MODELING TEACHERS' WAYS OF THINKING

ABOUT RATE OF CHANGE

by

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ABSTRACT

This study sought to develop and implement a method to explore high school teachers' ways of thinking about rate of change and affiliated ideas in function-based situations. Previous research has shown that rate of change is an integral idea in understanding functions, covariational reasoning, and topics of calculus. It is necessary, then, for teachers to have a well-connected system of meanings for rate. Although existing research has focused on the mathematical concept of rate, there is little information as to how teachers think about different types of rate of change and how those types interconnect in their thinking. The aim of the study was to move beyond definitions and procedures to develop a method of modeling systems of meanings for how teachers think about rate.

A constructivist approach provided the grounding for this study. Specifically, the researcher used Glasersfeld's idea of conceptual analysis to develop a method of generating models to describe how the teachers may have been thinking about rate of change. Three secondary mathematics teachers participated in two interviews approximately six months apart to establish stability in their ways of thinking. The interviews incorporated a number of rate of change tasks that were classified in type as definition, constant rate, average rate, or changing rate. For each teacher the researcher proposed ways of thinking for each type of rate and considered how each teacher interconnected those meanings.

The results indicate that the framework and method served as useful tools to describe teachers' ways of thinking about rate. The results evidenced that

each teacher had unique systems for thinking about rate. One teacher primarily employed definitions, the second teacher thought about rate graphically in terms of steepness, and the third teacher thought in terms of comparisons. Each teacher's system of meanings contained inconsistencies, and it was often during the average rate tasks that the teachers revealed them. Additionally, the teachers evidenced disconnected ways of thinking about constant, average and changing rate.

Dedicated to my wife and best friend, Christa, for her unending support and encouragement.

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CHAPTER 1: INTRODUCTION

One can find the topic of rate of change embedded throughout the secondary mathematics curriculum in the United States. Its presentation, though, may be varied. Introductory algebra introduces rate of change as something related to a line - a slope. At this level rate of change is constant. In calculus, rate serves as the foundation of the derivative as well as the fundamental theorem (Thompson, 1994a). There, rate of change can take different values. Somehow, between these two presentations, a person must develop a well-connected comprehensive understanding of rate.

Rate is an important topic in the mathematics curriculum. Researchers claim that an individual's success in higher-level mathematics depends on an understanding of rate (Monk, 1987; Thompson, 1994a; Zandieh, 2000; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). The question, though, of how teachers think about the interconnections of constant, average and changing rate has not been directly addressed.

In this study I develop a method to answer the question of how teachers think about rate of change and affiliated ideas in function situations. To explore this question in a comprehensive way, I consider how they think about constant, average, and changing rate. I propose a framework in which I ask the teachers about their definitions of these types of rate and I observe them working mathematical tasks within each type. To model a system of meanings, I hypothesize how each teacher thinks about each type of rate and I look for connections between those ways of thinking.

My approach for this task stems from a constructivist perspective using Glasersfeld's (1995) idea of conceptual analysis. That is, the models I develop seek to explain how it is these teachers may be thinking when they are thinking about rate of change. The models have two components to them: a written analysis and a map. The written analysis is lengthy and detailed. The map serves as a simplified organizer of each teacher's ways of thinking. This research method includes a pre and post interview for each teacher that allows me to focus on aspects of thinking that remain stable over a six-month period.

In Chapter 2, I establish my theoretical perspective and provide a review of literature pertaining to functions, covariational reasoning, and rate of change. In Chapter 3, I outline the method of the study and discuss conceptual analysis. In Chapter 4, I discuss the individual tasks. Following that, in Chapters 5, 6, and 7, I present the analysis and results of the three individual teachers. In Chapter 8, I summarize the individual results of the previous chapters and I will discuss similarities and differences between the teachers. I conclude Chapter 8 with a discussion of limitations and implications for research and practice.

CHAPTER 2: LITERATURE AND THEORETICAL PERSPECTIVE

Theoretical Perspective

This is a study of modeling teachers' ways of thinking about rate of change and affiliated ideas. To describe thinking, however, I must make claims about teacher knowledge.

Knowledge, to some, involves the holding in one's mind a structure that is isomorphic to some external, real thing. Some may think that, "knowledge is a fact and not a process" (Piaget, 1971, p. 1). Piaget claims knowledge is quite the opposite – it is a process, operating from the notion of what is viable in the mind of an individual. "The theory of knowledge is therefore essentially a theory of adaptation of thoughts to reality" (Piaget, 1971, p.24). To say someone knows something, according to Piaget's model, is to say that their thinking has adapted to his or her experiences. The person has made their thinking viable in the environment as they experienced it. To Piaget, knowledge is an active process in the mind of a learner (Piaget, 1970).

Piaget's perspective of knowledge conflicts with traditional epistemologies. It compels one to take a perspective that research cannot be about whether one has *right* knowledge. That is, research on human knowing must focus on what people know, not on whether what they know is correct in some absolute sense. The best a researcher can do is build models that provide coherent explanations of what a person might know. (Glasersfeld, 1995).

Knowledge is based in action (Piaget, 1971), where an action is an activity of the mind (Thompson, 1994b). According to Piaget:

> Our knowledge stems neither from sensation nor from perception alone but from the entire action, of which perception merely constitutes the function of signalization. The characteristic of intelligence is not to contemplate but to 'transform' and its mechanism is essentially operatory. Operations consists of interiorized and coordinated actions in group structures such as reversibility, and if we wish to consider this operatory aspect of human intelligence, we must begin with the action itself and [not] with perception alone. (Piaget, 1971, p. 67)

Mental operations, then, are interiorized actions that can be coordinated and reversed, though not every action is interiorized as a mental operation (Thompson, 1994b).

Glasersfeld's calls his extension of Piaget's theory of genetic epistemology *radical constructivism* (1992). In it, Glasersfeld focuses more on the logico-mathematical, or conceptual, aspect of Piaget's theory. That is, the action is more of an interiorized action than a sensory-motor action (Steffe, 1992). Thompson clarified what could otherwise be a confusing point about the nature and usage of constructivism:

> To pronounce Constructivism as a background theory is not to announce a commitment to a particular theory of learning or to a particular type of pedagogy. Instead, it is to announce a set of commitments and constraints on the kinds of explanations one may accept and on the ways one frames problems and phenomena. (Thompson, 2002, p.193)

Following in the direction of Thompson's constructivist thinking, a goal of research is to provide a coherent model to explain the possible ways a person may be thinking.

This type of research is a form of epistemological analysis (Thompson & Saldanha, 2000). For Thompson, the research aim of epistemological analysis is not so much the articulation of genetic stages applicable to all individuals (as was Piaget's goal). Instead, this analysis considers a more individual psychological approach; it considers knowledge on a case-by-case basis. Epistemological analysis utilizes what Glasersfeld called conceptual analysis (1995). I will discuss conceptual analysis further in the methods section of this paper.

Glasersfeld set a nice standard for this type of research:

> The test of anyone's account that purports to interpret direct experience or the writings of another, must be whether or not this account brings forth in

> the reader a network of conceptualizations and reflective thought that he or she finds coherent and useful. (Glasersfeld, 1995, p. 109)

Literature Review

Functions

To be successful in the study of higher level mathematics a student needs to have a strong grasp on the notion of function (Monk, 1987; Ferrini-Mundi & Lauten, 1993; Carlson, 1997; Carlson, 1998). This grasp, however, can be tenuous even among high-achieving students (Monk, 1987; Vinner & Dreyfus, 1989; Sierpinska, 1992; Carlson, 1998; Thompson & Thompson, 1994). Although a few researchers have examined teacher understanding of function (Cooney & Wilson, 1993; Wilson, 1993; Haimes, 1996) the number of studies in the last decade has been scant.

The function literature presents different frameworks for considering functions. Monk (1987) described how someone can have a “point-wise” or an “across-time” understanding. Students, he found, have difficulty adopting an across-time understanding, a requirement for success in calculus (Monk, 1987). While this is a helpful distinction, further research has been able to extend and clarify this idea. In a summary of research, Carlson and Oehrtman (2005), building on earlier research of Dubinsky and Harel, emphasize a conceptual division between action and process views of function. With an action view, students regard functions in terms of computations and lack ability to reason about the function across its domain. The process view, in contrast, involves a more dynamic understanding of function, where students can imagine the whole of the function at once. According to Carlson and Oehrtman's (2005) distinctions between action and process view of function, students with an action view:

- See functions as tied to rules, computations, or steps.
- Suppose the answer derives from a formula.
- See only one value at a time (input or output).
- See functions as static.
- See the graph as a geometric figure.
- Think through each action.

In contrast, students with a process view of function:

- See functions as a mapping process of input-output values.
- Suppose the process is not dependent on the formula.
- Run through all values in their minds.
- See functions as dynamic.
- See the graph as a mapping of input and output values.
- Imagine the entire process.

College students have difficulty conceptualizing function as process (Carlson, 1997). While it is important for students to be able to maintain a dynamic view of function, many have difficulty doing so (Monk, 1987; Carlson, 1998).

Coulombe (1997) distinguished between covariation and correspondence views. The covariation perspective emphasizes:

- Covariation between variables
- Dynamic relationship between two variables
- Qualitative descriptions of related changes in variables such as increasing, decreasing and constant. (Coulombe, 1997, p. 83)

The correspondence perspective emphasizes:

- Correspondence between variables
- Functions as static mathematical objects
- Functions as sets of ordered pairs (Coulombe, 1997, p.83)

Coulombe found that first-year algebra students generally held fragile views about covariation (1997).

Each of the above perspectives links strongly to the notion of covariational reasoning (Thompson 1994a; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Covariational reasoning is an important mathematical ability (Thompson, 1994a; Confrey & Smith, 1995; Saldanha and Thompson, 1998) and is grounded in a dynamic, process view of function (Carlson & Oehrtman, 2005). Covariational reasoning consists of “the cognitive activities involved in coordinating two varying quantities while attending to the way in which they change in relation to each other” (Carlson et al., 2002, p. 354). It involves “someone holding in mind a sustained image of two quantities’ values (magnitudes) simultaneously” (Saldanha & Thompson, 1998). Carlson et al. (2002) classified covariational reasoning across five mental actions, from a rough coordination to average rate of change to a continuous rate of change (Carlson et al., 2002):

- Mental Action 1 (MA1): Coordination of the variables.
- Mental Action 2 (MA2): Coordination of the direction of change.
- Mental Action 3 (MA3): Coordination of amounts of changes of the variables.
- Mental Action 4 (MA4): Coordination of average rate of change across uniform increments of the domain.
- Mental Action 5 (MA5): Coordination of instantaneous rate of change.

Carlson et al. (2002) found that calculus students were able to operate consistently at the MA1, MA2, and MA3 levels but encountered difficulty with MA4, the action dealing with average rate of change. This mirrors the finding of Thompson (1994a) who found that upper level and graduate mathematics students "lacked operational schemes for average rate of change" (p. 269).

Rate of change

While there are many aspects to understanding functions, one important dimension is that of rate of change (Stump, 1997; Coulombe, 1997; NCTM, 2000). It is the understanding of rate that provides a foundation for academic success in calculus and differential equations (Monk, 1987; Thompson, 1994a; Zandieh, 2000; Carlson et al., 2002; Stroup, 2002).

It is interesting to note that students struggle with covariational reasoning with MA4, the same point at which students are required to reason with average rate of change. Research has shown that average rate of change is difficult for calculus students (Bezuidenhout, 1998) and college algebra students find interpreting rate of change in a graph to be challenging (Carlson, 1997).

Thompson (1994b) built his definition of rate on reflective abstraction. A rate is a reflectively abstracted constant ratio, where a ratio is "the result of comparing two quantities multiplicatively" (p. 15). A ratio becomes a rate in the mind of a student when he conceives of a ratio as extending beyond its original "phenomenal bounds." From this perspective, a rate is a linear function (Thomson & Thompson, 1994). Thompson and Thompson (1992) characterized the mental process in moving from ratio to rate:

> Between ratio and rate are a succession of images and operations. We have identified, in principle, four levels in the development of children's ratio/rate schemes. The first level, *ratio*, is characterized by children's comparison of two taken-as-unchanging quantities according to the criterion "as many times as". The second level, *internalized ratio*, is characterized by children's construction of co-varying accumulations of quantities, where the accrual of the quantities occurs additively, but there is no conceptual relationship between ratio of accumulated quantities at iteration x and the ratio of accumulated quantities at iteration $x+l$. The third level, *interiorized ratio*, is characterized by children's construction of co-varying amounts of quantities, where the amounts vary additively but with

> the anticipation that the ratio of the accumulations does not change. The fourth level, *rate*, is characterized by children's conception of constant ratio variation as being a single quantity—the multiplicative variation of a pair of quantities as a measure of a single attribute. (Thompson & Thompson, 1992, p.8 [electronic])

For Thompson (1994a), the reflective abstraction of operations in understanding rate involves the interiorization of the multiplicative coordination of two unchanging quantities such that the ratio of accumulations (across multiple iterations, such as time) does not change. This, then, leads to an understanding of rate as a single quantity, constant in ratio. Thompson offers:

> a general scheme for rate entails coordinated images of respective accumulations of accruals in relation to total accumulations…the fractional part of any accumulation of accruals of one quantity in relation to its total accumulation is the same as the fractional part of its covariant's accumulation of accruals in relation to its total accumulation. (Thompson, 1994a, p. 233).

This articulates a multiplicative understanding of rate that involves segmenting of corresponding parts, in contrast to the sharing model for division.

Calculus students struggled with the fundamental theorem of calculus, and had difficulties stemming “from impoverished concepts of rate of change and from poorly developed and poorly coordinated images of functional covariation and multiplicatively-constructed quantities” (Thompson, 1994a). That is, they had weak schemes and held figural images of function. (Thompson, 1994a). For these calculus students “their actions outpaced their images because their actions were not coordinated” (Thompson, 1994a, p. 270).

Confrey and Smith (1994, 1995) also grounded the notion of rate in terms of ratio, but differently from Thompson. A rate for them is a unit per unit comparison, where a unit is “the invariant relationship between a successor and its predecessor” (Confrey & Smith, 1994, p. 142). For them, one can conceive of a rate as either additive or multiplicative in linear or exponential situations. This model, though, does not apply well to non-linear, non-exponential functions.

Hauger (1995) proposed a framework for analyzing student understanding of rate of change. Her framework provided for three different ways of conceptualizing rate: global, interval, and point-wise. Global, she claimed, is qualitative in nature while interval and point-wise are both quantitative. With a global view a student considers the overall properties of the graph: increasing, decreasing, and changing rates. With an interval view students address aspects of average rate of change. A point-wise view involves the student attending to instantaneous rate of change. Students used ideas of interval knowledge (average rate of change) to address instantaneous issues (Hauger, 1995). In this framework, interval knowledge was strictly quantitative and focused too much on computation and too little on understanding. Hauger defined Interval knowledge procedurally and without a conceptual meaning of the rate in question.

Stump studied understandings of slope with both students (2001) and teachers (1997). For the teachers, Stump looked at the various concept images that teachers have of slope. She based the study on five possible representations: geometric (slope formula), algebraic (*m* in slope-intercept), physical (roads, ramps), trigonometric (angle), ratio, and in the context of a dynamic view of function, “the rate of change in one variable that results from a change in the other variable” (Stump, 1997, p.2). Stump did not see slope as a representation of rate of change, for rate of change was one sort of perspective of slope. From a pedagogical view, her findings were profound and indicated that concept images of slope for teachers were primarily geometric. In interacting with students, however, the teachers most frequently considered physical representations. None of the teachers, though, had mentioned physical representations as necessary to understanding slope (Stump, 1997). Additionally, some teachers had difficulty with slope in a functional representation, but none of those teachers mentioned that representation as a source of difficulty for students (Stump, 1997). Students, too, had trouble with slope as rate of change as “few students used the slope of the line to measure rate of change involving the appropriate variables in functional situations”

(Stump, 2001). Although she considered teacher knowledge, Stump (1997) limited her exploration to slope and she did not consider the possibilities of the interrelations between different types of rate.

Meanings of Terms

It is important that I explain the key mathematical terms as I will use them.

Rate: Reflectively abstracted constant ratio (this is further explicated in the preceding literature review). A ratio extended beyond the original phenomenological bounds. For functions, the rate, or rate of change, involves a quantitative multiplicative relationship (not merely an act of division) comparing changes in the independent and dependent variables. The rate is therefore constant and can be unitized or extended.

Constant rate of change: Corresponding changes of the variables remain in constant proportion.

Average Rate of Change: The constant rate at which a second function needs to change, over a given interval, to produce the same change in output in relation to the same change in input.

Instantaneous Rate of Change: The rate of change at an instant, inherently paradoxical as it results in a division by zero. In calculus it is viewed as the limit of the multiplicative relation (as above) as the change in the independent variable approaches zero. A function does not have a changing rate of change, but rather the function may have different instantaneous rates of change for different values in the domain. To handle the paradox, one may imagine instantaneous rate of change as an average rate of change over a tiny interval.

Changing Rate of Change: This builds directly from the instantaneous rate of change. The notion of a changing rate of change does not imply that the rate of a function changes, but that the instantaneous rate of change *has different values* for different values within the domain. That is, the rate of change over a given interval is not constant.

Linear functions: Functions whose values change at a constant rate over any interval.

What does it mean to have a "quantitative conception" of a ratio? Thompson defines a ratio to be the result of a multiplicative comparison of two quantities (1994b). For this study it is necessary to determine what will constitute evidence of a multiplicative thinking.

While the multiplicative comparison is computed *numerically* through the operation of division, in some cases the action may be no more than knowing that division should be the numerical operation. This could be due to the idea that the mental operations have been well packed, or that no consideration is given as to the division as a relationship of two meaningful quantities.

The literature generally explicates two models for division: sharing and segmenting (Thompson & Saldanha, 2003) [or some such variant: segmenting and fragmenting (Steffe, 2002); partitive and quotitive (Harel, Post & Behr, 1988). An individual may employ either of these to conceive of a division quantitatively.

An application of the equal sharing model of division to the ratios in the rate of change tasks requires a perspective of conceptualizing $\Delta y / \Delta x$ as the quantity Δy being shared equally among Δx groups. That is, it has an effect of unitizing Δy *per unit* Δx. From that action, one can interpret the unitized quantity and scale it to operate with it as a rate.

An application of segmenting in the function rate of change tasks involves the "action of putting an amount into parts of a given size" (Thompson & Saldanha, 2003, p.106), and, for unlike units, a comparison of segmentations (1994b, p. 15). Consider the context where $\Delta y = 16$ and $\Delta x = 4$, where *y* is feet and *x* is in seconds. How might one conceive of $\Delta y / \Delta x$ in terms of segmenting? Imagine how far one would travel in one-half of the given time. As the rate is a linear function, the result would be one-half of the distance. That is, for a change in 2 seconds the position would change by 8 feet. Likewise, if we consider a time interval of $1/\Delta x$ seconds our position would change by $1/\Delta x$ of Δy (a segment of Δy).

CHAPTER 3: METHOD

Introduction

In this chapter I will build on the literature and the theoretical framework to describe the method of this study. In particular, I will explain the setting, participants, types of data, and collection methods as well as the analytical methods.

To begin, it is important to note that the constructivist perspective as a grounding theory provides the most significant influence on the method for this research (Glasersfeld, 1995; Thompson, 2000). Using the ideas of conceptual analysis, I attempt to develop a method to describe teachers' ways of thinking about rate and I implement this method with three teachers.

Participants / Setting

The three teachers, all experienced in their profession, brought differing backgrounds and experiences to the Teachers Promoting Change Collaboratively (TPCC) project. The first teacher, Becky, had taught for 22 years and served as department chair. She taught courses from the lowest level of the curriculum through AP honors calculus. She held a bachelor's degree in secondary mathematic education and a master's degree in secondary education with over 60 semester hours earned beyond her master's degree. The second teacher, Mary, with 11 years of teaching experience, held a bachelor's degree in mathematics education and a master's degree in educational leadership. She taught courses below the precalculus level. The third teacher, Peggy, held a bachelor's degree in mathematics and had taught for 14 years at the below-calculus level.

The three teachers involved in the study were all full-time, experienced teachers from a large public high school of roughly 3000 students in the Southwestern United States. Each of the teachers was involved in the Teachers Promoting Change Collaboratively (TPCC) professional development project sponsored by a university-based research center. As part of their involvement in the TPCC, the teachers received stipends, laptop computers, and the opportunity to register for graduate-level mathematics credit. The professional development opportunity for these teachers required that they participate in coursework and a weekly professional learning community (PLC) to reflect on practice. During the semester of the study, the teachers participated in a course on functions taught by two university researchers. The purpose of this course was to help the teachers develop a deeper understanding of functions. It is important to note that the course never focused directly on rate of change, though it did deal with action and process views of function as well as covariation.

The PLC met for one hour each week on the high school campus at the end of the school day. While this study focuses on three teachers, there were five participants in the PLC on a regular basis: the three teachers (the subjects of the study), one other teacher from the same department (who did not participate in the functions course), and myself. The purpose of the PLC was twofold: to help the teachers to reflect on their teaching practice and to help them strengthen a deeper understanding for their students.

As the facilitator for the group, I developed PLC meeting plans that involved relevant discussion that was largely pertinent to the corresponding functions course. I rarely gave direct responses to their mathematical questions, and I usually directed their questions to generate discussion. For example, in PLC #14 one teacher turned to me during a disagreement and asked how I would teach a certain troublesome topic. To promote discussion I gave the response, “Can they both be right?”

Methods of Data Collection

Prior to this study, I conducted a pilot study in spring, 2005, with teachers participating in a similar professional development project. During that semester I developed and revised an interview protocol with the continued input from two other researchers. The instrument underwent numerous revisions based on discussions of how the teachers may have interpreted the tasks.

As I selected and finalized the tasks, I also worked to refine the follow-up questions. These questions focused on various aspects of the mathematics and the teaching of the associated topics. In developing the questions I drew heavily from the work of An, Kulm and Wu (2004) in the area of pedagogical content knowledge. Initially, I assigned each of the tasks a follow-up question based on one of their categorizations. These four broad categories from their study involved building on mathematical ideas, addressing misconceptions, engaging students in learning mathematics, and promoting thinking.

The pilot study indicated that most of these tasks did indeed provide insight into how these three teachers thought about rate of change. Some of the tasks, however, proved to be of little value in exploring rate of change. Other tasks went through minor revisions to add clarification.

Informed by the findings of the pilot study, I classified each task by type of representation (analytic, graphical, numeric, or verbal) and type of rate of change (definition, linear, average, changing). Table 1 gives these categorizations for the final selection of tasks.

Table 1

Task types

	Task	Representation	Rate Of Change Type
1	linear plan	analytic / verbal	linear
2	decreasing at a decreasing rate	graphic	changing
3	exponential change	numeric / graphic	average
4	four slopes	analytic / verbal	linear
5	container	graphic	changing
6	ladder	open	changing
7	exponential plan	analytic / verbal	changing
8	parabola secant	graphical	average
9	avg velocity / func	analytic / numeric	average
10	open graph	graphic	changing
11	quickly with table	numeric	average
12	racetrack	graphic	changing
13	space shuttle	verbal	changing
14	definitions	verbal	definition
15	average rate, secant, tangent	graphic	average
16	why divide slope	verbal	n/a

I developed Tasks 1, 2, 3, 4, 7, 8, 10, 14 and 16, while I adapted Tasks 5, 9, and 11 from the Precalculus Concept Assessment Instrument (PCA) (Carlson, Oehrtman, & Engelke, in preparation). Pat Thompson created Tasks 13 and 15, and I borrowed Tasks 6 and 12 from Marilyn Carlson (2002).

I framed most of the tasks in some particular context of instruction. The exceptions were tasks 6, 12, 13, 14, 15 and 16. Tasks 6 and 12 focused on exploring a covariational view of function. Tasks 13, 14, and 15 were about definitions and probed what the teachers may have meant when they were using terms such as rate, average rate, instantaneous rate, tangent and secant. Task 16, a task found only on the post-interview, provided an opportunity for the teachers to address division as the appropriate operation for finding slope. I discuss each of these tasks in detail in chapter 4. Appendix A contains the entire protocol for the final interview[1].

The interviews served as the primary source of data. Graduate students, trained in the protocol, conducted the interviews at the beginning of fall, 2005. I conducted the post interviews in the middle of the spring, 2006, semester. This was subsequent to the end of our PLC semester and the functions-based course. The interviews themselves had an anticipated completion time of about 90 minutes, though constraints of teacher schedules required splitting some interviews across multiple sittings. Total interview times for the post interview ranged from one hour and twenty-six minutes to two hours and nineteen minutes.

In addition to the interviews, the body of research data also consists of initial lesson plans in the form of a lesson logic. A lesson logic is a two column form of a lesson plan that requires both the actions of the teacher as well as a

[1] There was one additional task in the initial interview that I did not include in the final interview, as it did not seem a prudent use of time given the length of the overall interview.

justification for each of the actions (Silverman, 2005). For the fall semester, I requested lesson logics for both linear and exponential function plans. Unfortunately, due to timing, the teachers did not complete lesson logics at the end of this study. The post interview, however, included questions about planning for the teaching of linear functions.

Videographers recorded each of the PLC meetings. Additionally, a graduate student summarized each meeting in the form of content logs (episodic summaries) as part of a larger research agenda. Although the overall focus of the PLC meetings was not specific to rate of change, many of the meetings did include discussions about rate. I additionally targeted some PLC activities to the topic of rate. For example, meeting number ten included a discussion of task 3 from the interview protocol.

Analytical Methods

My overall research goal is to develop a process of describing a system of meanings for each of the teachers. The development of this description is grounded in the ideas of conceptual analysis (Glasersfeld, 1995; Thompson, 2000; Thompson & Saldhana 2003). Conceptual analysis is "an instrument for thinking about knowing. It is useful to describe ways of knowing that operationalize what it is students might understand when they know a particular idea in various ways" (Thompson, 2000, p. 427).

Glasersfeld (citing Ceccato) developed his notion of conceptual analysis as exploring which "mental operations must be carried out to see that presented situation in the particular way one is seeing it" (1995, p. 78). Thompson (2000) pointed out that Glasersfeld used conceptual analysis in two different ways: to generate the models that might represent how someone is thinking about a particular thing or to "devise ways of understanding an idea that, if students had them, might be propitious for building more powerful ways to deal mathematically with their environments than they would build otherwise" (Thompson, 2000, p. 428). The purpose for this study focuses on the first type of conceptual analysis. Such a focus could well possibly extend to the second type.

The connection of conceptual analysis to a constructivist framework is a natural extension, as Thompson continued:

> As Steffe (1996) has noted, conceptual analysis (the conjoining of radical constructivism as an epistemology and a theory of understanding) emphasizes the positive aspect of radical constructivism – that knowledge persists because it has proved viable in the experience to the knower. Knowledge persists because it *works.* (2000, p.428)

Based on the notion that understanding is "assimilation to a scheme," such a description "requires addressing two sides of the assimilation – what we see as the thing a person is attempting to understand and the scheme of operations that constitutes the person's actual understanding" (Thompson & Saldanha, 2003, p. 99). I aim to develop a model of thinking about rate for each teacher, where I stand as a second-level observer (Steffe, 1992). "Doing conceptual analysis entails imagining students thinking about *something* in the context of *discussing* it" (Thompson, 2002, p. 198).

There is no specific, systematic method for conducting conceptual analysis. In this study, though, I propose a multi-phased analytical method to describe models of thinking. I will outline the phases I employed in the following paragraphs:

In the first pass, prior to creating the transcripts, I watched each of the pre-interviews in their entirety to become familiar with the data. No analysis took place during this phase other than to make general notes.

One of my goals was to become as familiar as possible with the data. To that end, in the second phase, I transcribed each of the interview videos. I also noted in the transcriptions what the teachers wrote as they completed the interviews tasks.

The third pass through the interview data involved my initial construction of detailed memos and the start of the imputation of my thinking as to how the teachers may be thinking about rate of change. In these memos, I sought to draw connections between the responses of each teacher across all of the tasks. During this phase I also described the patterns which emerged from the observable actions and statements given by the teachers.

In the fourth pass through the interviews I worked from the earlier memos, the transcripts, and the original data. To begin, I grouped the tasks according to the rate of change component: definition, linear, average, or changing. It was during this phase that I critically analyzed the memos that I made during the third pass, looking for consistency in what I had written as possibilities to explain the thinking of the teachers. In addition to the transcripts, I often consulted the video for extra details such as pauses and hand motions as teachers responded to the tasks. In some cases my earlier memos were confirmed, but in other cases I found that the memos I had recorded during the third pass were no longer viable when held against other data.

In the fifth phase of the analysis I developed conclusions based on the earlier memos, transcripts, written work, and occasional review of the interview videos. In this phase, I sought "to describe conceptual operations that, were people to have them, might result in them thinking the way they evidently do" (Thompson & Saldanha, 2000, p.4). To do so, I sought evidence that both confirmed and disconfirmed my conclusions for each individual interview, and I refined the conclusions until they were consistent and coherent. Following the initial set of conclusions for Becky, I attempted to develop maps to illustrate the interconnections of the conclusions. This endeavor was worthwhile as it compelled me to reflect further on the conclusions. The process of constructing the maps was helpful, so for the cases of Mary and Peggy, I worked to develop the maps in conjunction with the conclusions.

The maps that accompany each component of the interviews provide an unfolding pictorial interpretation of the conclusions of each interview. I began by developing an overall map for each interview to illustrate apparent ideas as explained by the teacher. In order to be included in the map an item had to be consistent with the overall conclusions of that interview. That is, I compared hypothesized connections and disconnections between the items to the body of conclusions. It took several iterations before the maps and conclusions became consistent, as the maps and conclusions could serve as reflective mechanisms for each other. It was helpful to have the two interviews as it allowed me compare the two maps to refine the ideas that remained consistent and stable. Those ideas became the core of the interview maps.

Following the development of the chapters themselves, I reviewed each chapter and spent additional time reflecting on each. I grounded this exploration for consistency and coherence more holistically and used my constructed understandings to lead to further refinements.

As my basis for each of these stages of the analysis was in the interview protocol itself, in the next chapter I will give an item-by-item inventory of each of the tasks.

CHAPTER 4: THE TASKS

As I noted in Chapter 3, there are four broad components of tasks: definitions, linear, average rate of change, and changing rate of change. What follows is a brief overview of the tasks in each component.

The Definition Component: Tasks 13, 14, 15, and 16.

Collectively, the tasks associated with the defining aspects of rate of change included questions about rate, rate of change, average rate of change, instantaneous rate of change, secant and tangent. The overall aim of these tasks was to provide a point of reference in the analysis of each individual teacher. That is, I used these tasks to make sense of the teachers' utterances; the tasks helped me to delineate what the teachers might have meant when they used specific terms.

In Task 13, known as the space shuttle task, I asked teachers to clarify what they meant by an instantaneous speed. The space shuttle task also presented an opportunity for the teachers to address the paradoxical nature of instantaneous rate of change:

> Task 13. When the Discovery space shuttle is launched, its speed increases continually until its booster engines separate from the shuttle. During the time it is continually speeding up, the shuttle is never moving at a constant speed. What, then, would it mean to say that at precisely

2.15823 seconds after launch the shuttle is traveling at precisely 183.8964 miles per hour?

Task 14 directly addressed the definition of rate. In the prompts I asked each teacher to explicate any differences in how they view rate and rate of change (recall that in the theoretical perspective I claim no difference). The third and final prompt asked directly about average rate of change. The task was as follows:

Task 14. What is a 'rate', mathematically speaking?

1. What do we mean when we speak of a "rate of change" in mathematics?
2. What do we mean by an "average rate of change"?
3. Where is rate and rate of change taught in the high school curriculum?

In Task 15 I asked a similar sort of question, though with this task I directly included secants and tangents. My purpose with this task was to observe graphical actions of the teachers. Additionally, while the differences between 'average rate of change' and 'average rate of change of a function over an interval' may initially seem ambiguous to teachers, it did allow an opening for the teachers to possibly articulate ‘average rate of change’ as an idea more broadly then ‘average rate of change over an interval.’ For instance, average rate of change could refer to an idea of the linearization of overall change while average rate of change over an interval may be more specific to a given instance – such as were I to travel at some constant speed between these two times I would travel the same distance. The item specifically stated:

> Task 15. How are average rate of change, average rate of change of a function over an interval, secant to a graph and tangent to a graph related?

Recognizing that the teachers could respond to the tasks without providing evidence of multiplicative thinking, I added a final task to the post interview to create a situation for the teachers to articulate a reason for using division to calculate slope:

> Task 16. Why do we use division to calculate slope?

The Linear Component: Tasks 1 and 4.

The tasks classified as linear involved an overt application to linear function situations. I framed Tasks 1 and 4 in the linear component. In Task 1, the linear functions lesson plan task, I expected that a teacher may or may not include rate as one of the elements of their lesson. Therefore, I was prepared with follow-up prompts that asked teachers to address rate through the meaning of *m* in $y = mx + b$.

Task 1. You are planning a review of linear functions early in a 2nd year algebra course…

1. What is mathematically important for knowing (and/or understanding) of linear functions?
2. What is the mathematical meaning of 'm' in $y = mx + b$?
3. What difficulties do students have in understanding linear functions?

I also framed Task 4 in the context of linear functions. This task placed the teacher in a teaching situation where four students are discussing the meaning of slope. This task allowed the teachers to reflect on how they might think about student responses related to slope.

Task 4. Four students are discussing the meaning of slope in a linear context. One student says it is $\frac{y_2 - y_1}{x_2 - x_1}$. Another says it is the angle of the line. A third student says it is the rate of change of the line. The fourth says simply that it is the number *m*.

1. Which of the students is correct? [Explain]
2. What student understandings are at play in these responses?

3. What would you like students to understand about slope in a linear context?
4. What do students find difficult about slope in a linear context?
5. What would you say to each of the students to guide them in their understanding? Why would you say those things?

The Average Rate of Change Component: Tasks 3, 7, 8, 9, and 11.

As mentioned in chapter two, average rate of change is an essential understanding for operating in higher levels of mathematics (Zandieh, 2000; Carlson et al., 2002). The concept of average rate of change, embedded in five interview tasks, played an important role in this study,

In Task 7, I asked teachers to sketch a plan to introduce exponential functions. This task, while not directly addressing average rate of change, allowed the teachers an opportunity to apply the idea of rate. One anticipated result was the utterance of language to describe an exponential function using ratio or rate. When such language did arise, the interviewers asked the teachers to explain how they used the words. The task was as follows:

Task 7. You are planning an introductory lesson on exponential functions (such as $y = ab^x$) in a second year algebra course…

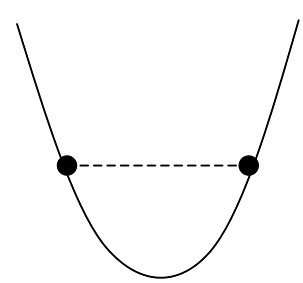
1. What is important in the knowing (and/or understanding) of exponential functions?

2. What is the meaning of 'b' in $y = ab^x$?

3. What difficulties do students have in understanding exponential functions?

In a second task on average rate of change I explored how the teachers worked with slope on a parabola. In Task 8, I asked the teachers if it were mathematically valid to apply the slope formula to a parabola. The aim of the task was to open an opportunity for the teachers to explain their meaning of slope in terms of rate of change for a segment that clearly did not align with the curvature of the given function:

Task 8. A student comes to you and says, "You know, when you apply the slope formula to opposite points on a parabola the slope is always zero."



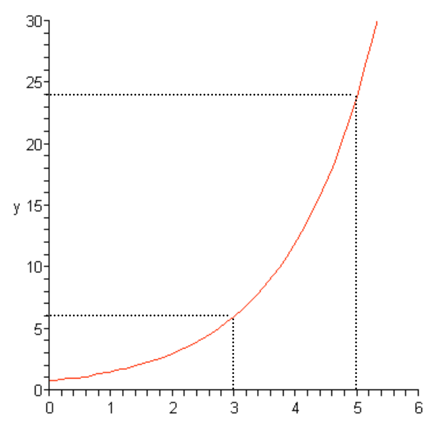
1. Is there mathematical validity in what the student observes? Explain
2. Is there any mathematical validity in applying the slope formula to a parabola?
3. What does it mean to understand the slope formula?
4. How would you build on the student's observation? Why would you do that?

5. What do students find complex about the slope formula?

In addition to the two tasks above, I included three other tasks for average rate of change. Using slightly different terminology, these tasks each asked about average rate of change in different ways. I presented Task 3 in the context of an exponential growth function dealing with 'how fast' a function was changing. For that task, the teachers were given a table of values, a graph, and a student response. I adapted Tasks 9 and 11 from the PCA (Carlson et. al. in preparation). In Task 9, I asked the teachers to work with average speed given a function relating position and time. Finally, in Task 11, I had the teachers work from a table of values that had no apparent pattern of decrease. The tasks were:

Task 3. You provide both a table and a graph and ask a student to find how fast the function is changing between x=3 and x=5. The student responds 18...

x	0	1	2	3	4	5	6
f(x)	.75	1.5	3	6	12	24	48



1. What must a student understand to answer this question?
2. What difficulties will students have with this question?
3. What might the student have been thinking?
4. What *should* the student have done?

5. What would you initially say to the student?
6. How would you guide the student? Why would you guide that way?

Task 9. Your objective for the day is to develop a lesson on average speed. Specifically, your students need to understand how to find the average speed of a car over a period of time. They should be able to answer the question, “What is a car’s average speed during the period from 2 seconds after it starts to 4 seconds after it starts, where it travels *s* feet in *t* seconds and *s* is given by $s = t^2 + t$ (with *t* measured in seconds)?

1. What mathematics is needed to understand this objective?
2. What common mistakes do students make with this objective?
3. What do students find complex about this?
4. How would *you* answer the example question?
5. Briefly sketch how you would plan this activity. Include your goals for the lesson. Why would you choose those goals?

Task 11. You are creating a lesson on how to find how quickly a function changes, given a table. For instance, given the table, students need to be able to compute how quickly the function *f* changes from $x = -2$ to $x = 3$.

x	*f(x)*
-2	103
-1	83
0	54
1	50
2	39
3	5

1. What is mathematically important for the students to understand in approaching this question?

   If no idea how to proceed:

   1a. What do you think is meant by “how quickly”?

   1b. Clarify :“we mean average rate of change, does that help?”

   If still no idea, end with:

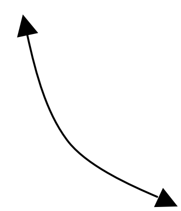
   1c. What would you say to a student who tries to apply the slope formula to this problem?

2. What should your lesson plan include? What are the goals of your lesson?
3. What real-life context could you provide for this example?
4. Describe student approaches to solving this that would be acceptable. Why would they be acceptable?
5. What difficulties will your students encounter?

The Change Component: Tasks 2, 5, 6, and 12[2].

Of the four tasks in the change component, two were related to teaching and two were problems requiring mathematical solutions. The first of the teaching related tasks, Task 2, introduced the teachers to an ambiguity where two students are debating whether a function was decreasing at a decreasing rate or decreasing at an increasing rate. My own experience in teaching has led me to find this issue a common point of debate and one that requires a methodical explanation in order to make a claim.

Task 2. Two students are having a debate about a function whose graph looks like the figure below. One student declares the function is “decreasing at a decreasing rate” while the other says it is “decreasing at an increasing rate…



1. What, do you believe, is mathematically relevant to understand language like “decreasing at a decreasing rate”?
2. Which student, do you think, is correct?

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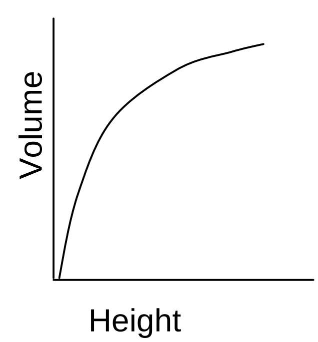
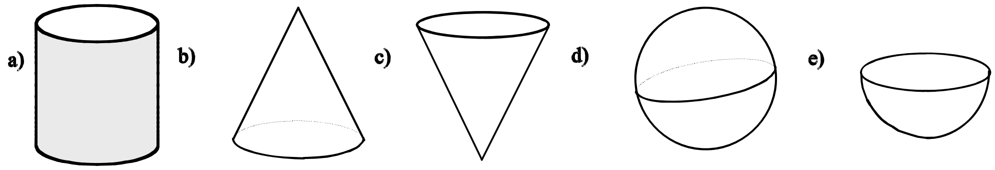
[2] I originally considered Task 10 to be a part of this component. During the study, however, I began to decide that the data was not meaningful and therefore dismissed it entirely from the final analysis. In hindsight, this was an error. I should not have predetermined importance of meaning outside of the whole of the data, as individual meaning unfolded with the analysis. What might have initially appeared meaningless could well have been meaningful to the analysis.

3. What is complex about understanding language like this?
4. What would you have the students focus on to promote the discussion? Why that?
5. What is the meaning of the second use of “decreasing” in the given situation? What is the meaning of the first usage?

In the second teaching-related changing rate task, Task 5, I asked the teachers to explain a covarying relation between two variables where neither of the variables was time. This issue of time was significant in that the teachers may intuitively operate with rate as grounded in time. In this task, teachers must attend to volume as a function of height as a bottle is filled. I adapted Task 5 from the PCA (Carlson et. al. in preparation).

Task 5. The following graph represents the volume of water as a function of the water’s height in a container. Which container goes with this graph?...





1. What must a student understand to answer this question?

2. How did you think through this question?

3. What is hard about this question?

4. Which container would you expect most students to select? Explain.

5. What would you say to those students to promote their understanding? Why?

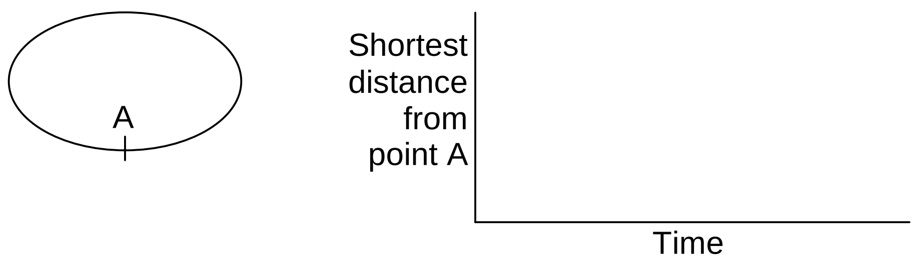
The two remaining changing rate tasks (6 and 12) come from Carlson (2002) and comprise problems requiring mathematical solutions. Both of these tasks relate to covariational reasoning. My hope in choosing these tasks was to observe the teachers working through problems that could incorporate different aspects of their thinking about rate.

In Task 6, the teachers were to imagine a ladder sliding down a wall. In this task they needed to attend to the rate of change in the vertical height with respect to time:

> Task 6. Suppose the foot of a ladder that is resting against a wall is pulled away from the wall at a constant speed. What can you say about the speed at which the other end of the ladder is dropping down the wall? (Please talk through your thinking.)

In Task 12, the teachers were given a task of constructing a graph relating distance as a function of time. This task, unlike the previous, does not directly address any particular aspect of rate and a teacher's solution may not include any mention of rate. The purpose of the follow-up prompt in this task was to observe how the teachers handled the runner's constant rate of change.

Task 12. A person is running around an oval race track at a constant speed. Construct a rough sketch showing the *shortest* distance between the runner and point *A* as a function of time. (Please talk through your thinking.)



Once the respondent has completed the question:

1. If the graph components are curved, ask "Should the graph contain curves or should it be straight? Explain"
2. If some components are linear, say "Tell me why you made some of the pieces straight."

Summary

Each of the tasks presented above served to provide insight into what teachers may mean when they refer to the mathematical idea of rate. In the next chapter I will begin the individual conceptual analysis for each of the three teachers in each of these domains.

CHAPTER 5: RESULTS AND ANALYSIS OF BECKY

Introduction to Becky

At the beginning of this study, Becky had taught for twenty-two years and this year she was teaching courses through calculus. When she first joined the TPCC she believed that she was already an effective teacher, but that she felt she had room for improvement. She entered the TPCC hoping to be able to help students better understand the mathematics she was teaching and wished to leave the project equipped with activities she could actually use in her daily classroom practice. During the post interview she pointed out that she had changed primarily through questioning the things she was teaching as to their overall importance in the curriculum; she was now attempting a more conceptually oriented form of teaching. The program, she claimed, had taught her "to think more deeply about certain things and why I teach them and how I teach them and how I can teach them better." She did not feel, however, that the program had been influential in changing her actual teaching practice. She wondered if whether the influence from the TPCC might have actually hurt her teaching in how she might try something new in a hurried fashion. She made this clear in a preliminary question for the second interview, as shown in Excerpt 1.

Excerpt 1

1. Becky: Because in fact sometimes I wonder if it has actually hurt the way, not so much the way I'm teaching, but, because I am trying to hurriedly try something in class it may not be presented as well as it could if I had more time to prepare it and think about it and practice it. So, and maybe it's just because I don't feel as comfortable with that presentation of it, the more traditional ways I've taught it or the ways I've taught it for many years that I'm feeling that way. Maybe it's not that way at all maybe it's I just feel that way

As I progress through my analysis of Becky, one theme that will emerge is that her thinking about rate tended to be closely related to her understanding of slope. Also, we shall see that the way that she understood slope was usually not graphical but rather quite formulaic in nature.

The analysis of Becky's first interview will focus on the broad components of rate as outlined in the previous chapter (definition, linear, average, and changing). Following that I will offer a similar analysis for her second interview. By looking across the interviews separated by six months it is then possible to see what is stable about the way Becky thinks about rate of change.

Becky: Interview 1

Becky: Interview 1, Definition Component

As part of the results and analysis discussion I will begin each component with the presentation of a pictorial map. The map illustrates the types of ideas and connections that might exist in the thinking of Becky for topics related to rate of change. A line in the map indicates evidence of some connection of ideas. An arrow indicates a connection that is directional while a dashed line indicates a weak connection. I developed the maps as a whole to be consistent with all of the conclusions of the analysis. The ideas illustrated in the definition component are therefore consistent with Becky's overall thinking across all tasks in Interview 1. The map for each component of this interview is a subset of the final map for the interview.

Becky viewed ratio, rate, and rate of change as separate ideas. She thought of rate of change in two primary ways: as a difference quotient or as a slope, where slope was about a relationship of changes or a graphical image. Becky strongly connected average rate of change to a graphical secant while she linked instantaneous rate of change to the slope of the tangent. She worked with the slope of the tangent as both a single value or a limit of secants. I provide a pictorial interpretation of these connections in Figure 1 while I provide my support for these claims in the following analyses.

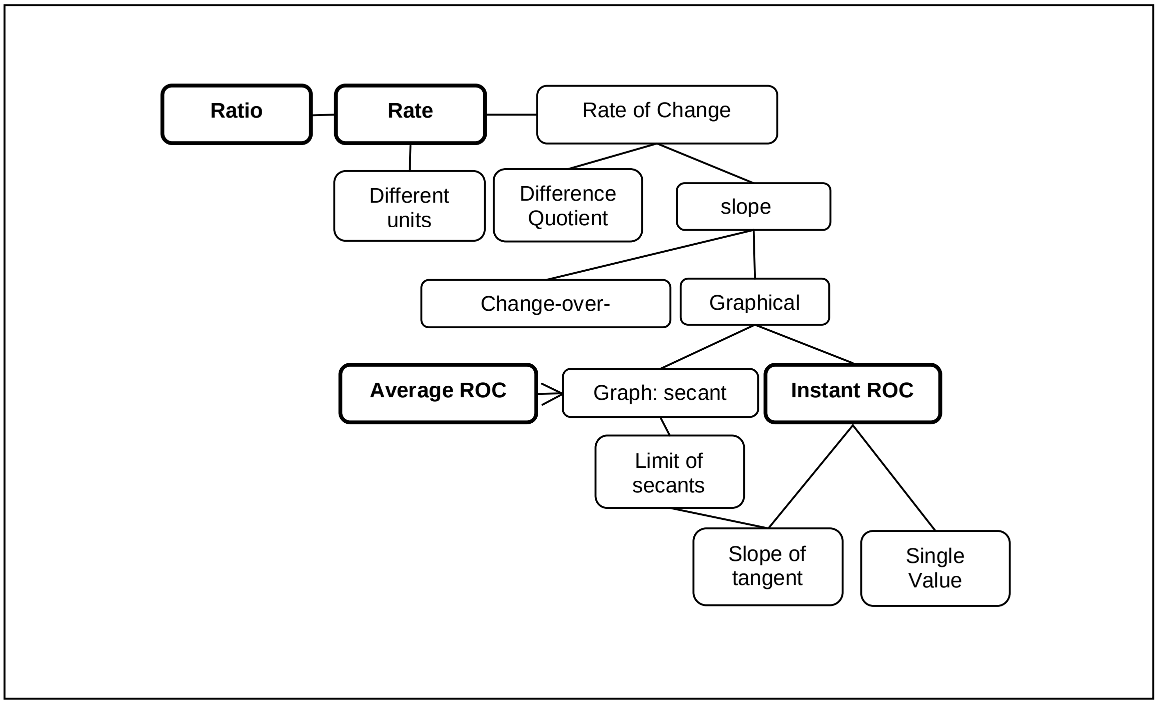


Figure 1. Becky, interview 1, definition component map

One common distinction people make between ratio and rate involves the idea that rates are fractions that relate quantities of "different natures" (Vergnaud, 1988, p. 158). Although this is not how I describe rate and ratio in this study, it was the way that Becky worked with them. Task 14 began with the interviewer asking Becky to define rate, and Becky clearly drew a distinction.[3]

Excerpt 2

1. Becky:	[reading the question] What is a rate, mathematically speaking?
2. Int:	What do we mean when we speak of a rate of change in mathematics?

[3] The inclusion of ellipsis indicates a pause, not omitted text.

3. Becky: OK a rate or rate of change? Because we can say a rate is a ratio of two measurements in different units. For example if we are saying something costs 99 cents per pound… and I guess you could still call that, think of that as a rate of change, if you want three pounds at 297 and five pounds at 495 and found a rate of change that you'd still end up with 99 cents per pound.

For this task, the interviewer inadvertently blended two ways of speaking, one asking about "rate" (as indicated in the text of the question presented to the teacher) and a second asking about "rate of change." Becky actually drew her own distinction between them and provided clarification for each as individual ideas. For her, a rate was a ratio with different units, while a rate of change involved taking two individual points and *calculating* the rate that their coordinates determine (the ratio of Δy to Δx). Becky evidenced this distinction in the next interchange, in which she defined "rate of change."

Excerpt 3

1. Int: So what do we mean when we speak of rate of change in mathematics?
2. Becky: The rate of change is the difference between two quantities measured in the same units over the corresponding something else of the same units. Or rate of change in general mathematics would be the slope of the line, how one quantity changes with respect to another.

I make the assumption that Becky was being consistent with the meanings she expressed earlier. Thus, "…over the corresponding something else of the same units" means that you have a ratio between two differences, where the quantities in each difference are in the same unit. She also indicated that rate of change is the "slope of the line," a potentially graphical descriptor. What we will see is that Becky's meaning of slope, while possibly graphical, tended to be more of a definitional application of a formula. Becky also drew a distinction between her meaning of slope and her meaning of "how one quantity changes with respect to another." These two meanings did not appear to involve a covariational relation as the variables were not necessarily changing in tandem. Rather, slope was the computation of two sets of changes resulting in a per-unit change, yielding a rate.

Becky's meaning of average rate of change was only slightly different from her meaning of rate of change, as shown in her remarks in the context of Task 14.

Excerpt 4

1. Int: And what do we mean by average rate of change?
2. Becky: The average rate of change would be talking about finding the slope between two points [sketches increasing, concave up curve with two points and secant between those two points].

Thus we see her meaning for average rate of change was similar to her meaning of rate of change, which, in Becky's meaning, appears equivalent to her idea of slope (calculating the ratio of two differences). For Becky, average rate of change is a calculation of slope in connection to a graphical picture of slope. This connection is illustrated by the relationship she described among average rate, secant and tangent in Task 15.

Task 15. How are average rate of change, average rate of change of a function over an interval, secant to a graph and tangent to a graph related?

Excerpt 5

1. Becky: … Well your average rate of change of the function over an interval is the same as the slope of the secant to a graph using those same two points… and I'm not sure how to distinguish between the average rate of change and average rate of change of the function over an interval, and the tangent to a graph [draws increasing concave up curve] so here is our average rate of change of a function over an interval [connecting the points with a secant] it's also the secant to the graph.

In Excerpt 5, Becky seemed to connect the idea of average rate of change of a function over an interval to the slope of a secant to the function's graph. What that connection meant to Becky is not clear.

When asked to explain instantaneous rate of change, Becky, as she did with average rate of change, made a connection to graphs. In the space shuttle task (Task 13) Becky immediately linked instantaneous speed to the tangent line.

Although the question did not ask about graphic ideas, she offered no meaning related to motion or rate other than the slope of the tangent.

Excerpt 6

1. Becky: If we were looking at exactly this 2.15823 what we would be talking about would be the slope of that tangent line and what that means is as we got closer and closer to this time that the, the speed would get closer to the 183.8964, it's the instantaneous rate of change would be the slope of that, that tangent line.

The "closer and closer" above seems not to refer to a limiting of secants, but rather to the idea that as the space shuttle gets closer to that particular time value, its speed will get closer to the given speed. It is worthwhile to note that Becky's way of thinking avoids the question of what it means to be traveling at a specific speed at a specific moment in time. This way of thinking about instantaneous rate of change did not involve a consideration of changes. In describing a tangent in Task 15, however, Becky indicated she did see the tangent to be a result of the limiting process of secants.

Excerpt 7

1. Becky: A tangent would be as you allow these two points to come together [marking tangent] so this is the average rate of change [secant line] and this becomes the instantaneous rate of change [pointing to tangent line], how it is changing as that interval shrinks as *a* comes in closer to *b.*

Becky's description in Excerpt 7 of a tangent as a limit of secants, in light of her earlier description of instantaneous speed in terms having nothing to do with average speed or secant, suggests that these meanings are interconnected in complicated ways in her system of meanings. In this first interview we see that Becky connected slightly different thinking to each of the ideas of rate, rate of change, average rate of change, and instantaneous rate of change. To Becky:

- a rate is simply a ratio, with different units on the top and bottom.
- a rate of change is the result, or what is calculated, when two differences are compared in a fraction form.
- an average rate of change is the slope of a secant.
- an instantaneous rate of change is the slope of a tangent, at a given point.

Becky: Interview 1, Linear Component

The linear tasks enable us to see Becky evidencing a stronger connection between slope and rate of change and between slope and the difference quotient than she did in the tasks in the definition component. We also will see Becky interpret rate of change situations as involving change over successive intervals.

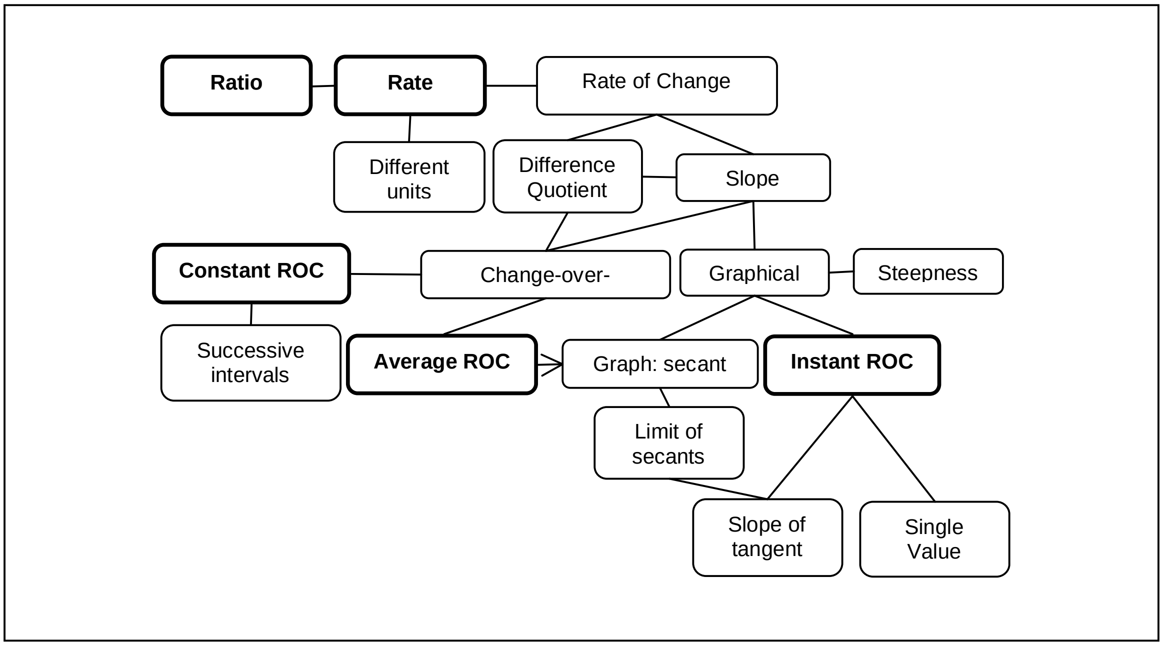


Figure 2. Becky, interview 1, linear component map

Slope, for Becky, was a ratio and existed only as a definition. When given the task of developing a lesson plan for linear functions, Becky described the meaning of *m*.

Excerpt 8

1. Int: What is the mathematical meaning of *m* in *y=mx+b*?
2. Becky: The mathematical meaning of m is called the slope and it is defined to be *y*2 minus *y*1 over *x*2 minus *x*1. I tell the students it gives us a measurement of the steepness of the line.

There is no evidence that “over” had a multiplicative meaning for Becky. She stated, “most of the time we need two points in order to find the slope.” I interpreted her statement to mean that slope is a characteristic of a line and is to be found based on a defined calculation. Excerpt 8 reveals that Becky also thought of *m* as providing a measurement of steepness; in addition, her lesson logic for linear functions reveals that she connected *m* to rate. We now discuss this connection.

In her lesson logic on linear functions Becky would begin with tables of linear patterns, then she would have students write equations that "generate" those tables. Following that, she would:

> Have students calculate the rate of change between successive pairs of coordinates. $rate\ of\ change = \frac{change\ in\ dependent\ values}{change\ in\ indep.\ values}$

Becky's intended focus here was on the idea of "successive pairs" of coordinates. Becky's meaning of constant rate of change (at least as she intended for her students) was a static consideration of successive interval computations and numerical results. By "static" I mean that Becky considered the intervals to exist as do pieces of wire and these intervals were not created through a variable's variation. It is in this sense that Becky meant that a rate of change is constant—the computations of rate for successive, static intervals yield the same numeric result.

We can see in Excerpt 9 (below) that Becky related *m* to rate of change as well as to slope by her response to Task 4. She stated that all of the students were correct but that some responses were better than others.

Task 4. Four students are discussing the meaning of slope in a linear context. One student says it is $\frac{y_2 - y_1}{x_2 - x_1}$. Another says it is the angle of the line. A third student says it is the rate of change of the line. The fourth says simply that it is the number *m*.

Excerpt 9

1. Int: OK, and what student understandings are at play in each of these responses?

2. Becky: [pointing to the fourth student response - *m*] This is probably the most basic level of understanding, calling it the number *m*, and then I would maybe say this is next step up [pointing to $\frac{y_2 - y_1}{x_2 - x_1}$] and rate of change and the angle of the line are pretty sophisticated levels of understanding.

Becky said further that the ideas of slope and rate of change are really the same.

Excerpt 10

1. Becky: Well I would hope in a linear context that they would be able to find a slope using this definition [pointing to $\frac{y_2 - y_1}{x_2 - x_1}$] and knowing that's how it can be found with two points, definitely recognizing *m* in the slope intercept form, hope that they begin to see that these are interchangeable [slope and rate of change].

For Becky, slope is what it is by definition. For her, even though slope and rate of change are “interchangeable,” they are not exactly equivalent, because she sees rate of change as the more sophisticated understanding (although it is unclear in what way it is more sophisticated). To Becky, a constant rate of change involves the idea that results of computations for successive pairs of points remains the same.

Becky: Interview 1, Average Rate of Change Component

While there will be much to say about Becky’s meaning for average rate of change, the map itself changes only by connecting “Units” to “Average rate of change.” In this section we will see that Becky connected average rate of change to a graphical image of the secant, though the connection was directional (meaning that when she thinks of average rate of change, she thinks of secant, but not vice-versa). Now, though, we will see that Becky linked average rate to change-over-change and to the difference quotient. That is, average rate of change in some tasks for Becky will not be graphical, even though she earlier defined it that way. Becky showed a particular strategy for solving these tasks involving a form of unit analysis, to be explained shortly.

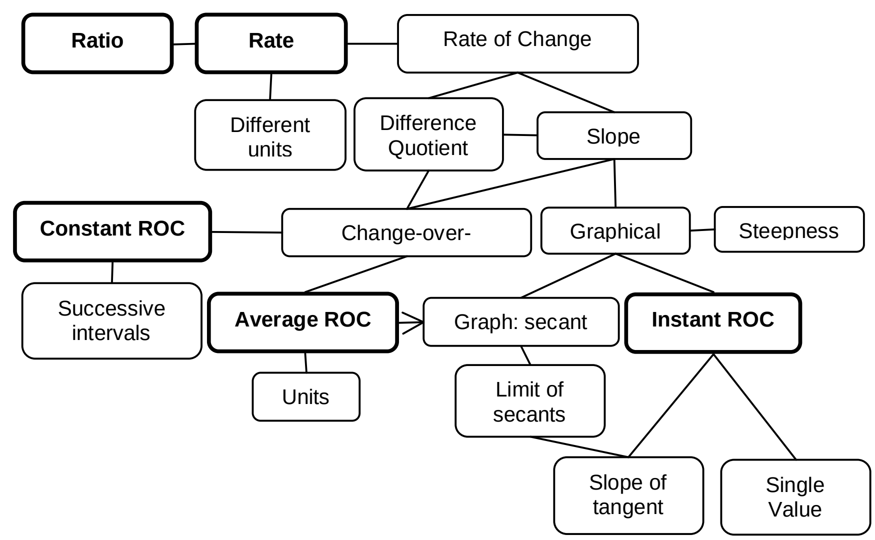


Figure 3. Becky, interview 1, average component map

While by average rate of change Becky meant the slope of a secant, she did not see the slope of a secant as meaning average rate of change, as shown in the parabola task (Task 8).

Task 8. A student comes to you and says, “You know, when you apply the slope formula to opposite points on a parabola the slope is always zero.”

Excerpt 11

1. Int: Is there mathematical validity in what the student observes?

2. Becky: There is validity in it as long as there is a quadratic function. If your axis of symmetry or line of symmetry is vertical than that would be

true. It would not be true if you have a problem that is not a function.

3. Int: Is there any mathematical validity in applying the slope formula to a parabola?

4. Becky: … The only thing I can think of is if I have the slope to come out to be zero that they found two points that are symmetric to the line of symmetry.

Task 8 involved a secant whose slope is zero, though Becky did not see it as opportunity to think about average rate of change. This directional thinking was in contrast to how she defined average rate of change as the slope between two points.

In Excerpt 12, below, she subsequently described the meaning of the slope formula in connection to steepness as measured according to a definition.

Excerpt 12

1. Int: What does it mean to understand the slope formula?

2. Becky: … In regards to quadratic ones or just in general?

3. Int: I think just in general.

4. Becky: Well it means that they understand that, how somebody a long time ago decided they would measure the steepness of a line [interruption]. I guess they, someone could have defined it differently but they have an understanding of the way that is accepted all over the world to define the slope.

While there was no evidence in Excerpt 12 that Becky connected slope to rate of change, in Task 11 she did reinforce a link between slope and rate of change. The link was not graphically-based. In the following excerpt, she talked about what her lesson plan would include.

Task 11. You are creating a lesson on how to find how quickly a function changes, given a table. For instance, given the table, students need to be able to compute how quickly the function *f* changes from *x*= -2 to *x* = 3.

x	*f(x)*
-2	103
-1	83
0	54
1	50
2	39
3	5

Excerpt 13

1. Int: So what is mathematically important for students to understand in approaching this question?

2. Becky: I guess the need to understand how this relates to slope to figure out how quickly the function, how fast something is changing that has to do with slope.

3. Int: What would your lesson plan include? What are the goals of your lesson?

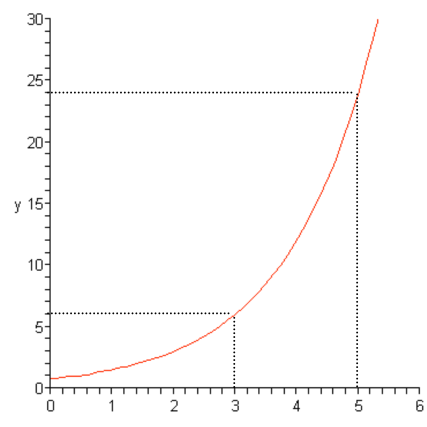
4. Becky: It would include a review of slope, some vocabulary of different ways you can say how quickly something is changing like rate of

change, what is the rate of change of the function, how fast is it changing, what is the slope, and then the goal of the lesson would just for the students to be able to understand those things and be able to apply it to the table.

Becky connected terms like "how quickly" or "how fast" (as in Task 3) to rate of change and then to slope. In Task 3, Becky viewed "how fast something is changing" as static, as shown in how she addressed a hypothetical student who merely found the change of the function rather than the rate of change.

Task 3. You provide both a table and a graph and ask a student to find how fast the function is changing between x=3 and x=5. The student responds 18...

x	0	1	2	3	4	5	6
f(x)	.75	1.5	3	6	12	24	48



Excerpt 14

1. Int: And what should the students have done?

2. Becky: The student should have taken (f(5)-f(3)) / (5-3) = (24-3)/(5-3)=18/2=9.

3. Int: What would you initially say to the student?

4. Becky: I think I would, I think I would have to reword this for them, maybe how fast the function is changing, like saying what is the rate of change of a function between x=3 and x=5.

One interesting aspect of Becky's treatment of these specific questions of "how fast" and "how quickly" is that she did not connect them to the secant. This is significant because, in the definition tasks, her responses suggested that she connected average rate of change with secant. It is therefore possible that she was not seeing these tasks as average rate tasks, but rather tasks for which she already had an established procedure. In Task 9 (below) Becky was asked specifically about average speed but she provided no application to a graphical interpretation using a secant (see Excerpt 15 and Excerpt 16, below). Becky solved that task using the difference quotient, but emphasized that knowing the units themselves indicated the placement of the values. This suggests that when Becky thought of *defining* average rate, she thought of a secant. But in the context of *finding* an average rate, Becky thought of a calculation.

Task 9. Your objective for the day is to develop a lesson on average speed. Specifically, your students need to understand how to find the average speed of a car over a period of time. They should be able to answer the question, "What is a car's average speed during the period from 2 seconds after it starts to 4 seconds after it starts, where it travels *s* feet in *t* seconds and *s* is given by $s = t^2 + t$ (with *t* measured in seconds)?

Excerpt 15

1. Becky: So what we are looking for here is the average rate of change because this is the distance the car travels in *t* seconds. The speed would be the change in the distance with respect to the change in time. So the average speed over this interval would be the average rate of change… on the interval from two to four [writing ((s(4)-s(2))/(4-2)].

In the next excerpt she added an emphasis dealing with units as providing guidance in placement of the numbers:

Excerpt 16

1. Becky: I think my goals would be to have the students to understand and know how to calculate the average rate of change. Go ahead and read it over again [interviewer rereads]… Well, I think again by

> looking at a table of values they're looking at some things that are going on in a table so they can construct a table. Look at the units of speed. So in this case would be feet per second, and what are we going to have to do to get feet per second. This is measured in feet [top of ratio] and this is measured in seconds [bottom of ratio] so that's going to give us what we want.

What seems to be the case for Becky in completing these average rate tasks is that she used a procedure of applying the difference quotient. I must note that to this point in the analysis there is no evidence that Becky directly connected her thinking on average rate of change to her thinking on constant rate of change.

Becky: Interview 1, Change Component

This component involves the tasks that deal with changing rate of change. Becky's actions for the changing rate of change tasks followed one of two patterns: slope of tangent or what I call *change-with-change*. Change-with-change, for Becky, is a way of thinking about changing rate of change that involves comparisons of loosely-held, corresponding changes.

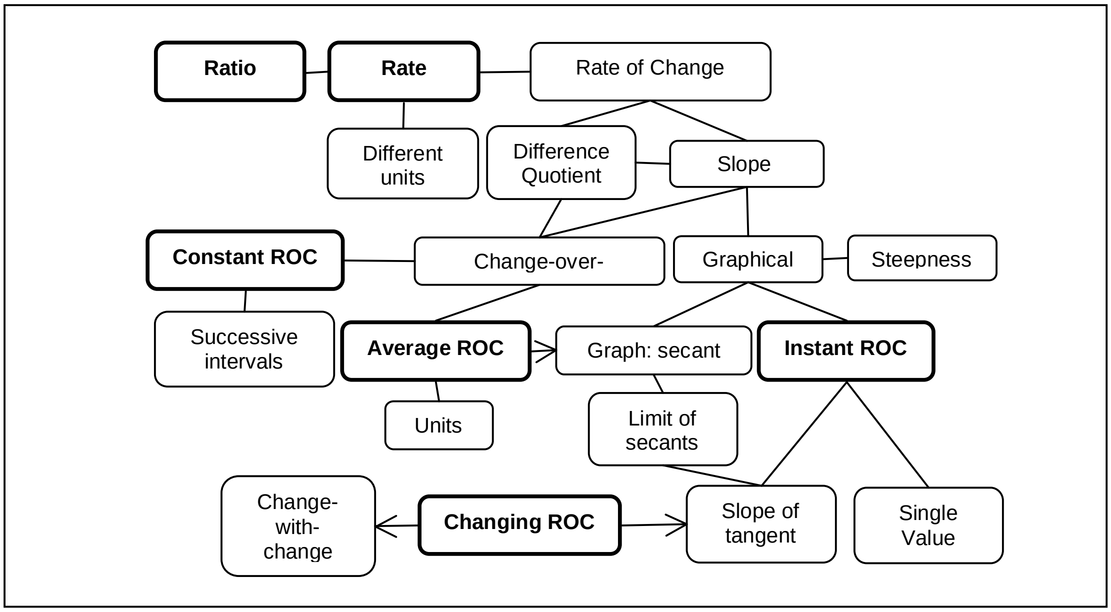
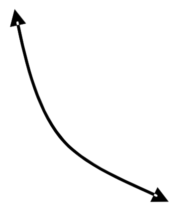


Figure 4. Becky, interview 1, final map

In Task 2 (below), which involved two students discussing a decreasing, concave up graph, Becky drew a conclusion by sketching "little tangent segments" to aid in her thinking. This action was compatible with her earlier definition of instantaneous speed where speed was a single value. In this case, though, Becky constructed a number of these segments and considered them from left to right. Becky seemed to give no attention to the spacing of the little tangents, or to their lengths, or to any other meanings of them or to the variables involved.

Task 2. Two students are having a debate about a function whose graph looks like the figure below. One student declares the function is "decreasing at a decreasing rate" while the other says it is "decreasing at an increasing rate…

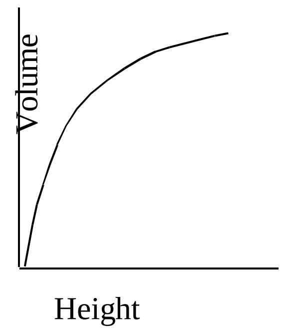


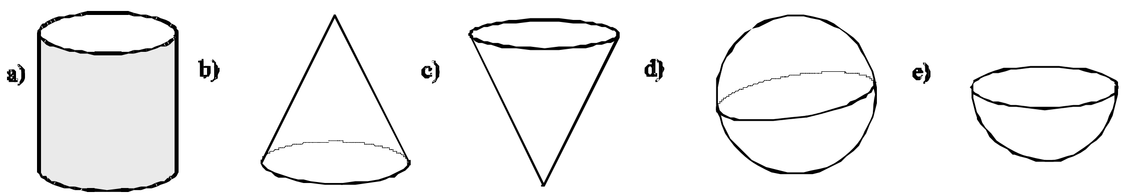
Excerpt 17

1. Becky: There is a more negative number in this one [moving pen across small tangent segments from left to right], these values are getting closer to zero and as they get closer to 0 they are increasing and I hope that they would see that, that the change in your *y* is getting greater there [makes horizontal and vertical component motions under each little tangent with pen].

In the container task (Task 5) Becky gave attention to the dynamic nature of the function as she described how the variables changed for different values across the domain.

Task 5. The following graph represents the volume of water as a function of the water's height in a container. Which container goes with this graph?





Excerpt 18

1. Becky: I think you need to understand that as, for example, if you are pouring water in here at a constant rate, how the, what's going to happen to this, as the height increases what is happening to the volume. So first of all they have to make sure they understand what two quantities we're looking at to figure out the relationship there.

In working this task, she identified container *B* as correct, though she did so without articulation of points separated on equal intervals. In Excerpt 19 (below) her analysis stemmed neither from uniform increments (as she did with the linear tasks) nor from a point and its estimated value (as she did with instantaneous rate). Instead, she used general, global language based on two different regions of the bottle. For example, for container *B* she worked from a standard of "big" and noticed that the volume was "not increasing very much." More interesting, though, was the quick analysis she made of container *C*, based on only one interval near the top of the container where "if it increases a little bit the volume increases a lot." Both of these approaches lack a comparison referent as to what "not very much" or "a lot" might mean.

Excerpt 19

1. Becky: Then this one [container *B*], as something is being poured into it the height is increasing, the volume is pretty big but as you're coming up here the height is increasing the volume is not increasing very much, so it's slowing down. It doesn't say anything about the rate it's coming in... All of them would be an increasing function so it's how it's increasing that we need to look at, so as the height increases [making two horizontal sweeps near top of container *C*], if it increases a little bit the volume increases a lot.

In Excerpt 20 Becky provided an explanation directed towards students. In this case she considered an approach using two contiguous intervals.

Excerpt 20

1. Int: And what would you say to those students to promote their understanding and why?

2. Becky: Well we could maybe have them take a certain height and try to think of the volume that is represented there [marking a disk about one-half of way up container] and then go up just a little bit higher and take some more volume and now the height is increased the volume has also increased [marking another disk a bit above the first] that as they continue up the amount of volume they're adding to that is not as much as what is being added before so it's not increasing as quickly as it was when the height was smaller numbers.

Becky thought the student should take two contiguous intervals and compare the changes in volume. There is still no indication, however, of whether those intervals should be equally spaced.

In the racetrack task (Task 12) Becky was able to convince herself that the components to the graph were not linear, but she provided no reasoning as to why the concavity behaved as it did. She tried an algebraic approach that did not work. She drew two possible sketches, but became confused and did not complete the task. The two sketches, (a concave down parabola and a concave up function that is increasing and then decreasing) are shown on the right side of Figure 5.

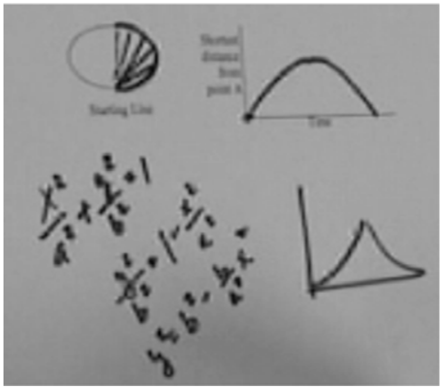


Figure 5. Becky, interview 1, task 12

As in the earlier tasks in this component, there was no indication as to how Becky spaced the points on the racetrack.

With respect to these changing rate of change tasks, Becky's meanings do not appear specifically linked to her earlier meanings of rate of change as "change-over-change" and the difference quotient. Rather, she linked her thinking strongly to a notion of a comparison of changes, a change in one thing as the other changes (what I will call *change-with-change*). In those applications she used references without referents.

Becky: Interview 2

The second interview took place six months after the first interview. This interview followed the completion of the semester for which I facilitated the professional learning community. Overall, we will see much throughout the second interview that supports the claims I made in the first interview. While there may be a few differences, we will see that the two interviews serve to compliment to each other.

Becky: Interview 2, Definition Component

In the definition component of Interview 2, Becky showed evidence of thinking similar in nature to her thinking in Interview 1. For example, for her a ratio was different from a rate since it involved different units. Also, although she claimed there was no difference between rate and rate of change, she still drew a subtle distinction. She also continued to hold two ways of thinking about rate of change: slope and the difference quotient. She continued to think of the difference quotient as change-over-change. Becky mentioned average rate of change in the definition component only in the graphical sense; and, by instantaneous rate of change, she meant the slope of the tangent. Figure 6 provides my map for Becky's definition component for Interview 2.

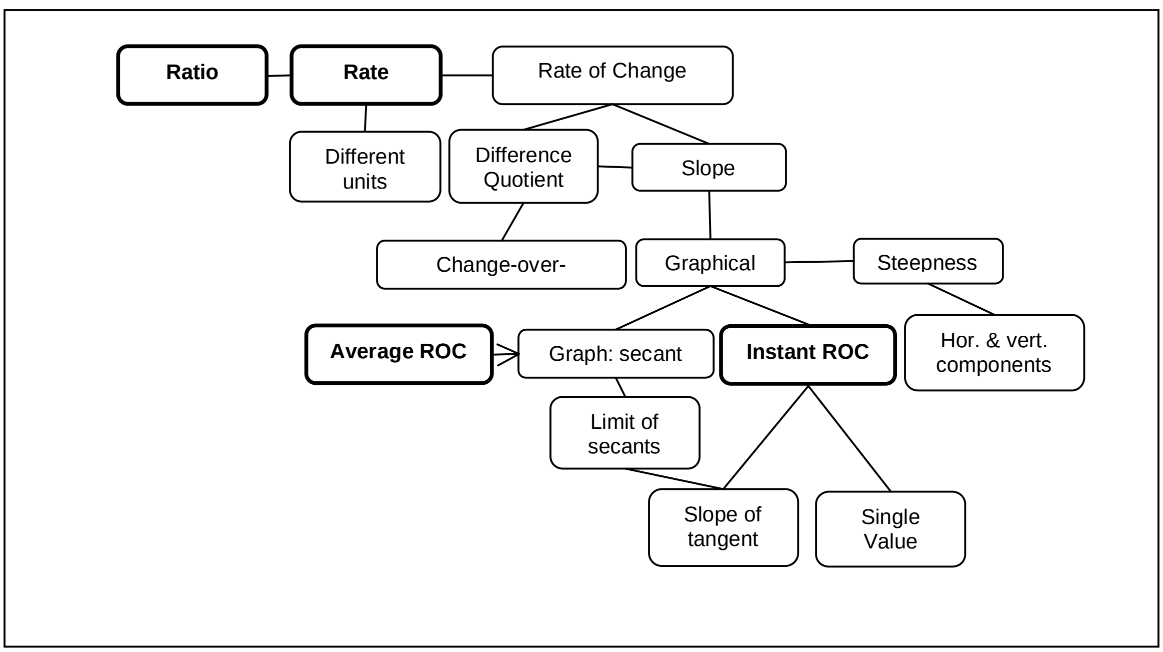


Figure 6. Becky, interview 2, definition component map

In Task 14, when I asked the meaning of rate, Becky spoke of a ratio involving two different types of units. This was the same distinction she made in Interview 1.

Excerpt 21

1. Becky: [writing “rate -> ratio”] We think of a rate as a ratio, and a lot of times when we think of rate in application problems we are thinking about dollars per pound we are talking about two different measurements, two different things measured in different units, think about miles per hour, degrees Fahrenheit per minute, all of those with the application ideas of rate.

Becky linked rate to the idea of unit rate. I noticed, however, that her response was merely a list of examples and so I pushed for further clarification.

Excerpt 22.

1. Int: So those are all examples, but what is a rate?
2. Becky: A rate is a ratio, the change in some value over the change in something else [writing $\frac{\Delta y}{\Delta x}$] the change in your dependent values over the change in your corresponding independent values.

For Becky, once again the notion of correspondence did not seem to include a notion of covariance, or values changing in tandem. I continued to probe with a question about rate of change.

Excerpt 23

1. Int: All right, so what do we mean when we speak of a rate of change in mathematics?
2. Becky: Well a rate of change again would be the change in the dependent over the change in the independent.
3. Int: Is there any distinction between rate and rate of change?
4. Becky: … I'm trying to think of something that might be an example of that, let's see, freshman students we teach them the unit rate but even that can be, could put it into a table of values and write an equation for and come up with a rate of change, the slope, representing that situation. So I would say no.

It appears in the above excerpt that Becky did not see a difference between rate and rate of change, yet she nonetheless was able to provide a subtle distinction between the two. Becky connected rate to a unit rate, which she could treat as a function to compute a rate of change - a slope. She did not speak in graphical terms as she saw slope as something to be computed.

The tasks in the definition component then moved to average rate of change.

Excerpt 24

1. Int: What do you mean by the average rate of change?
2. Becky: The average rate of change is the rate of change [laughter] the average rate of change is the slope a secant line, in other words

between two points [writing: avg rate of change = slope of a secant line (between 2 pts)]

3. Int: And how is that different than from a rate of change or what you said is a rate?

4. Becky: How is it what?

5. Int: How is that distinct from rate or rate of change?

6. Becky: I don't think it is.

She laughed about the two being the same, but then drew the same distinction as she did in Interview 1. She defined average rate of change using a graphical image of slope instead of defining it strictly by the difference quotient.

Becky also continued to view instantaneous rate of change as the slope resulting from a limiting of secants. In Excerpt 25 (Task 15), she explained that one comes from the other.

Task 15. How are average rate of change, average rate of change of a function over an interval, secant to a graph and tangent to a graph related?

Excerpt 25

1. Becky: The slope of a tangent to a graph is your instantaneous rate of change, in other words your slope as the interval around that, whatever value you are looking at, at that tangent, shrinks down to zero. It's the limit of that.

In the space shuttle task (Task 13) she provided an explanation that was more extensive than in the first interview (compare Excerpt 26 with Excerpt 6). The quotation from Task 3 (below) follows Becky's graphical explanation of the lengths of secants growing smaller around the point in question. She explains average speed as the slope of the secant line.

Excerpt 26

1. Becky: So as we started out looking at the change in the distance over the change in time [writing Δd and Δt on horizontal and vertical components between left and right points], we are letting the change in time approach 0 getting smaller and smaller until we are looking at [sketching small horizontal and vertical components from left point] these values and these [the horizontal and vertical components] until we finally bring that in to exactly 2.15823, so that is what we refer to as its instantaneous velocity.

Missing in this explanation is how those smaller component values approach exactly 2.15823. She did not confront the problem of division by zero. What is of significance here, for the remaining analysis of Becky, is that her use of horizontal and vertical components between the endpoints is a very typical action on her part. This may serve her as a means to visualize slope through steepness.

The second interview included an extra task, Task 16, to ask why division is used to calculate slope.

Excerpt 27

1. Becky: … Because you are finding out something per unit of time or per whatever that independent variable is standing for, you tell, you say how it is changing… I think that when you're dividing it is per something, this is the change per whatever,

2. Int: So that…ok, all right, because I was wondering sometimes when we compare things we use subtraction and so I was wondering why division? Why not subtraction?

3. Becky: You are doing subtraction but you are subtracting two different quantities you have two things changing at the same time if you're subtracting you are just looking at one quantity.

4. Int: So the two quantities are changing at the same time. Any other thoughts?

5. Becky: I guess I never really thought about that, I just accepted for how it is defined and go on with it, yeah if I just want to find out, if I'm just looking at, an example I was going to use is growing [inaudible] in time again, say how tall, how much someone had grown over a period of time that still throws in two different quantities in, so that would still be using division.

In Excerpt 27, Becky's initial response using the word 'per' prompted me to ask for clarification. Her justification for division was to subtract to get a comparison because there are two different quantities that are "changing at the same time."

And, as there are two quantities it is necessary to divide. Of course, she had not adequately justified the use of division. This notion of “at the same time” seems to carry no evidence of covarying or changing in tandem. Rather, there is a subtraction comparison for one and a subtraction comparison for the other and the two are then computed together “at the same time” using division. The insight for this analysis is that Becky admittedly had not considered division in the computation of slope as a quantitative operation.

Becky: Interview 2, Linear Component

In this component we will once again see Becky’s thinking about rate of change sometimes grounded in *change-over-change*. Figure 7 illustrates this connection.

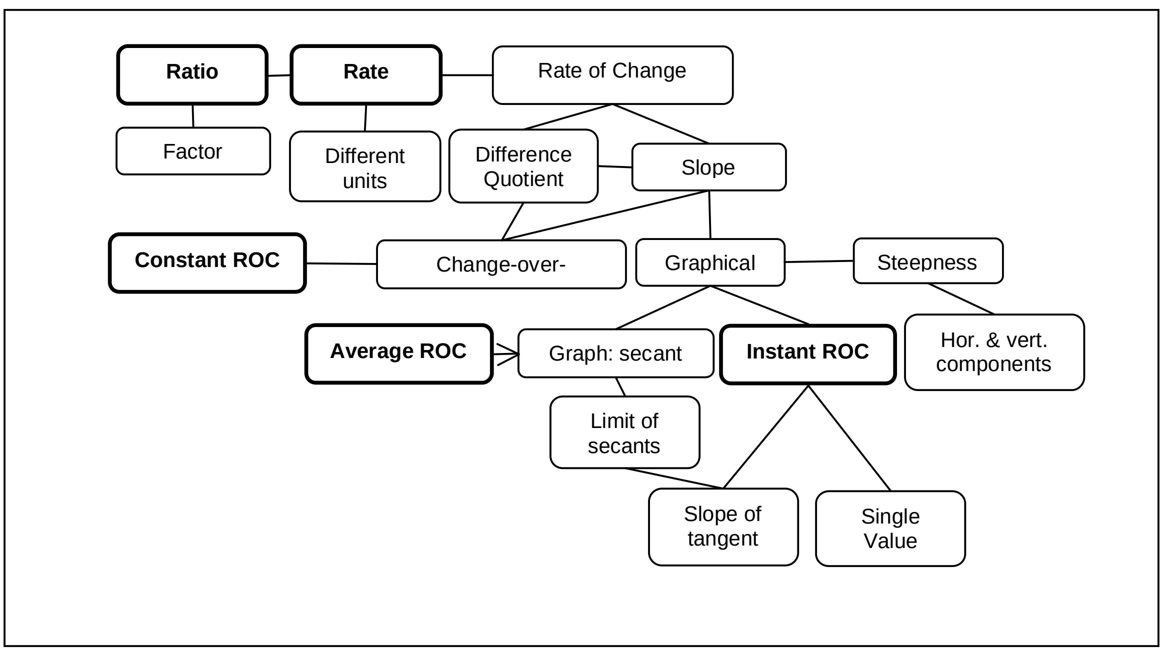


Figure 7. Becky, interview 2, linear component map

In Task 1, the linear lesson plan, Becky indicated that to understand linear functions it is important for students to know that rate of change is the slope defined as a difference quotient. This is the same change-over-change thinking she stated in the first interview.

Excerpt 28

1. Becky: Students need to understand the rate of change or the slope and they need to understand that for linear functions it is a constant rate of change [writes: rate of change – slope – constant], they would need to be able to understand different forms of a linear function. For example the slope intercept form. That is the one that they seem to be most familiar with. They need to understand the definition of slope or rate of change is the change in y over the change in *x*.

In the above excerpt we also see again her equivalence of meanings for slope and rate of change. She defined each by the same formula.

In Task 4, one of Becky's primary aims for her students in understanding the slope formula was for them to be able to carry out the procedure of calculating slope.

Task 4. Four students are discussing the meaning of slope in a linear context. One student says it is $\frac{y_2 - y_1}{x_2 - x_1}$. Another says it is the angle of the line. A third student says it is the rate of change of the line. The fourth says simply that it is the number *m*.

Excerpt 29

1. Int: So what would you like your students to understand about slope in a linear context?
2. Becky: I think all of those [laughter]. I certainly would let them know how to calculate it, maybe use this [referring to the formula] to calculate. This [pointing to the angle] gives them some more application, how can we apply it to some problems. Being able to discuss it as a rate of change is a more sophisticated idea then calling it slope, and certainly I would want them to recognize *m* in our traditional slope intercept form of the equation that we are letting *m* stand for the slope of the line.

Her above response sounds quite like her response in the first interview where there appeared to be a lack of clarity in how the slope could be the same as rate of change. In Excerpt 30 I provided her an opportunity to further explain her thinking.

Excerpt 30

1. Int: Anything else?
2. Becky: No… Probably just rate of change is a great way to be thinking of it, thinking about dollars per pound, miles per hour, gives them more applications.
3. Int: And what do students find difficult about slope in a linear context?

4. Becky: They don’t always make the connection I think between what I was just saying about dollars per pound, you know, in an application problem.

For Becky, rate of change is ”sophisticated” because it ties directly back to the idea of rate, where a rate is essentially a unit rate whose components are connected by the word “per.” For Becky, her idea of rate of change carried more meaning than slope in that a rate of change is something to be calculated and applied.

Becky: Interview 2, Average Rate of Change Component

The new elements in the map for the average rate of change component include the Becky’s emphasis on successive intervals of the domain for constant rate of change and an inclusion of a horizontal and vertical mode of thinking (see Figure 8). Additionally, Becky will evidence thinking about ratio as a factor, which is distinct from how she uses ratio as a fraction for rate. She will also emphasize the importance of units in working with average rate of change. In this section we will see more clearly how Becky thinks about (a) constant rate of change, (b) the important role of her change-over-change way of thinking, (c) the importance of time as a variable, and (d) the role of unitizing.

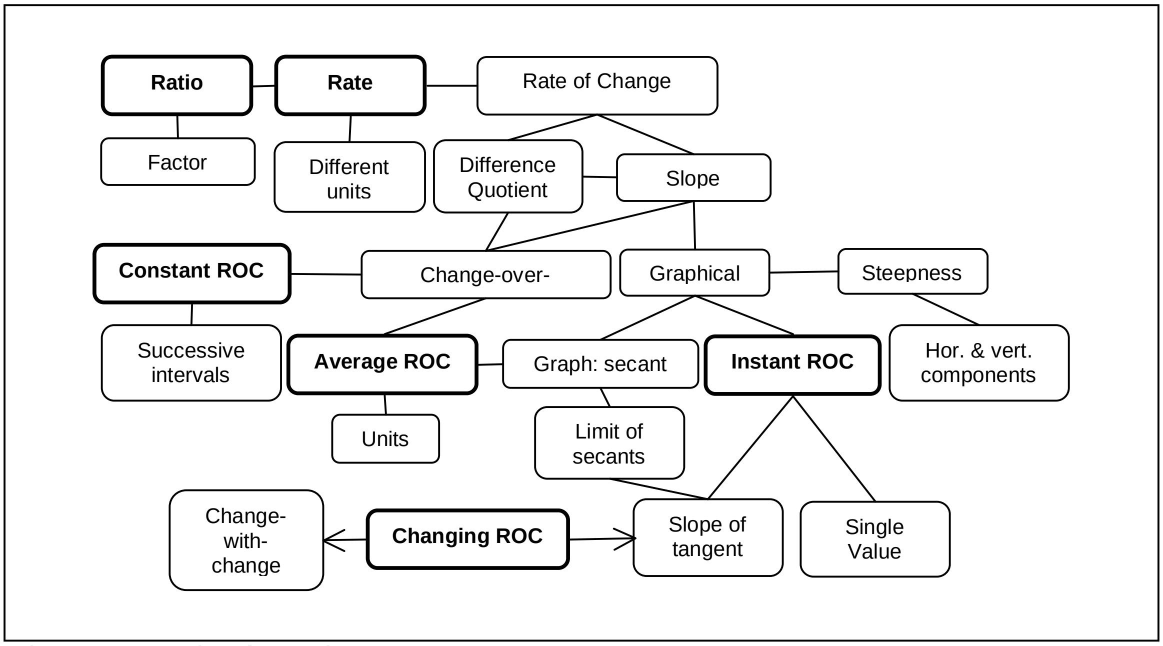
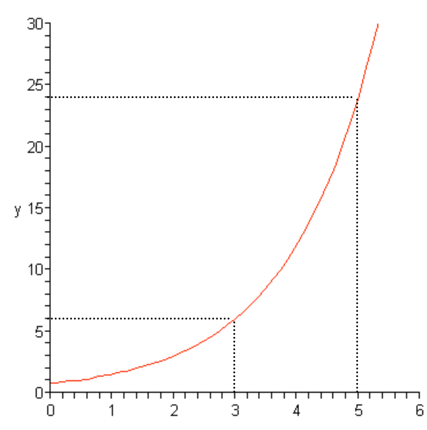


Figure 8. Becky, interview 2, average component map

Becky's thinking for the tasks on average rate involving "how quickly" along with "how fast" and "average speed" continued to be grounded in a thinking about change-over-change:

Task 3. You provide both a table and a graph and ask a student to find how fast the function is changing between x=3 and x=5. The student responds 18...

x	0	1	2	3	4	5	6
f(x)	.75	1.5	3	6	12	24	48



Excerpt 31

1. Int: And how should the student have done the problem?
2. Becky: The student should have taken f(5) minus f(3) over 5 minus 3 to get the rate of change, so that would be the change in your dependent values which is 18 divided by the change in the independent values which is two so they should have got nine, is how fast the function is changing between those two.

In this case she merged average rate of change and rate of change in her usage. Later she added:

Excerpt 32

1. Int: So how would you guide the student?
2. Becky: I would guide the student by telling them how fast means the rate of change which is asking for the slope and look at both graphically and from the table to find that rate of change or slope.

This was a change from the earlier interview where Becky gave no graphical connection for this average rate of change task.

Similarly, in Excerpt 33 from Task 11 she pointed out that her lesson plan to teach “how quickly” would reduce to slope as one computed change over another computed change, or change-over-change.

Task 11. You are creating a lesson on how to find how quickly a function changes, given a table. For instance, given the table, students need to be able to compute how quickly the function *f* changes from *x*= -2 to *x* = 3.

x	*f(x)*
-2	103
-1	83
0	54
1	50
2	39
3	5

Excerpt 33

1. Becky: My lesson plan would include talking about the average rate of change being, comparing it to something they are familiar with and that is the slope, the change in your independent variable, dependent variable over the change in your, excuse me the change in your dependent values over the change in corresponding dependent, independent values.
2. Int: What goals would you have for the lesson?
3. Becky: The goal would be to understand what it means by how quickly a function changes, the goal would be to the able to use average rate of change to, the slope to find that.

In Task 9, Becky initially struggled with her approach to the question but eventually she was able to reconcile the confusion by building a table of values. This table included all whole numbers between the endpoints and allowed her to consider the overall change in distance over the change in time. Becky applied a graphical approach to average speed. In Excerpt 34 Becky used horizontal and vertical components along with a secant line.

Task 9. Your objective for the day is to develop a lesson on average speed. Specifically, your students need to understand how to find the average speed of a car over a period of time. They should be able to answer the question, “What is a car’s average speed during the period from 2 seconds after it starts to 4 seconds after it starts, where it travels *s* feet in *t* seconds and *s* is given by $s = t^2 + t$ (with *t* measured in seconds)?

Excerpt 34

1. Becky:	The goal for the lesson would be able to have them understand what we mean by average speed, the goal would be to find the average rate of change in distance over a given time interval…
2. Int:	How might you get them there?
3. Becky:	Pretty much the same way I got there is to look at a table of values, and that if we want to look at the rate of change in distance with respect to *t* and that looking again at units that that would tell us the rate of change over that interval is our speed and show them that the average rate of change is what we mean by the

> average speed, we can look at a table, I mean a graph and looking at the slope [sketches increasing concave up curve] we would need to look at the slope of that curve looking at the slope of the secant line [marks points on curve, sketches horizontal and vertical components, connects points on curve with line extending through the points] we can talk about that being the average rate of change the change in the position, distance, with respect to time [writing Δs and Δt by components].

In Excerpt 34, Becky indicated that part of her orientation for average rate of change tasks was to work from the arrangement of the units themselves. She also worked from her definition of average rate of change as the slope of a secant line connecting two points. In addition, she used both the horizontal and vertical components connecting the endpoints of the secant to act as a sort of aid in helping to think about slope. This provided one more way for her to think about change-over-change.

In Task 11, Becky struggled with the notion of “how quickly” and she tried to relate it to “how fast” but determined the task would be easier if it were framed in the context of distance and time. She then proceeded to apply distance and time to work though the task, as the next excerpt shows.

Excerpt 35

1. Int: What do you think is meant by ‘how quickly’?

2. Becky: I think we're talking about either rate of change, or rate of change of the rate of change, let me think which one… I would think…. how… well I don't know, initially I want to say rate of change of a rate of change.

3. Int: What is causing you to rethink that?

4. Becky: Well I was trying to think of a concrete example, that would talk about, talk about a distance here and I said how quickly, I could replace quickly with how fast is this function changing, I just think I would be looking at the rate of change, how quickly it changes. If we look at *f*(3) minus *f*(-2) [writing *f*(3)-*f*(-2) / 3 + (+2)… 5 -103 / 5, …-98 / 5],

5. Int: What are you thinking?

6. Becky: What I should be doing, but I am not sure, like how quickly,

7. Int: Why is it unsure, you said how fast would be more comfortable. Why?

8. Becky: Well I don't know if it would be or not, if it was talking about distance and time, how fast might be more comfortable.

9. Int: What would the negative ninety-eight fifths mean?

10. Becky: The negative tells us it's decreasing at about 98 something for every 5 change, every five units in the independent variable, because if it was, if we were thinking of this being linear, which it is

not, it would represent the slope of the line between these two points.

11. Int: Could you justify that that is a 'how quickly'?

12. Becky: Yeah, I think I could. Say if we were talking about feet per second it's changing at 98 feet for every five seconds.

Although Becky did not use a graphical form of slope for her own thinking in these average rate of change tasks, she seemed to maintain a consistent pattern of seeking out a change-over-change.

At this time the reader should note that there are several important ways of thinking that emerge in her working through Tasks 3, 9 and 11: (a) the importance of time as the independent variable, (b) the idea of change-over-change, and (c) a form of unitizing to determine rate. Each of these ideas will become clearer through comparing her responses in from the tasks. In the next few paragraphs, I will explain these ideas more fully.

Time: In each of these three tasks, Becky used time as a means of making rate concrete. In the third paragraph of Excerpt 35 (Task 11) she described the 98/5 as "98 feet for every five seconds," while in the second paragraph of Excerpt 32 (Task 3) she indicated she would guide the student by telling them that "how fast" means "rate of change" and that they can then find the slope. As a follow-up, during Task 3 (Excerpt 36) I continued to question how she might clarify the "how fast" and "rate of change" connection for the students. She replied that it would be easiest to ground student thinking in time.

Excerpt 36

1. Int: Suppose they fumble on the connection between how fast and rate of change?
2. Becky: Well we could also go to a problem like we could give some physical meaning to this. Let this represents distance [*y* axis] and let this represent time [*x* axis]. And so we could say how fast, how fast is the car going if after three minutes it had traveled six, what kind of, at three minutes it was going around six miles after five minutes it has gone 24 miles so how fast was the distance changing between that. So, If I put some kind of physical meaning to the problem, they understand how fast when you're talking about the speed, of the change in the distance over the change in time.

Change-over-change: Becky repeatedly showed a strong recurrence of the idea of finding one computed change over another computed change. This change-over-change thinking often involved her finding two coordinated, though not covarying, values which are then divided. Additionally, her frequent use of the word "over" often refers not to a portion of an interval of the domain but rather to the quantity's position in the fraction. Consider the following quotations:

Excerpt 37

1. Becky: The student should have taken $f(5)$ minus $f(3)$ over 5 minus 3 (Task 3)

Excerpt 38

1. Becky: Ask them to find the slope of the line through those two points [drawing secant on giving graph] which we have a change in the function values [marking vertical and horizontal components] look at it this way for the sake of brevity, the change in *y* over the change in *x* (Task 3).

Excerpt 39

1. Becky: The change in distance *over* the change in time, 14 over 2 is 7 (Task 9).

Excerpt 40

1. Becky: Speed is the change in distance divided by the change in time" (Task 9).

Excerpt 41

1. Becky: That is the slope, the change in your independent variable, dependent variable *over* the change in your, excuse me, the change in your dependent values *over* the change in corresponding dependent, independent values (Task 11).

These provide evidence of Becky's usage of "over" to mean a position in a fraction. At other times, however, she may have intended "over" as a reference to an interval of the domain:

Excerpt 42

1. Becky: I think of the average speed as the average rate of change and the distance traveled for any time, over a period of time" (Task 9).

Excerpt 43

1. Becky: The negative tells us it's decreasing at about 98 something for every 5 change, every five units in the independent variable," (Task 11).

At other times, her usage of "over" is ambiguous:

Excerpt 44

1. Becky: If I put some kind of physical meaning to the problem, they understand how fast when you're talking about the speed, of the change in the distance over the change in time (task 3).

Excerpt 45

1. Becky: The goal would be to find the average rate of change in distance over a given time interval (task 9).

Unitizing: Excerpt 43 suggests that Becky did not think of the average rate of change as connected to a continuous function whose values covary, but rather as occurring over chunks of change (e.g. in chunks of 5). In this case, Becky employed an idea of a unitized rate, but it was not "per" unit of the independent variable. That is, the unit was not a single *x*-unit; it was an interval of 5 *x*-units.

By using this thinking she linearized the rate. In doing so she went beyond the bounds of the interval and claimed the rate of change is 98 *for every* 5 units. The unitizing came prior to the division in this case.

In Task 8 (the parabola task) Becky indicated that the secant "would be giving the rate of change between two points on a parabola." She referred to the secant as related to rate of change but she did not use that to extend student thinking. Her way of thinking about slope as a formula showed itself once again in Task 8:

Excerpt 46

1. Int: This is also a little different from what I asked earlier but what does it mean to understand the slope formula?
2. Becky: … I'm not sure what you are asking there, to understand the slope formula it is the change in, that somebody defined that slope is going to be the change in the dependent over the change in *x*. I mean somebody else could come along and define it the change in x over the change in *y* I guess, we'd have just different looking things but…

Slope, for Becky, is what it is because someone defined it that way.

In Task 7, Becky was asked to sketch a lesson plan to review an introduction to exponential functions. In the first interview, when Becky was asked about the meaning of *b* in $y = ab^x$ she said that it is "the ratio by which the function changes when the independent variable *x* increases by 1." This ratio, for

Becky, acted as a sort of rate of change, though it did so as a multiple of the preceding output when the outputs were generated sequentially from whole number inputs. In essence, Becky considered *b* to be the "constant factor" (language used in her lesson plan) of exponential growth or decay. In the analysis of her first interview, I hypothesized that language and definition influenced Becky's thinking. In this second interview I will support this claim by showing Becky's evident disequilibrium when I confronted her with the overlap in her definitions.

Excerpt 47

1. Becky: In comparison to linear functions, linear functions you had for a constant change in your independent value you had a constant change in your dependent value, but what's happening in exponentials is you have a constant ratio so I think that's what's important for students to understand there is a constant ratio.

Recall from the first interview that Becky viewed constant rate of change through contiguous segmentations of the domain. In the second interview, there is yet no reason to imagine that her thinking has changed. Her thinking for both linear and exponential, then, involves steps across the domain. For Becky, linear functions have a constant rate of change while exponential functions have a constant ratio. However, in her thinking rate and ratio are the same except that rates involve different units. The following interchange takes place when Becky was asked about the meaning of b in $y = ab^x$.

Excerpt 48

1. Becky: The meaning of *b* is the ratio. It’s that constant ratio that this function is growing by, a constant factor I should say. The constant factor its growing by…
2. Int: So what if a student said ‘well you told us that lines have constant rate of change and now you’re telling us that exponential functions have constant ratios? What’s the difference?’
3. Becky: The difference is how they are growing one is growing at a constant rate and one is not…you could also look at a graph of what is happening, look at the exponential growth [sketches increasing concave up graph, 4 points, horizontal and vertical components between them – see Figure 9] and we start looking at…compare with the linear functions to see how that [the horizontal and vertical components] is not staying the same.
4. Int: What do you think is the difference between rate and ratio?
5. Becky: Well they are the same. I think of a, a lot of times you think of a rate you think in a context, five pounds of apples per 4, I mean, five dollars for five pounds of apples so you know a dollar per pound, think of rate that way. But a ratio is nothing more than a fraction so it could be considered the same.

6. Int: You know, which led me to ask the question about the student who says ‘well you tell me that one is a constant rate and one is a constant ratio.’

7. Becky: Well if I looked at, if I was looking at the change in *y* over the change in *x* the same way we do a linear function then this is going to be equal to 5 and, lets see, this is going to be 10, and this one is going to be 20. I don’t know. I never thought about that or had a student ask that question… the rate of change of the line is going to be doubling each time we’re looking at this particular situation [sketching in secants between the four points], the slope of the line between, the slope of the secant line. Think about it that way, think of it as the rate of change between these two points [first two points on sketched curve] you want to make some kind of comparison between what you did with the linear function.

8. Int: Final word?

9. Becky: I guess I hope they don’t ask me that question, I have to think it through before, I have to say I’ll get back to you.

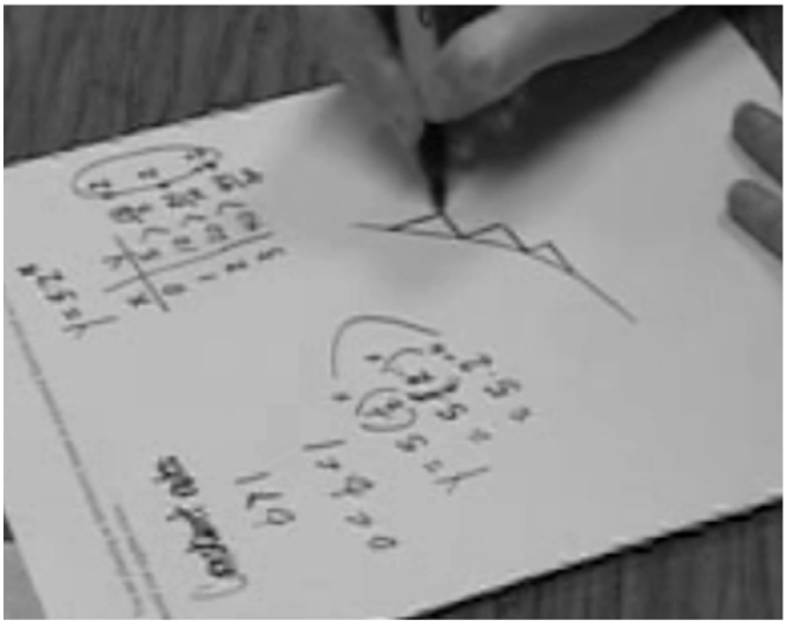


Figure 9. Becky, interview 2, task 7

Becky could not articulate a distinction between rate and ratio as used in linear and exponential situations. She indicated a need to make “some kind of comparison;” but since her notion of rate continued to work from a formulaic, unit-rate understanding, she remained at a loss as to what that comparison might be. She seemed to hold ratio in two ways: it could be either something that looks like a fraction or it could be a factor relating sequential values.

In Excerpt 48, when an algebraic approach did not work, Becky attempted a graphical approach. Becky viewed a linear function as having a constant rate of change when the horizontal components were the same. For Becky, rate of change now meant considering the steepness of the linear components across the successive intervals. Interestingly, she does not seem to think of these components as connected to average rate of change.

In Interview 2 there was still no evidence that Becky considered ratio to be a multiplicative relationship. Although she exhibited a propensity to unitize her efforts were not directed at making sense of that unit. Additionally, Becky seemed to unitize prior to performing the division itself. Most of the time, she thought of average rate of change as one change over another change. She did not describe rate itself in covarying terms as *over* often seemed to describe a position in a fraction.

Becky: Interview 2, Change Component

Becky employed two primary ways of thinking in the changing rate of change tasks. Her first type of thinking, similar to what we saw with the average component, involved the graphic consideration of the slope. Her second type of thinking involved a consideration of a coordination of changes, which I will once again call *change-with-change*. When Becky moved to the change-with-change type of thinking there was no apparent connection to any of her earlier ideas of rate. I have added these components to the final map for Interview 2 (see Figure 10).

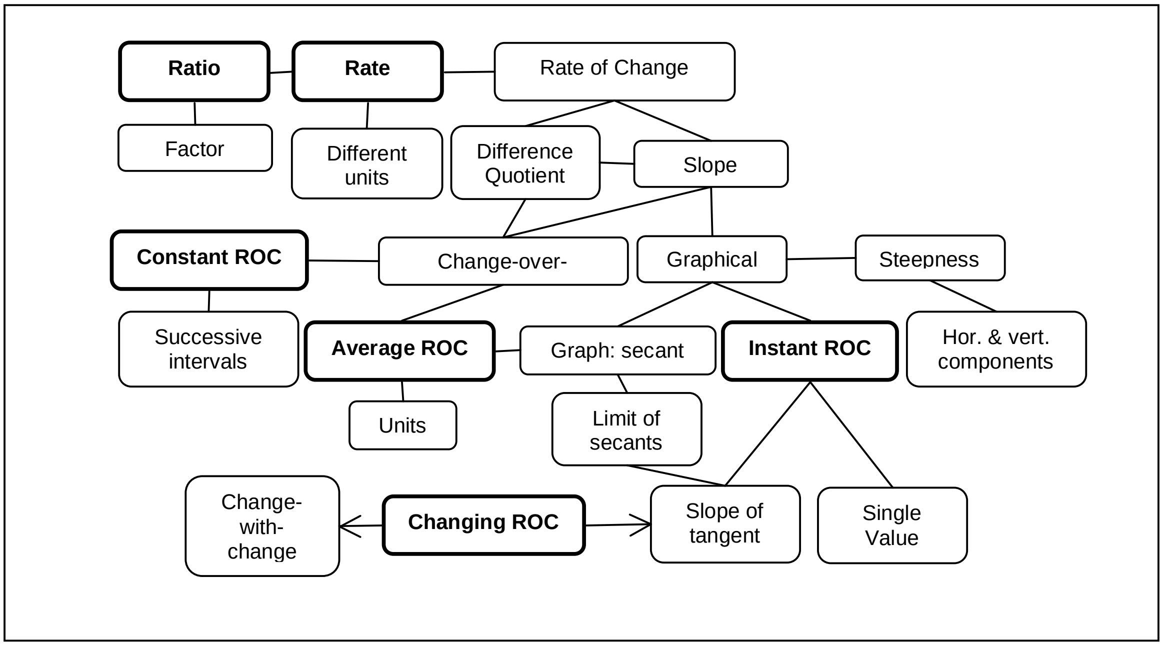
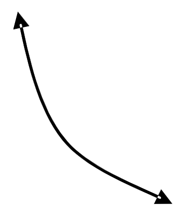


Figure 10. Becky, interview 2, final map

In the task about decreasing at a decreasing rate (Task 2) one of the most significant changes we see from Becky is that her discussion no longer centered on “little tangents” as it did before. Instead, she now focused on secants, where the secants act as estimators for the actual rate. She saw that rate that contained in the tangent.

Task 2. Two students are having a debate about a function whose graph looks like the figure below. One student declares the function is “decreasing at a decreasing rate” while the other says it is “decreasing at an increasing rate…



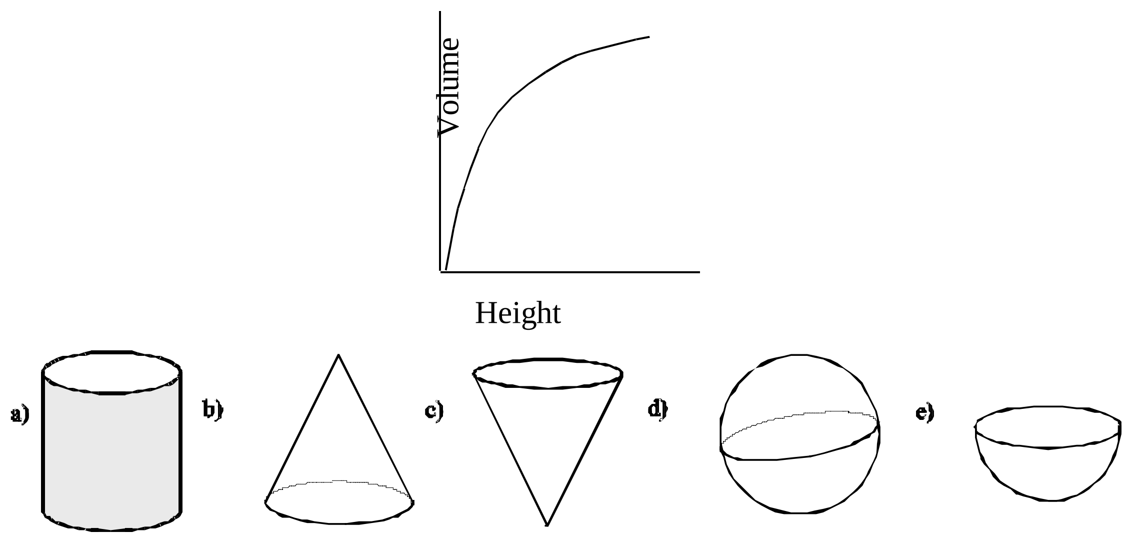
Excerpt 49

1. Becky: Well the rate again is the rate of change [writes rate of change] the rate of change or the slope of the curve meaning the slope of the tangent line at a given point which could be approximated by the slope of a secant line taking two points.

There is no evidence in Excerpt 49 to suggest that Becky thought of secants as having the change-over-change meaning that they had in the average rate of change tasks. Rather, in a graphical sense the secants were serving as approximations for the slope of the tangent. Of the four changing rate of change tasks, Task 2 (the task about decreasing at a decreasing rate) was the only one where Becky used the idea of secants in explaining how something changes with respect to something else.

In the container task (Task 5) Becky used another approach. Though she was able to proceed through the question rather effortlessly in the first interview, she encountered significant trouble as she seemed to want to introduce time as an underlying variable. She acknowledged early on that there was a covariational relationship between the height and volume.

Task 5. The following graph represents the volume of water as a function of the water's height in a container. Which container goes with this graph?



Excerpt 50

1. Becky: They would have to understand the relationship between the volume and the height and how as the height changes how is the volume changing so they would need to understand that covariational relationship.

After recognizing that all of the choices represented increasing functions, she began to consider choosing between options *B* and *C*. For container *B*, Becky attempted comparing two noncontiguous intervals, one near the top and one near the bottom. For container *C* she only considered a single height.

Excerpt 51

1. Becky: I think about this one [*B*] as the height increases the volume is going to increase slowly at first because the volume here [bottom] is much larger than it is, the change in volume say from this one to this one is much smaller I mean greater [between two drawn horizontal lines towards bottom] than it is if I am looking at a section up here [between two drawn horizontal lines towards top], if I'm looking here [*C*] again as the height increases than the bottom is still increasing [sketching a single horizontal slice – not thickness], it's going to increase quickly at first, which has behavior of this graph, and as it continues to increase it's going to slow down as we start reaching the top, so that's very promising there.

In Excerpt 51 we see global language where Becky's change-with-change thinking included using either two noncontiguous slices (of unknown matching thickness) or one point of reference around which she builds her case.

Figure 11. Becky, interview 2, task 5

Her analysis would have been consistent if the task were a function of height against time and if the rate of entering water were constant. She later recognized her mistake and attempted to reconcile her missteps.

Excerpt 52

1. Becky: I would feel more comfortable with against time than volume against height. I guess it really doesn't matter if it enters at a constant rate… maybe that's not, ok, [moving pen up container *B*] I don't know if… I think I will change my answer to *B*.

2. Int: Because?

3. Becky: Because I'm thinking as the height increases… this volume is, is, shoot, I don't know why I can't think of it… No, if the height was right here [bottom of container *C*] this volume would be very little so it starts out as a very little, if the height was right there [bottom

of container *B*], this volume [*B*] would be a lot bigger than this one [*C*] [laughter] can I go on to the next one?

The first point of conflict in this interchange involved that somehow, at the point of origin, bottle *C* has a smaller volume than bottle *B*. She seemed to confuse a starting value with an initial rate of change. The next excerpt continues from Excerpt 52:

Excerpt 53

1. Int: So you're torn between…
2. Becky: I am torn between and it is really bothering me. But this [the graph] doesn't fit with that [container *B*] either.
3. Int: Why not?
4. Becky: In my thinking, because your volume is getting quite a bit bigger [moving pen in container *C*], your change in your volume is small [moving pen along top of volume-height graph]

The "volume is getting bigger" in container C but it was unclear if she meant it was getting bigger over time, as height increased, or as height increased across contiguous or non-contiguous intervals. With "your change in your volume is small" she referred to the given graph and connected slope as rate of change. At this point she had left the conflict that she had near the origin and was now considering regions near the top of the containers. The next excerpt continues from Excerpt 53:

Excerpt 54

1. Int: Why do you think it can't be *B*?

2. Becky: … I don't remember why I said … I think it could be *B* because your change in your height, your change in your volume is small, and that's what you'll be having here [top of *B*] a small change in the height and you'd have a very small change in the volume, for here [top of *C*] for a very small change in your height you'd have a big change in your volume, so I'm going to say *B*.

Once again we see Becky using general language as she applies changing rate of change across a domain. Becky did not connect this change-with-change to her earlier notion of rate of change as she did not think in terms of slope or of change-over-change. She stated that for a "small change in the height and you'd have a very small change in the volume" and that "for a very small change in your height you'd have a big change in your volume." As she used small and big without referents, it is unclear precisely why they were small and big.

In the racetrack task, Becky considered the secants she drew from point *A* (see Figure 12) to increase until they reached 12 o'clock, at which point the distance was at a maximum. Becky's thinking in Interview 2 closely matched Interview 1 as evidenced by her drawing identical graphs (compare *Figure 12* to *Figure 5*). In Excerpt 55 we see that she compared the change in one variable as something else changed.

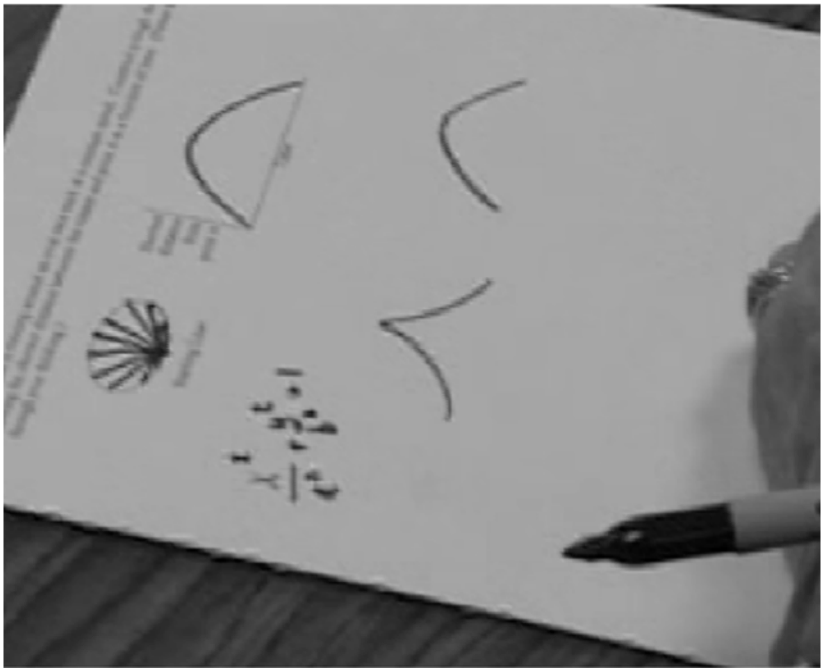


Figure 12. Becky, interview 2, task 12

Excerpt 55

1. Becky: I think it might be this one [the parabola] because as you're getting over here [one o'clock position] let's see this is our time [moving pen around right side of track] for a small change in time [pointing to one o'clock position] the shortest distance would not change that much. The same thing over here [pointing to parabola] so I think I would draw it like this [parabola]

This change-with-change thinking is very similar to the thinking that led to her final response in the container task. And, again, there was no real referent to what she meant by "small" or "not that much." Somehow, though, Becky translated “not much” into a concave down graph. Equal spacing of intervals clearly did not influence the construction of the secants that aided in her solution.

Excerpt 56

1. Int: Let me ask this, what was it that was driving your spacing of the secants that you drew in there?
2. Becky: I don’t know if I was aware of what I was using to drive those, because I can’t say they are equally spaced… so if I put some a little bit closer in there [4 – 6 o’clock region] it still is not, its increasing…
3. Int: You OK?
4. Becky: The concavity of it is what, [so you think it might be that or that – pointing to both sketches], well that was my initial…

After I asked about why her graph components were curved and not straight, Becky continued to try to reconcile her work with algebraic methods. She was, however, unable to achieve a satisfactory end. She then began to consider another way of solving the problem that included examining time intervals.

Excerpt 57

1. Becky: It might help to just take some time intervals just mark off some, I don’t know what they would be, and just do, you do some little pieces, takes some little string and do some things out here and plot some things may be that would help

2. Int: But you are pretty okay with what you've got then?

3. Becky: I'm not real confident in it, no

4. Int: But you can go with it?

5. Becky: I can go with it.

In Interview 2 Becky showed some consistency in how she approached changing rate of change problems. Frequently, she used a type of thinking that considered a relationship of one change to another. In these cases, though, the way of thinking was not as change-*over*-change, but rather change-*with*-change. Becky first sought to employ a global description for how she believes the function behaved, and, if necessary, she then moved to algebraic methods. Although Becky used secants in Task 2, there was no observable evidence in the remaining changing rate tasks to suggest that she thought about instantaneous rate as the slope of the tangent line. (The closest connection might have been a comment made during the container question when she mentioned that the top of the graph is where "your change in your volume is small.") When Becky used her change-with-change way of thinking she was unable to extend it to non-monotonic function situations.

Summary of Becky

Becky held two primary ways of thinking about rate: as a difference quotient and as a slope. In both cases, Becky usually thought about rate in terms of change-over-change. For Becky, slope had meaning in that it was the result of a definition (see Excerpt 12, p.56).

For constant rate of change (linear functions), I proposed that Becky did not think in terms of a dynamic relation of the variables changing in tandem. Rather, she thought in terms of a constant, computed result between successive intervals (see p. 51).

The findings show that Becky could think about average rate of change as change-over-change, or graphically as a secant (sometimes using horizontal and vertical components). In the average rate tasks, Becky unitized change-over-change but she did not coordinate it per unit of the independent variable. Rather, she based her unitization on the value that happened to be in the denominator of the change-over-change computation. For example, when she worked with (-98/5) she saw it as (-98) for every (5), rather than (-98/5) for every unit of the independent variable (see Excerpt 35, p. 82). This observation served to support the claim I made earlier about her constant rate thinking and how she thought in terms of successive intervals. Becky did not see any difference between constant rate and average rate of change (see Excerpt 24, p. 70).

Becky had a difficult time coordinating that a ratio and a rate were essentially the same and that linear functions had a constant rate while exponential functions had a constant ratio (see Excerpt 48, p.90). She defined rate and ratio similarly, and those definitions were not distinct enough to allow her to reconcile her disequilibrium.

For changing rate tasks, Becky did not primarily engage her change-over-change way of thinking. She solved Task 2 (the decreasing at an increasing rate task) using a graphical way of thinking about slope (little tangents in the first interview and secants in the second), indicating a possible connection in her thinking of changing rate with instantaneous rate (see Excerpt 49, p. 95). In the remaining tasks, however, she used a form of thinking that I called "change-with-change" (see Excerpt 19, p. 65, and Excerpt 54, p. 100). She appeared to disconnect her change-with-change way of thinking from her constant or average rate thinking in that she did not think about uniform increments or an application of slope. Becky's change-with-change way of thinking restricted her from being able to work through the tasks that were non-monotonic (see Excerpt 55, p. 101).

Overall, the maps I developed for Becky indicate a strong consistency in her ways of thinking about rate of change. The maps in Figures 4 and 10 show very few differences in the ways that she connected ideas in her thinking. In fact, the map for Interview 2 only contains two additional elements. One additional element is her stronger inclusion of using graphical horizontal and vertical components. This appeared to provide an additional way of thinking about slope that she perhaps connected back to her change-over-change way of thinking. A second element added to the map for Interview 2 reflects the struggle she evidenced in distinguishing ratio as rate and ratio as factor.

CHAPTER 6: RESULTS AND ANALYSIS OF MARY

Introduction to Mary

Mary had taught mathematics for 11 years prior to entering the Teachers Promoting Change Collaboratively project (TPCC). Her background included a bachelor's degree in mathematics education and a master's degree in educational leadership. To this point, Mary had been primarily teaching classes below precalculus.

Mary entered the program hoping to learn how to better connect mathematics to real life and she sought examples that focused on the patterns and relations in mathematics. She hoped she would be able to help her students gain a better understanding of the mathematics. She made these ideas evident during the preliminary questions, as shown in Excerpt 58.

Excerpt 58

1. Int: Can you give me an example to illustrate good understanding to you?
2. Mary: I would say that given a situation in an example, a life example, and be able to match to the correct function be able to explain the patterns in the function, the increasing and decreasing maximums and minimums, and what they mean. What's happening here when it's decreasing? What does this maximum mean? What does it mean, I guess, to real-life situation.

Mary held that a connection to the real world would provide deeper understanding for her students.

The interviews will also illustrate distinct ways of thinking for Mary. Chief among these is the idea that she thought strongly about the mathematics in a graphical sense. That is, Mary primarily thought about rate of change pictorially.

Overall, Mary's interview episodes were the shortest and she frequently responded to questions by indicating she simply did not know what to say. As before, this analysis will build through the components of types of rate of change.

Mary: Interview 1

Mary: Interview 1, Definition Component

In the definition component tasks, Mary stated that rate and rate of change have some distinctive properties, and she strongly linked rate of change to the slope of a line. However, she indicated no meaningful way of thinking about average rate. As such, I will introduce dashed lines in her map to indicate these weak connections. Instantaneous rate, for Mary, was the tangent line. We will also see her usage of the word "per" as one way she thinks about rate of change. The map in Figure 13 illustrates these ideas.

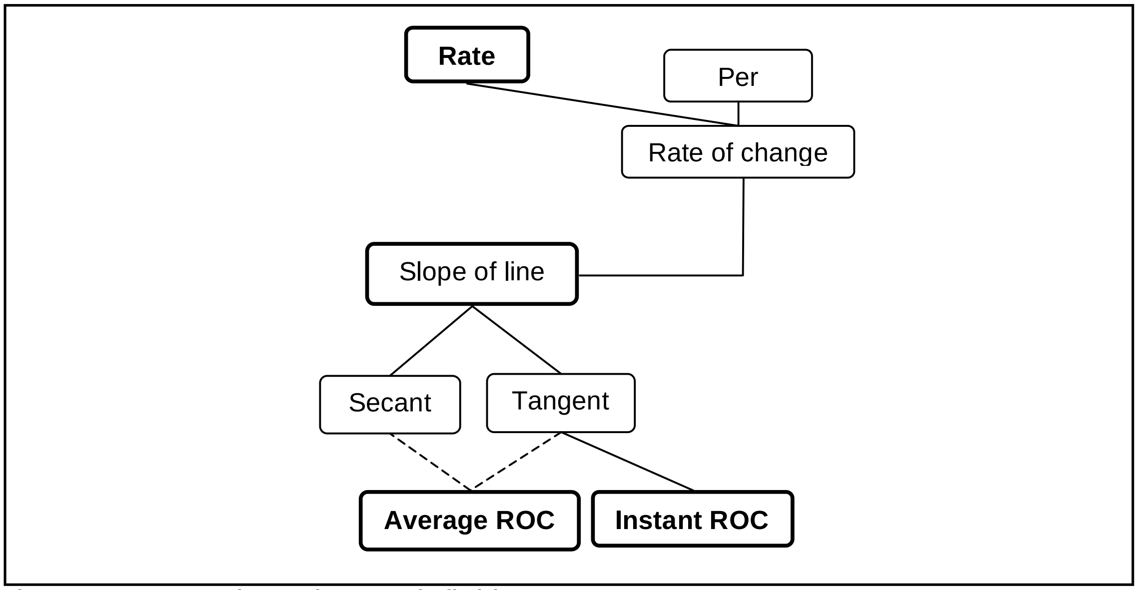


Figure 13. Mary, interview 1, definition component map

Mary's distinction between rate and rate of change was subtle. She aligned rate of change with the idea of "per" and slope.

Excerpt 59

1. Int:	What do you mean we speak of a rate of change in mathematics?
2. Mary:	…Rate of change is how fast something is changing, it's rate would be how fast it is changing.
3. Int:	You said how fast something is changing, elaborate on 'something' if you can.
4. Mary:	So you're talking about how fast gas prices are increasing in the state of Arizona, that's the rate, rate of change is always per something else, because its slope, so its per day or per month or per second, or [inaudible, something about dollars increasing].

When dealing with average rate of change, Mary found herself at a loss. She was unable to say what she meant by average rate of change. At one point in the interviews (Task 11) she jokingly suggested that average rate of change had to do with finding the average of a number of rates, but her accompanying laughter suggested she knew that to be not quite right. In Task 14, when asked about the meaning of average rate, she responded by addressing it as the slope between two points. Mary appeared to hold this idea with no additional meaning.

Excerpt 60

1. Int: What do you mean by an average rate of change?
2. Mary: You already asked me that [laughter] you confused me when you asked me it. Before you asked me I thought average rate of change is just, I mean, that's where all that other problem, but it's kind of between two points what would be the slope of that line between any two given points.
3. Int: OK, is there a conceptual understanding to average rate of change and rate of change or the difference between the two or similarities?
4. Mary: I don't know.

Mary was likely thinking about this connecting of points pictorially. In the next task (Task 15) she exhibited confusion with average rate of change being either the slope of a secant or that of a tangent.

Task 15. How are average rate of change, average rate of change of a function over an interval, secant to a graph and tangent to a graph related?

Excerpt 61

1. Mary:	[rereading question, slowly] I would say if you have some graph, average rate of change is at one point, so that would be your tangent [sketching an increasing curve, marking tangent at one point], and I'm totally guessing, average rate of change over an interval you're talking from here to here [marking two other points on curve], so that could be your secant [drawing secant between those two points], so that this one and the secant are the same [circling function over an interval and secant], and then the tangent and the average rate of change are the same.
2. Int:	Anything else?
3. Mary:	I have no clue what I'm doing, no, I'm done

Although she equated average rate and tangent in the above excerpt, Mary did not have a problem thinking about instantaneous rate of change as the slope of a tangent in Task 13.

Task 13. When the Discovery space shuttle is launched, its speed increases continually until its booster engines separate from the shuttle. During the time it is continually speeding up, the shuttle is never moving at a constant speed. What, then, would it mean to say that at precisely 2.15823 seconds after launch the shuttle is traveling at precisely 183.8964 miles per hour?

Excerpt 62

1. Int:	And I want you to do is just tell me what you are thinking of with that question.
2. Mary:	OK, if that is continually increasing it's doing this [draws increasing concave up curve] and so at precisely at 2.15 seconds you add the tangent line [sketches tangent at a point], that would be the slope or the miles per hour of that line.

In this case Mary provided a graphical response even though the context was not graphical. The line itself had a slope and therefore it had a "miles per hour." We saw earlier that Mary made some sense of rate of change through the language of "per." Here we see her connecting the language to a graph. For Mary, "per" is a language device that accompanies a graph.

Mary: Interview 1, Linear Component

In the linear component tasks Mary spoke of slope both as a difference quotient and as rise over run. Of greater importance, though, is that we will also see the start of her emphasis on the steepness of the line to serve as rate of change. With respect to linear functions themselves, Mary showed that she placed an emphasis on looking at graphs using what she referred to as correlations. Figure 14 shows these new connections.

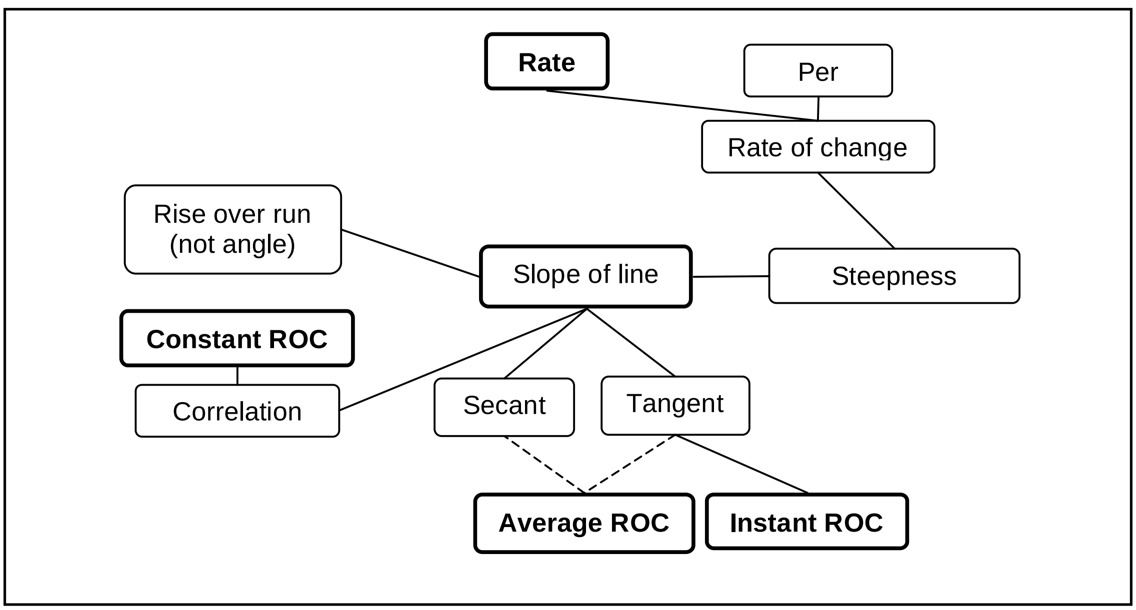


Figure 14. Mary, interview 1, linear component map

For Mary, functions are graphs that are infinite collections of points. She has an action-based perspective of functions (Carlson & Oehrtman, 2005). This turns out to be a rather easy claim to make as she later stated this perspective during a PLC meeting. In that meeting I had asked the teachers to read part of the Carlson and Oehrtman (2005) paper about action and process views as a take-home activity. In her written reflection Mary made her position clear:

Excerpt 63

> "Also, I believe that I understand functions at an action level. Most of the article made no sense to me, because I don't conceptualize it either. If I don't conceptualize it, how can I teach my students to?!!
>
> Finally, I feel that some of the suggestions such as at the bottom of page 10 about describing transformations are just stupid. The language is so complex I can't understand it. It seems like we are changing vocabulary to make things confusing." (from Mary's reflection dated 10/11/05)

I do not believe that Mary was being at all disingenuous with these statements, for her handling of functions is explainable from an action view. Given this, Mary appeared to think of functions as geometric figures rather than a mapping that relates inputs and outputs in a dynamic way.

In Task 1, the task about planning a lesson on linear functions, Mary claimed to view slope as a "correlation." Excerpt 64 shows her response when the interviewer asked her about the meaning of slope.

Excerpt 64

1. Mary: Like positive correlation, negative correlation, how it's again relates to a real world problem, that if it is increasing by two feet

per second but that's more than one foot per second, what a *y*-intercept usually means is a starting point in relationship to a word problem, that's it.

In Task 4, the interviewer asked Mary what students need to understand about slope. She based her response in the graph as a geometric figure. In Excerpt 65 we see that her meaning of correlation involved a notion of visual inspection.

Excerpt 65

1. Int: What would you like students to understand about slope in a linear context?
2. Mary: I feel like you asked me that question already [laughter].
3. Int: A couple of these are very similar.
4. Mary: Just in a linear context they need to understand what positive slope is, they need to go look at something and tell me if it has positive slope or positive correlation or negative correlation, one difficulty for students is 0 slope or undefined, understanding the difference a horizontal line has your zero slope in a vertical has undefined, kind of understanding why because in undefined you are dividing by 0 when you find your slope. And again applying it to where it occurs in the real world that it is rate of change, how fast things are changing.

The missing piece from earlier, how she connected slope to the secant or tangent, is seen her explanation of the meaning of *m* in *y=mx+b*. In Excerpt 66 (from Task 1) we see that for Mary, slope was steepness.

Excerpt 66

1. Int: I think you may have already addressed this but it's one of the things on my list, but what is the mathematical meaning of *m* in *y=mx+b*?
2. Mary: *m* is your slope, how steep a line is

Steepness, for Mary, acted as her primary way of thinking in dealing with rate of change. This steepness was of a specific sort. She evidences a particular view of slope in how she would respond to the student who referred to slope as an angle.

Excerpt 67

1. Mary: But the meaning of slope is your rate of change of the line, the second student, the angle of the line, they are probably thinking the larger the angle the steeper it is, but slope really isn't in degrees so they are missing, they are wrong in what they say

By "rate of change of a line" Mary again means steepness. The student in the above excerpt, however, did not use the right sort of steepness.   This explanation continued in the excerpt below.

Excerpt 68

1. Int: What would you want the student to say back to you for that question?

2. Mary: How steep the line is, how fast it's changing, I would hope that I've done enough from old examples but they could give me an example, how your salary increases is the only example I can think of right now. Just give me a specific example and describe it more. The fourth one, again I'd start with what they know, where did you come up with them? Where do you see it in this formula? But what does it mean in this relationship? Is it where it starts, how steep the line is? Just leading in the same direction with all of them and eventually do the same questions.

By "how fast it's changing," Mary is speaking directly about steepness. By "fast," then, Mary was not thinking of comparing two quantities, changing in tandem. We see that Mary viewed functions as geometric figures, and rate of change as slope. Slope, for Mary, is a pictorial steepness.

Mary: Interview 1, Average Rate of Change Component

In this component we will continue to see Mary's emphasis on her importance of the graph. She held that average rate of change meant to apply the slope formula, but she struggled with ignoring the middle of the graph. Additionally, while she might not make connections to "faster" and "quickly" she did have an alternative way of thinking about rate in terms of speed that did not involve any graphical interpretation; Mary might solve a task relating distance, rate, and time using a formula. Lastly, and of most importance, was her connection between steepness and “fast.” In Figure 15 I refer to this way of thinking as “flatter-slower-smaller, steeper-faster-larger.”

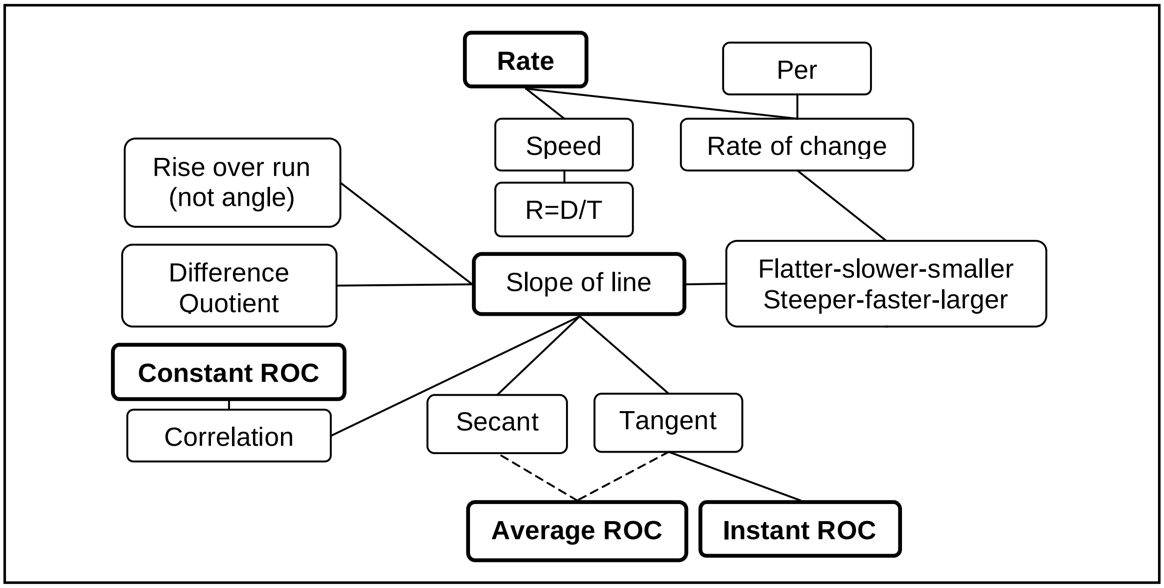


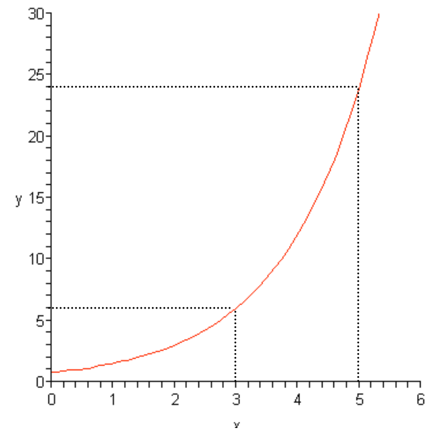
Figure 15. Mary, interview 1, average component map

During Task 3 Mary evidenced her thinking that the graph itself *was* the function. In

Excerpt 69, Mary related how she had difficulty interpreting the task.

3. You provide both a table and a graph and ask a student to find how fast the function is changing between x=3 and x=5. The student responds 18...

x	0	1	2	3	4	5	6
f(x)	.75	1.5	3	6	12	24	48



Excerpt 69

1. Int: What must the students understand to answer this question?

2. Mary: …I think the first thing to understand is what you are asking about how fast the function is changing. They need to be able to… Very simply they need to be able to see the relationship between the graph and the table that they are, just *the table is just some points on the red line on this graph,* that again how fast the function is changing, I believe what you're asking for is the slope, how fast, how much it changes between the point (3, 6) and the point (5, 24) [italics added].

Mary added that the student should have taken the two points "and found the slope because rate of change is your slope and that's how fast it is changing."

In Excerpt 70, Mary emphasized the perspective of the graphical in how she would respond to the student who provided the change in output values of the function. In the excerpt below she pictorially described the change in outputs as both "vertical" and "high."

Excerpt 70

1. Mary: I would agree with them that that is the vertical change, how high it changed it started at six and went to 24 but when we are talking about how fast function is changing that that's rate of change and rate of change is slope, so we need to look at *x* and *y* and ask what is the formula for slope and lead them through it
2. Int: Can you elaborate more on the fast part, how you're interpreting how fast the function is changing?
3. Mary: I guess I'm just seeing it as slope I feel like I'm doing everything as slope but I would just say that from here to here [drawing secant between the two points on the graph] what is that rate of change?

In a companion task to this one, Task 11, Mary did not know how to determine how quickly a function changed. She began by looking for a pattern; she recognized none. Her trouble came with the middle portion between the two points in question.

Task 11. You are creating a lesson on how to find how quickly a function changes, given a table. For instance, given the table, students need to be able to compute how quickly the function *f* changes from $x = -2$ to $x = 3$.

x	*f(x)*
-2	103
-1	83
0	54
1	50
2	39
3	5

Excerpt 71

1. Mary: OK... I guess that's why I have a hard time answering this question, because I'm not sure what you mean by how quickly the function changes. Because if you want to ignore the middle and go from here to here you're just talking slope, how fast it dropped, and you are ignoring what's happening in the middle [moving pen along middle of graph]. There's different rates that it's changing all the way through and that after it and before it's going to continue, I mean if this pattern continues.
2. Int: What if we mean average rate of change, instead of how quickly?
3. Mary: Then they need to be able to find the slope between those two points

For Mary, slope was about two points. With average rate, though, Mary was not sure about ignoring the middle of the graph (between the two endpoints). This is why I used a dashed connecting line in her map (see Figure 15). Mary could follow a procedure to determine “how quickly” a function changed over an interval, but she could not provide meaning over the entire domain. In Excerpt 72, below, Mary could not articulate any difference between rate and average rate.

Excerpt 72

1. Int: What’s the difference between rate of change and average rate of change? Is there a difference?
2. Mary: That’s a very good question and I don’t know.

In this analysis, then, we have seen Mary refer to average rate as the tangent and to the slope of the tangent as an instantaneous rate. Additionally, in the above excerpt she stated that she saw no difference in meaning between rate of change and average rate. Mary did not have a clear way of thinking about rate of change in these different contexts.

Mary’s goals for the students in Task 11 remained consistent with her graphical perspective, as shown in Excerpt 73.

Excerpt 73

1. Int: What are the goals of the lesson
2. Mary: We need to be able to calculate rate of change, they need to be able to describe it like on a graph. If you have these two points

[drawing secant between endpoints of interval], see what's happening, describe I mean, that it's gone down this many feet [marking vertical from left and point to x-axis] in this many feet [marking from that intersection point to other end of interval] which you can do without the graph but to see how it relates to the graph.

At the end of Task 11, she once again expressed confusion as she moved her pen across the middle of the graph. She knew that average rate involved slope, which she could picture. She was still unsure, however, as to how average rate had any meaning.

Excerpt 74

1. Mary: If they are just finding rate of change they can take these two points [circling -2,103 and 3, 5 on table] and do their slope formula [writing difference quotient] and tell me how fast it's changing. They could graph it and figure out the change in *x* and the change in *y* [marking along horizontal and vertical]. I guess I'm just still confused about [traces pen across middle of graph].

In Task 9 (below), the interviewer asked Mary about finding average speed when given a formula of distance as a function of time. I found her response to this task interesting, as she did not begin by connecting it to slope. Rather, she chose to approach the task from a formulaic perspective where *d=rt*.

Task 9. Your objective for the day is to develop a lesson on average speed. Specifically, your students need to understand how to find the average speed of a car over a period of time. They should be able to answer the question, “What is a car’s average speed during the period from 2 seconds after it starts to 4 seconds after it starts, where it travels *s* feet in *t* seconds and *s* is given by $s = t^2 + t$ (with *t* measured in seconds)?

Excerpt 75

1. Int: What mathematics is needed to understand this objective?

2. Mary: They need to be able to … they need to be able to, if you are given how many seconds to be able to calculate how many feet it has traveled to begin with. So like if it’s two seconds it is two squared plus two which is six, so at two seconds it has gone six feet, and then be able to calculate the second one at four seconds. At four seconds it went 20 feet. They need to understand what speed is, which is usually rate and we use distance equals rate times time [writes “*d=rt*”] so it’s your distance divided by time [writes “*d/t*”], is that enough? [laughter]

She continued with the problem by using a quotient, but she did not give any evidence of this as related to the formula for slope. That is, her thinking here did not appear connected to her earlier ways of thinking about average rate or rate of change. Mary did not approach the task graphically or even using “per.” Rather, she used a formula. The next excerpt followed her comments regarding student mistakes.

Excerpt 76

1. Int: This just kind of reiterates what we just talked about but what do students find complex about this problem?
2. Mary: I think they are going from one situation that they are dealing with speed and feet and you need to transfer it, it’s not really telling you a formula you need to transfer that yourself to, if you know what speed is, that it means rate, and be able to use that formula.
3. Int: How would you answer the example question?
4. Mary: I would do this first [points to numbers already calculated] and then I would do the change in distance over the change in time [writes (20-6)/(4-2)] which I think that its average speed is seven feet per second

For Mary, this seemed to reinforce her distinction between rate and rate of change. Rate had to do with speed. Rate of change had to do with slope, which she used to address how “fast something is changing” (see Excerpt 59). Here, the task addressed speed, which she connected to rate, to the formula of *d=rt*. She summarized her method in Excerpt 77.[4]

Excerpt 77

1. Mary: If I'm asking about speed what am I going to have to use, and refer back to the distance equals rate times time

Mary simply applied a formula when she had the right pieces: distance, rate, and time.

In Task 8, the secant with the parabola, Mary did not address the applicability of the slope formula to the parabola as given. In Excerpt 78 she spoke of how the tangents had slopes and that those slopes meant something. She was not exactly sure of that meaning.

Excerpt 78

1. Mary: Again when you start thinking about tangent lines and what's happening, the slope of the tangent line at certain points, [sketching tangent near left point] you can talk about how fast it's decreasing or increasing you know your slope at a minimum or maximum would be zero [sketching tangent along bottom of

---

[4] She did briefly mention rate of change, but corrected herself. Her second goal " was to be able to go ahead and find rate of change or what were we asked to find, the speed of the car, the average speed, so I think in the beginning of the lesson I would review distance equals rate times time."

parabola] that's kind of important because it is bottoming out and turning around, so I haven't really done it but I guess you can. I think it's in higher classes.

Her statement was vague because there was little meaning for her in these tangents. Her meaning was limited to the idea that something that appears steeper is somehow faster.

In Task 8, Mary provided one other glimpse into how she understood the slope formula.

Excerpt 79

1. Int: OK, what does it mean to understand the slope formula?
2. Mary: I think more than just being able to crunch numbers and it is. Understand what it means with respect to the line, in respect to a situation, kind of what we talked about before, how fast it is decreasing or increasing? How fast the line is at certain points. If you're talking about tangent line at certain points slope is really linear so you have to always do it to the tangent line, or a line

For Mary, the difference quotient explains a graphical property of a line. Mary understood slope as something that someone does to a line. This explains the difficulty she had in ignoring the middle in the earlier task. To Mary, a line had a property of a "fast" increase or decrease, or the line *itself* is fast, which, I propose, Mary based in steepness.

Mary: Interview 1, Change Component

My analysis of the final component in this interview continues to strengthen the claim of how strongly Mary holds the “flatter-slower-smaller, steeper-faster-larger” way of thinking about steepness. In addition to this, we will also see Mary attempt in two tasks to connect changing rate of change to some comparison of changes, though in both cases she was unable to follow that thinking to the end. Figure 16 reflects her of changing rate of change as a form of steepness.

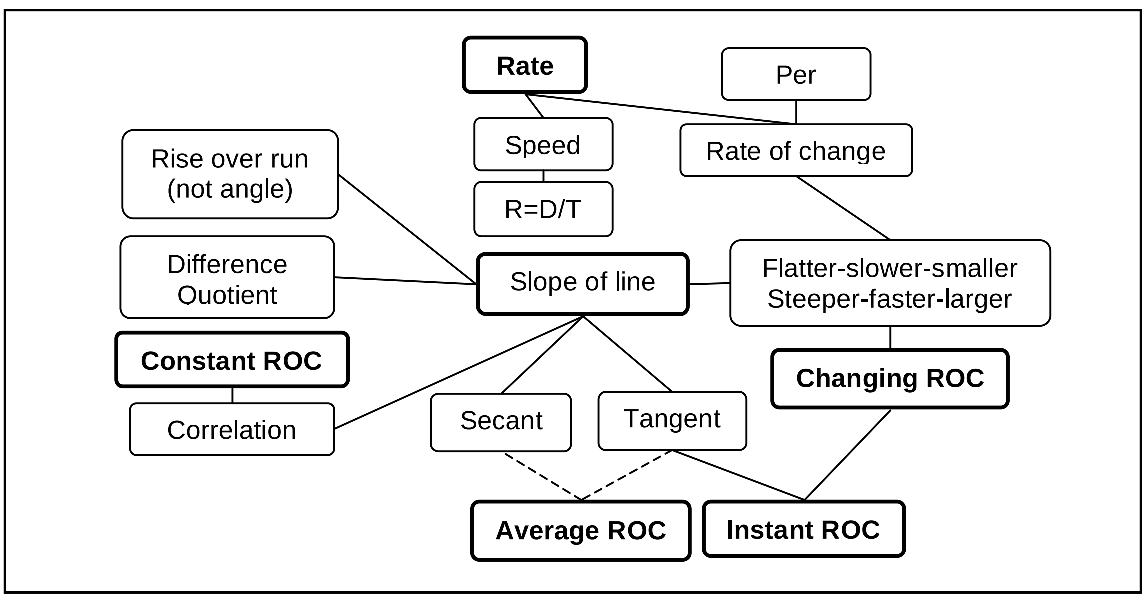
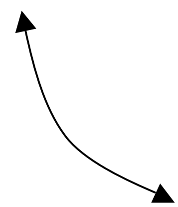


Figure 16. Mary, interview 1, final map

In Task 2, the task asking about decreasing at a decreasing rate, Mary supported her response using her “flatter-slower-smaller, steeper-faster-larger” way of thinking.

Task 2. Two students are having a debate about a function whose graph looks like the figure below. One student declares the function is “decreasing at a decreasing rate” while the other says it is “decreasing at an increasing rate…



Excerpt 80

1. Int: What do you believe is mathematically relevant to understand language like decreasing at a decreasing rate?

Mary: To begin with they’re both correct that it’s decreasing, as you go from right to left it’s getting it’s going down, for a decreasing rate they need to understand slope, that that’s your slope, that the smaller the slope, it’s getting, your slope at different points on the line, your tangent line at different points on the line, why am I fumbling?

Excerpt 80, Mary related the “smaller” slope to the flatter part of the provided graph. For Mary, a less steep slope was slower. She used the tangent lines to interpret slope as a rate, which she had earlier connected to being the instantaneous rate of change. After a bit she continued.

Excerpt 81

1. Int: Which student do you think is correct? You had said that they are both correct?

2. Mary: No, they are both correct that they are decreasing, but the question there are they decreasing at a decreasing rate or at an increasing rate… Did this on the test the other day… OK this is decreasing at a decreasing rate [pointing to given question] because as you come down here [right end of curve], tangent line this has a smaller slope than this one [pointing to tangent sketched on left end of curve] this would be an example of decreasing at an increasing rate [pointing to self drawn decreasing concave down curve] because it gets, falling faster and faster and faster

3. Int: Can you explain on that one with the tangent lines?

4. Mary: OK here the slope would be close to zero [left end of her sketched curve – see Figure 17] but here [right end] you are getting closer to larger and larger, I mean if you doing absolute value ignoring the negatives, it getting more and more vertical so it’s a larger slope.

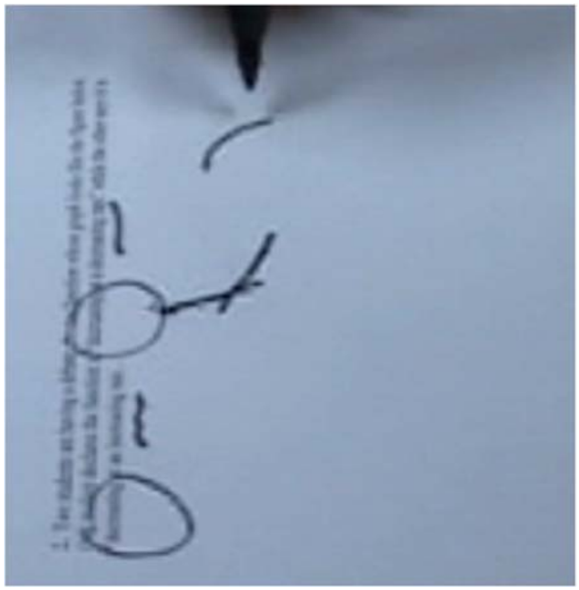


Figure 17. Mary, interview 1, task 2

Mary appeared to attend to the shape of the graph as she traced it from left to right. Rate, to Mary, was not a coordination of two quantities. Rate was a single idea: a graphical slope. This is clear in the following excerpt.

Excerpt 82

1. Int: What is the meaning of the second use of decreasing in the given situation? And what is the meaning of the first usage?
2. Mary: OK, the second one is basically, how I see it is the slope of the tangent line, it slows down in how fast it's going down, the first one is the general overall picture of entire graph not, because this one can be decreasing [sketches decreasing line] where this would be increasing [sketches increasing line] or this would be increasing from left [sketches another increasing line] to right, so this is generally what is happening from left to right whereas this is

[pointing to second 'decreasing'] more specifically looking at the curve and what's happening to the curve.

In the bottle question, Task 5, she again based her thinking of slower and faster in steepness rather than a coordination of quantities. In the following excerpt she began to orient herself to the task and provided a contrast between "larger" and "slowing down."

Excerpt 83

1. Mary: On this one [*A*] as you're filling it the rate of change will be constant because it's the same width apart all the way across. *B* will start slower and then increase [crossing it out as a choice] because the distance here [pointing to bottom of container] is longer is further apart is bigger and *B* gets smaller and smaller, so as you are pouring water it's going to get faster and faster. This one [*C*] will start faster and slow down but then I just started thinking about *E* so I'll come back to that. *D* will start faster and become slow but then it would get fast again, so speed back up because it's getting skinny at the top. Okay I can't remember why I ruled out *E* earlier…[going back to *C*]… now I'm confused myself and I'm not sure which one it is.

2. Int: OK let's go through a couple, what must the students understand to answer this question?

3. Mary: That the rate of change in the beginning is much larger than the rate of change at the end, it's going to start slowing down.

After this, the interviewer specifically asked Mary to use the terms volume and height to explain what she meant. She stated "it would be the change of your volume over the change of your height," and wrote the fraction "volume/height." This, though, did not assist her as she returned her focus to faster and slower. In Excerpt 84 she tried at first to create equally spaced intervals but she then confused the width of the first interval.

Excerpt 84

1. Mary: The volume is going faster and then slower, OK, and height, this is evenly spaced so like let's say this is height 1, 2, 3 [on container *C* – see Figure 18] and this would be 1, 2, 3 [on graph height (horizontal) axis] if you evenly mark them … And the volume increases quickly at first which means it has to be wider and it's skinnier here [the first interval on the height axis], I've confused myself, I've forgotten your question.
2. Int: What must the students understand to answer this question
3. Mary: … I don't know.

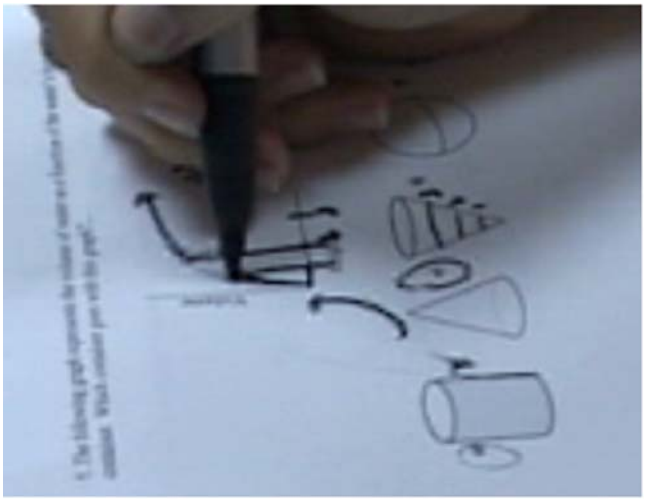


Figure 18. Mary, interview 1, task 5

Mary was unable to complete the task as she was not operating with a coordination of the variables themselves. In explaining her thinking she stated, “we always stick with distance and time and so I automatically jumped to distance and time and I don’t truly compare that to other situations.”

In Task 12, the racetrack task, she once again focused on steepness as reflective of whether the straight-line distance from the start was changing rapidly or slowly. She began by drawing distances as secants that connected points on the track to the starting point. In Excerpt 85 she spoke of the distance getting “faster.”

Excerpt 85

1. Mary: I think the distance, I really am not sure, I think the distance gets faster [sketching up and right from origin what will become a parabola]. It’s increasing, you’re getting further, the distance

between them [pointing to endpoints of secants] is getting larger and larger over time and then it kind of slows down [sketching vertex]. And then it's getting shorter and shorter and then as you get closer it speeds up again [completing parabola],

2. Int: Can you maybe

3. Mary: Not speeds up but it gets shorter faster.

Of interest in the above excerpt is that Mary mentioned that the "distance gets faster" and later "speeds up again." Although she clarified both, it indicates that her initial way of processing rate of change while she drew was as a single variable – steepness.

Overall, Mary had a challenging time with the tasks related to changing rate of change. Her action view of functions and strong graphical connection to steepness and speed were not sufficient for her to deal with dynamic situations. She tended not to connect the variables to each other, but preferred to operate graphically, in the form of tangents (as in the decreasing at a decreasing rate task) or, more generally, with faster as a property of steepness.

Mary: Interview 2

Mary: Interview 2, Definition Component

Mary's second interview took place six months after the first interview. During her time in the TPCC she stated that she had developed a sense of frustration. She knew that she needed to change her teaching practice, but she did not know what changes she should be making. In the introductory questions for the second interview she stated, “There’s a better way to do it, I know there’s a better way to teach, but I’m not sure how to get there.”

Overall, in this component we will see that Mary sometimes spoke in terms of change compared to change. Mary continued to struggle with average rate of change. Again, I will indicate those struggles in her map with dashed lines. For Mary, instantaneous rate of change was about the slope of a tangent line. She did not appear to involve a comparison of changes when speaking of instantaneous rate. In Figure 19 “change compared to change” refers to her way of thinking about vertical and horizontal changes on a graph.

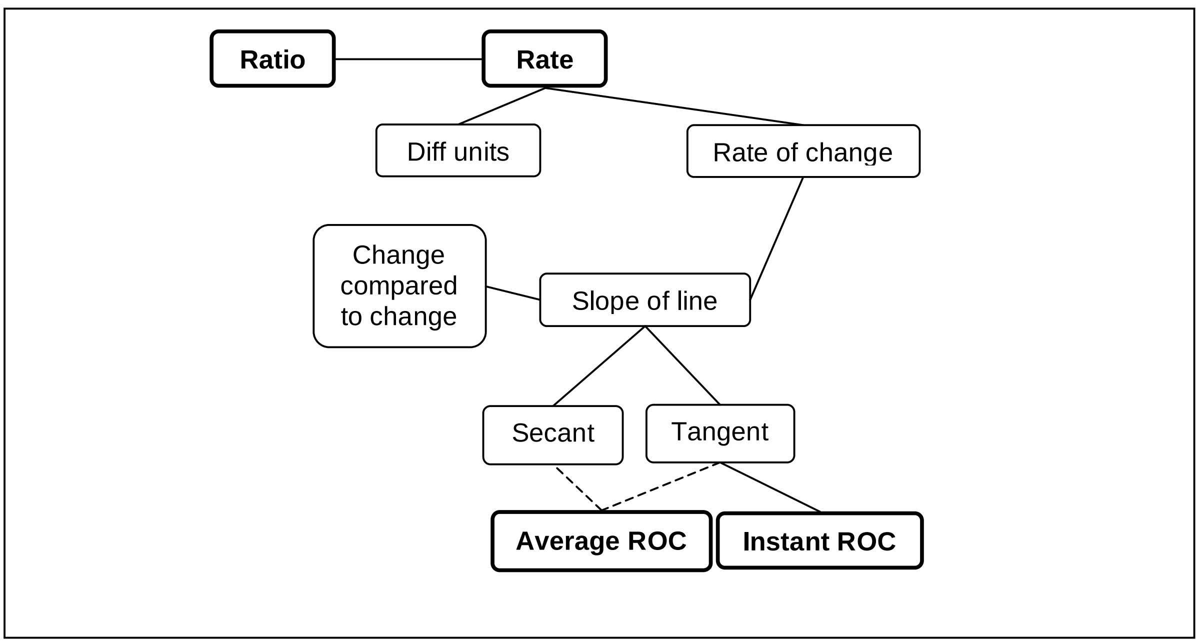


Figure 19. Mary, interview 2, definition component map.

Mary indicated that the difference between a ratio and a rate is that a rate has different units. Excerpt 86 shows this distinction.

Task 14. What is a 'rate', mathematically speaking?

Excerpt 86

1. Mary: Usually it's a ratio comparing two things or a fraction or whatever you, so a rate of 4 to 5 [writes ratio 4/5] or a ratio of 4 to 5, usually a rate compares two different units, miles to hours [anything else] no.
2. Int: What do we mean we speak at a rate of change in mathematics?
3. Mary: It's the ratio of the change in one value [circles four] as compared to the change in another value [circles five].

In the above excerpt, Mary continued to draw her own distinction between rate and rate of change. For her, a rate was a comparison while a rate of change was a comparison of changes. While she may define rate of change that way, we will see that she primarily thought about those changes in a graphical sense.

For Mary, her meaning of rate of change continued to remain graphical. When asked about where rate of change is taught in the high school curriculum, she said, "I guess you talk about rate of change whenever you are graphing."

Mary's meaning for average rate of change, even at the definition level, remained unclear. I will explain this more completely in the component for average rate of change. For now, it is worthwhile to note that Mary's meaning of average rate of change remained unconnected from her meaning of rate of change. For example, she said the following about average rate of change when I asked how average rate related to secant and tangent in Task 15.

Task 15. How are average rate of change, average rate of change of a function over an interval, secant to a graph and tangent to a graph related?

Excerpt 87

1. Mary: I think that the average rate of change is the tangent, if you're talking about a curve it's the, so the average rate of change is the tangent. You can find the average rate of change of a curve by finding the slope or the rate of change of the tangent line and the average rate of change of a function over an interval would be finding the change of the secant. So if you want to know from here to here you'll be finding the slope, or the rate of change, of the secant.

As in Interview 1, Mary's meaning of average rate of change relates to both secant and tangent. That is, she continued to ground her understanding in a graphical way of thinking. Interestingly, she rather consistently described ideas related to instantaneous rate of change as directly related to the tangent rather than in connection to average rate of change. In the space shuttle question (Task 13) she directly described her notion of instantaneous rate of change.

Excerpt 88

1. Mary: Well if its speed is increasing continually then it's never at a constant speed so if you're increasing continually and they want to know at precisely this they are finding the tangent line [sketches straight line] to that curve.

In Excerpt 88, Mary did not answer the question. Rather, she spoke of how to find instantaneous rate. For her this involved finding a tangent line. Later, in the same task I returned to Mary's meaning of a tangent line.

Excerpt 89

1. Int: So how does the tangent line help specifically?
2. Mary: Well if it was a graph of distance to time [sketching new graph], distance is increasing faster and faster and faster so at a certain time I can tell you a certain distance, but if you found the rate of change of that tangent line you would be getting his speed at that certain time, from time zero to that point.

For Mary, instantaneous rate of change was the slope of the tangent line. It had nothing to do with a comparison of changes.

In Task 16, I asked why one would use division to calculate slope, Mary's answer (Excerpt 90, below) focused on her idea of comparison rather than on a sharing or segmenting form. Additionally, though, Mary also explained that the reason we use a ratio is that the comparisons would remain invariant as the values change.

Excerpt 90

1. Int: One last question, so you've noticed there is a lot of talk about slope in here, tell me why we use division to calculate slope.
2. Mary: Because you want to compare that's a change per something else as…
3. Int: What are you thinking?
4. Mary: Well when you multiply two units then you're not, you don't break it down into a ratio of the change in something over the change in something else, you lose the ratio aspect of it.
5. Int: Are you kind of wondering what other thing would I use?
6. Mary: Yeah [laughter].
7. Int: A lot of times if I ask students to compare something I might use subtraction. Why is it fair game to use division?
8. Mary: … I don't know. It's just a ratio. It's a rate of change, it's a change vertically over a change horizontally, and so you want to keep the ratio of that change. It needs to be shown and so you do that with division. Because if I go up six steps or up five feet in two seconds or if I go up 10 feet in four seconds that ratio is kept, but if I was subtracting the change is three and than the change is no longer three, it looks like an example.

Mary focused the invariant ratio changes on horizontal and vertical components of a line. She focused on graphical aspects of change rather than on an operation on quantities. She also used a pictorial term of “up” to describe the first change in the ratio.

Mary: Interview 2, Linear Component

In the linear component Mary continued to express linear functions as having a constant rate of change and she continued to base them in a graphical perspective. We will see that she graphically used steepness to connect the difference quotient to rise-over-run. Mary also introduced a way of thinking with respect to a graphical form of unitization (which I develop in greater depth in the following sections). We will also see the emphasis she placed on steepness as an intermediate connection between slope of a line and rate of change. Figure 20 illustrates these ideas.

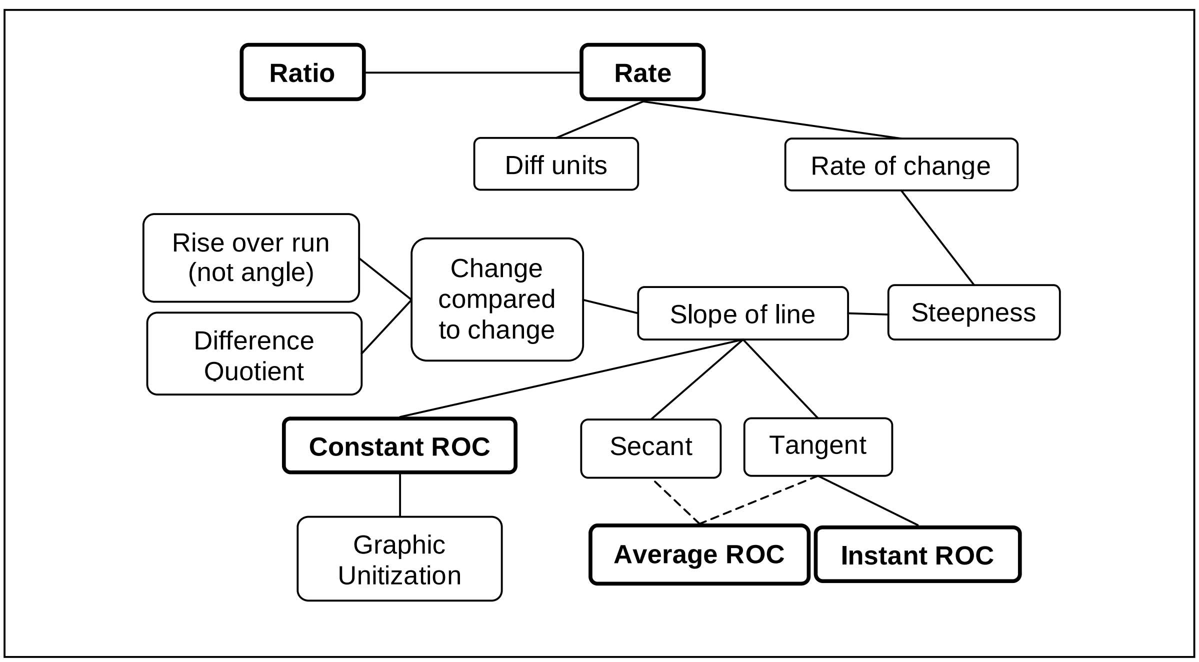


Figure 20. Mary, interview 2, linear component map.

One thing that remained the same between the interviews was Mary's idea that slope pertained to a line's steepness, but not to the angle it forms with the horizontal axis. The following exchange was from Task 4.

4. Four students are discussing the meaning of slope in a linear context. One student says it is $\frac{y_2 - y_1}{x_2 - x_1}$. Another says it is the angle of the line. A third student says it is the rate of change of the line. The fourth says simply that it is the number *m*.

Excerpt 91

1. Mary: Slope is how steep a line is but it's not connected to the angle, if you have a 60 degree angle it has nothing to do with that slope [sketching angle], mean the slope is not 60 degrees, it does have something to do with it but, I wouldn't say it's the meaning of slope,

She revisited the notion later in the task.

Excerpt 92

1. Mary: I try to keep my kids away from saying it is the angle of the line [underlining that phrase in the question]. So I'm trying to figure out why I do that, because its rise over run it's not an angle

Slope, then, was rise over run. This was again a graphical interpretation. She stated this in explaining how she wanted students to understand slope.

Excerpt 93

1. Int: Okay what would you like students to understand about slope in a linear context?
2. Mary: I think they need to just really understand that it's rate of change, that, and emphasizing in context [circling 'rate of change']. Or first especially, pictorially, that as I'm moving along the *x* axis, how much I'm changing vertically, if it's linear, I want them to see that it's relationship between the two it's not just your vertical change, it's not just your horizontal, but it's, you know, the change in the two, a ratio of the vertical change over the horizontal change.

For her, this emphasis is a declared change in her teaching. To Mary, this thinking of vertical change to horizontal change somehow carried more meaning than the notion of rise-over-run or the difference quotient. She clarified this to some degree in reflecting on her own change in thinking.

Excerpt 94

1. Mary: I guess I did change a little bit as I taught slope this year. I tried to really emphasize that it was the vertical change over the horizontal change, it took many days discussing that and when I finally gave them the formula they were all really mad at me. Why didn't you just teach us that it's just a formula? And if I ask them about the vertical every time I mentioned slope, we talk about vertical

change and horizontal change and they just roll their eyes at me, or rise over run and they just want to memorize the formula.

In Excerpt 95 Mary summed up her graphical emphasis well when explaining how she would respond to the fourth student, the one who merely states that slope is *m.*

Excerpt 95

1. Mary: The fourth student says it's the number m, ask them where they got the m from again and always coming back to, so if you have a slope of such and such what does it mean? Again I always have to go back to the graph

One of the more interesting exchanges during Task 4 (the four students) occurred when she described how she would respond to the first student, the one using the difference quotient.

Excerpt 96

1. Mary: … The first student, hmm, … I'd maybe ask them why are they doing that formula and what does it relate to. Maybe give them the points on a graph [sketches coordinate axes] you know two points [sketches and connects two points] and so we just found the slope and so the slope is two, and what does that mean according to this graph? And try to have them explain that *it went up two, for every, because it went over one* [sketches vertical and horizontal components between points], the vertical change was two and the

> horizontal change was one, trying to bring it back pictorially to exactly what it means instead of just a number. [Italics added]

What is of interest is where Mary focused on the student explanation. I believe that her use of "for every," which she then changed to "because it went over" is significant – it supports my claim that Mary grounded her invariance of the comparison in the graphical aspects, not as a constant numerical ratio. As we saw in the definition component, Mary coordinated the use of division through a comparison that remained invariant. She also worked strongly in the graphical perspective. The idea of something going "over" was more explanatory then a more generic sort of rate. This way of thinking will continue to unfold in the following sections.

Mary: Interview 2, Average Rate of Change Component

Mary, as before, did not have a way to think about average rate of change that made sense to her. However, she evidenced a graphic way of thinking that might be an indicator of a connection of average rate of change to constant rate of change. As before, we will see that Mary applied a formula when the task involved distance, rate, and time.

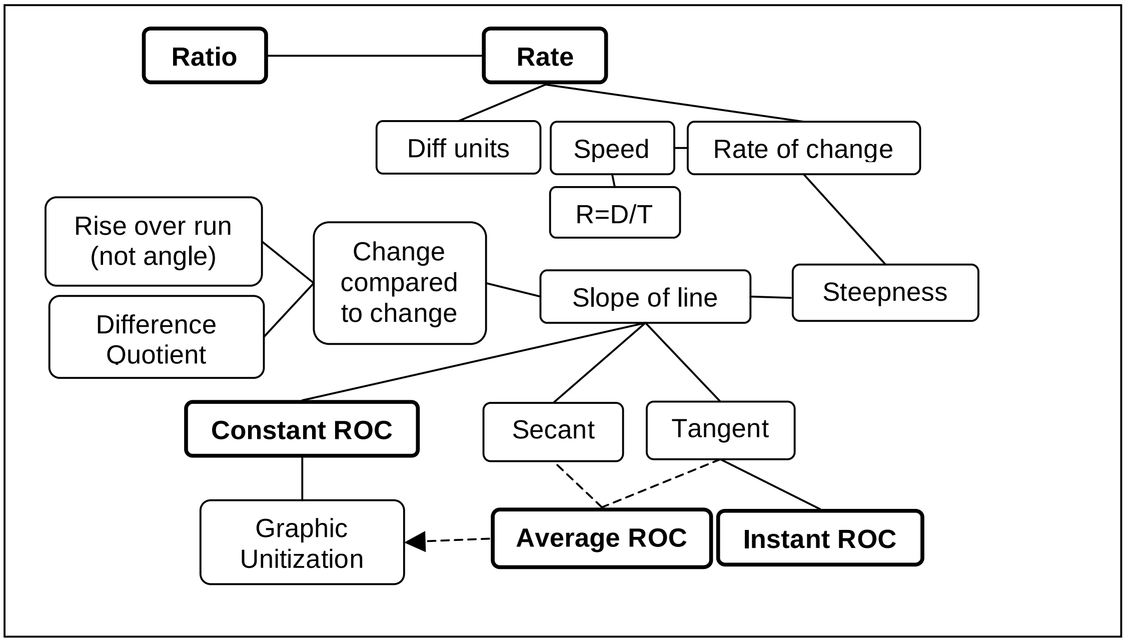


Figure 21. Mary, interview 2, average rate of change map

In Task 8 (the parabola task) she continued her emphasis on the graphical aspect. Although she did not directly answer the question in Excerpt 97 (below), she placed her focus on vertical and horizontal changes.

Task 8. A student comes to you and says, “You know, when you apply the slope formula to opposite points on a parabola the slope is always zero.”

Excerpt 97

1. Int:	What do think it means for a student to understand the slope formula?
2. Mary:	That you can find the rate of change between any two points, how much it’s vertically changed or how much it’s horizontally changed.

In Task 7, the exponential function plan, Mary mentioned the word “rate” in another context: the exponential function $y=ab^x$.

Excerpt 98

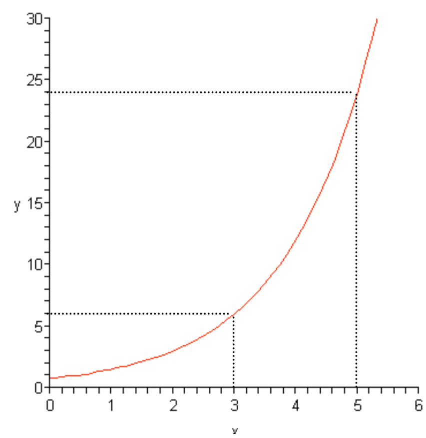
1. Mary: The *b* is…[long pause]… I don’t know how to explain it, it is what is being taken to the exponent so it is what’s being raised to the power. So in the example if you double your allowance, two, the first week is two to the zero, two to the first, two to the second… When you are investing it is your percentage rate and so I’m wondering what it has to do with a rate so, I don’t know that’s all I can think of.
2. Int: So it has something to do with the rate?
3. Mary: Yeah, it has to when you are investing it is your percentage rate that you earn… but I’m trying to tie that to a rate of change and I’m failing in where that…
4. Int: Because that is a word you had been using in previous questions?
5. Mary: Exactly, so I’m not sure exactly how that ties in or if it does.
6. Int: Nothing is coming to mind?
7. Mary: No.

Mary’s primary thinking of rate of change was in the graphical perspective and based in steepness. Because of this, she could not make sense of how a percentage rate, as might be used in an exponential function, could be a rate.

In the three tasks that similarly dealt with average rate of change (Tasks 3, 9, and 11) Mary continued to ground her thinking in the graphical, with one exception. This exception was the same as in Interview 1 where Mary approached Task 9 (the average speed task) from a formulaic perspective, completely avoiding *any* graphical interpretation. In Task 3 (how fast a function changes) she recalled that "how fast" might mean rate of change.

3. You provide both a table and a graph and ask a student to find how fast the function is changing between x=3 and x=5. The student responds 18...

x	0	1	2	3	4	5	6
f(x)	.75	1.5	3	6	12	24	48



Excerpt 99

1. Mary: I think we had this problem and we discussed what do you mean by how fast the function is changing. They need to understand what you mean by that, and I think what the question is implying is that the rate of change between the point when *x* is three and the point when *x* is five would be what we are looking for.

Note here that she did not mention average rate of change. The student, she indicated, should have approached the question graphically.

Excerpt 100

1. Int: What should the student have done?
2. Mary: They should have found the rate of change between these two points [sketching secant] which increased 18, had a vertical change of 18 and horizontal change of two so the rate of change or how fast it's changing is nine whatever for every one step [moving pen across *x* axis] there's no units, but,
3. Int: How would you, sorry, go ahead.
4. Mary: It goes up nine and over one for every one step you go over [sketches right triangle up from *x*=3 to point on secant at *x*=4].

In this we see another circumstance where Mary unitized the vertical and horizontal components. For her, unitization is in terms of "up" and "over" and is repeatable. She did not ground this unitization in a coordination of the variables. Rather, I propose she thought of this unitization as a scaling (or shrinking) of her pictorial image of slope to a single *x*-unit.

For Task 11 (how quickly a function changes), Mary used similar graphical thinking. Even though this task did not include a graph, Mary wanted students to view the table as points that exist on some graph.

Task 11. You are creating a lesson on how to find how quickly a function changes, given a table. For instance, given the table, students need to be able to compute how quickly the function *f* changes from *x*= -2 to *x*=3.

x	*f(x)*
-2	103
-1	83
0	54
1	50
2	39
3	5

Excerpt 101

1. Int: What is mathematically important for students to understand in approaching this question?

2. Mary: … Basically they need to understand that these are specific points on the graph. They're not all the points but they are specific points and that you can figure out how fast it is changing between any two points. You can find your rate of change by finding the slope.

Even though the task included no graph, we see that for Mary the graph *was* the function. The slope, as she would find it on the graph, is the rate of change. The explanation for how she wishes student to approach this task provides support for this claim:

Excerpt 102

1. Int:: What do you think your lesson plan should include and what would the goals be?
2. Mary: I would probably introduce it with looking at graphs and asking them to describe how quickly it changes between two points so they visually can see what they are doing.

As she progressed through the task, I asked her to think of a real-life example for the function. She found this challenging, as she wanted to include time as the independent variable, but that would necessitate using negative numbers for time. She finally decided to describe a person with a parachute jumping from an airplane. In Excerpt 103 I probed further, asking about how students should work through the task. Mary provided an example of how she constructed her graphic form of unitization.

Excerpt 103

1. Int: So describe student approaches to solving this that would be acceptable.
2. Mary: I guess finding the slope between these two points [circling (-2, 103) and (3, 5) on table], I didn't like this problem in the last interview either.
3. Int: So what would that slope represent?
4. Mary: How fast he changed from whenever *x* is negative two to whenever *x* is three. Repeat your question.

5. Int: What would the slope mean in your parachute question?

6. Mary: In the parachute question [draws secant connecting endpoints on her sketched graph] how much he had dropped [moving pen down] for every one second that he went over [moving pen across] if you reduced it down to a rational number.

While Mary recognized average rate of change tasks as opportunities to apply constant rate of change thinking, she really did not know why that was the case. For Mary, rate of change was a property of lines and in average rate of change tasks the line does not match up to the function. In Excerpt 104 we see that Mary has a procedure to work with average rate. She is, however, not clear as to how the procedure has meaning.

Excerpt 104

1. Int: What difficulties do think your students will encounter?

2. Mary: I remember last time just even myself I again have a hard time finding rate of change between two points when there’s a lot of things going on in between [circling middle of graph] because I want to consider what’s happening in between, not just the beginning and end result.

3. Int: So how come your answer this time is different than last time?

4. Mary: Because I asked after the interview and he said I could do that

5. Int: He, who?

6. Mary: The interviewer. No, anyway, maybe it wasn't this problem. It was finding average rate of change and I was struggling between can you just find it between the first and the last one, or do have to find the rate of change [moving pen down table] and average them all? And we had this discussion in the PLC and that he told me that I had to average them all and you said no, I didn't, after a PLC, so I can just find the rate of change from the beginning to the end, so I was very frustrated because I got two different answers from two different people.

7. Int: So which one do you think is right?

8. Mary: I think you are right because even though what's happening here, we want to know how much he fell [sketching vertical and horizontal components between endpoints] for every second he was falling total, I mean what happened in between yes he slowed down and he increased but in the end how much had he fallen for every second [moving pen down vertical and across horizontal components].

9. Int: But then we are ignoring everything in the middle.

10. Mary: No, well he's still falling here and it's still taking more time here so it kind of averages out

11. Int: It kind of averages out?

12. Mary: [laughter]

I assume that Mary used the same sort of thinking that she did in Task 3 (Excerpt 100) where I claim she employed a graphic form of unitization. In that case she was able to make sense of finding “how fast” a function changed by thinking about vertical and horizontal components that had the same graphical direction as the secant connecting the endpoints. What is different here is that she used the same sort of language though she did not draw the smaller horizontal and vertical components. Her trouble arose in that she could not determine how this was in any way an “average.” In light of this, it seems clear that Mary can make some sense, graphically, of an idea of average rate of change. That connection seems to come in a form similar to her graphic unitization she used in the linear tasks.

In Task 9, I asked directly about average speed. During this task, Mary made no distinction between speed and average speed, as she treated the two synonymously. Also, she approached the task with a method similar to her first interview, using a formula and not a graph.

9. Your objective for the day is to develop a lesson on average speed. Specifically, your students need to understand how to find the average speed of a car over a period of time. They should be able to answer the question, "What is a car's average speed during the period from 2 seconds after it starts to 4 seconds after it starts, where it travels *s* feet in *t* seconds and *s* is given by $s = t^2 + t$ (with *t* measured in seconds)?

Excerpt 105

1. Int:	So what mathematics is needed to understand this objective?
2. Mary:	…I guess you need to understand what average speed is, so your speed is your change in your distance, or *s,* over your change in your *t.* and you're just asking for the rate of change from two seconds to four seconds

She later stated "average speed is your rate of change and that it's the change in distance over a change in time" and then wrote down "$\frac{D}{t}$". Her answer to the question appeared to be a difference quotient, but she made no connection to a difference quotient as slope, algebraically or graphically.

Excerpt 106

1. Int:	So how would you answer the question?

2. Mary: I would find the change in distance, actually I would find a distance at two seconds, so my distance at two seconds is 6 [writes s(2)=6], my distance at four seconds is 20. so my change in distance was 14 and my change in time was two [writes 14/2], so my change was 7 ft in 1 second, so seven feet per second from two seconds to four seconds. So across those two seconds it increased seven feet for every one second

This is a case where she did not ground her unitization in terms of "over" and "up." She had a formula for resolving this task and followed it, and as she saw a comparison of changes she referred to this as a form of rate of change. It is because of this mention that I connected speed to rate of change in Figure 20. Although she was thinking in terms of what she calls a rate, this sort of comparison is clearly not her primary way of thinking about rate of change.

This task concluded by my asking why one uses division to calculate average rate of change. Her response indicated that she hoped students would somehow see the unitization in the ratio, though the reason for that was unclear.

Excerpt 107

1. Int: So what if somebody asks why do I divide?
2. Mary: …Well what is speed to you? Give me an example of speed. If you're going 50 mph what does that mean? For every 50 miles I go one mile [difficult to hear on tape, she may have said "one more hour"] so I went six feet in three seconds, so how far, how many

> feet did I go in one second, hopefully help them see that you would divide it.

Overall, then, Mary's completion of the average rate of change tasks provided greater clarity to the developing concept map. She gave no indication that she understood how an average rate of change was an average. She could employ a unitization procedure to complete the tasks. The closest she came to describing her thinking in situations where she did not have a ready formula involved a form of graphic unitization. This unitization, though similar to how she handled constant rate, was different as she could not explain the role of the points between the endpoints.

Mary: Interview 2, Change Component

In this component of the analysis we will see how Mary primarily thought about changing rate of change using a graphical way of thinking. We will also see how her "flatter-slower-smaller, steeper-faster-larger" thinking served to confound her covariational thinking.

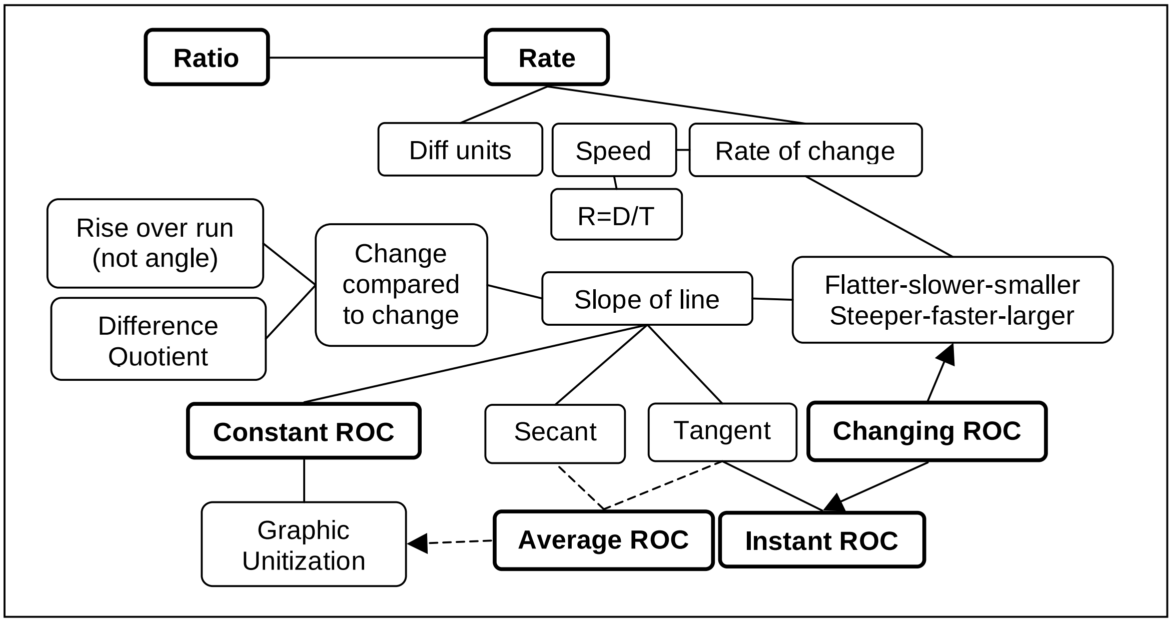


Figure 22. Mary, interview 2, final map

In Task 2, the decreasing at an increasing rate task, Mary was able to recall the PLC discussion in which we addressed the idea of changing rate. Most interestingly, though, is how she was not comfortable with what she believed to be the correct approach.

Excerpt 108

1. Mary: The decreasing rate is what confused me this year and I taught it wrong and I understand the definition and why they say that this is decreasing at an increasing rate and I understand what they are, because your rate of change or your slope is getting larger and larger because it's becoming closer to zero, its starting from the negative and growing larger .

2. Int: So you remembered how you answered it before?

3. Mary: Yeah, and I taught it wrong [laughter]. I mean I don't know how I answered it in the interview before. How I understood it before, it is that, wait, yeah, I understood it before as decreasing at a decreasing rate because the rate of change was changing less and less and less and I wasn't taking into account the sign and I still think my way is better [laughter].

As she could see a numerical justification for the alternative response, she admitted to teaching it wrong. More importantly, for some reason she still believed that her way was better. Her thinking involved a graphical consideration of slope, and steepness in particular. It is surprising that she held this view as she was able to continue describing two other ways of thinking to support the other response in Excerpt 109.

Excerpt 109

1. Int: So which student do you think is correct?

2. Mary: Mathematically correct [laughter] it is true that it is decreasing at an increasing rate as it is concave up, and that helps you with your increasing rate.

3. Int: How's that?

4. Mary: That's just the visual, that's what Becky ended up saying it's just a summary that when it is concave up its at an increasing rate because anything that's concave up, I can't think of the other,

increasing [drawing increasing concave up sketch], that one also is increasing at an increasing rate because if it is concave up its an increasing rate, this one is very clear to me its getting larger and larger and larger but like I said before it's just because the slope at any tangent points is, this [pointing to tangent at right end of curve] is a greater negative number it's to the right on the number line I don't know how to explain it but the slope would be [marking on given curve] maybe negative four, for this maybe negative two, and so it's at an increasing rate.

Mary remembered that if a function is concave up then it has an increasing rate. She could even make sense of that idea numerically. Yet, as she said in Excerpt 108, she thought her way of thinking was better. For changing rate of change, Mary did not consider the tandem relation of the variables, or the dynamic nature of the function, or a comparison of changes. Rather, she coordinated successive estimated values of steepness of tangents. This is how Mary thought about instantaneous rate.

Excerpt 110

1. Int: What is the meaning of the second use of decreasing, like right there? What is the meaning of the second use in that situation?
2. Mary: It means that the rate of change is decreasing that the slope again at each of those tangent points is… is getting smaller in this case the slope at each of those points is getting smaller.

3. Int: What is the meaning of the first usage of the word decreasing? that one right there.

4. Mary: As your independent variable is increasing so as you are moving in this direction [moving pen horizontally under graph] the dependent is getting smaller [tracing pen down curve itself].

In Task 5, the container task, Mary's way of thinking connecting steeper to faster served to undermine her solution strategy. In Excerpt 111 she began by using language to summarize the situation.

Excerpt 111

1. Mary: … They first need to focus that we are comparing height and volume, so that as the height, to me they need to understand that we want to look at the height at equal intervals [marks off equal intervals across height axis] and what's happening to the volume. so during the first interval the volume increased a lot, during the second interval the same, if this was one inch [labeling interval, not tick mark, on height axis as one inch] and the next inch of height that the volume increased a lot less, and so on, that volume increases at a decreasing rate.

She began her approach by making equally spaced marks on the horizontal (height) axis of the graph. The language she used appeared correct; during the first interval the volume increased more than it did during the second.

In going through the choices and reflecting on how she would answer the question, her language made a slight change.

Excerpt 112

1. Mary: I knew that one [*A*] would be wrong and this one [crosses out choice *B*] I kind of see it as the height. Take this idea and do it here [drawing horizontal lines across container] it would take a lot more water to fill the first one and then the second one so that it's going to take longer [pointing to widest interval] it's faster as you go up because it's going to take less water to fill that next height. So it wouldn't be this one it would be like that [sketching increasing concave up curve] and then this one [*C*] is the opposite that I just like to picture the height, look at it equal intervals and that it would take less water at first and so the volume would be increasing and then decrease [following given curve].

Figure 23. Mary, interview 2, task 5

She indicated that it would take a lot more water to fill the first interval for bottle *B*, which matched what she said earlier when reflecting on the graph. She then claimed it was going to take “longer” which led to faster “as you go up.” For Mary, faster tied to steeper and, in this case, concave up. She sketched the concave up function to the right of the given graph and dismissed option *B*, though she essentially described both the graph and the bottle using the same language. Her use of the idea of “faster” overpowered how she was thinking through the task. This led her to confusion in the coordination of variables.

In Task 6, the ladder task, Mary constructed the diagram shown in Figure 24. In it she constructed a number of ladder positions and drew a smooth curve across the tops of these positions. In Excerpt 113 she struggled with her interpretation of what she had drawn. She had no way to make sense of her diagram numerically, graphically, or formulaically.

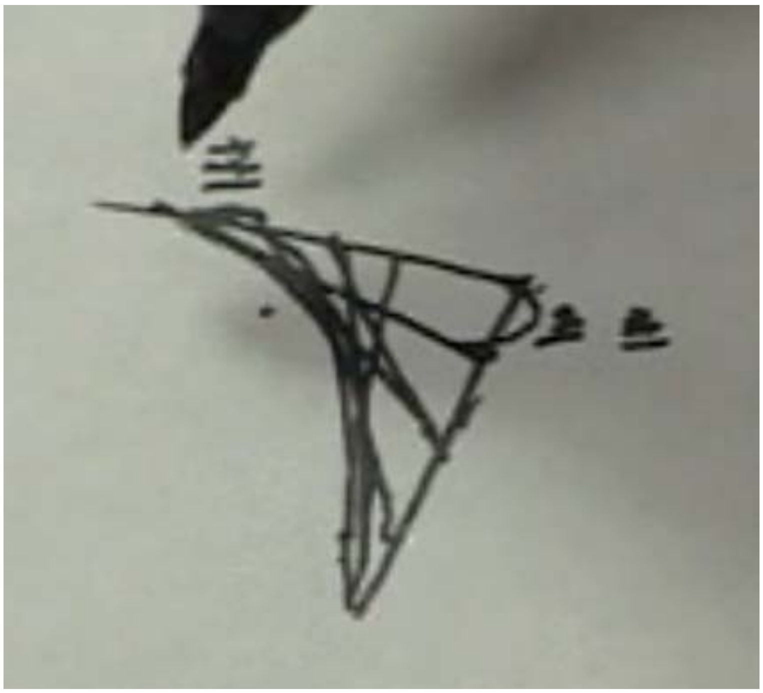


Figure 24. Mary, interview 2, task 6

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Excerpt 113

1. Int: So I hear you saying there is perhaps some significance to the shape of that?

2. Mary: Yes because the ladder would be along it the point of tangency along it, that’s what I’m picturing, and so that the rate of change…[talking to self] so the rate of change here [top part of curve] is greater than the rate of change down here [lower part of curve]

3. Int: So the rate of change up here versus, explain that a little bit more.

4. Mary: The point of tangent, again I'm talking about, that's a much greater slope of this line, so if I'm, I don't know, something's missing,

5. Int: What would you say are the units when you say rate of change here?

6. Mary: Well that's what I'm, why I just went back… [to self] that's why I'm mixing speed with distance that's why, the distance is related to speed but, …[writes "*D=rt*"]… going in a constant speed, I don't know.

In paragraph two of the above excerpt Mary worked in her mode of flatter-slower-smaller, steeper-faster-larger. The steeper component had a greater rate of change. She briefly considered returning to speed in the form of *D=rt*, but ended up not being able to find a method of sense-making that was stronger than steeper meaning faster.

The same sort of flatter-slower-smaller, steeper-faster-larger thinking showed up in Mary's solution to Task 12, the racetrack question. As her solution developed in stages on the same graph, I will include progressive snapshots from the video along with the explanation. To begin, Mary considered lengths of secants and plotted four points, along with the origin, as shown in Figure 25.

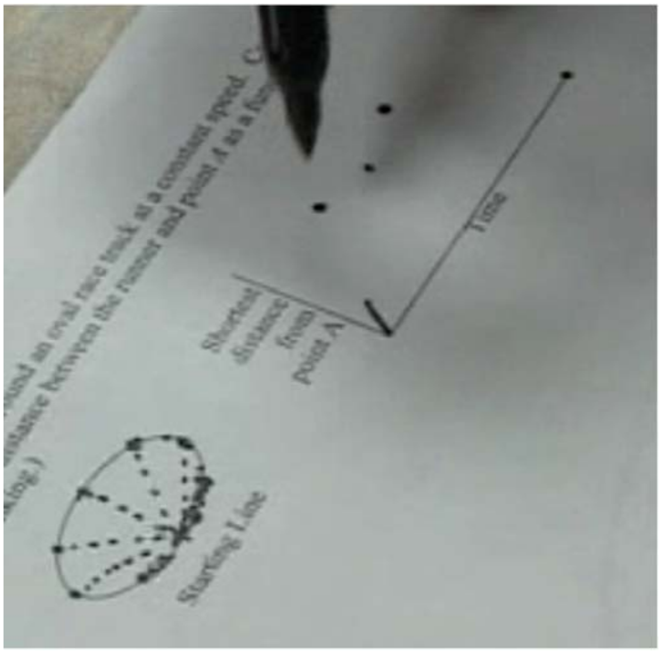


Figure 25. Mary, interview 2, task 12a

After this, she worked to connect the points. She felt that the component departing form the origin should be concave up, as it was “increasing more quickly at first.” She then constructed the rest of the graph and used the language attached to each component as shown in Figure 26.

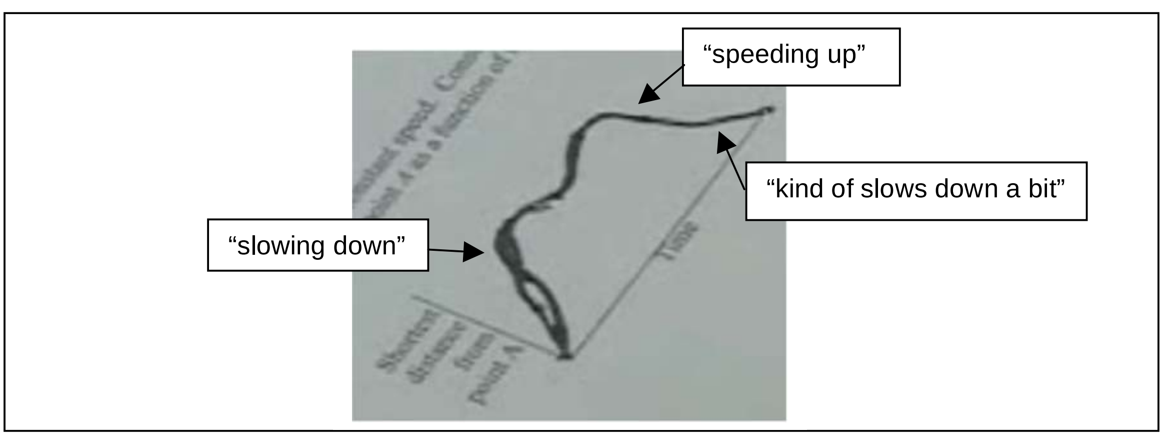


Figure 26. Mary, interview 2, task 12b

In the middle of the graph she only referred to the components as decreasing and increasing, with no dynamic sort of language. In talking through the question one more time, she moved her pen around the track and noted as she neared the starting line that “it increases faster and faster.” For Mary, this was a cue for steeper and resulted in a portion that is concave down. This compelled her to make a concavity correction for the right tail of the graph, as shown in figure Figure 27. Upon completion of that tail, she treated the graph symmetrically and did the same to the left end of the graph as well, though she gave no evidence for how this interacted with a change in her original thinking.

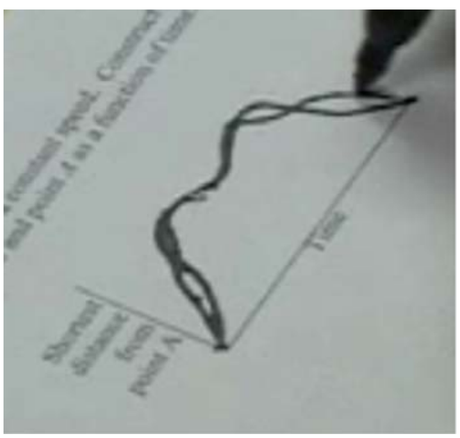


Figure 27. Mary, interview 2, task 12c

The concavity change resulted in a graph with cusps, which she seemed compelled to smooth out. This resulted in the graph in Figure 28, which she described as though it “seems really complicated for that racetrack.”

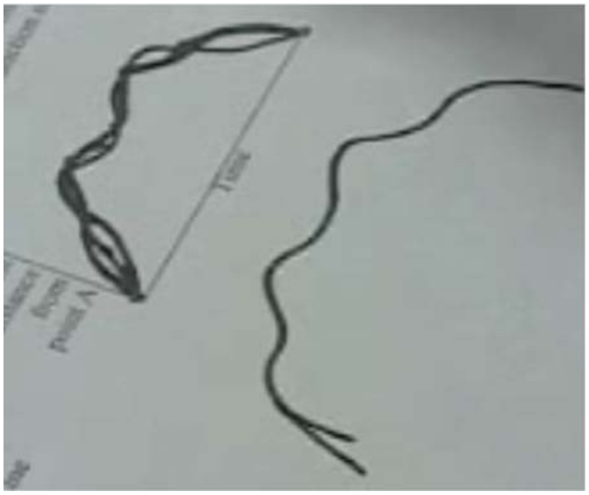


Figure 28. Mary, interview 2, task 12d

For Mary there was no connection to rate of change except through her use of the words faster or slower and their connections to the steepness and concavity of the graph.

Mary evidenced that she handled changing rate of change tasks not from a constant or average rate of change perspective, nor from a numeric perspective for support. Rather, she graphically approached these tasks by thinking about rate of change though steepness. To her, this is an understanding that she sees as sufficiently coherent to employ it as her primary way of thinking about rate of change.

Summary of Mary

Mary primarily held one way of thinking about rate: she considered rate as a measurement of steepness and the steeper the graph appeared the faster the graph was going. With the exception of Task 9, Mary viewed all rate of change tasks graphically.

For Mary, the notion of slope had to do with a graphical comparison remaining invariant. That is, the pairing of the horizontal and vertical components continued in the same graphical direction (see Excerpt 90, p. 139). The data evidenced that Mary held to an action view of function (see Excerpt 63, p. 113). She appeared to view the graph as the actual function rather than as a representation of a dynamic relationship between two varying quantities (see Excerpt 69, p. 118, and Excerpt 101, p. 149. In Interview 2, she focused on the vertical and horizontal changes as a sort of pictorial comparison of those changes (see Excerpt 100, p. 148). This thinking helped me to make sense of her thinking about constant rate of change in terms of graphic unitization. That graphic unitization may indicate a connection between her thinking about average and constant rate, but she did not make that connection evident.

The data supports that Mary did not have a meaningful way to think about average rate; nor did she possess a structure for reasoning about average rate of change situations. She expressed trouble with ignoring the middle of the function. In the first interview, she had no coherent definition of average rate (e.g. see Excerpt 74, p. 122).

In Interview 2, Mary still evidenced no coherent definition of average rate, but she did show a potential connection in thinking between the average rate tasks and the linear tasks. Specifically, although she was still uncomfortable ignoring the middle, she used her form of graphic unitization to make some sense of rate in the average rate tasks (see Excerpt 100, p. 148).

In neither interview did Mary approach Task 9 graphically. I claim that this was due to the fact that the task provided her with all of the necessary parts to make her formula $d = r\,t$ work (distance and time were provided in a task asking about speed).

In the changing rate tasks we saw evidence for Mary's primary way of thinking about rate. In both of the interviews Mary employed a way of thinking that I called "flatter-slower-smaller, steeper-faster-larger." Although she sometimes used language to appropriately describe a task, her thinking in terms of steepness could overpower what she had said (see Excerpt 112, p. 161).

So, in contrast to Becky who employed multiple ways of thinking to help her understand rate, Mary employed a single way of thinking: graphical steepness founded on "flatter-slower-smaller, steeper-faster-larger." Mary held to that way of thinking even though she knew that other ways were better (see Excerpt 108, p. 157).

Overall, as the fully developed maps illustrate (Figures 16 and 22), most of Mary's ways of thinking remained stable between the two interviews.

CHAPTER 7: RESULTS AND ANALYSIS OF PEGGY

Introduction to Peggy

Peggy had a bachelor's degree in mathematics and had taught mathematics for fourteen years prior to the semester of this study. In recent years, Peggy had been teaching classes below the calculus level. Her hope in entering the TPCC was to increase her own confidence in doing mathematics in order to improve her teaching. Following her first semester in the TPCC, Peggy stated that she had indeed changed in a significant way. This change was to focus less on "things that don't feel so much like concepts they feel more just like practice." She now felt more like a teacher than a trainer. An additional significant change, however, was that she changed her overall attitude to teaching. In Excerpt 114 she spoke of how she now viewed her job as positive and challenging.

Excerpt 114

1. Int:	How useful has it been to your teaching practice?
2. Peggy:	It's been useful to the point where a year ago I was wondering if I was even going to be able to stay in this profession until retirement, seriously, I just can't stomach this anymore. What I'm doing is just so unfulfilling and so, you know, I have never really felt that there is strong support from our culture there has never been that support, we say we support teachers we say we support

> education but, really, when you look at the actions we are just saying that we do because the actions don't show that, and for many years I thought no, I'm a teacher, I love teaching and then I was reaching a point where I just thought, you know, I don't know if I love it that much. This just doesn't feel like what I want to keep doing, and now I really am back to where, no, I'm back to this is challenging. There are certain things that I know are right and there are certain things I know I'm not going to be doing and I, I just feel a little bit more, I think I can see myself now standing up for what I know is right where for about three years I just thought this AIMS, whatever, just tell me what I'm going to need to do next. You just tell me I'll go in and do it. I'll teach it just like you tell me to. And now I'm a little bit more kind of I'm not going to do your thing, whoever, you know, this is the way I'm going to do it.

Peggy: Interview 1

Peggy: Interview 1, Definition Component

In the definition component of the first interview, Peggy basically considered rate to be a comparison. What we will see throughout the interviews, however, is that she had compartmentalized different ideas of that comparison and she had difficulty when the different forms interacted with each other.

In this section on definitions, Peggy described her basic definitions for rate, ratio, and rate of change. For Peggy, a rate is a ratio with different units.

However, Peggy often thought of rate in terms of comparisons. Rate of change involved a ratio comparison of changes while changing rate of change involved a sequential comparison of imagined values. Peggy attempted to link instantaneous rate of change to the slope of a tangent, and she kept instantaneous rate of change distinct from a ratio comparison of values. Peggy, like Mary, did not have a meaningful way to operate with average rate of change. I show my pictorial interpretation of the connections of her ideas in Figure 29.

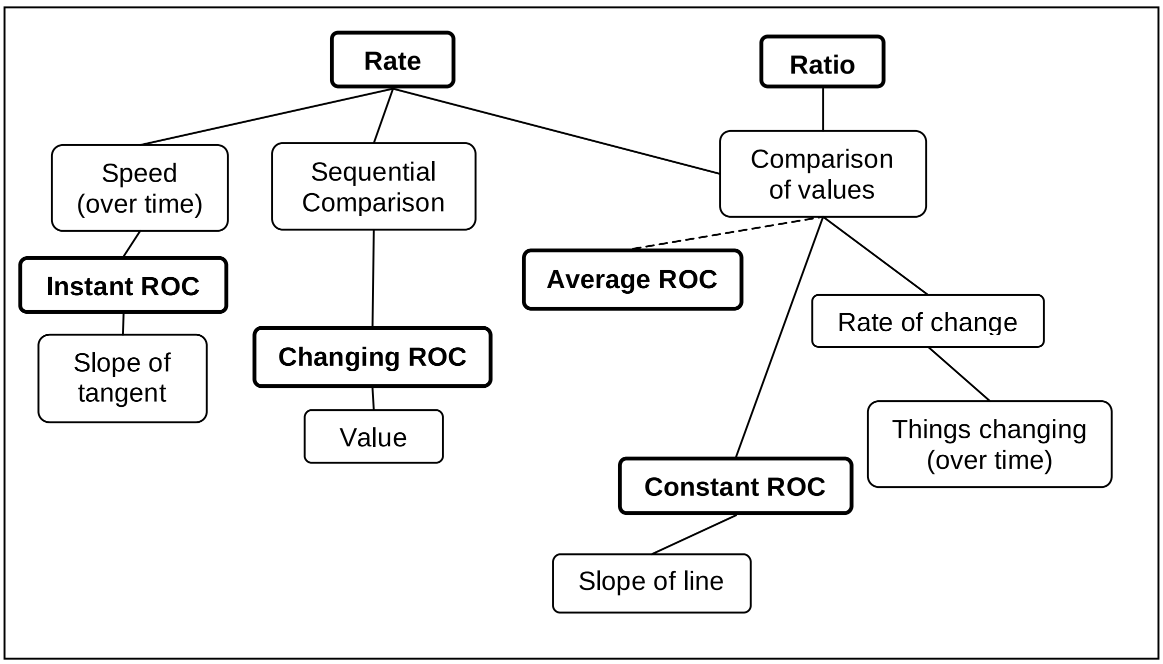


Figure 29. Peggy, interview 1, definition component map

In Excerpt 115 from Task 14 Peggy explained her distinctions between rate and ratio.

Excerpt 115

1. Peggy: A rate is a ratio which is a comparison of two quantities and what distinguishes a rate from what…what sets rate apart from just any

old ratio is that we are comparing quantities but those quantities have different units, so that leads into where one type of quantity is affected by what happens to the other type of quantity. We look at unit rate, where whatever comparison we are making through division you can divide and get the unit rate, and then that also leads into rate of change looking at the rate at which something is making a change, with linear functions we tie that into steepness on the graph, how fast is it changing or the rate of change the bigger the rate of change or the higher the rate of change or the, I guess faster something is changing, the steeper it appears on the graph

In indicating that one variable is affected by the other, we also see an implication of a dependence type of relationship between the variables. Peggy also connected her thinking of steeper, in a graphical sense, to faster. However, here she limited that thinking to linear functions. As she worked through Task 14 she reflected on her earlier work in Task 11 (finding how quickly a function changes given a table). In Excerpt 116 her reflection included a dependency between the variables and a comparison restricted to linear functions.

Excerpt 116

1. Peggy: Yeah, rate of change, that how quickly is something, there's that word quickly [laughter] how quickly is something changing with each comparing the, hmm, okay so in that one problem I guess

that would mean that if the change from negative two to negative one causes [writes down ratio (103-83)/(-2-(-1))=20/-1=-20] …so if something is changing 20 units or decreasing 20 units, the rates of change there is different from [writes down ratio (83-54)/(-1-0)=29/-1=-29]. So these values aren't from a linear model it's not from a table of order pairs that result in a linear function. But I have a sense that there's the rate of change, there's still a change taking place and so the rate of change is happening more quickly from the -1 to 0 then it is for the negative 1 values, or from the negative two to the negative 1, I'm not sure how to really work in that area and have confidence in it, but what I know of rate of change is mostly from a linear perspective.

Her trouble with average rate of change may be due to her restriction of her ratio comparison to linear functions. In Excerpt 117 (still from Task 4) she evidenced that she had no operational understanding of average rate of change. Peggy applied rate of change in linear tasks and she applied average rate of change in non-linear tasks.

Excerpt 117

1. Int: So what do we mean by an average rate of change?
2. Peggy: … Boy, that's something that is difficult for me because I know, I know about averages, averaging things, but…you know if there is, suppose you have decrease and then some increase and then

some more decrease overall its decreasing but [sketches decreasing-increasing-decreasing function] there's something in my mind that says well you can't just average the end points, it seems too simple, there has to be something that takes a look at well, it's decreasing but then it increased and then it decreased again, that has to be averaged in and I'm not sure how to do that, so I'm not real sure how to respond that.

In Task 14, Peggy also spoke (Excerpt 118, below) of how graphical aspects and numerical results were not the primary importance of slope.

Excerpt 118

1. Peggy: In terms of slope, but it's not always about, you know, with linear functions it's not just about graphing a line, it's not just about finding a slope number that there's, there are all these other meanings that go along with linear functions that don't necessarily have to do with graphing lines.

The above excerpt is interesting in that it contrasts Peggy's thinking of slope to Becky and Mary. For Mary, slope was about a graph while for Becky slope was about a computation. In Task 15, Peggy made clear that she did not see a strong connection between average rate of change and the slope of a secant. She indicated, however, that there might be some sort of connection. She began the task by sketching a concave-down parabola in Quadrant 1.

Task 15. How are average rate of change, average rate of change of a function over an interval, secant to a graph and tangent to a graph related?

Excerpt 119

1. Peggy: … So I have an interval [between *x*=1 and *x*=3], I can look at the change that is taking place from the end points of the interval [tracing the curve, marking intersection points of vertical lines at *x*=1 and *x*=3 and sketched parabola] and I know that the function is increasing and perhaps this is even a maximum point, I don't, if I'm only looking at the interval I'm not sure because I don't know what's happening next, or what's happening to the right of that point…[rereads question] it seems like secant to a graph maybe there's some connection there [draws secant between x=1 and x=3]…because you have, you have the idea that secant intersects with two values and in this case my secant line is intersecting with two values, the end points of the domain…

As she continued her response to Task 15 she spoke of instantaneous rate of change as related to the tangent. For her, the slope of a tangent gives a steepness that connects to speed in some contexts. She did not describe this speed as a comparison of changes. Instead, in Excerpt 120 she mentioned another sort of comparison to think about changing rate of change: a sequential comparison.

Excerpt 120

1. Peggy: Looking at the tangent to the graph it's like a little snapshot of what's happening just at that value in the domain for that independent value right there there's some little point and the tangent line that belongs to that point has a steepness, so there's slope. For something like the tangent line that runs through the maximum point [sketching another upside-down parabola with horizontal tangent at maximum] the slope would be zero, I remember that's how you tell that that's a max or a min…what else? And it seems like it has to do with velocity…there's increasing, the function is increasing and depending on the slope can kind of give you insight to the slope of that tangent line for any point on that function can give you insight to how much it is increasing or you could compare the increase at one point with the increase at another point. And if this were about …so if this were about speed, velocity [pointing to a point on the second parabola to the left of the maximum] I'm kind of thinking that this [moving pen over tangent drawn at that point] would represent accelerating, that the speed is accelerating. And then at this point, the maximum point, there is no velocity, it's like a moment where

> at that point it is not increasing but it is not decreasing either, so the only thing you could be doing is having no value at all so it would be zero.

She clearly has some confusion in how position, velocity, and acceleration interrelate.

Peggy: Interview 1, Linear Component

We have seen that Peggy applied a ratio comparison in linear contexts. In this component we will see a clarification of her thinking about constant rate of change and how it involved constant patterns. In the tasks for this component she used horizontal and vertical comparisons. We will also see her emphasis on thinking in a context as being of greater importance than thinking graphically. Figure 30 illustrates these ideas.

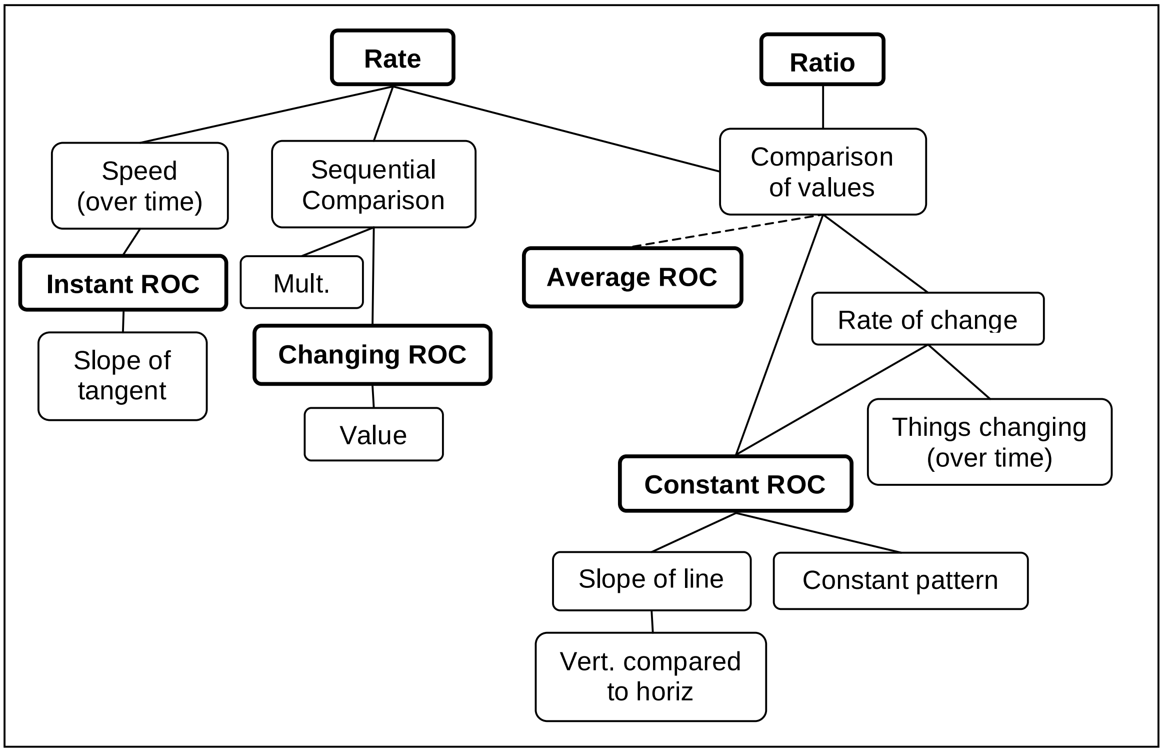


Figure 30. Peggy, interview 1, linear component map

In Task 1 Peggy described *m* as a rate and as a comparison.

Excerpt 121

1. Int: So what is the mathematical meaning of *m* in the equation *y=mx+b?*

2. Peggy: It's a rate and we talk about it as slope, but it really is a rate because it's not always about slope. It's a comparison of two numbers or two quantities.

In Task 4, the four students discussing slope, she continued using comparison as a means of thinking about rate of change. In Excerpt 122 she mentioned comparisons numerous times.

Excerpt 122

1. Peggy: Well the meaning of slope in a formula sense, we look at the change in the *y*'s and we compare that to the change in the *x*'s. and then if you look at a line that has steepness compared to a horizontal it does form an angle, I guess sometimes I have a hard time with what is an angle, is it the stuff inside then also to have this sense of a sweeping out motion that there's some arm this terminal arm that sweeps out and lands somewhere, so in a way I could see students say angle of line because I myself see it that way compared to the horizontal it forms an angle. And then rate of change again that's what the formula is about is comparing how much is, how much did *y* change compared to how much did your vertical position change compared to how much your horizontal position changes.

In linear situations she wished for students to understand that the rate of change is constant. Specifically, this meant that the difference quotient will always lead to the same answer.

Excerpt 123

1. Int: What would you like students to understand about slope in a linear context?
2. Peggy: That the rate of change is, is like a constant thing. When I compare these changes in the *y* with the changes in *x* [pointing to

the difference quotient] that there is this constant pattern, or constant number that keeps happening and that's what causes it to be linear. That there is no, it can be increasing, it can be decreasing or it can be zero. That there's no increasing at an increasing rate type stuff going on or decreasing at an increasing rate, none of that's taking place that it's a constant type of comparison.

In considering student understandings, Peggy clarified that the first student, the one who provided the difference quotient, needed to also understand the context, the comparison, and that the formula was of less importance.

Excerpt 124

1. Peggy: …In the first one they are understanding that there's a formula that if you have two points on a line the way we work with it in the classroom, if you're given two points then you can plug it into a formula and you can find a slope. I think what I'd like them to do is be able to know that in a real contextual, application way or giving it context that comparing changes in the dependent variable with the changes in the corresponding independent variable…you can use this formula also.

Peggy seemed more comfortable with the third student, the one who responded with rate of change. In the case of Excerpt 125, Peggy tied rate of change more directly to a context.

Excerpt 125

1. Peggy: And then the third, rate of change to me implies that there's more of a sense of having application or having a context. It comes from something from data from something real from something that connects with connects with some field in science or whatever.

Peggy: Interview 1, Average Rate of Change Component

Peggy's responses in this component clarify and support many of my earlier proposals for her ways of thinking. In this component, Peggy will evidence her inability to operate with average rate of change. In particular, she found working with the function between the endpoints to be a challenge. Additionally, Peggy's thinking of rate as a comparison of sequential values will become clearer.

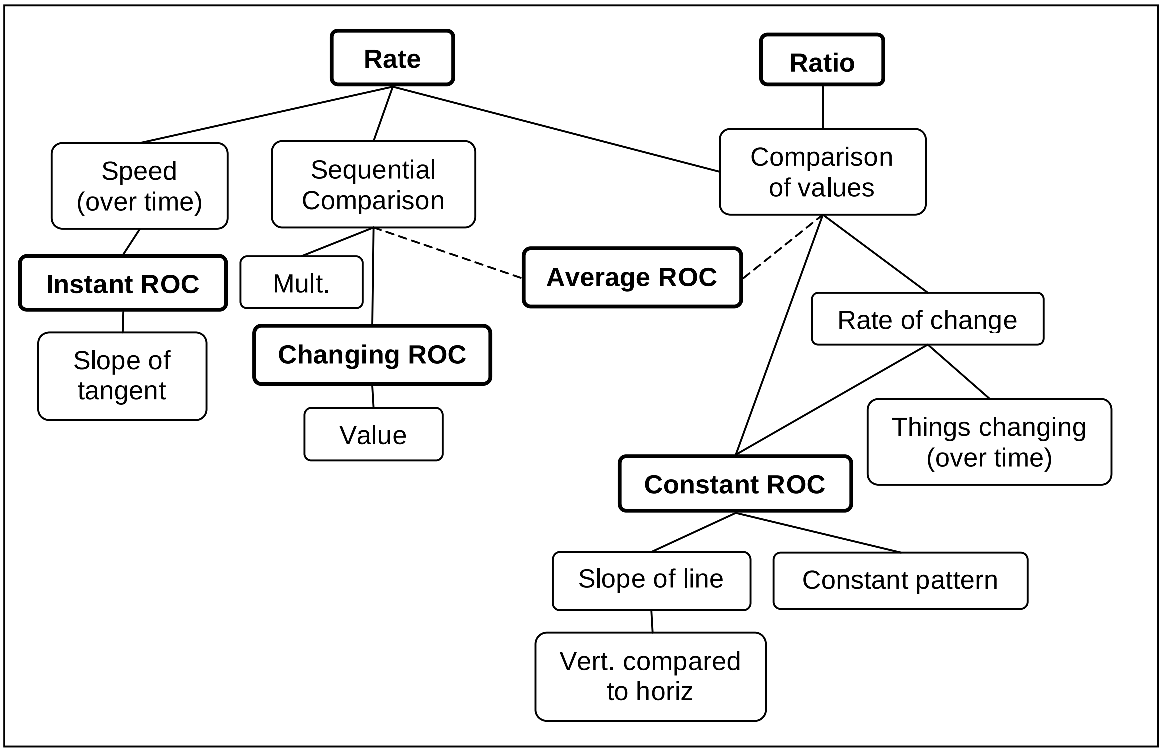


Figure 31. Peggy, interview 1, average component map

When the interviewer asked Peggy about developing a lesson plan for exponential functions (Task 7), she stated that she did not view the graph as the function itself. Rather, she generally used graphs when she did not have a clear understanding.

Excerpt 126

1. Peggy: We do get into the graph probably not as much in this area as I rely on the graphs more in areas where I really don't get the concept, and I try to get as much out of the graph so that partly for myself and for my students, because I assume if I'm really weak in this concept maybe the graph can help us know more. I use graphing in the exponential, with the exponential function probably

as the last thing. Usually we talk about the graph and then we go back over some of the types of problems we've been doing and we make connections. Well what's happening on the y axis? Where do you see that in the equation? Where is that in the real problem, in the application problem?

Peggy had trouble working through the three similar average rate of change tasks (Tasks 3, 9 and 11). In Task 3, Peggy saw the sequential values doubling and thought about a multiplicative sequential comparison. She became stuck in that sort of thinking and felt there was not enough information to answer the question. In response to how she would guide students, she really did not know what to say.

Task 3. You provide both a table and a graph and ask a student to find how fast the function is changing between *x*=3 and *x*=5. The student responds 18...

Excerpt 127

1. Peggy: I want to focus on the fact that the values for the dependent variable are doubling… and I'm not even sure myself but I'm thinking if we're claiming that *f(x)* represents [whispers speed], I don't think I've ever worked on exponential function where, with

problems dealing with speed, with velocity. So I honestly have to say I'm not sure. I would have to take this to Becky.

Peggy also had trouble in Task 9 (Excerpt 128, below) as she was not sure how to reconcile that "distance equals speed times time" as the question posed distance defined as a quadratic. In describing what students would find complex about the task, she indicated that she found the ratio comparison complex. She did not know how to apply her ratio comparison way of thinking to the task.

Task 9. Your objective for the day is to develop a lesson on average speed. Specifically, your students need to understand how to find the average speed of a car over a period of time. They should be able to answer the question, "What is a car's average speed during the period from 2 seconds after it starts to 4 seconds after it starts, where it travels *s* feet in *t* seconds and *s* is given by $s = t^2 + t$ (with *t* measured in seconds)?

Excerpt 128

1. Peggy: I don't know how to explain it. It's like something travels *s* feet in *t* seconds, I think my students who are juniors have had some exposure to, they drive, they know 55 mph means you are traveling this many feet or miles in this many seconds or hours. They kind of get that's a ratio comparison of feet and seconds.

> And then how do you connect that with that *s* is *t* squared plus *t* I think that's difficult that's difficult for me to think through.

In her last statement for this task she stated, "I honestly don't know. I know I would get some help."

The interview time ran out as Peggy worked Task 11. The interviewer, therefore, postponed part of the task until they met again three days later. Unfortunately, during the first eleven minutes of the second session the microphone did not work. However, it appears that those first minutes of the interview repeated the portions of the task discussed during the first interview session.

She began Task 11 demonstrating that student understanding involves something more than slope.

Task 11. You are creating a lesson on how to find how quickly a function changes, given a table. For instance, given the table, students need to be able to compute how quickly the function *f* changes from $x = -2$ to $x = 3$.

x	*f(x)*
-2	103
-1	83
0	54
1	50
2	39
3	5

Excerpt 129

1. Int: So what is mathematically important for students to understand in approaching his question?
2. Peggy: …They need to know that, I think they need know that this isn't linear, they need to have the skills to know not to assume that this is just that there is some way of finding a slope or rate of change they need to know that we got to move beyond that.

I am not sure what "beyond that" might mean to Peggy. What we do know is that Peggy, as indicated earlier, connected slope to linear functions. She later indicated that what was happening between the points was significant. In Excerpt 130 she spoke more about the region between the endpoints.

Excerpt 130

1. Peggy: And at, is it just from two to three can I just look at from these two, the beginning and the end, and not care about what's in between? If we're just talking about from here to here how quickly changes [from negative two to positive three] … how quickly [reflecting to herself], OK time where does the time fit into this? Is time the *x* where it's changing from negative two timewise to three or is *f(x)* the dependent variable? So assuming that the independent variable is time, how quickly it changes [again reflecting], yeah I don't know it's just really, depends on where you see time in this where does it fit in

2. Int: So that would be in your lesson, time?

3. Peggy: What do you mean by how quickly, how fast, how many minutes? How many seconds? I'm thinking time so, I guess.

Peggy did not resolve the issue about the region in-between the endpoints. Unfortunately she had to stop the interview at this point.

As the audio returned during the continuation session Peggy appeared to be in the midst of trying to establish a context in which to situate the task. She was, however, unable to come up with one. In the following excerpt, Peggy acknowledged she was still having difficulty ignoring the points between the endpoints.

Excerpt 131

1. Int: Describe student approaches to solving this that would be acceptable to the original question. And why would they be acceptable? How would you want your students to solve it?

2. Peggy: I would want them to suffer through it just the way you've seen me suffer through it, I would want them to at least think of everything they do know about this. What do I know about this table? I would want for them to just at least put out everything they do know and even if they come up with a dead end I would want them to at least try and question, given that there is no application here I don't know why we're only looking from negative two to positive three and I don't know if my students would even care to know, I don't

think that that would even occur to them. And then I'm also wondering is, I connected this with a graph but is there any reason why I should even connect that? I mean these could just be separate pieces of data with nothing in between or no reason to justify having it be continuous, so.

3. Int: So what difficulties do you think your students will encounter?

4. Peggy: I don't think they would have as much difficulty with it as I am having because they would be satisfied to say it changed from 103 to five, down to five, decreased. And they see that as 98 compared to a horizontal change of five and that would be good with them. To them they figure out that number and that would be it.

This last paragraph in the above excerpt is informative. Peggy tried to make sense of average rate of change, yet she could not connect it to any of her existing ways of thinking. She could not easily justify using thinking reserved for linear functions to describe properties of nonlinear functions. Her students, however, could simply compare the changes and move on. She later indicated that such a solution would leave the meaning "hanging in the air."

In Task 8 (the parabola task) Peggy thought the student performed the calculation correctly. She did not, however, have any way to explain why the computation had mathematical validity.

Excerpt 132

1. Int: Is there mathematical validity in what the student observes, and explain.
2. Peggy: Yes. Because it doesn't matter if it's on a parabola it doesn't matter, you know…if we are just simply comparing from one location to another…we can talk about slope, it really doesn't matter that it's a parabola…the rate of change from this point to this point is 0 [pointing to each intersection point on parabola]
3. Int: Is there any mathematical validity in applying the slope formula to a parabola?
4. Peggy: I don't know…I found myself pointing out to students that now that we are working with quadratics we are not going to be taking the slope anymore we won't be finding that… That the coefficient of the *x* in the linear function formula represented slope, but now were working with a very different kind of function and so the coefficients have different meaning. I've never worked with slope formula at all with quadratics.

Slope, for Peggy, once again only had meaning when restricted to linear functions.

In Excerpt 133, Peggy explained the meaning of the slope formula again in terms of a comparison of changes. When asked about the meaning of the slope formula in Task 8 she clarified there was more that she needed to know.

Excerpt 133

1. Int: What does it mean to understand the slope formula?

2. Peggy: The slope formula? To understand the slope formula to me is to know that there is, you are comparing change with change there is a change in…[asks for, and has question repeated]…it means to understand rate of change…you know we, what I use, how I use it in my teaching is basically for steepness but I know there's a lot more that is going to be learned I just don't go there. I know that in taking derivatives and slope that's a connection but I have not done any calculus ever in my teaching.

Peggy did not meaningfully apply the slope formula, or rate of change, to average rate of change situations.

Peggy: Interview 1, Change Component

In the changing rate tasks, Peggy did not operate from her ratio comparison way of thinking. In this component we will see occasions where she initially worked in that direction, but that she was unable to follow her thinking to the end. Instead, Peggy acted with respect to her other type of comparison thinking: a sequential comparison of values. In this case she often set the comparison in a consideration of equal intervals. The only change in the map for this component (shown in Figure 32) is a possible loose connection in Peggy's thinking about average rate and changing rate.

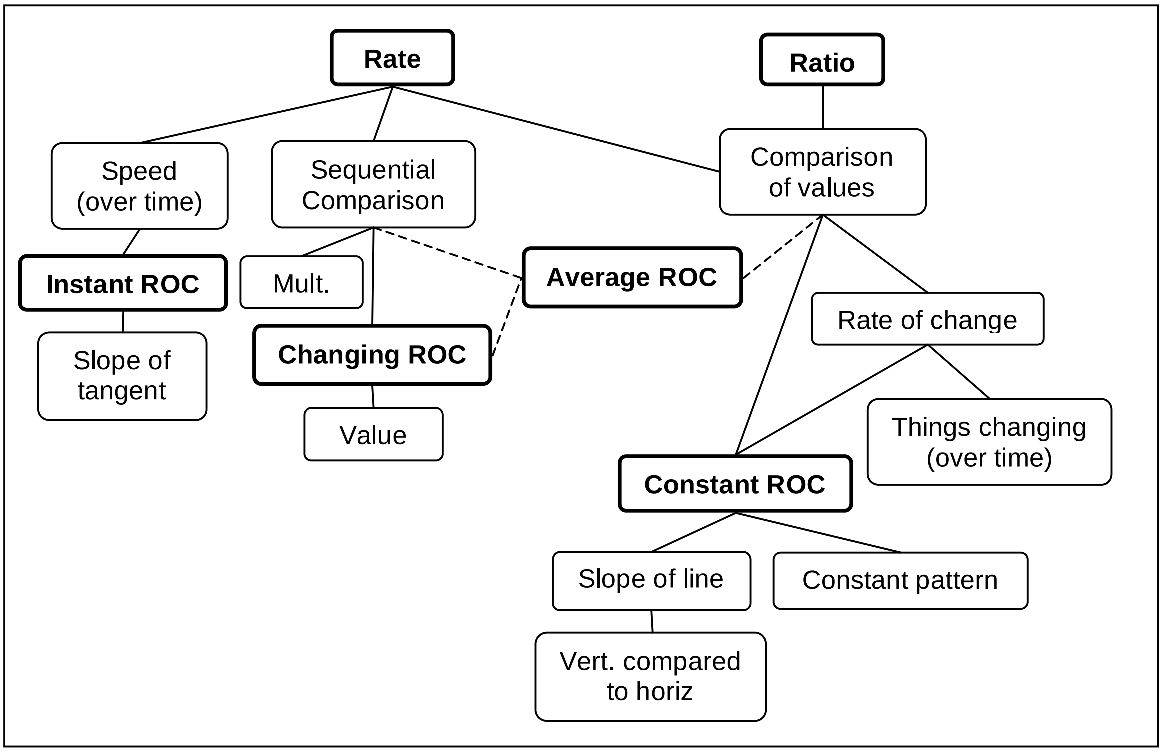
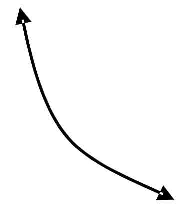


Figure 32. Peggy, interview 1, final map

Peggy evidenced a different sort of language in the changing rate tasks. For example, in Task 2 she mentioned a sort of coordination of variables to describe ideas relevant to understanding the language in the question.

Task 2. Two students are having a debate about a function whose graph looks like the figure below. One student declares the function is "decreasing at a decreasing rate" while the other says it is "decreasing at an increasing rate…



Excerpt 134

1. Peggy: I think we have had a tendency to kind of simplify the language we'll say something is getting bigger or its getting smaller or we'll use baby talk in a way and when we talk about increasing, something is increasing, students need to know that you are looking at what's happening with the x at *the same time* you're looking at, or you're looking at what is happening with the independent variable *at the same time* you are looking at what is happening with the dependent variable. So I think that's real key to understanding that. [Italics added]

Her language of comparison is absent in the above excerpt. She did not mention that she would compare the two variables' changes. Rather, she would consider the variables "at the same time." Her analysis sounded pictorial and her later comments support that notion.

Excerpt 135

1. Int: Can you explain your thinking?
2. Peggy: As the value for the independent is getting, is increasing, the dependent variable is drastically increasing and then as you continue on with that there's a change in the pattern [moving finger from left to right across graph] and it's not decreasing so drastically so it's decreasing at a decreasing rate.

It was in dealing with students in Excerpt 136 that her sequential comparison became clear:

Excerpt 136

1. Int: What would you have the students focus on to promote discussion and why would you have them focus on that?
2. Peggy: …I would want them to focus on the fact that because this is a function there is, you've got to keep your eye on what's happening as the independent variable is increasing what's happening and what rate is it happening, with each movement to the right what is happening to that. We can *see* its decreasing, but what rate is it decreasing at different places.

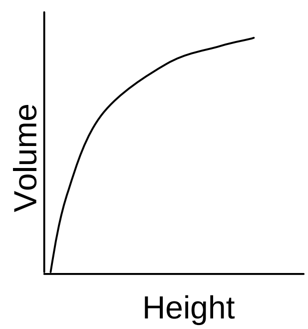
In this case, Peggy connected rate to a sequential comparison of steepness values. Her comments in the next excerpt support this claim. Of interest in Excerpt 137 is that Peggy began by trying to draw a connection between rate and comparison. She did not complete the connection.

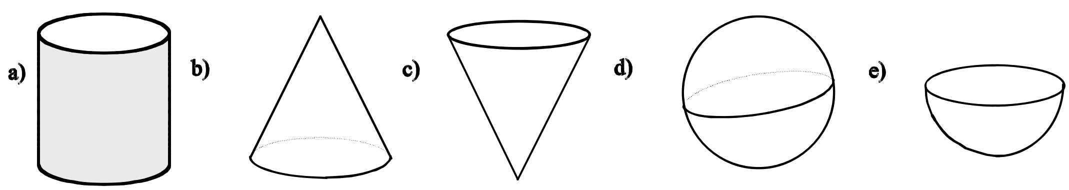
Excerpt 137

1. Peggy: It means that the rate, the comparison between the…I do not know how to say this…I kind of think of it as it's not picking up speed it's slowing down in its approach. It's decreasing so it's approaching south or its approaching downward
2. Int: By "it" you mean?
3. Peggy: But it's doing it at, [responding to interviewer] The graph or the relationship between whatever. There's no context for this so I don't know how to talk about where did this come from? All I know is this is little graph. So it kind of, kind of leaves me with just you know, we know its decreasing. We could look at certain pieces of sections of the graph and compare what's happening, it's steeper here than it is over here.

Peggy further described her connection of steepness to fastness in Task 5 (the container question). Again, she did not use her thinking of a ratio comparison of values.

Task 5. The following graph represents the volume of water as a function of the water’s height in a container. Which container goes with this graph?...





Excerpt 138

1. Peggy: OK…so first of all as the height is increasing the volume is increasing and in the beginning of this as we look at these height values its, the volume is increasing quickly, or it has a lot of steepness to it so I see it as increasing faster than it is over on this other half [moving finger from left to right across graph]

Becky made an effort to coordinate the variables, but she was unable to do so accurately. She narrowed her choice to containers *C* or *E*. In describing how a student should approach the question she tried to link back to a comparison. Once again, though, she did not know how to make that connection.

Excerpt 139

1. Int: What must a student understand to answer this question?
2. Peggy: I think they need to understand that…as the height is increasing if you want to know, if you want to be able to make a judgment it's good to use uniform segments that…you want to compare…something that's comparable…and you know I'm not even sure if that's true, I just know that's how I do it.

Time also played an important role in Peggy's thinking about rate of change. In Task 5 her thinking about time served to confound her solution. In Excerpt 140 (below), she noted, "With each increase in the height there is a decrease in the rate at which this volume is changing." She did not make sense of the task as time appeared to get in her way. In the following excerpt we see her struggle and her application of a multiplicative sequential comparison.

Excerpt 140

1. Peggy: In the second one, in *B*, I'm thinking of [sketches horizontal disks] as I fill this up… now there is this time, kind of like time gets in the way, it takes longer to get this height and I think that's the challenge there and that's what makes it abstract for me anyway is, I want time to be a part of this I'm thinking well it takes longer to fill this up and then I have to remind myself its volume. I'm putting more, adding more volume right here than I am at any other time in this filling process [pointing to lower two discs] and in fact, oh

man, so what's happening is this is the volume is increasing at a …constant rate? You guys. Go easy on me. With each increase in the height there is a decrease in the rate at which this volume is changing or increasing [still on *B*]…then there is some question in my mind about well this has, the way it is shaped, is there some pattern? There's some pattern to that, that, I kind of wonder if it comes into play because the amount of decrease is kind of like lessening by the, I don't know, by some, I want to believe there's some fractional amount like half and that half of that and half of that half of that as it to the top, but I don't know that.

She ended up having second thoughts about her initial selection of *C* or *E*, but she never directly changed her answer. Peggy could not coordinate her multiplicative sequential comparisons with her rate of change language.

In the ladder task, Task 6, Peggy attempted to make a coordination of the changes for one interval of ladder movement. She based her thinking on one interval alone and she was not able to succeed. In Excerpt 141, Peggy attempted to compare values but she could not find any mathematical meaning. She seems to have presented two different answers.

Excerpt 141

1. Peggy: I look at that and I think the speed at which the foot of the ladder is constant so the speed at which the top of the ladder is changing should also be constant. And I'm thinking that…the amount of

change for the foot of the ladder isn't going to be as much as the amount of change for the top of the ladder and its change in position, but I really don't know why, I don't really know why, how I would explain that in a mathematical way.

After this, she imagined that the top of the ladder stayed fixed to a point on the wall while the bottom of the ladder was able to swing away from the floor. Although she tried, she could not use comparison with this model to explain her overall thinking.

Excerpt 142

1. Peggy: … I'm just seeing, let's see…see I can't connect, I can't go anymore than that, I'm like OK, so where's that connecting thought about this length [one segment of vertical near top of axis] compared to that length [moves across whole horizontal axis] being bigger, but for some reason I, some reason I just can see that as being true.

Following this, the interviewer removed the task. She grabbed it back and continued to work.

Excerpt 143

1. Peggy: [she begins to read task 8] Wait a minute I'm not done with that [grabs back task 7]. So if that were true, which is still on hold for me because I want a way to prove that that's true, if that were the case then that would mean that the speed at which the bottom of

> the ladder is changing would be greater than the speed for which the top of the ladder would be changing and so, I'd have to think through that, too, because it seems that if you move the bottom the top should change at the same rate. Okay I'm ready for this one now [returns to task 8].

Even though she felt one answer was better than the other she could not mathematically resolve which was correct. That is, she could not articulate how the comparisons made sense. However, this loose comparison of changes over a single interval may be an indication of a weak connection between her changing rate thinking and her average rate thinking

Finally, in the racetrack task, Peggy seemed at home with the task as it involved distance and time. She was able to draw a reasonable graph, but she did not attend to the concavity as she seemed to simply estimate values and plot points. It appeared that the concavity of the graph occurred in how she naturally connected the points. Her response included no attention to her earlier ideas of rate of change (either as ratio comparisons, sequential comparison or steepness). She used language such as decreasing, or shorter, as one distance compared to the next, but she did not utilize language such as decreasing at a decreasing rate. Rate did not seem to play a pivotal role in her construction. When asked about the curvature she gave no mention of changing rate.

Excerpt 144

1. Peggy: The distance is, there's an increase in the distance, and then there is a decrease in the distance from point *A*. And so there has to be some curves, we lead back to no distance at all.

Peggy: Interview 2

Peggy: Interview 2, Definition Component

In the definition component of Interview 2, Peggy shared many of the same ideas she shared in Interview 1. Peggy continued to think of rate of change as a ratio comparison of changes. She evidenced no strong meanings for average rate change. These findings are illustrated in Figure 33.

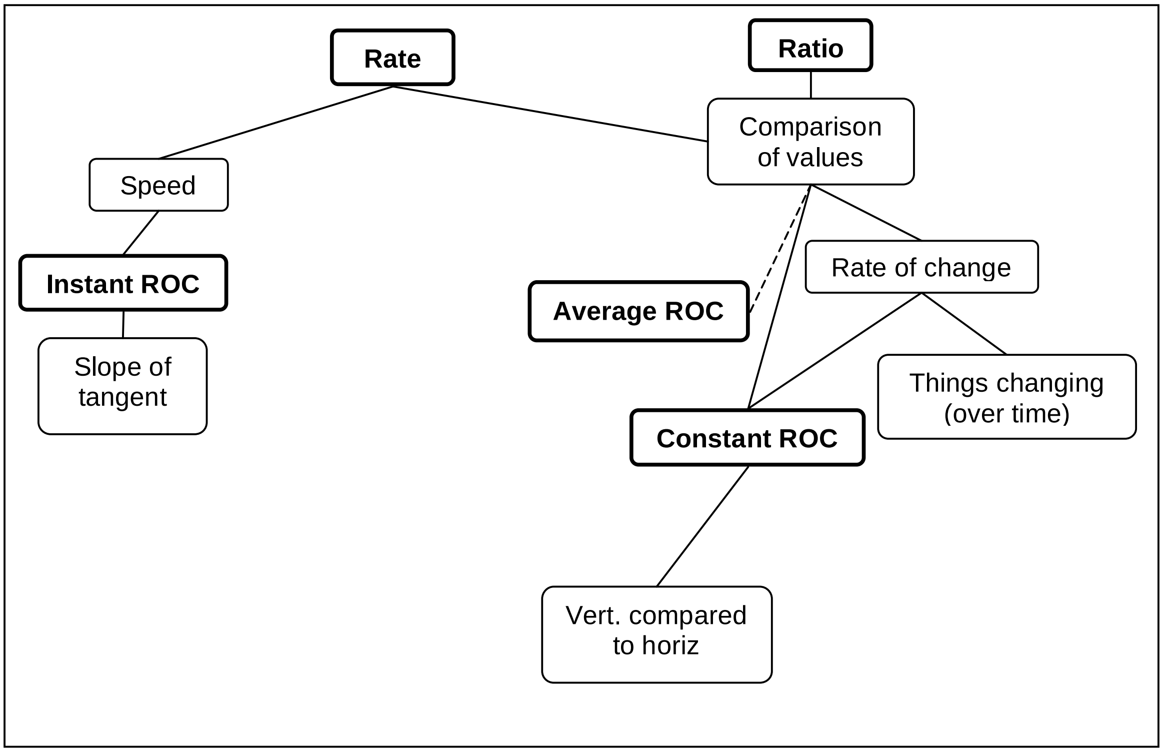


Figure 33. Peggy, interview 2, definition component map

In Task 14, Peggy provided the same rate/ratio distinction as she did in Interview 1: a rate was a ratio comparison of two quantities with different units.

Task 14. What is a 'rate', mathematically speaking?

Excerpt 145

1. Peggy: What is a rate? Haven't I told you that before? What is a rate? It is a ratio. It is a comparison. It is two things that are changing together, well actually I guess they don't have to be changing because…a rate is a comparison of 2 quantities and that's it. Two quantities with different units is how we define it.

In Excerpt 146 (also from Task 14), we see that with rate of change the quantities represented changes. For Peggy, rate of change, as a ratio comparison of changes, describes how fast something is changing.

Excerpt 146

1. Int:	What do we mean when we speak of a rate of change in mathematics?
2. Peggy:	That's when we're looking at how two quantities are changing and comparing the change
3. Int:	Elaborate a little bit.
4. Peggy:	Compared to rate of change, when you have two things that are varying or changing together…rates of change is…describes how things are changing, how they are changing what their relationship is, regarding the change
5. Int:	A little more specific.
6. Peggy:	OK, so well on the cut and dried definition is, you know, looking at vertical change to horizontal change [writes ratio "vertical chg / horiz chg"] but that is kind of meaningless. So that when you start to look at how dependent, dependent variable change compares to the independent variable change [writes in fraction form], then it is a little more meaningful, so rate of change is how fast or, let's see, how fast something is increasing, or how it is decreasing, how fast it's decreasing.

Continuing in Task 14, Peggy spoke about average rate and how she related it to the end points without considering what takes place in between.

Excerpt 147

1. Int: So what do we mean by an average rate of change?
2. Peggy: It means that…well I always think about if I say if I traveled 300 miles and the trip took me five hours then I think I must have averaged 60 miles per hour although I know that what took place in those five hours was stopping and starting and a lot of different speeds in between, average rate of change is just looking at from end points or from start to finish and not considering what took place in between, an overall change

There was a subtle distinction in Excerpt 147 in that besides the endpoints, average rate of change accompanied something that was actually happening. Something actually was changing.

In Excerpt 148 (also from Task 14), she held a sort of dynamic view for rate of change while average rate of change was more static.

Excerpt 148

1. Int: How does that [average rate of change] differ from rate of change?
2. Peggy: Because rate of change is more like looking at how things are changing. Is something increasing at an increasing rate? Or how fast it's changing, or taking a look at how, see this is hard for me because I don't ever talk about those things when I initially teach

rate of change. We only look at pretty much, I mean we look at average rate of change because we are only looking at constant rate of change. What did I just say? We are only looking at constant rate of change so we don't really get into any of the increasing or decreasing that can take place because we are all linear. But the way I, the way I, what I know of the difference of the average rate of change and rate of change is when we start to look at increasing at an increasing rate and increasing at a decreasing rate those kinds of things.

As before, Peggy reserved average rate of change for non-linear situations. The connection in her thinking of average rate to constant rate, if any, is unclear.

In Task 15 Peggy connected average rate graphically to a secant, but she still expressed some uncertainty with the middle of the function.

Excerpt 149

1. Peggy: So the tangent, the slope of this tangent tells me how fast the function is increasing right here at this point, and the secant through this curve tells me like the average, the change from this point to this point over this interval…which I guess would be the same as average rate of change of the function over this interval, but it doesn't describe anything that's really taking place in between those two points, that's all I know.

I will clarify the significance of her uncertainty in the section on average rate of change.

In Excerpt 149, we also see her connecting instantaneous rate to the slope of the tangent. In the space shuttle task (Task 13) she explained instantaneous rate of change only as instantaneous velocity. She did not include a ratio comparison of changes even though in Excerpt 146 she explained that such a comparison explains how fast something is changing. It did not appear that she connected velocity to any of her other ways of thinking about rate of change.

For Task 16 I asked Peggy about using the operation of division to calculate slope. She viewed the division as a comparison. Peggy struggled, however, to provide a reason for using division to compute slope. For her, division had to do with sharing, and this caused no small trouble in her analysis.

Excerpt 150

1. Int: Why do we use division to calculate slope?
2. Peggy: Oh, man, I never really thought about that. Why do we use division? Because it is a ratio so if I compare say, 5 to 10, [writes "5:10"], that's five out of 10 or five of ten or five divided by 10 [writes as fraction "5/10"] , because slope is the ratio and this ratio is the same as a 1 to 2 ratio [writes "1:2"] which is the same as one out of two [writes "1/2"]. I never thought of it as a division operation as much as a comparison, but it is a division…let me think, ok…

She talked some more, drawing a linear graph passing through the points (0,0), (1,2), (2, 4) and (4,8). She ended up with a segmenting sort of model. Her model, though, was not a segmenting of components of corresponding changes in the dependent and independent variables; rather, it was a segmenting of the line itself.

Excerpt 151

1. Peggy	…I don't know I've never really thought of it as the division operation, but I think of it this way, that this one goes on I can compare 8 to 4 [extending linear points] and so I guess in a sense you can say you're dividing this up into segments looking at it…a change of eight compared to four, or four compared to two or two compared to one and so on and so on, and so I guess I see division that way but I don't see it as a, the way I view the operation for division is kind of different than I view it here, as a comparing type thing
2. Int:	How do you usually think of division?
3. Peggy:	It usually depends on the context, if I'm dividing the class up I'm dividing it up into groups. It is the inverse of multiplying. I have so many groups. But…in this sense I just see it as more of a, I don't know, that's really weird. I've never thought of that before, because you are, I can divide this in half [pointing from (4,8) to (2,4)] and compare for the same way, four compares to two the

> same way that eight compares to four, and I can divide that segment in half so I see division that way. Which I guess is the same way as dividing the class into groups, really. Thank you, Ted, for giving me this enlightenment.

I, of course, gave only the question. She provided her own enlightenment. Peggy, whose view of rate of change was comparison, could not quantitatively explain why such a comparison uses the operation of division.

Peggy: Interview 2, Linear Component

In the linear component Peggy continued to primarily view rate of change as grounded in some sort of context rather than as a formula or graphical (in contrast to Becky and Mary). This is not to say that those ideas were not present, simply that they were not her primary way of thinking. Linear functions, for Peggy, change in a constant, predictable way. Rate continued to be a ratio comparison of changes. Figure 34 shows Peggy's map with these added elements.

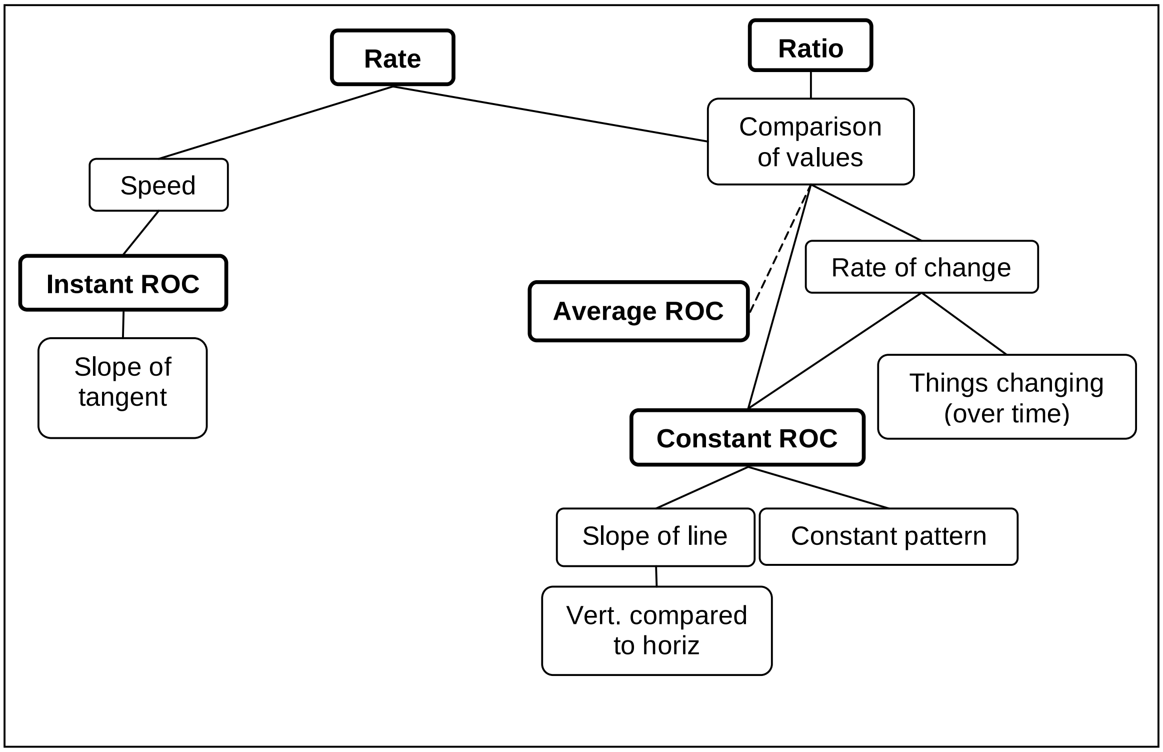


Figure 34. Peggy, interview 2, linear component map

Peggy continued to view a function as a dependency relation where two things vary together in context. She made this clear in task one, the linear function lesson plan.

Excerpt 152

1. Int: What is mathematically important for knowing or understanding linear functions?

2. Peggy: Knowing that it's about 2 things that vary, and that as one of those variables, the relationship between the two variables are such that there is a constant rate of change, that the function, is changing in a constant rate so that it happens to make, you know, on the graph it makes a, the points are lined up [motioning with hands] but it's

important to make sure that you don't get too much into the graph that it's just a line [explain] you don't want them to, you don't want students to just think linear functions are all about lines you want them to think about two things that are varying, and one, as one varies the other one also varies in a linear way, or in a constant, at a constant rate, and that and that it just so happens that if you took all of those points they would show a line on the graph, but really that's all that is, not losing sight that I think it's important to not get too much into linear functions are lines and we need to keep our eye on the real-life part of it, that there are these types of real-life situations where as something changes the other I don't know the other variable, the effect that it has on the other variable changes in a constant way, a predictable way, … once you recognize, you know how those two variables work together then you can make predictions about things, …

For Peggy, linear functions are functions that vary in a predictable, constant way. This involved the idea of a constant comparison in ratio form. She made this clear when I asked her about the meaning of *m* in *y=mx+b*.

Excerpt 153

1. Peggy: What is the meaning of it? It's the…it's the change in the, well it's a ratio, it's the vertical compared to the horizontal and it's a ratio,

which is going to end up being a rate most of the time when you apply it to anything applicable

Of interest here is how she drew a distinction between what she meant by rate and rate of change. The slope, *m*, is a rate as it will be a ratio comparison of quantities, not necessarily quantities of change.

In Task 4 (the four students) Peggy echoed her distinction between rate and rate of change.

Task 4. Four students are discussing the meaning of slope in a linear context. One student says it is $\frac{y_2 - y_1}{x_2 - x_1}$. Another says it is the angle of the line. A third student says it is the rate of change of the line. The fourth says simply that it is the number *m*.

Excerpt 154

1. Int: What would you like your students to understand about slope in a linear context?
2. Peggy: That it's a rate so I would want them to most importantly, I would want them to know it as a rate of change

Continuing in Task 4, Peggy noted that the first student, the one who responded with the difference quotient, needed to also view it as the "change in vertical compared to the change in the horizontal." She then spent time debating the correctness of the student who referred to slope as an angle and determined that the student may have been on to something. In the two following excerpts from this task she clarified her way of thinking about rate of change that contrasted with both Mary and Becky. In Excerpt 155 she claimed that the slope was something beyond the graph. She based her explanations in two ordered pairs she had written down: (-2, -3) and (3, 1).

Excerpt 155

1. Peggy: I'd like them to just see it, well, as a change from a negative three to one, well, that's a positive, we are going up, not just on the graph but it is increasing and that's happened in comparison to this change of five [pointing to negative two and three] and that's also an increase.

In Excerpt 156 she clarified that slope was more than simply a formula. Slope provides information on a comparison

Excerpt 156

1. Peggy: Well I would like them to know that this is a formula for slope, [circling difference quotient] that it isn't in itself slope that it's a formula for slope and of the reason it works is because you are looking at what happened, what is the difference between where you were [pointing to -3] and where you ended up [pointing to 1] versus the other variable. What is going on there, making that comparison.

In Excerpt 157 (also from Task 4) she stated that slope was something to be applied to lines.

Excerpt 157

1. Peggy: The third student says it is the rate of change of the line, well wait a minute, what about this line thing? [Underlines the words "the line" in task]. I'd like them to not necessarily assume it's about line, that rate of change, I don't know, slope goes with lines, and…[informed that there is one minute of tape remaining] OK, rate of change of the line…well slope represents steepness of the line or, I don't know, I don't know what else to say to this student because that is what slope is for, lines.

Peggy once again restricted slope to linear functions. Slope, for her, represented a rate of change as it was a ratio comparison of changes. In linear situations that ratio is constant and predictable. Unlike Mary, Peggy did not see the graph as integral to interpreting rate of change.

Peggy: Interview 2, Average Rate of Change Component

Peggy's thinking that slope only applied to lines led her to disequilibrium in the tasks on average rate of change. In this component, Peggy continued to view average rate of change as an analysis of endpoints. Additionally, as in the first interview, she primarily held two distinct and irreconcilable views of rate.

One view was about a ratio comparison of changes while the other involved a multiplicative sequential comparison of values.

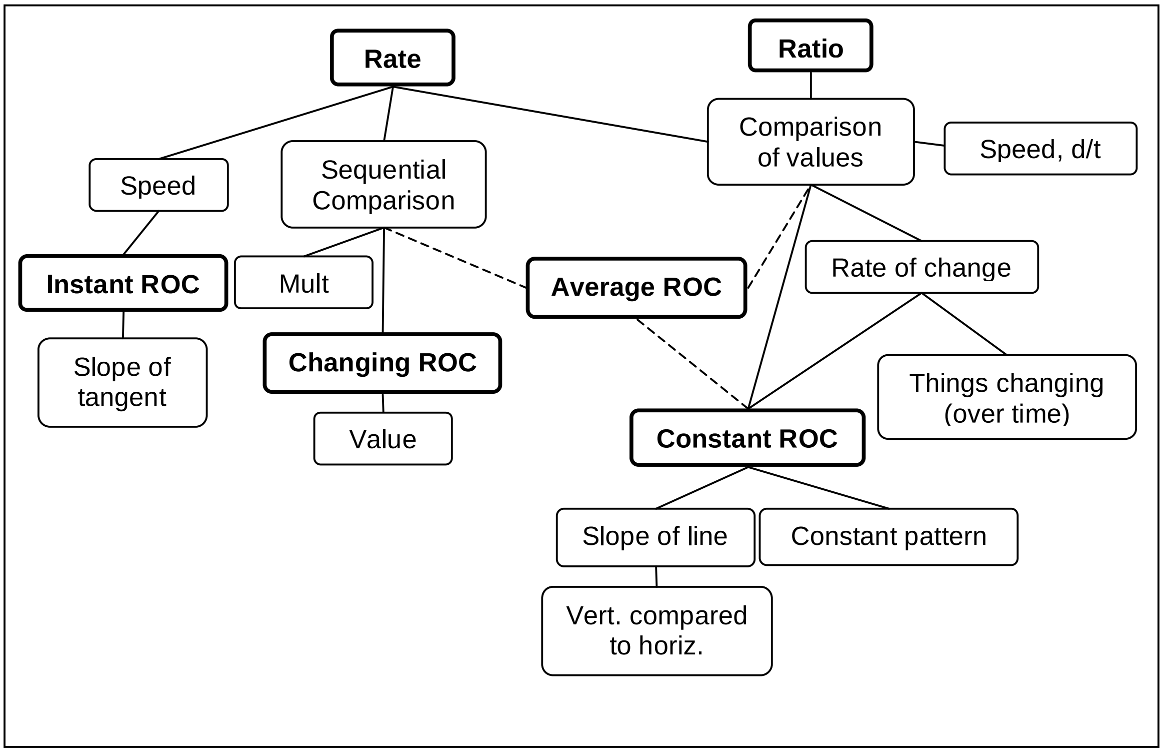


Figure 35. Peggy, interview 2, average component map

Peggy applied rate thinking to the exponential lesson plan task (Task 7). While the following excerpt is long, it serves to explicate her two distinct ways of thinking about rate. In this case her disequilibrium arose from my asking about her simultaneous references to constant rate and changing rate.

Task 7. You are planning an introductory lesson on exponential functions (such as $y = ab^x$) in a second year algebra course…

Excerpt 158

1. Int: So what is the meaning of *b*?

2. Peggy: *b* is the growth or decay factor and that that's found by taking 100 percent of what you started with, well that's not really a good explanation, that it's the base, a base made up of the rate of increase or decrease taken from or added to 100 percent.

3. Int: Does the rate change?

4. Peggy: It does when it's, when you look at it as a power, because of the exponent it is increasing or decreasing. The rate itself doesn't change.

5. Int: Because you just said that sometimes these functions increase at an increasing rate

6. Peggy: I did, didn't I. OK, so it's not that. OK, it's the amount is what's going to increase. I guess I don't know how to word that. It's going to increase, increasingly over time. I don't know. Because the graph of, you know growth graph [sketching increasing concave up exponential type function], it blows up so the rate of change for this amount or over time, if this is, if x is representing time and y is the amount, because you're raising the rate to some exponent, this power is going to increase as time increases.

7. Int: So do you think you used rate in two different ways there?

8. Peggy: Probably. Let me go back over that what I said, I said that…

9. Int: I'm going back and comparing this rate [pointing to written "rate"] and the increasing at an increasing rate [pointing to sketched curve].

10. Peggy: Yeah they are kind of, I guess I did, I mean I guess I would have used rate, there is some rate of increase or decrease that's a constant [pointing to written word "rate"], and then at the same time you have this growth that [pointing to sketch of curve] I mean I view that as a rate of increase that's increasing.

11. Int: So what is the difference?

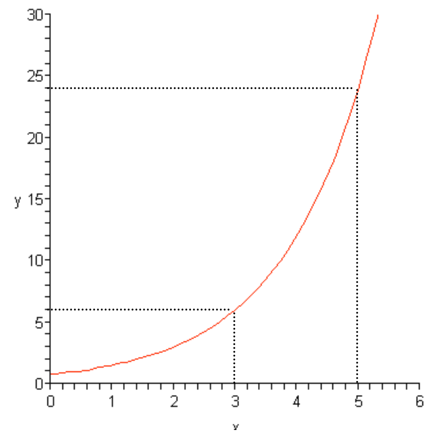
12. Peggy: One is a constant and one isn't. And this is a given rate [referring to the word "rate"] that doesn't change but when its combined with, when you put it into this power [pointing to written b^x] that as x increases the whole thing increases… at an increasing rate. [laughter] And that's all I have to say about that [laughter].

Directly after this she pointed out that student difficulties might arise "[p]robably that their teachers are talking about things in doublespeak." For her, the rate '*b*' in '100+/-*b*' remained constant as *b* was fixed. However, her usual thinking when seeing an increasing concave up graph was to see it as a function that increases at an increasing rate. Peggy was unable to reconcile how this was both a constant and a changing rate. She was unable to reconcile that she was treating rate both as a sequential comparison and a ratio comparison of values.

In Task 3, Peggy continued her focus on endpoints alone for average rate. In Excerpt 159 she described what students need to know about average rate.

Task 3. You provide both a table and a graph and ask a student to find how fast the function is changing between x=3 and x=5. The student responds 18...

x	0	1	2	3	4	5	6
f(x)	.75	1.5	3	6	12	24	48



Excerpt 159

1. Peggy: They need to understand that it's not just about the increase from this height to this height, that there's, that that change has taken place over, that vertical change has taken place over a change of two for the horizontal, in order to understand that they need to know that's how fast is about taking the average, you've got to divide the change in the vertical by the change in the horizontal.

2. Int: How is that the average?

3. Peggy: 18? You mean 9?

4. Int: You just used average. I'm wondering how you use that word.

5. Peggy: Because I'm not looking at the curve in between [pointing to sketched secant] I'm just looking at what happened over, it doesn't tell you that this is time or anything, I'd just see it as average rate of change because you're not looking at what's happening in between you're just looking at overall what, I don't know, the average…okay I think about it this way if I'm driving and it takes me two hours to go 1800 miles, whatever, it takes me 2 hours to drive 18 something, I could, I would say well my average speed was 9. Who knows what I did in between if I was accelerating, decelerating, stopping, starting, we don't look at that in between. How likely is it that you just go constant? So that's why I say average.

Peggy seemed to connect average rate of change to constant rate of change in that she used a ratio comparison of changes.

Peggy's follow-up response in Excerpt 160 showed her working with time as one of the variables. This allowed her to consider the task formulaically.

Excerpt 160

1. Int: And what should the student have done?

2. Peggy: They need to make a connection with how fast I mean if we are talking about speed how fast that there's a time that goes with that that, you know, if we are covering distance than 18 would be your distance, for your distance versus time, 18 is your distance that

you covered, but it didn't ask about distance it asked about how fast, so there's this ratio, this rate that enters in, I mean how fast 18 in two days or two hours or whatever?

I asked her how she would handle a student that noticed this was not a question about distance and time. Her response involved a movement from a ratio comparison for rate to a sequential comparison.

Excerpt 161

1. Peggy: [audible sigh] What else could it be?…[marking points again] I guess we would do some brainstorming…could look at growth of something over time….You know if you look at, if you get students to focus more on the table of values and kind of go away from the graph then you can kind of start to really talk about you know what is this ¾ or 75 hundredths at zero what could that mean? And then what's happening as x is increasing, x is increasing by one each time, and then they could kind of get them to notice that there's this doubling…I don't know, I think from there could probably get some ideas from your students on what could this represent, and I think students have enough, I think they have enough awareness that there is something about this that has to do with growth. Maybe you could probably extend this out and have them, what's going to happen at seven [extending table], and if you could get, if you focused on this is growth then getting back to how fast what

does that mean, you know how fast is something doubling. How fast is something growing, gosh, how fast is something growing…I mean not looking at the graph [covers graph with hand] and not looking at the question, I want to say how fast is something growing? It's doubling every minute or second or whatever, and to me that's how fast.

There was a reason that Peggy covered the graph with her hand. The graph itself represented a contrasting thought process that she could not reconcile. The following excerpt clarified this claim. In Excerpt 162, I asked about a student who gave an answer using a multiplicative sequential comparison. After some back-and-forth to understand my question, she made the following statements.

Excerpt 162

1. Int: I was hearing you're saying that it multiplied by two for instance so to go from here to here it would multiply by two, right?
2. Peggy: Not with the *x*'s
3. Int: Well, just from the *y*'s
4. Peggy: Oh, doubling, OK
5. Int: And so this is what I'm looking at. Well, so following that, would it mean that the student would say from 6 to 24 it was multiplied by 4? So could that be a fair answer for the student?

6. Peggy: Okay, I see what you're saying. So how fast…4 times as fast…Oh, I don't know. I don't know what I would say to that. OK it multiplied by four but it did it over two units of time or two units of whatever *x* is representing so now getting back down to that rate how much per…I guess pointing out that we are interested in the unit rate. If you multiply by four over two units that would give you two. But then back to this nine [tracing secant on graph]….the average would be nine, I just I don't know I just don't get how the nine, and I remember I didn't understand this before, how does the nine connect to that growth function…

7. Int: So you see you see it over here [referring to the graph], but not necessarily over here [referring to the table]

8. Peggy: Uh-uh, no I don't.

9. Int: But is nine the right answer?

10. Peggy …I don't know how to make, yeah, I don't know, taking the average of…looking at it as the average rate of increase here or average rate of change you get 9 [talking about graph]. I just, I still don't know how that fits in with this picture [pointing to table].

11. Int: The table?

12. Peggy: Right, and to how things are relating here [referring to the table].

Peggy was in a state of complete disequilibrium at this point. She had two different ideas of rate that she could not reconcile. She knew there was some sort of reason that nine was correct, but she could not state a reason why it should be different from her notion of doubling. She saw the nine in the graph but not in the table.

In Task 9, Peggy referred to the terms rate of change and average rate of change rather synonymously. As that task involved speed and distance, she had little trouble developing a comparison.

Task 9. Your objective for the day is to develop a lesson on average speed. Specifically, your students need to understand how to find the average speed of a car over a period of time. They should be able to answer the question, “What is a car’s average speed during the period from 2 seconds after it starts to 4 seconds after it starts, where it travels *s* feet in *t* seconds and *s* is given by $s = t^2 + t$ (with *t* measured in seconds)?

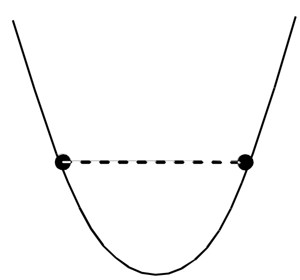
Excerpt 163

1. Int: What mathematics is needed to understand this objective?
2. Peggy: I think…they need to know that speed is distance, well that speed is a rate. It is distance compared to time [writes “speed = distance / time”]. And that to find the average you’re going to take the average rate of change or the rate of change over those two points [pointing to the endpoints].

Speed had a place in her thinking as that relation involved a change in distance compared to a change in time - values that can be computed. She had no need to apply a secant. She wanted to help her students “disconnect from this thing that has always been linear [pointing to d=rt] for them and help them see that it also applies to something like a curve.” Absent, however, was any recognition that her computation of applying the difference quotient assumed a form of linearity (as in considering it a secant in earlier average tasks).

In Task 8 (the parabola question) Peggy applied the slope formula and constructed a line through the points. She was not at all sure of the role of her constructed line to any idea of rate of change.

8. A student comes to you and says, “You know, when you apply the slope formula to opposite points on a parabola the slope is always zero.”



Excerpt 164

1. Int: So let me ask it this way, is there any mathematical validity in applying the slope formula to a parabola?

2. Peggy: Well don't you…if you see this as a secant line for this curve…isn't there some validity to that? Or if you, we take the slope of tangent lines of the curve [sketching horizontal tangent at vertex] somebody does that, [laughter].

3. Int: So there might be, or there is?

4. Peggy: Yes there is. If I take the slope of a tangent line on a curve and its slope is zero than that would be a maximum or a minimum point, for a secant I can't remember for sure, didn't we do that the other day when we were, it seems like, this, there was that average that's a secant line, taking the average of these two [sketches increasing concave up curve with two points and connects points with a line] to answer how fast something was changing, so that's all I know about it.

For Peggy, there was no strong meaning for the constructed secant line. When I asked directly after this for her meaning of the slope formula, she went directly to rate of change. Her answer, in the excerpt below, evidenced a strong connection to her thinking of rate as a ratio comparison of values.

Excerpt 165

1. Int: Well let me ask you a few more questions, what does it mean to understand the slope formula?

2. Peggy: What does it mean to understand the slope formula? It means to understand that, it means to understand rate of change. What's,

> how is the function changing over time or in regards to the independent variable…What does it mean to understand the slope formula? It means that you are always going to compare the vertical change to the horizontal change or the difference in what takes, the difference between the change for a dependent variable compared to what's taking place at the same time with the independent variable.

Overall, in Interview 2, Peggy continued to provide evidence that she did not have a meaningful way to think about average rate. Although there may have been some connection in her thinking about average rate and constant rate, that connection was not clearly defined. Peggy was unsure if she should use her ratio comparison way of thinking about rate or her sequential comparison.

Peggy: Interview 2, Change Component

In the tasks for the changing rate component, Peggy primarily viewed rate of change as sequential comparisons of values. In the changing rate tasks I propose that Peggy coordinated the variables to find individual values and compared those values sequentially. Peggy's final map is shown in Figure 36.

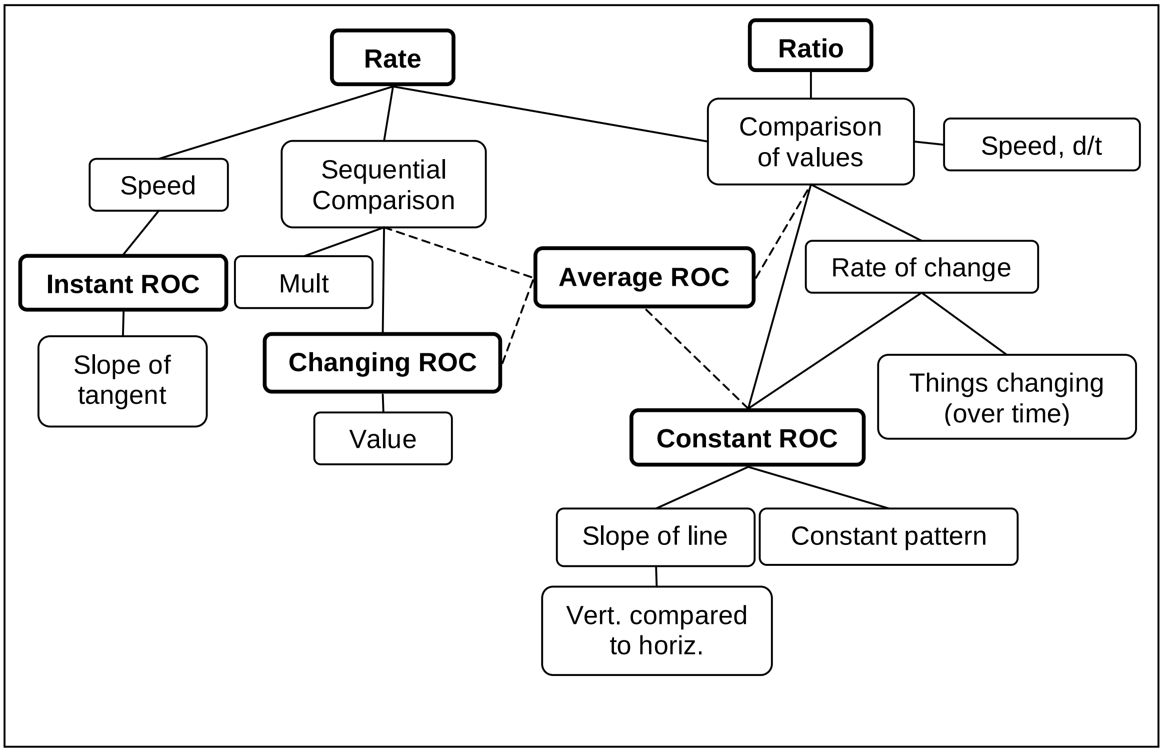


Figure 36. Peggy, interview 2, final map

In Task 2, Peggy explained changing rate in terms of a comparison of sequential secant slope values. She did not explain these values as situated in any sort of ratio comparison of changes. In this case, she explained the importance in using equal intervals.

Task 2. Two students are having a debate about a function whose graph looks like the figure below. One student declares the function is "decreasing at a decreasing rate" while the other says it is "decreasing at an increasing rate…

Excerpt 166

1. Peggy: I always think of the independent variable as just changing kind of one at a time or certain segments. I don't know how to explain that, but suppose that *x* changed by point-one and I look along here to see what's happening to y. I mean the function is all about *y* because *x* is just that they're going along at a hundredth or point-one or whatever and looking at making comparisons at how things are changing, if you keep the, if you keep focused on [pointing to curve] something constant with that [shaking hand] am I making sense?
2. Int: Constant with respect to the scale on *x*.
3. Peggy: Yes, OK … so keeping your eye on what's happening to the vertical, I don't know what else to say as far as mathematically. Tell me what you asked me again.
4. Int: What do you believe is mathematically relevant to understand language like decreasing at a decreasing rate?
5. Peggy: Mathematically relevant…well just that your vertical is changing, it's either changing faster, its decreasing faster or its starting to slow down in how much its decreasing, I don't know what else.

With respect to her idea of the function being "all about *y*" we will see Peggy's thinking of rate here is simply as a sequential comparison of outputs. Her comments in Excerpt 167 support this claim as she thought about which student was correct.

Excerpt 167

1. Peggy: OK. The first student says decreasing at a decreasing rate because there's a sense of slowing down [moving pen along right end of curve] it's the, the vertical is not changing or decreasing as drastically as you move to the right [steps pen down right side of curve] and the second student is viewing it as it's decreasing but there is… increasing, something about moving to, moving to the right and it being more positive [it being?] the *x*. Decreasing at an increasing rate, well I don't know, I'd have to say the first one would be correct because I'm not able to relate with this one decreasing at an increasing rate…

She was able to quickly see validity in the first student's response but she struggled to understand the second student's thinking. She eventually made a case for that student using a comparison of secant slope values originating from the same point rather than across adjacent intervals. Given her struggle, this was clearly not her usual way of thinking about this sort of rate of change situation. What is of interest here, though, is that this she did not connect her thinking back to instantaneous rate of change through the slope of the tangent.

She showed this again in how she would promote student discussion on comparisons.

Excerpt 168

1. Peggy: I probably want them to do some comparing with breaking up, well we would use graph paper and [sketching tick marks on horizontal axis] [plots some points] I'd probably want them to make comparisons about the slope between some points [sketching secants] and then help them keep in mind that something like -4 it is less than -2 that type of thing.

Her explanation in Excerpt 168 represents a departure from her earlier interview. Although it was not her initial thinking for the problem, as her explanation involved secants, it shows a possible connection between her changing rate of change and average rate of change thinking. As we saw in the last component, though, her thinking of average rate as secant was not strongly connected.

In the racetrack task (Task 12) Peggy 's work serves to reinforce my earlier claim of her thinking about a comparison of sequential output values. She approached this task by (1) coordinating the variables, (2) finding values and (3) comparing those values. In this task she was able to quickly coordinate the variables and plot a few points. She then smoothly connected the points to construct a graph virtually identical the one she constructed in the first interview. Figure 37 shows her response to the task.

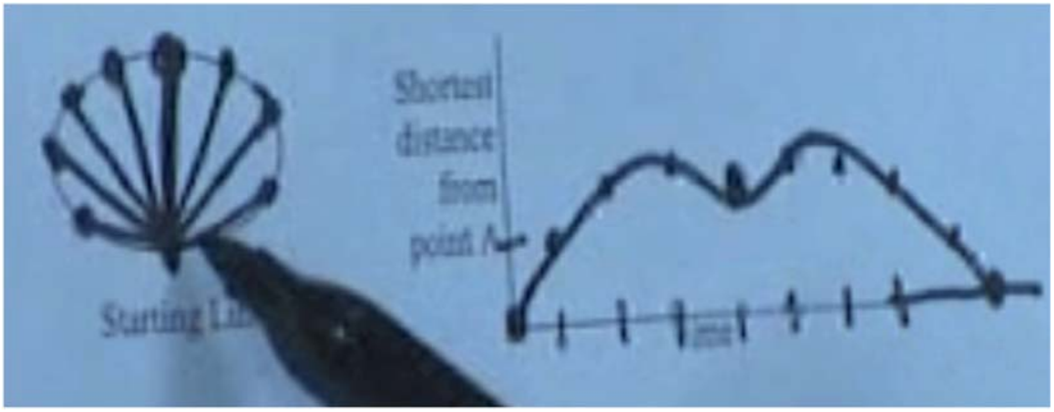


Figure 37. Peggy, interview 2, task 12.

As she worked through the task, Peggy used no increasing-at-an-increasing-rate sort of language. She did use words such as “shorter,” “increase,” and “decrease” when constructing the points. Her use of these reflected a sort of comparison between consecutive points and their estimated values. She provided the following when I asked whether what she drew as curved should be straight:

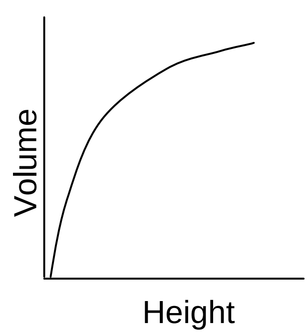
Excerpt 169

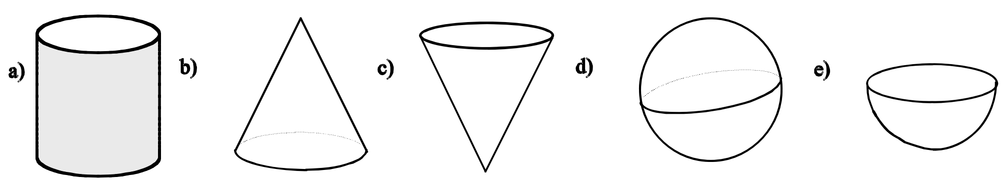
1. Peggy: It should be curves because the distance is increasing…let’s see, increasing….I think it’s like that increasing at an increasing rate thing or increasing, and it starts to increase at a decreasing rate so there should be curves.
2. Int: OK.
3. Peggy: It shouldn’t be like lines.
4. Int: You are pretty sure of that?
5. Peggy: I’m pretty sure, I’m positive.

Her reflection here did not truly provide any reasoning as to why or how the “increasing at an increasing rate thing” behaves as it did in this situation. Unfortunately, I did not follow up in this direction.

In Task 5 (the container question) Peggy coordinated the variables and narrowed down her own response to bottle *B*. She appeared to work through the task with ease.

Task 5. The following graph represents the volume of water as a function of the water’s height in a container. Which container goes with this graph?...





Excerpt 170

1. Peggy: How would I think through it? I would start with that as the height is increasing [stepping pen across horizontal axis] the volume is increasing at a faster rate in the beginning and then the rate kind of tapers off [moving pen along top right of graph] or slows down

> so the rate of change for volume is, volume is increasing at a decreasing rate is the way I want to look at that and so, looking at the containers as the height is changing, let's see, it almost kind of seems backwards it's like I want it to be the volume is changing or the height is changing as the volume changes but, OK I need to keep focused on, if the volume is changing, so as the height increases by, say the height increases by an inch at a time here [container *A*], the volume is increasing at a constant rate, in *B* [container *B*] as the height, say the height is increasing at one inch at a time here the volume is increasing but it is increasing at a decreasing rate. In this one [*C*] as the height is increasing for each inch of height increase there's going to be more volume each time so that's increasing at an increasing rate

I propose that her way of thinking in this task is the same as her earlier tasks in this changing rate component. That is, in Excerpt 170 somewhere around her saying "OK," she began to turn her thinking from a dynamic image of the function to the function consisting of a few individual points that have values. Her comparison then became one of those values. The last sentence of the above excerpt supports this claim.

Overall, then, it appeared that Peggy's ways of thinking allowed her to navigate some of the changing rate tasks. As she did not speak of speed or the slope of a tangent line, her thinking in these tasks did not seem to relate to how she defined instantaneous rate of change. She might connect this changing rate of change to average rate of change in the sense that she did consider endpoints of intervals. She approached none of the tasks, however, from the same sort of ratio comparison of values used when she spoke of rate of change. It is possible that her individual segmentations of the domain do indeed act as an important component of the comparison, though I would need more data to make that claim. Given that, I will concede that what I have proposed in this final component might not be the best description of her thinking, but it is consistent within itself and with her two primary ways of thinking about rate of change.

Summary of Peggy

The data suggests that Peggy had two primary ways of thinking about rate: a ratio comparison of changes and a sequential comparison of values. These two ways of thinking remained stable from the first interview to the second. She used her ratio comparison of changes thinking when she worked with tasks on constant and average rate. When she worked with the changing rate tasks, she used her sequential comparison of changes.

Peggy saw rate of change as something applied to linear situations (see Excerpt 116, p. 173, and Excerpt 148, p. 204). This thinking, however, led to her difficulty in developing a meaningful way to think about average rate in the context of nonlinear situations (see Excerpt 131, p. 188, or Excerpt 158, p. 214, or Excerpt 162, p. 220). In contrast to Mary, Peggy saw functions as entities grounded in a real-life context. She did not prefer to operate graphically (see Excerpt 126, p. 183).

In the linear tasks. Peggy evidenced ideas related to steepness and vertical and horizontal components, but she operated with them using her ratio comparison of values. For Peggy, a constant rate of change meant that the comparison of changes would result in a constant pattern (see Excerpt 123, p. 180). However, Peggy was unable to quantitatively explain why division was an appropriate operation in the computation for slope (see Excerpt 151, p. 207).

In Peggy's second interview, the data showed she thought that average rate had something to do with the endpoints. However, her lack of having a solid definition of average rate led to confusion about how she should think about the average rate tasks. She did not know whether she should think about them in terms of a ratio comparison or a sequential comparison. In one exponential task, Peggy wanted to apply multiplicative sequential comparisons to the table of values while she preferred ratio comparisons for the graph. She could not reconcile which she should use (see Excerpt 161, p. 219, and Excerpt 162, p. 220). She applied both types of rate thinking in the task on developing an exponential lesson plan, but when I asked about her double usage of rate she could not provide reconciliation (see Excerpt 158, p. 214).

Peggy applied her sequential comparison thinking in the changing rate tasks. That is, the comparison made was of sequential values rather than a ratio of changes (see, for example, Excerpt 166, p. 227). In these cases she based her comparisons on slope values of steepness (see Excerpt 168, p. 229) or output values of the function (see Excerpt 170, p. 231). She continued to separate her thinking about instantaneous rate from her other types of rate.

As with the cases of Becky and Mary, the maps that I developed for Peggy reveal stability in her ways of thinking about rate. My conclusions in Interview 2 essentially served to support the conclusions of Interview 1. Another addition to the map for Interview 2 is a possible connection between her average rate and constant rate thinking. Although her overall conception of average rate was not coherent, there is evidence that she may have developed some connection in her understanding.

CHAPTER 8: SUMMARY AND CONCLUSIONS

This research developed a method to model ways of thinking. To accomplish this, I developed a method for implementing a conceptual analysis. Specifically, I createed a means to construct models of how teachers think about rate of change. I worked from the assumption that each teacher had a unique way of thinking about topics of rate and that I could, through my own perspective, create a model to describe that thinking. I began by considering how each teacher thought about rate in specific contexts and sought to determine how those ideas may interconnect in the mind of the teacher. I developed these models through the creation of my own hypothesized conclusions of their thinking and imagined how these ideas related using graphic maps. My conclusions and maps served to inform each other.

General Conclusions

The conceptual analysis yielded explanatory models of thinking about rate of change. That is, using this method I was able to describe ways of thinking about rate for three teachers that made their answers to tasks reasonable given they possessed those ways of thinking. Each teacher's model was unique. This uniqueness provides an opportunity to consider that an overall understanding of rate of change may not be as uniform as anticipated in earlier research. The models themselves indicate how ideas may be connected, but also suggest why some ways of thinking may be disconnected.

None of the teachers appeared to have a quantitative understanding of rate of change. None could clearly explain the use of division to calculate slope; there was no evidence of a quantitative understanding of the ratio. Also, none of the teachers considered rate in terms of accumulations. The teachers based their unitizations largely in static computations between points.

The teachers also evidenced a struggle to make sense of average rate. Becky had the most connected thinking for average rate, but she still encountered trouble with the tasks. She was limited by her understanding of ratio. Peggy could not decide how to connect average rate to her other types of rate thinking. She could not decide between her thinking of ratio comparisons and her thinking of sequential comparisons. Not one of the teachers evidenced a fully coherent model of thinking that allowed them to work with the average rate tasks.

I view this difficulty with average rate to be highly significant because average rate is essential in covariational thinking (Carlson et al., 2002) and calculus (Monk, 1987; Thompson, 1994; Zandieh, 2000; Carlson et al., 2002; Stroup, 2002). If an understanding of average rate is essential to a robust, increasingly complex understanding of functions, the present study may serve as a warning that the topic is indeed non-trivial. The ways that these teachers thought about constant, average, and changing rate did not fit into a smooth progression of understandings. Although existing research has established that students have difficulty working with average rate of change (Thompson, 1994a; Bezuidenhout, 1998; Carlson et al., 2002), this research begins to provide a framework for understanding the disconnection in understanding average rate. It may be that in an individual's thinking there is no meaningful way to think about average rate. That is, a person may simply have no meaningful way to connect it with other their other understandings of rate.

As each of the teachers struggled with average rate in some form, the covariation framework (Carlson et al., 2002) may help to explain the difficulties the teachers experienced with the changing rate tasks. In order to reason covariationally, this framework proposes, one must first be able to coordinate average rates of change.

Implications

Although this study did not focus on the development of these teachers' ways of thinking about rate of change, there are implications in how these teachers may develop the topic with their students. If the teachers, for instance, have disconnected ways of thinking about rate it is unlikely that they will be able to promote coherent understandings of rate of change in their students.

There is a need, then, for a clear, comprehensive curriculum for rate of change. The general curriculum separates rate of change topics from each other over a number of courses. As different types of rate are included in the traditional curriculum in distinct places with distinct applications, it is not surprising that teachers' ways of thinking may follow those compartmentalizations.

At the algebra level, it may seem sufficient to think about rate of change in terms of a static, graphic computation of slope. At that level, rate of change describes a slope between two points and fits well into the slope-intercept or point-slope forms used to answer the required questions. At this level there may be no need to interpret the value as anything other than a relation between two points. Thinking of rate in terms of steepness as a fastness will serve to solve the problems presented at that level. There is no need at this level of the curriculum to consider the ideas of average rate of change or changing rate of change. It may simply be one change divided by another change. Here, slope is limited to lines and not necessarily as a rate describing the behavior of the function in a dynamic way. It is not necessary to describe slope as a quantitative, multiplicative relationship.

At levels directly below calculus, we must draw a distinction between linear and non-linear functions, particularly exponential functions. The earlier thinking about slope from the algebra level is still compatible with certain types of problems but it may not extend to everything at this level. As we have seen, one may view average rate of change as something applied to non-linear situations. That is, average rate may simply find the slope between the two points. This may or may not be held with meaning. It may or may not have any connection back to constant rate of change from the algebra level. This would be the case if during the algebra experience there was no understanding of how rate quantitatively described the behavior of the function over that portion of the domain. This difficulty of having a meaningful understanding of average rate can particularly evidence itself in trying to describe exponential functions. We saw that without an understanding for the meaning of average rate (based on constant rate), it may become overly confusing to distinguish a difference between a rate and a ratio (as commonly used in the curriculum).

It is in calculus that the curriculum focuses on changing rates. In theory, the changing rate analysis is based on the limit of a secant, which is a form of an average rate of change, which, in turn, is a form of a constant rate of change. By definition, and by calculation, this may make sense. However, this is a long chain of understandings, possibly held without quantitative meaning. As we have seen, without the strong connections between the types of rate of change, working with nonlinear functions and describing their changing rate properties will require alternative ways of thinking that are not based in an evolved understanding of rate. Such ways of thinking may not be reliable.

A meaningful way of thinking about rate of change must start at the elementary algebra level. The meaning should develop from an understanding of rate, building from a quantitative understanding of the operation of division in a dynamic, function situation. With the proper foundation, I propose, the other types of rate will become much more meaningful.

Limitations and Contributions

When I began planning for this study I was interested in finding a way to describe how teachers understand ideas related to rate of change. As my thinking continued to develop, I became specifically interested in exploring how teachers think about rate of change as a comprehensive idea. My research, I determined, would stem from a framework of connecting constant, average and changing rate of change to consider how teachers interrelate those ideas. This question was best asked from a qualitative methodological perspective and best answered using a conceptual analysis. However, there were two significant issues in employing conceptual analysis: the absence of a systematic method and that the building of models involves a large amount of time and effort. My hope is that I have contributed to a better way of employing conceptual analysis in order to lessen the time it might take for others to utilize the method. It is only a first step, but I hope that this study can contribute to a more efficient application of developing models of thinking.

In this analysis the support for my claims arose through successive refinements in the development of hypotheses. I generated those hypotheses from teacher utterances and actions and tested them against the other responses of the teacher and my other claims as to how the teacher was thinking. This process, though tedious, was open to bias on the part of the researcher. In addition, in the later stages of analysis, I began to ground my further abstractions of each teacher's thinking on already-generated conclusions and maps. This could also introduce bias.

I intend that the models are "coherent and useful" (Glasersfeld, 1995, p. 109), but they are indeed my own interpretations of another individual's thinking. Similarly, the elements and connections on the maps represent my best attempts at accurately describing the results of my thinking about their thinking. I intend the maps to be an efficient means to communicate the overall model, but they, too, are subjective.

The developed models, though, are not (and I do not intend them to be) absolute proclamations of how these teachers precisely thought about these ideas. I propose them merely as possible ways of thinking. There may be many other possible systems of meanings, and this research provides a starting point for understanding those systems. We have seen that a comprehensive way of thinking about rate is complex. This research serves as a beginning for our understanding of the complexity. The next steps from this study would have the goal of refining this method so that it can be implemented with much less effort to find information sufficient to generate a model of teachers' meanings and connections. An efficient, refined method would then position the field to determine a distribution of ways of thinking about rate of change among the population of middle and high school mathematics teachers.

In addition to the methodological limitations, there were limitations on the construction of the study itself. First, I must hope that I asked enough questions across sufficient tasks to invite the opportunities for the teachers' thinking to manifest itself. It was for this reason that I included so many tasks with such overlap. I do wish, however, that I had included more opportunities for the teachers to explain their constant rate thinking; that particular part of the analysis could have been richer. I also wish more of the tasks had focused on establishing the overall view of functions.

With this particular trio of teachers and this method, I have described possible complex ways of thinking that challenge our thinking about how mathematics teachers understand rate of change. However, another limitation in the study was that I based some of the conclusions on how certain ideas did not appear to be connected. Further information might well have led to additional connections of some ideas. Similarly, it was possible that a teacher may have been operating with such a well-packed scheme that she sometimes did not see a need to describe a task in any way other than procedural.

These limitations notwithstanding, this study provides a launching point for further exploration with these particular teachers. In the interviews these teachers also provided much explanation about their teaching strategies. It would be interesting to return to that data to explore how their strategies might interact with their now-established ways of thinking about rate of change. These teachers also participated together in a year-long professional learning community (PLC). It would be of value to the mathematics education community to explore the relationship between their established understandings and how these teachers interacted with each other. There were many times during the PLC meetings that the topics of action and process view of function and covariation were part of the discussion. Finally, these teachers eventually participated in a course on rate of change. It would be interesting to see how that course shifted their understandings.

I also hope that this study will rekindle an interest in research about rate of change. While it may seem apparent that average rate of change should be a well-connected idea to a complete understanding of rate of change, we have seen that such understanding may not be easily developed. There is much more to learn about the complex relationships that make up this mathematical concept and this study can act as a framework for further research. This work can influence further research into establishing how other teachers think about rate and it can serve as a foundation for professional development programs that target the development of a coherent understanding of rate. We need to be careful to understand that rate of change is a complex idea. After all, if experienced teachers have such difficulty developing a coherent model of related ideas then students will likely face similar challenges.

This study supports the notion that it is unwise to assume that two individuals are thinking the same way simply because they are using the same words. This, however, is not only true between teachers, but it is also true for communication between teachers and students. This study serves as a reminder that teachers must probe beyond initial responses to establish how a student may be thinking about a mathematical concept.

REFERENCES

An, S., Kulm, G. & Wu, Z. (2004) The pedagogical content knowledge of middle school, mathematics teachers in china and the U.S. *Journal of Mathematics Teacher Education, 7,* 145-172.

Bezuidenhout, J. (1998). First-year university students' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology, 29(3),* 389-399.

Carlson, M. P. (1997) Obstacles for college algebra students in understanding functions: What do high performing students really know? *AMATYC Review, 19(1),* 48-59.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. *Conference Board of the Mathematical Sciences, Issues in Mathematics Education, 7,* 114-163.

Carlson, M. P. (2002). Physical enactment: A powerful representational tool for understanding the nature of covarying relationships. In F. Hitt (Ed.), *Representations and Mathematics Visualization* (pp. 63–77). North American Chapter of the International Group for the Psychology of Mathematics Education.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*, 352–378.

Carlson, M. P. & Oehrtman, M. (2005). Key Aspects of Knowing and Learning the Concept of Function. *MAA Online Research Sampler, 9.* Retrieved July 7, 2005, from http://www.maa.org/t_and_l/sampler/rs_9.html

Carlson, M. P., Oehrtman, M., and Engelke, N. (in preparation). The development of an instrument to assess precalculus students' conceptual understandings: The precalculus concept assessment instrument.

Confrey, J. & Smith E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, *26*, 135-164.

Confrey J. & Smith E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, *26(1)*, 66-86.

Cooney, T. & Wilson, M. (1993). Teachers' thinking about functions: Historical and research perspectives. In T. Romberg, E. Fennema, & T. Carpenter (Eds.), *Integrating Research on the Graphical Representation of Functions* (pp. 131-158). Hillsdale, NJ: Lawrence Erlbaum Associates.

Coulombe, W. (1997). First year students thinking about covariation. Unpublished doctoral dissertation, North Carolina State University, Amherst (UMI No. 9736819)

Ferrini-Mundi J., & Lauten, D. (1993). Teaching and learning calculus. In Wilson, P. (ed.), *Research ideas for the classroom. High school mathematics* (pp. 155-176). New York: Macmillan.

Glasersfeld, E. von (1992). *Aspects of radical constructivism and its educational recommendations*. Retrieved June 26, 2006, from University of Massachusetts, Scientific Reasoning and Research Institute Web site: http://srri.nsm.umass.edu/vonGlasersfeld/onlinePapers/html/195.html

Glasersfeld, E. von (1995). *Radical constructivism: A way of knowing and learning*. London: The Falmer Press.

Harel, G., Post, T., & Behr, M. (1988). An assessment instrument to examine knowledge of multiplication and division concepts and its implementation with in-service teachers. In M. Behr, C. Lacampagne, & M. Wheeler (Eds.), *Proceedings of the Ninth Annual Conference of PME-NA* (pp. 411-417). DeKalb, IL: PME. Retrieved September 5, 2006, from http://www.education.umn.edu/rationalnumberproject/88_5.html

Haimes, D. H. (1996). The implementation of a “function” approach to introductory algebra: A case study of teacher cognitions, teacher actions, and the intended curriculum. *Journal for Research in Mathematics Education*, *27(5),* 582-602.

Hauger, G. (1995, April). *Rate of change knowledge in high school and college students.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA. (ERIC Document Reproduction Service No. ED 392598)

Monk, S. (1987). *Students' understanding of functions in calculus courses.* (Unpublished manuscript, University of Washington, Seattle, WA).

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Piaget, J. (1971). *Psychology and epistemology* (A. Rosen, Trans.). New York: Orion Press. (Original work published 1970)

Piaget, J. (1970). Genetic Epistemology (E. Duckworth, Trans.). New York: Columbia University Press.

Saldanha, L. & Thompson, P. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education, North America.* Raleigh, NC: North Carolina State University.

Sierpinska, A. (1992). On understanding the notion of the concept of function. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp. 25-58). *MAA Notes, 25*. Washington, D.C.: Mathematical Association of America.

Silverman, J. (2005). *An investigation of content knowledge for teaching: Understanding its development and its influence on pedagogy*. Unpublished doctoral dissertation, Vanderbilt University, Nashville, TN.

Steffe, L. (1992) Schemes of action and operation involving composite units [Electronic version]. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L. (2002) A new hypothesis concerning children's fractional knowledge [Electronic version]. *Journal of Mathematical Behavior, 20,* 267-307.

Stroup, W. (2002). Understanding qualitative calculus: A structural synthesis of learning research. *International Journal of Computers for Mathematical Learning*, 7, 167-215.

Stump, S. (1997, March) *Secondary mathematics teachers' knowledge of the concept of slope*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL. (ERIC Document Reproduction Service No. ED 408193)

Stump, S. (2001). High school precalculus students' understanding of slope as measure. *School Science and Mathematics, 101(2),* 81-89.

Thompson, P. W. (1994a). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*, 229-274.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NY: SUNY Press. Retrieved June 30, 2006, from http://www.pat-thompson.net/PDFversions/1994ConceptSpeedRate.pdf

Thompson, P. W. (2000). Radical constructivism: Reflections and directions. In L. P. Steffe & P. W. Thompson (Eds.), *Radical constructivism in action: Building on the pioneering work of Ernst von Glasersfeld* (pp. 412-448). London: Falmer Press. Retrieved June 30, 2006, from http://www.pat-thompson.net/PDFversions/2000Constructivism-Ref's&Dir's.pdf

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing and Modeling In Mathematics Education*. Dordrecth, The Netherlands: Kluwer. Retrieved June 30, 2006, from http://www.pat-thompson.net/PDFversions/2002DidacticObjs.pdf

Thompson, P. W. & Saldanha, L. (2000). Epistemological analyses of mathematical ideas: A research methodology. *Proceedings of the Twenty-second Annual Meeting of the International Group for the Psychology of Mathematics Education*. Tucson, Arizona. Retrieved June 30, 2006 from http://www.pat-thompson.net/PDFversions/2000EpistAnalysis.pdf

Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95-114). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W. & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education, 25*(3), 279-303. Retrieved June 30, 2006, from http://www.pat-thompson.net/PDFversions/1994Talking%20about%20Rates%201.pdf

Thompson, P. W., & Thompson, A. G. (1992, April). *Images of rate*. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA. Retrieved June 30, 2006, from http://www.pat-thompson.net/PDFversions/1992Images%20of%20Rate.pdf

Vergnaud, G. (1988) Multiplicative structures. In J. Hiebert & M. Behr (Eds.), Number concepts and operations in the middle grades (pp. 141-161). Reston, VA: National Council of Teachers of Mathematics.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20(4),* 356-366.

Wilson, M. (1993, April). *One preservice teachers' evolving understanding of mathematical functions.* Paper presented at the annual meeting of the American Educational Research Association. Atlanta, GA.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education IV.* (pp. 103-127). Providence, RI: American Mathematical Society.

APPENDIX A

PROTOCOL FOR INTERVIEW 2

Interview Instructions:

1. Whenever a respondent uses the words “understand”, “understanding”, or “knowledge” provide a follow-up prompt with “what do you mean by understanding (or knowing) __?”
2. Watch carefully for the use of the word “it.” When spoken, be sure there is no ambiguity in what “it” refers to. If there is, simply ask “what do you mean by ‘it’?”
3. If the respondent says they would “ask questions,” probe for specificity in those questions. Have them provide an example.
4. More information can often be gathered by following up their response with the question “anything else?”

Introductory Questions:

You are participating in a professional development program aimed at helping you to improve your teaching.

(a) how have you changed so far this year? Be specific and give examples
(b) mathematically, how has this program been worthwhile?
(c) how useful has it been to your teaching practice?

“I am now going to present you with a number of tasks. Most of them will have follow up questions that will be asked once you have become familiar with the situation. You are welcome to write on the paper and talk through your responses as you see fit. As I place each task before you I’d like to ask that you read it aloud.”

Tasks:

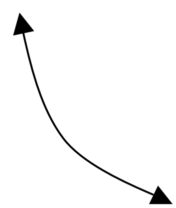
1. You are planning a review of linear functions early in a 2^{nd} year algebra course…

1. What is mathematically important for knowing (and/or understanding) of linear functions?

2. What is the mathematical meaning of ‘m’ in $y = mx + b$?

3. What difficulties do students have in understanding linear functions?

2. Two students are having a debate about a function whose graph looks like the figure below. One student declares the function is “decreasing at a decreasing rate” while the other says it is “decreasing at an increasing rate…



1. What, do you believe, is mathematically relevant to understand language like “decreasing at a decreasing rate”?

2. Which student, do you think, is correct?

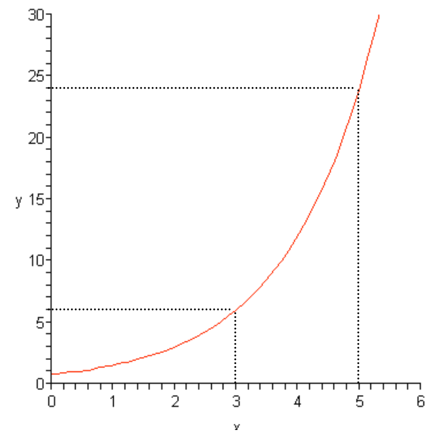
3. What is complex about understanding language like this?

4. What would you have the students focus on to promote the discussion? Why that?

5. What is the meaning of the second use of “decreasing” in the given situation? What is the meaning of the first usage?

3. You provide both a table and a graph and ask a student to find how fast the function is changing between x=3 and x=5. The student responds 18...

x	0	1	2	3	4	5	6
f(x)	.75	1.5	3	6	12	24	48



1. What must a student understand to answer this question?
2. What difficulties will students have with this question?
3. What might the student have been thinking?
4. What *should* the student have done?
5. What would you initially say to the student?
6. How would you guide the student? Why would you guide that way?

4. Four students are discussing the meaning of slope in a linear context. One student says it is $\frac{y_2 - y_1}{x_2 - x_1}$. Another says it is the angle of the line. A third student says it is the rate of change of the line. The fourth says simply that it is the number *m*.

1. Which of the students is correct? [Explain]

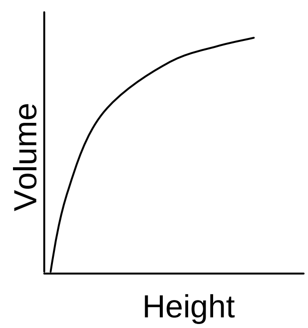
2. What student understandings are at play in these responses?

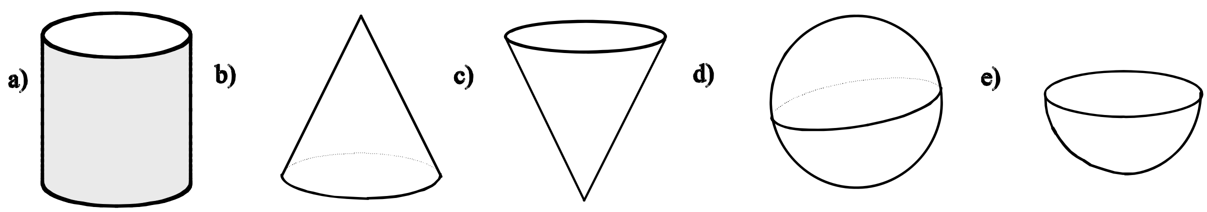
3. What would you like students to understand about slope in a linear context?

4. What do students find difficult about slope in a linear context?

5. What would you say to each of the students to guide them in their understanding? Why would you say those things?

5. The following graph represents the volume of water as a function of the water’s height in a container. Which container goes with this graph?...





1. What must a student understand to answer this question?

2. How did you think through this question?

3. What is hard about this question?

4. Which container would you expect most students to select? Explain.

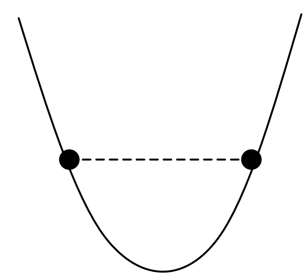
5. What would you say to those students to promote their understanding? Why?

6. Suppose the foot of a ladder that is resting against a wall is pulled away from the wall at a constant speed. What can you say about the speed at which the other end of the ladder is dropping down the wall? (Please talk through your thinking.)

7. You are planning an introductory lesson on exponential functions (such as $y = ab^x$) in a second year algebra course…

> 1. What is important in the knowing (and/or understanding) of exponential functions?
>
> 2. What is the meaning of ‘b’ in $y = ab^x$?
>
> 3. What difficulties do students have in understanding exponential functions?

8. A student comes to you and says, “You know, when you apply the slope formula to opposite points on a parabola the slope is always zero.”



1. Is there mathematical validity in what the student observes? Explain
2. Is there any mathematical validity in applying the slope formula to a parabola?
3. What does it mean to understand the slope formula?
4. How would you build on the student’s observation? Why would you do that?
5. What do students find complex about the slope formula?

9. Your objective for the day is to develop a lesson on average speed. Specifically, your students need to understand how to find the average speed of a car over a period of time. They should be able to answer the question, "What is a car's average speed during the period from 2 seconds after it starts to 4 seconds after it starts, where it travels *s* feet in *t* seconds and *s* is given by $s = t^2 + t$ (with *t* measured in seconds)?

1. What mathematics is needed to understand this objective?

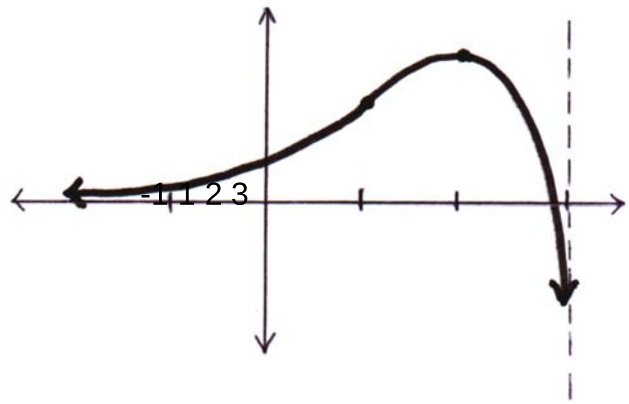
2. What common mistakes do students make with this objective?

3. What do students find complex about this?

4. How would *you* answer the example question?

5. Briefly sketch how you would plan this activity. Include your goals for the lesson. Why would you choose those goals?

10. Consider the graph to the right:



1. What is noteworthy on this graph?

2. How would you want students to describe the graph (at the pre-calculus level)?

3. What terms are essential? Why?

4. (Based on response, look for any terms that imply a dynamic view of function or covariation, such as ‘rate’, ‘changing’, ‘growing’, ‘limit’ or ‘approaching’):

   How do go about developing the term … with your classes?

5. Suppose now that the units on the y-axis are profits (in dollars) and the units on the x axis are years. How would you now like students to describe the graph?

6. What is happening at year 1?

7. What mathematical understandings would be necessary to make such a description?

11. You are creating a lesson on how to find how quickly a function changes, given a table. For instance, given the table, students need to be able to compute how quickly the function *f* changes from $x = -2$ to $x = 3$.

x	*f(x)*
-2	103
-1	83
0	54
1	50
2	39
3	5

1. What is mathematically important for the students to understand in approaching this question?

If no idea how to proceed:
1a. What do you think is meant by “how quickly”?
1b. Clarify :“we mean average rate of change, does that help?”

If still no idea, end with:
1c. What would you say to a student who tries to apply the slope formula to this problem?

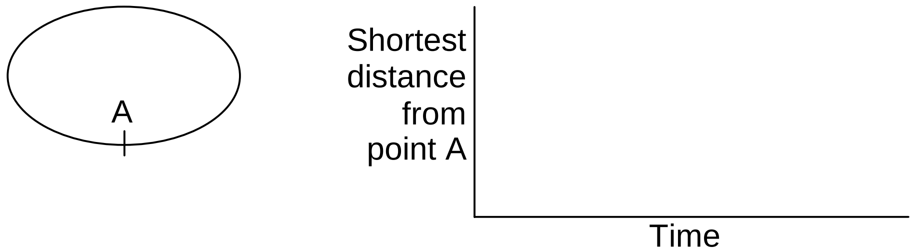
2. What should your lesson plan include? What are the goals of your lesson?

3. What real-life context could you provide for this example?

4. Describe student approaches to solving this that would be acceptable. Why would they be acceptable?

5. What difficulties will your students encounter?

12. A person is running around an oval race track at a constant speed. Construct a rough sketch showing the *shortest* distance between the runner and point *A* as a function of time. (Please talk through your thinking.)



Once the respondent has completed the question:

1. If the graph components are curved, ask “Should the graph contain curves or should it be straight? Explain”

2**.** If some components are linear, say “Tell me why you made some of the pieces straight.”

13. When the Discovery space shuttle is launched, its speed increases continually until its booster engines separate from the shuttle. During the time it is continually speeding up, the shuttle is never moving at a constant speed. What, then, would it mean to say that at precisely 2.15823 seconds after launch the shuttle is traveling at precisely 183.8964 miles per hour?

14. What is a ‘rate’, mathematically speaking?

1. What do we mean when we speak of a “rate of change” in mathematics?

2. What do we mean by an “average rate of change”?

3. Where is rate and rate of change taught in the high school curriculum?

15. How are average rate of change, average rate of change of a function over an interval, secant to a graph and tangent to a graph related?

16. Why do we use division to calculate slope?