cubic feet)/foot". Of these, five students included a number in front of the unit, five students wrote "cubic feet" (the unit for the volume only), and eight students wrote an answer similar to "volume/height". Eighteen students (20%) gave an answer similar to "(cubic feet)/foot" which included six students who changed (cubic feet)/foot to square feet. The tendency to change cubic feet/foot to square feet was either a slight error due to the emphasis on reducing fractions in their previous mathematics courses or they didn't understand the concept of average rate of change of volume with respect to height in this situation. Thirty-eight students (42.2%) gave no response to this question. Perhaps they did not understand what a unit was or perhaps they were not able to determine the unit in this atypical situation.

Performance on Posttest

Of the 57 students who were present for the posttest, 18 students (31.6%) gave an answer other than “(cubic feet)/foot”. Of these, two students included a number in front of the unit, two students wrote “cubic feet” (the unit for the volume only), and 10 students wrote an answer similar to “volume/height”. Twenty-two students (38.6%) gave an answer similar to “(cubic feet)/foot” which included 10 students who changed “(cubic feet)/foot” to “square feet”. Seventeen students (29.8%) gave no response to this question.

Comparison of Pretest and Posttest Performance

Of the 57 students who took both tests, cells (0,0), (1,1) and (N,N) of Table 6a show that 27 students (47.4%) were consistent in their answers from pretest to posttest. Of those who gave an unsatisfactory response on the pretest, cell (0,1) shows that 10 people (17.5%) gave a correct response on the posttest, and cell (0,N) shows that four students (7.02%) gave no response on the posttest. Of those who gave a satisfactory answer on the pretest, cell (1,0) shows that two students (3.51%) gave an incorrect answer on the posttest and cell (1,N) shows that three students (5.26%) gave no response on the posttest. Of those who gave no response on the pretest, cell (N,0) shows that eight students (14%) gave an incorrect response on the posttest and cell (N,1) shows that three students (5.26%) gave a correct response on the posttest.

Table C6a

Rows are levels of **C6a**
Columns are levels of **PC6a**
No Selector

	N	**cni**	**cuf**	**cuf/f**	**nmbr**	**sqf**	**su/su**	**total**
N	10 17.5	2 3.51	1 1.75	2 3.51	0 0	2 3.51	4 7.02	21 36.8
cni	2 3.51	0 0	0 0	0 0	1 1.75	0 0	2 3.51	5 8.77
cuf	0 0	1 1.75	0 0	2 3.51	0 0	1 1.75	1 1.75	5 8.77
cuf/f	2 3.51	0 0	0 0	3 5.26	0 0	0 0	1 1.75	6 10.5
nmbr	2 3.51	0 0	1 1.75	1 1.75	0 0	0 0	1 1.75	5 8.77
sqf	1 1.75	0 0	0 0	0 0	0 0	6 10.5	0 0	7 12.3
su/su	0 0	0 0	0 0	5 8.77	1 1.75	1 1.75	1 1.75	8 14.0
total	17 29.8	3 5.26	2 3.51	13 22.8	2 3.51	10 17.5	10 17.5	57 100

table contents:
Count
Percent of Table Total

N	No answer.
cni	Could not interpret.
cuf	Cubic feet or ft^3. These students only gave the unit for the volume
cuf/f	(Cubic feet)/foot. This was not the same as square feet.
numbr	Number in front of the unit.
sqf	Square feet. Changed "(cubic feet)/foot" to "square feet".
su/su	(Some unit)/(some unit). The unit in the numerator or denominator were incorrect

Categories of Inferred Thought Processes

The nature of Problem 6a was such that performance was essentially equivalent to inferred thought processes. It should be noted, however, from Table C6a that seven students on the pretest and 10 students on the posttest changed (cubic feet)/foot to square feet. Also, eight students on the pretest and 10 students on the posttest realized that a unit for average rate of change is (some unit)/(some unit) but the unit in the numerator and/or denominator were incorrect. For example, five students on each test wrote "volume/height".

Summary of Interviews

During the interview with the six students, two students did not know what a unit was. Two students changed "(cubic feet)/foot" to "square feet" and did not appear to understand that you cannot treat a rate as you would a rational number. For example, it does not make sense to say that water flowing into a tank at a rate of 50 cubic feet/foot is equivalent to a rate of 50 square feet, since square feet is not a rate. The remaining two students gave cubic feet/foot as their answer and began to change their answer to square feet but realized that this was not a rate. Changing (cubic feet)/foot to square feet was a strong impulse for these students which may have come from an emphasis on reducing fractions in their earlier mathematics courses.

Problem 6b

A spherical storage tank stood empty one morning and then was filled to capacity with water. The water's volume increased as its height increased. A supervisor, who had a dip stick but no clock, measured the water's depth repeatedly as the tank filled. The graph at the right represents the water's volume, in cubic feet, as a function of its height above the tank's bottom. The tank is 8 feet high and holds 268 feet of water.

(b) What, approximately, was the water's average rate of change of volume with respect to its height after the tank was filled?

Intent of Problem 6b

The intent of Problem 6b was to determine if students could correctly compute the average rate of change of volume with respect to its height from the given information or from the given graph. This problem closely parallels the intent of Problem 5a, but now the student must apply reasoning to a rate involving unfamiliar units (volume and height) rather than the more familiar units (distance and time). Students without a clear understanding of rate might not be able to transfer reasoning used in Problem 5 to this new context.

Criteria for Assigning Scores

A "1" was assigned an answer similar to "268/8 cubic feet/foot" or "33.5 cubic feet/foot". A "0" was assigned to all other answers.

Table 6b

Rows are levels of **6b**
Columns are levels of **P6b**
No Selector

	0	**1**	**A**	**N**	**total**
0	9 10	6 6.67	12 13.3	6 6.67	33 36.7
1	2 2.22	9 10	4 4.44	1 1.11	16 17.8
N	5 5.56	4 4.44	17 18.9	15 16.7	41 45.6
total	16 17.8	19 21.1	33 36.7	22 24.4	90 100

table contents:
Count
Percent of Table Total

*Frequencies of students who gave an answer other than "268/8 cu ft/ft" or "33.5 cu ft/ft" (**0**), who gave an answer similar to "268/8 cu ft/ft" or "33.5 cu ft/ft" (1), who were absent from the posttest (**A**), or who gave no response (**N**). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.*

Performance on Pretest

Table 6b shows that 33 students (36.7%) gave an answer other than "268/8 cu ft/ft" or "33.5 cu ft/ft". These 33 students included 11 students who gave 0 as the answer to this problem, perhaps interpreting the statement to mean that the tank was no longer being filled. One other of these students gave "(225-50)/(6-2)" as his answer perhaps focusing on the "straight" part of the graph. Table 6b also shows that 16 students (17.8%) gave an acceptable answer and 41 students (45.6%) gave no answer to this problem on the pretest.

Performance on Posttest

Of the 57 student that took the posttest, 16 students (28.1%) gave an answer other than "268/8 cu ft/ft" or "33.5 cu ft/ft" including 10 students who gave 0 as an answer and two other students who gave “(225-50)/(6-2)” as their answer. Nineteen students (33.3%) gave an acceptable response and 22 students (38.6%) gave no response to this problem on the posttest.

Comparison of Pretest and Posttest Performance

By examining the (0,0), (1,1) and (N,N) cells of Table 6b, of the 57 students who took both tests, 33 students (57.9%) were consistent from pretest to posttest. Of those students who gave an unacceptable response on the pretest, cell (0,1) indicates that six students gave a correct response on the posttest and cell (0,N) indicates that six students gave no response on the posttest. Of those who gave an acceptable response on the pretest, cell (1,0) indicates that two students gave an unacceptable response on the posttest and cell (1,N) indicates that one student did not give a response on the posttest. Of those who did not answer this problem on the pretest, cell (N,0) indicates that five students gave an unacceptable response on the posttest and cell (N,1) indicates that four students gave an acceptable response on the posttest.

Table C6b

Rows are levels of C6b
Columns are levels of PC6b
No Selector

	N	cni	cv/ch	wv/wh	zero	total
N	15 26.3	1 1.75	4 7.02	1 1.75	3 5.26	24 42.1
cni	3 5.26	2 3.51	1 1.75	1 1.75	1 1.75	8 14.0
cv/ch	1 1.75	0 0	11 19.3	0 0	1 1.75	13 22.8
wv/wh	0 0	0 0	1 1.75	0 0	0 0	1 1.75
zero	3 5.26	0 0	3 5.26	0 0	5 8.77	11 19.3
total	22 38.6	3 5.26	20 35.1	2 3.51	10 17.5	57 100

table contents:
Count
Percent of Table Total

N *No answer*.
cni *Could not interpret.*
cv/ch *(Change in volume)/(change in height).*
wv/wh *(Wrong volume)/(wrong height)*. They wrote an answer similar to “(225-50)/6-2)”
zero *Zero*. They assumed the tank was full.

Categories of Inferred Thought Processes

Performance was essentially equivalent to students’ inferred thought processes on Problem 6b. Most students used the information given in the problem rather than the graph in order to determine a solution for this problem and simply divided 268 cubic feet, the volume given in the problem, by 8 feet, the height given in the problem.

Summary of Interviews

During the six interviews, one student, who had answered this problem correctly on both tests, made the observation that at any point along the graph, if the volume were divided by its height at that point, this would give the average rate of volume with respect to height from the start. Another student also made an insightful observation. He stated that “what went on while filling the tank can be ignored, only the amount of cubic feet of water and the height at the end is important” in order to compute the average rate of volume with respect to height after the tank was filled. He compared the computation of average rate of change of volume with respect to height to the computation of a class average. He stated that the class average is “the number of points that each person would get equally and not how many points they received individually.”

However, other students did not have the insight and understanding of the rate of change of volume with respect to height as did the students above. One student when asked about the rate of change of volume with respect to height replied “I don’t see how you can do it without time...I guess I think in terms of rate of change as being time.”

Another student divided 268 by 8, which he computed to be 33.5 cubic feet/foot and thought this result was the amount of water in every foot. He then stated that “the tank is cylindrical”. When he was reminded that the tank was spherical he was unable to explain what the 33.5 cubic feet/foot meant. Later it was explained to him what “average” rate of change of volume with respect to height meant, which in this case was 33.5 cubic feet/foot, and he appeared to understand this concept. Another student who had focused on the “straight” part of the graph in order to compute the average rate of change of volume with respect to height stated that “a normal tank would be constant from top to bottom” and, hence, thought that there was something wrong near the endpoints of the graph where the graph curved.

Problem 6c

A spherical storage tank stood empty one morning and then was filled to capacity with water. The water’s volume increased as its height increased. A supervisor, who had a dip stick but no clock, measured the water’s depth repeatedly as the tank filled. The graph at the right represents the water's volume, in cubic feet, as a function of its height above the tank's bottom. The tank is 8 feet high and holds 268 feet of water.

(c) What, approximately, was the water's average rate of change of volume with respect to its height after the water's height varied from 3 to 5 feet?

Intent of Problem 6c

The intent of Problem 6c was to determine if students could correctly compute, from the graph, the average rate of change of volume with respect to height for a specific interval of the height. This computation was not a matter of dividing one given number by another, as in Problem 6b, but rather involved obtaining the needed information from the graph in order to divide a change in volume by a change in height for the specific interval of height. During the interviews, one goal was to determine if the students realized that the average rate of change of volume with respect to height from 3 to 5 feet in this problem was not the same as the average rate of change of volume for the entire spherical tank (computed in Problem 6b). The only way to answer this was by interpreting the given graph during the appropriate interval (3 to 5 feet).

Criteria for Assigning Scores

A "1" was assigned an answer approximately equal to "(175-75)/(5-3) cubic feet/foot" or "50 cubic feet/foot". This answer, the average rate of change of volume with respect to height from 3 to 5 feet, was derived from the information in the graph and was computed from the change in volume as compared to the change in height from 3 to 5 feet. A "0" was assigned to all other answers.

Table 6c
Rows are levels of **6c**
Columns are levels of **P6c**
No Selector

	0	**1**	**A**	**N**	**total**
0	3 3.33	8 8.89	9 10	4 4.44	24 26.7
1	2 2.22	10 11.1	5 5.56	1 1.11	18 20
N	3 3.33	11 12.2	19 21.1	15 16.7	48 53.3
total	8 8.89	29 32.2	33 36.7	20 22.2	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who gave an answer which was not approximately "50 cu ft/ft" (0), who gave an answer approximately equal to "50 cu ft/ft" (1), who were absent from the posttest (A), or who gave no answer (N). Rows are levels of performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 6c indicates that 24 students (26.7%) gave an answer that was not approximately equal to “(175 - 75)/(5 - 3) cu ft/ft” or “50 cu ft/ft”, the (change in volume)/(change in height) from 3 to 5 feet. Of these 24 students, it appears that six students computed only the change of volume from 3 to 5 feet from information given in the graph, that is, they wrote “100 cubic feet” as their answer to this problem in the pretest. Two other students, for the average rate of change of volume with respect to height from 3 to 5 feet, wrote “(75/3 + 175/5)/2 cu ft/ft” which perhaps is the arithmetic average of the average rates of change of volume with respect to height from 0 to 3 feet and from 0 to 5 feet. Table 6C also shows that 18 students (20%) gave an answer similar to “50 cu ft/ft” and 48 students (53.3%) gave no answer to this problem on the pretest.

Performance on Posttest

Table 6C indicates that, of the 57 students who took both tests, eight students (14%) gave an answer which was not similar to “50 cu ft/ft”, 29 students (50.9%) gave an answer similar to “50 cu ft/ft” and 20 students (35.1%) gave no answer to this problem on the posttest.

Comparison of Pretest and Posttest Performance

Table 6C shows how consistent students’ performance was from pretest to posttest by examining the (0,0), (1,1) and (N,N) cells. Of the 57 students who took both tests, these cells indicate that 28 students (49.1%) were consistent. Of those who gave an unsatisfactory answer on the pretest, cell (0,1) shows that eight students then gave a satisfactory answer on the posttest, and cell (0,N) shows that four students then gave no answer on the posttest. Of those who gave a satisfactory answer on the pretest, cell (1,0) indicates that two students gave an unsatisfactory answer on the posttest and cell (1,N) indicates that one student gave no response. Of those who did not give a response on the pretest, cell (N,0) indicates that three students gave an unsatisfactory answer on the posttest and cell (N,1) indicates that 11 students gave a satisfactory answer on the posttest.

Table C6c
Rows are levels of **C6c**
Columns are levels of **PC6c**
No Selector

	N	**cni**	**cv/ch**	**cvol**	**total**
N	15 26.3	3 5.26	11 19.3	0 0	29 50.9
aaver	1 1.75	0 0	1 1.75	0 0	2 3.51
cni	2 3.51	2 3.51	3 5.26	1 1.75	8 14.0
cv/ch	1 1.75	0 0	9 15.8	2 3.51	12 21.1
cvol	1 1.75	1 1.75	4 7.02	0 0	6 10.5
total	20 35.1	6 10.5	28 49.1	3 5.26	57 100

table contents:
Count
Percent of Table Total

N *No answer*.
aaver *Arithmetic average*. They calculated the arithmetic average of the average rates of change of volume with respect to height from 0 to 3 feet and 0 to 5 feet.
cni *Could not interpret.*
cv/ch *(Change in volume)/(change in height).*
cvol *Change in volume*. They calculated only the change in volume from 3 to 5 feet.

Categories of Inferred Thought Processes

The nature of Problem 6c was such that performance was essentially equivalent to inferred thought processes. Hence, no further discussion will be given for Problem 6c.

Summary of Interviews

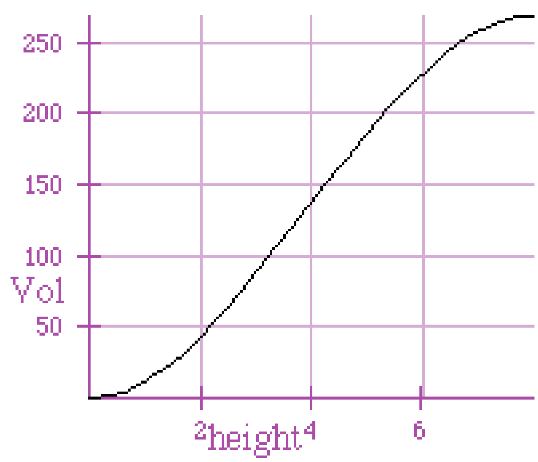
Of the six students interviewed, four students were able to answer this problem from the information given in the graph and compute the change in volume as compared to the change in height from 3 to 5 feet, although one student made an arithmetic error. All four of these students seemed to have a strong understanding of what their calculations represented also. However, none of these students compared their answer in Problem 6c to their answer in Problem 6b. One student was later asked to make this comparison and discuss the reasons why the answers to Problem 6b and 6c were different.

One student, who had difficulty answering this problem, stated that this problem involved a constant rate of change of volume with respect to height of 268/8 = 33.5 cubic feet of water for every foot of the tank which she had calculated in Problem 6b. She then drew a straight line on top of the given graph to indicate a constant rate of change. In her discussions of Problems 5a to 5e, she indicated that she knew the formula for the average speed of the ball during a certain time interval, that is, average speed = (change in distance)/(change in time). However, she did not apply this knowledge to the solution of Problem 6c which involved a (change in volume)/(change in height) in order to calculate the average rate of change of volume with respect to height during a certain interval of height. Her knowledge of average rate of change seems to be limited to having memorized the formula for average speed.

Another student, who had difficulty answering this problem, ignored the given graph, and wrote "[(5-3) feet]/[8 feet] = 1/4", and stated that the units canceled each other. After asking him to explain what the question was asking, he understood that he was to compute the average rate at which volume of water increased as the height increased from 3 to 5 feet. However, he focused on the units and stated that his answer, 1/4, was incorrect since his answer "had no units but the units of the answer should be cubic feet/foot". Still ignoring the given graph, he was unable to correctly answer Problem 6c. The interviewer then pointed out that, according to the graph, the answer would be approximately (175 - 75)/(5-3) = 50 cubic feet/foot but he had difficulty comprehending this. Not until he was asked to draw a spherical tank and explain why this answer was greater than the answer in Problem 6b, 33.5 cubic feet/foot, did he comprehend these answers. He pointed out, on a drawing of a spherical tank 8 feet high, that the spherical tank is widest in the middle and, hence, the average rate of volume of water with respect to the height between 3 and 5 feet would be greater than the average rate of volume of water with respect to height for the entire spherical tank.

Problem 6d

A spherical storage tank stood empty one morning and then was filled to capacity with water. The water's volume increased as its height increased. A supervisor, who had a dip stick but no clock, measured the water's depth repeatedly as the tank filled. The graph at the right represents the water's volume, in cubic feet, as a function of its height above the tank's bottom. The tank is 8 feet high and holds 268 feet of water.



(d) Suppose someone claimed that the water was poured into the storage tank at a constant rate of 85 cubic feet per minute. Would that claim be consistent with the above graph? Explain.

Intent of Problem 6d

The intent of Problem 6d was to determine if students, when presented with this situation involving three changing quantities (volume, height and time), could recognize that the average rate of change of volume with respect to height was not affected by a constant rate of change of volume with respect to time. In other words, could the students recognize that a change in a quantity (in this case volume) with respect to time had no bearing on the given graph, which did not involve time.

Criteria for Assigning Scores

A “1” was assigned to the answer “yes” which included an explanation for this answer similar to the following: “The given graph, which represents volume as a function of height, would be consistent with the claim that water was being poured into the tank at a constant rate of 85 cu ft/min or any other constant rate of volume with respect to time since the average rate of change of volume with respect to height was not affected by the amount of time it took the water to reach a particular height.” A “0” was assigned to all other answers.

Table 6d

Rows are levels of 6d
Columns are levels of P6d
No Selector

	0	1	A	N	total
0	17 18.9	3 3.33	10 11.1	9 10	39 43.3
1	2 2.22	2 2.22	4 4.44	0 0	8 8.89
N	7 7.78	1 1.11	19 21.1	16 17.8	43 47.8
total	26 28.9	6 6.67	33 36.7	25 27.8	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who gave an answer other than yes with an acceptable explanation for this answer (0), who answered yes with an acceptable explanation for this answer as described above (1), who were absent from the posttest (A), or who gave no answer (N). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 6d shows that 39 students (43.3%) stated that the graph was not consistent with a constant rate of 85 cubic feet/minute. Eight students (8.89%) stated that the graph was consistent with a constant rate of 85 cubic feet per minute and gave an acceptable explanation for their answer. Forty-three students (47.8%) gave no answer to this problem on the pretest.

Performance on Posttest

Of the 57 students who took both tests, 26 students (45.6%) stated that the graph was not consistent with a constant rate of 85 cubic feet/minute or stated that the graph was consistent with a constant rate of 85 cubic feet/minute but gave an unacceptable explanation for this answer. Six students (10.5%) stated that the graph was consistent with a constant rate of 85 cubic feet/minute and gave an acceptable explanation. Twenty-five students (43.9%) gave no answer to this problem on the posttest.

Comparison of Pretest and Posttest Performance

Cells (0,0), (1,1) and (N,N) of Table 6d show that 35 students (61.4%), of the 57 students who took both tests, were consistent from pretest to posttest. Of those who gave an unsatisfactory response on the pretest, cells (0,1) an (0,N) indicate that three students gave a satisfactory response and nine students did not give a response on the posttest. Of those who gave a satisfactory response on the pretest, cells (1,0) and (1,N) indicate that 2 students gave an unsatisfactory response and none failed to give a response on the posttest. Of those who gave no response on the pretest, cells (N,0) and (N,1) indicate that seven students gave an unsatisfactory response and one student gave a satisfactory response on the posttest.

Table C6d

Rows are levels of **C6d**
Columns are levels of **PC6d**
No Selector

	N	**calc**	**cnst**	**notgr**	**nwk**	**stln**	**total**
N	16 28.1	1 1.75	1 1.75	3 5.26	1 1.75	2 3.51	24 42.1
calc	1 1.75	2 3.51	0 0	1 1.75	0 0	0 0	4 7.02
cnst	1 1.75	1 1.75	1 1.75	1 1.75	0 0	1 1.75	5 8.77
notgr	3 5.26	1 1.75	0 0	2 3.51	2 3.51	1 1.75	9 15.8
nwk	0 0	0 0	0 0	1 1.75	1 1.75	0 0	2 3.51
stln	4 7.02	1 1.75	1 1.75	2 3.51	2 3.51	3 5.26	13 22.8
total	25 43.9	6 10.5	3 5.26	10 17.5	6 10.5	7 12.3	57 100

table contents:
Count
Percent of Table Total

N *No answer.*
calc *Tried a calculation.*
cnst *A constant rate of 85 cu ft/min was consistent with the graph.*
notgr *Time was not in the graph.* These students stated that they couldn't determine the answer since time was not one of the variables in the graph.
nwk *Stated an answer but gave no explanation for their answer.*
stln *Straight line.* They stated that the graph of volume as a function of height would be a straight line if the constant rate of flow of water was 85 cu ft/min.

Categories of Inferred Thought Processes

Table C6d shows that responses, for the 57 students who took both tests, fell into six main categories based on inferred thought processes. Thirteen students on the pretest and seven students on the posttest indicated that the given graph was not consistent with a constant rate of 85 cubic feet of water per minute since "a constant rate would refer to a graph with a straight line." Twenty-four students on the pretest and 25 students on the posttest gave no answer. Five students on the pretest and three students on the posttest gave an acceptable response, as described above. Nine students on the pretest and 10 students on the posttest either stated that the graph was not consistent with a constant rate of 85 cu ft/min or could not determine the answer to this problem since time was not a variable of the given graph. One of these students wrote, “It has no relation. One is time, the other is height”. Four students on the pretest and six students on the posttest stated that the graph was not consistent with a constant rate of 85 cubic feet of water per minute since it would take 3.15 minutes to fill the 268 cubic foot tank but the graph shows it was filled at 8. Perhaps these students confused height and time or simply replaced height with time on the horizontal axis. One student on the pretest and one student on the posttest stated that the answer to Problem 6d was “No, because during the steepest part of the curve, the average is only 50 cu ft/min”. They had changed “foot” to “minutes” in “50 cubic feet/foot”, their answer to Problem 6c, when giving their answer to this problem.

Summary of interviews

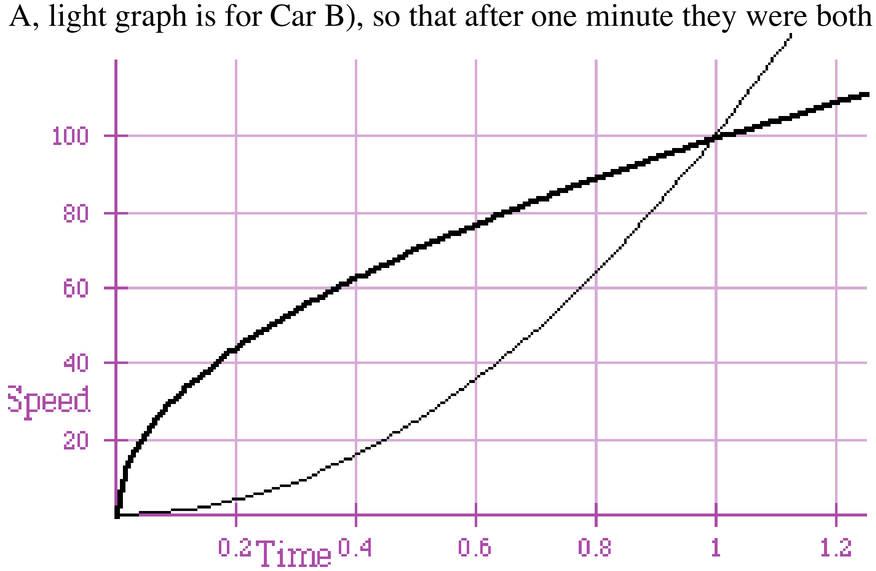
During the six interviews, when Problem 6d introduced the unit “cubic feet/minute”, it appeared that two students thought that the horizontal axes then represented time rather than height. These students stated that it would take 3.15 minutes to fill the tank at a rate of 85 cu ft/min, not 8, and one of the students added that the graph should be a straight line since the water was flowing in at a “constant” rate. Three other students also stated that the given graph was inconsistent with a constant rate of 85 cubic feet per minute since “constant” implied a straight line, and the graph was not a straight line.

Another student, also thought the graph was inconsistent with a constant rate of change of volume with respect to time. He stated that the given graph showed “what the volume is with respect to how far the tank is filled up, not how long it took to fill the tank up...they are unrelated and the units cu ft/ft and cu ft/min are unrelated". When asked to describe the graph of a constant rate of 85 cubic feet/minute if the horizontal axis were changed to time, he was not able to answer this question.

It is interesting to note that none of the students, during their interview regarding Problem 6d, made a real effort to understand why the given graph, representing the volume of water as a function of height of the spherical tank, had "curves" at both ends although they appeared to understand that "The water's volume increased as its height increased." They also had difficulty understanding that the time it took the water to reach certain heights would not affect the given graph which represented the volume as a function of the height. Hence, they had difficulty understanding that the given graph was consistent with a constant rate of change of 85 cubic feet per minute.

Problem 7a

Two cars, Car A and Car B, started from the same point, at the same time, and traveled in the same direction. Their speeds increased, as shown in the graph (heavy graph is for Car A, light graph is for Car B), so that after one minute they were both traveling at 100 mph.



(a) Was the distance between the cars increasing or decreasing 0.8 minutes after they started? Explain.

Intent of Problem 7a

The intent of Problem 7a was to determine if students, given the graphs of the speeds of two cars as a function of time, could describe the change in the distance between the two cars at a particular moment in time. Students who confuse speed with distance (in the graph) might expect the distance between the cars to decrease as the difference between their speeds decreases. Students with a clearer understanding of the graph and of relative speed would recognize that, since Car A is going faster than Car B throughout the first minute, the distance between the cars is continually increasing throughout the time interval from 0 to 1 minute.

Criteria for Assigning Scores

A "1" was assigned to any answer which suggested that the student thought the distance between the two cars was increasing at 0.8 minutes since Car A's speed was still faster than Car B's speed at this particular moment in time. All other answers were assigned a "0".

Table 7a

Rows are levels of **7a**
Columns are levels of **P7a**
No Selector

	0	**1**	**A**	**N**	**total**
0	34 37.8	7 7.78	20 22.2	3 3.33	64 71.1
1	5 5.56	3 3.33	3 3.33	1 1.11	12 13.3
N	3 3.33	0 0	10 11.1	1 1.11	14 15.6
total	42 46.7	10 11.1	33 36.7	5 5.56	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who gave an answer other than increasing with an acceptable explanation (0), who gave an answer of increasing with an acceptable answer (1), who were absent from the posttest (A), or who gave no response (N). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 7a shows that 64 students (71.1%) gave an incorrect answer or explanation. Twelve students (13.3%) gave the correct answer and explanation as described above and 14 students (15.6%) gave no answer to this problem on the pretest.

Performance on Posttest

Of the 57 students who were present for the posttest, 42 students (73.7%) gave an incorrect answer or explanation. Ten students (17.5%) gave a correct answer and explanation as described above and five students (8.77%) gave no answer to this problem on the posttest.

Comparison of Pretest and Posttest Performance

By examining the (0,0), (1,1) and (N,N) cells of Table 7a we can see that 38 students (66.7%), of the 57 students who took both tests, were consistent from pretest to posttest. Of the 41 students who gave an unsatisfactory answer or explanation on the pretest, cell (0,1) shows that seven students gave a satisfactory answer and explanation on the posttest and cell (0,N) shows that three students gave no response on the posttest. Of the nine students who gave a satisfactory answer and explanation on the pretest, cell (1,0) shows that five students gave an unsatisfactory answer or explanation on the posttest and cell (1,N) shows that one student gave no answer. Of the four students who gave no answer on the pretest, cell (N,0) shows that three students gave an unsatisfactory answer or explanation on the posttest and cell (N,1) shows that none of these students gave the correct answer on the posttest.

Table C7a

Rows are levels of **C7a**
Columns are levels of **PC7a**
No Selector

	N	**axsgr**	**cni**	**ind**	**jstdc**	**jstgr**	**total**
N	0	0	2	0	0	0	2
	0	0	3.51	0	0	0	3.51
axsgr	0	12	3	2	4	2	23
	0	21.1	5.26	3.51	7.02	3.51	40.4
cni	0	0	3	1	1	1	6
	0	0	5.26	1.75	1.75	1.75	10.5
ind	1	1	2	3	2	0	9
	1.75	1.75	3.51	5.26	3.51	0	15.8
jstdc	0	1	1	0	2	0	4
	0	1.75	1.75	0	3.51	0	7.02
jstgr	0	4	0	3	1	5	13
	0	7.02	0	5.26	1.75	8.77	22.8
total	1	18	11	9	10	8	57
	1.75	31.6	19.3	15.8	17.5	14.0	100

table contents:
Count
Percent of Table Total

N *No answer.*
axsgr *Focused on the axes and graph.* They stated that Car B was increasing its acceleration at a faster rate than Car A at 0.8 minutes, hence, the distance between them was decreasing.
cni *Could not interpret.*
ind *Increasing distance at 0.8 minutes.*
jstdc *Just wrote "decreasing".*
justgr *Focused just on the graph.* Wrote "decreasing" since the lines of the graph came closer together at 0.8 minutes.

Categories of Inferred Thought Processes

Table C/a indicates that the responses of the 57 students who took both tests fell into six categories based upon the inferred thought processes of these students. It was presumed that there were several reasons why many of these students incorrectly decided that the distance between the cars was decreasing, rather than increasing, at 0.8 minutes. Twenty-three students (40.4%) on the pretest and 18 students (31.6%) on the posttest, although they realized that this was a graph of the speed of the cars as a function of time, gave a reason similar to the following, “B was increasing its acceleration and also A but at a slower rate so B was getting closer to A at 0.8 minutes”. These students perhaps had confused acceleration with speed. Although Car B’s acceleration was greater than Car A’s acceleration at 0.8 minutes, as the tangents to the graph at this moment in time would indicate, Car B’s speed is still slower than Car A’s speed at this time. Hence, the distance between the cars is still increasing at 0.8 minutes. Thirteen students (22.8%) on the pretest and eight students (14.0%) on the posttest either thought the axes represented distance and time rather than speed and time or ignored the axes and believed the graphs were “paths” the cars had followed, hence, at 0.8 minutes the distance between the cars would be decreasing.

Two students (3.51%) on the pretest and one student (1.75%) on the posttest gave no answer to Problem 7a. Six students (10.5%) on the pretest and 11 students (19.3%) on the posttest gave an answer which the investigator could not interpret. For example, one student wrote “25 miles apart” and another wrote “This problem is confusing because I do not have any idea of where the cars are in space relative to each other. I am only presented with data pertaining to the relative speeds of both cars.” Four students (7.0%) on the pretest and 10 students (17.5%) on the posttest wrote “decreasing” but gave no explanation. Nine students (16.1%) on both tests gave an acceptable answer and explanation; the distance between the cars is increasing at 0.8 minutes since Car A’s speed was still greater than Car B’s speed at that moment in time. They were able to relate the speed of the cars to the change in distance between the cars at a particular moment in time.

Summary of Interviews

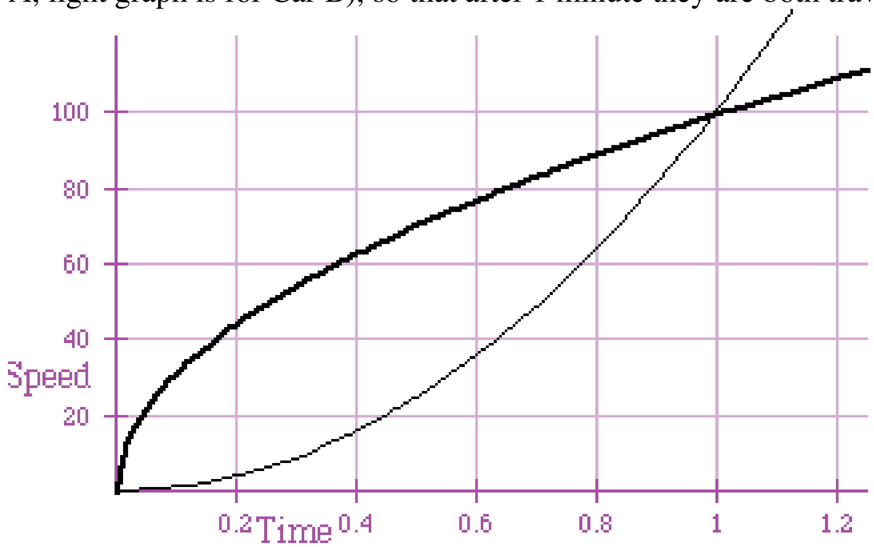
During the six interviews, one student stated that he was "confused" and could not answer this problem since he was only presented with a graph which pertained to the cars' speeds and not with the distance the cars had traveled or where the cars were located. He was unable to relate the distance between the cars given the graph of the speeds of the cars as a function of time.

Three other students stated that the distance between the cars was decreasing since they assumed the graph represented the "paths" the cars traveled and at 0.8 minutes the distance between the paths was decreasing. However, one of these three students then described the speeds of the cars from 0 to 1 minute and stated that Car A's speed was always greater than Car B's speed but he still could not relate this understanding to the distance between the cars at 0.8 minutes.

Two other students came to the correct conclusion that the distance between the cars was increasing at 0.8 minutes but their thought processes differed. One student, who had achieved the highest score overall on both tests, gave a reason which was unique. He stated that the areas below the speed as a function of time graphs represented the distances the cars had traveled and at 0.8 minutes Car A was increasing its area more than Car B. Hence, the distance between the cars was increasing at 0.8 minutes. This student added that he had learned this technique before he enrolled in Math 150. The other student, who on his pretest had written that the distance between the cars was "decreasing" at 0.8 minutes and on his posttest had written that the distance between the cars was "increasing" at 0.8 minutes, during the interview stated that he agreed with his answer on the posttest. He stated on his posttest and during the interview that "Car A was still pulling away from Car B at 0.8 minutes since Car A's speed was still greater than Car B's speed at 0.8 minutes."

Problem 7b

Two cars, Car A and Car B, started from the same point, at the same time, and traveled in the same direction. Their speeds increased, as shown in the graph (heavy graph is for Car A, light graph is for Car B), so that after 1 minute they are both traveling at 100 mph.



(b) Describe the cars' relative positions 1 minute after they started. Explain.

Intent of Problem 7b

The intent of Problem 7b was to determine if students, given the graphs of the speeds of two cars as a function of time, could compare the distances the two cars traveled during an interval of time. This question examined whether students' understood that the car with the greater speed traveled further than the slower car for the same time interval. Because the distance between the cars was continually increasing throughout the time interval 0 to 1 minute, at 1 minute the cars were farthest apart.

Criteria for Assigning Scores

A "1" was assigned to an answer which suggested that the student thought that Car A had traveled further than Car B during the same time interval, from 0 to 1 minute, since the graph indicated that Car A's speed was always greater than Car B's speed during this time interval. A "0" was given to all other responses.

Table 7b
Rows are levels of **7b**
Columns are levels of **P7b**
No Selector

	0	**1**	**A**	**N**	**total**
0	28 31.1	7 7.78	18 20	4 4.44	57 63.3
1	1 1.11	10 11.1	4 4.44	2 2.22	17 18.9
N	4 4.44	0 0	11 12.2	1 1.11	16 17.8
total	33 36.7	17 18.9	33 36.7	7 7.78	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who gave an answer other than Car A traveled further than Car B since Car A's speed was always greater than Car B's speed from 0 to 1 minute (0), who answered that Car A traveled further than Car B since Car A's speed was always greater than Car B's speed from 0 to 1 minutes (1), who were absent from the posttest (A), or who gave no response (N). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 7b shows that 57 students (63.3%) gave an incorrect response or explanation. For example, the student may have stated that the cars had traveled the same distance or only discussed the speeds of the two cars and not the cars' relative positions.

Seventeen students (18.9%) gave the correct response and explanation and 16 students (17.8%) gave no response.

Performance on Posttest

Of the 57 students who took the posttest, 33 students (57.9%) gave an incorrect response or explanation, 17 students (29.8%) gave the correct response and explanation and seven students (12.2%) gave no response.

Comparison of Pretest and Posttest Performance

Cells (0,0), (1,1) and (N,N) of Table 7b show that 39 students (68.4%), of the 57 students who took both tests, were consistent from pretest to posttest. Of the 39 students who gave an incorrect answer or explanation on the pretest, cell (0,1) shows that seven students gave a correct answer and explanation on the posttest, and cell (0,N) shows that four students gave no answer on the posttest. Of the 13 students who gave a correct answer and explanation on the pretest, cell (1,0) shows that one student gave an incorrect answer or explanation on the posttest and cell (1,N) shows that two students gave no answer on the posttest. Of the 5 students who gave no answer on the pretest, cell (N,0) shows that four students gave an incorrect answer or explanation on the posttest and cell (N,1) shows that none of these students gave a correct answer and explanation on the posttest.

Table C7b

Rows are levels of **C7b**
Columns are levels of **PC7b**
No Selector

	AgrBd	**N**	**cni**	**jstac**	**samed**	**samev**	**total**
AgrBd	9 15.8	2 3.51	0 0	0 0	1 1.75	0 0	12 21.1
N	0 0	0 0	2 3.51	0 0	1 1.75	0 0	3 5.26
cni	1 1.75	0 0	2 3.51	1 1.75	2 3.51	0 0	6 10.5
jstac	0 0	0 0	0 0	3 5.26	1 1.75	0 0	4 7.02
samed	6 10.5	0 0	4 7.02	4 7.02	13 22.8	1 1.75	28 49.1
samev	0 0	0 0	0 0	1 1.75	2 3.51	1 1.75	4 7.02
total	16 28.1	2 3.51	8 14.0	9 15.8	20 35.1	2 3.51	57 100

table contents:
Count
Percent of Table Total

AgrBd *Car A's distance was greater than Car B's distance*.
N *No answer*.
cni *Could not interpret*.
jstac *Car A's acceleration was decreasing, Car B's acceleration was increasing*. They did not discuss the cars' relative positions.
samed *Same distance*. They stated that the cars had traveled the same distance during the time interval from 0 to 1 minute.
samev *Same velocity*. They stated that the cars were traveling at 100 miles per hour but they did not discuss the cars' relative positions.

Categories of Inferred Thought Processes

For those 57 students who had taken both tests, responses to Problem 7b fell into six categories based on students' thought processes. Table C7b shows that 28 students (49.1%) on the pretest and 20 students (35.1%) on the posttest stated that the cars had traveled the same distance. Of these students, six students on the pretest and one student on the posttest, thought of the graphs as "paths" the cars traveled along. These students stated that the "cars ran into each other" or "there was a collision" at 1 minute. Three other students on the pretest also thought the cars traveled the same distance but considered the speeds of the cars in the time interval 0 to 1 minute. They discussed the average acceleration of the cars when they stated that "The cars are at exactly the same distance from the start. Car A started out fast, but slowed down. Car B started slow but sped up rapidly because its slope is steep. The curves are identical except for them being reversed from each other." The remainder of the students thought that the cars, having reached the same speed at 1 minute, traveled the same distance at 1 minute. This can be seen in statements similar to "They are now traveling at the same speed, 100 mph, and so they are both beside each other having traveled the same distance."

Three students (5.26%) on the pretest and two students (3.15%) on the posttest gave no response to this problem. It appeared that these students were not persistent in their attempt to find the relative distance between the cars at 1 minute given a graph of the cars' speeds as a function of time. Six students (10.5%) on the pretest and eight students (14%) on the posttest gave an answer which the investigator could not interpret, such as, "There was no distance." Twelve students (21.1%) on the pretest and 16 students (28.1%) on the posttest gave the correct answer and an acceptable explanation. One of the students, who had given a correct answer and acceptable explanation, on the posttest stated that "Car A is ahead because his average speed with relation to time is better." Four students (7.0%) on the pretest and two students (3.5%) on the posttest only stated that both cars were traveling at 100 mph at 1 minute. These students did not discuss the cars' relative positions. Four students (7.0%) on the pretest and nine students (15.8%) on the posttest only stated that Car A was slowing down and Car B was speeding up. These students also did not discuss the cars' relative positions.

Summary of Interviews

During one of the six interviews, a student described her thoughts regarding the given graph in Problem 7b as follows: “Car A took off fast and kept at a steady pace but Car B took off slowly and gradually increased so they both meet at 1 minute. For some reason they look like they are on a little path.” It is interesting to note that, from her first comment, she was able to accurately interpret the graph and compare the speed of Car A to Car B. However, she states that “they both meet at 1 minute” since the cars “look like they are on a little path”. She appears to be viewing the graph at two levels. One is at a naive level (the graph represents a path on which the cars travel), and the other is at an interpretive level (the graph represents speed as a function of time which must then be related a situation regarding their relative positions). When it was pointed out to her that Car A’s speed was always greater than Car B’s speed during the time interval 0 and 1 minute, she still insisted that the cars were traveling next to each other at 1 minute. She again interpreted the graph correctly in terms of their speeds but also looked at the graph as the “paths” the cars traveled. She also insisted throughout the interview that since the cars reached the same speed at 1 minute they had traveled the same distance at 1 minute.

Another student said the cars were traveling next to each other at 1 minute since “their average velocity was the same” during the time interval 0 to 1 minute. He drew a straight line from the origin through the intersection point of the graphs at 1 minute to represent this “average velocity”. He did not realize that this straight line represented the average acceleration of each car and not the average velocity of each car.

Two other students correctly stated that Car A was ahead of Car B at 1 minute but gave different reasons for this answer. One student looked at the area under each graph from 0 to 1 minute, which he said represented the distance each car traveled, and concluded that Car A had traveled further than Car B during the first minute since the area under its graph was greater. The other student related the graphs of speeds as a function of time to the distances the cars traveled. He concluded Car A traveled further than Car B since Car A's speed was always greater than Car B's from 0 to 1 minute after which time "Car B will begin to catch up".

Two other students during their interview stated that they could not determine the relative positions of the cars from the graph which represented the speeds of the cars as a function of time. They only commented that the cars were both traveling at 100 mph at 1 minute. However, after it was pointed out to them that the graph showed that Car A's speed was always greater than Car B's speed from 0 to 1 minute, they were able to relate Car A's position to Car B's position, and stated that Car A was ahead of Car B at 1 minute.

Problem 8

When the Discovery space shuttle is launched, its speed increases continually until its booster engines separate from the shuttle. During the time it is continually speeding up, the shuttle is never moving at a constant speed. What, then, would it mean to say that at precisely 2.15823 seconds after launch the shuttle is traveling at precisely 183.8964 miles per hour?

Intent of Problem 8

Problem 8 was intended to explore students' understanding of an accelerating object given a specific situation with specific numerical values. Students with a clear understanding of instantaneous velocity would be aware that if the shuttle were to stop accelerating at precisely 2.15823 seconds after the launch, it would continue to travel at a constant velocity of 183.8964 miles per hour. One reason this question was of interest was the fact that instantaneous velocity is a term used frequently in many calculus texts.

Criteria for Assigning Scores

A "1" was assigned to an answer which suggested that the student understood that if the object stopped accelerating at precisely 2.15823 seconds the object would continue to travel at the constant velocity 183.8964 miles per hour or if they expressed instantaneous velocity as a limit of average velocities over intervals of decreasing lengths (no students actually gave this explanation, however). A "0" was assigned to all other answers including an answer similar to "instantaneous velocity". "Instantaneous velocity" is a term used to describe the given situation but does not indicate that the student understood the concept "instantaneous velocity" or the situation in the given problem.

Table 8

Rows are levels of **8**
Columns are levels of **P 8**
No Selector

	0	**1**	**A**	**N**	**total**
0	41 45.6	1 1.11	22 24.4	3 3.33	67 74.4
1	1 1.11	0 0	0 0	0 0	1 1.11
N	9 10	0 0	11 12.2	2 2.22	22 24.4
total	51 56.7	1 1.11	33 36.7	5 5.56	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who gave an answer that did not include stopping the acceleration of the shuttle at a precise moment (0), who gave an answer that did include stopping the acceleration of the shuttle at a precise moment after which it would continue at a constant velocity (1), who were absent from the posttest (A), or who gave no response (N). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 8 shows that 67 students (74.4%) gave an answer which did not explain the meaning of the phrase "at precisely 2.15823 seconds after launch the shuttle is traveling at precisely 183.8964 miles per hour." Of these 67 students, 27 students either repeated the previous phrase or stated that the previous phrase meant "instantaneous velocity" and gave no further explanation. Two students tried to solve for a distance using the formula "distance = velocity x time" substituting the numbers given in the problem for velocity and time. There was only one student (1.11%) who gave an acceptable answer to Problem 8. This problem had the least number of correct answers overall on the pretest. Twenty-two students (24.4%) gave no answer.

Performance on Posttest

Of the 57 students that took the posttest, 51 students (89.5%) gave an answer which did not explain the phrase "at precisely 2.25823 seconds after launch the shuttle is traveling at precisely 183.8964 miles per hour." Of these 51 students, 37 students repeated the previous phrase or stated that the phrase meant "instantaneous velocity" and gave no further explanation. Two students tried to solve for distance using the formula d = velocity x time, substituting the given velocity and time into this equation. Only one student (1.75%), the least number of correct answers overall on the posttest, gave an acceptable answer to Problem 8. Five students (8.77%) gave no answer to this problem on the posttest which was approximately one-third of the number of the students who gave no answer to this problem on the pretest.

Comparison of Pretest and Posttest Performance

Cells (0,0), (1,1) and (N,N) of Table 8 show that 43 students, of the 57 students who took both tests, were consistent from pretest to posttest. Of the 41 students who gave an unacceptable answer on the pretest, cell (0,1) shows that one student gave an acceptable answer on the posttest, and cell (0,N) shows that three students gave no answer. One student gave an acceptable answer on the pretest; this student gave an unacceptable answer on the posttest (saying only "instantaneous velocity" on the posttest). Of the 22 students who gave no answer on the pretest, cell (N,0) shows that nine students gave an unacceptable answer on the posttest and cell (N,1) shows that none of these students gave an acceptable answer on the posttest.

Table C8

Rows are levels of **C8**
Columns are levels of **PC8**
No Selector

	N	**cni**	**d=rt**	**instv**	**stpac**	**total**
N	0 0	1 1.75	0 0	5 8.77	0 0	6 10.5
cni	0 0	13 22.8	1 1.75	6 10.5	0 0	20 35.1
d=rt	0 0	0 0	1 1.75	1 1.75	0 0	2 3.51
instv	1 1.75	2 3.51	0 0	24 42.1	0 0	27 47.4
stpac	0 0	0 0	0 0	1 1.75	1 1.75	2 3.51
total	1 1.75	16 28.1	2 3.51	37 64.9	1 1.75	57 100

table contents:
Count
Percent of Table Total

N *No answer.*

cni *Could not interpret.*
d=rt *Distance equals rate (or velocity) x time.*
instv *Wrote "instantaneous velocity" but gave no further explanation.*
stpac *Stopping the acceleration of the shuttle at a precise moment in time yields a constant velocity for the shuttle after that moment in time.*

Categories of Inferred Thought Processes

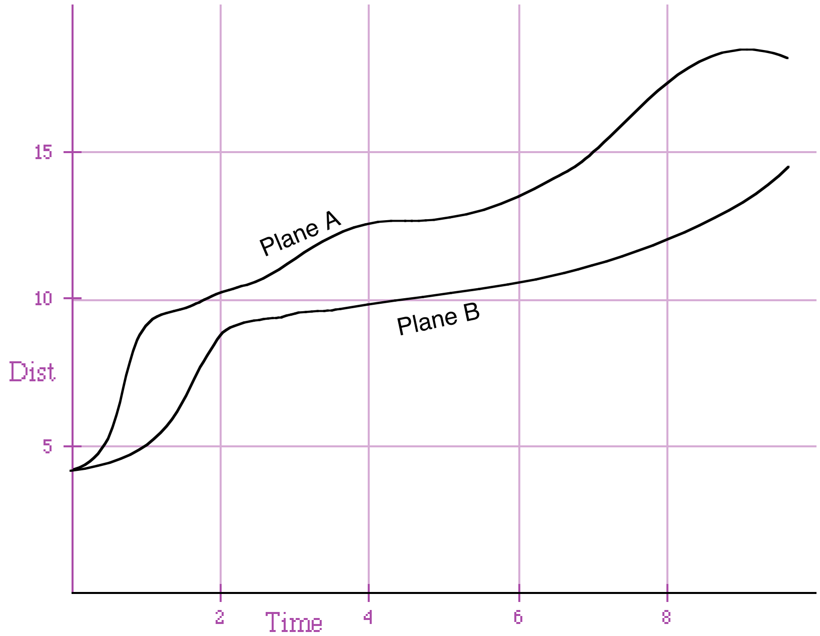
The nature of Problem 8 was such that performance was essentially equivalent to inferred thought processes. It was interesting to note terms, other than "instantaneous velocity", which students used to describe the situation in the given problem and yet had not explained what these terms meant. For example, "the tangent" was given as an answer. The investigator assumed they were referring to "the slope of the tangent to the graph of distance as a function of time at precisely 2.5823 seconds was precisely 183.8964 miles per hour". The term "derivative" was also given as an answer. Again, the investigator assumed they were referring to "the derivative of the distance as a function of time at precisely 2.5823 seconds was precisely 183.8964 miles per hour."

Summary of Interviews

During the six interviews, five students recognized that this problem referred to a situation that involved "instantaneous velocity", a term discussed in first semester calculus, but they were unable to explain what this term meant. Two of these students added that at a time other than 2.15823 seconds the shuttle would be traveling at a speed other than 183.8964 miles per hour. The one student who had given an acceptable response to this problem on the posttest, during the interview, also described taking a photograph at precisely 2.15823 seconds and, at that precise moment in time, the shuttle would be going precisely 183.8964 miles per hour. This was another description of "instantaneous velocity" which did not indicate a clear understanding of the fact that if the shuttle had stopped accelerating at precisely 2.15823 seconds after the launch, it would continue to travel at a constant velocity of 183.8964 miles per hour.

Problem 9

Assume that two planes, A and B, are flying away from San Diego and that their distances from the San Diego airport are continually monitored. The planes' distances from San Diego are shown by the graph given below for a 10 second period of their trip.



Compare the planes' speeds 1.5 seconds after the beginning of this period of time.

Intent of Problem 9

The intent of Problem 9 was similar to Problem 7, except this problem was designed to determine if students, given the graphs of the distances traveled as a function of time for two planes, could compare the planes' speeds at a particular moment in time. This problem extended problem 7 because in Problem 7 one velocity was greater than the other throughout the entire time interval; thus, the students could solve Problem 7 without the concept of instantaneous velocity. In order to solve Problem 9, an understanding of instantaneous velocity was essential.

Criteria for Assigning Scores

A “1” was given a response similar to “Plane B’s speed was greater than Plane A’s speed at 1.5 seconds”. A “0” was given all other responses.

Table 9
Rows are levels of **9**
Columns are levels of **P 9**
No Selector

	0	**1**	**A**	**N**	**total**
0	31 34.4	3 3.33	18 20	4 4.44	56 62.2
1	3 3.33	12 13.3	3 3.33	0 0	18 20
N	2 2.22	0 0	12 13.3	2 2.22	16 17.8
total	36 40	15 16.7	33 36.7	6 6.67	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who gave an answer other than Plane B’s speed was greater than Plane A’s speed at 1.5 seconds (0), who answered that Plane B’s speed was greater than Plane A’s speed at 1.5 seconds (1), who were absent from the posttest (A), or who gave no response (N). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 9 shows that 56 students (62.2%) gave an answer other than Plane B’s speed was greater than Plane A’s speed at 1.5 seconds. Eighteen students (20%) stated that Plane B’s speed was greater than Plane A’s speed at 1.5 seconds, and 16 students (17.8%) gave no answer.

Performance on Posttest

Of the 57 students who took the posttest, 36 students (63.1%) gave an answer other than Plane B's speed was greater than Plane A's speed at 1.5 seconds. Fifteen students (27.4%) said Plane B's speed was the greater, and six students (10.5%) gave no answer to this problem on the posttest.

Comparison of Pretest and Posttest Performance

From Table 9, cells (0,0), (1,1) and (N,N) indicate that 45 students (78.9%), of the 57 students who took both tests, were consistent from pretest to posttest. Of the 31 students who gave an unacceptable answer on the pretest, cell (0,1) indicates that three students gave an acceptable answer on the posttest and cell (0,N) indicates that four students gave no answer on the posttest. Of the 18 students who gave an acceptable answer on the pretest, cell (1,0) shows that three students gave an unacceptable answer on the posttest and cell (1,N) shows that all these students gave an answer on the posttest. Of those who gave no response on the pretest, cell (N,0) shows that two students gave an unacceptable answer on the posttest and cell (N,1) shows that none of these students gave an acceptable response on the posttest.

Table C9

Rows are levels of **C9**
Columns are levels of **PC9**
No Selector

	BgrAv	**N**	**averv**	**cni**	**jstd**	**jstgr**	**total**
BgrAv	14 24.6	0 0	2 3.51	0 0	0 0	1 1.75	17 29.8
N	0 0	0 0	1 1.75	0 0	0 0	0 0	1 1.75
averv	3 5.26	3 5.26	15 26.3	1 1.75	2 3.51	2 3.51	26 45.6
cni	0 0	0 0	0 0	5 8.77	0 0	0 0	5 8.77
jstd	0 0	0 0	1 1.75	1 1.75	0 0	0 0	2 3.51
jstgr	1 1.75	0 0	2 3.51	0 0	2 3.51	1 1.75	6 10.5
total	18 31.6	3 5.26	21 36.8	7 12.3	4 7.02	4 7.02	57 100

table contents:
Count
Percent of Table Total

BgrAv *Plane B's velocity was greater than Plane A's velocity at 1.5 seconds.*
N *No answer.*
averv *Compared the planes average velocities during the interval 0 to 1.5 seconds.*
cni *Could not interpret.*
jstd *Compared only the distances traveled by each plane at 1.5 seconds.*
jstgr *Compared the graphs and ignored the axes.* They concluded that Plane A's speed was greater than Plane B's speed at 1.5 seconds since its graph was "higher".

Categories of Inferred Thought Processes

Table C9 shows that responses fell into six categories based on students' thought processes. Of the 57 students who took both tests, 17 students on the pretest and 18 students on the posttest gave the correct response. These students gave several different reasons for their response, based on their conception of the information in the given graph. Several students focused only on the graph which represented Plane A's distance as a function of time. They indicated that the increase in distance for Plane A during a small interval of time which included 1.5 seconds was close to zero. They were then able to conclude that Plane A's speed was close to zero, hence, Plane B's speed was greater. Several other students compared the graphs of both planes during a interval of time which included 1.5 seconds. They concluded that "Plane B is traveling faster than Plane A because Plane B is covering a greater distance in the same time as Plane A". Other students, on both tests also, compared the slopes of the tangents of the graphs at 1.5 seconds. They concluded that since the slope of the tangent for Plane B was greater than the slope of the tangent for Plane A at 1.5 seconds, Plane B's speed was greater than Plane A's speed at this moment in time.

Of the 57 students that took both tests, five students on the pretest and seven students on the posttest gave an answer which the investigator could not interpret. For example, several students gave an answer similar to "How can I, there is no mention of speed in the graph." Other students drew a vertical line at 1.5 seconds and stated that the planes were traveling the same speed at this moment in time.

Six students on the pretest and four on the posttest appeared to only focus on the graphs and concluded that, since the graph for Plane A was "higher" than the graph for Plane B, Plane A's speed was greater at 1.5 seconds. Several of these students stated that Plane A's altitude was greater than Plane B's since Plane A's graph was higher. Twenty-six students on the pretest and 21 students on the posttest calculated the average speed of the planes from 0 to 1.5 seconds. From their calculations, they concluded that Plane A's speed was greater than Plane B's speed at 1.5 seconds. Two students on the pretest and four students on the posttest did not give an answer regarding the speeds of the planes but only discussed the fact that Plane A had traveled further than Plane B during the time interval from 0 to 1.5 seconds.

Summary of Interviews

During the six interviews, two students, after they were asked to determine which plane was traveling faster at precisely 1.5 seconds, calculated the average speed of both planes during the time interval from 0 to 1.5 seconds and concluded that Plane A had the higher speed. When they were asked why they calculated the average speed of the planes they replied that "we wanted speed which is distance over time". They appeared not to understand the distinction between the concept of average velocity and instantaneous velocity in this problem.

Another student was asked if she had seen problems similar to this problem in Math 150 and she replied "not like this". The investigator asked this question because during the interviews, many students did not appear to understand the concepts involved in this problem, that is, they were to compare the instantaneous velocities, not average velocities, of the two planes. Obviously this student had studied instantaneous rate of change and average rate of change in developing the concept of the derivative in Math 150 yet, as her reply appeared to indicate, she could not transfer what she had learned to this problem.

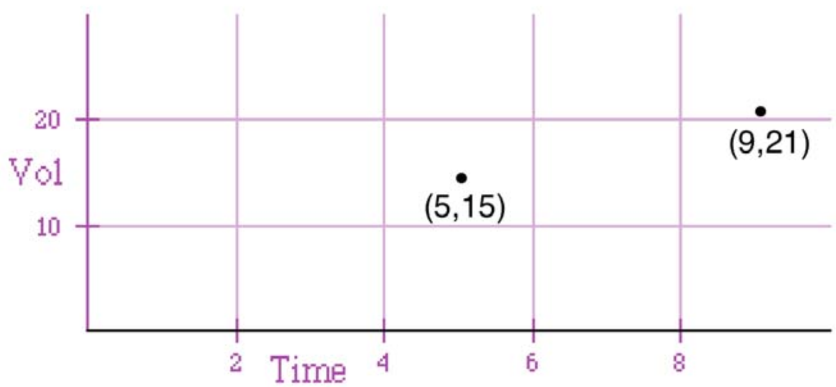
A fourth student was asked what the slope of the tangent line to each graph at 1.5 seconds represented. She replied that she "had studied this at the very start of Math 150 but its not coming back". She did not comment on this question any further.

Another student, at first, appeared to ignore the axes of the given graph which indicated that this graph represented the planes' distances as a function of time. He appeared to think the graph represented velocity as a function of time when he stated that "about 1.5 seconds, Plane A's velocity remains almost constant but is still greater than Plane B's velocity." He then stated that "You could take the derivative, but I'm looking at the overall graph." When asked to explain his last statement, he seemed to indicate that taking the derivative of the given graph left the graph unchanged but changed the axes from distance as a function of time to velocity as a function of time. He was then told to compare the distances the planes traveled during a small interval of time which included 1.5 seconds. From this, he was able to give the correct answer and explanation; Plane B's speed was greater than Plane A's speed at 1.5 seconds since Plane B traveled a greater distance than Plane A during this small interval of time.

During the interview with the sixth student, he compared the slopes of the slopes of the tangents of the graphs at 1.5 seconds which he stated represented the speeds of the planes at 1.5 seconds and concluded that Plane B's speed was greater than Plane A's speed at that moment in time.

Problem 10a

Water flowed into a tank at a constant rate with respect to time. The water's volume was measured (in gallons) after flowing for 5 seconds and again after flowing for 9 seconds. This information is given in a graph, below.



(a) How much water was in the tank when water began flowing into it?

Intent of Problem 10a

The intent of Problem 10a was to determine if students, given the volumes of water in a tank at two moments in time, could use the fact that water was flowing in at a constant rate in order to calculate the water's volume at another moment in time (in this case at t = 0 seconds). In this type of problem, students often assume that the initial volume of water in the tank is zero since most textbook problems encourage this assumption. This problem explores their ability to solve a problem having an initial volume that is not zero.

Criteria for Assigning Scores

A "1" was assigned only to an answer equal to 7.5 gallons, the volume of water in the tank when water began flowing into the tank. A "0" was assigned to all other answers, even though the student may have demonstrated a correct thought process in solving Problem 10a. As will be discussed, although a student may have possessed a correct thought process he may not have been able to arrive at the correct answer or any answer at all.

Table 10a

Rows are levels of **10a**
Columns are levels of **P10a**
No Selector

	0	**1**	**A**	**N**	**total**
0	25 27.8	1 1.11	15 16.7	6 6.67	47 52.2
1	4 4.44	11 12.2	11 12.2	3 3.33	29 32.2
N	5 5.56	1 1.11	7 7.78	1 1.11	14 15.6
total	34 37.8	13 14.4	33 36.7	10 11.1	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who gave an answer other than 7.5 gallons (0), who gave the answer 7.5 gallons (1), who were absent from the posttest (A), or who gave no response

(N). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 10a shows that 47 students (52.2%) gave an answer other than 7.5 gallons, 29 students (32.2%) gave the answer 7.5 gallons and 14 students (15.6%) gave no answer to Problem 10a on the pretest.

Performance on Posttest

Of the 57 students that took the posttest, 34 students (59.6%) gave an answer other than 7.5 gallons, 13 students (22.8%) gave the answer 7.5 gallons and 10 students (17.5%) gave no answer to Problem 10a on the posttest.

Comparison of Pretest and Posttest Performance

Examining cells (0,0), (1,1) and (N,N) of Table 10a, we see that, of the 57 students who took both tests, 37 students were consistent from pretest to posttest. Of the 32 students who gave an answer other than 7.5 gallons on the pretest, cell (0,1) indicates that one student gave the answer 7.5 gallons on the posttest and cell (0,N) indicates that six students gave no answer on the posttest. Of the 18 students who gave the answer 7.5 gallons on the pretest, cell (1,0) indicates that four students gave an answer other than 7.5 gallons on the posttest and cell (1,N) indicates that three students gave no answer on the posttest. Of the 14 students who gave no answer on the pretest, cell (N,0) indicates that five students gave an answer other than 7.5 gallons on the posttest and cell (N,1) indicates that one student gave the answer 7.5 gallons on the posttest.

Table C10a

Rows are levels of **C10a**
Columns are levels of **PC10a**
No Selector

	N	**cni**	**nwk**	**stln**	**zero**	**total**
N	0 0	3 5.26	0 0	1 1.75	1 1.75	5 8.77
cni	1 1.75	2 3.51	2 3.51	0 0	1 1.75	6 10.5
nwk	1 1.75	2 3.51	2 3.51	1 1.75	1 1.75	7 12.3
stln	2 3.51	4 7.02	3 5.26	14 24.6	0 0	23 40.4
zero	0 0	0 0	3 5.26	3 5.26	10 17.5	16 28.1
total	4 7.02	11 19.3	10 17.5	19 33.3	13 22.8	57 100

table contents:
Count
Percent of Table Total

N *No answer.*

cni *Could not interpret.*

nwk *Stated that there was not enough information to answer the question.*

stln *Drew a straight line through the points and calculated the volume of water at 0 seconds to be 7.5 gallons.*

zero *Zero gallons.* They assumed the tank was empty when water began flowing into it.

Categories of Inferred Thought Processes

Responses for Problem 10a fell into five categories based on students' thought processes. On both pretest and posttest, almost all the students understood that the problem was asking for the volume of water in the tank when water began flowing into it, at time t = 0 seconds. Table C10a shows that 23 students on the pretest and 19 students on the posttest calculated the volume correctly but thought about the problem in basically two different ways. One way was to draw a line through the two points on the graph, since the water flowed into the tank at a constant rate, find the equation for this line and determine the y-intercept. The second way that students thought about this problem was to consider the change in volume as compared to the change in time from the point (5, 15) to the point (9, 21). They determined that the volume increased by 6 gallons when the time increased by 4 seconds, that is, the rate of change of volume with respect to time was 1.5 gallons per second. They then worked "backwards" and calculated the volume at $t = 0$ seconds. Several of the students, who considered the rate of change of volume with respect to time, however, were unable to realize that (6 gallons)/(4 seconds) = 1.5 gallons/second. Hence, they were only able to determine that at 1 second there was 9 gallons of water in the tank. They arrived at the point (1, 9) by working "backwards" 4 seconds and 6 gallons from the point (5,15). Perhaps they could not determine the amount of water in the tank at 0 seconds since they only thought in terms of the fixed increment "for every 4 second change there is a 6 gallon change in the volume of water in the tank." These students had used a correct approach but could not determine the answer. It is interesting to note that almost all the students who used a correct thought process in solving this problem worked backwards from the point (5,15) after determining that the rate of flow was 1.5 gallons/second rather than calculating the y-intercept from the equation of the line through the two points. Also, a few students, after finding the rate of flow, 1.5 gallons/second, did not work "backwards" from the point (5,15) but rather "forwards" from the point with first entry 0 seconds. Perhaps they asked themselves how much water would be in the tank at time $t = 0$ seconds if after 5 seconds, during which time 7.5 gallons = (1.5 gallons/sec)(5 sec) of water was added to the tank, there would be 15 gallons of water in the tank.

Sixteen students on the pretest and 13 students on the posttest assumed that the tank was empty when water began flowing into the tank. Hence, they answered that the tank had 0 gallons in it when water began flowing into the tank. However, a few students, although they had drawn a curved line that passed through both points and the origin, indicating that the tank was empty at the start, calculated the correct answer using one of the methods discussed above and were included in the straight line category. The investigator assumed they realized that the line they had drawn was incorrect, drawing the line through the origin because of their past experiences.

Five students on the pretest and four students on the posttest gave no answer. Seven students on the pretest and seven students on the posttest stated that there was not enough information to answer the question. Six students on the pretest and 11 students on the posttest gave answers other than 7.5 gallons or 0 gallons with no calculations or explanation, hence, no inferred thought process could be determined for these students.

Summary of Interviews

During the six interviews, two students assumed the tank was empty when water began flowing into it and answered 0 gallons to Problem 10a. Another student answered 0 gallons to this problem because he interpreted the phrase "water flowed into a tank at a constant rate with respect to time" meant that, although the volume of water increased as time increased, at 0 seconds the volume of water in the tank also had to be 0 gallons. That is, he thought "water flowed into a tank at a constant rate with respect to time" could be expressed as the proportional relationship $g = kt$ where g represents the gallons of water in the tank, k is a constant, and t was time.

During the interview with another student, she first drew a line through the points stating that "it" was constant. She then noticed that at 0 seconds the y-intercept was not 0 gallons, indicating that there was some water in the tank when water began flowing into it. To calculate the original amount of water in the tank at 0 seconds, she first calculated the rate of flow. She stated that in 4 seconds the volume increased 6 gallons so the rate of flow was 1.5 gallons/second. She then multiplied 1.5 gallons/second by 5 seconds to determine that 7.5 gallons of water had been added to the tank during the first 5 seconds. She noticed that there was 15 gallons in the tank at 5 seconds and stated that "you must have had 7.5 gallons to begin with."

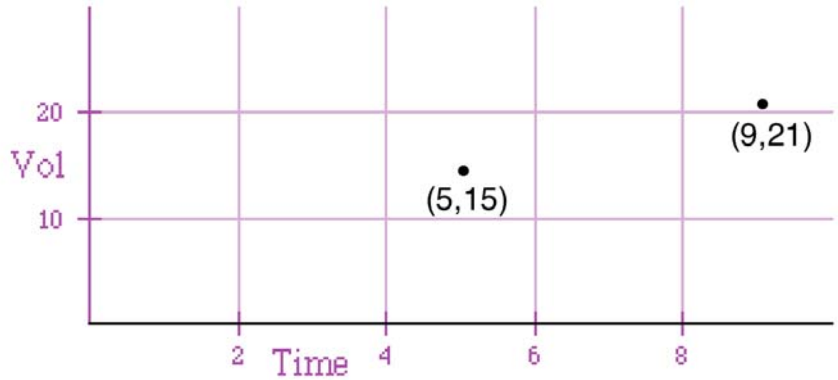
Another student derived the equation of the line (using the point-slope form) through the two given points from which he substituted 0 seconds for the time and calculated the y-intercept of the line. He understood that this y-intercept represented the amount of water in the tank at 0 seconds.

Another student had written "9x + k = 21 and 5x + k = 15" in solving this problem on his pretest. He explained during the interview that k was the initial amount of water in the tank and x was the rate at which water was flowing into the tank (the rate of change of volume of water with respect to time). He stated that the rate at which water flowed into the tank, x, was 3/2 gallons per second and then he solved for k, the initial amount of water in the tank. On his posttest he had written that the rate was (9-5)/(21-15) = 4/6 = 2/3 gal/sec but realized his mistake during the interview.

Another student during the interview went "backwards" 4 seconds and 6 gallons from the point (5,15) and stated that at 1 second there would be 9 gallons in the tank. However, he could not determine how many gallons were in the tank at 0 seconds, although he understood that this was the solution to the problem. He never realized that since the rate of flow was constant, if he were to go "backwards" only 1 second (1/4 of 4 seconds) from the point (1, 9) he would go "backwards" 1.5 gallons (1/6 of 6 gallons) to the point (0, 7.5). Hence, there would be 7.5 gallons of water in the tank when water began flowing into it.

Problem 10b

Water flowed into a tank at a constant rate with respect to time. The water's volume was measured (in gallons) after flowing for 5 seconds and again after flowing for 9 seconds. This information is given in a graph, below.



(b) At what rate did water flow into the tank?

Intent of Problem 10b

The intent of Problem 10b was to determine if students could solve for the rate of flow of water into the tank since they may have used a highly calculational approach to solve Problem 10a. For example, a student could calculate the slope, m, from the two given points as 3/2. Then this slope and the values (x, y) from one of the given points could be substituted into the equation $y = mx + b$ to solve for b, the y-intercept, at $t = 0$ seconds. In Problem 10b the student must understand what these calculated values represent within the context of this real life situation. Alternatively, students could use a non-calculational approach by recognizing that water flowed in at a constant rate of 6 gallons every 4 seconds (by focusing on the 4 second interval $t = 5$ to $t = 9$ seconds).

Criteria for Assigning Scores

A “1” was a assigned to the answer 1.5 gallons per second or equivalent to 1.5 gallons per second, the rate of flow of water into the tank. A “0” was assigned to all other answers.

Table 10b

Rows are levels of **10b**
Columns are levels of **P10b**
No Selector

	0	**1**	**A**	**N**	**total**
0	11 12.2	5 5.56	11 12.2	4 4.44	31 34.4
1	6 6.67	19 21.1	15 16.7	4 4.44	44 48.9
N	3 3.33	2 2.22	7 7.78	3 3.33	15 16.7
total	20 22.2	26 28.9	33 36.7	11 12.2	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who gave an answer not equal to 1.5 gallons per second (0), who gave the answer 1.5 gallons per seconds or an answer equal to 1.5 gallons per second (1), who were absent from the posttest (A), or who gave no response (N). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 10b indicates that 31 students (34.4%) gave an answer other than 1.5 gallons per second or not equivalent to 1.5 gallons per seconds, the rate at which the water flowed into the tank. Forty-four students (48.9%) gave the answer 1.5 gallons per second or equivalent to 1.5 gallons per second, and 15 students (16.7%) gave no answer to this problem on the pretest.

Performance on Posttest

Of the 57 students that took the posttest, 20 students (35.1%) gave an acceptable answer, 26 students (45.6%) gave an acceptable answer, and 11 students (19.3%) gave no answer to this problem on the posttest.

Comparison of Pretest and Posttest Performance

Cells (0,0), (1,1) and (N,N) of Table 10b indicate, that of the 57 students that took both tests, 33 students (57.9%) were consistent from pretest to posttest. Of the 20 students who gave an unacceptable answer on the pretest, cell (0,1) indicates that five students gave an acceptable answer on the posttest, and cell (0,N) indicates that four students gave no answer on the posttest. Of the 29 students who gave an acceptable answer on the pretest, cell (1,0) indicates that six students gave an unacceptable answer on the posttest, and cell (1,N) indicates that four students gave no answer on the posttest. Of the eight students who gave no answer on the pretest, cell (N,0) indicates that three students gave an unacceptable answer on the posttest and cell (N,1) indicates that two students gave an acceptable answer on the posttest.

Table C10b

Rows are levels of **C10b**
Columns are levels of **PC10b**
No Selector

	N	**cg/cs**	**cnfsu**	**cni**	**nwk**	**onept**	**total**
N	0 0	3 5.26	0 0	1 1.75	0 0	1 1.75	5 8.77
cg/cs	2 3.51	20 35.1	1 1.75	1 1.75	2 3.51	5 8.77	31 54.4
cnfsu	1 1.75	1 1.75	0 0	0 0	0 0	0 0	2 3.51
cni	0 0	0 0	1 1.75	3 5.26	2 3.51	2 3.51	8 14.0
nwk	0 0	1 1.75	0 0	0 0	4 7.02	0 0	5 8.77
onept	0 0	4 7.02	0 0	0 0	0 0	2 3.51	6 10.5
total	3 5.26	29 50.9	2 3.51	5 8.77	8 14.0	10 17.5	57 100

table contents:
Count
Percent of Table Total

N *No answer.*
cg/cs *(Change in gallons, the volume)/(Change in seconds, the time).*
cnfsu *Transposed the values in the given points, confusing the units volume and time.*
cni *Could not interpret.*
nwk *Gave an answer other than 1.5 gallons per second with no calculation or explanation for their answer.*
onept *Calculated the rate from one of the given points since they assumed the tank was empty when water began flowing into it.*

Categories of Inferred Thought Processes

Responses fell into six categories based on students' thought processes. Table C10b indicates that, of the 57 students who took both tests, 31 students on the pretest and 29 students on the posttest gave an acceptable answer. Except for two students on the pretest and one student on the posttest, all these students reduced 6 gallons per 4 seconds, an acceptable rate of flow of water into the tank calculated from the two given points, to 1.5 or 3/2 gallons per second. Two of the students had written slope or m = 3/2 and were given credit since it was assumed that they understood the slope represented the rate of flow, which was the quantity asked for in the problem.

Six students on the pretest and 10 students on the posttest stated that the rate of flow was 3 gallons per second calculated from the point (5,15) or 7/3 gallons per second calculated from the point (9,21) and perhaps assumed the tank was empty when water began flowing into the tank. One of these students stated that the rate of flow was 3 gallons per second for the first 5 seconds then 7/3 gallons per second after the first 5 seconds. Two other students averaged 3 gallons per second and 7/3 gallons per second, perhaps trying to calculate a constant rate.

Five students on the pretest and three students on the posttest gave no answer. Eight students on the pretest and five students on the posttest gave an answer with an explanation which the investigator could not interpret. Five students on the pretest and eight students on the posttest gave an answer with no calculation or explanation for their answer, hence, it was difficult to determine their thought processes. Two students on both tests transposed the values in the given points in their calculations, confusing time and volume, and stated that the rate of flow of water was 2/3 gallons per second.

Summary of Interviews

During the six interviews, one student, who had used the point-slope form on both tests to find the y-intercept, had also calculated the slope of the line through the two given points. He stated that “the slope, 3/2, was the rate of flow, 3 gallons every 2 seconds, since the top number is in gallons and the bottom number is in seconds.” He had used the units, gallons and seconds, in this problem to help him understand and explain how the slope of the line through the given points represented the rate of flow of water into the tank.

Another student, rather than trying to understand what the given values represented within this real life situation, tried a highly calculational approach by trying to recall some equation which she could use to help her solve this problem. She began by talking about the slope of the line through the given points but then stated "I'm not sure if that would give you the rate." Apparently, this student and several other students can calculate the slope of the line through the given points as well as the derivative of certain functions by following certain algorithms, but they do not have a clear understanding of what these calculations represent.

Another student, during her interview and on both tests, wrote the only equation she knew which involved rate; "distance = rate x time" which led her to "rate = distance/time". She stated that she then substituted "volume" for "distance" in her equation since volume and time were the given quantities in this problem. She calculated the difference between the values for the volume and the values for the time, again since they were given in the problem, and substituted these values into her equation in order to solve for the rate. This highly calculational approach led to a correct answer, however, it appeared that she had little understanding of the real life situation as presented in this problem. She did not discuss the situation as presented in this problem but simply substituted values into the only equation she knew which involved rate. The remaining three students, who were interviewed, had discussed the answer to this problem during their discussion of Problem 10a.

Problem 10c

Water flowed into a tank at a constant rate with respect to time. The water's volume was measured (in gallons) after flowing for 5 seconds and again after flowing for 9 seconds. This information is given in a graph, below.

(c) Write a formula which gives the amount of water in the tank after water has flowed for t seconds.

Intent of Problem 10c

The intent of Problem 10c was to determine if students could combine information from Problem 10a for the initial amount of water in the tank with information from Problem 10b for the rate at which water flowed into the tank in order to represent the volume of water in the tank at any given time. They would need to recognize that the formula would be a linear equation. Again, students would be required to relate their calculations to this specific context.

Criteria for Assigning Scores

A “1” was assigned any answer equivalent to $v(t) = 7.5 + 1.5t$ where t represents time, in seconds, and $v(t)$ is the volume of water in the tank as a function of time, in gallons. A “0” was assigned to all other answers.

Table 10c

Rows are levels of **10c**
Columns are levels of **P10c**
No Selector

	0	**1**	**A**	**N**	**total**
0	22 24.4	1 1.11	15 16.7	9 10	47 52.2
1	3 3.33	5 5.56	6 6.67	2 2.22	16 17.8
N	6 6.67	1 1.11	12 13.3	8 8.89	27 30
total	31 34.4	7 7.78	33 36.7	19 21.1	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who gave an answer not similar to v(t) = 7.5 + 1.5t as described above (0), who gave an answer similar to v(t) = 7.5 + 1.5t as described above (1), who were absent from the posttest (A), or who gave no response (N). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 10c indicates that 47 students (52.2%) gave an answer not equivalent to v(t) = 7.5 + 1.5t where t represents time, in seconds, and v(t) represents the volume of water in the tank as a function of time, in gallons. Sixteen students (17.8%) gave an answer equivalent to v(t) = 7.5 + 1.5t and 27 students (30%) gave no answer to this problem on the pretest.

Performance on Posttest

Of the 57 students who took both tests, Table 10c indicates that 31 students (54.4%) gave an answer not equivalent to $v(t) = 7.5 + 1.5t$. Seven students (12.3%) gave an answer equivalent to $v(t) = 7.5 + 1.5t$, and 19 students (33.3%) gave no answer to this problem on the posttest.

Comparison of Pretest and Posttest Performance

Cells (0,0), (1,1) and (N,N) of Table 10c indicate, that of the 57 students that took both tests, 35 students (61.4%) were consistent from pretest to posttest. Of the 32 students who gave an answer not equivalent to $v(t) = 7.5 + 1.5t$ on the pretest, cell (0,1) indicates that one student gave an answer equivalent to $v(t) = 7.5 + 1.5t$ on the posttest, and cell (0,N) indicates that nine students gave no answer on the posttest. Of the 10 students who gave an answer equivalent to $v(t) = 7.5 + 1.5t$ on the pretest, cell (1,0) indicates that three students gave an incorrect answer on the posttest, and cell (1,N) indicates that two students gave no answer on the posttest. Of the 15 students who gave no answer on the pretest, cell (N,0) indicates that six students gave an incorrect answer on the posttest and cell (N,1) indicates that one student gave an answer equivalent to $v(t) = 7.5 + 1.5t$ on the posttest.

Table C10c

Rows are levels of C10c
Columns are levels of PC10c
No Selector

	N	cni	jstb	nopts	nwk	sumab	total
N	4 7.02	2 3.51	3 5.26	0 0	0 0	1 1.75	10 17.5
cni	0 0	5 8.77	0 0	1 1.75	2 3.51	1 1.75	9 15.8
jstb	4 7.02	1 1.75	3 5.26	1 1.75	4 7.02	1 1.75	14 24.6
nopts	0 0	0 0	1 1.75	0 0	1 1.75	0 0	2 3.51
nwk	1 1.75	3 5.26	2 3.51	0 0	4 7.02	0 0	10 17.5
sumab	1 1.75	0 0	2 3.51	2 3.51	2 3.51	5 8.77	12 21.1
total	10 17.5	11 19.3	11 19.3	4 7.02	13 22.8	8 14.0	57 100

table contents:
Count
Percent of Table Total

N *No answer.*
cni *Could not interpret.*
jstb *Wrote v = 1.5t from information obtained in Problem 10b and ignored their non-zero initial amount of water in the tank computed in Problem 10a.*
nopts *Wrote an equation which related the information from Problems 10a and 10b, however, the given points did not satisfy the equation.*
nwk *An incorrect answer with no calculations or explanation for this answer.*
sumab *Wrote an equation which related the information from Problems 10a and 10b and the given points satisfied the equation.*

Categories of Inferred Thought Processes

Responses fell into six categories based on students' thought processes. Table C10b indicates that, of the 57 students who took both tests, 10 students on both tests gave no response. Fourteen students on the pretest and 11 students on the posttest focused on the result calculated in Problem 10b and ignored the initial non-zero amount of water in the tank in Problem 10a. These students wrote v = (3/2)t , v = 1.5t or v = 3t (if 3 was their answer in Problem 10b) and either were not aware of the fact that both given points did not satisfy their equation or if they were, couldn't write an equation that would satisfy both given points. One of these students wrote "volume = mt" and ignored the rate of flow, 3/2 gallons per second, which he calculated in Problem 10b. Perhaps he did not understand that the rate of flow found in Problem 10b, 3/2 gallons per second, was the same as the slope, m, of the line through the points on the graph. Recall the opposite scenario, during an interview with a student regarding her answer to Problem 10b, when she stated "I'm not sure if that (the slope m) would give you the rate". The responses from these students seem to indicate that there is little connection between the constant rate of flow of water given in the real life context of the problem and the slope of the line between the two given points.

Twelve students on the pretest and eight students on the posttest gave a correct equation which combined the information in Problems 10a and 10b and the given points satisfied this equation. They appeared to understand that 7.5 gallons, the initial amount of water in the tank, had to be added to 1.5t, the constant rate at which water flowed into the tank multiplied by the time t. However, although two students on the pretest and four students on the posttest understood that the equation for Problem 10c involved combining information from their responses to Problems 10a and 10b, due to the fact that one or both of their answers to Problems 10a and 10b were incorrect, the given points did not satisfy their equation. These students perhaps were also either not aware of the fact that both given points did not satisfy the equation or if they were, couldn't write an equation that would satisfy both given points.

Ten students on the pretest and 13 students on the posttest wrote equations but gave no explanation for their answer. For example, students wrote "t = rv", "f(t) = t/t^2", "t + st = volume", "w = v/t", "3x/2y", "v(t) = ve^(kt)", or "v= d/t". The investigator assumed that in the last example, "v = d/t", (usually an abbreviation for velocity =

distance/time), the student intended the "v" to represent the volume of water in the tank, rather than velocity. Three of these students wrote "rate = (change in volume)/(change in time)" which they may have learned from solving previous problems but this was the answer to Problem 10b, not the answer to Problem 10c. One other student used a highly calculational approach, the point slope form, to find the equation of the line through the two given points, however, his answer to Problem 10b, the rate of flow of water into the tank, was 5 seconds. Perhaps his understanding of rate was very limited, as seen by his answer to Problem 10b, although he was able to solve this problem. Only on the posttest do we see answers involving the derivative. One student wrote "water in tank = dv/dt" and another student wrote "dx/dt(t^2) + dt/dx(v^3) = 1".

Summary of Interviews

During the six interviews, one student stated that she was "... not sure what formula to use". She proceeded to answer the question using the point-slope form of the line through the two given points and not in terms of the rate which she had calculated in Problem 10b or the initial amount of water in the tank calculated in Problem 10a.

Another student, almost immediately upon reading Problem 10c, gave as her answer "7.5 gallons + 1.5t gal/sec = amount of water after t". She explained that the sum of initial amount of water and the rate of flow of water multiplied by the time t gave the volume of water in the tank at time t. She seemed to have a clear understanding of the real life situation presented in Problems 10a, 10b, and 10c and was able to then answer these problems correctly. She may have had a clear understanding of these problems because they involved a constant rate and the rate was with respect to time, with which most students are familiar.

Two other students had already answered this problem during their discussion of Problem 10a. One of these students stated that he had learned to solve these types of problems prior to his enrollment in first semester calculus. The remaining two students interviewed could not write the equation that gave the amount of water in the tank at time t either from the information they obtained in solving Problems 10a and 10b nor from their attempt to use the point-slope form of the line through two given points. The investigator informed one of these two students that the answer was in the form y = mx +b, where x represented the time t and y represented the volume of water in the tank at time t. The student then, using the point slope form of the line and the two given points, calculated the slope m and the y-intercept b. After stating that the equation was “y = 3/2 x + 7.5” he was not able to relate the 3/2 in his equation to the 3/2 calculated in Problem 10b (one was slope, the other was the rate of flow of water into the tank). He was also not able to relate the 7.5 in his equation to the 7.5 calculated in Problem 10a (one was a y-intercept, the other was the initial amount of water in the tank). He could not relate his calculations in Problem 10c to the real life context of the problem, although the question specifically refers to the real life context of the problem. The other student, who also could not solve this problem briefly discussed what the derivative meant to him. He stated that “the derivative is the tangent to a curve at a point which is the instantaneous velocity at that point”. He did not state that the derivative was the “slope” of the tangent to a curve at a point nor that the derivative was the instantaneous “rate of change” at a point (instead saying the derivative was an instantaneous velocity).

Problem 11

A “pitching machine” is a machine that shoots baseballs to simulate a baseball pitcher. Patrick Henry’s pitching machine shoots baseballs at a speed of 90 ft/sec, and it is set to shoot balls every 2 seconds. One day the machine was loaded with “gravity resistant” baseballs - they don’t fall to the ground, they just go in a straight line at 90 ft/sec - firing one every 2 seconds. The machine shot gravity resistant balls at Jose, the center fielder, while he ran straight at the machine at a speed of 10 ft/sec. He caught each ball fired at him, and then dropped it as he continued running. Jose's coach thought that the time intervals between catches should get smaller as Jose got closer to the machine, since successive balls had smaller distances to travel. Jose's father thought that Jose should catch one ball every 2 seconds, since that is how rapidly the machine fired them. What is your opinion? Explain.

Intent of Problem 11

The intent of Problem 11 was to determine if students could solve a problem which involved two constant, opposing rates. One correct method of reasoning would be to recognize that if Jose was not moving, he would catch a baseball every two seconds regardless of his distance from the pitching machine. While moving towards the pitching machine at a constant velocity, the time interval between catches is less than 2 seconds (as long as he continues to move towards the pitching machine). His running towards the machine at 10 ft/sec is equivalent to his standing still with the baseball traveling towards him at 100 ft/sec (instead of 90 ft/sec). Thus, the net effect is that the time interval between catches is constant, but less than 2 seconds. To correctly solve this problem, the student needed to think about the effect that the two constant (but opposing) velocities had on each other.

Criteria for Assigning Scores Performance on Pretest

A "1" was assigned to any answer which suggested that the student was able to determine that the interval between catches was constant but less than 2 seconds. The student did not need to calculate the constant time interval between catches, which was (2 sec)(90 ft/sec)/(90 ft/sec + 10 ft/sec) = 1.8 seconds, in order to receive a "1" for his answer. A "0" was assigned to all other answers.

Table 11

Rows are levels of **11**
Columns are levels of **P11**
No Selector

	0	**1**	**A**	**N**	**total**
0	41 45.6	2 2.22	22 24.4	2 2.22	67 74.4
1	2 2.22	0 0	1 1.11	0 0	3 3.33
N	7 7.78	0 0	10 11.1	3 3.33	20 22.2
total	50 55.6	2 2.22	33 36.7	5 5.56	90 100

table contents:
Count
Percent of Table Total

Frequencies of students who gave an answer other than the time interval between catches was constant but less than 2 seconds (0), who gave an answer similar to the time interval between catches was constant but less than 2 seconds (1), who were absent from the posttest (A), or who gave no response (N). Rows are levels of problem performance on the pretest; columns are levels of performance on the posttest; cell entries show numbers of students and percents of all students who received respective scores on pretest and on posttest.

Performance on Pretest

Table 11 shows that 67 students (74.4%) gave an answer which suggested that they were not able to determine that the time interval between catches was constant and less than 2 seconds. Three students (3.33%) gave an answer which suggested that they were able to determine that the time interval between catches was constant and less than 2 seconds each and 20 students (22.2%) gave no answer to this problem on the pretest.

Performance on Posttest

Of the 57 students who took the posttest, 50 students (87.7%) gave an answer which suggested that they were not able to determine the time interval between catches was constant and less than 2 seconds. Two students (3.5%) gave an answer which suggested that they were able to determine that the time interval between catches was constant and less than 2 seconds and five students (8.8%) gave no answer to this problem on the posttest.

Comparison of Pretest and Posttest Performance

Cells (0,0), (1,1) and (N,N) of Table 11 indicate that, of the 57 students who took both tests, 44 students were consistent from pretest to posttest. Of the 45 students who gave an answer indicating that the time interval between catches was not constant or not less than 2 seconds on the pretest, cell (0,1) indicates that two students gave an acceptable answer on the posttest, and cell (0,N) indicates that two students gave no answer on the posttest. Of the two students who gave an answer indicating that the time interval was constant and less than 2 seconds on the pretest, cell (1,0) indicates that both gave an unacceptable answer on the posttest. Of the 10 students who gave no response to this problem on the pretest, cell (N,0) indicates that seven students gave an unacceptable answer on the posttest and cell (N,1) indicates that none gave an acceptable answer to this problem on the posttest.

Table C11

Rows are levels of C11
Columns are levels of PC11
No Selector

	N	cni	cnsti	coach	fathr	shtch	total
N	0	2	0	3	0	0	5
	0	3.51	0	5.26	0	0	8.77
cni	0	7	0	3	1	0	11
	0	12.3	0	5.26	1.75	0	19.3
cnsti	0	0	1	0	0	0	1
	0	0	1.75	0	0	0	1.75
coach	0	1	1	12	5	1	20
	0	1.75	1.75	21.1	8.77	1.75	35.1
fathr	1	2	1	3	7	0	14
	1.75	3.51	1.75	5.26	12.3	0	24.6
shtch	1	0	0	3	0	2	6
	1.75	0	0	5.26	0	3.51	10.5
total	2	12	3	24	13	3	57
	3.51	21.1	5.26	42.1	22.8	5.26	100

table contents:
Count
Percent of Table Total

N *No answer.*
cni *Could not interpret.*
cnsti *Constant time interval between catches less than 2 seconds each.*
coach *Coach was correct.* They agreed with the coach, the time intervals between catches become smaller since successive balls have smaller distances to travel.
fathr *Father was correct.* They agreed with the father, Jose should catch a ball every 2 seconds since that is how rapidly the machine fired them.
shtch *An answer which suggested that they only thought about the time intervals between a catch and when the pitching machine fired the ball, not the time interval between catches.*

Categories of Inferred Thought Processes

Responses for the 57 students who took both tests fell into six categories for Problem 11. Table C11 shows that 20 students on the pretest and 24 students on the posttest agreed with Jose’s coach, that is, the intervals between catches get smaller as Jose gets closer to the machine since successive balls have shorter distances to travel. Some of these students apparently concluded that the time intervals between catches would get smaller as Jose got closer to the machine since the time intervals between catches and the firing of the ball from the pitching machine got smaller as Jose had shorter distances to travel.

Other students thought in terms of a limit. They argued that the time interval between catches, as Jose approached the pitching machine, decreased but could never be less than 2 seconds since "If he is standing right in front of the machine he catches them every 2 sec". These students were correct in saying that if Jose were standing still, whether in front of the machine or not, he would catch the ball every 2 seconds. However, they perhaps thought the decrease in distance between Jose and the machine affected the time interval between catches or the time intervals between catches and when the machine fired the balls. As this change in distance approached 0 and finally reached zero when Jose was standing directly in front of the machine, these students assumed that the time interval between catches reached its limit when Jose was standing directly in front of the machine.

Another student thought in terms of this limit when he stated that "Eventually, he would get so close that it would take a fraction of a second for the ball to reach Jose's glove. He would eventually catch a ball every 1.9999... seconds but never reaching 2".

Fourteen students on the pretest and 13 students on the posttest agreed with Jose's father, that is, Jose should catch one ball every 2 seconds since that is how rapidly the machine fired them. One of these students, on the posttest, added that "since Jose's speed is constant he would catch each ball every 2 seconds, but if he were accelerating, he would catch each ball in less than 2 seconds." It is interesting how this last statement relates to the correct statement "if Jose were standing still he would catch each ball every 2 seconds, but if his speed is greater than 0 he would catch each ball in less than 2 seconds." Another student, on the posttest, added that "the interval is constant at 2 sec since the balls slow down but the distance decreases.” Evidently, to this student the speed of the ball is directly related to the decrease in distance between Jose and the machine rather than the time interval between catches.

Six students on the pretest and three students on the posttest gave an answer which suggested that they only thought about the time interval between a catch and when the pitching machine fired a ball rather than the time interval between catches. One student on the pretest and three students on the posttest gave a correct response which suggested that they thought the time interval between catches was constant and less than 2 seconds. Eleven students on the pretest and 12 students on the posttest gave a response which did not relate to the question. For example, several students were concerned about how far Jose was from the pitching machine when Jose began running toward the machine or stated that “As Jose gets closer to the machine, the balls will seem to come faster.” One student stated that “both Jose’s coach and father were correct if the distance was 90 feet.” Five students on the pretest and two students on the posttest gave no response to this problem. It was interesting to note than none of the students on the tests or in the interviews tried to calculate the constant time interval between catches.

Summary of Interviews

During the six interviews, one student began by stating that “the closer Jose got to the machine the faster the ball would be coming to him.” This comment implied that she was not thinking in terms of summing the two constant rates, which would yield a constant rate of 100 ft/sec for the ball, but rather that the ball was actually accelerating toward Jose as he approached the machine. She added that “Right in front of the machine Jose would catch the ball every 2 seconds but the further away he got from the machine, it would take longer”. Perhaps she assumed that the time interval between catches was directly affected by the distance Jose was from the machine. This suggests she was thinking about the amount of time a ball traveled rather than the amount of time between catches. She concluded that Jose's coach was correct, the time interval between catches would be getting smaller as Jose got closer to the machine.

Another student stated that the time interval would get smaller as Jose approached the machine. When it was pointed out to him that there were two time intervals involved in this problem, one involving the time interval between catches and the other involving the time interval between a catch and when the machine fired a ball, he understood these. However, he still felt very strongly that the time interval between catches was decreasing since the distance to the machine was decreasing as Jose approached the machine.

Another student stated that if Jose were standing still, the time interval between catches would be 2 seconds and as he ran at a constant velocity toward the ball "he would catch them at equal intervals less than 2 seconds, but I don't know how I could figure this out using calculus". This comment indicated that she felt the calculation of the actual time interval between catches was important, although this was not necessary to answer the question.

Another student compared this problem to the Doppler effect (involving wave theory) studied in a physics class in which he was enrolled. He was able to answer this question correctly based on his understanding of this theory.

Another student stated that "the time interval would get smaller". He was asked to explain this answer since it was not clear to which time interval he was referring or if the time interval between catches was constant but smaller than 2 seconds. He explained that if Jose stood still he would catch the ball every 2 seconds and as Jose ran toward the machine at a constant speed, the time interval between catches would be the same but less than 2 seconds. This was correct, however, he stated that he couldn't believe the time interval between catches could be less than every two since the machine fired a ball every 2 seconds.

The last student interviewed gave a unique answer to this problem on both tests and during his interview. He stated that the “interval between catches was increasing as Jose approached the pitching machine”. He explained that “When Jose is far from the machine, by the time Jose catches a ball the machine may have shot several balls, so he would have to catch them quickly. As Jose gets close to the machine, after Jose catches a ball he has to wait for the machine to shoot another ball.” The investigator then asked him to explain how often Jose would catch the ball if Jose stood still and he replied “every 2 seconds”. Then he was asked how often Jose would catch the ball if Jose were running at a constant speed toward the machine and he replied “since he is meeting the ball, he would catch them before every 2 seconds and since he is running at a constant rate, the time interval between catches stays the same”. Hence, this student, who had given a uniquely incorrect interpretation of the situation in this problem, was able to understand and solve the problem after a little direction.

CHAPTER V

SUMMARY OF RESULTS

Understanding the concept of rate of change is foundational to the understanding of the derivative, the instantaneous rate of change. This study was intended to investigate first semester calculus students' understanding of rate of change and how their understandings were affected by instruction on the derivative. By examining their understandings of rate of change at the beginning of the first semester calculus course and soon after their typical study of the derivative, we can gain insight into the reasons why some calculus students have difficulty acquiring a conceptual understanding of the derivative. The investigator designed and administered a written examination in order to investigate first semester calculus' students understanding of rate of change. Ninety college students were given this written examination at the beginning of their first semester calculus course and then 57 students of the original 90 students were given this same examination soon after completing their study of the derivative. These written examinations were followed up by interviews with six students regarding their understanding of the concepts presented on the examinations.

The written examinations were analyzed from two perspectives: performance and process. Performance data was determined by scoring students' answers according to whether they were correct. Process data was determined by analyzing students' solution methods. Performance was scored 0 (not correct), 1 (correct), A (absent for posttest) or N (no response). Solution processes were categorized from students' responses according to their methods of reasoning. For example, in Problem 1 which states that "A car traveled 5 miles across town in 15 minutes. Is it reasonable for the teacher to ask how far the car went in the first 2 minutes?", a student may have reasoned that the car traveled the 5 miles across town at an average velocity of 20 miles per hour, hence, the teacher's question was not reasonable or the student may have reasoned that the car traveled the 5 miles across town at a constant velocity of 20 miles per hour, hence, the teacher's question was reasonable. In both performance and process cases, data from the pretest and posttest were analyzed by constructing contingency tables in order to examine pretest and posttest differences. A summary of these differences will now follow and possible reasons for these differences are addressed later in the chapter.

Comparison of Pretest and Posttest Performance

In order to analyze the differences between pretest and posttest performance the percent increase or decrease of correct responses from pretest to posttest will be examined. Table 12 shows the percent of correct responses for each problem on the pretest and posttest based on performance as well as the percent increase and decrease for each problem. Also included in Table 12 is the average of the percent of correct responses on the pretest and the average of the percent of correct responses on the posttest, as well as the percent increase from pretest to posttest of these averages.

Table 12
Comparison of Percentage Correct on the Pretest and Posttest based on Performance

Problem Number	% Correct on Pretest	% Correct on Posttest	Gain
5a	51%	22%	-29%
5e	45%	35%	-10%
5b	36%	28%	-8%
5c	36%	28%	-8%
10a	36%	28%	-8%
10c	24%	18%	-6%
4b	58%	53%	-5%
2	7%	4%	-3%
10b	59%	57%	-2%
5d	39%	38%	-1%
4a	28%	28%	0%
8	2%	2%	0%
11	4%	4%	0%
9	28%	29%	1%
7a	17%	19%	2%
1	30%	33%	3%
6d	12%	19%	7%
7b	25%	34%	9%
3	18%	30%	12%
6a	39%	55%	16%
6b	36%	54%	18%
6c	46%	78%	32%

Problems having little or negative improvement in performance.

The analysis of the performance data revealed that, in certain problems, there was little difference between the percent of correct answers on the pretest as compared to the posttest. On all problems except 3 and 6(a-c) there was little or no positive change in performance. Problem 5(a-e) had a consistent negative change, with 5a showing a large negative change in performance.

Problem 5 was about average rate of change of a falling object over various intervals of time, with several parts asking students to either produce or interpret a difference quotient. The consistent negative change in performance on Problem 5 is particularly surprising, as this problem more than any other resembled standard calculus content.

Problems having positive improvement in performance.

Only on Problem 3 and Problem 6 was there a noteworthy increase in performance. Problem 3 involved the ability to reason imagistically and schematically about the speed of two runners in relation to distance and time. Unfortunately, all but one of the interviewees answered Problem 3 incorrectly on the posttest, so there is no information on why 6 people in the total group changed from “distance equals time” to “time further than distance.”

Problem 6a involved the ability to determine the unit for average rate of change of volume with respect to height. Problem 6b asked for the average rate of change of volume with respect to height from given information or from the given graph. Problem 6c asked for the average rate of change of volume with respect to height from a given graph for a specific interval of the height. Problem 6d asked whether the average rate of change of volume with respect to height would be affected by the rate of change of volume with respect to time. Possible reasons for the large increase in correct performance in Problem 6 are discussed in Chapter 5.

Comparison of Pretest and Posttest Process

The methods of solution for each problem, based on students’ thought processes rather than correct answers, from pretest to posttest were compared to determine how their understanding of rate of change was effected by instruction on the derivative. In order to analyze the differences between pretest and posttest process the percent increase or decrease of correct methods from pretest to posttest will be examined. Table 13 shows the percent of correct methods for each problem on the pretest and posttest based on process as well as the percent increase and decrease for each problem. Also included in Table 13 is the average of the percent of correct methods on the pretest and the average of the percent of correct methods on the posttest, as well as the percent increase from pretest to posttest of these averages.

Table 13

Comparison of Percentage Valid Processes on the Pretest and Posttest

Problem Number	% Valid on Pretest	% Valid on Posttest	Gain
5a	45.6%	19.3%	-26.3%
10a	40.4%	33.3%	-7.1%
10c	21.1%	14.0%	-7.1%
4a	28.1%	21.1%	-7.0%
2	8.8%	3.5%	-5.3%
6b	8.8%	5.3%	-3.5%
6d	8.8%	5.3%	-3.5%
10b	54.4%	50.9%	-3.5%
5d	17.5%	15.8%	-1.7%
8	3.5%	1.8%	-1.7%
4b	52.6%	52.6%	0.0%
5e	12.3%	12.3%	0.0%
7a	15.8%	15.8%	0.0%
5b	19.3%	21.1%	1.8%
9	29.8%	31.6%	1.8%
5c	24.6%	28.1%	3.5%
11	1.8%	5.3%	3.5%
1	22.8%	28.1%	5.3%
7b	21.1%	28.1%	7.0%
3	17.5%	29.8%	12.3%
6a	10.5%	22.8%	12.3%
6c	21.1%	49.1%	28.0%

Problems on which students showed little or negative improvement in process.

The data revealed that, in most problems, there was little improvement or negative improvement between the percent of valid methods used on the pretest and on the the posttest. The greatest difference is on Problem 5a, which involved finding a falling object's average speed over the duration its fall. The decrease in valid processes used in Problem 5a seemed to be due largely to inappropriate use of the derivative—using the derivative to find the velocity at the end of 9 seconds to answer a question about an average rate of change over a period of 9 seconds. No other problem showed the dramatic decrease in valid processes; all others except Problem 3, 6a, and 6b showed only marginal decreases or increases. This common absence of increase or decrease is noteworthy, and will be discussed more fully in Chapter 5.

Problems that showed improvement in process.

Correctness and validity were essentially the same for Problems 1, 2, 3, 4a, 5b, 6a, 6b, 6c, 8, and 10a, according to the researcher's scoring criteria, so no information is gained beyond what was already concluded from the discussion of correctness.

There was a slight increase in the percent of valid solution methods from pretest to posttest for problems 7b, 3, and 6a, and large increase for problem 6c. Problems 3, 6a and 6c were discussed previously. The slight increase for Problem 7b does not warrant discussion.

Chapter VI

DISCUSSION

This study was intended to investigate first semester calculus students' understanding of rate of change and how their understandings were affected by instruction on the derivative. By examining their understandings of rate of change at the beginning of the first semester calculus course and soon after their typical study of the derivative, we can gain insight into the reasons why some calculus students have difficulty acquiring a conceptual understanding of the derivative. The investigator designed and administered a written examination in order to investigate first semester calculus' students understanding of rate of change. Ninety college students were given this written examination at the beginning of their first semester calculus course and then 57 students of the original 90 students were given this same examination soon after completing their study of the derivative. These written examinations were followed up by interviews with six students regarding their understanding of the concepts presented on the examinations.

The written examinations were analyzed from two perspectives: performance and process. Performance data was determined by scoring students' answers according to whether they were correct. Process data was determined by analyzing students' solution methods. Performance was scored 0 (not correct), 1 (correct), A (absent for posttest) or N (no response). Solution processes were categorized from students' responses according to their methods of reasoning.

In both performance and process, data from the pretest and posttest were analyzed by constructing contingency tables in order to examine pretest and posttest differences. A summary of these differences will now follow and possible reasons for these differences are addressed later in the chapter.

Conclusions

Overall Results

Ninety first semester calculus students who were tested before any instruction on the derivative had an overall average percent score based on performance of 20.7% and an overall average percent score based on process of 22.1%. After completing their study of the derivative, 57 remaining students of the 90 original were given the same test and had an average percent score based on performance of 23.7% and an average percent score based on process of 22.5%.

Overall, the scores were quite low for both the pretest and posttest and the percent increases indicate there was little difference between the pretest and posttest results. This result is rather disturbing. Many problems on the written examination covered material presented at the eighth and ninth grade level in a modern curriculum. Moreover, the concepts addressed by these problems are certainly germane to the concept of the derivative as "instantaneous rate of change." If students do not understand average rate of change, it is hard to imagine they have anything but a superficial understanding of instantaneous rate of change.

Also, one might expect a greater percent increase in correct responses from pretest to posttest than what was found, especially in light of these facts:

- The average pretest score for students who dropped the class was lower than the average pretest score for those who remained ($t = 2.62$, $p<.10$). It was the stronger students who remained.
- Those students who took the posttest had seen these exact problems on the pretest.
- Several students mentioned during their interview that they had thought about these problems after the pretest and worked some more on them after the pretest..

Several possible factors may have contributed to students' poor performance. The students may not have had enough motivation to function at their highest abilities. On the first page of the examination it stated that "performance on this set of questions will not be part of your grade." Also, the percentage of correct performance and process may have been low because many students may have left items blank in order to finish the examinations early. However, the questions which received no response were coded with an N and not considered wrong answers or wrong processes on the part of the student. The most compelling reason that students performed poorly seems to be that they did not understand rate of change deeply enough to reason appropriately.

As shown in Chapter 4, most problems showed little difference in performance from pretest to posttest. Performance on problem 5 (average speed of a falling object) went down; performance on problems 3 (Fred and Frank) and 6 (spherical storage tank) went up substantially. The investigator suspects that the change in performance on problem 3 was incidental to instruction on the derivative, and that changes in performance on problems 5 and 6 were due to more directly to what students' assimilated from instruction. In problem 5, students tended to use derivatives, but used them nonsensically. In problem 6, evidently more students read the graph as representing a functional relationship when seeing it on the posttest than when seeing it on the pretest. Perhaps their cumulative experience with functions and graphs oriented them to a more interpretive stance the second time. Or, they may simply have "learned" by taking the test. Also, problem 6c asked for an average rate of change of volume with respect to height over an interval of on which the graph looked approximately straight. This may have fit with their notion of rate-as-slope, which would explain why they improved substantially on problem 6c but did not improve on any part of problem 5.

It is noteworthy that students did not become "skilled" at any of these problems. In fact, they scored quite poorly. Even more, they also failed to change substantially their reasoning about rate-like situations. Whatever was "the content" of their course, what students assimilated was largely irrelevant to their understanding of rate of change.

Limitations

Throughout the course of this research several factors that could have widened the scope of this study were not taken into account. First, the investigator did not compare and contrast the manner of calculus instruction that occurred between the pretest and the posttest. Second, the extent of the students' concept of function was not measured during the pretest nor during the posttest. As a result, no inference should be made as to the effect that different styles of calculus instruction might have on calculus students' understanding of rate of change.

Different instruction methods could possibly be related to students' understanding of rate of change. For example, an instructor may supplement abstract theory involving rate of change with numerous real life examples. Or, an instructor may concentrate on the abstract presentation of the subject matter and give no real life examples that involve rate of change. In another instance, an instructor may show that the derivative of $f(x) = x^2$ is $f'(x) = 2x$ by instructing the students to mechanically move the exponent 2 in front of x, thereby, arriving at 2x. The student then may fall into the misunderstanding that the definition of derivative is merely a mechanical operation and he does not make the connection between rate of change and the derivative.

The concept of a functionis surely related to the understanding of rate of change. To understand rate of change, one must envision two variables co-varying systematically, which entails the idea of functional relationship. This study was not meant to determine the relationship of the students' understanding of function to their understanding of rate of change. Several students who were interviewed, even though the interviewer attempted to draw them out, did not demonstrate an understanding that "$d(t) = 16t^2$ " was a function that would completely describe the distance the object traveled at any given time. Also, students did not appear to understand the fact that for any given amount of time there is a unique distance value and for any change in time there is a corresponding change in distance (assuming the rate is not zero). By relating a change in one variable to a corresponding change in another, the concept of a function is entailed in a student's conceptual understanding of rate of change.

As a result, it may be beneficial in the future to research how the factors of classroom instruction and students' concept of function relate to students' understanding of rate of change. However, this investigation only concentrated on the influence that a course's treatment of the derivative in the first semester calculus had on students' understanding of rate of change.

The results of this study suggest that most of the students currently entering first semester calculus have a weak understanding of rate of change, and this university's calculus classes do not improve it. This is an alarming situation, as an understanding of rate is foundational to the concept of the derivative.

These first semester calculus students had completed their study of the derivative and showed no significant increase in their understanding of rate of changeThough it would seem the study of the derivative in first semester calculus would greatly improve students' understanding of rate of change, this appears not to be the case.

Implications

This study has implications for elementary and secondary math education, teacher education, and for calculus curriculum and pedagogy.

Elementary and Secondary Math Education The concept of rate of change which enables understanding of the calculus does not come from nowhere, and it cannot be built when studying the calculus itself. Repeated and early exposure to the concept of rate of change is needed if students will understand the concept of the derivative when studying calculus. Teachers at the elementary and secondary level should understand that proportional reasoning is not acquired all at once and comparing ratios is an advanced idea .

Knowledge of Piagean stages might help with the timing of the introduction of rate and proportional concepts. First, it is fundamental that children understand what is meant by the concept of change or difference in both additive and multiplicative terms. Next, children should be taught that multiplication is not just an abbreviated version of repeated addition but also arises as scaling, magnification or growth. Hart (1983) stated that “the main error children commit in solving ratio and proportion problems is the use of an incorrect addition strategy.”

Riedesel (1967, p. 287) stated that “because rate and ratio are such important mathematical concepts, and because students have difficulty learning these concepts, it is essential that [elementary and secondary] teachers place more emphasis on developing these concepts in students.”

The elementary and secondary mathematics curriculum should help children construct intuitive knowledge about fraction and ratio equivalence. For example, in a situation modeled by $a/b = c$, if a stays the same and b increases then c decreases, or if both a and b increase, then it cannot be determined whether c increases, decreases, or stays the same in value. Although this example is generally not included in the elementary mathematics curriculum, the reasoning developed in this type of question transfers to other situations including rate of change rather than just directly taught reasoning abilities. Also, rational number skills once developed, are not being used or applied to real life situations and the concept of ratio is typically not studied until fifth grade.” (Reys, Suydam & Lindquist, 1989, p.201) However, Van den Brink and Streefland (1979) give evidence that children as young as 6 and 7 years old have intuitive knowledge about ratios and proportions. Young children develop intuitions and insights into splitting that are either ignored or are inhibited by our current curriculum. Karplus et al. (1977) have noted that “It is clear that a substantial fraction of students between 13 and 15 years of age lack the ability to articulate proportional reasoning and/or control of variables.” (p. 416). Hence, many beginning secondary school students are not prepared to deal with proportional concepts. Also, their results showed that 20% of college students were still not able to use proportional reasoning effectively.

Teacher Education

Required changes in elementary and secondary mathematics curriculum and instruction will not be possible if teachers are not positioned to make these changes. Post, Harel, Behr, and Lesh (1988) found that a minority of middle school mathematics teachers understand ratio and rate as well as we wish school students to understand them. They also found that an even smaller percentage of middle-school mathematics teachers can construct coherent explanations of *their own* solutions to ratio or rate problems, even when those solutions are correct. Is it reasonable to expect more of middle-school students than their teachers themselves can deliver? It will be essential to address this problem if we are to expect entering calculus students to understand rate of change and functional covariation.

Calculus Curriculum and Pedagogy

If college calculus students resemble themselves as high school mathematics students, then they have a strong expectation that "what matters" in a mathematics class is "what you are supposed to do." They await directions as to what symbolic procedures to memorize, and they filter-out discussions of "why" procuedures work as they do, since these discussions are immaterial to memorizing the procedures themselves. Thus, if instructional experiences in class and tasks asked of them by texts are not different enough from "memorize and apply a procedure," students will attempt to understand them as expecting them to memorize and apply procedures.

Put another way, students succeeded in previous courses by memorizing procedures; if they think they can succeed in calculus by doing the same thing, then they will pay little attention to discussions or tasks which, from students' perspective, do not fit their expectations. Discussions and tasks which are highly symbolic from the beginning, and fit the mold of "symbol work leads to answer," will probably be assimilated by students as something to memorize instead of something to understand and connect with other understandings. We should not expect progress in students' conceptual understanding when everywhere they turn they find further support for a non-understanding engagement with tasks and evaluations.

The ideas of first semester calculus emerge through the study of graphs. This occurs through a transformation from the static mathematics of the function of the graph where one quantity is calculated when another is known, to the dynamic mathematics of the function, which considers how one thing changes with another. The student first learns to draw graphs of functions and then how to find rates of change. This method of understanding functions or graphs is one of the chief ways in which mathematics can deal with real life problems. Calculus students should first learn about rate of change from everyday situations then the more formal language of the calculus can be introduced.

When Newton discussed the derivative, he did not feel the need to give a formal definition of a fluxion (a quantities rate of change) but made an appeal to our familiar intuitions of motion. The famous book Principia Mathematica Philosophiae Naturalis of 1687 was concerned with velocity, acceleration, tangents and curvatures which are largely handled now by the methods of calculus but Newton presented them in the form of geometrical demonstrations with an almost complete lack of analytical calculations. For a long time English mathematicians clung to the idea of velocity in dealing with problems involving the derivative which was probably due to a desire for an intuitively satisfying conceptual background as presented by their great predecessor Newton whose method of fluxions remained essentially geometrical. Perhaps this satisfying conceptual background is absent when calculus is presented as a formal theory of functions without reference to concrete examples, diagrams or geometrical representations.

Apollonius, Oresme, Viete, Descarte, and Fermat also realized that functions could be depicted graphically, so that any problem having to do with accumulation could be represented as the determination of an area and that any problem having to do with rate of change could be represented as the determination of tangency (Boyer, 1957). That is, the initial development of the ideas of the calculus was done by mathematicians who had a strong understanding that even though they were focusing explicitly on tangents to curves or areas bounded by curves, they were in fact looking for general solutions to any problem of accumulation or change that could be expressed as a function.

Dreyfus states that "visualizing" a function through its graph pictures the underlying structure. Before a student can achieve high level thinking, he must "first operate in concrete terms which allows him to visualize what is happening, then use operational thinking which is carried out in thought". Students abstract concepts in their mind through the replacement of concrete phenomena. If students are taught to make analogies between graphical and real life situations, they will have superior problem solving skills. In advanced math classes instructors should be aware that students have difficulty working with the high order thinking of functions, limit and infinity. Peterson (19XX, p. XX) states that "meaning and understanding should be emphasized rather than rote learning to facilitate high order thinking. If students do not apply meaning or understanding to their math, math will become meaningless symbol manipulation." We must avoid letting students perform the operations of differentiation mechanically, without any understanding of what they are doing. It should be our aim to develop the concepts of differentiation so as to give students an intelligent comprehension of the processes they are using.

Form A

Rates of Change

Calculus is about the mathematics of change. This set of questions is intended for us to better understand *your* understanding of rates of change and representations of them. We hope that by gaining this information we can design school and college mathematics courses to better prepare students for the calculus.

This is not a course test! Your performance on this set of questions will not be part of your grade, and your answers will be kept completely confidential. Your answers and your name will be stored separately.

We will share total results of students' performance with calculus instructors, but individual examinations will not be shared. That is, ***your individual performance will be kept completely confidential.***

Even though this set of questions will not influence your grade, we hope that you will give your very best effort. Also, please note that each question has a place for you to write notes about anything you find confusing. It will help us tremendously if you will jot a brief note trying to explain any confusions you have.

Thank you very much for assisting us in our efforts to improve our mathematics courses.

Your name: ____

Year: Freshman Sophomore Junior Senior (circle one)

High School Math Courses (circle those that apply):

Algebra I Algebra II Geometry Trigonometry Precalculus

Calculus

A textbook stated: "A car traveled 5 miles across town in 15 minutes." Is it reasonable for the teacher to ask how far the car went in the first 2 minutes?

(a) Yes. It went ____ miles.

(b) No. Because … (explain)

Is this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)

A car went from San Diego to El Centro, a distance of 93 miles, at 40 miles per hour. At what average speed would it need to return to San Diego if it were to have an average speed of 65 miles per hour over the round trip?

Is this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)

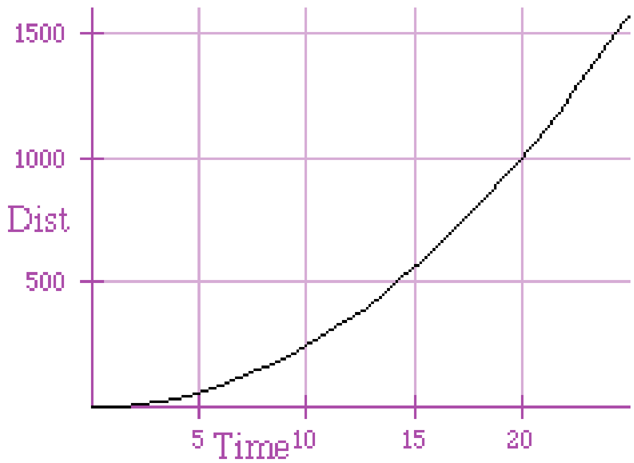
Fred and Frank are two fitness fanatics. On a run from A to B, Fred runs half the way and walks the other half. Frank runs for half the time and walks for the other half. They both run and walk at the same speed. Who finishes first?

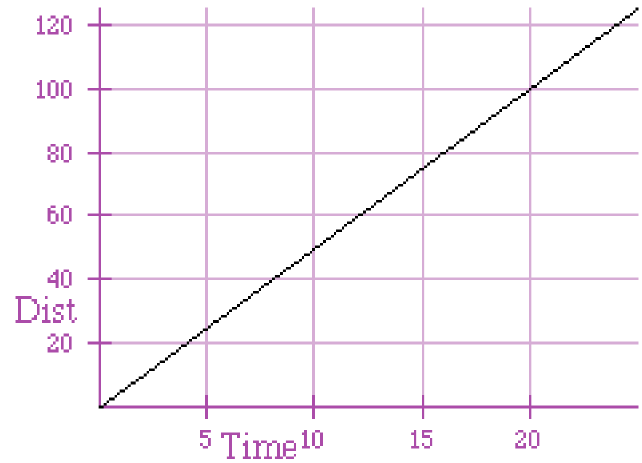
Is this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)

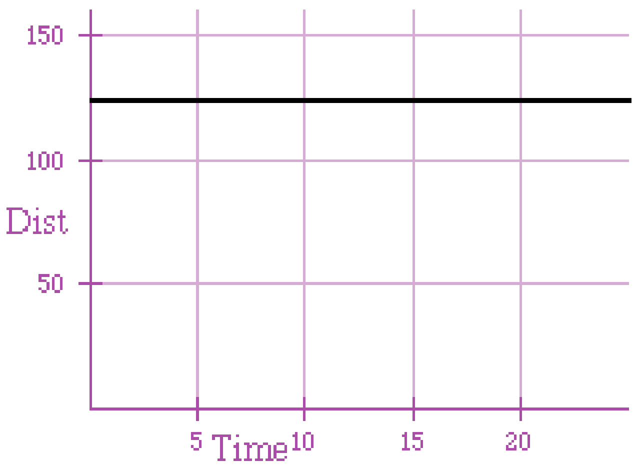
A car's speed from a standing start increases at the rate of 5 ft/sec/sec over a 25 second interval.

(a) What does 5 ft/sec/sec mean in this situation?

(b) Circle the graph that represents the car's distance over this 25 second interval.

i


ii


iii


iv *None of these represents the car's distance*

(c) Explain your selection in Part (b).

Is any part of this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)

Joe dropped a ball from the top of a building. It took 9 seconds for the ball to hit the ground. The distance the ball fell in t seconds after it was released is given by the function d(t), where $d(t) = 16t^2,\ 0 \le t \le 9$.

a. What was the ball's average speed for the time between when it was released and when it hit the ground?

b. Write an expression that represents how far the ball fell during the period between t seconds and t+2 seconds after it was released.

c. What was the ball's average speed during the period between $\frac{1}{2}$ second and $2\frac{3}{4}$ seconds after it was released?

d. Write an expression (a formula) for the ball's average speed during the period between u and $u+h$ seconds after it was released, where $h>0$ and $u+h \le 9$?

e. Suppose that t and w represent numbers of seconds and d is the function defined above. What does the expression $\frac{d(t+w) - d(t)}{w}$ represent about the falling ball?

Is any part of this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)

A spherical storage tank stood empty one morning and then was filled to capacity with water. The water's volume increased as its height increased. A supervisor, who had a dip stick but no clock, measured the water's depth repeatedly as the tank filled. The graph at the right represents the water's volume, in cubic feet, as a function of its height above the tank's bottom. The tank is 8 feet high and holds 268 cubic feet of water.

a. What would be the unit for "average rate of change of volume with respect to height"?

b. What, approximately, was the water's average rate of change of volume with respect to its height after the tank was filled?

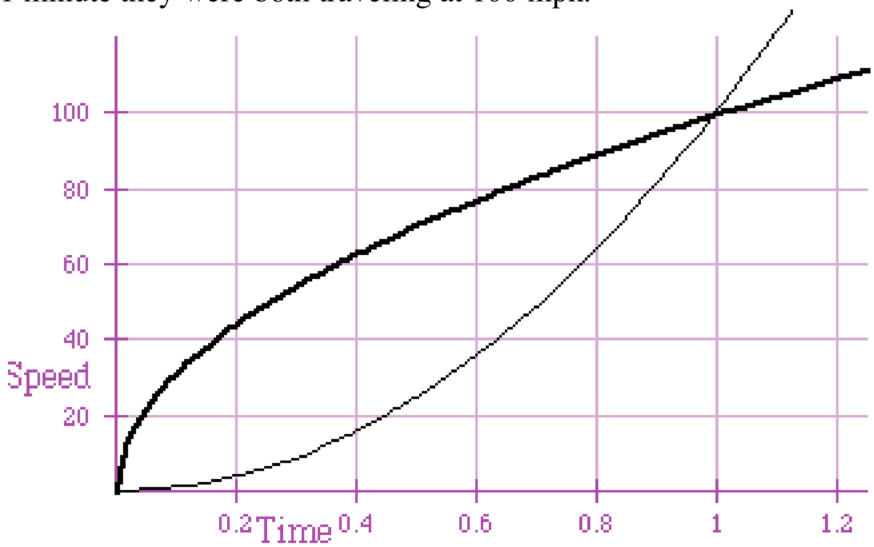
c. What, approximately, was the water's average rate of change of volume with respect to its height after the water's height varied from 3 feet to 5 feet?

d. Suppose someone claimed that the water was poured into the storage tank at a constant rate of 85 cubic feet per minute. Would that claim be consistent with the above graph? Explain.

e. What does "average rate of change of volume with respect to height" mean to you?

Is any part of this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)

Two cars, Car A and Car B, started from the same point, at the same time, and traveled in the same direction. Their speeds increased, as shown in the graph (heavy graph is for Car A, light graph is for Car B), so that after 1 minute they were both traveling at 100 mph.



(a) Was the distance between the cars increasing or decreasing 0.8 minutes after they started? Explain

(b) Describe the cars' relative positions 1 minute after they started. Explain

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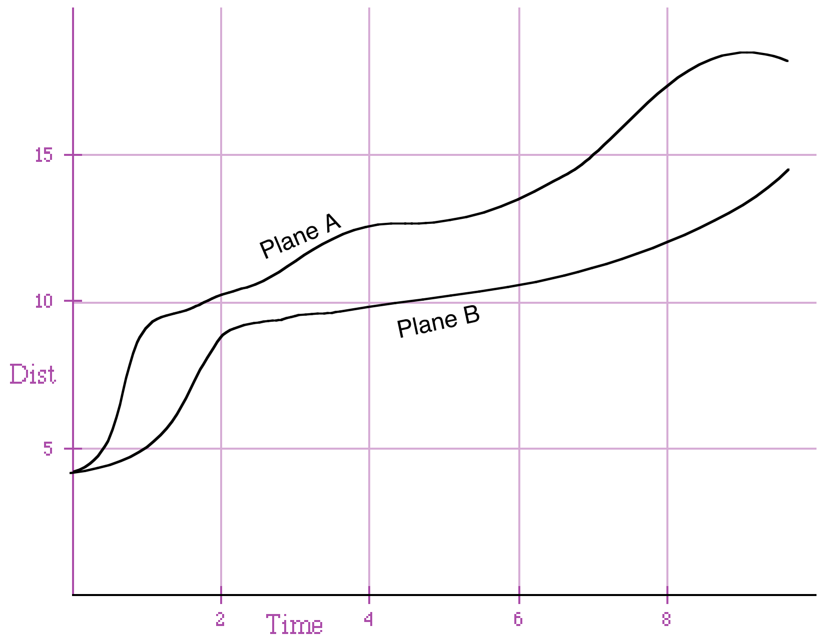
Is any part of this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)

When the Discovery space shuttle is launched, its speed increases continually until its booster engines separate from the shuttle. During the time it is continually speeding up, the shuttle is *never* moving at a constant speed. What, then, would it mean to say that at precisely 2.15823 seconds after launch the shuttle is traveling at precisely 183.8964 miles per hour?

Is this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)

Assume that two planes, A and B, are flying away from San Diego and that their distances from the San Diego airport are continually monitored. The planes' distances from San Diego are shown by the graph given below for a 10 second period of their trip.



Compare the planes' speeds 1.5 seconds after the beginning of this period of time.

Is this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)

Water flowed into a tank at a constant rate with respect to time. The water's volume was measured (in gallons) after flowing for 5 seconds and again after flowing for 9 seconds. This information is given in a graph, below.

(a) How much water was in the tank when water began flowing into it.?

(b) At what rate did water flow into the tank?

(c) Write a formula which gives the amount of water in the tank after water has flowed for *t* seconds.

Is any part of this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)

A “pitching machine” is a machine that shoots baseballs to simulate a baseball pitcher. Patrick Henry’s pitching machine shoots baseballs at a speed of 90 ft/sec, and it is set to shoot balls every 2 seconds. One day the machine was loaded with “gravity resistant” baseballs—they don’t fall to the ground, they just go in a straight line at 90 ft/sec—firing one every 2 seconds. The machine shot gravity resistant balls at José, the center fielder, while he ran straight at the machine at a speed of 10 ft/sec. He caught each ball fired at him, and then dropped it as he continued running.

José’s coach thought that the time intervals between catches should get smaller as José got closer to the machine, since successive balls had smaller distances to travel. José’s father thought that José should catch one ball every 2 seconds, since that is how rapidly the machine fired them.

What is your opinion? Explain.

Is this problem confusing to you? If so, what is your confusion? (Please use the back of this sheet if you need additional space.)