

Exploring Teachers' Capacity to Reflect on Their Practice: An
Investigation Into the Development of Mathematical
Knowledge for Teaching

by

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ABSTRACT

Although the idea of pedagogical content knowledge (PCK) has been elaborated in numerous studies, there has been little clarification of what constitutes it or research into its development. Furthermore, studies that have investigated PCK, or mathematical knowledge for teaching (MKT) as introduced by Thompson and Thompson, have historically focused on pre-service teachers at the elementary level. This study contributes to filling these voids by investigating in-service secondary school teachers' ways of thinking that supported or constrained their capacity to reflect on their practice as they engaged in activities designed to promote powerful mathematical knowledge for teaching as proposed by Silverman and Thompson.

The study's participating teachers expressed understandings and ways of thinking that constrained their capacity to reflect productively. There were very few occasions where teachers' behaviors could be characterized as reflective. Teachers kept their personal meanings private, operated with tacit assumptions unquestioningly and unawarely, operated freely with underlying incoherent meanings, and were unperturbed when their meanings were misaligned with their students or with their colleagues. Teachers did not question their own or their colleagues' meanings, did not question their assumptions, did not question the coherence of their meanings, and did not question whether their meanings were aligned or even compatible with those of their students or colleagues. I propose an orientation toward learning and teaching of mathematics (called an empirical orientation) as a potential way of thinking that accounts for teachers' ways of

operating and helps to explain teachers' reticence to reflect. An empirical orientation hinders productive reflection because it constrains teachers' capacity to take their own actions and thoughts as objects of thought, and it constrains teachers' ability to take the points of view of their students and of their colleagues.

DEDICATION

To my wife, Teri: Thank you for making my dream, our dream. I could not have accomplished this without your constant encouragement and unending support. Are we finally at the “fun” part? Thank you for taking this journey with me...I cannot wait to see what’s next.

To my daughters, Briana and Payton: Thank you for believing in me, putting up with me, and tolerating the frequent times that I was mentally or physically absent. Throughout this journey, I have always attempted to make both of you aware that you and your mom are the most important people in my life. Things should settle down now...do you want to play?

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CHAPTER 1

STATEMENT OF PROBLEM

Stigler and Hiebert (1999), drawing on the conclusions of the Third International Mathematics and Science Study (TIMSS), highlighted the necessity for reform in mathematics education in the United States. In the years since *The Teaching Gap* (Stigler & Hiebert, 1999) there has been one generally agreed upon theme—that students are not developing a satisfactory level of mathematical proficiency (e.g., Baldi, Jin, Skemer, Green, & Herget, 2007; Gonzales, et al., 2000; Gonzales, et al., 2004).

Although the mathematics performance of elementary and secondary students has shown some improvement over the past decade on tests such as the National Assessment of Educational Progress (NAEP), this improvement has not been in all grades assessed and is not equal for all groups of students (Baldi, et al., 2007; Hall & Kennedy, 2006). Several documents have indicated the important role that teachers play, not only in what students learn, but also in any mathematics reform effort (e.g., Borasi & Fonzi, 2002; College Board, 2006; Lampert, 2001; National Council of Teachers of Mathematics, 2000; National Mathematics Advisory Panel, 2008; Stigler & Hiebert, 1999; Wenglinsky, 2002, February 13). Therefore, in order to most propitiously influence student learning, it is of paramount importance to identify characteristics of effective teaching and determine how best to develop these characteristics in the minds of teachers.

In the past two decades a number of researchers have attempted to identify and explore the characteristics of mathematics teaching and teachers. Several researchers have investigated teacher beliefs (e.g., Cooney, 1985; Ernest, 1989; Leder, Pehkonen, & Torner, 2002; A. G. Thompson, 1984, 1992); whereas, others have placed a focus on identifying the structure of teacher knowledge in an attempt to identify a knowledge base for teaching (e.g., Ball, 1988; Ball & Bass, 2000; Ball, Hill, & Bass, 2005; Rowland, Huckstep, & Thwaites, 2005; Shulman, 1986, 1987). In addition, various researchers have commented on the difficulty inherent in attempting to distinguish between knowledge and beliefs (Fennema & Franke, 1992; Grossman, Wilson, & Shulman, 1989; A. G. Thompson, 1992).

According to Ball (2003), “The quality of mathematics teaching and learning depends on what teachers do with their students, and what teachers can do depends on their knowledge of mathematics” (p. xv-xvi). Although several studies have shown that teachers’ content knowledge influences their teaching practices (Fernández, 1997; Sowder, Philipp, Armstrong, & Schappelle, 1998; Swafford, Jones, & Thornton, 1997), attempts to quantify a link between teacher subject matter knowledge and student achievement have been largely inconclusive (Begle, 1972, 1979; Byrne, 1983; Eisenberg, 1977; Monk, 1994; Rowan, Chiang, & Miller, 1997).

According to Thompson and Thompson (1996):

The most plausible explanation for these puzzling results is that the two variables, teacher subject matter knowledge and student learning, were inadequately conceptualized. These early attempts suggest that both teacher

knowledge and student knowledge and the relationship between the two are complex theoretical constructs that defy simplistic definitions and conceptualizations. (p. 2)

Although there is general agreement amongst the mathematics education community that effective mathematics teaching is dependent upon teachers' knowledge of content, the "nature of the knowledge required for such teaching is poorly specified, and the evidence concerning the nature of the mathematical knowledge that is needed to improve instructional quality is surprisingly sparse" (Ball, 2003, p. xvi).

Beginning with Shulman's (1986, 1987) seminal work regarding *pedagogical content knowledge* (PCK), several researchers have focused on the form, nature, organization, and content of teachers' mathematical knowledge (Ball, 1990; Lampert, 1991; Leinhardt & Smith, 1985; Marks, 1987; Steinberg, Marks, & Haymore, 1985; A. G. Thompson, 1984; A. G. Thompson, Philipp, Thompson, & Boyd, 1994; Wilson, Shulman, & Richert, 1987). This work has highlighted the significant influence that a teacher's mathematical understandings have on their "pedagogical orientations and decisions—on their capacity to pose questions, select tasks, assess students' understanding, and make curricular choices" (A. G. Thompson & Thompson, 1996, p. 2). According to Schoenfeld (2000), "Such studies indicate ways in which teachers' knowledge shapes what the teachers are able to do in the classroom—at times constraining their options, at times providing the support-structure for a wide range of activities" (p. 247).

In 1996, Thompson and Thompson coined the term “*mathematical knowledge for teaching* (MKT)” to refer to understandings that “cut across the types of knowledge typically embraced by phrases such as ‘content knowledge’ or ‘pedagogical content knowledge’” (p. 19). Silverman and Thompson’s (2008) framework for mathematical knowledge for teaching, which builds from Thompson and Thompson’s (1996) conception, involves powerful understandings and ways of thinking that allow a teacher to act spontaneously in ways that promote conceptual teaching. Conceptually oriented teachers express themselves “in ways that focus students’ attention away from thoughtless applications of procedures and toward a rich conception of situations, ideas, and relationships among ideas” (A. G. Thompson, et al., 1994, p. 86).

In Silverman and Thompson’s (2008) view, the development of mathematical knowledge for teaching involves developing significant personal understandings of a particular mathematical topic and transforming these personal understandings to understandings and ways of thinking that are pedagogically powerful. According to Silverman and Thompson (2008), both the personally powerful understandings and pedagogically powerful understandings develop via a process that Piaget (2001) called *reflective abstraction*.

Although, the idea of pedagogical content knowledge has been elaborated in numerous studies (e.g., Ball & Bass, 2003; Ball, Hill, et al., 2005; Ball, Thames, & Phelps, 2008; Carpenter, Fennema, Peterson, & Carey, 1988; Grossman, 1990; Ma, 1999), there has been little clarification of what constitutes it or research into its

development. In addition, the majority of studies related to mathematics teacher change have focused exclusively on teachers at the elementary level (Carpenter, Fennema, Franke, Levi, & Empson, 2000; Farmer, Gerretson, & Lassak, 2003; Jaberg, Lubinski, & Yazujian, 2002), although a limited number have included in-service secondary school mathematics teachers (Olson & Kirtley, 2005; Simon & Schifter, 1991), or pre-service secondary mathematics teachers (Cooney, 1999; Cooney, Shealy, & Arvold, 1998; Eggleton, 1995; Even, 1993). Absent from the literature has been an exclusive and detailed focus on teacher change involving in-service secondary school mathematics teachers. The current study contributes to filling these voids by exploring in-service secondary school mathematics teachers' cognitions as they engage in activities designed to promote powerful mathematical knowledge for teaching as proposed by Silverman and Thompson (2008).

If we assume that the development of mathematical knowledge for teaching, as indicated by Silverman and Thompson (2008), occurs via reflective abstraction, then investigating teachers' cognitions as they reflect on their practice would seem a propitious area of research. Therefore, from a research perspective, the current study attempts to identify the understandings and ways of thinking that support or constrain teachers' capacity to reflect on their practice. However, reflecting on one's practice entails thinking about more than the mathematics that will be taught. It also entails thinking about the mathematical realities of students who will learn that mathematics and the tasks that will be used in teaching. As such, the study attempts to answer the following research questions:

- 1) In what ways do teachers' mathematical understandings and ways of thinking support or constrain their capacity to reflect on their practice?
- 2) In what ways do teachers' images of their students' mathematics support or constrain their capacity to reflect on their practice?
- 3) In what ways do teachers' images of the tasks they will employ in their teaching, in relation to their mathematical understandings and their understandings of their students' thinking, support or constrain their capacity to reflect on their practice?

CHAPTER 2

LITERATURE REVIEW

Current educational reform movements in the United States have set ambitious goals for student learning (e.g., National Council of Teachers of Mathematics, 2000; National Mathematics Advisory Panel, 2008). Although many factors contribute to achieving these goals, Borko (2004) asserts that, “The changes in classroom practices demanded by the reform visions ultimately rely on teachers” (p. 3). Several documents have highlighted the critical role that teachers play, not only in what students have the opportunity to learn, but also in any mathematics reform effort. The Final Report of the National Mathematics Advisory Panel (2008) proclaims that, “Teachers are crucial to students’ opportunities to learn and to their learning of mathematics” (p. 35). The Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000) state that, “The improvement of mathematics education for all students requires effective mathematics teaching in all classrooms” (pp. 16-17).

Although improving classroom teaching has received renewed attention in the search for ways to improve student learning (e.g., Lampert, 2001; National Commission on Mathematics and Science Teaching for the 21st century, 2000; Stigler & Hiebert, 1999), the “question of how to support practicing teachers in implementing the reforms it envisions looms ever larger” (Farmer, et al., 2003, p. 331). According to Senger (1999), reform documents in mathematics education have “placed a great responsibility for the success of reform on the teacher ... [and]

many of these [reform] suggestions are new to the typical classroom teacher and require change at some level to make them a reality” (p. 200). Furthermore, Senger (1999) asserts that, “Many teachers’ images and beliefs about mathematics and what learning mathematics entails may be incompatible with current research and reform efforts in the field” (p. 200). Although these statements make clear the important role that teachers play both in student learning and reform efforts, questions pertaining to what specific characteristics “make-up” an effective teacher and effective teaching, and how best to promote changes in a teacher’s current conceptions, remain largely unanswered.

Early Research on Teaching

According to Shulman and Quinlan (1996), early research on teaching distinguished among four kinds of variables in the study of teaching: presage (e.g., teacher attributes that might influence their effectiveness, such as background, training, and verbal ability), context (e.g., those aspects of the settings in which teaching occurred that might influence effectiveness), process (the observable classroom interactions between teachers and students), and product (any student outcome measures of either intellectual or affective consequences of instruction). From these four classes of variables, various combinations could be formed to produce different paradigms for studying teaching (Shuell, 1996). According to Shulman and Quinlan (1996), the most widely practiced approach was “process-product,” which seeks to “document the connections between teachers’ behavior (process) and student academic outcomes (product)” (p. 409).

This early research focused almost exclusively on broad aspects of teaching (e.g., classroom management, teacher planning) and attempted to link teacher characteristics, teaching behaviors, or features of teacher reasoning to student achievement gains (Ball, Thames, et al., 2008). According to Shulman and Quinlan (1996), this research “essentially ignored the role of subject matter as a central feature of teaching and learning” (p. 409). Shulman (1986) dubbed this problem “the missing paradigm” (p. 6) and argued that subject matter “must return to its proper place in research programs on teaching, [and] ... needed to be included as a central feature of any studies of teaching” (Shulman & Quinlan, 1996, p. 409). Although these earlier studies were conducted in classrooms where specific subject matter was being taught (e.g., mathematics, reading), “attention to the subject itself and to the role it played in teaching or teacher thinking was less prominent” (Ball, Thames, et al., 2008, p. 390). What then of the role of subject matter? How important is a teacher’s subject matter knowledge?

Subject Matter Knowledge in Teaching

Several researchers have discussed the need for teachers to possess a deep understanding of the mathematics that they teach (Ball, 1988, 1993; Grossman, et al., 1989; Ma, 1999; Post, Harel, Behr, & Lesh, 1991; Shifter, 1995). Although it seems incontrovertible that “the quality of mathematics teaching and learning depends on what teachers *do* with their students, and what teachers *can do* depends on their knowledge of mathematics” [author’s italics] (Ball, 2003, pp. xv-xvi), several studies have indicated that the mathematical knowledge of teachers is not very good

(Ball & McDiarmid, 1990; Carré, 1993). Therefore, it would seem indubitable that improving the mathematical knowledge of teachers should improve student achievement, but this has historically not been the case. According to Fennema and Franke (1992), “In spite of the beliefs in the importance of mathematical knowledge and the evidence that some teachers do not have adequate knowledge of mathematics, research has provided little support for a direct relationship between teachers’ knowledge and student learning” (p. 148).

Darling-Hammond (2000), reports that studies of teachers' subject matter test scores on the National Teacher Examinations (NTE) have found “no consistent relationship between the measure of subject matter knowledge and teacher performance as measured by student outcomes” (Previous Research section, ¶ 4). Begle’s (1979) summary of findings of the National Longitudinal Study of Mathematical Abilities (NLSMA) indicated that there was no evidence to suggest a significant positive relationship between student achievement and a teacher’s subject matter knowledge. Monk (1994), using data from the Longitudinal Study of American Youth (LSAY), found that teachers' mathematics content preparation (measured by mathematics coursework) was positively related to student achievement in mathematics, but the relationship was curvilinear, and indicated that beyond five undergraduate mathematics courses there was minimal additional effect on their students’ mathematics performance. Finally, a study by Rowan et al. (1997) found a positive relationship between student achievement and teachers’ knowledge of subject matter, but the effect size was very small. In summary, past

research to determine a relationship between teacher knowledge and student achievement has been largely inconclusive (Mewborn, 2003).

Fennema and Franke (1992) argue that “before one totally dismisses the belief that teacher knowledge of mathematics is a valuable area of study, consider how most of these studies measured knowledge” (p. 148-149). In the view of De Corte, Greer, and Verschaffel (1996), the lack of a strong relationship “could be attributed to the fact that only global and superficial measures of teachers’ mathematical knowledge were used, such as the number of college mathematics courses completed or scores on standardized mathematics tests” (p. 519). According to Ferrini-Mundy, Senk, and Schmidt (2005), many of the common indicators of teachers’ knowledge (e.g., college major, college GPA, subject matter credits, teacher certification) “are proxies for knowledge...[which ignore] the question of whether this knowledge actually gets drawn on in practice” (p. 3). In a review of more than 57 high-quality research reports concerning teacher preparation, Wilson, Floden, and Ferrini-Mundy (2001) concluded that research that attempts to examine the relationships between teacher knowledge and student achievement “must be designed for and sensitive to the factors that make that relationship tenuous at best” (p. 34), and should employ measures of teacher knowledge, skill, and practice that are theoretically most likely to relate to student learning (Ferrini-Mundy, et al., 2005).

Subject Matter Knowledge for Teaching

Thompson and Thompson (1996) assert that, “Most people agree that teachers’ knowledge of mathematics is essential to their ability to teach effectively. Yet, historically, researchers have had great difficulty elucidating the roles that mathematical knowledge plays in effective mathematics teaching” (p. 2). In addition, for the little research that has been conducted, “knowledge” is largely taken to mean declarative knowledge or process knowledge. Research has not placed a focus on *meanings* or *ways of thinking* about the mathematical ideas that might be productive for interacting with students in ways that engender rich and connected meanings in students. Although teachers’ mathematics knowledge plays a significant role in shaping the quality of their teaching, the mathematics knowledge it takes to teach is inadequately understood. In the view of Ball, Bass, Sleep, and Thames (2005), “Little is known about *what* mathematical knowledge teachers need, *where* in teaching this knowledge is useful, and what it takes to make this knowledge *usable* in practice” [author’s italics] (p. 1).

The Conference Board of the Mathematical Sciences (2001) asserts that, “The mathematical knowledge needed by teachers at all levels is substantial, yet quite different from that required by students pursuing other mathematics-related professions” (p. 7). According to Ball, Bass, and Hill (2004), “Over the past 20 years...the discourse about teachers’ knowledge of mathematics [has demonstrated that] the mathematical knowledge needed for teaching is different from that needed by mathematicians” (p. 1). In addition, research has indicated that the teacher

knowledge required for teaching differs from the knowledge that a teacher has acquired in their formal education (Ball & Bass, 2003; Cooney & Wiegel, 2003). How then, does a teacher's subject-matter knowledge differ from both that of the subject-specialist and that developed through content-specific coursework?

Pedagogical Content Knowledge (PCK)

Ball (1988) developed a conceptual framework for exploring teachers' subject matter knowledge. In this framework, Ball (1988, 1991) distinguished between knowledge of mathematics and knowledge about mathematics. Ball (1991) defines knowledge *of* mathematics as "understandings of particular topic (e.g., fractions and trigonometry), procedures (e.g., long division and factoring quadratic equations), and concepts (e.g., quadrilaterals and infinity), and the relationships among these" (p. 10). Knowledge *about* mathematics includes understandings about the nature of mathematical knowledge, "where the knowledge comes from and how it is justified, what it means to do mathematics, [and] what the connections are between mathematics and other domains" (Ball, 1988, p. 21).

Although Ball's (1988) knowledge of mathematics clearly identifies the topics and procedures that could serve to identify behavioral instructional objectives, it does not address the mental processes that express highly structured thinking (P. W. Thompson, 1985). In addition, Ball's formulation gives the impression that there exists some "universal knowledge of mathematics" that is to be understood by teachers and taught to students. This view does not appear to consider that an individual's (whether this individual be a teacher or her students)

actual understandings are highly idiosyncratic, and that different ways of understanding even fairly specific ideas (e.g., constant rate of change, average rate of change) can have vast implications for how learnable those ideas are by students of the teacher having those understandings.

Shulman (1986) distinguished among three categories of content knowledge in teaching: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. Shulman (1986) identified *pedagogical content knowledge* (PCK) as “subject matter knowledge *for teaching*” [author’s italics] (p. 9). For Shulman (1986), pedagogical content knowledge consists of “the ways of representing and formulating the subject matter that make it comprehensible to others” (p. 9), such as analogies, illustrations, examples, explanations, and demonstrations; an “understanding of what makes learning specific topics easy or difficult” (Shulman, 1986, p. 9); and, common student conceptions and preconceptions.

According to Shulman (1987), “Pedagogical content knowledge identifies the distinctive bodies of knowledge for teaching” (p. 8), and represents the “blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction” (Shulman, 1987, p. 8). Furthermore, pedagogical content knowledge is “the category most likely to distinguish the understanding of the content specialist from that of the pedagogue” (Shulman, 1987, p. 8). In Shulman’s (1987) view, the key to distinguishing the knowledge base of

teaching lies at the “intersection of content and pedagogy, in the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students” (p. 15).

Research on pedagogical content knowledge. According to Shulman and Quinlan (1996), a central feature of many of the early research studies contributing to the explication of pedagogical content knowledge (e.g., Grossman, 1990; Marks, 1990; Wilson, 1992; Wineburg, 1991), was the argument that “excellent teachers transform their own content knowledge into pedagogical representations that connect with prior knowledge and dispositions of learners” (p. 409). The capacity to teach effectively, therefore, is not composed of a “generic set of pedagogical skills, [but]... is highly dependent on both content knowledge and pedagogical content knowledge, on how well one understands the subject matter and how well one understands ways of transforming the subject matter into pedagogically powerful representations” (Shulman & Quinlan, 1996, p. 409).

Although Shulman and Quinlan’s (1996) assertion indicates that there are not generic, or “universal” pedagogical skills, they do give the impression that there is a set of “universal” ways of understanding particular ideas, and that an individual’s understanding of these universal ideas can be assessed. According to Thompson (personal communication, April, 2009), “It is not ‘how well’ a person understands an idea, but what a person ‘understands an idea to be’ that is the important consideration in its learnability by students.”

In the past 20 years there have been many attempts to not only characterize a teachers' knowledge base, but to precisely define and expand upon Shulman's construct with respect to mathematics (e.g., Even & Tirosh, 1995; Grossman, 1990; Leinhardt, Putnam, Stein, & Baxter, 1991; Ma, 1999; Mason & Spence, 1999; Rowland, et al., 2005; Schoenfeld, 2000). Much of this work can be distinguished between claims that the study makes about the *form* of teacher knowledge and claims about the *content* of teacher knowledge (Sherin, Sherin, & Madanes, 2000). According to Sherin et al. (2000), claims about the form of knowledge are "concerned with the specific structures through which knowledge is organized and represented" (p. 364) and provide ways of thinking about how an individual holds their mathematical knowledge. Examples include knowledge packages (Ma, 1999), and agendas, scripts and routines (Leinhardt, et al., 1991). Claims about the content of knowledge "have to do with what the knowledge is for, or what it is about" (Sherin, et al., 2000, p. 364). Examples include substantive and syntactic knowledge (Grossman, et al., 1989), and "knowing that" and "knowing why" (Even & Tirosh, 1995). Although there are instances in which a researcher makes a claim about teacher knowledge that says something about both form and content (e.g., Leinhardt, et al., 1991), there has been little research into how this knowledge develops.

Several recent studies have shown that teachers' content knowledge influences their teaching practices (Fernández, 1997; Sowder, et al., 1998; Swafford, et al., 1997). Results from Fernández (1997), obtained through the analysis of lesson

observations of nine secondary mathematics teachers, indicate that the teachers' strong content knowledge enabled the teachers to: generate counterexamples to uncover errors in students' thinking, apply student methods to simpler or related problems, and understand and incorporate students' alternative problem solving approaches into instruction.

De Corte et al. (1996) note that, "Researchers have started to provide detailed descriptions of individual teachers' knowledge of particular mathematics topics and to relate these descriptions to their instructional actions" (p. 519). In addition, several studies have provided evidence that teachers' practice not only influences students' mathematical learning, but also influences what students have the opportunity to learn about math as a discipline (Ball, 1991, 1992; Fennema, et al., 1996; Gearhart, et al., 1999; Sanders & Horn, 1998). According to De Corte et al. (1996), for most of the aspects of pedagogical content knowledge, "there is empirical evidence of the existence of differences among mathematics teachers, as well as of the impact of these differences on teachers' behavior and student outcomes" (p. 519). Marks (1990) asserts that although the concept of pedagogical content knowledge is "difficult to pin down theoretically, ... [it does represent] a class of knowledge that is central to teachers' work and...is meaningful and useful, helping teacher educators focus on what teachers ought to know and how they might learn it" (p. 9).

Mathematical Knowledge for Teaching (MKT)

One recent research trend is to acknowledge that most of what Shulman (1986, 1987) had in mind is really mathematical knowledge that has special characteristics that support the teaching of mathematics. As such, the phrase *mathematical knowledge for teaching* (MKT), initiated with the work of Thompson and Thompson (1996), is now often used in place of pedagogical content knowledge (e.g., Adler & Davis, 2006; Ball & Bass, 2003; Ball, Bass, et al., 2005; Ball, Thames, et al., 2008; Ferrini-Mundy, et al., 2005; Silverman & Thompson, 2008). The work of these researchers expands upon the earlier work on pedagogical content knowledge by identifying and categorizing the types of mathematical knowledge that is employed in teaching (e.g., Ball & Bass, 2003), and clarifying what constitutes mathematical knowledge for teaching and how it develops (Silverman & Thompson, 2008). Although the work of these researchers is complimentary, their perspectives entail fundamentally different foci. The focus of Ball, Hill, and colleagues has been on mathematics at the elementary school level (i.e., the K-6 mathematics curriculum), whereas, Silverman and Thompson focus on secondary mathematics (i.e., grades 7-12). The current study investigated mathematical knowledge for teaching within the Silverman and Thompson (2008) framework. The differences between these frameworks and the reasons for the current study's choice will be subsequently addressed.

MKT in the framework of Ball, Hill, and colleagues. The most active work being done on defining and assessing MKT has been by Ball, Hill, and colleagues

(e.g., Hill & Ball, 2004; Hill, Ball, Blunk, et al., 2008; Hill, Ball, & Schilling, 2008; Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004). Ball, Hill et al. developed a meaning for mathematical knowledge for teaching, “which includes both the mathematical knowledge that is common to individuals working in diverse professions and the mathematical knowledge that is specialized to teaching” (Hill, Ball, Blunk, et al., 2008, p. 430), through studies at the elementary level. Ball, Hill et al. developed their framework for MKT by focusing on the “mathematical work of teaching,” in an attempt to answer the question: “What do teachers do in teaching mathematics...and how does what they do demand mathematical reasoning, insight, understanding, and skill?” (Ball, Thames, et al., 2008, p. 395). Ball and Bass (2003) define “mathematical work of teaching” as a special kind of mathematical problem solving enacted in the practice of teaching. Therefore, rather than starting with the mathematics content knowledge and attempting to determine where this knowledge resides in the teaching practice (a “top down” approach), Ball, Hill et al. have identified the “work” that a teacher does and seek to find where the mathematics lies in this work (“bottom up” approach).

Examples of mathematical work include (Ball & Bass, 2003): making mathematical ideas available to students; attending to, interpreting, and handling students’ work; giving and evaluating mathematical explanations; and, establishing and managing classroom discourse. Mathematical work also includes: (1) the ability to “unpack” (or “decompress”) ideas—the “ability to deconstruct one’s own mathematical knowledge into [a] less polished and final form, where elemental

components are accessible and visible” (Ball & Bass, 2000, p. 98); and, (2) an understanding of the “connectedness” of mathematical ideas, “both across mathematical domains at a given level, and across time as mathematical ideas develop and extend” (Ball & Bass, 2003, p. 11). In sum, mathematical work involves all of the regular day-to-day, moment-to-moment demands of teaching (Ball & Bass, 2003).

It should be noted that although Ball and Bass (2003) assert that the “mathematics as it is being learned...requires a kind of decompression, or ‘unpacking,’ of ideas” (p. 11), and that teachers’ opportunities to learn mathematics should “include experiences in unpacking familiar mathematical ideas, procedures, and principles” (Ball & Bass, 2003, p. 13), they do not address a teacher’s need to reflect on her “unpacked” ideas when she finds that her constituent meanings are not coherent. This is in direct contrast to Silverman and Thompson’s (2008) perspective in which “unpacking” is viewed as a very reflective activity.

Ball, Hill et al. have specified six domains, or categories, of mathematical knowledge that together compose MKT. These categories of mathematical knowledge emerged first through conceptual analyses of classroom teaching and were subsequently refined as Ball, Hill, and colleagues developed and implemented measures of mathematical knowledge for teaching. The six categories are distributed between two higher order domains: Subject Matter Knowledge and Pedagogical Content Knowledge (Figure 1).

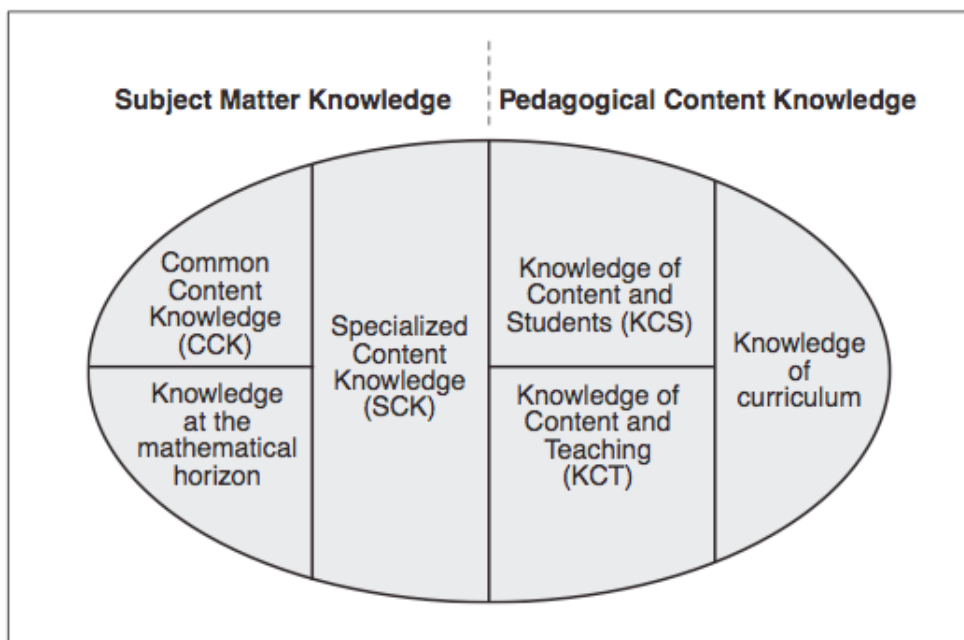


Figure 1. Ball, Hill, and colleagues' domain map of MKT (Hill, Ball, & Schilling, 2008).

Subject Matter Knowledge entails Shulman's (1986) content knowledge, and is partitioned into three categories, or sub-domains:

- (1) Common content knowledge (CCK)—“common” knowledge of mathematics that any well-educated adult should have (Ball, Hill, et al., 2005; Ball, Thames, et al., 2008); that is, mathematical knowledge that is not likely to be unique to teachers, such as the ability to correctly calculate $\frac{5}{6} \div \frac{1}{3}$ (Ball, Bass, et al., 2008);
- (2) Specialized content knowledge (SCK)—mathematical knowledge that is “specialized” to the work of teaching and that only teachers need know (Ball, Hill, et al., 2005; Ball, Thames, et al., 2008), such as the knowledge employed

in determining whether the following calculation is a fluke, whether it works

in general, and why it works: $\frac{5}{6} \div \frac{1}{3} = \frac{10}{12} \div \frac{4}{12} = 10 \div 4 = 2\frac{1}{2}$ (Ball, Bass, et al.,

2008); and,

- (3) Knowledge at the mathematical horizon—knowledge of “how mathematical topics are related over the span of mathematics included in the curriculum” (Ball, Thames, et al., 2008, p. 403).

Pedagogical Content Knowledge entails an elaboration of Shulman’s (1986) construct of the same name, and is partitioned into three categories, or sub-domains:

- (1) Knowledge of content and students (KCS)—utilized when answering questions such as: “What are common errors students make when dividing fractions? What difficulties do students typically have interpreting the answer to a division of fractions problem?” (Ball, Bass, et al., 2008);
- (2) Knowledge of content and teaching (KCT)—called upon when considering: “Which representation would you use to introduce the meaning of division of fractions? What sequence of problems would you use to begin work on division of fractions?” (Ball, Bass, et al., 2008); and,
- (3) Knowledge of curriculum.

According to Ball, Bass et al. (2005), knowledge of content and students (KCS) and knowledge of content and teaching (KCT), are “closest to what is often

meant by ‘pedagogical content knowledge’—the unique blend of knowledge of mathematics and its pedagogy” (p. 4-5).

The preceding examples are characteristic of the mathematical ideas contained within the perspective of Ball, Hill et al.—ideas that rarely focus on meanings and their roles in reasoning and understanding, ideas that are central to the Silverman and Thompson perspective. For example, regarding division of fractions, the Silverman and Thompson perspective would highlight teachers’ knowledge of what fractions are and what it means to divide them, and then how that knowledge might express itself in teachers’ instructional decision-making.

Fennema and Franke (1992) argue that teacher knowledge is not monolithic, but rather “a large, integrated, functioning system with each part difficult to isolate” (p. 148). Ball, Hill et al. have attempted to address issues concerning the isolation of categories, what they have classified as a “boundary problem,” by asserting that “it is not always easy to discern where one of our categories divides from the next [which]...affects the precision (or lack thereof) of our definitions” (Ball, Thames, et al., 2008, p. 403). The overlap and fluidity between domains in the framework of Ball, Hill et al. can be illustrated by considering the example of analyzing a student error. According to Ball, Thames et al. (2008), “Recognizing a wrong answer is common content knowledge (CCK), whereas sizing up the nature of the error...typically requires...specialized content knowledge (SCK)” (p. 401). Conversely, “familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students

(KCS)” (Ball, Thames, et al., 2008, p. 401). Furthermore, the category that is drawn upon might differ depending on the teacher. According to Ball, Thames et al. (2008), in determining what went wrong through the mathematical analysis of a student error, one teacher might consider: “What steps were taken? What assumptions [were] made?” (p. 403)—a utilization of specialized content knowledge (SCK); whereas, another teacher might determine the error “because she has seen students do this before with this particular type of problem” (Ball, Thames, et al., 2008, p. 403)—a utilization of knowledge of content and students (KCS).

Finally, Ball, Thames et al. (2008) assert that the issue as to whether these categories are the right ones is not what is most important, that “likely they are not...[and] will continue to need refinement and revision” (p. 403). It must be emphasized that the main issue with the framework of Ball, Hill, and colleagues is not with the boundaries, so their caution that a focus on the categories is unproductive fails to remove the problem—that their categories, wherever one draws a boundary, fail to include a concern for the actual meanings that teachers and students have and by which their thinking operates.

The work of Ball, Hill, and colleagues to categorize MKT has had some empirical backing. Using factor analysis, Hill, Ball et al. (2008) found that their instrument for assessing MKT does contain a scale for assessing teachers’ knowledge of content and students (KCS). In a study of pre-service elementary mathematics teachers, Morris, Hiebert, and Spitzer (2009) confirmed that the participants in their study held specialized content knowledge (SCK) that was

separate from knowledge of content and students (KCS). Furthermore, Hill, Ball, Blunk, et al. (2008) determined that MKT, as measured by their assessment instrument, is related positively to quality of instruction.

Although Ball, Hill, and colleagues' focus on identifying and classifying the mathematical reasoning, insight, understanding and skill needed in teaching mathematics is important as a means for explicating best practices in both teacher education and professional development programs, their work does not address questions pertaining to what constitutes this knowledge or how it develops. In addition, the perspective advanced by Ball, Hill et al. does not focus on mathematical meanings or mathematical ideas. In contrast to the perspective of Ball, Hill, and colleagues, the Silverman and Thompson (2008) framework focuses on characterizing elaborate connections of meanings and ways of thinking that are expressed in teachers' long-range instructional plans and in their in-the-moment decision making.

The Knowledge of Algebra for Teaching (KAT) project (Burrill, Ferrini-Mundy, Senk, & Chazan, 2004; Ferrini-Mundy & Senk, 2006; Ferrini-Mundy, et al., 2005), is a secondary level counterpart to the work of Ball, Hill, and colleagues. Burrill et al. (2004) have worked to examine the mathematical knowledge needed for teaching algebra and to develop a framework to describe algebra knowledge for teaching, where "algebra" refers to "both algebra as a strand in school mathematics and algebra as a mathematics course" (p. 1). The project has developed a conceptual framework for teaching algebra at the secondary level (Figure 2).

Tasks of teaching	Categories of knowledge					
	Core content knowledge	Representation	Content trajectories	Applications & contexts	Language & conventions	Mathematical reasoning and proof
Analyzing students' mathematical work and thinking						
Designing, modifying and selecting mathematical tasks	Bridging: Making connections across topics, assignments, representations, & domains					
Establishing and revising mathematical goals for students						
Accessing and using tools and resources for teaching			Trimming: Removing complexity while retaining integrity			
Explaining mathematical ideas and solving mathematical problems						
Building and supporting mathematical community and discourse				Decompressing: Unpacking complexity in ways that make it comprehensible		

Figure 2. Knowledge of algebra for teaching conceptual framework (Ferrini-Mundy & Senk, 2006).

This two-dimensional matrix consists of categories of knowledge (e.g., core content knowledge, representation, language and conventions) and tasks of teaching (e.g., analyzing students' mathematical work; thinking, designing, modifying, and selecting mathematical tasks)(Ferrini-Mundy & Senk, 2006). Ferrini-Mundy and Senk (2006), also identify three overarching categories bridging (i.e., making connections across topics, assignments, representations, and domains),

trimming (i.e., removing complexity while retaining integrity), and decompressing (i.e., unpacking complexity in ways that make it comprehensible), which overlay the framework.

As with the work of Ball, Hill et al. for which this project parallels, there is a clear omission regarding both a focus on mathematics as a system of meanings, and the realization that each person has an idiosyncratic system of meanings that constitutes his or her mathematics. With such a focus, as illustrated in the Silverman and Thompson perspective, a teacher's mathematics (system of meanings) might be incoherent, making it un-learnable by students. In addition, a student's mathematics (system of meanings) might be such that it "is incompatible with the teacher's mathematics, and therefore the student cannot learn the teacher's mathematics without the teacher becoming aware of the incompatibility and making appropriate adjustments in her actions" (Thompson, personal communication, April, 2009).

The Silverman and Thompson perspective promotes an informed interventionist model of teaching in which the teacher has "models of what they hope learners will learn [and] place themselves in positions to be interpreted in ways they intend by the persons they wish to affect" (P. W. Thompson, 2000, p. 427). Thompson (2000) asserts that:

The effect may be circuitous, in that ... teachers might start by saying something they know will be interpreted by students in ways that differ predictably from what they intend students understand, but which will

provide springboards for moving the discussions in directions not possible without the initial (mis)interpretations. (p. 427)

The projects of Ball, Hill et al. and Ferrini-Mundy et al. have mainly involved pre-service teachers and teacher education programs. A literature search has found little research investigating mathematical knowledge for teaching through professional development involving in-service teachers. One such study, the QUANTUM project (Adler & Davis, 2006) investigates the amount and kind of mathematics that middle school and senior school teachers need to know and know how to use “in order to teach successfully in South Africa’s diverse classroom contexts” (p. 271). This study also explores how, and in what ways, mathematics teachers preparation and support programs can and do provide opportunities for learning mathematics for teaching (Adler & Davis, 2006). Building on Ball and Bass’s (2000) notion of “unpacking” in the mathematical work of teaching, Adler and Davis (2006) assert that, “Across the range of courses, and particularly mathematics courses designed specifically for teachers, compression (in contrast to unpacking) of mathematical ideas is dominant” (p. 270).

Silverman and Thompson’s (2008) model for mathematical knowledge for teaching, which expands MKT to the high school level, builds from Thompson and Thompson’s (1996) conception of teaching for conceptual understanding, and on Simon’s (2006) construct of *key developmental understanding* (KDU), Silverman’s (2005) construct of *key pedagogical understanding* (KPU), and Piaget’s (e.g., 2001) construct of *reflective abstraction*. Silverman and Thompson (2008) focus on

developing powerful personal and pedagogical mathematical understandings that allow a teacher to act spontaneously in ways that promote conceptual teaching. According to Silverman and Thompson (2008), the impetus for their focus derives from the observation that students (at least in the U.S.), and therefore teachers, “seldom experience coherence or generativity in mathematics and that the work of teaching for understanding is predicated on coherent and generative understandings of the big mathematical ideas that make up the curriculum” (p. 501).

MKT in the framework of Silverman and Thompson. Silverman and Thompson (2008) developed their framework as an elaboration of the earlier work of Thompson, Philipp, et al. (1994) and Thompson and Thompson (1996). This elaboration comes through studies at the post-elementary level, with both pre-service secondary mathematics teachers (Silverman, 2005; Silverman & Thompson, 2005) and in-service secondary mathematics teachers (P. W. Thompson, Carlson, & Silverman, 2007).

The Silverman and Thompson (2008) framework for MKT is both developmental and cognitive, and is grounded in the idea that “teachers teach what they know” (P. W. Thompson, 1994a, p. 3), where “to know” means to have a scheme of meanings (i.e., mathematical ways of thinking and mathematical ways of understanding) that express themselves in action. Therefore, for Silverman and Thompson, “to know” entails both a teacher’s personal and pedagogical understandings of the mathematics that they teach and their “images of mathematical activity and beliefs about the enterprise of learning and teaching

mathematics” (P. W. Thompson, 1994a, p. 3). Furthermore, a teacher’s pedagogical understandings are greatly influenced by their personal understanding, because a teachers’ personal understanding of mathematical ideas “constitutes the most direct source for what they intend students to learn, and what they know about ways these ideas can develop” (Liu, 2005, p. 1). Therefore, “how” a teacher understands the content that they teach critically influences not only their pedagogical orientations, but their ability to make instructional, curricular, and assessment decisions (Liu, 2005).

Accordingly, for Silverman and Thompson (2008), “to understand” means to assimilate to a scheme, and “to assimilate” means to imbue with meaning. Moreover, “to know,” means to have a scheme to which a situation might be assimilated. Silverman and Thompson’s (2008) framework addresses the occasions when teachers have powerful understandings and ways of thinking that allow them to act spontaneously in ways that promote conceptual teaching. For Silverman and Thompson, whatever a teacher understands is at the core of his or her “mathematical knowledge for teaching”. Therefore, the phrase “mathematical knowledge for teaching” cannot refer to just good knowledge, it must refer to *the knowledge a teacher has*.

In Silverman and Thompson’s (2008) view, powerful mathematical knowledge for teaching involves pedagogical understandings that are grounded in significant, coherent personal understandings, which, in turn produce robust ways of thinking about mathematics and learning mathematics. This type of mathematical

understanding allows the mathematics to “do work” in teaching. Powerful personal understandings are defined as *key developmental understandings*, or KDUs (Simon, 2002, 2006), and describe understandings “that carry through an instructional sequence, are foundational for learning other ideas, and play into a network of ideas that does significant work in students’ reasoning” (P. W. Thompson, 2008a, p. 46). Therefore, key developmental understandings are useful goals of mathematics instruction (Simon, 2002) and can act as “powerful springboards for learning” (Silverman & Thompson, 2008, p. 502). An individual (e.g., a mathematics teacher or mathematics student) has developed a KDU when they have constructed a scheme of meanings that proves foundational for understanding a broad array of mathematical ideas and methods. The scheme of meanings comprised by number naming, counting, adding, subtracting, and whole number numeration is a KDU in arithmetic. Other KDUs include:

- Schemes of meanings comprised by measurement, multiplication, division, and fractions (P. W. Thompson, 1988, March, 1994a; P. W. Thompson & Saldanha, 2003).
- Schemes of meanings comprised by quantity, variation, covariation, constant rate of change, average rate of change, and quantitative relationships (formulas) (Carlson, 1999; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Persson, & Smith, 2003, July; A. G. Thompson & Thompson, 1996; P. W. Thompson, 1988, March, 1994b, 2008a; P. W. Thompson & Thompson, 1994).

KDUs need not be as large as these examples and are not mutually exclusive.

As schemes, KDUs can participate in other KDUs. Further examples of key developmental understandings include:

- rate as a scheme that entails two images of quantitative change: (1) a quantity that accumulates in multiplicatively-structured bits and (2) the bits themselves accumulate by way of two quantities changing in relation to each other (P. W. Thompson, 1994b; P. W. Thompson & Thompson, 1992, April);
- exponential growth as happening by a quantity whose rate of change at every moment is proportional to the quantity's size at that moment (P. W. Thompson, 2008a);
- an understanding of area as a multiplicative quantity derived from two linear measurements (Silverman & Thompson, 2008) as distinct from area as a one-dimensional quantity that is made by stringing a number of units in a pattern (P. W. Thompson, 2000);
- an understanding that equal partitioning creates specific units of quantity (Simon, 2006);
- a covariational conception of a function (Silverman, 2005);
- arc length as angle measure (P. W. Thompson, 2008a; P. W. Thompson, et al., 2007); and,
- an understanding of graphs as emergent aspects of covariation (P. W. Thompson, 2002).

These personal understandings (key developmental understandings) are then transformed from an understanding having pedagogical potential to an understanding that is pedagogically powerful (Silverman & Thompson, 2008). Pedagogically powerful understandings, defined as *key pedagogical understandings*, or KPUs (Silverman, 2005), involve a teacher's transformation of a KDU, from a way of personally understanding a particular mathematical concept, to a way of understanding how this KDU could empower their students' learning of related ideas were they to have it (Silverman, 2005).¹ Silverman (2005) asserts that, "It is against the background of the images that a teacher holds with regard to their own understandings and of the understandings they hope students will have that they select tasks, pose questions, and make other pedagogical decisions" (p. 3).

Thompson (2010) describes the process of developing a KPU as follows:

- 1) Phase 1 – the teacher has developed a rich understanding of an idea (i.e., a KDU); at this stage, student thinking is not an issue.
- 2) Phase 2 – the teacher is oriented to student thinking, but tacitly assumes that information is all that students need; the teacher projects herself (i.e., attributes her thinking onto her students) by default.

¹ The use of the term *key developmental understanding* employed here is Silverman and Thompson's interpretation of Simon's construct. Simon (personal communication, April, 2010) asserts that his original conceptualization of the term involves both a teacher's personal and pedagogical understandings of an idea (from an observer's perspective), which is more aligned with what Silverman and Thompson have identified as a *key pedagogical understanding*.

- 3) Phase 3 – the teacher becomes aware that students think differently than she anticipates they do, but the teacher is overwhelmed by a seeming cacophony of student thinking. Note that throughout Phases 1-3, the students are constructing an image of the teacher's thinking and expectations.
- 4) Phase 4 – the teacher begins to imagine different epistemic ways of thinking among students; these ways of thinking are grounded in the teacher's ways of thinking.
- 5) Phase 5 – the teacher begins to imagine how different ways of thinking among students will lead to different interpretations of what she (i.e., the teacher) says and does; the teacher begins to develop a mini-learning of actions that might help students think the way she intends.
- 6) Phase 6 – the teacher adjusts both her understanding of the mathematical idea as she adjusts her image of the ways students' think about that idea, and her understanding of how students might think about the mathematical idea as she adjusts her understanding of it.

Thompson (2010) asserts that this process (i.e., Phases 1-6) is both iterative and recursive, in that, even the development of a key developmental understanding involves considerations of others' meanings.

Therefore, in Silverman and Thompson's (2008) view, powerful mathematical knowledge for teaching involves developing significant personal understandings of a particular mathematical topic (i.e., a KDU) and transforming these personal understandings to understandings and ways of thinking that are

pedagogically powerful (i.e., a KPU). In order for this KDU to be transformed into a KPU requires the teacher to: become reflectively aware of the KDU, realize the benefit that such an understanding could provide for her students' future learning (i.e., the work that the KDU could do for her students), and build a way of thinking about how to support the development of such an understanding in the minds of her students (A. G. Thompson, et al., 1994).

In Silverman and Thompson's view, powerful MKT is constituted by a structure of key pedagogical understandings, which orient a teacher's actions to be driven by:

- an image of a coherent system of ideas and ways of thinking that she intends for the students to develop (A. G. Thompson & Thompson, 1996; P. W. Thompson, et al., 2007);
- an image of how these ideas and ways of thinking can develop in a learner (A. G. Thompson & Thompson, 1996);
- images about what her students might understand;
- an image of ways of knowing that might be problematic to students' understanding (A. G. Thompson & Thompson, 1996);
- ideas related to materials and activities that promote and support reflective discourse around the desired mathematical ideas which itself might support students developing those ideas (P. W. Thompson, 2002); and,
- an image of the students' passage through an entire curriculum (P. W. Thompson, 1985).

Thompson and Thompson (1996; 1994) contrast two teachers' mathematical knowledge for teaching. The two-part report (A. G. Thompson & Thompson, 1996; P. W. Thompson & Thompson, 1994) describes how two teachers interacted with a sixth-grade student during a four-day teaching experiment designed to teach concepts of speed. In Part II of the report, Thompson and Thompson (1996), illustrate how one of the author's (Patrick Thompson) mathematical knowledge for teaching helped resolve a dysfunctional learning situation between the first teacher (Bill) and the student (Ann).

For six months leading up to the experiment, Bill had received instruction pertaining to the sequence of images that the research team intended for the student to construct in her development of an image of speed. In addition, Bill had been advised on various pedagogical aspects of the experiment, including: situations and questions that he would initiate, possible difficulties that Ann might have, and pedagogical strategies that Bill might take in relation to these difficulties (P. W. Thompson & Thompson, 1994).

Part I of the report (P. W. Thompson & Thompson, 1994) describes Bill's attempts to teach concepts of speed to Ann and the resulting dysfunctional communication, which concluded with an "emergency call" asking Pat (Thompson) to take over the instruction. Thompson and Thompson (1994) examined how Bill's way of knowing mathematics was "reflected in the language he used in teaching" (p. 279) and how his "conceptualizations of rate, although strong and elaborate, were encapsulated in the language of numbers and operations" (p. 279). This way of

knowing ultimately undermined Bill's efforts to help Ann understand rates conceptually (P. W. Thompson & Thompson, 1994).

In contrast to Bill, Pat's instructional actions were directed by how he had conceptualized rate and his analysis of the student's (Ann) conceptions. Pat exhibited clear images of understanding rate conceptually, "how those images might be expressed in discourse, and what benefits might accrue to students by addressing the conceptual sources of their difficulties" (A. G. Thompson & Thompson, 1996, p. 3).

Prior to the teaching experiment, Pat had developed a clear *image of the mathematics that he intended for the student to develop*. According to Thompson and Thompson (1996):

The image of speed we intended Ann to construct through this unit is composed of these items, which themselves are constructions:

1. Speed is a quantification of motion.
2. Completed motion involves two completed quantities—distance traveled and amount of time required to travel that distance (this must be available to students both in retrospect and in anticipation).
3. Speed as a quantification of completed motion is made by multiplicatively comparing distance traveled and amount of time required to go that distance.
4. There is a direct proportional relationship between distance traveled and amount of time required to travel that distance. That is, if you go m distance units in s time units at a constant speed, then at this speed you will go

$\left(\frac{a}{b}\right) \cdot m$ distance units in $\left(\frac{a}{b}\right) \cdot s$ time units. Described imagistically, to say that there is a direct proportional relationship between distance and time means that one “sees” that partitioning a traveled total distance implies a proportional partition of total time required to travel that distance, and partitioning the total time required to travel a distance implies a proportional partition of the distance traveled. (p. 3)

In addition, Pat had a clear *image of how these constructions develop*. Building upon Piaget’s (1970a) theory of the epigenesis of speed, Thompson and Thompson (1996) assert that:

This image develops through a progressive internalization of measuring total distances in units of speed-lengths—distance traveled in one unit of time. When children have internalized the measurement of a total distance in units of speed-length they can anticipate that traveling a distance at some constant speed will produce an amount of time. This implies that children first conceive speed as a distance and time as a ratio (total length speed length). With this anticipation they can reason about their image of completed motion, thinking about corresponding segmentations of accumulated distance and accumulated time. Their internalization of the dual measurement process provides a foundation for their conceptualizing constant speed as a rate. (p. 3)

In preparation for his session with Ann, Pat reviewed videotape taken during Day 2 of the experiment. During this review session Pat constructed an *image of what the student understood and the essential features that were missing from Ann's images* of the situations that she had been discussing with the initial teacher (Bill). Specifically, Pat identified that Ann's conceptions were missing:

(a) an image of ... motion as entailing the simultaneous accumulation of distance from its starting point and time since it began; (b) an image of completed motion as entailing both a completed distance and a completed amount of time required to move that distance; and (c) the anticipation of ... completed motion so that she could construct a proportional correspondence between accumulated distance and accumulated time in relation to anticipated total distance and anticipated total time. (A. G. Thompson & Thompson, 1996, p. 5)

After reflecting on the images that the student understood, including those images that the student's conceptions were lacking, and the images that he intended the student to construct, Pat developed an agenda for his session with Ann. This agenda entailed *images of how to move Ann's current constructions to those that were intended* and *images of the activities that would promote and support that development*. Accordingly, Pat designed activities that would help Ann to think of:

(a) distance and time as covarying quantities; (b) the idea that going all the distance will take all the time, and conversely, that going all the time will cover all the distance; and (c) the proportional correspondence between

accumulated distance and accumulated time. (A. G. Thompson & Thompson, 1996, p. 5)

Finally, Pat's actions as he interacted with Ann were "highly image-oriented and his language was deliberately chosen to help Ann in two ways: to become oriented likewise and to form, in fact, those images" (A. G. Thompson & Thompson, 1996, p. 19). Thus, Pat's actions were constantly informed by his images of the mathematics that he intended for Ann to develop, images of Ann's developing understandings, images of how best to influence Ann's constructions, and images of the types of activities and conversations that would move Ann's conceptions toward the intended constructions. In short, Pat interacted via image-informed schemes that oriented him to strive for conceptual coherence, both in his pedagogical actions and in the student's conceptions (A. G. Thompson & Thompson, 1996).

In Silverman and Thompson's (2008) view, the development of powerful mathematical knowledge for teaching involves developing coherent personal mathematical understandings of a particular topic and transforming these personal understandings "to an understanding of: (1) how this key developmental understanding could empower their students' learning of related ideas; and (2) actions a teacher might take to support students' development of it and reasons why those actions might work" (p. 502). According to Silverman and Thompson (2008), both the personally powerful understandings and mathematical knowledge for teaching develop via Piaget's notion of reflective abstraction, where the development of a personal mathematical understanding "involves a first abstraction

via reflecting abstraction and the development of MKT involves a second abstraction (where the development of a [personal understanding] is the first-order abstraction)" (p. 503).

Reflection and the Reflective Practitioner

Reflection plays a central role in Silverman and Thompson's (2008) model, and, in light of the frequent use of the word in teacher education literature (e.g., Artzt, Armour-Thomas, & Curcio, 2008; Jaworski, 1994; Schön, 1983, 1987; Zeichner & Liston, 1996), I must explicate what various researchers mean by "reflection." In the Silverman and Thompson model of mathematical knowledge for teaching, the word "reflection" and the phrase "to reflect" are used in the sense of Dewey and Piaget. Although research on the learning and teaching (of mathematics) has indicated the importance of "reflection," not only in learning, but in teacher change (Artzt & Armour-Thomas, 1999; Cooney & Shealy, 1997; Jaworski, 1994; Kemmis, 1985; Schön, 1983), the meaning of the word "reflect" is not only different from that of Dewey and Piaget, but is not even consistent within the literature.

According to Mewborn (1999), the past two decades has seen "a proliferation of theoretical and philosophical writing on the topic of reflective thinking [and] many writers have used Dewey's work as the basis for their ideas, and they have added new layers to his work" (p. 316). Unfortunately, a significant feature of the literature on reflective thinking "is the lack of consensus about what constitutes reflective thinking; there is little agreement on its content or on the nature of the contexts that promote it" (Mewborn, 1999, pp. 316-317). In addition, although

“reflective teaching approaches to professional training and development have been associated with notions of growth through critical inquiry, analysis, and self-directed evaluation” (Calderhead, 1985, p. 43), what reflective teaching amounts to is not so widely agreed.

What others have termed “reflection.” Teacher education programs that embrace “reflective practice” usually have one of the following aims, which they label as “reflection” (Gimenez, 1999):

- to enable teachers to analyze, discuss, evaluate and change their own practice, adopting an analytical approach toward teaching;
- to foster teachers’ appreciation of the social and political contexts in which they work, helping teachers to recognize that teaching is socially and politically situated and that the teacher’s task involves an appreciation and analysis of that context;
- to enable teachers to appraise the moral and ethical issues implicit in classroom practices, including the critical examination of their own beliefs about good teaching;
- to encourage teachers to take greater responsibility for their own professional growth and to acquire some degree of professional autonomy;
- to facilitate teachers’ development of their own theories of educational practice, understanding and developing a principled basis for their own classroom work; or,

- to empower teachers so that they may better influence future directions in education and take a more active role in educational decision-making. (p. 133)

In a review of literacy literature related to varying perspectives on reflection, Risko, Roskos, and Vukelich (2002) determined that, at least three instructional activities emerge from the existing literature as a means to invite “reflective thinking” in beginning teachers: journal writing, dialogue with a coach (e.g., analyzing or critiquing excerpts of a teaching performance), and inquiry activities (e.g., action research, writing autobiographies). An example of a “reflective” journaling activity involved answering the question: “Look back through your journal and select a few representative entries. As you wrote, what connections were you making to your experiences?” (Risko, et al., 2002, p. 156). In this example, “to reflect” is something that a teacher can be told to do.

Therefore, not only is there little agreement on the meaning of the term “reflection,” the term is liberally applied to a variety of activities, most of which have nothing to do with reflection in the sense of Dewey or Piaget. It is reflection in the sense of Piaget (and Dewey) that is germane to the current study.

Dewey’s “reflective thought.” For Dewey (1910), reflection constituted the, “active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of grounds that support it, and the further conclusions to which it tends” (p. 6). Therefore, reflection acts as the “primary activity by which people gain coherence in their thinking ... [and arises]... under fairly definite conditions and in fairly specific forms” (P. W. Thompson, Milner, & Liu, 2004, p. 2).

According to Dewey (1910), “Reflection involves not simply a sequence of ideas, but a consequence—a consecutive ordering in such a way that each determines the next as its proper outcome, while each in turn leans back on its predecessors” (p. 2-3). These successive phases “grow out of one another and support one another” (Dewey, 1910, p. 3). Each phase is a step from a term of thought to a term of thought and “each term leaves a deposit which is utilized in the next term” (Dewey, 1910, p. 3). Finally, “the stream or flow becomes a train, chain, or thread” (Dewey, 1910, pp. 2-3). Dewey (1910) distinguished between five distinct steps in reflection:

(i) a felt difficulty; (ii) its location and definition; (iii) suggestion of possible solution; (iv) development by reasoning of the bearings of the suggestion; (v) further observation and experiment leading to its acceptance or rejection; that is, the conclusion of belief or disbelief. (p. 72)

Piaget’s “reflective abstraction.” According to Thompson et al. (2004):

Piaget’s notion of reflection, which he described in the active form as *reflective abstraction*, bears striking similarities to Dewey’s ideas. But there is remarkably little overlap between the theoretical foundation taken by investigators of teacher reflection and the construct of reflective abstraction that is so central to Piaget’s genetic epistemology [authors’ italics]. (p. 4)

In Piaget’s theory of knowledge development, reflective abstraction is the mechanism by which new cognitive structures are built from old structures (Piaget, 2001), and functions to abstract “properties of our processes of knowing, our action

coordinations” (Campbell, 2001, p. 4). According to Montangero and Maurice-Naville (1997), “Reflective abstraction is an endogenous process of knowledge formation [which] leads to the construction of new forms of knowledge by drawing them from knowledge or know-how that the [person] possessed already” (p. 60). Furthermore, reflective abstraction always involves two inseparable parts: a projection of something borrowed from a lower level of operating onto a higher level of operating, and a cognitive reconstruction or reorganization of what has been projected (Piaget, 1985).

According to Piaget (1985), abstraction itself engenders the operatory levels by means of alternating projection and cognitive reconstruction (or reorganization) such that only a single mechanism is involved. For Piaget (1985), the focus of reflective abstraction is on the idea of how “actions and operations become thematized objects of thought” (p. 49), where “thematization is a matter of making comparisons based on correspondences” (Piaget & Garcia, 1989, p. 12). It is these thematizations that give knowledge its coherence and generativity.

The main distinction between Dewey and Piaget’s view of reflection is the context in which they talk about reflection. For Dewey, reflection is a more or less conscious acts, where for Piaget (2001), reflecting abstraction “need not involve or lead to conscious thought, but reflected abstraction [reflecting on reflections] does” (p. 23). In addition, Dewey, in effect, is speaking about adults, whereas Piaget talks about reflection as how an individual gets from being an infant to an adult intellectually—reflection as a process that is at play from infancy to adulthood.

Piaget's notion of reflective abstraction will be expanded upon in the following chapter.

The role of perturbation. For Dewey (1910), the idea of a perturbation or cognitive conflict plays a significant role in reflective thought—reflective thinking begins with “a felt difficulty” (p. 72). Furthermore, for Dewey (1910), the “demand for the solution of [this] perplexity is the steadying and guiding factor in the entire process of reflection” (p. 11). McDuffie (2004) asserts that, “Inherent to all discussions of reflection is a problematic or puzzling event triggering reflection” (p. 35). In addition to being central to Dewey's conception of reflective thought, C. L. Thompson and Zeuli (1999) have identified the necessity of a high level of cognitive dissonance in teacher learning programs as a requirement for “changes in deeply held beliefs, knowledge, and habits of practice” (p. 342).

According to Dewey (1910), “There is a double movement in all reflections: a movement from the given partial and confused data to a suggested comprehensive (or inclusive) entire situation” (p. 79), and back from “this suggested whole—which as suggested is a *meaning*, an idea—to the particular facts, so as to connect these with one another and with additional facts to which the suggestion has directed attention” [author's italics] (Dewey, 1910, p. 79). Dewey (1910) equated meaning and understanding, and by each meant a “scheme of connections among items of thought that renders them sensible and coherent as a whole, taking the position that a concept is just an established meaning” (P. W. Thompson, et al., 2004, p. 3).

Therefore, for Dewey, the aim of reflection, although initiated by a felt confusion or

perplexity, is “to eliminate it by developing a new, inclusive meaning for the conflicting thoughts” (P. W. Thompson, et al., 2004, p. 3).

Perturbation also plays a central role in Piaget’s theory of knowledge development. In Piaget’s (1985) view, perturbation is “the most influential factor in acquiring new knowledge structures” (p. 33). Furthermore, Glasersfeld (1996) asserts that, “Knowledge has the function of eliminating perturbations” (p. 29). The role of perturbation in Piaget’s theory of knowledge development will be further expounded in the following chapter.

CHAPTER 3

THEORETICAL PERSPECTIVE

In this chapter, I will make explicit the study's underlying theoretical perspectives. Such a discussion will help to orient the reader to those perspectives that ground the study's design and analysis. A major component of the theoretical perspectives underlying the study is Silverman and Thompson's (2008) framework for the development of powerful mathematical knowledge for teaching, a framework described in detail in the previous chapter. The current chapter will consist of discussions involving: (a) a theory of knowledge development, (b) decentration, (c) mathematical realities, and (d) intersubjectivity. Throughout each of these sections an effort will be made to connect theory to practice, by examining how each perspective relates to both mathematical knowledge for teaching in the Silverman and Thompson (2008) framework and to the study itself.

Silverman and Thompson (2008) propose what Piaget called reflective abstraction as the process by which teachers create powerful ways of thinking about managing the mathematics they want students to know, and powerful ways of thinking about managing their actions in helping their students develop that knowledge. Therefore, considering the significant role that reflective abstraction plays in the Silverman and Thompson perspective of mathematical knowledge for teaching, I believe that it is necessary to situate reflective abstraction within a larger context—within a theory of knowledge development (i.e., a theory of knowing). It is not my intent to offer an exhaustive explication of either Piaget's genetic

epistemology, Glasersfeld's radical constructivism, or any theory of learning, but to highlight the ideas that are central to the current study.

I believe that such a discussion will serve four purposes. In the first place, a theory of knowledge development will act as the lens through which my comments, explanations, and assertions must be viewed. Secondly, a theory of knowing will provide a context for which to construct models of teachers' understandings as they reflect on their practice. Thirdly, a discussion of Piaget's theory of knowing will advance the need to explicate several notions that play significant roles both in the development of mathematical knowledge for teaching and in the analysis of the study's data corpus: scheme, imagery, memory, and understanding. Lastly, such a discussion will raise questions pertaining to the underlying theories of knowledge development to which the study's participating teachers subscribe.

In the following discussion I will employ Piaget's genetic epistemology and Glasersfeld's radical constructivism. Piaget's genetic epistemology deals with both the development and the meaning of knowledge. Radical constructivism is a theory of knowing that emerged from Glasersfeld's (1995) elaboration of Piaget's genetic epistemology and the cybernetic concept of viability.

In the graduate mathematics education course from which the study's data was collected, a main goal of instruction was for the teachers' to transform their practices by re-thinking their teaching, re-conceptualizing their students, and re-conceiving their mathematics. A teacher's images of mathematics, teaching, and their students are all part of a teacher's ways of operating—their cognitive

structures (or schemes). Therefore, a goal of instruction was for the teachers' to transform their ways of operating with mathematically and pedagogically powerful schemes of meanings.

In Piaget's theory of knowledge development, a person constructs new conceptions by transforming their old conceptions through reflective abstraction. From this perspective, a teacher's potential to transform their practice—to re-think their teaching, re-conceptualize their students, or re-conceive their mathematics—is contingent on their capacity to reflect. Therefore, the current study attempted to operationalize the teachers' understandings and ways of thinking and explore how these ways of operating supported or constrained their capacity to reflect on their practice. To do so required that I take the teachers' ways of operating as "data" and to look for occasions that could be indicative of reflection (e.g., exhibiting two different ways of operating, or expressing unexpected ways of operating). In order to discuss what I mean by taking the teachers' ways of operating as "data," it will be necessary to make explicit what I mean by conceptions, such as knowledge, knowing, reflecting, and understanding.

Theory of Knowledge Development

Piaget's epistemology is genetic in the sense that it describes the temporal development of cognitive structures and the general rules that govern their development (Dubinsky & Lewin, 1986). According to Piaget (1970b), "The essential functions of intelligence consist in understanding and in inventing, in other words, in building up structures by structuring reality" (p. 27). Therefore, knowledge does

not represent or depict an independent reality, but is a collection of inventions that fit the world as the experiencer experiences it (Glaserfeld, 1997).

In Piaget's view, coming to know an object does not mean making a mental copy of it. Rather, coming to know an object it means to act on it and with it (Piaget, 1971a). That is to say, in order to know objects, a person "must act upon them, and therefore transform them: he must displace, connect, combine, take apart, and reassemble them" (Piaget, 1976b, p. 12). From the most elementary sensorimotor actions (e.g., pulling, pushing) to the most complex intellectual operations, which are "interiorized actions carried out mentally (e.g., the anticipation of joining together, putting in order, putting in one-to-one correspondence), knowledge is constantly linked with actions or operations, that is, with transformations" (Piaget, 1976b, p. 12). Therefore, in Piaget's view, knowledge neither arises from objects nor from the individual (i.e., person), but from interactions between the individual and those objects (Piaget, 1976b).

In his elaboration of Piaget's theory of cognitive development, Glaserfeld (1995) formulated two basic principles of radical constructivism:

- 1) Knowledge is not passively received either through the senses or by way of communication...[but] is actively built up by the cognizing subject.
- 2) The function of cognition is adaptive...tending towards fit or viability...[and] serves the subject's organization of the experienced world, not the discovery of an objective ontological reality. (p. 51)

Glaserfeld distinguished this “experiential world” from ontological reality by defining it as the structure that a person creates in the flow of their otherwise unorganized experience (Saldanha, 2004). The activity of “creating this structure—the building up of personal knowledge by organizing one’s experiential world—is characterized as ‘operating’” (Saldanha, 2004, p. 30).

The graduate mathematics education course from which this study’s data was generated was designed to promote conceptually oriented ways of operating and to allow for the development of these ways of operating to be explored. As such, the current study investigated and attempted to identify the teachers’ ways of operating that supported or constrained their capacity to reflect on their practice.

In Piaget’s view, the building (constructing) of knowledge (cognitive structures) is genetically determined as a function, but what this function produces (knowledge), is influenced by what the individual already knows and by the specific constraints of the environment (Glaserfeld, 1997). The important thing is that the mind builds knowledge that allows it to fit (i.e., be viable) within the constraints of the environment. Therefore, what we call “knowledge” is the result of our own construction, and these constructions arise by engaging in purposive goal-directed activity (Glaserfeld, 1995), in which a person strives to make sense of and organize their personal experiences (Saldanha, 2004).

In Piaget’s view, an action is always purposive, and therefore is undertaken by a person in the context of attaining some goal. But, an action is not the same as an observable behavior. To Piaget, an action could “range in complexity from the most

basic, and directly observable, sensorimotor actions (e.g., the rooting reflex) to the most sophisticated imaginative actions (e.g., the group transformations of the square) that need not be expressed in any observable behavior” (Saldanha, 2004, p. 34). According to Piaget (1971a), “Any piece of knowledge is connected with an action and that to know an object or a happening is to make use of it by assimilation into an action scheme” (p. 6). This statement brings into play two indispensable concepts in Piaget’s theory of knowledge development: the notions of scheme and assimilation. These ideas are important to the current study, because it is the teachers’ schemes, their ways of operating, which are being investigated.

Assimilation and scheme. According to Glasersfeld (1995), assimilation must “be understood as treating new material *as an instance of something known*” [author’s italics] (p. 62). Therefore, from the point of view of an observer, a person perceives (assimilates) only what it can fit into the conceptual structures (schemes) it already has (Glasersfeld, 1995); whereas, when a person assimilates, she remains “unaware of, or disregards, whatever does not fit into the conceptual structures [she] possesses” (Glasersfeld, 1995, p. 63). Thompson (2000) asserts that, “In Piaget’s genetic epistemology, assimilation is constitutive—a scheme’s activation constitutes the subject’s experience of the ‘things’ assimilated to it” (p. 413). Therefore, to assimilate an object to a scheme is “simultaneously to tend to satisfy a need and to confer on the action a cognitive structure” (Piaget, 1976a, p. 71). These last statements suggest a need to provide meaning to the term “scheme.”

For Piaget and Inhelder (1969), a scheme is “the structure or organization of actions as they are transferred or generalized by repetition in similar or analogous circumstances” (p. 4). Therefore, a repeated action engenders a “scheme,” which is characterized by “whatever is repeatable and generalizable in an action” (Piaget, 1971b, p. 42). Moreover, assimilation is the source of schemes, with the “exception of the original reflex and hereditary schemes which orient the first assimilations” (Piaget, 1976a, p. 71). According to Piaget (1976b), “The basic property of a scheme...is to ensure ‘assimilations,’ which ... may be correct or wrong” (p. 33). This recognition that assimilations can be “correct or wrong” (from the point of view of an observer) will have implications regarding the models of teachers’ “understandings” (which will be subsequently defined) that the current study attempted to construct. Furthermore, Piaget (1971b) asserts that schemes can consist of sub-schemes and can be coordinated with one another, thus “implying the general coordination of actions” (p. 42).

For Piaget, the operative aspects of cognitive functions (i.e., actions, operations) act on transformations of a person’s experiential reality (Inhelder, 1976). Consequently, schemes depend on the operative aspects of knowledge (Inhelder, 1976). Conversely, figurative aspects of cognitive functions (i.e., perception, imitation, image) “are limited to the representation of states without reference to the transformations that result in those states” (Inhelder, 1976, p. 102). According to Inhelder (1976), “The distinction between operative and figurative is essential for understanding the distinction between a *scheme* and a *schema*”

[author's italics] (p. 102). A scheme "indicates the general structure of actions and operations (e.g., the scheme that permits one to arrange elements in an ordered series)" (Inhelder, 1976, p. 102); whereas, a schema "is merely a simplified imagined representation of the result of a specific action" (Inhelder, 1976, p. 102).

Glaserfeld (1995), elaborated the notion of scheme into "action scheme." This elaboration, initiated with the work of Cobb and Glaserfeld (1983), suggests a global structure of goal-directed activity that, together with the processes of assimilation and accommodation (which will be subsequently defined), explains how a person might come to know whatever is "repeatable and generalizable" in an action (Saldanha, 2004). According to Glaserfeld (1995), an action scheme consists of a tripartite pattern:

- 1) Recognition of a certain situation;
- 2) a specific activity associated with that situation; and
- 3) the expectation that the activity produces a certain previously experienced result.

(p. 65)

Glaserfeld (1995) asserts that the "recognition" that occurs in part 1 is always the result of assimilation. Glaserfeld (1989) offers the following example to illustrate the ideas of assimilation and scheme: "An infant quickly learns that a rattle it was given makes a rewarding noise when it is shaken, and this provides the infant with the ability to generate the noise at will" (p. 127). Glaserfeld (1989) asserts that Piaget would consider this the construction of a scheme, and relates the infant's activities to his tripartite pattern:

- (1) Recognition of a certain situation (e.g. the presence of a graspable item with a rounded shape at one end);
- (2) association of a specific activity with that kind of item (e.g. picking it up and shaking it);
- (3) expectation of a certain result (e.g. the rewarding noise).

It is very likely that this infant, when placed in its high-chair at the dining table, will pick up and shake a graspable item that has a rounded shape at one end. We call that item a spoon and may say that the infant is assimilating it to its rattling scheme; but from the infant's perspective at that point, the item is a rattle, because what the infant perceives of it is not what an adult would consider the characteristics of a spoon but just those aspects that fit the rattling scheme. (p. 125)

The current study attempted to identify and characterize the schemes expressed by the teachers as they reflected on their practice by engaging in the course activities. During these engagements, the teachers attempted to assimilate their experiences to their schemes, which, at times, resulted in an unexpected outcome, thus producing a need to modify their current schemes or construct new ones. As indicated in the previous chapter (p. 47), the schemes that instruction intended for the teachers to develop were highly imagistic.

Imagery. According to Saldanha (2004), "Piaget's notion of imagery was broad and tightly bound up with action schemes" (p. 36). Vuyk (1981) asserts that, for Piaget and Inhelder, a mental image can be a representation of "an object or

situation that was experienced at an earlier moment or an anticipation of what is going to happen” (p. 95). In the Silverman and Thompson (2008) perspective, mathematical knowledge for teaching is grounded in the idea that “to know” means to have a scheme of meanings (i.e., mathematical ways of thinking and mathematical ways of understanding) that express themselves in action.

As indicated in the previous chapter (p. 29), for Silverman and Thompson (2008), “to understand” means to assimilate to a scheme, and “to assimilate” means to imbue with meaning. Moreover, “to know,” means to have a scheme to which a situation might be assimilated. Therefore, when Silverman and Thompson speak about “ways of understandings” and “ways of thinking,” what they have in mind are “image-informed schemes” (A. G. Thompson & Thompson, 1996, p. 21). These image-informed schemes are what allow a teacher to act spontaneously in conceptually orientated ways. Furthermore, in Silverman and Thompson’s (2008) framework, powerful mathematical knowledge for teaching develops via reflection abstraction—via the construction and reorganizations of schemes of meanings.

According to Saldanha (2004), “To Piaget, images were not like static mental pictures, nor were they data structures produced by perceptual processes...[but, rather] dynamic and reconstitutive of experiences, and as entailing vestiges of mental operations that constituted them” (p. 36). Therefore, “coordinated” images can “consolidate in mental operations and can come to provide the conceptual substance by which students constitute situations” (P. W. Thompson, 1996, p. 281). According to Thompson (1996), “Piaget focused on images as the products of

acting...[and considered] images [to be] residues of coordinated actions, performed within a context with an intention" [author's italics] (p. 270).

Piaget (1967) distinguished among three types of images:

- 1) "An internalized act of imitation, a copy or transfer, not of the object as such, but of the motor response required to bring action to bear on an object ... a *schema of action*" [author's italics] (p. 294).
- 2) "In place of merely representing the object itself, independently of its transformations, this image expresses a phase or an outcome of the action performed on the object" (Piaget, 1967, p. 295)...[but] "the image [cannot] keep pace with the actions because, unlike operations, such actions are not coordinated one with the other" (Piaget, 1967, p. 296).
- 3) "[An image] that is dynamic and mobile in character ... entirely concerned with the transformations of the object. ... [The image] is no longer a necessary aid to thought, for the actions which it represents are henceforth independent of their physical realization and consist only of transformations grouped in free, transitive and reversible combination" (Piaget, 1967, p. 296).

According to Thompson (1994b), "Internalization refers to an assimilation, an initial 'image having'" (p. 181); whereas, "interiorization refers to the progressive reconstruction and organization of actions so as to enable them to be carried out in thought, as mental operations" (P. W. Thompson, 1994b, p. 181).

Thompson (1996) characterizes image as "being constituted by experiential fragments from kinesthesia, proprioception, smell, touch, taste, vision, or hearing"

(p. 267). In addition, for Thompson (1996), images may “entail fragments of past affective experiences, such as fearing, enjoying, or puzzling, and fragments of past cognitive experiences, such as judging, deciding, inferring, or imagining” (p. 268). Therefore, a person’s images can be drawn from many sources, and therefore tend to be highly idiosyncratic (P. W. Thompson, 1994c).

In 1994, Thompson restated Glasersfeld’s action scheme elaboration using language that emphasizes that a person’s recognition capacities are due to internal states which are the result of their own activity and actions and relates schemes to imagery (Saldanha, 2004). More specifically, Thompson (1994b) stated that, “A scheme is an organization of actions that has three characteristics: an internal state which is necessary for the activation of actions composing it, the actions themselves, and an imagistic anticipation of the result of acting” (p. 182). Furthermore, according to Thompson (1994b), “The imagistic anticipation need not be iconic. It could just as well be symbolic, kinesthetic, or any other re-presentation of experience—although the specific characters of its actions and anticipations may affect the generality with which a scheme is applied” (p. 182).

However, Thompson came to believe that his acceptance of Cobb and Glasersfeld’s (1983) characterization, stated above, was a mistake (P. W. Thompson, 2009). Thompson now reserves “scheme” for describing systems of mental operations, action schemata, and imagery that are larger than what Cobb and Glasersfeld (1983) described. For example, Thompson (2009) describes having a “graph scheme” that is not just about producing graphs, but rather entails the

person's having coordinated such ideas as: using coordinate systems to locate points, the role of scale in creating axes, the representation of quantities' magnitudes by putting measures on an axis, and the role of covariation in producing a graph. All these would compose a person's "graph scheme" (P. W. Thompson, 2009).

The ideas of "assimilation" and "scheme" also play significant roles with regards to the notions of understanding and memory, both of which were central to the methodological analysis of the current study's data corpus.

Understanding. Thompson and Saldanha (2003) define "to understand" as "to assimilate to a scheme" (p. 99). This definition purposely omits Skemp's (1987) use of the word "appropriate," in his definition of understanding—"to understand something means to assimilate it to an appropriate schema" (p. 29). This omission allows for a person to have understandings that someone else may judge to be inappropriate (P. W. Thompson & Saldanha, 2003). According to Thompson and Saldanha (2003), "To describe an understanding requires addressing two sides of the assimilation—the thing a person is attempting to understand and the scheme of operations that constitutes the person's actual understanding" (p. 99). This view played an important role throughout my analysis of the data corpus, because the descriptions of and rationale for course activities, including the evolving nature of instruction, served to describe what I saw as the "things" the teachers attempted to understand, and the models that I constructed of the teachers understandings and ways of thinking constituted the teachers' actual understandings.

Therefore, when I speak of a teacher's "understanding," I followed Thompson and Saldanha (2003) to mean that which results from the teacher's interpretation of "signs, symbols, interchanges, or conversations—assigning meanings according to a web of connections the [teacher] builds over time through interactions with...her own interpretations of settings and through interactions with other people as they attempt to do the same" (p. 99). The notions of assimilation, scheme, and "to understand" were central to my interpretations of the teachers ways of operating, because I attempted to identify the teachers' schemes and explore how the teachers might have modified or replaced their schemes as a result of their engagement with instruction (Steffe, 2007).

Memory. Two activities that the teachers in the study engaged in were explicitly designed for them to reflect on their experiences, specifically on activities for which they had participated as students of mathematics. Piaget's conception of memory played a significant role in the analyses of these episodes.

For Piaget (1968), memory "depends on the subject's operations... is modified during development, and depends at any given moment on the subject's operational level" (p. 2). Piaget (1968) showed samples of children, ages 3-7 years, an ordered configuration of 10 sticks, varying in size from about 9-15 centimeters, arranged in the order of increasing length. The children were asked to look at the configuration so that they could draw or describe verbally what they were shown at a later date (one week later and six months later). Children's recollections of the display were gathered one week after presenting it. Then children were asked to

recreate the display six months later (most recalled the original event). A large majority (74%) of the children had a more accurate recollection of the configuration after six months than they did after one week.

For Piaget (1968), these results indicated that a person's memory-images are constituted by the person's current operations and schemes. Children's seriation operations were more highly developed six months later, so their theory of the display aligned more accurately with what they were shown after six months than after one week. This is pertinent to the two activities indicated earlier (p. 58), because the teachers were attempting to re-construct a lesson in which they had just participated as students. They were attempting to re-construct the lesson with the cognitive structures that they engaged the lesson with. Therefore, the activity of re-construction gives insight into how the teachers assimilated the lesson (i.e., it helps to reveal the teachers' understandings and ways of thinking).

Accommodation. In the current study, as the teachers engaged with instruction, an instructional goal was for the teachers to modify or replace both their curricular knowledge and their pedagogical conceptualizations (i.e., their schemes) so that their images of what-is-to-be-learned would provide conceptually oriented guidance for their instructional actions (P. W. Thompson, 1994a). During these engagements, instruction was designed with the intent that the teachers' would experience perturbation when attempting to assimilate their experiences to their current ways of operating (to their current schemes).

According to Campbell (2001), when a person applies a cognitive structure to an experiential situation, or assimilates the experiential situation to the structure, the “structure may work without modification; it may fulfill the [person’s] goals as expected. But in some cases assimilation will be unsuccessful; applying the scheme will not lead to attaining the goal as expected” (p. 3)—there will be a perturbation. When unsuccessful assimilation has taken place, the person may need to modify the scheme, in order to accommodate it to the experiential situation. In Piaget’s view, all accommodation is triggered by a perturbation, by the person’s realization that something is amiss, does not work, or is in some way a surprising result (Glaserfeld, 1995).

There are several forms of accommodation, not all of which involve the modification of schemes. One form of accommodation involves the avoidance of the situation. For example, students (and teachers), when confronted with the experience of not understanding what they are doing, may simply decide to give up (Thompson, personal communication, June 15, 2010). Another form of accommodation by students, one that involves students’ accommodating to teachers’ expectations that they must perform despite incoherent instruction, is to avoid trying to make sense of instruction—to just memorize the performance upon which one will be judged (Thompson, personal communication, June 15, 2010). According to Thompson (personal communication, June 15, 2010), “I suspect that teachers retain this image of coping with instruction, which reveals itself in their

lack of expectation for students to understand anything and their explicit expectation that they remember and do what they've been shown.”

Accommodation may mean putting restrictions on the scheme, or differentiating it into one or more sub-schemes (Campbell, 2001). On occasion, “entirely new schemes will need to be constructed if successful accommodation is to take place” (Campbell, 2001, p. 3). Therefore, accommodation is the modification of one’s cognitive structures and is driven by one’s efforts to assimilate situations (Saldanha, 2004). According to Saldanha (2004), “In this sense, accommodation is an operation that enables [a person’s] continued assimilation of encountered situations” (p. 35).

Glaserfeld (1989) offers the following example to illustrate accommodation, which is an extension of the earlier example pertaining to an infant’s attempt to assimilate a spoon to their rattling scheme:

Shaking the spoon, however, does not produce the result the infant expects: the spoon does not rattle. This generates a perturbation (“disappointment”), and perturbation is one of the conditions that set the stage for cognitive change. In our example it may simply focus the infant’s attention on the item in its hand, and this may lead to the perception of some aspect that will enable the infant in the future to recognize spoons as non-rattles. That development would be an accommodation, but obviously a rather modest one. Alternatively, given the situation at the dining table, it is not unlikely that the spoon, being vigorously shaken, will hit the table and produce a

different but also very rewarding noise. This, too, will generate a perturbation (we might call it “enchantment”) which may lead to a different accommodation, a major one this time, that initiates the “spoon banging scheme” which most parents know only too well. (p. 125-126)

Thompson (1994b) offers a more mathematical illustration of accommodation in his elaboration of a teaching experiment done to investigate a fifth grader’s (JJ) construction of the concept of speed as a rate and the components of a conception of average rate. The teaching experiment utilized a computer microworld (called OVER & BACK), which presented two animals (a turtle and a rabbit) who ran along a number line. Thompson (1994b) describes how JJ’s initial scheme for speed “did not involve conceiving quantified motion as a quotient—it was not a ratio between distance and time” (p. 224). Rather, JJ’s initial conception of speed “was that it was a *distance*, and her initial conception of time was that it was a *ratio*” [author’s italics] (p. 224). At one point in the experiment JJ stated that the “distance rabbit ran, divided by the number of feet he ran each second, gave you a number that you must then turn into a number of seconds. JJ knew that you should get a number of seconds, but she did not” (Thompson, personal communication, June, 2009). JJ needed to perform an extra step in order to get what she knew she should get.

Thompson (1994b) describes how JJ eventually accommodated that scheme by developing a new way of thinking about distance traveled and the amount of time needed to travel that distance. According to Thompson (personal correspondence,

June, 2009), “She developed proportional correspondence as an accommodation of her original way of thinking that could give you any of time, distance, and speed just by thinking of distance and time in relation to each other.” Furthermore, “Proportional correspondence was JJ's thematization of her different, loosely related, ways of thinking about distance from speed and time, time from distance and speed, and speed from distance and time” (Thompson, personal correspondence, June, 2009).

Equilibrium (equilibration and perturbation). According to Campbell (2001), “Piaget believed that schemes tend to get applied, so assimilation and accommodation happen naturally” (p. 3). In Piaget’s view, development is driven by processes that tend toward a balance, or equilibrium, between assimilation and accommodation—the process of equilibration. Equilibration refers to “a process that leads from a state of near equilibrium to a qualitatively different state at equilibrium by way of multiple disequilibria and reequilibrations” (Piaget, 1985, p. 3).

According to Dubinsky and Lewin (1986), “There must always be assimilation in any given equilibration, for all construction begins with existing cognitive structures, even if they prove inadequate for the task at hand” (p. 60). On the other hand, there is also always accommodation, for “even in the most trivial and automatic cognitive acts, the particular experience under consideration is understood only as cognitive structures are applied to it” (Dubinsky & Lewin, 1986, p. 60). For example, Dubinsky and Lewin (1986) assert that, “Even in cognitive acts

as simple as recognition, that which is recognized must be 're-cognized,' that is, situated within an already existing system of cognitive identifications" (p. 60).

For Piaget (1985), "Reequilibration often amounts to nothing more than returning to previous states of equilibrium without creating new equilibrated forms" (p. 3); however, the "equilibrations most fundamental to development result in equilibria that are not only new but are also better than previous equilibria (Piaget, 1985, p. 3). Cognitive development, then, can be characterized as expanding equilibration—by an "increase in the range of perturbations the subject is able to eliminate" (Glaserfeld, 1995, p. 67). Piaget (1972b) asserts that, "Learning in terms of experience ... is not due to pressure passively felt by the subject but to the accommodation of its...schemes" (p. 108). In Glaserfeld's view (1995):

The learning theory that emerges from Piaget's work ... [is] that cognitive change and learning in a specific direction take place when a scheme, instead of producing the expected result, leads to a perturbation, and perturbation, in turn, to an accommodation that maintains or re-establishes equilibrium. (p. 68)

Montangero and Maurice-Naville (1997) characterize Piaget's equilibration model as follows, "Cognitive perturbations cause disequilibrium (cause or triggering mechanism of the equilibration process) which, in turn, produces regulations (the means by which equilibration takes place). The regulations are supposed to compensate the perturbations, but in so doing, they generate new constructions" (p. 108). Perturbations can be external or internal in origin and prevent a person from

attaining a goal or from using or incorporating new elements into a scheme (Montangero & Maurice-Naville, 1997)—they are obstacles to assimilation.

According to Simon, Tzur, Heinz, and Kinzel (2004), when a teacher attempts to provoke perturbation in her students, there are three possible results: (a) the students experience the intended perturbation, resulting in the students' development of the intended understandings or ways of thinking; (b) the students experience the intended perturbation, but the perturbation does not result in the students' development of understandings or ways of thinking that are compatible with that intended by the teacher; (c) the students are not perturbed.

This is particularly germane to the current study, because the study's participating teachers engaged in several activities designed to perturb their ways of operating—designed with the intent that the teachers' modify or replace their ways of operating with mathematically and pedagogically powerful schemes of meanings. Therefore, in alignment with Simon et al. (2004), in order for the participating teachers to experience the intended perturbation, resulting in the teachers' development of the intended understandings and ways of thinking (intended by the course instructor, Pat Thompson), the teachers must have schemes that “allow for a compatible interpretation (assimilation) of the situation to that of the [instructor] (i.e., a recognition of the conflict) *and* schemes that can be (and are) accommodated to construct a new understanding compatible with that of the [instructor]” [authors' italics] (Simon, et al., 2004, p. 308).

But equilibration does not explain how we come to know about our own processes of knowing, or about the coordinations of our own actions, our reasoning (Campbell, 2001). Although the theory of equilibration explains how cognitive structures can be accommodated in order to “fit” with experiences, it is not sufficient for explaining how new, more advanced cognitive structures develop out of existing structures. According to Dubinsky and Lewin (1986), equilibration “is the organizing principle of cognitive development” (p. 60), one that can be described as the process by which a person “attempts to understand a given item of information by situating that item in the [person’s] overall system” (Dubinsky & Lewin, 1986, p. 55). This “situating” occurs as the person “cognitively constructs an understanding of the item of information through the process of ‘reflective abstraction’” (Dubinsky & Lewin, 1986, p. 55). Piaget calls upon abstraction as the mechanism whereby new cognitive structures are developed. According to Silverman (2005), “Abstraction should also be thought of as a (more advanced) means of equilibration” (p. 10-11).

In Piaget’s theory of knowledge development, a person constructs new conceptions by transforming their old conceptions through reflective abstraction. Therefore, a teacher’s potential to transform their practice is contingent on their capacity to reflect.

Abstraction. According to Piaget (1980), “All new knowledge presupposes an abstraction” (p. 89), but not all abstractions are the same. Piaget distinguished among three types of abstraction: empirical, pseudo-empirical, and reflective. The focus of empirical abstraction is on objects and their properties, whereas for

pseudo-empirical empirical, the focus is on actions on objects and the properties of those actions (Gray & Tall, 2007). Reflective abstraction occurs through “mental actions on mental concepts in which the mental operations themselves become new objects of thought” (Gray & Tall, 2007, p. 23). As indicated earlier (p. 39, 46), the notion of reflective abstraction plays a central role in the development of mathematical knowledge for teaching in Silverman and Thompson’s (2008) model. Therefore, I will give only a brief overview of empirical and pseudo-empirical abstractions, and spend the bulk of the discussion on reflective abstraction.

Empirical, or simple, abstraction draws its information directly from objects (mental constructs) and consists of “merely extracting the inherent properties of the object” (Piaget, 1967, p. 25). Empirical abstraction involves the separating of the object or object’s composition into similarities and differences (P. W. Thompson, 1985) and refers to the generalization of properties of objects. Although empirical abstraction may be a transformation of a previous cognitive structure, it does not result in the creation of new knowledge, but rather involves focusing attention on particular attributes of the object (Piaget, 2001).

Empirical abstraction consists of acting on objects to extract knowledge by abstraction based on the objects themselves, such as when “picking up solids, the child will notice by physical experience the diversity of weight, its relation with volume of equal density, the variety of densities, and so forth” (Piaget, 1972b, pp. 70-71). Piaget (1971b) offers the following example to illustrate empirical abstraction:

A child, for instance, can heft objects in his hands and realize that they have different weights—that usually big things weigh more than little ones, but that sometimes little things weigh more than big ones. All this he finds out experientially, and his knowledge is abstracted from the objects themselves. (p. 16)

Pseudo-empirical abstraction is intermediate between empirical and reflective abstraction (Dubinsky, 1991), and teases out properties that the person's operations have introduced into objects (Piaget, 1985). According to Piaget (2001), pseudo-empirical abstraction occurs “when the object has been modified by the subject's actions and enriched by the properties taken from their coordinations” (p. 303). While the abstraction “proceeds with objects and current observables, as is the case in empirical abstraction, what it truly notes includes the products of the coordination of the subject's actions” (Piaget, 2001, p. 303).

According to Thompson (personal communication, June, 2009), “A young child's realization that addition is commutative is most often pseudo-empirical. They simply notice that they get the same answer no matter which way they add.” The notion of commutativity, as a necessary property of addition, comes later when the child realizes that you can count from left to right or from right to left. If you have $n = a + b$ objects lined up from left to right, from the left you are going to count a objects then b objects; from the right, you will count b objects then a objects. When the child “sees that it is the same set of objects either way, she realizes that $a + b$

must be the same as $b + a$. That is a reflective abstraction” (Thompson, personal communication, June, 2009).

In contrast to empirical abstractions, by means of which categories are formed and generalized, reflective abstraction concerns patterns derived from actions or operations (Glaserfeld, 1995). Reflective abstraction is knowledge abstracted from coordinated actions; where “the emphasis is on the transformations these actions bring about and on that which remains constant when they are performed” (P. W. Thompson, 1985, p. 196). According to Thompson (1994b), “Reflective abstraction ... is the motor of interiorization—the process whereby actions become organized, coordinated, and symbolized” (p. 183). Piaget (Piaget, 1972a) characterizes reflective abstraction as:

The mode of abstraction that derives its knowledge not from objects as in the case of simple abstraction, but from actions and from the subject’s operations. It is reflective in the twofold sense of the word, in a quasi-physical sense in that it reflects (like a light-ray) on to a higher level what it derives from a lower level, and in the cognitive sense of mental reflection. Thus defined, reflective abstraction is necessarily constructive and enlarges and enriches the structures from which it starts, reconstructing them on a conceptual level, and in this way ends at new structures, which are now operational in character, and are no longer due to the simple reflection from which they originated. (p. 221)

Therefore, reflective abstraction consists in acting on objects, but with abstraction of knowledge based on action and not on the objects themselves (Piaget, 1972b).

The following example from Thompson (2008b), involving linear functions and rate, further illustrates and distinguishes between pseudo-empirical abstraction and reflective abstraction. A student is given the following problem to solve:

I am traveling at a constant speed of 0.53 mi/min. 5.2 minutes after I first looked at my watch I was 7.3 miles from home. Define a function whose graph shows how far from home I was at every moment during the walk.

The student reasons as follows:

- After 5.2 minutes I was 7.3 miles from home
- Over that 5.2 minutes my distance increased at a rate of 0.53 mi/min
- 5.2 minutes ago I was $(0.53)(5.2)$ miles closer to home
- I started $7.3 - (0.53)(5.2)$ miles from home
- My distance to home after x minutes of traveling is $(7.3 - (0.53)(5.2)) + 0.53x$ miles

Therefore, the requested function is: $y = 0.53x + (7.3 - (0.53)(5.2))$

The student is given several similar problems to solve and solves these by reasoning in a manner similar to that described in the initial problem. The student is then given the following problem, which requires abstraction on the part of the student (the student must abstract the problem):

I am thinking of a function that has a constant rate of change that for now I will call r and its graph passes through a point that for now I will call (x_1, y_1) .

In a little while I will tell you what r , x_1 , and y_1 are. Right now, I want you to define a function that will pass through that point with a rate of change of r , so that the graph will be correct as soon as you type values for x_1 and for y_1 .

The student reasons as follows:

- Reflecting on the results of the previous problems:

$$y = 0.53x + 7.5 - (0.53)(5.2)$$

$$y = 4.3x + 8.7 - (4.3)(2.4)$$

$$y = 3.7x + 9.5 - (3.7)(8.1)$$

- The student abstracts the following from the problem (Figure 3):

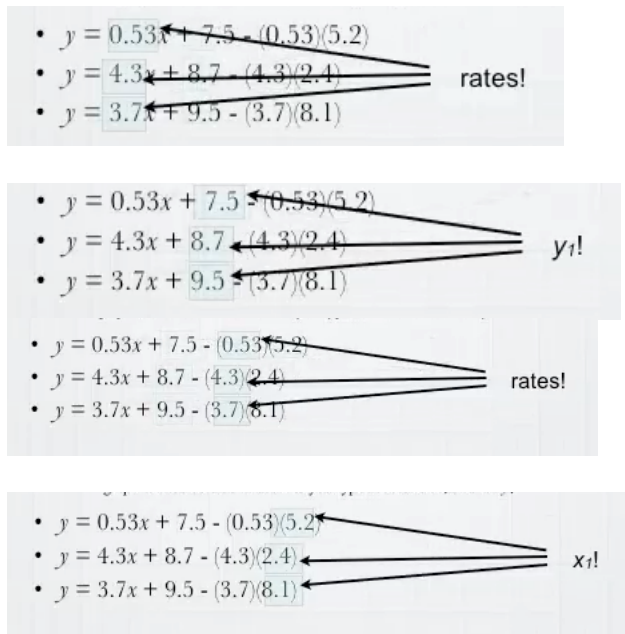


Figure 3. Pseudo-empirical abstraction.

Through pseudo-empirical abstraction, the student identified properties in the products of the coordination of her actions. Put another way, the student teased out properties of the *products* of her reasoning, but without teasing out properties of her actual reasoning. In this way she arrived at her abstraction, that $y = rx + y_1 - rx_1$ gives the general answer to the original question.

The move to reflective abstraction involves the student finding patterns in her actual reasoning, as in:

- If you change x by Δx , then you will change y by $r(\Delta x)$
- The graph passes through (x_1, y_1) with a rate of change of r . Suppose that you are at $(x, y) = (x_1, y_1)$. If you change x by $-x_1$ to make x have a value of 0, then y will have a value of $y_1 + r(-x_1)$, so $y = rx + y_1 + r(-x_1)$.

This is generalized thinking and illustrates reflective abstraction—the student’s mental operations themselves were her objects of thought, and her resulting abstraction was based on her reasoning and not just on the products of her reasoning.

According to Thompson (1994b), “Reflective abstraction is perhaps the most subtle, important, and least understood of Piaget’s constructs” (p. 183). Simon, Tzur, Heinz, and Kinzel (2004) note that the French term *abstraction réfléchissante* used by Piaget, has been translated as “reflective abstraction,” “reflecting abstraction,” and “reflexive abstraction.” I will use the term reflective abstraction, as do Silverman and Thompson (2008) in their framework for mathematical knowledge

for teaching, except in cases where I directly quote the work of others who use one of the equivalent terms.

To further complicate matters, Piaget proposed a higher-order refinement of the process of reflective abstraction, called *abstraction réfléchie* (reflected abstraction), which, in turn, has its own higher-order refinement, called *métaréflexion* (Campbell, 2001). Harel and Lesh (2003) note that the definitions given to these terms are inconsistent and in some cases conflicting in the literature. Vuyk (1981) translates *métaréflexion* as “reflexive abstraction” and asserts that, Piaget sometimes used reflected abstraction and reflexive abstraction as synonyms, and sometimes distinguished between the two. In this study, I will employ the translations given by Campbell (2001), which are consistent with the interpretations given by Steffe, Cobb, and Glasersfeld (1988), Thompson (1994b), and Silverman and Thompson (2008).

According to Piaget (1985), reflective abstraction always involves two inseparable features: “*réfléchissement* or projecting onto a higher level something borrowed from a lower level...[and a] more or less conscious *réflexion* in the sense of cognitive reconstruction or reorganization of what is transferred” (p. 29). Thompson (1994b) asserts that the first aspect of reflective abstraction is the “reconstruction of actions so as their activation is progressively less dependent on immediate experience (p. 183); whereas, the “second is an assimilation to higher levels of thought—to schemes of operations” (P. W. Thompson, 1994b, p. 183). The first is closely allied with what we normally think of as learning, while the second is

closely allied with what we normally think of as comprehending (Thompson, 1994b).

Hadamard's (1945) notion of "incubation" is highly compatible with what Piaget's means by reflective abstraction. According to Hadamard (1945), the phenomenon of mathematical invention, although marked by sudden illumination, consists of four separate stages: preparation, incubation, illumination, and verification. The preparation phase "consists of deliberate and conscious work [and constitutes] a person's voluntary, and seemingly fruitless, engagement with a problem" (Liljedahl, 2004, p. 250). Liljedahl (2004) characterizes the preparation stage as an attempt to solve the problem by "trolling through a repertoire of past experiences ... [and] creates the tension of unresolved effort that sets up the conditions necessary for the ensuing emotional release at the moment of illumination" (p. 251).

Unable to come to a solution, the person's conscious preparation sets in motion a mechanism that begins to work on the problem at an unconscious level. Hadamard (1945) refers to this as the "incubation" stage of the inventive process, an unconscious mechanism that searches for the solution. It is this unconscious work on a problem that resembles Piaget's notion of reflective abstraction. Thompson (personal communication, April, 2009) describes incubation as follows:

There is a process that's going on, it is non-rational, you're not thinking about it, you aren't deliberating, you aren't even aware that it's going on but there is this drive for coherence, for things to fit together and make sense.

To complete Hadamard's stages, after the period of incubation may come the sudden emergence of a solution, referred to as "illumination," where the solution is "transmitted from a lesser or greater depth of unconsciousness to fringe-consciousness" (Hadamard, 1945, p. 81). According to Liljedahl (2004) the phenomenon of illumination, "Is the manifestation of a bridging that occurs between the unconscious mind and the conscious mind, a coming to (conscious) mind of an idea or solution" (p. 251). Finally, in the fourth and final stage ("verification") the idea or solution (the illumination) is evaluated for correctness and to possibly act on its consequences (Liljedahl, 2004). Another way to think of Liljedahl's characterization of verification is that it describes thinking via reflected abstractions—the uses of the products of reflective abstraction in subsequent thinking.

Silverman and Thompson (2008) illustrate reflective abstraction as follows:

As a teacher thinks about the content to be taught, she envisions a student (other than the teacher) working through the material, easing through some problems and stumbling over others. The entire time, the teacher must ask herself "what must a student understand to create the understanding that I envision?" and "what kinds of conversations might position one to develop such understandings?" The prospective teacher must put herself in the place of a student and attempt to examine the operations that a student would need and the constraints the student would

have to operate under in order to (logically) behave as the prospective teacher wishes a student to do. This is reflective abstraction. (p. 508)

The graduate mathematics education course from which the current study's data was gathered was designed to provoke the teachers' to reflect with the intent that they re-conceptualize their ways of operating. Therefore, the current study attempted to identify those occasions where reflection occurred. In addition, the study attempted to determine the teachers' ways of thinking that either supported or constrained their capacity to reflect, and thus their capacity to re-conceptualize their ways of operating.

Mathematical Realities

From a radical constructivist perspective, individuals construct their own reality on the basis of their experiences, and as an observer, I have no direct access to those constructions (Glaserfeld, 1995). Therefore, I attributed understandings and mathematical realities (Steffe & Thompson, 2000b) to the study's participating teachers that were independent of my own understandings and mathematical realities. In addition, in order to speak about a teacher's understandings and mathematical realities, required that I construct a model that describes or explains what I believe the teacher understands.

Consistent with the work of Steffe and Thompson (2000b), I will use the phrase "teachers' mathematics" to refer to "whatever might constitute [teachers'] mathematical realities" (p. 268), and the phrase "mathematics of teachers" to refer to my interpretations of teachers' mathematics. Teachers' mathematics is something

I attribute to teachers independently of the researchers' interactions with them, and is "indicated by what they say and do as they engage in mathematical activity" (Steffe & Thompson, 2000b, p. 268). Furthermore, teachers' mathematics is not considered as "knowable independently of the [researcher's] ways and means of perceiving and conceiving" (Steffe, 2007).

Therefore, a goal of the current study was to construct models of teachers' mathematics—to construct "mathematics of teachers." These models, which include the modifications teachers make in their ways of operating as a result of their engagements with activities (Steffe & Thompson, 2000b), serve as my interpretations of the teachers' mathematical realities. Occasions where the teachers expressed conflicting or unexpected conceptions or ways of thinking were of particular importance, because these occasions were indicative of times where the teachers were reflecting.

Since I had no access to the teachers' actual understandings, the best that I could do is to employ an analytic method that allowed me to determine the viability of these models of teacher's conceptions in hopes that the models became increasingly stable. A discussion of the analytical methods that were employed in the study will be given in the next chapter, pertaining to methodology.

Decentration (Decentering and Egocentrism)

According to Silverman and Thompson (2008), in the development of MKT, the transformation of a personal key developmental understanding of a particular mathematical concept to a key pedagogical understanding requires decentering—

the “attempt to imagine one’s experience from another perspective” (Steffe & Thompson, 2000b, p. 4). In order to make this transformation, the teacher must attempt to see the mathematical understanding from the perspective of another (their students) and construct an image of: “(1) how this KDU could empower their students’ learning of related ideas; (2) actions a teacher might take to support [their] students’ development of it and reasons why those actions might work” (p. 502).

According to Montangero and Maurice-Naville (1997), “The notion of decentering accounts for the direction of cognitive development (p 94). Piaget (1962) asserts that:

A [person] whose perspective is determined by his action has no reason for becoming aware of anything other than its results; decentering, on the other hand, i.e., shifting one’s focus and comparing one action with possible others, particularly with the actions of other people, leads to an awareness of “how” and to real operations. (p. 9)

Therefore, since cognitive activity is under the control of the person’s actions, knowledge will be dependent on the person’s point of view (Montangero & Maurice-Naville, 1997). Centration limits knowledge growth due to a “cognitive fixation on the immediate goals of isolated personal activity, a focus on one’s own point of view and on what is imposed by perception” (Montangero & Maurice-Naville, 1997, p. 96). Decentering frees a person from these limitations, so that they

can situate their own point of view within a set of possible points of view (Montangero & Maurice-Naville, 1997).

Therefore, decentering is the process whereby a person liberates herself from egocentrism (Montangero & Maurice-Naville, 1997). Montangero and Maurice-Naville (1997) assert that the concept of egocentrism has been a source of misunderstanding since Piaget introduced the term in 1920. According to Piaget (1962), "Cognitive egocentrism, as I have tried to make clear, stems from a lack of differentiation between one's own point of view and the other possible ones, and not at all from an individualism that precedes relations with others" (p. 9). Therefore, egocentrism does "not express a deliberate choice in favor of one's own point of view, because it results from a deficiency in the grasp of consciousness of the very idea of point of view" (Montangero & Maurice-Naville, 1997, p. 101).

The idea of decentration can play a role in the mathematics classroom when a teacher "sets their own mathematical ways of operating temporarily to the side in order to focus on students' ways of operating (Hackenberg, 2005, p. 47). The development in teachers of a "schema of decentering" (Piaget, 1962), as way of thinking, can have the affect of sensitizing "them to the reasoning of others and encourages decentering from their own thought processes in an attempt to understand the thinking [of their students]" (Steffe, 1990, p. 179). According to Steffe (1990), "This is an essential attitude for a mathematics teacher if he or she is to encourage the mathematical activity of his or her students" (p. 179).

One of the goals of the graduate mathematics education course from which the current study's data was gathered was to promote (in the teachers) ways of operating that orient the teachers to decenter. Therefore, exploring the teachers' ways of operating can help to identify those ways of operating that supported or constrained the teachers' capacities to decenter. In addition, decentering played an important role in my construction of models of teachers' understandings. The construction of these models required that I decenter, or "suspend" my own mathematical ways of thinking in an attempt to adopt the view of the teachers' mathematical ways of thinking—I attempted to decenter and imagine what the teachers' experiences might have been like, as I tried to understand their assimilated situations (Steffe, 1990).

Since the graduate mathematics education course from which the study's data was gathered involved many whole-class and group activities, the idea of learning as a social phenomenon must be addressed. Such a discussion necessitates the introduction of Glasersfeld's intersubjectivity, for which the notion of decentration is central. According to Steffe and Thompson (2000a), "It is no accident that Piaget considered decentering—the attempt to imagine one's experience from another perspective—so central to both socialization and to reflective abstraction. Egocentric thought cannot support the construction of intersubjective knowledge" (p. 196-197). Montangero and Maurice-Naville (1997) assert that "the process of coordination of actions and operations is the most important instrument of decentering ... and this coordination is both individual and social" (p. 96).

Social Interaction (and Intersubjectivity)

According to Carlson and Thompson (2005), the core problems of radical constructivism from its beginning, “entailed the question of how physically disconnected self-regulating organisms could influence each other to end in a state where each presumes there is essential common agreement on what is their shared environment” (p. 2). From a radical constructivist perspective, what we take as “collective activity is constituted by interactions among individuals each having schemes by which they generate their activity and by which they make sense of other’s actions” (Carlson & Thompson, 2005, p. 3). This is in contrast to a social constructivist perspective, where “collective activity and social interaction are given, predating any individual’s participation in it. The individual accommodates to social meaning and practice” (Carlson & Thompson, 2005, p. 3).

Throughout the Functions 3 classroom activities, one of the instructor’s (Pat) goals was to have teachers “make their mathematical knowledge explicit and to find limits in their ways and means of operating” (Steffe & Thompson, 2000b, p. 290). Although there were some occasions where the teachers expressed themselves individually through written assignments, the majority of the activities involved interactions. These interactions arose during instructor-whole class, instructor-group, instructor-teacher, or group activities. These mutual interactions are important, because they were also occasions where the teachers expressed themselves; occasions that generated artifacts (through audio and video-tape), and which offered the possibility of constructing models of the teachers’ understandings

and ways of thinking. These interactions also serve another purpose, one involving the notion of intersubjectivity.

Carlson and Thompson (2005) assert that social interactions “emerge by way of individuals’ mutual interpretation of what each perceives as other-oriented action” (p. 3). Therefore, according to Carlson and Thompson (2005):

As two people interact we understand both to be predicating their actions on an image they have of the other and implications that image has for their actions. Each person acts self-reflexively; each person imagines and acts, sometimes reflectively but always through imagery built from past interactions. (p. 3)

According to Glasersfeld (1995), “To say two people communicate successfully means no more than that they have arrived at a point where their mutual interpretations, each expressed in action interpretable by the other, are compatible—they work for the time being” (as cited in Thompson, 2002, p. 194). Thompson (2002) defines intersubjectivity as “the state where each participant in a socially-ongoing interaction feels assured that others involved in the interaction think pretty much as he or she anticipates they do” (p. 194). Therefore, intersubjectivity is not a claim of identical thinking, but, rather, a “claim that no one sees a reason to believe others think differently than he or she presumes they do” (P. W. Thompson, 2002, p. 194).

The identification of those instructor-teacher or teacher-teacher interactions where intersubjectivity has been “punctured” were important to the current study

for at least two reasons. Firstly, during instructor-teacher interactions, places where Pat modified his instructional plan indicated occasions where he had identified something in the teachers' understandings that needed to be addressed. These occasions often signaled places where constructions of models of teachers' mathematics were propitious toward getting at their ways of operating. Secondly, during teacher-teacher interactions, the identification of intersubjective "puncture" often established those occasions where one, or more of those in the interaction were reflecting.

CHAPTER 4

METHODOLOGY

The goal of this chapter is to make explicit the methodology employed in both the design of the course from which the data was gathered and the analysis of the data. As such, it will be necessary to describe the background and context of the study, so that the prior chapter's discussion pertaining to the underlying theoretical perspectives will have practical significance. The chapter will be divided into two main sections: the experimental (course) methodology and the analytical methodology. The experimental methodology section will describe the graduate level mathematics education course for secondary school teachers from which the study's data was gathered. Included in this description will be an outline of the larger project for which the course is situated and the design methodology of the course itself. The analytical methodology section will describe the methods that I employed to analyze data from both a global and local perspective, and will include discussions of those methods that contributed to the study's analytical design: (a) conceptual analysis, (b) grounded theory, and (c) retrospective analysis of classroom video.

Experimental Methodology

As indicated above, the data for the study was generated from a graduate level mathematics education course for in-service secondary school teachers. Prior to making explicit the design methodology for the course, it will be necessary to

describe the background and context of the study itself, including situating the course within its larger parent project.

Background and context of the study. This study is part of a NSF sponsored research project, the Teachers Promoting Change Collaboratively (TPC²) Project, conducted by Professor Patrick Thompson and his research team at Arizona State University. The project participants consisted of local high school mathematics teachers from a large district in the southwestern United States.

Project description. The larger project goal was to help teachers move from a very teacher-centered orientation to a very student-centered orientation. In addition, the project attempted to develop a model of teachers who are invested in students' understanding and a model for influencing teaching practices. These goals were grounded in the underlying project belief that:

- Students count most;
- Student learning can be influenced by influencing teacher practices;
- All students can develop deeply coherent, richly connected, powerful mathematics; and,
- It takes twelve years to learn calculus.

The teacher-participants were involved in three project-specific components: Professional Learning Communities (PLCs), three graduate-level mathematics education courses taken over three years, and weekly Reflecting on Practice Sessions (RPSs). The mathematics professional learning communities (PLCs)

consisted of groups (three to five) of high school mathematics teachers whom cooperated toward the following common goals:

- Improve student learning by enhancing their teaching and enhancing the mathematics they teach;
- Dig deeply into mathematical ideas and how students come to understand them; and,
- Work together to plan instruction and assess student learning.

The Reflecting on Practice Session's (RPSs) were designed to support the PLC members in working together to use the knowledge learned in the courses to improve their instructional practices, and to promote the member's reflection on their instructional practices and the effects of their practice on student learning. Artifacts generated from the Reflecting on Practice Sessions were used to understand the process of change, train other PLC facilitators, and plan future RPSs.

The artifacts specific to this dissertation were generated from the last graduate course (of three) that many of the teachers participated in as part of the Teachers Promoting Change Collaboratively (TPC²) Project.

Courses for teachers: Background. As indicated earlier (p. 84), many of the participating teachers enrolled in three graduate-level mathematics education courses as part of their work in the project. These courses were taken over three years and were designated "Extended Analysis of Functions I, II, and III" or "Functions 1, 2, and 3". The third course was officially referred to as "Functions 2/3," rather than "Functions 3," because one of the larger teacher cohorts were

taking the class as their second course in the project; whereas, the other teacher cohorts were taking the class as their third course. The “Functions” courses involved activities that were intended to transform the participating teachers’ personal and pedagogical understandings of the mathematics they teach. All three of the Functions courses were about mathematics—teaching it, doing it, and thinking it. In addition, each Functions course focused on two qualitatively different kinds of activities: content activities and pedagogical activities. Each of these activities was grounded in the notion that to teach for understanding, teachers must have an image of what they want students to understand.

Content activities were aimed at the teachers as students of mathematics. The activities were designed to enrich the teachers’ images of algebra, geometry, and calculus. Pedagogical activities were aimed at the teachers as teachers of mathematics. The activities were designed to promote the development of images of teaching that are conducive for students to build powerful mathematical understandings. Pedagogical activities often employed artifacts created from other areas of the project to promote the teachers’ reflections on the practice of teaching. The artifacts included material created from a case study of one TPC² teacher’s transformation of her curriculum and her instructional practices (reformed-Algebra 1 case study), and included material from the teachers’ Reflecting on Practice Sessions (RPSs).

The design of Functions 3 was in the context of the project’s acknowledgment that, for a large majority of teachers, no matter the quality of what

teachers learned in Functions 1 and 2, it was not having a major influence in their instruction. By and large, they taught the same mathematics even though they attested to large changes in their own mathematical understandings. Outcomes from the Functions 1 and 2 courses indicated that, while beneficial, these courses did not affect teachers' curricular knowledge sufficiently to have them see these new ways of thinking as replacements for what they already taught; and, that future courses need to give explicit attention to teachers' images of how one actually affects students' thinking. As such, Functions 3 course activities were designed explicitly to create contexts for which the teachers would reflect on their own activity in a way that might translate into them doing things differently in their own classrooms.

The project attempted to affect the teachers' mathematical content knowledge in Functions 1 and 2. The Functions 3 course attempted to get the teachers to reflect on their own mathematics understandings, and, through reflection, develop ways of thinking about how their own students might develop similar understandings. In addition, Functions 3 was designed to allow for this transformation to be explored. Thus, the Functions 3 course was designed to both promote and investigate the development of mathematical knowledge for teaching in the Silverman and Thompson (2008) framework.

The Functions 3 course. The Functions 3 course met for a total of 14 class sessions (7 summer meetings, 7 meetings during the fall semester, all sessions lasting 2.5 hours). A listing of all class meetings, and their corresponding meeting dates is displayed in Table 1.

Table 1

Functions 3 Class Meetings

Functions 3 Class Meetings	
Summer	
Class Number	Meeting Date
1	June 18, 2007
2	June 20, 2007
3	June 21, 2007
4	June 25, 2007
5	June 27, 2007
6	June 28, 2007
7	July 2, 2007
Fall Semester	
8	August 20, 2007
9	August 27, 2007
10	September 17, 2007
11	October 15, 2007
12	November 5, 2007
13	November 19, 2007
14	December 3, 2007

Sixteen teachers enrolled in Functions 3, two of which participated in only the summer portion of the course (Classes #1-7). Five of these teachers took Functions 1 in the fall of 2005 from Pat Thompson, eight took Functions 1 from another instructor in the summer/fall of 2006, and six took Functions 2 from Pat Thompson in the summer/fall of 2006. A full description of Functions 1 and Functions 2 will not be provided here, but, since the focus of this study is on Functions 3, this will not be essential for the discussion.

The main goal of the Functions 3 course was for the teachers to transform their current ways of operating with personal and pedagogically powerful schemes

of meanings. More specifically, the Functions 3 course was designed to address: (a) teaching for student understanding, (b) developing mathematical knowledge for teaching (MKT), and (c) re-conceptualizing classroom practices to incorporate conceptual conversations. Functions 3 was conducted as much as a teaching experiment as a course, with the aim of understanding teachers' thinking about issues (a) through (c) while at the same time trying to affect their thinking positively in regard to them.

Functions 3 was given in three phases: (1) Self-discovery and reflection on relationships between teaching and learning; (2) Analysis of videos from the reformed-Algebra 1 case study component of the project; and, (3) Re-conceptualization of instruction on specific topics from the viewpoint of conducting classroom conceptual conversations.

Experimental (course) design. Since promoting the development of MKT was a primary goal of the Functions 3 course, and at the core of the Silverman and Thompson (2008) model is reflection, the course was designed to give the teachers opportunities to reflect on their own activity, both as learners and as teachers. Therefore, course activities were designed explicitly to create contexts for which the teachers would reflect on their own activity in a way that might translate into them doing things differently in their own classrooms.

There are five aspects to the course design that influenced the potential for collecting data on the teachers' ways of operating: (a) teaching experiment

methodology, (b) the *Didactic Triad*, (c) conceptual conversations, (d) reflective discourse, and (e) didactic objects and didactic models.

Teaching experiment methodology. In a constructivist teaching experiment, researchers seek to understand and explain how students operate mathematically and how their ways of operating change in the context of teaching (Steffe & Thompson, 2000b). In Piaget's theory of knowledge development, a person constructs new conceptions by transforming their old conceptions through reflective abstraction. From this perspective, a teacher's potential to transform their practice—to re-think their teaching, re-conceptualize their students, and re-conceive their mathematics—is contingent on their capacity to reflect. Therefore, as a teaching experiment, Functions 3 was designed and conducted to investigate the teachers' ways of operating that supported or constrained their capacity to reflect on their practice.

According to Steffe and Thompson (2000b), "Researchers' imputations to students of mathematical understandings and operations are constrained by the language and actions they are able to bring forth in students" (p. 267), and by students' essential mistakes—"mistakes that persist despite researchers' best efforts to eliminate them" (Steffe & Thompson, 2000b, p. 267). Since the sources of essential mistakes reside in students' current mathematical knowledge, a goal of the researcher in a teaching experiment is to experience these constraints in attempts to build models of students' mathematical understandings and to establish boundaries to these models (Steffe & Thompson, 2000b).

Since the instructor for Functions 3, Pat Thompson, was also the principle-investigator for the project, he has able to enact the role of “Teacher as researcher and model builder” called for in Thompson and Thompson (1996). Therefore, instruction unfolded as Pat built models of the teachers’ understandings and established boundaries for these understandings by determining where their knowledge “broke down.” In addition, throughout instruction, Pat shaped the context of what the teachers were reflecting on by making them “confront” the issues that arose in their conceptions.

The didactic triad. A major focus of the Functions 3 course involved discussions and activities centered around the *Didactic Triad* (P. W. Thompson, 2009). The *Didactic Triad*, is a characterization of the inter-relationships among (1) creating a clear statement of the mathematical ideas and ways of thinking one intends that students learn from instruction, (2) creating tasks and materials designed to support students’ learning in the context of instruction, and (3) the design of instruction aimed at supporting students’ learning while engaging with tasks and materials designed to support that learning. The *Triad*, when taken as an object of discussion and reflection, provides a means of support that re-orient teachers productively as they design instruction, and emphasizes the interrelationships amongst three aspects of instruction: learning goals—understandings or ways of thinking that the teacher intends for their students to develop; tasks/materials—tasks, activities, and materials that the teacher chooses or creates to support the development of the intended ideas; and, teaching—

choreographing activities and conversations about those activities in a manner that enhances the likelihood that students develop the intended understandings or ways of thinking.

Effective teachers tend to consider each aspect in relation to the others. The TPC² project has found that teachers who successfully changed their instruction to emphasize students' mathematical thinking tended to use the *Didactic Triad* in a very special way. Rather than looking at each component in isolation, these teachers remained mindful of the other aspects of the *Triad* when focusing on any one of them.

Conceptual conversations. The course was designed so that a normative behavior of all course participants was the expectation that they express themselves with coherence and meaning. This expectation involved the teachers' active participation in conceptual conversations (A. G. Thompson, et al., 1994; A. G. Thompson & Thompson, 1996). Thompson (2006) defines a conceptual conversation as "one that has a diminished emphasis on technique and procedure, and an increased emphasis on images, ideas, reasons, goals, and relationships" (§ 1). According to Thompson (2006), "People conversing conceptually speak in ways that make their meanings clear to others in the conversation; they speak about ideas and ways of thinking" (§ 1).

Therefore, as participants in the Functions 3 course, it was expected that the teachers would be fully committed to understanding what others were trying to say and to making themselves understood in the way that they intended. To do so

required the teachers to “speak sensibly and coherently about their reasoning, their judgments, and the context giving rise to them” (P. W. Thompson, 1994a, p. 3).

The idea of conceptual conversations played an important role in the analysis of the data, because a teacher who makes her meanings clear to others in a conversation, increases the possibility of making herself clear to an observer of the conversation. This played a particularly important role in regards to building models of the teachers’ images and understandings.

Finally, conceptual conversations are pedagogically important in that each participant is transformed by their attempts to engage in a process that involves everyone participating in ways that make their meanings clear to others in the conversation (P. W. Thompson, 2006). It is in this sense that conceptual conversations assisted me, as an observer of the conversations, in identifying points where the teachers reflected and re-organized their mathematical understandings during discourse. This brings up the idea of reflective mathematical discourse.

Reflective discourse. Throughout the Functions 3 course, instruction was designed so that the teachers engaged in activities that supported reflective discourse. Cobb, Boufi, McClain, and Whitenack (1997) define reflective discourse as being “characterized by repeated shifts such that what the students and teacher do in action subsequently becomes an explicit object of discussion” (p. 258).

Cobb et al. (1997) investigated the nature of reflective discourse and its relationship to participants’ cognitive development and contended that reflective discourse constitutes conditions for the possibility of mathematical learning, but

does not inevitably result in the learning of each participant. Therefore, although instruction was designed to support reflective discourse, I followed Liu's (2005) assertion that I cannot assume that each teacher reflected and reorganized their mathematical understanding in the discourse. In addition, as described by Liu (2005), this prevented me from taking for granted that teachers' understandings were in line with the intended instructional goals, or what appeared to be collectively understood.

Didactic objects and didactic models. The design of instruction for the Functions 3 course was organized around the notions of didactic objects and didactic models (P. W. Thompson, 2002). Although the course was designed to give the teachers opportunities to reflect on their practice, and thus promote the development of powerful mathematical knowledge for teaching as described by Silverman and Thompson (2008), the desire for the teachers to undergo this development is of secondary importance to the study. More pertinent to the study is exploring how the teachers responded to instruction designed to provoke the teachers to reflect on their practice. In order to generate models of the teachers' understandings of the ideas addressed in instruction as they reflected on these ideas, it was necessary for my analysis to take into account the teachers' experiences within instructional interactions (Saldanha, 2004).

Thompson (2002) describes a didactic object, as "a thing to talk about" (p. 198), that exists at the boundary between original and reflective conversations, and is designed with "the intention of supporting reflective mathematical discourse

involving specific mathematical ideas or ways of thinking” (P. W. Thompson, 2002, p. 211). As Thompson (2002) notes “objects cannot be didactic in and of themselves; rather, they are didactic because of the conversations that are enabled by someone having conceptualized them as such” (p. 198). Thus, a didactic object is a tool that “the teacher [or instructional designer] envisions as having the possibility of engendering productive discourse” (Silverman, 2005, p. 19). Silverman (2005) asserts that the “knowledge necessary to conceive of a didactic object is a key pedagogical understanding” (p. 20).

Didactic objects played a key role in the design and implementation of the Functions 3 course, because the course was designed to provide an environment such that the teachers would “purposively [participate] in conversations engineered and choreographed to foster reflection on some mathematical thing—an object, an idea, or a way of thinking” (Saldanha, 2004, p. 39). In addition, didactic objects played a key role in generating data to be analyzed. According to Saldanha (2004), “Didactic objects are useful tools for helping develop conceptual analyses; when employed in teaching experiments to produce environments that foster reflective mathematical discourse, they help generate observable information” (p. 40). As such, didactic objects “facilitate researchers’ formulation of hypotheses about students’ understandings and development in relation to their engagement with instruction” (Saldanha, 2004, p. 40). In the current study, the use of didactics objects, and the expectation that teachers interact via conceptual conversations,

helped to generate observable data (the teachers' ways of operating) that facilitated my formulation of hypotheses related to the teachers' capacities to reflect.

The creation of a didactic object requires an "image of all that needs to be understood for someone to make sense of the didactic object in the way he or she intends" (P. W. Thompson, 2002, p. 212)—it requires a framework for thinking about the purpose and aim of the didactic object. A didactic model (P. W. Thompson, 2002) is an image that guides "decisions concerning how the didactic object will be used, such as what conversations to have around them and what issues to raise in those conversations" (Saldanha, 2004, p. 40). Thompson (2002) notes that the didactic model is "not a model for students of something they have experienced" (p. 212), but a model for instructional designers of "what they intend students will understand and how that understanding might develop" (P. W. Thompson, 2002, p. 212). Silverman (2005) hypothesizes that key pedagogical understandings are "necessary but not sufficient for one to develop a didactic model" (p. 21).

Thompson (2002) describes this view of instructional design as creating "a particular dynamical space, one that will be propitious for individual growth in some intended direction, but will also allow for a variety of understandings that will fit with where individual students are at that moment of time" (p. 194). This form of instructional design allows for "the possibility that successful implementation of an instructional sequence might lead to students' learning something very different from what was intended" (P. W. Thompson, 2002, p. 213). This separation between descriptions of instructional sequences and descriptions of learning is the main

distinction between this form of instructional design and Simon's (1995) hypothetical learning trajectory. In this study, the ideas of didactic objects and didactic models will be reflected in my descriptions of instructional activities, and will help in identifying those occasions that were propitious for teacher reflection.

Analytical Methodology

As indicated earlier, the data corpus analyzed for this study came from artifacts generated during a graduate level mathematics education course for in-service secondary mathematics teachers. Data included video recordings of all Function 3 class meetings made with two cameras (14 class meetings x 2.5 hours per class = 35 hours), the teachers' written work (including group white-boarding assignments), and research documents made throughout the course (e.g., individual class content logs, and graduate student observation reflections). In addition, 10 of the 14 teachers submitted video of their own classroom instruction (as part of an out of class assignment during the fall semester), which were used as didactic objects to discuss in Functions 3.

The course was designed to both promote and investigate the development of mathematical knowledge for teaching in the Silverman and Thompson (2008) framework. As such, the course tasks and activities were specifically designed to provide opportunities for the participating teachers to reflect on their practice, which includes their understandings of the mathematical meanings they expect students to learn.

A recurring theme throughout the discussion of the theoretical perspective and its relation to the current study was the necessity to construct models that represented the teachers' understandings and ways of thinking—their ways of operating. In addition, it was necessary for my analysis to allow for the determination of the viability of these models in hopes that the models become increasingly stable. According to Steffe (2007), a model is considered as viable or as fitting with the researcher's experience (including observed phenomena) of teachers so long as it is not countermanded by what teachers say or do. Therefore, the methodology employed to analyze the data corpus was such that it supported the creation of models of teachers' ways of operating and possessed a means for determining the viability of these models.

Conceptual analysis and a modified version of grounded theory met these requirements. In addition, although my analysis of the data focused on those occasions where the teachers were provided with the opportunity to reflect on their practice, it was necessary for me to construct models of teachers' ways of operating throughout the larger data corpus—so that I could explore how these ways of operating exhibited themselves during the reflective episodes. Therefore, it was necessary for me to employ an analytical methodology that provided a focus on those occasions for which reflection was provoked, as well as the instructional engagements in general (i.e., the “non-reflective” episodes).

Conceptual analysis. I employed Glaserfeld's (1995) method of conceptual analysis to create models of teachers' thinking as they participated in Functions 3.

Glaserfeld (1995) combined Ceccato's operational analysis and Piaget's genetic epistemology to "devise a way to talk about reasoning and communicating as imagistic processes and of knowledge as an emergent aspect of them" (P. W. Thompson & Saldanha, 2000, p. 4). The aim of conceptual analysis is to "describe conceptual operations that, were people to have them, might result in them thinking the way they evidently do" (P. W. Thompson & Saldanha, 2003, p. 99); that is, to describe the teachers' ways of operating.

The product of conceptual analysis of this form is the construction of models of teachers' understandings and ways of thinking. These "second-order models are models the observer constructs of the subject's knowledge in order to explain their observations or experience of the subject's states and activities" (Steffe & Kieren, 1994, p. 721). Therefore, these models served as my interpretations of the teachers' mathematical realities, and were considered as viable so long as they were not countermanded by the teachers' actions or inactions (Steffe, 2007).

Since I had no access to teachers' mathematical realities, the most that I could hope for is to establish viable ways and means of thinking that fit within the experiential constraints that were established as the teachers' interacted with instruction (Steffe & Thompson, 2000b). Although conceptual analysis provided a means with which to construct models of the teachers' understandings, it was necessary to ascertain the viability of these models. For this, I employed a method consistent with grounded theory and that developed by Cobb and Whitenack (1996).

Grounded theory. The need to make viable and stable hypotheses of the teachers' conceptions through the comparison of teachers' current models with later ones as the instruction moves forward, made grounded theory a likely choice as an analytical methodology. In addition, considering my inability to verify the claims that I made in the study, due to the nature of the data, grounded theory's search "for coherence, fittingness, and usefulness, or relevance" (Smith, 2008, p. 7), served to provide an appropriate level of validity.

Grounded Theory, with its focus on building theory from the context of the data, constant comparison, and the ability to be simultaneously systematic and creative (Strauss & Corbin, 1998), served as a means to check the viability and stability of the models of understanding that I constructed. Although grounded theory offered clear benefits to the analysis of my study, there were problems with triangulation because the data corpus did not include teacher interviews. In addition, my analyses needed to include all forms of teacher expressions, not just their words, since much of the action was contained on whiteboards and in the teachers' actions and inactions. Therefore, analysis needed to come from more than just the line-by-line analysis of the transcripts. A final issue that arose in an attempt to employ grounded theory came from Charmaz and Mitchell's (2001) assertion that, "All variants of grounded theory include...simultaneous data-collection and analysis" (p. 160). Since all of the data collection for this study had already occurred, and the analysis was completely retrospective, a purely grounded theory approach was unattainable.

Retrospective analysis of classroom video. A second candidate that I considered for an analytical methodology, one that is compatible with grounded theory, is the methodology developed by Cobb and Whitenack (1996). This methodology can be employed in retrospect to analyze large quantities of qualitative data (e.g., video recordings, transcripts), and involves tracking regularities and patterns in the data to test the viability of claims (Cobb & Whitenack, 1996).

One drawback to this method, as with grounded theory, involves the use of student interviews to provide a means of triangulating inferences. As mentioned earlier (p. 99), there were no interviews conducted during the Functions 3 class. In addition, Cobb and Whitenack (1996) assert that one of the considerations that “contribute to the reasonableness of and justifiability of analyses” (p. 225) to this approach is prolonged first-hand experience observing and interacting with the subjects (i.e., students) over an extended period of time. They describe observing 27 class episodes over a year’s time. This was not the case for the Functions 3 course. I was present for 12 of the 14 class meetings, which were held over seven summer meetings and seven classes throughout the fall semester, and at no time did I interact with the teachers. In addition, at the time of my observations, I did not possess a theoretical framework that might have allowed me to make sense of what was happening.

In addition, the methodology of Cobb and Whitenack (1996) involves identifying regularities in the small group relationship that remain “stable over

time, and relating them to analyses of the individual's...mathematical development" (p. 215). During the Functions 3 course, the majority of the activities were either whole-class or involved teachers working in groups of three to four people. During this group work, the bulk of the interactions between group members were difficult or impossible to follow. At times the microphones were unable to pick up specific discourse, while at other times there were too many different group conversations being recorded simultaneously to discern the discussion of any individual group. Although the groups did present their work during class discussions, any models of understanding that I constructed could not be attributed to all members within that group. Although it can be assumed that all members were involved and assented to the group's final report, I could not take group board work as being indicative of any individual member's understanding.

In addition, during activities involving whole class discussions, several of the teachers participated throughout the discussions as the ideas of the lesson were developed. Therefore, I was unable to track any one individual teacher's understanding throughout the engagement with these activities, and was only be able to construct models of distinct teacher's understandings at various points during these engagements. Therefore, it was not possible to indentify stable small group regularities and relate them to analyses of the individual's mathematical development.

Data analysis from a global perspective. From a global perspective, the analytical approach that I employed for generating descriptions and explanations is

consistent with Cobb and Whitenack's (1996) method for retrospective analysis of qualitative video data and grounded theory's (Strauss & Corbin, 1998) iterative process of continual review, constant comparison, and regeneration. This analytical approach consisted of three levels of analysis.

First level analysis. The first level of analysis involved a review of the entire collection of videotaped class meetings with the intent of identifying those lessons or episodes either within a lesson or across lessons that were explicitly designed to provoke reflection on the part of the teachers. Specifically, I attempted to identify those occasions where instruction was designed for the teachers to transform their practices via reflective abstraction. The focus of an episode might be that teachers re-think their teaching, re-conceptualize their students, or re-conceive their mathematics. But salient selection criterion will be that the episode is designed such that the teachers were provoked to reflect on their ways of operating.

A search of the data corpus resulted in the identification of four such occasions or episodes. A reflective episode from Class #3 (June 21, 2007) was designed to motivate the teachers to re-conceive the mathematics they teach. Specifically, the lesson attempted to transform the teachers' conceptions of ideas related to the concept of speed. A second reflective episode, from Class # 10 (September 17, 2007), was designed to provoke reflection, on the part of the teachers, on the structure of the discourse that the teachers employ in their classrooms. This lesson was designed to promote (in the teachers) an image of

instruction that generates conceptual conversations that focus on student reasoning and understanding.

Two distinct reflective episodes were constructed to motivate the teachers to re-think their teaching. Each of these lessons was designed to promote (in the teachers) an image of teaching that focuses on meanings and the development of significant mathematical ideas in the minds of their students. In each lesson, the teachers first engaged with instruction as students of mathematics. After experiencing the lesson as students, the teachers were required to re-construct the lesson (that they had just experienced) in terms of their (the teachers') conceptions of the meanings and ideas that instruction intended to promote, and the activities that were designed to promote the development of these meanings and ideas—the logic of the lesson. The first such reflective episode, from Class #12 (November 5, 2007), involved the teachers re-constructing a lesson involving ideas related to the concept of speed. The second such reflective episode, from Classes #13 and 14 (November 19 and December 3, 2007), involved the teachers re-constructing a lesson involving data analysis.

Second level analysis. The second level of analysis occurred in three phases: 1) provide descriptions of and rationale for each reflective episode, 2) employ conceptual analysis to construct models of teachers' ways of operating as they engaged with instruction during the reflective episodes, and 3) employ conceptual analysis to construct models of teachers' ways of operating as they engaged with instruction during the non-reflective episodes.

Once “reflection provoking” occasions or episodes had been identified, each was analyzed in order to address the “two sides of assimilation” described by Thompson and Saldanha (2003), that is, “the thing the [teacher] is attempting to understand and the scheme of operations that constitutes the [teacher’s] actual understanding” (p. 99). This analysis involved two phases, each having a distinct purpose and perspective.

Phase 1. During the first phase, analyses provided descriptions of and the rationale for each reflective episode. This served to address what I interpreted as the understandings and ways of thinking that instruction intended the teachers to develop. These descriptions gave context to the analysis and provided models of intended instructional outcomes; thus, allowing for comparisons between the anticipated ways of operating and those actually expressed by the teachers.

Phase 2. In Phase 2, conceptual analysis was employed to construct preliminary models of the teachers’ ways of operating as they engaged with instruction. These models served as my understanding of the scheme of operations that constituted the teachers’ actual understandings. In addition, these models assisted me in identifying what it was that the teachers took away from a lesson.

In each of the four reflective episodes, Pat employed didactic objects to motivate reflective discourse around the intended meanings and ideas. For example, in the reflective episode from Classes #13-14, Pat utilized unorganized data sets and the *TinkerPlots* exploratory data analysis software to support reflective discourse around the importance of organizational display when attempting to analyze data.

As stated earlier (p. 90-91), the Functions 3 course was designed so that instruction unfolded as a result of the teachers' engagements with the activities, and Pat's interpretations of the teachers' statements and actions in terms of the meanings, commitments, and values that he saw as implied by them. Therefore, in light of the important role that reflective discourse played throughout instruction, in concert with the emergent nature of instruction, my analyses depended on the discussions that emanated during instruction.

In addition, an analytical focus on the classroom interactions assisted in capturing the subtle nuances that occurred as Pat sought to provoke reflection on the part of the teachers and to establish his understandings of the teachers' understandings. Therefore, lesson descriptions and models of teachers' understanding were generated through analyses of the class discussions and the teacher conceptions embedded within them. Each of the reflective lessons can therefore be described and analyzed in terms of the lessons' activities and discussions. Specifically, each reflective episode can be characterized as unfolding in a sequence of activities, and each individual activity can be characterized as unfolding in a sequence of discussions.

Phase 3. Once I had constructed models of the teachers' ways of operating as they engaged with instruction during the reflective episodes, I moved my analyses to the larger data corpus. Specifically, those lessons or parts of lessons not explicitly designed to promote reflection. I reviewed the remaining videos and written work,

those not associated with the reflective episodes, with the intent of developing models of individual teachers' ways of operating as they engaged with instruction.

Early into this process, it became apparent that due to the nature of the data corpus, models of individual teacher's ways of operating could not be viably constructed. For example, the majority of the small-group discussions were difficult both to track and to hear. At times segments of group discussions overlapped or blended together. In addition, there did not always arise opportunities in either the whole-group or small-group discussions for all teachers to contribute or to extensively articulate and elaborate their ideas and ways of thinking. Therefore, I refocused my attention on developing models of epistemic ways of operating via conceptual analysis. These epistemic models emerged as thematic summaries of teachers' ways of operating as I viewed and re-viewed the data.

Third level analysis. Once I had developed models of epistemic ways of operating via conceptual analysis of the larger data corpus (i.e., the non-reflective episodes), my analyses moved to the final level of analysis. Whereas, the first level of analysis partitioned the data corpus into two distinct categories (those lessons or parts of a lesson that explicitly involved reflection, and those lessons that did not), the third level of analysis looked at the data corpus as a coherent whole.

This third level of analysis involved re-viewing the four reflective episodes with an eye toward modifying or adjusting those epistemic models constructed in Phase 3 (of the Second Level Analysis). Specifically, this level required that I coordinate those models of epistemic ways of operating (developed in Phase 3) with

the preliminary models that I had developed in Phase 2 of the second level of analysis, while re-viewing the reflective episodes. In essence, I “ran” the epistemic models developed in Phase 3 of the second level of analysis through the reflective episodes in an attempt to develop viable and robust models of teacher’s ways of operating.

Data analysis from a local perspective. From a local perspective, the main component of my analysis involved the construction of models of teachers’ ways of operating. The construction of these models entailed focusing my attention on the teachers’ statements, actions, and inactions, in an attempt to identify important aspects of their meanings, commitments, and imagery, as the teachers engaged with instruction.

My analyses of the classroom video recordings generated from the Function 3 course involved making several passes through the videos, each pass involving a more focused level of attention on specific ways of operating. A major focus of the Functions 3 course involved discussions and activities centered on the *Didactic Triad* (p. 95). As such, the initial coding scheme that I employed for use with *Studiocode* video analysis software highlighted the three “legs” of the *Didactic Triad*: Learning Goals, Tasks and Materials, and Teaching.

Early into coding of segments of the reflective episodes and the larger data corpus, two themes emerged: 1) my analyses required that I employ annotated transcriptions of the pertinent lessons, and 2) my coding scheme needed to be expanded. Specifically, my codes needed to differentiate between the teacher’s

personal mathematical understandings and the teacher's models of their students' understandings (i.e., mathematics of students), and required the addition of codes for Meanings, Decentering, and Perturbation. These additional codes helped orient me toward those occurrences requiring further inspection and possible transcription. A sample video analysis using Studiocode and the stated coding scheme is illustrated in Figure 4.

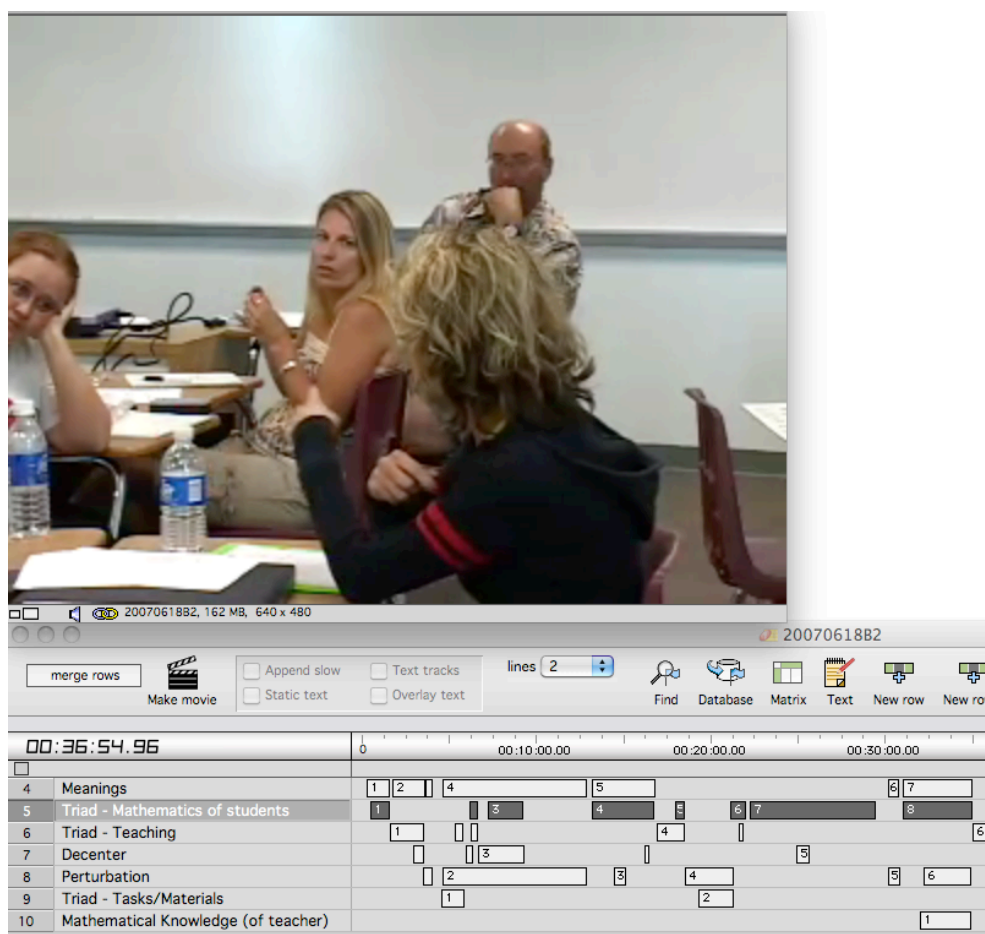


Figure 4. Studiocode sample analysis.

Subsequent passes through the videos involved coordinating both the expanded coding scheme (highlighting those video segments pertinent to the coding

scheme via Studiocode), with the construction of annotated transcripts of the relevant video segments. Annotated transcripts were designed as Word documents, and included a transcript of significant portions of the video under consideration, my initial impressions of that segment, and any pertinent video images (e.g., group white-boarding) or relevant written work embedded in the document.

After the coding and annotated transcripts were completed for two of the four reflective episodes, and several hours of the larger data corpus (i.e., those lessons not explicitly designed to promote reflection on the part of the teachers), I introduced another analytical tool to assist in keeping track of the emerging models of teachers' ways of operating. Specifically, I constructed a matrix using butcher-block paper, which contained the individual teachers as columns and individual lessons as rows of the matrix. In each individual cell, I listed my impressions of that teacher's ways of operating as they engaged in the activities for that lesson. This matrix allowed me to track emerging ways of operating both across teachers and across lessons. In addition, this matrix allowed me to "stand back" and track emerging epistemic ways of operating.

Once all of the annotated transcripts were completed, the construction of the teachers' ways of operating began in earnest. The construction of these models of teachers' ways of operating consisted of the following iterative three-step procedure, adapted from Saldanha (2004):

- 1) Line-by-line analysis of the pertinent lesson. Formulate initial impressions (e.g., descriptions and hypotheses) about teachers' ways of operating as they engaged with instruction.
- 2) Test viability of initial impressions by searching the related data for supporting or contradictory evidence, as in step 1.
- 3) Adjust initial impressions on the basis of evidence obtained in step 3; refine, elaborate, abandon, or reconstruct initial impressions.

Analyses generated by iterating this process throughout the Functions 3 lessons were aimed at developing increasingly stable and viable hypotheses and models of teachers' ways of operating as they engaged with instruction. This iterative process was employed for both those lessons explicitly designed to provoke reflection on the part of the teachers (i.e., the reflective episodes), as well as the larger data corpus.

Those passes involving video generated from reflective episodes employed both the annotated transcripts, and the reflective episode descriptions constructed in Phase 1 of the second level of analysis. This allowed me to coordinate emerging models of teachers' understandings with anticipated course objectives, and highlighted those occurrences where Pat either focused the teachers' attention in specific directions or appeared to modify instruction due to his interpretations of the teachers' statements and actions in terms of the meanings, commitments, and values that he saw as implied by them. Finally, the number of passes that I made through any particular piece of data was dependent upon the quality of the data,

where I employ “quality” to mean usefulness in helping me to construct, modify, or replace models of teachers’ ways of operating.

CHAPTER 5

TEACHERS REFLECTING ON ASPECTS OF THEIR PRACTICE (PART I)

In this chapter, I will describe and analyze two of the four reflective episodes that the teachers engaged in during the Functions 3 course. Specifically, these reflective episodes were designed for the teachers to re-conceive their mathematics (Reflective Episode #1, Classes #3 and #4), and re-think their teaching and re-conceptualize their students (Reflective Episode #2, Class #10). These episodes will serve to orient the discussion toward the issues that arose as the teachers engaged with activities designed to promote reflection, and highlight those understandings and ways of thinking expressed by the teachers that impacted their capacity to reflect.

Reflective Episode #1: Building Meaning for the Idea of Speed

A reflective episode that spanned Classes #3 and #4 (June 20 and 21, 2007) was designed to motivate the teachers to re-conceive the mathematics they teach. Specifically, the lesson attempted to transform the teachers' image of ideas related to speed; that is, to move the teacher's from a conception of speed as one identified by the relationship $\text{speed (i.e., average speed)} = \Delta \text{distance} / \Delta \text{time}$ —a conception devoid of meaning, to a scheme of meanings (i.e., a KDU), including:

- Speed as a quantification of motion.
- Completed motion involves two completed quantities—distance traveled and amount of time required to travel that distance.

- Speed as a quantification of completed motion is made by multiplicatively comparing distance traveled and amount of time required to go that distance.
- There is a direct proportional relationship between distance traveled and amount of time required to travel that distance. That is, if you go m distance units in s time units at a constant speed, then at this speed you will go $\left(\frac{a}{b}\right) \cdot m$ distance units in $\left(\frac{a}{b}\right) \cdot s$ time units
- Average speed is the constant speed at which another object would need to travel in order to travel the same distance in the same amount of time as the object in question.

In addition, the episode was designed to provide the teachers with the opportunity to experience a lesson grounded in meanings and the development of significant mathematical ideas; a lesson designed for meanings to “do work” for the teachers’ learning of related ideas.

The reflective episode had two phases, where Phase 1 consisted of activities and conversations that anticipated the development of the idea of constant speed (Class #3), and Phase 2 consisted of activities and conversations that were intended to develop and build meaning to the idea of constant speed (Class #4). A chronological overview of the episode’s phases is given in Table 2. It should be noted that the events unfolded in a manner quite different from that anticipated by the course instructor (Pat Thompson). Pat had intended for the teachers to first

engage with what I have identified as Phase #1, and then engage in the finger tool activities (see Appendix A).

Table 2

Chronological Overview of Reflective Episode from Classes #3 and #4

Class Meeting (Date)	Phase	Duration (minutes)
	Phase 1 – The Idea of Constant Speed (Setting the Stage)	
Class #3 (6/20/07)	Phase 1.1 – Orienting the Discussion: Getting the Idea of Speed on the Table	1.5
Class #3 (6/20/07)	Phase 1.2 – Making Comparisons: Anticipating a Need for a Meaning for Speed	1.5
Class #3 (6/20/07)	Phase 1.3 – Move to Make the Teachers’ Conceptions of Speed Explicit	2.5
Class #3 (6/20/07)	Phase 1.4 – What Shall We Mean by “Faster Than”?	6
Class #3 (6/20/07)	Phase 1.5 – Division is Not a Meaning for Speed	6
Class #3 (6/20/07)	Phase 1.6 – Comparing Speeds: Making the Relationship Between Distance Traveled and Time Elapsed Explicit	4.5
Class #3 (6/20/07)	Phase 1.7 – Speed is about Scaling	8.4
Class #3 (6/20/07)	Phase 1.8 – Retrospective Discussion of Activities and Conversations	3.5
Between Class Assignment	Phase 1.9 – Finger Tool Activity	n/a
	Phase 2 – Developing a Meaning for Constant Speed	
Class #4 (6/21/07)	Phase 2.1 – Discussing the Finger Tool Activity	4.7
Class #4 (6/21/07)	Phase 2.2 – Getting Teachers’ Conceptions of Speed Out on the Table	4
Class #4 (6/21/07)	Phase 2.3 – What Shall We Mean by Speed?	15.3
Class #4 (6/21/07)	Phase 2.4 – Speed at an Instant	8.8
Class #4 (6/21/07)	Phase 2.5 – Orienting the Discussion Toward Constant Speed	16
Class #4 (6/21/07)	Phase 2.6 – Developing a Meaning for Constant Speed	9
Class #4 (6/21/07)	Phase 2.7 – Retrospective Discussion of Activities and Conversations	12.5

The finger tool activities involved the animated characters Clown (a clown) and Dude (a “frog prince”), from Jim Kaput's computer program called *SimCalc*

Mathworlds. Both Clown and Dude walk back and forth across the computer screen at varying speeds. The screen displays a timer (which displays time from start, in seconds), a ruler (measured in feet), and controls to start, stop, pause, and move either character in “steps.” Dude starts at the far left of the screen, walks all the way to the right, turns around, and moves back to the beginning, with a small variation in speed throughout the walk. Clown moves similarly to Dude, but makes smaller left to right movements throughout the walk, and has a more dramatic variation in speed.

In the actual lesson, Pat first had the teachers engage in the finger tool activities, and only realized that the events were out of order when the teachers were on the second (of three) activity. Pat informed the teachers that he had skipped over something that he had planned to do, which had really changed the coherence of the activity. Pat stated that they were going to “roll back time” to before Activity 1 of the finger tool. A discussion of the finger tool and the associated activities will be provided in the next chapter (p. 238), and the activities themselves are provided in Appendix A.

Phase 1 – The idea of constant speed (setting the stage). Phase 1 consisted of 9 sub-phases; each designed to orient the teachers’ focus on ideas related to speed. The activities and conversations were intended to push the teachers to make their conceptions of speed explicit and for the teachers to reflect on the coherence of these conceptions. In addition, the engagements were designed

for teachers to experience a lesson grounded in meanings and the development of significant mathematical ideas.

Phase 1.1 – Orienting the discussion: Getting the idea of speed on the table. Pat initiated the reflective episode (Class #3, June 20th) by displaying the Clown animation (Figure 5), and asking, “What is Clown doing?” The intent of this phase of the activity was to get the idea of speed out in the open, as something with which to have a productive discussion.

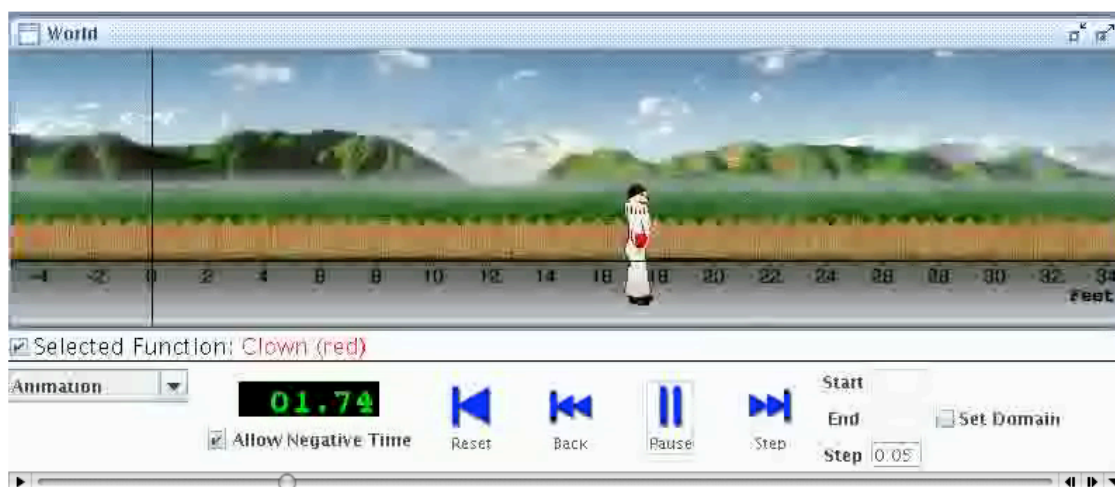


Figure 5. Clown animation.

After his initial question, Pat attempted to get the conversation to focus on speed, as illustrated in the following excerpt.

Excerpt 1, Class #3

Annie: Walking back and forth.

Pat: He's walking back and forth; can you say anything more precise about what he's doing?

Johnny: His pace is changing as he's walking back and forward...²the speed is changing.

Pat: First, what you mean by that, and second, how do you know?

Johnny: The fact that...the speed...his elapsed time or how far he traveled in a certain amount of time gets shorter.

Pat: Alyce can you repeat what Johnny said...whatever you understood him to say...

Alyce: Well, we can't control time so time has to keep elapsing at the same speed and we can see him moving back and forth at a faster...covering more distance in the same amount of time or less distance in the same amount of time, you can tell he's going faster or slower.

In the preceding excerpt, neither Johnny nor Alyce provided a meaning for speed. Johnny's comment, and Alyce's elaboration, suggests that each was thinking about speed as the number that you obtain when you divide distance by time, rather than as a multiplicative relationship between two covarying quantities. In addition,

² Note that in the excerpt sections, "Pat" stands for utterances made by Pat, the course instructor; "Several" stands for utterances made by at least two teachers simultaneously. Also note that the symbol "..." signifies that either the speaker paused during the utterance, or that the speaker did not complete the utterance due to an interruption or because they simply stopped speaking. The symbol "[...]" signifies that text has been omitted from the transcript. As a general rule, omitted text consists of a small number of utterances within a segment of a discussion excerpt; however, on certain occasions, entire intervening segments are omitted due to their irrelevance to the issue or analysis at hand.

their comments suggest that in order to answer the question (did Clown's speed change?) they need only focus on the distance Clown covers in a set amount of time (allowing for time to fall into the background)—they were not covarying.

Phase 1.2 – Making comparisons: Anticipating a need for a meaning for speed. Pat continued to push the teachers to make their conceptions of speed explicit and for the teachers to begin to think about providing meaning for what it was they were comparing (i.e., speeds).

Excerpt 2, Class #3

Pat: It appears that he speeds up and slows down...

Johnny: Right

Pat: And that means what?

Tami: That his rate is changing.

Pat: Well, if you say that his rate changes that's the same thing as saying his speed changes...

Tami: He's going to travel farther when he's going at a faster rate and as his rate...or speed slows down, he's actually traveling a shorter amount of distance [...]

Alyce: You can't just say he's traveling more distance or he's traveling less distance, because that's not necessarily true. Traveling more distance in the same amount of time or less distance in the same amount of time...if he goes slow from one side of the screen to the other and then fast for just a little bit, he's not traveling more distance or less distance.

Tami: It would be relative to the amount of time elapsed.

Pat: Yeah, so you're saying...that you can't ignore...the relation, that it's the amount of distance covered in relation to a certain amount of time...

In the preceding excerpt, Tami initially focused on describing a way to make the comparison (e.g., faster or slower), rather than providing a meaning for what it was that was being compared (i.e., speeds). In addition, Tami initially did not emphasize the two variables that were involved in the relationship (distance traveled and time elapsed). Her focus on the comparison (the outcome), had allowed for time to fall into the background. Tami's focus was only on comparing distances traveled (e.g., more distance or less distance).

Although Alyce's comment that speed involved both distance traveled and time elapsed did bring the focus on the two quantities, she did not allow for time to vary when describing the relationship (e.g., "traveling more distance in the same amount of time). In addition, Alyce did not identify a relationship (e.g., multiplicative) between the two quantities.

Phase 1.3 – Move to make the teachers' conceptions of speed explicit. Pat next attempted to get the teachers to make their meaning for speed explicit. Pat stated that if they were comparing speeds, then what was it specifically that they were comparing (i.e., what meaning for speed were they working with) and what would it mean for one speed to be faster than another speed?

Excerpt 3, Class #3

Pat: And if we say that he has gone faster...in comparison to something else, then what are we comparing?

Faye: Rates.

Pat: No, I mean what are we comparing to make that determination?

Alyce: What he did before?

Pat: Well, the distance he's traveled and...

Johnny: A specific time.

Pat: Yeah, you're picking a specific amount of time...and one way to do it is make it the same amount of time in both cases, right? Do we need to have the same amount of time in both cases?

Gwen: No

Pat: [manipulating the Clown animation] Could you say, let's look how far he went in...right here, let's get him back to the beginning...Let's look how far he goes in the first second...it looks like about 11 feet, right? Now let's look how far he goes in the next 2 seconds, can we do that?

Johnny: Yeah, it just makes it a little....you just gotta do a little bit of conversion. That's why it's nice to have the same timing, because then you don't have to do that conversion.

Johnny's comment suggests a focus on making the comparison (an anticipation of the outcome). Johnny described a way to make the comparison, rather than indicating a meaning for what was being compared. Not only were the

teachers not talking about a meaning for speed, they provided no indication that they conceived of a distinction between constant speed and average speed. The teachers appeared to be focused on the products of their reasoning (calculating the “speed” and comparing “speeds”), rather than their meanings and their reasoning.

Pat next moved the focus of the conversation on the two covarying quantities (distance traveled and elapsed time), and the need for the teachers to provide some meaning for what is was that was being compared, by asking how the teachers would handle the situation in which they were not the one’s collecting the data—a situation in which they needed to make a judgment based on the data that somebody else had given them.

Pat determined (using the timer and ruler on the Clown animation, see Figure 5) that in three-tenths of a second Clown moved about 1.5 feet, and in the next eight-tenths of a second Clown moved 8 feet. After being provided with the two sets of data, several of the teachers appeared to be making calculations on their paper. These teachers appeared to be comparing the values obtained through a calculation for average speed ($\text{average speed} = \Delta\text{distance}/\Delta\text{time}$) using the two sets of data.

Pat next asked whether Clown went the same speed both times, to which several of the teachers answered, “No.” Pat emphasized the idea that if you are not able to make the elapsed times the same (equal amounts of time), then you need some way to make the comparison—reinforcing the need for a meaning for speed.

Phase 1.4 – What shall we mean by “faster than”? Pat next moved the teachers to make explicit their meaning for “faster than,” as illustrated in the following excerpt.

Excerpt 4, Class #3

Pat: What are we going to say that we mean when we say that he’s [Clown] going faster in interval A than he is in interval B? [Quiet for seven seconds]

Pat: What are we going to say that we’re going to mean? [Quiet for six seconds]

Tami: The amount of distance traveled relative to the amount of time elapsed is greater [...]

Liz: The ratio of the distance traveled to the time that occurred is greater.

Pat: [writes Tami’s response on the whiteboard and labels it #1, Figure 6]
What was the question that I asked?

Tami: If you have two different intervals A and B, how can you tell what it means by faster or slower?

Pat: I asked the question, “What shall we mean.” I didn’t ask how do you find, I asked what shall we mean.

Story: He traveled the same amount of distance in less time.

Pat: Is that okay? [Writes Story’s response on the whiteboard and labels it #2, Figure 6]

Marcy: Yeah.

What shall we mean when we say that clown traveled faster in interval A than interval B?

1. The amount of distance traveled relative to the amount of time elapsed is greater.
2. He traveled the same amount of distance in less time.

Figure 6. Question and suggested meanings for “faster than.”

Both Tami’s and Liz’s responses suggest that they are focused on what is obtained after dividing (or “taking the ratio”) distance traveled by elapsed time. Neither Tami nor Liz provided a meaning for “faster than” or for speed—their focus was on the object obtained after performing the calculation, without consideration to what the object means.

Pat’s assertion that they were attempting to determine a meaning for speed suggests that he had interpreted Tami’s response as indicating that she was thinking about how to calculate whether interval A or B is faster (a focus on the comparison, the product of reasoning), rather than thinking about meanings.

Story’s response suggests that she had let distance fall to the background (she was not covarying) and only focused on the elapsed time. Pat’s question was an attempt to get the teachers to think about a meaning for speed that makes sense for any distance and any elapsed time (emphasizing the need to focus on both quantities, distance traveled and elapsed time, simultaneously, and a meaning for the relationship between the two).

Pat continued to push the teachers to focus on the two covarying quantities and the relationship between them, as illustrated in the following excerpt.

Excerpt 5, Class #3

Pat: Where does it say that he traveled the same amount of distance?

Story: It goes back to the ratio thing.

Pat: Okay, but...there's no just throwing in a technical term and saying, "Okay, everyone knows what a ratio is so let's use it," and define it from there.

Pat's comment attempted to make the teachers aware that they were employing technical terms without providing meaning to these terms, and that Story was simply bringing in another undefined technical term to define an undefined technical term (i.e., faster). Pat next continued to push the teachers to focus on meaning.

Excerpt 6, Class #3

Alyce: Well, Johnny said we should divide and get the same...

Johnny: Well again, he's saying that...

Story: The same time.

Johnny: Well, but Pat's saying that...we're not asking "how to do it" ...

Pat: We're asking, "What shall we mean" [...]

Alyce: But we're saying that the answer you get after you have divided is...this.

Alyce's comment made her conception of speed explicit, specifically that speed is the value that you obtain after you have divided (a focus on the product of her reasoning, an outcome). In addition, Alyce's comment suggests that her

conception of speed might have meaning for her, for how she interprets the term “meaning”—it might not be clear to her what Pat is indicating by “a meaning.”

Pat next moved the teachers to think about how a focus on the operation that is performed does not provide a meaning for speed.

Excerpt 7, Class #3

Pat: Well, why don't you add then...the answer that you get after you add is bigger. [Quiet for 16 seconds]

Pat: So, here we have two proposed meanings, all right?

Pat: [pointing to the first meaning] Does this one work? [Quiet for 14 seconds]

Johnny: See both...both of those seem to be operations and that's what you're trying to get away from...

Pat: No...does this convey a meaning?

Marcy: Well, can we write another one?

Pat: Sure...we can keep adding, but at some point we have to come back and say is that sufficient [points to #1], is that sufficient [points to #2]...

Story: And the first one I think is saying if the time amount is the same, then the distance is greater.

Pat: The amount of distance traveled relative to the amount of time elapsed is greater.

Story: So, it's saying if this time's the same [moves left hand] and that time's the same [moves right hand], then this distance is greater [moves left hand].

Story's comment suggests that she conceived of only two possible ways to make a comparison involving speed: make the times the same and compare the two distances, or make the distances the same and compare the two times. Neither possibility focuses on two covarying quantities or the relationship between these quantities.

Gwen asserted that she did not believe that the first definition was talking about the times being the same, to which Story disagreed and asserted that we could "just word it in there." This reinforces the idea that for Story one of the quantities (either distance or time) must be the same in both intervals under comparison.

Pat attempted to make clear that they were talking about two intervals (A and B), so that neither the times nor the distances needed to be the same. Furthermore, Pat indicated that they were looking for a meaning that would work no matter what intervals were chosen.

Phase 1.5 – Division is not a meaning for speed. Pat next made explicit his awareness that the teachers were (in general) thinking about speed as distance divided by time, and attempted to provide the teachers with a sense of how such thinking was problematic (e.g., that thinking distance divided by time did not make any sense).

Pat stated that when you divide you get a quotient and that when you look at what you obtain when you divide twelve by four, you get the number of fours in twelve. Pat next stated that if you were to divide distance by time, as in 10 miles divided by 3 hours, then are you saying how many 3 hours are in 10 miles, which

does not make any sense. Pat asserted that, “You can’t just say distance divided by time because what does that mean? I know how to divide numbers, but I don’t know how to divide distance by time.”

Alyce asked whether they could use ratio in their definition, to which Pat replied that they could as long as they say what they meant by ratio.

Excerpt 8, Class #3

Alyce: Because if we take the ratio of distance to time and we simplify both ratios, then whichever one is greater is the greater...

Pat: Okay, then the reason that I’m going to object is because you’re using ratio as a substitute for division.

Although Pat attempted to perturb the teachers’ conceptions of speed, Alyce’s comment suggests that (for her) the problem could be rectified by replacing the word “division” with an equivalent term (but implying that she was still thinking about dividing distance traveled by elapsed time). This suggests that Alyce was not thinking about ratio as a multiplicative relationship between two quantities, but as the value obtained by dividing one quantity by another quantity.

Next, Liz offered up a different possible meaning for speed. Although Liz’s suggestion moved away from employing division, the focus was still on obtaining values (via some operation) to compare. The focus was still on outcomes, rather than meanings and reasoning.

Excerpt 9, Class #3

Liz: The relationship of the distance traveled to the time that elapsed in that interval compared to the distance traveled and time elapsed in another, just whatever relationship they have, whether if you want to divide them, to multiply them, I don't care, but a relationship...

Pat: Add them, subtract them.

Liz: But the relationship they had, as long as you're comparing like the same type of a relationship to the two, you could get to where I would say that this one's greater...by doing that same type of a relationship to the two separate intervals.

Marcy: That's true, but it would matter whether you add, subtract, multiply, or divide, you're not going to get the same relationship when you do that.

Janine: It does matter.

Marcy: So, it's kind of what do you mean by a relationship...I agree with what you said, but when you say a relationship and you say it doesn't matter what, it does matter what you do.

Although Liz did not stress the need to divide distance by time, her comment did focus on the product of performing an operation. Liz suggested that regardless of the operation that you perform with distance and time, as long as you perform the same operation for each interval, then comparing the products of the operations will allow for a comparison. This suggests that Liz's focus was not on the relation, but on the products of an operation. Marcy's comment suggests that she did not have an

issue with Liz's focus on the product of performing an operation, only with the need for the operation to be division.

The teachers' comments employed a variety of pronouns (e.g., whatever relationship "they" have, when you say a relationship and you say "it" does not matter what, "it" does matter what you do). Throughout the preceding exchange, the teachers did not attempt to make one another articulate their meanings (their pronouns), nor attempted to ascertain whether others' interpretations were aligned with their own.

As Pat moved to discuss the need to be explicit about the relationship involved, Story brought up the idea of proportionality, an idea that Pat intended to develop.

Excerpt 10, Class #3

Pat: What does proportionality have to do with what we're talking about?

Alyce: But they're not...they couldn't be proportional because we're comparing them, but you do need it to be...your operation that you're doing has to be a multiplication or a division, because your final result...what your comparison between your distance and your time is is a ratio, which is a half of a proportion. So whether you're multiplying to find the relationship between them or dividing it's because of the proportion.

Marcy: But using your ratios you could create proportionality in such a way that then you could compare them...is what you're saying.

Alyce: Yeah.

- Pat: But what does proportionality have to do...
- Story: Because it's a distance to a time.
- Marcy: Yep.
- Story: Compared to a distance to a time [...]
- Pat: Oh, so you're talking about a proportion as an equality of two ratios [...]
You're talking about proportion as a cross-multiply kind of thing.
- Alyce: Yeah [...]
- Pat: What does cross-multiplying have to do with speed?
- Story: I thought that when you compared one number with another number you would usually divide them.

Alyce, Marcy, and Story's comments suggest that, although several of the teachers have indicated a conception of speed as the value obtained when you divide distance by time, that they understand Pat as requiring that they not perform the division. Therefore, they are left with two objects of the form d_1/t_1 and d_2/t_2 to compare, objects which the teachers had indicated that they conceive as ratios. Alyce's comment suggests that she believed that these two objects did not form a proportion, because they were not equal (rather they were being compared).

The teachers were still not attempting to provide a meaning for speed; rather, they were focused on performing some form of operation to obtain values that they could then compare (the focus was on making the comparison). Pat used

Story's last comment regarding comparisons to bring out the idea of how to mathematically compare quantities.

Phase 1.6 – Comparing speeds: Making the relationship between distance traveled and time elapsed explicit. Pat asserted that there were two kinds of comparisons, additive comparisons and multiplicative comparisons. Pat stated that the teachers needed to not lose sight of the question, “What shall we mean when we say that clown is travel faster in interval A than in interval B?” Pat next moved the teachers to focus on the need to provide a meaning for speed that would allow for the comparisons of speeds with different distance and/or time intervals.

Pat asked what the teachers would do if the intervals that they were using to make the comparison were not the same, and whether they could we make one situation exactly like the other. The teachers suggested that they could either unitize or get a common denominator. Pat responded by stating that if one interval was 7 seconds and the other interval was 5 seconds, could they make the interval of 7 seconds look like the interval of 5 seconds. Story suggested making them both 35, which moved Pat to push the conversation toward thinking about proportionality, but in the sense that there is a direct proportional relationship between distance traveled and amount of time required to travel that distance. That is, if you go m distance units in s time units at a constant speed, then at this speed you will go $\left(\frac{a}{b}\right) \cdot m$ distance units in $\left(\frac{a}{b}\right) \cdot s$ time units.

Pat next moved to make this meaning, of a proportional relationship between distance traveled and amount of time required to travel that distance, “do work” for the teachers, and to anticipate the need to discuss constant speed. Pat stated that if Clown were moving at a constant speed for seven seconds, then at some point in Clown’s trip, he had to have passed by five seconds. Pat then asked if Clown were moving at a constant speed for 7 seconds, how far would he have traveled in 5 seconds of that 7 second trip. Several of the teachers responded that Clown would have travelled five-sevenths as far, to which Pat added that if Clown was going five-sevenths of the time, then he would have travelled five-sevenths of the distance—that at a constant speed you can use proportionality to say how far Clown goes in any number of seconds.

Excerpt 11, Class #3

Pat: That’s where the proportionality comes in...you can scale any comparison into any other comparison.

Story: Kids won’t like that, they freak out with fractions, I’m sorry [...]

Pat: You’re not supposed to assume that, “Oh, okay, let’s spring it on them” ...you’re supposed to stop and think, “Well, what kind of teaching do we have to engage in, so that this makes sense?”

Alyce: If it makes sense, then they’ll [students] do it...it’s the fact that fractions don’t make sense, so that’s why they’re afraid of them.

Story: Because I talk meaningfully about fractions...and kid’s are like, “Ahhh” ...it’s over their heads.

Pat: Then you're not talking meaningfully...

Story: Well, I know what I'm talking about. [Several of the teachers laugh]

Up until this point, the activity had focused on the teachers' conceptions of speed. Story's comment suggests that her personal development of a scheme of meanings (i.e., a KDU) might be entangled with her conceptions of her students' development of similar understandings. Story indicated that her students would not be open to the idea of a proportional relationship between distance traveled and amount of time required to travel that distance, because the relationship "involves fractions." Subsequently, Story appeared unwilling to reflect on her own conceptions, because of how she viewed her students reaction to a conception involving "fractions"—she thought about her students mathematical conceptions in a very non-cognitive sense. In addition, Story's assertion that she "talks meaningfully about fractions" to her students, in a manner that had meaning to her (although apparently stated in jest), suggests that she is not mindful of her students' mathematics or how she might be able affect it through instruction.

Pat asserted that meanings have to do work for kids, that if meanings do not work for kids then it is just more vocabulary for them to attempt to remember.

Phase 1.7 – Speed is about scaling. Pat next moved the conversation to discussing the idea that to say that objects travel at the same speed means that when you "scale" one trip in the other, they travel the same distance in the same amount of time. In addition, Pat described how he would talk about the idea of scaling with students.

Pat drew three line segments on the whiteboard: a distance-line (segment) labeled “17 ft,” below that, a smaller line time-line labeled “5 sec,” and below that, another time-line labeled “7 sec” (Figure 7).

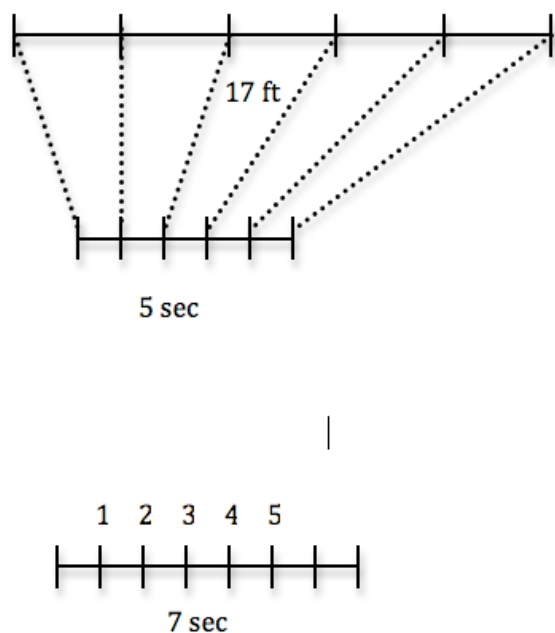


Figure 7. Scaling one speed into another.

Pat partitioned the five-second line segment into five congruent pieces, and the seven-second line segment into seven congruent pieces. Pat stated that he wanted to draw a distance line above the seven-second line and turn the 17 feet/5 second situation into some distance/7 seconds.

Excerpt 12, Class #3

Pat: How am I going to talk about the distance that goes with 7 seconds, so that these two [gestures to line segments] have...

Story: $7/5$ ths of 17 feet.

Pat: Yeah...okay, so, right here we have...

Pat next partitioned the 17 feet segment into five pieces and drew lines that connected each one-fifth time segment with each one-fifth distance segment (Figure 7).

Excerpt 13, Class #3

Pat: So, each fifth of the five seconds goes with one-fifth of the 17 feet...you cut up the time into five parts, you have to cut up the 17 into five equal parts. So, one-fifth of the time goes with one-fifth of the 17 feet. [Marks off five segments of the seven-second time line, Figure 7]

Pat: That would be like we've gone up to here [points to the segment labeled "5 sec"], how far have we gone? [Quiet for four seconds]

Johnny: 17 feet

Pat: 17 feet...now, we're going to travel how much more time?

Marcy: Two-fifths.

Alyce: Two-fifths of 17 [...]

Johnny: Two-fifths of the 17 feet.

Pat: Oh yeah, but I was asking my question about time.

Johnny: Oh, okay.

Pat: We're going to travel two more seconds, but those two seconds are what fraction of five seconds?

Marcy: Two-fifths.

Pat: Two-fifths of 5 seconds.

Pat: So, two-fifths of 5 seconds, goes with [gestures to drawing] two-fifths of 17 feet...you see you have to come back to here [gestures from 7 second line to 5 second line] to get to the 17 feet [gestures to 17 feet line] [...]

Pat: The idea is that these two seconds [gestures to 7 second line] are a special two seconds. They are two-fifths of these 5 seconds that we've already used in establishing a relationship. And we've already established that in those two-fifths of 5 seconds, he's [Clown] going to travel two-fifths of 17 feet.

Pat asserted again that you cannot go directly from the seven-second line to the 17 feet line, that you must first move to the five-second line, because the 17 feet line is the distance that is in a multiplicative relationship with the five-second line. In the preceding excerpt, the teachers were focused on obtaining the answer, rather than thinking about their reasoning. The teachers jumped directly to the answer without clarifying how they were thinking. Specifically, the teachers did not express their reasoning, what meanings did they coordinate that enabled them to answer "Two-fifths of the 17 feet."

In addition, the teachers appeared to be focus only on their role in answering the question, and not about how their students would need to reason in order for the situation to make sense (to their students). Pat had prefaced the discussion by asserting that he was going to describe how he would discuss the notion of "scaling" with students. The teachers appeared to be so focused on their own experience in

the discussion (in obtaining a solution) that they were constrained in imagining their leading a similar discussion with their own students—how they might manage the conversation to focus their students' attention productively (i.e., make reasoning an object of discussion and possible reflection).

Pat asserted that speed was really about scaling. That when making comparisons between two intervals of different lengths it is necessary to have a common scale, scaling one into the other or scaling both into the same scale, and that kind of working knowledge is very powerful.

Phase 1.8 – Retrospective discussion of activities and conversations.

During the last few minutes of class Pat pointed out to the teachers that the discussion related to proportional relationships and scaling had come up, because it was part of his agenda.

Excerpt 14, Class #3

Pat: Where did the discussion of proportional relationship, scaling, and all that, where did it come from?

Alyce: Because we said the clown went faster or slower.

Pat: Because you said the clown went faster or slower and what? I mean it didn't come from just because you said it went faster or slower [...]

Tami: We're trying to determine what that means.

Pat: It was because, that's right...and where did this issue come from?

Tami: The relationship between distance and time...

Pat: No, why did we even talk about...why did we even talk about that question?

Sheila: We looked at the movement of the clown.

Tami: Well, we noticed different speeds.

Pat: Yeah, but why the discussion of what do we mean by different speeds?

Faye: Because there were different interpretations...

Pat: Because I asked [...] It didn't come out spontaneously, it did not emerge naturally, right? Because it was part of an agenda.

In the preceding excerpt, the teachers' comments suggest that they were focused on their role in the discussion, not on how such a discussion might be managed—a focus on what they did, on their actions, not on meanings, and reasoning. This apparent lack of awareness of a teacher's role in promoting the development of ideas, and engaging students in reflective discourse, might suggest that the teachers awarded agency to tasks and activities.

Pat next stated that he believed the idea of proportionality was important to get out in the open and to make part of the discourse. Pat asserted that the idea that proportionality is part of the intrinsic nature of understanding speed helped to orient him to the types of questions that he had asked during the lesson.

Phase 1.9 – Finger tool activity. As an out of class assignment, the teachers were requested to complete the third Clown and Dude activity. The activity involved only Clown, and led the teachers through four stages (Appendix A):

- (1) Track Clown's total distance along the horizontal axis using your right index finger;
- (2) Track Clown's distance from the start along the vertical axis using your left index finger;
- (3) Do steps 1 and 2 together, keeping your left finger along the vertical axis and your right finger along the horizontal axis;
- (4) Do steps 1 and 2 together, keeping your left finger directly above your right finger. Pretend that you've dipped your distance finger in pixie dust (trace the pixie trail with your pencil).

The teachers were also asked to pick three particles of pixie dust and to determine what each of these particles represented.

Phase 1 – Summary. Throughout Phase 1, the teachers' propensity to focus on the products of their reasoning, rather than their reasoning, constrained their capacity to reflect productively. In addition, the teachers' conception of speed as distance traveled divided by time, proved to be highly resistant to Pat's attempts at perturbation. The teachers expressed ways of thinking in which meanings were not objects of thought, conceptions in which meanings appeared to not even be on their radar.

Phase 2 – Developing a meaning for constant speed. Phase 2 consisted of 7 sub-phases; each designed to promote the development of a scheme of meanings related to the idea of speed. As in Phase 1 of the episode, the activities and conversations were designed for teachers to experience a lesson grounded in

meanings and the development of significant mathematical ideas. In addition, the activities and conversations were intended to push the teachers to reflect on the incoherence of their current conceptions related to the idea of speed and to build a coherent scheme of meanings.

Phase 2.1 – Discussing the finger tool activity. Pat initiates the second part of the reflective episode (Class #4, June 21) by engaging the teachers in a discussion of the third Clown and Dude activity that the teachers were required to do as an out of class assignment (Phase 1.9). There was general agreement amongst the teachers that Activity 3 was more difficult than either Activity 1 or 2 (Appendix A). These activities involved tracking Clown's total distance and elapsed time both individually and simultaneously (Activity #1), and Clown's distance from start and elapsed time both individually and simultaneously (Activity #2).

Pat asked the teachers why they believed it was harder to track total distance versus distance from start than it was to track either total distance versus time, or distance from start versus time. Bernardo indicated that the difficulty arose because each quantity was a distance and was thus affected by Clown's speed. Johnny asserted that tracking total distance versus distance from start was more difficult, because it was not something that he had previously done. Janine asserted that tracking time was easier (finger tool Activities #1 and #2) because it was continuous, suggesting that she did not need to attend to time—that she was not covarying.

Pat stated that the difficulty had to do with attending to two things simultaneously and that in order to covary you must attend to the values of two quantities as they vary simultaneously. Pat asserted that if you allow one variable to drop into the background, then you are not attending to two variables, which is not covariation. Pat stated that, "You have the feeling that it is easy, but it is not easy for your kids." Pat continued by asserting that, "You have the feeling that you can do it a little bit and you got it, so move on, but the kids do not get it, they're not covarying. Instead the kids start to focus on shapes."

This comment indicated that Pat believed that the teachers were focused on what the finger tool did for them (the teachers, their own experience), rather than what the finger tool could do for their students.

Phase 2.2 – Getting teachers' conceptions of speed on the table. In the next activity Pat attempted to get the teachers to make their conceptions of speed explicit and to perturb these understandings. Pat displayed a video of Clown (Figure 5) moving back and forth across the video screen.

Excerpt 15, Class #4

Pat: Now, watch the clown [...] [runs the clown movie for about 15 seconds]

Pat: How fast did he go?

Tami: However far he walked divided by...

Story: How many seconds.

Tami: However many seconds of time.

Tami offered a definition for speed that does not have meaning, but is rather based on an operation—speed is the result of dividing a distance by a time. This is the conception of speed that the teachers continually expressed during Phase 1 of the reflective episode. This conception focused on Clown’s observed distance traveled, and the observed time that had elapsed to travel that distance, and emphasized the result of dividing distance traveled by elapsed time—a pseudo-empirical abstraction.

Rather than discussing speed as a direct proportional relationship between distance traveled and amount of time required to travel that distance, Tami’s comment suggests that she was still thinking about speed as what you obtain when you divide distance traveled by elapsed time—highlighting the resistive nature (to transformation) of such a focus.

Excerpt 16, Class #4

Gwen: Well that would be an average if he was going that same exact speed the entire time, which I don’t think is necessarily true...So, I have a problem with the question, because...

Pat: Okay

Gwen: I don’t think he was a constant rate the entire time.

Story: Is he going a constant rate?

Gwen disagreed with Tami’s definition by asserting that she believed that the Clown traveled at different speeds throughout the segment and that what Tami would be calculating would be an average speed. Although Gwen did not provide a

meaning for speed (constant or average), her comment suggests that she did have distinct understandings for average speed (change in distance divided by change in time) and constant speed. In addition, both Gwen and Story's comments brought up the idea of constant speed (although neither indicated a need to provide meaning to the term), an idea that will become central to the discussion.

Pat asked the teachers to watch Clown again, and set Clown in motion from the starting point (marked as 0 on the ruler). Clown walked 20 feet to the right, then moved left until reaching the 6 foot marker, then moved right until the 30 foot marker, then left again. At 5.79 seconds (on the timer), the clown was moving left and was at the 24-foot marker (on the ruler). The intent of the activity was to perturb the teachers' conceptions of speed by first making the teachers think about the distance that they employed in their conception (i.e., what distance were they using in their calculation).

Excerpt 17, Class #4

Pat: So, I ask this question again, how fast did he go?

Gwen: Fast enough to travel those distances.

Bernardo: Well, it took him 5.79 seconds to get from zero to however many feet that is up there...

Story: Well, no...because he went up and back and up and back.

Bernardo's comment suggests that he conceived of speed as: $\text{speed} = (\text{distance end} - \text{distance start}) / \text{total time elapsed}$; the distance that Bernardo employed in his calculation was the Clown's distance from start. Story's comment

suggests that she conceived of speed as: $\text{speed} = (\text{total distance traveled}) / \text{total time elapsed}$; the distance that Story employed is the total distance that Clown had traveled in the elapsed time. Neither Bernardo nor Story indicated that they were thinking about a direct proportional relationship between distance traveled and amount of time required to travel that distance (either distance from start or total distance) as had been discussed the previous class period (Phase 1); rather, they were thinking about what you obtain when you divide distance (either distance from start or total distance) by elapsed time.

Excerpt 18, Class #4

Bernardo: No but to end up...he walked different speeds at different times...from zero to 5.79 seconds...

Story: Right.

Bernardo: And changed direction.

Story: Yeah, but are you talking about total distance or just distance from the start?

Janine: The total distance.

Bernardo: Well, I know that's what we're talking about...'cause if all I'm interested in is from where he started to where he stopped, then and he stopped [...]
Then he went 30 feet in 5.79 seconds...what he did between my pictures, I don't know...

Pat: All right, Story, you were thinking about it differently, how were you thinking about it?

Story: I was thinking the total distance he traveled in that amount of time.

Bernardo: That's another way to look at it.

Story: And you didn't specify.

Although both Story and Bernardo agreed that there was more than one way to calculate the distance (i.e., total distance traveled or distance from start), neither indicated how the "speed" that would be calculated would be different (i.e., what speed is each calculating) depending on the distance used in the calculation—only that Pat had not specified which distance they should use. In addition, neither teacher provided a meaning for the speed they were discussing; rather each indicated that speed was determined by the ratio distance/time. Furthermore, as Pat will point out, each teacher talked about "the speed" as if that was the speed that Clown moved during the entire trip, anticipating the need to discuss and differentiate between constant speed and average speed.

Excerpt 19, Class #4

Pat: All right, but how does...each of you are talking about a speed that he went as if that somehow characterizes his whole trip.

Story: Oh, no...that's the average of it.

Pat: I didn't ask what's his average speed...anybody hear the word average?

Story: You asked how fast is he going [shrugs arms and shoulders].

Pat: I said what was his speed.

Story: Well it was varied...you can't get...

Gwen: What ...

Story: It was varied...

Janine: You could get his speed at, I mean...how, how far down do you want to break the seconds, because his speed may change every billionth of a second, so how can you get...unless you assume that he's walking at a constant rate from this second to that second...well, I just think that it's kind of implied that when you ask that question we have to give you an average, because how else could we?

Janine's comment suggests that she did differentiate between constant speed and average speed, although she did not articulate a meaning for either. Janine indicated that since Clown was not walking at a constant speed that it would be necessary to use the average speed; that they must be speaking about the average speed. In addition, she indicated that although the speed could be determined for smaller and smaller increments of time, it would still be an average speed.

Janine next asserted that, "I'm just giving you an average because otherwise it is too hard to see it. His rate could be changing every second so how else should we answer the question?" Alyce asked whether there needed to be "an" answer, to which Janine responded that there were "a lot" of answers—implying that the average speed could be calculated for many different increments of time, and that to get "the answer" would require a specified time increment. Although the teachers appear perturbed by Pat's question, no one has indicated that what they need is a meaning for speed, rather the teachers appeared to indicate that they what they need is to be told what to divide.

Phase 2.3 – What shall we mean by speed? Four minutes into the activity, Adriana asked, “What shall we mean by speed?” Even after the previous day’s (Phase 1) emphasis on meanings and Pat’s assertion that speed involved a direct proportional relationship between distance traveled and amount of time required to travel that distance, the teachers continued not to focus on a meaning for speed. Rather, the teachers continued to indicate that they were thinking about the object obtained after dividing distance traveled by elapsed time. Pat moved the discussion toward addressing Adriana’s question.

Excerpt 20, Class #4

Pat: So, what shall we mean when we say speed?

Janine: Average.

Alyce: I don’t like the average, because he never went the average, well he might have, but he probably didn’t.

Janine: So, when you talk about how fast a car’s going, how fast did you go over that trip, when you went to your mother’s how fast were you going?

Alyce: On the freeway I was going this [gestures with hand], on...

Janine: How do you know?

Alyce asserted that she would like to use a speed that Clown actually traveled at some point during the trip, and that she believed the average speed did not address this. Janine implied that all they can know is the average speed. Pat stated that in order for the conversation to move forward they needed two things:

1. What shall we mean when we talk about something’s speed.

2. What shall we mean when we talk about something's average speed.

Pat asked the teachers to observe the Clown animation again, as he moved Clown frame by frame, and to indicate when they believed that had Clown changed speeds, using whatever meaning for speed that they had (i.e., using their intuition). After several of the teachers had asserted that Clown had jumped, or hopped, Pat asked what that meant. Rachel explained that, "[Clown] covers a greater distance in the same [time] increment." Pat pointed out that Rachel had brought in the notion of time, which had yet to be mentioned. As they watched Clown move again Pat commented that in one-tenth of a second Clown went about 1 foot, and later in two-tenths of a second Clown went about 2 feet; the teachers agreed that clown had not sped up, that the "hop" that Clown took was a mirage.

They continued to watch the clown animation (frame by frame) until at one point they found that Clown had travelled 2 feet in one-tenth of a second, turned around and travelled 3 feet in one-tenth of a second; the teachers agreed that Clown had sped up. Pat asked, "As you answered that question: 'Did he speed up?' What definition of speed did you use to answer that question?" Rachel stated that, "We look at the distance he travels in that increment in less amount of time or more amount of time ... [or] within a set increment of time he covers more of less distance."

Rachel's comment suggests a focus on perception, she "observed" Clown moving a certain distances in a specific amount of time. Rachel's focus is on the observed quantities. In addition, Rachel's response indicates that she is focused on

making a comparison (answering the question), rather than thinking about what meaning for “speed” would make sense no matter what distance and time increments were given. Pat asserted that often, when making comparisons, it is common not to say anything (i.e., provide meaning) about the things that are being compared. Pat next moved the teachers to derive a meaning for speed.

Excerpt 21, Class #4

Pat: What shall we mean by an object’s speed? [Quiet for seven seconds]

Annie: Distance traveled divided by time. [Quiet for 15 seconds]

Pat: Annie offered something is that it, somebody offered something, so we should go on? [...]

Alyce: Say it again Annie, we didn’t hear it over here.

Annie: The distance traveled in a set amount of time or the ratio of the distance traveled to a set amount of time. [Quiet for 17 seconds]

The teachers continued to hold on to their conception of speed as the object obtained after dividing distance by time—they are constrained in their capacity to productively reflect. Pat asserted that he had hoped that someone would have stated that, “What we’re really talking about is constant speed”; which was at the root of making comparisons. Pat stated that, “The claim that something is going at a speed means [...] you’re implying that it’s not changing, you’re implying that it’s constant.”

Pat suggested that the teachers attempt to articulate what they meant by constant speed, because otherwise there is often a tendency to think about speed at

a moment (what is Clown's speed right now). Pat asked what the Clown's speed was when the animation was stopped, an action that anticipated a discussion of speed at an instant.

Excerpt 22, Class #4

Story: Zero.

Gwen: He's not going anywhere.

Pat: What does that mean? It means as time passes he doesn't move. Do you see time passing?

Johnny: Nope.

Story: No.

Pat: So, how can his speed be zero?

Johnny: It's your relative frame of reference...

Pat: See your using it within your experiential time, you're just saying, "Look, time's passing and I'm getting older and he's not moving so..."

Story: It's undefined then.

Gwen: Zero's the denominator.

Pat: So, what's his speed right now?

Gwen: You can't find it.

Pat: Well...we're tempted to say zero, he's not moving right?

Gwen: Right, but even then if you we were to find a speed he'd have to be moving, he'd have to be covering a distance in order for us to find a speed...and since he's not covering a distance, then we can't find that, so

that's why we need to take small increments that are really close together, to have him move just a little bit, so that we could find that speed.

Pat: Do we really have to take small increments?

Gwen: If you want to find it really close [gestures with hand] to that spot.

Pat: What is this idea of speed at an instant?

In the preceding excerpt, Story initially asserted that Clown's speed was zero—she had perceived Clown as not moving. After Pat asserted that time had not passed, Story indicated that Clown's speed was “undefined”; suggesting that she was thinking about dividing zero by zero. Gwen indicated that attempting to determine Clown's speed when the animation was stopped was problematic due to the fact that time would be zero; that obtaining the “speed” would require that you divide by zero. Both Story and Gwen appeared focused on making the calculation, rather than thinking about a need for a coherent meaning for speed.

The conversation next turned to making the teachers' conceptions of speed at an instant explicit and an object for discussion, and to anticipate a discussion of constant speed.

Phase 2.4 – Speed at an instant. Pat asserted that when driving, “Don't you look at your speedometer and say right now I'm going 75 mph?” Janine stated that this speed was an average. Rachel asserted that this was the “speed at that one second.” Alyce stated that the speed was the speed that you “were going right before that, that's what is registered.”

Excerpt 23, Class #4

Pat: So, are you saying that there's no such thing that a speed in an instant?

That at a moment in time you can't be going any speed.

Story: Yeah, if you freeze time, then you're not moving.

Pat: So, you're not going any speed.

Gwen: I'd have to say yes, you have to have small little pieces of something going some distance in order to find, to find...

Alyce: Because you have to be going someplace.

Pat: So, does that mean...you're driving from here to home, you're on the freeway...at precisely 5 o'clock you are not traveling at any speed...that's what you are saying.

Alyce: Yes [...]

Faye: I think you would have a particular speed at a certain clock reading.

In the preceding excerpt, the teachers appeared uncertain of their conceptions. Story and Gwen asserted that there must be some time elapsed in order for there to be movement, and that "speed" required that there be some perceived movement (e.g., required something "observable")—this suggests that the teachers might have been reflecting on their conceptions.

Pat asserted that the conversation should center around what made sense and that the teachers should be aware that they were trying to come up with constructs that were coherent to them. Pat stated that, "Let's take that as evidence that some people think there is such a thing as a speed at an instance," but that such

a conception would not work for the teachers unless they could somehow make such a conception make sense to them.

Excerpt 24, Class #4

Pat: So, is there speed at an instant?

Marian: I want to say no...just for the fact that... at that moment that's where you are at, you're at a certain position...I don't know...

Bernardo: Well, when we take a speed of something, aren't we comparing two things...in order for speed at an instant to make sense, it's almost like we have to have another speed from somewhere else to compare it, to see if it does make sense.

Janine: But you don't have any distance at that instant [...]

Bernardo: If I'm going 70 mph on the way back home, at exactly 5 o'clock all of the sudden I careen to a halt?

Janine: But at the beginning of your instant and at the end of your instant, was there change?

Bernardo: Was there motion?

Janine: Did your distance change?

Bernardo: Let's assume yes.

Janine: It's a very small instant, but...

Bernardo: Sure...

Pat: No, at an instant means no amount of time has elapsed.

Janine: And then there's no distance, and so...

Gwen: So, you don't know if you're moving.

Janine: Then I agree with you, you need an interval...and therefore...you are always assuming it's an average.

Janine's comment suggests that (for her) speed is always about an average speed, and that although you can make your time interval smaller and smaller, that you are still talking about an average speed. Although Pat intended for the discussion to promote a focus on constant speed, Janine apparently interpreted the discussion as indicating that there was only average speed (to which she did not provide a meaning). After a brief discussion on how radar worked, Pat introduced the "speed" pictures, in an attempt to perturb the teachers' focus on the need to observe motion (i.e., to perceive movement).

Phase 2.5 – Orienting the discussion toward constant speed. The "speed" pictures involved a variety of vehicles (e.g., van, motorcycle) at the instant they had crossed through the two white crossing lines (denoting a pedestrian crosswalk) at an intersection. The teachers were informed that they knew the camera (shutter) speed for the photos and that the two white lines painted on the street were 10 feet apart. For the photo of the motorcyclist (Figure 8), Pat asked whether the guy was moving. Several teachers responded, "Yes."



Figure 8. Photo of motorcyclist.

Pat then asked whether the motorcyclist looked like he was moving. The majority of the teachers responded, “No.” Rachel stated that the motorcyclist had to be moving, because otherwise he would fall over. Pat displayed a second photo (of a sports utility vehicle, SUV), to which the teachers said that they could not tell if there was movement. Pat zoomed in on the photo and stated that close up you could see a little line between the doorframe that indicated movement (Figure 9). This picture indicated how far the vehicle traveled during the time that the camera’s shutter was open.



Figure 9. Photo of sports utility vehicle indicating movement.

Pat asserted that the photo was taken at a shutter speed of $1/1000^{\text{th}}$ of a second, which offered an estimated speed of a little less than 60 miles per hour. Pat asserted that since there was no evidence of movement in the motorcycle picture, that they had no way to estimate the speed, but that in the photo of the SUV, when they zoomed in there was evidence of motion.

Pat gave a scenario where the time was one-sixtieth of a second and the car went 2.5 feet, and asked how fast the car was moving. This question moved the conversation toward a discussion of the relationship between distance traveled and time elapsed, and anticipated a discussion of constant speed.

Excerpt 25, Class #4

Pat: We have this information, how can now we start thinking about how fast he was going? [Several of the teachers appeared to be making calculations on their papers]

- Pat: I'm not asking for you to give me a calculation to do...listen to my question, "How can we think about how fast he was going?"
- Faye: 2.5 feet in one-sixtieth of a second.
- Pat: Well, okay, but how do we turn that into a speed then? Conceptually how do we turn that into a speed? [...] It's a speed if we assume what? [...]
- Pat: That it moved during that time. It's not like, well [...] The door is 2.5 feet long and one second went by while I looked at it.
- Alyce: There's no change.
- Pat: There's no change, so the idea of speed is not just a distance and a time, like a door that is 2.5 feet long and we happened to glance at it for one second, right? There's a distance and a time, isn't there...so what's that missing, say it again Alyce?
- Alyce: Change.
- Pat: Change in what sense?
- Alyce: A change in distance over elapsed time.
- Pat: A change in location. It's not a change in distance; it's a change in location. So, the distance that it traveled is the change in location; it's the distances between the changes in location...
- Pat: We get the speed from it, not by mentioning a distance and mentioning a time, although that's part of it...it's because we're also imagining that distance being covered.

In the preceding excerpt, Pat had intended for the teachers to focus on speed as the quantification of motion, and the need for a change in location (an imagined distance being covered). The teachers' comments indicated that they were still focused on speed as the object distance/time, where any required change involved a perceptual "change in distance over elapsed time." Pat next pushed the idea of constant speed.

Excerpt 26, Class #4

- Pat: When we said that speed is a little under 60 miles an hour...the one where we said it was an inch in $1/1000^{\text{th}}$ of a second...so, what else are we assuming when we say that? [Quiet for five seconds]
- Pat: It traveled an inch in $1/1000^{\text{th}}$ of a second, okay, so we're assuming there's motion. Right? Well, what else are we assuming? [...]
- Story: That it was maintaining that for quite some time.
- Pat: Well, let's not talk about before or after right now. Let's talk about that $1/1000^{\text{th}}$ of a second. Are we saying, "Well, you know, it might have traveled that inch in the last one millionth of a second?" Are we allowing that...is that one of the things we're entertaining?
- Johnny: The fact that speed is constant throughout the whole...
- Pat: But now you're being circular...what are we assuming about that motion?
- Liz: That it's constant.
- Story: Yeah.

Pat: Okay, it's constant, but that's one of the things that we have yet to say what we mean. What do we mean by constant, other than it doesn't change? [Quiet for 25 seconds]

Pat: There is something called speed, we don't know what it is, but it doesn't change. So...whatever concept of speed kids have, which could be the most off the wall thing that you can possibly imagine, and you say that's fine, just understand that constant speed means that stupid thing you have in your head doesn't change...are we okay with that?

In the preceding excerpt, the teachers appeared uncertain as to the issue that Pat was raising. The teachers indicated that if a car traveled one inch in $1/1000^{\text{th}}$ of a second, then they were assuming that his speed was constant for that $1/1000^{\text{th}}$ of a second. This contradicted an earlier assertion (made by the teachers) that there was a difference between average speed and constant speed, and what the teachers had calculated (as with the earlier case) was what they had identified as an average speed. This might indicate that since the teachers were asked initially to imagine the situation, they were unable to perceive the motion (it was not observable)—they assimilated the situation to one involving constant speed.

Pat attempted to move the teachers to consider their students' conceptions of speed, and the need for some coherent meaning to be established.

Excerpt 27, Class #4

- Pat: It would be nice to if we could say what we would like them to mean, wouldn't it? We don't want to just say, whatever you happen to have in your head think of it not changing. [Quiet for 16 seconds]
- Pat: I'm not going to say a word. You can look at me all day long. I'm not going to say a word [...]
- Janine: You mean the average over that interval of time?
- Marian: The average rate of change.
- Johnny: It's not average.
- Pat: But...I know how to average two numbers.
- Alyce: Are we defining constant...
- Rachel: I don't know.
- Janine: What are we trying to define though?
- Gwen: Speed.
- Pat: Constant speed.

In the preceding excerpt, the teachers continued to struggle “seeing” the issue that Pat attempted to raise. The teachers continued to hold onto a meaning of speed as “distance divided by time”—which constrained their capacity to reflect on what they might mean by “constant speed” so that it spoke about motion.

Phase 2.6 – Developing a meaning for constant speed. Eve brought up the idea of “cutting up” the intervals, as Pat had done during the previous class meeting (Phase 1.7, pp. 133-136). Pat employed the idea of “cutting up” the intervals to

promote the idea of a proportional relationship between distance traveled and elapsed time.

Excerpt 28, Class #4

Eve: If we cut that amount of time into intervals it will cover the same distance.

Pat: Does that work? Can you say it again?

Eve: In that amount of time, if we cut the time into the same intervals, in those intervals of time it will cover the same distance.

Pat: So if you have some interval of time and you cut it up into smaller amounts of time...

Johnny: Then the distance traveled is always the same in each of the split up time intervals [...]

Pat: Okay, we're getting somewhere now; we're not just saying that this is about one interval of time. We're saying that it's also about any subinterval of time in there, right?

Pat: Eve's way of saying it was that if you cut up the amount of time that it travels interval into...say 153 pieces...then what about the amount of distance that go with each of those...equal sized 153 pieces.

Story: You get 153 of them.

Pat: Well, you're certainly going to get 153 pieces...

Johnny: They're all the same size.

- Pat: So, that sounds like we're having a criterion for constant speed. If no matter how you cut up the interval that was traveled...can we say this a shorter way? [Quiet for seven seconds]
- Pat: Like maybe those smaller pieces remain in the same proportion to the distances traveled? [Quiet for nine seconds]
- Pat: See...what Eve said was that constant speed entails a proportional relationship.
- Johnny: The proportion stays the same no matter how you...
- Pat: So...to say that some object traveled at a constant speed over some interval of time [...] Is to say that if you travel some...fraction of that amount of time what does that imply about the distance that it's going to travel? You're going to travel that same fraction of the distance.
- Pat: So, if you travel $1/100^{\text{th}}$ of that time, you'll travel $1/100^{\text{th}}$ of the distance. If you travel for 99.99% of that time, you'll travel 99.99% of the distance.
- Pat: It's saying that within that motion is a proportional relationship. All subunits of time will be in the same proportion to their corresponding amounts of distance traveled.

Pat indicated that this was what they had worked on in the previous class (which the teachers appeared not to recall) and that constant speed means that there's an invariant proportional relationship between amount of time traveled and the amount of distance covered.

The teachers next worked to construct a sentence that denoted their working definition for constant speed, and ultimately decided on: Constant Speed—the change in distance compared to the change in time stays in the same proportion for any interval chosen. Pat asserted that the notion of constant speed as an invariant proportional relationship between distance and time was something that could do a lot of work for students.

Phase 2.7 – Retrospective discussion of activities and conversations. In the last part of Phase 2, Pat asked the teachers how they ended up getting a working definition of constant speed. As with Phase 1, the teachers focused their attention on their participation in the activities, rather than indicating an awareness of Pat’s involvement in managing the conversations.

Pat asserted that they had spent quite a lot of time developing a working definition for constant speed (as a proportional relationship between distance traveled and time elapsed), and asked, “What does that buy us that we don’t get by just saying well, in the same amount of time he always travels the same distance? Do we get anything more?”

Excerpt 29, Class #4

Faye: So, we could make predictions, or...compare with other distances or other intervals of time...to predict...maybe how long it took him to go 20 feet or 20 inches, with proportional reasoning we can look at ...

Pat: Okay, so now with this notion of proportional reasoning being explicitly a part of the idea of constant speed, then you can rely on it explicitly to answer other questions...anything else?

Adriana: I know that...in the past when I've been working with students and we throw out 60 miles and hour, I've never really emphasized or ever brought out this idea that there's this change going on [...] I really don't think I've stressed that there's a change involved.

Pat: That there's motion involved [...]

Johnny: When I talk about that with my students I ask them what does it mean when you look at a speed limit, when you see 65 milers per hour when we have that discussion. I'm trying to get the kids this to get this understanding of this idea that for every incremental hour that goes by I've gone an additional 65 miles farther. Trying to get them to see it that way, this proportionality.

Pat: But the way that we were talking about it brings more than that...I'd like us to be explicit about what more does it bring? I'm sure that you all do it very well, getting across the idea, that if the kids don't already know it, that if you go an additional hour your going to go an additional 65 miles, or something like that. Do you ever talk about how far they're going to go in a millionth of a second?

Several: No.

Pat: That's what you get from this constant proportional relationship, is now you can talk about how far you're going to go in a millionth of a second, because you can cut up any amount of time into any number of pieces. And what you can do is you can build up a sense of a continuum.

Faye's comment focused on the types of experiences that Pat had led the teachers through during the lesson. Johnny's comment indicated that he tended to focus his students' attention on distance changing per unit increment of time, not in terms of a proportional relationship. In addition, the teachers' comments suggest that they did not view their "working definition" of constant speed as providing anything more than what they had experienced or how they had promoted the idea to their students prior to the lesson—they did not indicate that the "working definition" was radically different from how they had conceived of speed before the lesson.

Pat concluded the episode by asserting the importance of emphasizing continuous quantities (i.e., continuous variation) to students. That, otherwise, students can develop a conception that the "world is made up of whole numbers," a conception deleterious to an understanding of functional relationships. Pat asserted that this was why the idea of constant speed was so important; that instruction must leverage ideas of constant speed, constant rate of change, changes in little bits, and thinking about small changes to get students to think about continuous change

Phase 2 – Summary. Throughout Phase 2, the teachers' conception of speed (expressed as distance traveled divided by elapsed time), continued to be highly

resistant to Pat's attempts at perturbation. The teachers' enduring meaning of speed as "distance divided by time" appeared to constrain their ability to recall the prior day's discussions about proportionality and to connect activities and conversations from Phase 2 with those that they had engaged with during the previous class period (Phase 1). In addition, the teachers appeared uncertain as to what issues, regarding meanings, Pat had attempted to highlight—meanings appeared not to be a focus of the teachers' ways of thinking. Finally, the teachers appeared to focus solely on their participation in the activities, emphasizing those objects that they could perceive, thus constraining their capacity to reflect productively.

Reflective Episode #2: Managing a Conversation

The second reflective episode that I will describe was designed to orient teachers to re-conceive their students and to re-think their teaching. Specifically, the episode was designed to move the teachers to focus on their students' understandings and on the instructional actions that the teachers might take to support student development of intended meanings and ideas.

The episode occurred during the tenth class meeting, and involved the discussion of a classroom video provided by one of the participating teachers (Bernardo). The video consisted of Bernardo leading his geometry (identified as "Math 3-4, Period 1") students in a discussion of problems from a recent homework assignment. The particular geometry class was part of the reformed-Geometry case study component of the TPC² project.

Prior to Class #10 (as an out-of-class assignment), the teachers were asked to: review Bernardo's pre-lesson learning goals (Appendix B), watch the video, and provide written comments regarding Bernardo's success at meeting his two learning goals. In addition, teachers were asked to be prepared to discuss the video and to share their comments. The homework problems that were being discussed (in the video) can be found in Appendix C, and an annotated transcription of the portion of the video that the teachers viewed during the reflective episode can be found in Appendix D.

In his pre-lesson learning goals (Appendix B), Bernardo stated that he intended to spend the majority of the lesson discussing a homework problem that asked students to "construct a point equidistant from three non-collinear points." The problem was placed in the context of "three friends wanting to start an Elvis club, but no friend wanted to walk farther than the other two" (Problem #6, Appendix C). Bernardo also indicated that his specific learning goals included that:

- Students take ownership of the given problem, and through classroom discussions, develop a method for solving the problem that focuses on explaining their reasons for doing things.
- Students see how they can use seemingly unrelated information to use to solve new types of problems.

Prior to the "reflective" portion of the lesson, the teachers provided Bernardo with feedback, as to whether they believed that he had achieved his goals. Bernardo led these initial activities, with the Functions 3 course instructor (Pat) playing the

role of observer. The teacher's initial comments (which were essentially verbal restatements of what they had written for the out-of-class assignment) were all very positive and focused on praising Bernardo for having met his stated pre-lesson learning goals.

Although Bernardo had stated that a learning goal was for students to explain "their reasons for doing things," in the actual lesson, Bernardo focused his students' attention on describing what it was that they or another student had done—a focus on the steps that the students had performed. None of the teachers commented that Bernardo (in the reformed-Geometry case study video) had not focused his students' attention on meanings and reasoning (either the student's own reasoning or the reasoning of other students). Rather, Bernardo had focused students' attention on the steps that were performed to solve the problem.

Tami commented that she liked the "different methods" that the students had displayed, but focused her attention on the differences in the students' diagrams, rather than the differences in the students' reasoning and rationale. Annie asserted that she "liked how [Bernardo] had the students take ownership...where the kids had to ask each other questions, they were not asking you questions...it was pretty much student driven." Marian stated that she liked how Bernardo had consistently asked the students to clarify themselves, or to clarify what other students had said. What neither Annie nor Marian made explicit was *what* the students were questioning or clarifying (i.e., were they explaining steps or discussing their reasoning), or Bernardo's role in how the conversation unfolded (i.e., how Bernardo

managed the conversation). In actuality Bernardo had been at the center of all remarks. Bernardo had conducted the class as if having a private conversation with each student to whom he spoke, though he did speak to a large number of students.

Next, Bernardo reiterated that one of his learning goals had been for students to take ownership of the problem and that he “step back and let them just basically have at it.” Bernardo stated that they were going to re-watch the video (as a group) with the intent of identifying the path of the conversation (i.e., to check whether Bernardo was the “fount” of all knowledge). After watching the video (see Appendix D for an annotated transcript of the video), Bernardo requested that the teachers review their diagrams (of the path of the conversation). The maps showed Bernardo at the center of all conversations.

Excerpt 30, Class #10

Bernardo: So, take a look at...your diagram, given my goal of me stepping back and letting the students do it. Did I achieve that goal?

Liz: Not so far.

Rachel: Not at the beginning, you were starting to let it happen more as it progressed [...]

Bernardo: I was definitely the fount of all knowledge was I not? [Several teachers laugh]

Johnny: Yeah, it seemed like you were the one that was...asking the probing questions [...] the only time you heard the students speak is when they

were up there at the board, and it was kind of in a response to what you had asked.

Bernardo: Right, I agree.

Alyce: But you did a good job of not clarifying or correcting [...] I have a hard time just letting the conversation keep going whenever I can see that clearly she doesn't know what she's doing, and for you just to go, "Oh, yeah, okay, anybody else? [Several laugh]

Bernardo: Part of it is that I'm trying to get them to explain their reasoning, and hopefully to let them see where their reasoning doesn't make sense, themselves, rather than me say, "You're wrong."

None of the teachers expressed much concern with the fact that Bernardo had been the center of all conversations. In fact, Alyce praised Bernardo for not "correcting" the students, and simply allowing the conversation to continue, whether it was moving anywhere productive (for student learning) or not. In addition, Bernardo's comment suggests a focus on the products of reasoning (whether the solution, or each step in a solution, is correct or not), rather than on meanings and reasoning.

As the conversation continued, the discussion began to focus on reasons why the teachers believed that Bernardo's students had not taken a more active role in the discussion. Specifically, the conversation began to highlight a focus on non-cognitive aspects of the teachers' conceptions of their students, such as the students' comfort level with the material, with the teacher, or with one another. In addition,

none of the participating teachers identified Bernardo's role in how the discussion had unfolded, or how he might have managed the conversation so that things had unfolded differently—they were not focused on the teaching.

Pat intervened at various points during the discussion, mainly in an attempt to keep the teachers focused on the student thinking and the teaching that they witnessed in the video. After 44 minutes of Bernardo facilitating the discussion, Pat intervened and began the reflective episode. The reflective episode had two main phases with four sub-phases in the first and eight sub-phases in the second. A chronological overview of the lesson's phases (Class #10, September 17th, 2007) is given in Table 3.

Table 3

Chronological Overview of Reflective Episode from Class #10

Class Meeting (Date)	Phase	Duration (Minutes)
	Phase 1 – Managing a Conversation: Setting the Stage	
Class #10 (9/17/07)	Phase 1.1 – Orienting the Discussion: Focus on the Teaching	2
Class #10 (9/17/07)	Phase 1.2 – Focus on Students’ Reasoning	3.7
Class #10 (9/17/07)	Phase 1.3 – Focus on the Interactions	1.3
Class #10 (9/17/07)	Phase 1.4 – Anticipating a Discussion on Managing a Conversation	6.5
	Phase 2 – Managing a Conversation: A Focus on Students’ Mathematics and Instructional Actions	
Class #10 (9/17/07)	Phase 2.1 – Managing the Conversation	7.8
Class #10 (9/17/07)	Phase 2.2 – Focus on Perception or Logic?	4.2
Class #10 (9/17/07)	Phase 2.3 – Imagining an Alternative Discussion (Part 1)	7.5
Class #10 (9/17/07)	Phase 2.4 – Making Students’ Reasoning Explicit	4
Class #10 (9/17/07)	Phase 2.5 – Imagining an Alternative Discussion (Part 2)	9.8
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Phase 1 – Managing a conversation: Setting the stage. The intent of Phase 1 of the reflective episode was for teachers to make their conceptions of their students and their teaching explicit. Specifically, the discussions were designed to move the teachers to articulate their conceptions of students’ mathematics and of instruction designed to engage students in meaningful mathematics.

Phase 1.1 – Orienting the discussion: Focus on the teaching. Pat initiated Phase 1 of the reflective episode by asking the teachers why they felt so compelled to be nice when analyzing the video. This instructional action attempted to move the discussion away from one about Bernardo (i.e., the teacher) toward one about what

happened (i.e., the interactions and the students' thinking, the teaching). Story stated that since all of the teachers did not want to be "ripped" when it was their turn to display video, that she believed the teachers were setting a non-judgmental atmosphere from the beginning.

Excerpt 31, Class #10

Pat: You can't learn the lessons unless you make it about the teaching...because if it's always about the teacher, then everybody does wonderful...and nobody learns anything.

Annie: Well, I think that also says a little bit about why the students might not have asked questions, because when they were up there they wouldn't want students to ask them questions...

Pat: I would argue that there might be another reason why there weren't more student questions, that had to do with the structure of the discourse.

Tami: But he did have certain people getting up there who were willing to try.

Marcy: Yeah, that was nice.

Pat: Okay, that's fine, that's wonderful.

Tami: I think that it takes quite a lot of courage to get up in front of 20 other people.

Pat: That's wonderful, but it's beside the point.

Johnny: I still saw...most of it was a conversation between the teacher and whoever was on the board and that was it...and periodically you had

questions fielded, “What do you think of this or that,” but it was always toward a person...we’re always taught to don’t, “Ask what do you guys think of,” so then you always ask pointed questions to one person, so you don’t get stuck in, “What do you think of this, or how many agree,” so...

Tami’s comment is focused on speaking (i.e., getting students to talk), not about *what* is being spoken about. There is a difference between students being willing to speak versus providing students with something to talk about, and Tami is focused on the former. Johnny’s comment suggests that he focused on the number of people involved in the discussion, rather than on *what* is being discussed. The focus is on bringing as many students into the discussion as possible—an attempt to get all of the students to perceive (experience) the situation. It is worth noting that Johnny also brought up the issue of the way Bernardo managed the conversation—that Bernardo always made the conversation between himself and whomever was at the board. He did not shape the conversation so that other students felt invited to participate.

Phase 1.2 – Focus on students’ reasoning. Pat moved to the next phase of the reflective episode by bringing in the idea of student reasoning, specifically, whether the teachers had noticed students expressing their reasoning during the lesson (in the video). This action attempted to move the discussion away from who was conversing (or how many were conversing), and toward what was being discussed.

Excerpt 32, Class #10

Pat: Let me ask you this, you watched the video closely, right... how many times did you hear a student express his or her reasoning?

Liz: That's what I was telling Marian...I said he got up there and he drew all these lines and says, "So, it's this point here." He never explained what his circles or lines or anything meant.

Pat: Well...whatever they meant...how many times did you hear a student express his or her reasoning?

Liz's comment suggests that she believed the focus (in the video) should have been on the students explaining their drawing, more specifically, why they drew what they drew. This suggests that Liz might intend that the focus of the task (in the video) to be on having the students explain their (or one another's) diagrams, rather than on using the diagrams as a tool to help students explain their (or another's) reasoning.

To Liz, Bernardo's and his students' focus appeared to be on the diagram, rather than the logic—the focus was on making the steps in the students' solutions visible (i.e., perceivable), rather than getting the students' reasoning (i.e., logic) on the table as objects for discussion and possible reflection. Pat next moved the discussion toward getting the teachers' to separate the students' physical actions (i.e., what the students had drawn) from their reasoning.

Excerpt 33, Class #10

Pat: Okay...it doesn't take things to get very complicated, before nobody really understands what to do...right? Just get a few line segments up there and a couple of circles, and you don't know what you're supposed to pay attention to [...] I mean there's so much going on up there, you don't know what you're supposed to pay attention to, and you don't have any idea why anybody's doing what they're doing...what's to talk about then...what is there to talk about?

Liz: What did you do?

Pat: Liz?

Liz: Well, if that's all they've done then all you can talk about is, "What did you do?" There's no reasoning behind it...

Marcy: Instead of, "Why did you do?"

Liz: I made a circle...there's no reasoning behind it, it's just something somebody did for who knows why.

Johnny: Procedure.

Liz: Yeah.

Johnny: Pretty much.

Janine: There must have been a reason...

Johnny: But they don't say it...they don't say I'm going to draw a circle, for this reason.

Pat: Why? What does a circle get you?

Johnny: Exactly, but they don't even say that though.

Bernardo: And I never asked him.

The previous excerpt suggests that there was some agreement among the teachers that the students (in the video) were not expressing their reasoning—although Janine's comment suggests that she believed that the reasoning was implicit. What is not clear is whether the teachers were talking about getting the students to explain their steps (e.g., why I did this step, and next I need to do this step), or having the students focus on meanings (e.g., what would it mean for a point to be equidistant from three non-collinear points?) and coordinating their meanings.

In addition, although the focus of the discussion was on getting student reasoning out on the table, the teachers did not indicate why they believed getting students' to express their reasoning would be important (e.g., is it important for the students' learning or for the teacher to gauge for understanding?) or how the teacher could manage the conversation so that the students' reasoning would be made explicit.

Pat next moved to make explicit his earlier comment that the lack of conversation (in the video) was due more to the nature of the discourse than the nature of the students.

Excerpt 34, Class #10

Pat: See Annie, if there isn't a lot to talk about then there is no reason to speak.

[Quiet for eight seconds]

Tami: But likewise, we don't know what he did previous to this...is this a process that you've done many times?

Pat: It doesn't matter...suppose, yes...suppose the answer is yes, then what Tami?

Tami: Then we might be assuming that more students know what is going on and the reason behind that circle.

Pat: So, you're saying that once meaning is established then you can slip into speaking meaninglessly and not worry about it.

Tami: You might just understand why they...

Pat: But it's not about you understanding, you see that's the whole point. It's not about you understanding, but the students' understanding.

Tami's comment suggests that she found the conversation (in the video) unproblematic—that having students display their visible solutions, without discussing their reasoning, to be unproblematic. In addition, Pat's comment suggests that he interpreted Tami as indicting that once a teacher has assessed that students understand, then the articulation of meanings (apparently for the benefit of the teacher) is not required (or necessary). Such a focus does not allow for student reasoning to be made explicit, and does not move students to reflect on their own reasoning, or the reasoning of one another.

Liz next asserted that the only time she had heard a student's reasoning being questioned was when one student asked another, "Why did you do that?" Pat asked Bernardo to replay the portion of the video that Liz was referring to and

informed the teachers to “listen closely,” that he wanted the teachers to be able say what happened in the video clip.

Phase 1.3 – Focus on the interactions. In the video (of the Geometry class), a student (Amber) had drawn a dotted line from point B to point A (Figure 10) on the diagram that another student (Tina) had earlier drawn to indicate her solution to problem #6 on the homework (Appendix C).



Figure 10. Amber’s addition to Tina’s diagram.

A student (Deb) asked Amber about the dotted line. Bernardo directed Larry to answer Deb’s question and then asked another student Peter to summarize what he believed had transpired. An annotated transcription of the discussion (from the Geometry class) is provided in the following excerpt (and in Appendix D).

Excerpt 35, Reformed-Geometry Case Study

47. Deb: I have a question.

48. Bernardo: Guys, if you've got a question just speak up, holler out.

49. Deb: Okay, I have a question. I don't know what you were doing with the dotted line, like...

50. Bernardo: Okay now, be specific...

51. Deb: I know you're trying to find the center, but why is there one coming off of B? Like where is it going, does it just stop? [...]

52. Bernardo: Larry...is that a question?

53. Larry: Okay, I don't think it's done yet, because you have to draw the circle around B [...]

54. Bernardo: I'm kind of lost here. Peter...could you kind of get us caught up here?
What do you think we're trying to do, 'cause everybody had some good comments there?

55. Peter: They're trying to find the distance from B and S [...]

56. Bernardo: What distance from B and S?

57. Peter: Rephrase that [...]

58. Bernardo: I heard you say you're trying to find the distance from B and S...what distance from B and S?

At this point Bernardo stopped the video and Pat asked the teachers, "What happened?"

Excerpt 36, Class #10

Alyce: Nothing.

Johnny: Well, she never answered...

Pat: A lot happened.

Liz: Honestly, Bernardo stepped in and stopped the girl from ever being able to put any meaning as to why she thought she did it. Bernardo intervened and said, "Oh, somebody else might want to answer," and let somebody else join in, and he had, "Oh, it's wrong, you need to do something else."

Pat: Bernardo, were you aware at the time that that was what was happening?

Bernardo: Umh...that was a knee jerk reaction from me...I saw Larry raise his hand...Liz is absolutely right...I didn't let Amber answer.

Pat asserted that he was trying to push the teachers to not focus on what was right and what was wrong, but to focus on what happened.

Excerpt 37, Class #10

Liz: Do you think...if you sit there long enough the students will eventually ask, like the one, "Okay, fine, I'll ask" ...if you sit there long enough do you think that those kid's would have finally taken over? Instead of every time there was a pause, you threw a question in.

Bernardo: Umh...yeah, well, it's gonna depend on the topic.

Marcy: Yeah.

Bernardo: With what we're doing now, they have more of a tendency to jump in and just do it...if that's because they're more comfortable with just jumping in

and talking, or if it's because of the subject matter that we're talking about. I'm not sure...but the nature of this class is they're very quiet, and so eventually they'll do it, I think.

Liz: I know that's where I find a lot of mine, they'll finally go, "What are we thinking about? I don't know what are we thinking about, you tell me", and that's when another kid will finally step in, "We're supposed to be trying to do..." and then they'll start going against themselves...I know it has to be quiet for a while and the students finally get brave enough to do, "I don't know what I'm thinking about," because they think they're supposed to be doing something...but it takes another student to tell them that...

Liz continued by asserting that had Bernardo been less concerned with getting as many students involved as possible, and allowed the students to ask questions (of one another), then there would have been a better chance that the students would question "why" another student was "doing something."

Liz's comments suggest that she was not thinking about how the teacher could *manage* the discussion in a manner where meanings played a prominent role, where reasoning is made explicit and employed as objects for discussion—where meanings "do work" for students' thinking. Liz's comment that the students believed they should be "doing something" suggests a focus on action sequences, where reasoning is limited to describing the steps in the sequence.

In addition, the preceding excerpts focused on non-cognitive aspects of students, such as students will get tired of the quiet, or of not understanding, and eventually begin to question one another's reasoning. Bernardo next asserted that this was a very "quiet" class, to which Pat responded, "Why do you think that is, Bernardo?" This action allowed Pat to anticipate a discussion centered about the idea of managing a conversation.

Phase 1.4 – Anticipating a discussion on managing a conversation.

Bernardo commented that during a recent lesson, no one from the project had been present (no cameras, no researchers), and he anticipated that the students would be very vocal. Bernardo stated that he was surprised by how quiet the students were and asserted that perhaps he was "overwhelming them."

The ensuing discussion focused on identifying possible reasons why there was not much discussion in class, such as the students' prior experience with similar courses that focused on students needing to be vocal, class dynamics, or the subject matter. Although Bernardo did assert that the students *could* have a conversation, and that they *could* be engaged, he did not provide any indication that he had considered his role in getting them to converse or engage. The focus of the discussion continued to be on non-cognitive aspects of the students and the need for the students to be more vocal, rather than on *what* the students were being asked to discuss (i.e., meanings and reasoning), or the teacher's role in getting and maintaining the focus on meanings and reasoning.

Phase 1 – Summary. Throughout Phase 1, the teacher expressed conceptions that focused on non-cognitive aspects of their students and a lack of emphasis on their own role (as teachers) in moving students to engage productively in classroom discussions. In addition, the teachers expressed a focus on getting students to perceive, rather than getting them to reason.

Phase 2 – Managing a conversation: A Focus on students’ mathematics and instructional actions. The intent of Phase 2 of the reflective episode was for teachers to transform their image of what it meant to engage students in meaningful mathematical activity, by interacting with students in ways that promote reasoning and reflective discourse. Rather than focusing on the number of students who talk, or non-cognitive reasons why students do not talk, Pat managed the conversation so that the teachers were moved to focus on students’ mathematics, and how the teachers imagined that they might engage students to promote specific ways of thinking.

Phase 2.1 – Managing the conversation. After a short break, the class resumed. Pat stated that they were going to watch the video again (as a group), but with a different perspective. Pat stated that he wanted the teachers to look for significant moments, moments where it was important that what actually happened happened, and moments where it might have been important that something else had happened. More specifically, Pat asserted that he wanted the teachers to look at what happened and to also think hypothetically about how things might have unfolded differently, and with different consequences. Pat actions were intended to

move the teachers to focus on student thinking and how their instructional actions (as teachers) could move student thinking in desired directions—to manage the conversation.

Pat stated that he was going to simply play the video and that he would rely on the teachers to inform him when they wanted him to stop and discuss whatever they believed was important about the situation. Pat indicated that the expectation would be that the teacher would then articulate why they believed the event was important and how they believed it might have unfolded differently, in a “parallel universe.”

The teachers watched the video for 38 seconds before Alyce instructed Pat to stop. An annotated transcription of the discussion (from the Geometry class) that Alyce focused on is provided in the following excerpt (and in Appendix D).

Excerpt 38, Reformed-Geometry Case Study

12. Bernardo: Tina, is that what you did?

13. Tina: Yeah.

14. Bernardo: Yeah?

15. Tina: It’s really bad, like on here [gestures to her homework paper] it’s more straight.

16. Bernardo: Oh, okay.

17. Tina: So, it doesn’t really kind of like curve or anything, but...

The following excerpt picks the conversation up (from the Functions 3 course) after Pat had stopped the video per Alyce’s request.

Excerpt 39, Class #10

Alyce: I would have told her to go back up and fix your diagram.

Sheila: Explain it...explain what's going on.

Alyce: Use a straight edge, tell us what you're doing while you're doing it...since that's not at all what it looks like on your paper...because her paper looks like a pretty decent diagram (Figure 11) [...]

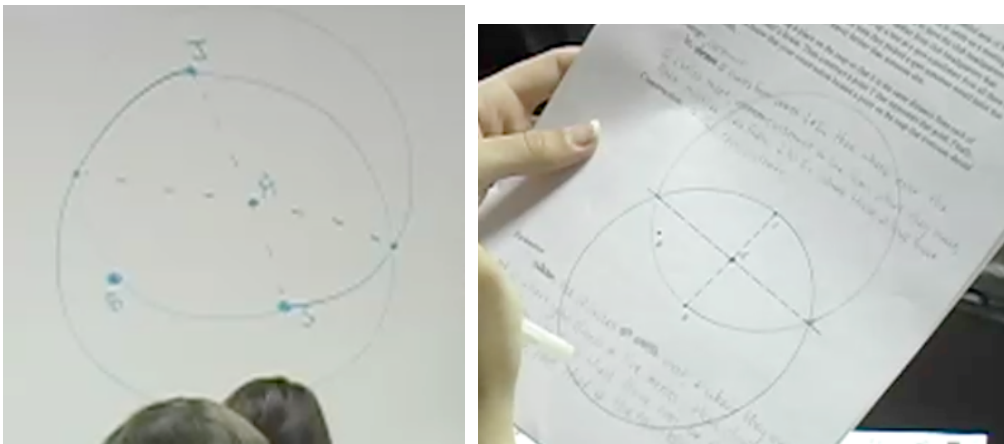


Figure 11. Tina's diagram for problem #6 on the board and on paper.

Pat: Is the point of her going back to make the diagram more accurate or is the point for her to go back and explain the logic of what she was doing?

Alyce: Both...because I don't think that she can explain it and that other students can understand without a more accurate diagram.

Pat: Do you need an accurate diagram to explain a logic?

Several: No [...]

Alyce: I think a little bit yeah...whenever she says this is a perpendicular bisector, if it looks like this [not perpendicular] (Figure 12)...then that might explain...why [she] thought that perpendicular was any two crossing lines...because that's what her perpendicular bisectors looked like [...]



Figure 12. Alyce's depiction of Tina's perpendicular lines.

Sheila: I think that a misleading diagram can cause misconceptions; I think that you're right about that, but on the other hand I think it's really good for kids to explain what they're doing even though the picture doesn't look like it.

Alyce: I agree.

Sheila: I will intentionally mislead my kids sometimes in a diagram, because I will say that's a 100 degree angle, but it's acute...and all they see is the picture versus what information is there and what am I asking you

about...I think it's important for them to be able to take the knowledge that they have no matter what it look like.

Alyce: But then at the same time we teach kids to say, "Well, does that answer make sense" ...and so if they get an answer that doesn't make sense with what their seeing...

Sheila: And that's the hazard of the misconception of what it looks like...I don't disagree.

Pat: The issue is which should rule.

Alyce: I want my kids to be able to use common sense versus this is the procedure and this is what I'm supposed to do...this is 100, so we have to just go with 100

Pat: No...not procedure...I meant logic versus perception, which do you want to rule.

Alyce: Logic.

Pat: If every diagram is perfect, then the issue of logic versus perception never arises...and so they get to trust perception [...]

Bernardo: Had I had her explain the sketch as she was doing it, then I think the lopsided circles and stuff like that would have been a non-issue.

Sheila: How about having the gentleman...he was explaining what she had done and you turning to her and say is that what you've done, how about having him go up and explain the diagram.

Bernardo: That would have been another option too.

Sheila: If he's pointing to things, I think this is what she did then... [...]

Alyce's initial comments suggest a focus on perception, rather than logic.

Alyce asserted that she would have had the student "tell us what you're doing while you're doing it," as in explaining the steps in a procedure. As Pat (and Sheila) pushed Alyce to think about whether she intended to emphasize perception or logic, Alyce indicated that she would emphasize logic, although it was unclear if she was thinking about maintaining the focus on making the steps in the students' solutions visible (i.e., perceivable), or getting the students' reasoning (i.e., logic) on the table as objects for discussion and possible reflection. Sheila's comment at the end of the excerpt suggests a focus on having students explain another's steps in a procedure (where the emphasis is on the diagram, rather than the logic). Pat used this discussion to move the conversation toward getting the teachers to imagine what they would consider a satisfactory explanation to the problem.

Phase 2.2 – Focus on perception or logic? Pat asserted that the phrase "explain the diagram" had cropped up several times during the discussion, and asked the teacher what they would consider to be a satisfactory explanation of the diagram, "What would a satisfactory explanation of the diagram have looked like, or sounded like...what would have been a satisfactory explanation of the diagram?" Bernardo asked if Pat could display the student's drawing from the sheet of paper, and Pat answered that he would rather not (and did not). This instructional action pushed the teachers to imagine the logic, rather than focusing on why the actual (perceived) diagram was right or wrong.

Excerpt 40, Class #10

Liz: I believe that part of that, it would have to be, so you can say, "Make a circle about A," but not just do that, but explain why you would do that or how that promotes you to where you want to go...being able to say what the point of that was...

Alyce: But she didn't make a circle about the point, she included the point on the circle.

Liz: I don't mean hers in particular...I just meant an accurate description of a diagram, of being able to explain it...

Liz's comments suggests a focus on logic in what she would consider a hypothetically satisfactory explanation, whereas Alyce's comment moved the conversation back to the diagram that the girl had actually drawn. In addition, Liz's comments included several pronouns (e.g., that, it) that might have allowed for teachers to interpret Liz's comments differently from what Liz had intended, although neither Alyce, nor any of the other teachers indicated that they believed that Liz's meanings could have been different from how they had interpreted her.

The ensuing conversation moved away from what the teachers envisioned as a satisfactory explanation and toward a discussion of the actual diagram. In addition, this conversation included a brief discussion of whether the girl's drawing "actually worked." Although the teachers concluded that the drawing did not work, they neither indicated what they meant by "work," nor what they envisioned a satisfactory answer to be.

Rachel next questioned Bernardo as to why he had asked someone else to explain what the girl had done, rather than have the girl explain what she had done.

Excerpt 41, Class #10

Alyce: To see if anybody was paying attention.

Bernardo: To see if they were attentive and because I thought...yeah, to see if they were attentive.

Rachel: Attentive to what she had drawn or just...

Bernardo: To what she had drawn.

Bernardo's response suggests that he intended for students to focus on the diagram, something the students could perceive. The discussion next moved to how the diagram made it unclear for the teachers to know what the girl had understood, and how the ensuing student explanations (in the video) had only added to the confusion. These comments focused on the teachers' capacity to ascertain the student's understanding, rather than how the diagram (and explanations) assisted in student learning.

The discussion concluded with Pat asserting that there were some things about the girl's construction that they did not know; specifically, they did not know the construction's logic. Pat next moved the conversation away from what had actually happened and toward how the teachers imagined the lesson unfolding differently. This allowed Pat to push the teachers to consider how their instructional actions could motivate student reasoning.

Phase 2.3 – Imagining an alternative discussion (part 1). Pat initiated the next by stating that they had the student's diagram, that this was as far as they gotten in the lesson, and asking, "Where else might this conversation have gone?" It was very quiet for 18 seconds, at which time Pat repeated the question. An additional seven seconds passed before any of the teachers responded.

Excerpt 42, Class #10

Alyce: She might know more than it looks like she knows.

Story: Mhm [Quiet for 20 seconds]

Annie: We really have no idea what she knows, though, based on this information that's up here right now...and that there's no feedback really from her...except that it looked better on my paper.

Alyce: And it did [...]

Johnny: Well, and I think because of that...granted she can draw them...because we're not asking those questions, "Why are you doing this?" We don't have a clue of what she does know or where she's trying to go with it...or somehow, we haven't asked those types of questions, to find all those out.

Annie: Well, Bernardo asked the questions, "How do you know," a lot, but I don't remember hearing a lot of their responses how they actually did...like, I don't remember what her response was to that question, "How do you know that that's the point equidistant from the three points?" ...I don't remember her answer to that.

Bernardo: She may not have, it's very possible that I cut her off.

Alyce: She just said well it looks better on my paper.

The teachers appeared to be constrained in their capacity to think about an alternative conversation. The teachers' comments indicate that they were focused on what they could perceive—on the diagram and the conversation. The teachers' comments suggest that they were unable to comment on how the student *might* have been thinking, because she was never asked how she was thinking.

Pat asserted that the teachers were still in the “old universe,” and that they were headed off in a “parallel universe”; that he intended for the discussion to not be about where they went, but where else they might have gone.

Excerpt 43, Class #10

Faye: You might have asked, did anybody solve it in a similar way...or come up with something similar...to what she did? [Pat writes Faye's suggestion on chalkboard: Did anyone solve it similarly?]

Pat: And where do you think that might have gone then...with that question, where might it...

Faye: Students might've said, no, I didn't do that, I didn't have this part or...it might get into her logic and comparing her logic with other students.

Pat: Okay, did anyone solve it similarly? [Raises hand] I did it differently.

Faye: Well you'd need to go...after further questioning...and then whoever maybe talks about it differently, could talk about what they did and why they did it.

Pat: Okay.

Liz: And I'd say it would possibly be, "Yeah mine was similar, I had circles and those lines, but mine might have been right, but I still had circles and lines."

Pat: Okay, but Faye...what's important is, were you...asking that question because it might head somewhere and the somewhere is...?

Faye: To look at her reasoning and compare it to the other students' reasoning?
[Pat writes on the chalkboard "To compare her reasoning with another students".]

Sheila: I would say too that if other students do say they did solve it similarly, then you can find misconceptions or misunderstandings that maybe the majority of the class has and so you can clarify them.

Pat: So you're saying that's another thing that might come out from...Faye...so, remember now we can head of in other universes too.

Both Faye and Sheila's comments suggest that Faye's instructional action of asking, "Did anyone solve it similarly," would be for benefit for the teacher, not the students. The teacher could "compare" students' reasoning, and find "misconceptions or misunderstanding." The teachers did not indicate that they had conceived of how such a discussion could benefit the students, how it might promote students to think about their own or another's reasoning.

Pat continued to push the teachers to consider how they might have moved the conversation in a different direction and the reasons for such a move by asking, "Anybody have another question that might take us in another direction?"

Excerpt 44, Class #10

Marian: I'd try to find out where her lines are coming from.

Pat: So what question would you...

Marian: I'd say, "Why did you make the dashed line from, I don't know what that point is..."

Liz: JS. [Pat writes: "Why did you make the dashed lines?" on the chalkboard]

Marian: Because if she's thinking of some kind of a reason for a perpendicular bisector, I have to have some segment between my points, then maybe we know she's on the right path, and then we could figure out what's that other line come from then...

Pat: But...when you ask a question, you have to...anticipate that they're going to answer the question that you asked.

Marian: Okay.

Pat: All right?

Pat: Answer your question [points to board]..."Why did you make the dashed lines," like instead of solid?

Marian: No.

Several: Laugh [...]

Pat used Marian's question to push the point that the teachers needed to attempt to think like their students, and that the teachers needed to consider how the questions that they posed to their students might be interpreted by their

students. In addition, Pat moved to get the teacher's to make explicit the reasons for their question—the logic behind their “alternative” questions.

Excerpt 45, Class #10

Liz: More like why did you construct the line JS or the segment JS.

Marian: OK, right.

Pat: So it's not that you're interested in why she made her lines dashed...

Marian: Why between J and S and where'd the other one come from?

Sheila: It looks like the other one came from the intersection of the two circles.

Marian: And then we could ask her, “Why the heck are you intersecting two circles?”

Alyce: Well, when asked what she constructed she said a dotted line, that's what she constructed. [Pat writes: “Why did you construct the line between J and S?”]

Liz: Yeah, did you say when asked what she constructed?

Alyce: Yeah.

Pat: You're thinking your question will head in what direction...your reason for asking the question?

Marian: To see if she is looking for the segment for to bisect, a segment to bisect...looking for equidistance. [Pat writes: “Is she looking for equidistant points”]

Pat: So, is she looking for equidistant points, is that right?

Marian: Yeah.

Pat: Okay...anybody else? [Quiet for 14 seconds]

Marian's questions appear to be focused on getting the teacher to understand what the students was thinking, rather than getting the students to reflect on their own and one another's reasoning. Pat appeared to take into consideration the teachers' difficulty in envisioning alternative conversations, especially conversations that did not simply allow for the teacher to gauge the students' level of understanding, with his next instructional action.

Phase 2.4 –Making students' reasoning explicit. Pat next provided a sample question of his own for discussion. This action enabled Pat to focus the teachers' attention on the importance of getting *students* to think about their own reasoning, and the reasoning of their classmates.

Excerpt 46, Class #10

Pat: Here's one that occurred to me...this is..."Tina, could you explain what you've shown us?" [Writes question on chalkboard] [...]

Pat: Now if you're going to engage someone in a conversation...I mean [...] a conceptual conversation...you know, where the intellects are engaged...then it's good to have a demarcated domain of discourse [...] if you pick something really particular to talk about then you're talking about something that's really particular...that was my only motive for suggesting that third one, Tina, could you explain what you've shown us.

Pat: We don't know what she sees up there, we just don't know...so, until we get some idea what she's trying to show us, we can't talk to her about what she's shown.

Pat: Remember last time we were together we talked about some of the characteristics of what we call a good conversation...a genuine interest.

Johnny: Mhm [...]

Pat: You want to get a question out to the person that you asked to go to the board, right? And so, you engage them in a way that gets their thinking out in public, conveys that you value their thinking, but also gives them the freedom to really say what they were thinking, and not [just] talk about what you've decided you want them to talk about.

Pat next attempted to move the discussion back to the video, back to the teachers imagining alternative conversations with the students, and their reasons for those conversations.

Phase 2.5 – Imagining an alternative discussion (part 2). Pat restated that he would run the video until someone told him to stop the video. The teachers watched the video for 1.4 minutes before Annie asked for the video to be stopped. An annotated transcription of the discussion (from the Geometry class) that Annie focused on is provided in the following excerpt (and in Appendix D).

Excerpt 47, Reformed-Geometry Case Study

8. Ted: Umh...S and J [...]

9. Bernardo: So, how do you know she's getting the point in the middle of S and J?

10. Ted: Because she drew the circle with radius B and S, and then she did the one with J and B, and then she put a dotted line between them, and where they crossed she put a dot labeled A.
11. Bernardo: Okay [...]
12. Bernardo: Tina, is that what you did?
13. Tina: Yeah.
14. Bernardo: Yeah?
15. Tina: It's really bad, like on here [gestures to her homework paper] it's more straight.

The following excerpt picks the conversation up (from the Functions 3 course) after Pat had stopped the video per Annie's request.

Excerpt 48, Class #10

- Annie: Right there...you have to do more than just accept a one-word answer..."Yup, that's what I did, that's what's on my paper."
- Alyce: Well...she's going on to explain it.
- Annie: But if I remember right it was like that's what I got on my paper.
- Liz: On my paper it looks better. [Pat plays the video from line 15 to 22, Appendix D]
- Liz: The kid, will interpret what you see, "Well, she drew a circle that did this, and then this," well then when you ask her is that what you did..."Yeah," there was nothing else for her to say, how could she say anything else, "Yeah that's what I did...I did make a circle here and a circle there."

- Pat: And what's the "that," okay, say it again.
- Liz: I'm not sure what her "that" is.
- Pat: No, no, but you said that's what I did, what's the "that"?
- Liz: The kid, however he said, she made a circle here and a circle there, is that what you did, "Yeah that's it."
- Pat: Okay.

Pat's question to Liz emphasized the need for the teacher's to be more clear with their comments, that they needed to limit the use of pronouns. Liz's comments suggest that she is focused on what transpired, that all the student could respond to is what was "done," rather than the reasoning involved—Liz was not thinking about an alternative conversation, only the conversation that she had perceived.

Excerpt 49, Class #10

- Liz: Her response is to...she did step 1, step 2, step 3, is what I see and all she could say is "Yes, I did that."
- Rachel: In the...when she was doing and showed the picture and stuff, the stuff that she did, at least from what I saw...it doesn't even address the third point and...
- Bernardo: It looked like she ...
- Rachel: She ...
- Bernardo: Constructed a bisector of just...
- Rachel: Of one...
- Bernardo: Mhm.

Rachel: I mean cause her picture there is really nice for one...and I don't know if you'd asked did you include the third point, I mean...something where...going back and saying, "Is it the center of all three?" I mean they've been doing...with just the one, I mean cause she had it for the one, but never addressed that third point. [Quiet for eight seconds]

Pat: And your point is...

Rachel: I'm just trying to figure out...did she...just change the problem? And say, "I don't know how to do it with three, so I'm just going to look at it with, one, with two points."

Janine: I think so.

Rachel: I mean we don't know why she totally ignored that.

Pat: Yeah, we don't know.

Rachel's comments focused on an attempt to figure out why the student (in the video) had drawn what she did. The focus is still on what was what had transpired (i.e., what Rachel perceived), rather than attempting to imagine an alternative discussion.

Pat next asked Liz to restate her point, so that it could be used as their point of departure into a parallel universe. Liz stated that a student (Ted) had been asked what he believed Tina had done in her diagram. Liz asserted that Ted responded that, "I saw that she made a circle here, and a circle here, and so on." Liz then stated that Bernardo had then asked Tina, "Is that what you did?" To which, Tina had replied, "Yes."

Liz informed Pat that she would like to pick up, in an alternative conversation, from the point where Ted was talking. In the video (of the Geometry class) Tina had just completed drawing her diagram and Bernardo asked Ted to comment on his understanding of what Tina had done; as indicated in the following excerpt (and in Appendix D).

Excerpt 50, Reformed-Geometry Case Study

5. Bernardo: Ted, do you know what Tina's doing? [...]

The following excerpt picks the conversation up (from the Functions 3 course) after Pat had stopped the video per Liz's request.

Excerpt 51, Class #10

Liz: Instead of "what do you see that Tina did," perhaps asking, "What meaning do you make out of what Tina did?"

Pat: Asking that question of Ted...okay [writes: Ted, what meaning does Tina's diagram have for you?]

Pat: And the direction you see this going is? [...]

Liz: Well, I hope that he would address that...in his mind he saw as being a perpendicular bisector of one side of the triangle.

Pat: Okay, you're hoping for a particular interpretation.

Liz: I know...I mean I would hope that eventually down the road...I don't know exactly what response I'd get [...]

- Pat: See...okay, like when I said Tina can you explain what you've shown us?
I'm not hoping for a particular answer...I'm trying to create space for a particular conversation [...]
- Marcy: It's well focused.
- Alyce: Well, asking the question this way, you will find out if there is meaning or if it's just a procedure, you know, if they're just trying to reconstruct diagrams that they've seen in class or if it actually means something to them...and you also can then go back and ask Tina, "Did you have that same meaning," and she can have an answer versus, "Yes, I did those things," or "No, I didn't do those things."
- Pat: Yeah, instead of steps it's more about the ideas that are at play, and...I suspect that when you ask Ted that question, he'll say, "It doesn't mean anything to me," or something like that. Right? [Several teachers laugh]
- Marcy: I'm not sure what it means; I just know what she did.
- Pat: Yeah...but, it's an opening...
- Marcy: But at least you're addressing whether he understands.
- Pat: That's an opening...then you can say, "Well, Tina...this is your responsibility Tina...so, help Ted out here."

In the preceding example, Pat attempted to push the point that it was important for the students to get meanings out on the table, that the conversation should be focused on getting students to articulate their meanings. Both Alyce and Marcy appeared to interpret this as indicating that it was a way for the teacher to

check for understanding—where the focus on meanings was beneficial for the teacher (to ascertain “understanding”), rather than the students.

Phase 2.6 – Providing a space for students to discuss meanings. Pat next moved the discussion to the importance of getting meanings out on the table, but emphasized the importance for students. Pat stated that Liz’s hypothetical question (“Ted, what meaning does Tina’s diagram have for you?”) created the opportunity for a discussion of meaning, interpretation, and attention. Pat next asked for other alternative conversations.

Excerpt 52, Class #10

Tami: You might ask did anybody have anything to add to the diagram that might lend clarification [...]

Pat: Okay...are we at the “Ted” moment?

Tami: The girl with the [...] Amber.

Tami indicated that she was picking the conversation up (from the Geometry class) from the point after Tina had returned to the whiteboard to answer question from her classmates regarding her diagram. The following excerpt describes the conversation that unfolded after Tina had returned to the whiteboard and up to the additions that Amber made to Tina’s diagram (also in Appendix D).

Excerpt 53, Reformed-Geometry Case Study

27. Bernardo: Okay, you’re now saying that point A is the same distance from point B as it is from J and S?

28. Tina: Yes, on my paper it is.

29. Bernardo: Okay, let's talk about this then. Umh...how do you know that A is the same distance from all three?
30. Tina: Umh, because...Oh, because, umh, you kind of find the distance between J and S with this line [gestures to dotted line segment JS] and the distance between B and the rest with this line [gestures to second dotted line]
31. Bernardo: Okay, I don't [...] Amber do you see how Tina's finding the distance between B and S?
32. Amber: Not really, I'm kind of confused.
33. Tina: If you saw my paper you wouldn't be confused, this is so crooked.
34. Bernardo: Well, if you're confused guys you need to ask a question.
35. Amber: Well, I know how she's trying to find the middle of J and S, like the center point of that, but I don't know how she found that it's equal to B also.
36. Bernardo: Okay, how would you find the center point of B and S?
37. Amber: Umh, draw another segment...
38. Bernardo: Okay, why don't you try it. Thank you Tina. Everybody give Tina a hand. It's not easy being up there folks.
31. Bernardo: Okay, I don't [...] Amber do you see how Tina's finding the distance between B and S?
32. Amber: Not really, I'm kind of confused.
33. Tina: If you saw my paper you wouldn't be confused, this is so crooked.

34. Bernardo: Well, if you're confused guys you need to ask a question.

35. Amber: Well, I know how she's trying to find the middle of J and S, like the center point of that, but I don't know how she found that it's equal to B also.

36. Bernardo: Okay, how would you find the center point of B and S?

37. Amber: Umh, draw another segment...

38. Bernardo: Okay, why don't you try it. Thank you Tina. Everybody give Tina a hand. It's not easy being up there folks.

The following excerpt picks the conversation up (from the Functions 3 course) after Pat had stopped the video per Tami's request.

Excerpt 54, Class #10

Pat: So, now it's not so much that Ted said anything; you're...saying at this moment we can't speak to Ted, but speak to the whole class.

Alyce: But why would you need to add something to it? Tina solved it perfectly correct.

Marcy: Uhh...she didn't.

Tami: To add clarification in case people didn't understand what she did, just to try to bring out more students to respond.

Alyce: So, you're saying adding understanding, not adding to the diagram? I mean obviously something needs to be added to the diagram, because she didn't solve it correctly, but if you say that then it's implying that she did it wrong.

Tami: I think just to add understanding, or add clarification to what's going on here. [Pat writes on board: Can anyone add to this diagram?]

Pat: And Tami, just for posterity's sake...you saw this question heading in what direction?

Tami: Toward understanding.

Pat: Well...

Story: Improvement, fixing.

Tami: Clarification...just clarifying what's happened...really just to draw more students into the...discussion, or to get more responses to different students [...]

Liz: That's where the class went though [...]

Tami: Well, if they're saying it's in the middle, somebody might come up and show...something...to add to it.

Tami indicated that her question was intended to add clarification and to get more students involved in the discussion. What is not clear is whether she intends for the focus (the clarification) to be of the drawing or the logic. Story and Alyce appeared to suggest that the focus should be on making the drawing correct. In addition, Tami did not make clear her reasons for getting more students to respond—is her intent to provide the teacher with more opportunities to ascertain student understanding, or to get the students' meanings out on the table as object of thought?

As the conversation began to return to what had actually occurred in the video, rather than hypothetical discussions, Liz suggested that Bernardo should have asked Larry to construct his diagram of the problem, but to leave Amber's diagram on the board. In the video (from the Geometry class), Amber had just finished attempting to justify her addition to Tina's original diagram. Bernardo asked Peter to summarize what it was that Amber had described drawing (a dotted line from point B to A, Figure 10).

Excerpt 55, Reformed-Geometry Case Study

54. Bernardo: I'm kind of lost here; Peter...could you kind of get us caught up here?

What do you think we're trying to do, 'cause everybody had some good comments there?

55. Peter: They're trying to find the distance from B and S [...]

56. Bernardo: What distance from B and S?

57. Peter: Rephrase that? [...]

58. Bernardo: I heard you say you're trying to find the distance from B and S...what distance from B and S?

Tina raised her hand.

59. Bernardo: Does anybody have like a picture of what we're trying to find? Okay [...]

Larry. That's right I promised that you could go up next. Amber, thank you very much. Larry why don't you go up there and...

As Amber moved back to her seat, the class clapped, and Larry moved to the whiteboard.

60. Larry: Can I redraw this?

61. Bernardo: Okay, go right ahead [...]

The following excerpt picks the conversation up (from the Functions 3 course) after Pat had stopped the video per Liz's request.

Excerpt 56, Class #10

Liz: So, it would have at least got back to similarities, can you compare the meanings behind them.

Bernardo: Side by side.

Liz: And you could have found her misconception [...]

Liz: Is one more right than the other?

Marcy: Is one more clear?

Although Liz asserted that the drawings could be used to compare the “meanings behind them,” the teachers’ comments suggest that the intent would be for comparisons of clarity or correctness (of the diagrams), not to create a space to discuss meanings and reasoning. Pat moved the discussion to that of a criterion for choosing alternative conversations in an attempt to bring the focus back on the students’ meanings and reasoning.

Phase 2.7 – Criterion for choosing among alternative conversations. Pat asked the teachers what they believed the criterion should be for deciding among alternatives conversations; that they needed a way to determine, among competing alternatives, which to choose.

Excerpt 57, Class #10

Johnny: Does it help clarify the situation.

Pat: To whom?

Johnny: I think the class in general [...]

Alyce: The one that's the most inviting to the class and the least offensive to the student on the spot, so that you don't shut them down and they won't ever want to do something in your class again. [...]

Pat: Inviting in what sense? Sheila, do you have a sense for what she means?

Sheila: Well, I would take inviting to mean...okay, if I'm a student they're to respond to, that's how I would take that word inviting...non-threatening.

Pat: So, inviting to engage? In other words, get involved...like, "I got an opinion on that!" ...That kind of inviting? [Writes on the chalkboard: Among competing options, choose the one that is the most inviting to the students—inviting to engage]

Sheila: Which is good, but if it doesn't lead you to what they are thinking, I mean I could ask a lot of inviting questions that won't get me anywhere as far as knowing what they know [...]

Pat: Ah, okay, so...so, you're saying that there needs to be another consideration...I mean inviting is important...

Sheila: Yes.

Pat: Everybody grants that, inviting is important...but not every inviting question is...is a good option...inviting but [...]

In the preceding excerpt, Alyce focused on making students comfortable with participating in the discussion, whereas Sheila emphasized the importance of getting student thinking out in the open. The subsequent discussion focused on getting students' interpretations, reasoning, or meanings out in the open.

Excerpt 58, Class #10

Pat: So, what's the principle? [...] I've heard several people say something about meaning or interpretation, so getting people involved with, "How are you interpreting this, what meaning are you giving," is that right?

Marcy: Asking for further interpretation or meaning.

The conversation next moved back to the video and how Ted had been asked to explain Tina's diagram.

Excerpt 59, Class #10

Marcy: I'm not sure sometimes that when you ask someone to explain what someone else means that...I don't know...It's all right, I guess, depending on what you're doing...with volunteers, I'm not sure...

Pat asserted that there could be different reasons for asking one student to explain another's reasoning (e.g., to get meanings out in the open, to check for student engagement). Pat moved to the chalkboard and added to the list of alternatives: Asking for further interpretation or meaning (who you ask depends on circumstances and motive)

Pat next asserted that the statement "Asking for further interpretation or meaning (who you ask depends on circumstances and motive)" was something that

qualified the statement “Among competing options, choose the one that is the most inviting to the students—inviting to engage.” Pat asked whether the teachers had another idea at the level of “Among competing options, choose”?

Excerpt 60, Class #10

Tami: Could you maybe ask someone to start from scratch, and explain exactly what they’re trying to find and how they’re going to approach it, before they actually go through the procedure.

Pat: Ah...okay, so it sounds like, Tami, you’re saying, “Ask about strategy.”

Tami: Explain where they’re going, what their intent is.

Pat: So, strategy is more “big picture,” right? [...]

Pat spent the next seven minutes describing a strategy for solving the problem, and, in particular, how a proof (using Geometer’s Sketchpad) would require a sophisticated argument. Pat next returned to Tami’s suggested alternative, and Bernardo’s difficulty in getting his students to understand what was meant by “strategy.” Bernardo asserted that the student’s initially thought that by “strategy” he meant what they did, until someone pointed out that strategy meant their “plan.” Pat highlighted the importance of moving the conversation away from having students discuss what they were doing, and toward a focus on the students’ reasoning.

Phase 2.8 – Retrospective discussion of the activity. In the last few minutes of Phase 2, Pat asserted that the activities that they had engaged in (e.g., a focus on students’ mathematics, designing conversations to promote particular ways of

thinking and ways of reasoning) involved the kinds of lenses that he hoped the teachers would employ in their future observation of video; that video was nothing more than an opportunity to speculate about what happened, why it happened, and how it might have happened differently.

Phase 2 – Summary. Throughout Phase 2, the teachers expressed a focus on that which could be perceived, either their own perceptions (of what they had seen in the video), or on making what they intended students to understand visible (i.e., perceivable) to the students, or making student understanding visible (i.e., perceivable) to themselves as the teacher. A focus on their own perceptions appeared to constrain the teachers’ capacity to envision alternative conversations with students or from the students’ point of view. A focus on the visible appeared to constrain the teachers’ capacity to envision instruction designed to make meanings and reasoning explicit, as objects of thought.

Summary of Reflective Episodes #1 and #2

As illustrated in the two reflective episodes, the teachers exhibited an orientation that focused on the “visible”—that which could be perceived, either by themselves or by their students. Such an orientation expressed itself through personal and pedagogical actions that did not focus on meanings, but rather focused on objects or the products of the teachers’ (or their students’) reasoning (i.e., the outcomes). In addition, such an orientation expressed itself through the teachers’ focus on their own perceptions (i.e., egocentrism), thus obstructing Pat’s attempts to get the teachers to decenter. Such an orientation appeared to make it natural for the

teachers not to reflect—in essence limiting the teachers' to empirical or pseudo-empirical abstractions. In the following chapter, I will develop two epistemic ways of operating as indicated by my analysis of the data corpus, and propose an orientation toward learning and teaching mathematics that has the potential to explain the teachers' reticence to reflect.

CHAPTER 6

EPISTEMIC WAYS OF OPERATING

In this chapter I will develop and demonstrate the robustness of two epistemic ways of operating that expressed themselves throughout the Functions 3 course (i.e., throughout the data corpus). Specifically, teachers' disinclination to: (1) articulate their meanings, and (2) be mindful of others' mathematics. In addition, I will propose a way of thinking—an orientation toward learning and teaching mathematics—that would explain the teachers' reticence to reflect.

Although my descriptions of the teachers' ways of operating characterize how the teachers operated, I understand that these descriptions might be viewed as a deficit model (i.e., "things" the teachers cannot do), and, as such, could be considered as not being very helpful to the research and professional development community. More specifically, it seems self-evident that if productive reflective abstraction requires that a person take their reasoning as objects of thought, then a disinclination to think about meanings would hinder a person's capacity to reflect on their reasoning. Furthermore, If we assume that in order for a teacher to transform her personal understandings into powerful pedagogical understandings requires the teacher to decenter, as indicated by Silverman and Thompson (2008), then a disinclination to be mindful of others' mathematics would hinder teachers' ability to decenter and thus hinder their creation of key pedagogical understandings.

Such characterizations are not very helpful when attempting to develop and implement engagements that promote teacher transformation (i.e., how to get teachers “inclined” or at least less “disinclined”). In addition, these descriptions of the teachers’ ways of operating do nothing in the way of accounting for these ways of operating—what schemes must the teachers be operating with so that they indeed act (more specifically, be “disinclined to act”) in the ways that I have identified? Rather than developing a deficit model, I would like to make clear that what I am describing is the way the teachers actually operated (i.e., the way they were), so that I can talk about how it was that the way they operated got in the way of other things—how the way they operated made some things not possible, such as reflection.

Several of the activities in the Functions 3 course were designed to provide the teachers with opportunities to express and reflect on their personal and pedagogical understandings of the mathematics that they teach. During engagements with these activities, the teachers’ understandings, meanings, and ways of thinking expressed themselves through their actions. These meanings were sometimes indicated verbally, by what the teachers said, sometimes were counter-indicated by what they did not say, and by the teachers’ actions or inactions.

Articulation of Meanings

An activity from the initial class meeting (Class #1) will serve to illustrate what I mean by an activity designed for teachers to express and reflect on their

meanings. Furthermore, a discussion of this activity will provide a means for me to initiate the construction of models of the teachers' ways of operating.

One of the initial class activities required the teachers (in groups) to examine an Algebra 1 quiz entitled "Interpreting/Creating Graphs" (Appendix E). The quiz consisted of four contextual situations each involving variable quantities. The quiz was designed to provide Algebra 1 students with the opportunity to reason about ideas involving rate of change (e.g., rate, proportion, constant rate of change), covariation (e.g., a focus on varying two quantities simultaneously, a focus on covariation over small intervals), graphs, and the students' coordinations of their meanings regarding these ideas. Furthermore, the quiz was designed to initiate instruction (e.g., in a subsequent lesson) on ideas related to average rate of change, non-constant rate of change (i.e., increasing and decreasing rate of change), and linear functions.

The teachers' task was to articulate what they believed to be the quiz's learning goals and to describe instruction that they believed would prepare their students to be successful on the quiz. The activity served as an introduction to the *Didactic Triad*, which had been briefly described prior to the activity. The teachers worked together in groups of three to four and were required to write their consensus thinking on white-boards and present their thinking to the class.

It is possible to hear the conversations for two of the six groups as they discussed the quiz. In the following discussion, I will focus on one of these groups (Group #1, consisting of Alyce, Gwen, and Eve). The following excerpt occurred

immediately after the group members had read the first problem (describe a person's biking behavior given a graph of his distance from start relative to time; Appendix E).

Excerpt 61, Class #1

Gwen: I already have been thinking a lot about this, because not just this quiz, but...I already know that in order for them to be successful on something like this, you have to teach them like this the entire year.

Alyce: Yeah

Gwen: You can't just spring...oh here's a...because that's what we've tried the last two years [...]

Alyce: I would like to teach all the time like this

Gwen: I would love to, but then we have district things and...

From the start of the activity, Alyce and Gwen repeatedly spoke about (what I take to be) substantive mathematical issues in the quiz as if they were unproblematic. Specifically, they did not address what the "this" is to which Gwen referred. Although Alyce agreed with Gwen, indicating that she also had a meaning for "this," Alyce did not attempt to ascertain whether her meaning for "this" was compatible with Gwen's meaning for "this." Whatever meaning either teacher had for teaching "this" and "teaching like this" (since neither teacher stated her meaning), there was no indication that either teacher was aware that their meanings could be different.

This excerpt illustrates a common occurrence throughout the activity—the teachers' in Group #1 were disinclined³ to articulate their meanings regarding matters that the research team considered nontrivial. After Group #1 had discussed their solution to the initial problem on the quiz, the conversation changed to learning goals.

Excerpt 62, Class #1

Gwen: So, what's the learning goal?

Alyce: I think this is testing rate of change

Gwen: Yeah, oh yeah...definitely [...]

Gwen: So, working on rate of change...I think we're also looking at working on graph interpretation.

Alyce: Uh-huh.

Gwen: Graphical...

Alyce: You could just say graph interpretation...

Gwen: Interpretation.

Alyce: Because we gave the representation.

Although both Alyce and Gwen asserted that rate of change was a learning goal, both kept their meaning for rate of change implicit. They did not address the meaning of rate of change they believed the quiz to assess. By a "meaning" for rate of

³ I use "disinclined" advisedly—it summarizes a persistent pattern of teachers not doing something that includes occasions where they were urged to do it, but still did not. So, a "disinclination to make meanings explicit" points to many occasions where they did not make meanings explicit, and to occasions where they were urged to say what they meant but still did not.

change I mean a conceptualization, such as: constant rate as a scheme that entails two images of quantitative change: (1) a quantity that accumulates in multiplicatively-structured bits and (2) the bits themselves accumulate by way of two quantities changing in constant proportion to each other (P. W. Thompson, 1994c; P. W. Thompson & Thompson, 1992, April).

Neither Gwen nor Alyce said what they meant by “graph interpretation” or what it meant that students learn it. Finally, neither Alyce nor Gwen showed any awareness that their meanings for rate of change or for interpreting a graph might not align with others’ meanings.

Gwen next read the second problem on the quiz (coordinating distance from start with distance walked around a track; Appendix E) out loud, and the group attempted to identify additional learning goals.

Excerpt 63, Class #1

Alyce: So, it will be the graphical interpretation.

Gwen: It’s not just that, but it’s...graph creation.

Alyce: Uh-huh.

Gwen: They created a graph...you could put graphic interpretation and just put and creation.

Alyce: Yeah.

As with the preceding excerpt, neither Alyce nor Gwen said what they meant by “graph creation”; or, considered it possible that their meanings did not align.

Furthermore, Gwen’s emphasis on the “created” graph,” suggests that she is focused

on the solution to the problem (i.e., the outcome) rather than thinking about the meanings, understandings, and reasoning involved in creating the graph. A meaningful way to think about graphs involves more than simply producing graphs, but rather entails the person's having coordinated such ideas as using coordinate systems to locate points, the role of scale in creating axes, the representation of quantities' magnitudes by putting measures on an axis, and the role of covariation in producing a graph (P. W. Thompson, 2009).

Throughout Group #1's discussion of the quiz, members frequently used the term "rate of change" without discussing what they meant, the phrase "graph creation" without explicating a way of thinking about graphs, and spoke of "graphical interpretation" without discussing ways of thinking that students might use to create an interpretation. The preceding excerpts illustrate Alyce and Gwen's disinclination to make their mathematical understandings and ways of thinking (i.e., their understanding of the mathematics) explicit.

Group #1's disinclination to articulate their meanings was not limited to their mathematical conceptualizations, but also included the types of instruction that the teachers believed would prepare students for the quiz. The following sequence of excerpts, which occurred during Group #1's discussion of instruction, illustrate this point.

Excerpt 64, Class #1

Alyce: Okay, how do we get there? [...]

Eve: We speak with meaning

Gwen: Well, write that, speak with meaning, that's good...we need to speak with meaning.

Alyce: And force, but we have to come up with a better word than force, then to...

Gwen: Compel?

Alyce: Require they're answers to also be...because not only do we need to speak with meaning, but they have to be able to speak with meaning.

Gwen: Yeah.

Eve: So, encourage them to speak with meaning.

Alyce: Uh-huh...yeah, teachers and students [...]

Alyce: Teach with meaning.

Gwen: Context questions.

Alyce: Yeah, that's what I wanted to say, context questions.

It is somewhat ironic that the group identified "to speak with meaning" as a form of instruction to prepare students for the quiz, but did not make explicit what they would be speaking about—what meanings did they have in mind?

Alyce introduced the idea of "to teach with meaning," which Gwen apparently understood as employing contextual questions (which Alyce agreed is what she meant). Although the teachers identified "context questions" as a form of instruction that they believed would prepare students for the quiz, they did not say how they envisioned contextual questions assisting student learning. Specifically, what mathematical meanings would the context bring out and what ideas could be

developed? As the discussion continued, “reform-oriented” instructional methodologies were introduced into the conversation.

Excerpt 65, Class #1

Alyce: Umh...student discussions.

Gwen: Student discussions...but I also think...

Alyce: Hands on...or discovery activities.

Gwen: There we go...how else do we get them there?

Alyce: How about individual practice that expands their...extends their classroom...it sounds good anyway.

Gwen: (Gives thumbs up) [...]

Gwen: How about that extends their classroom experience? (Eve writes the suggestions on the white-board)

Gwen: Awesome, I love it!

The group added student discussion, discovery activities, and individual practice to their list of instruction without providing any indication as to what they envisioned these providing to the development of the learning goals. The group appeared to be simply running through a litany of “reform” terminology pertaining to instruction.

Throughout their discussion of instruction, the teachers in Group #1 did not connect their learning goals to instruction. The teachers provided no sense for how instruction would move their students to a point where they had developed the learning goals.

The disinclination to articulate mathematical meanings, ways of thinking, and instruction was not limited to Group #1; rather, it appeared to be a general way of operating within all of the groups. The group presentation by Group #4 (Liz, Annie, and Story) provides another illustration of the general lack of meaning expressed by the teachers during the activity. Liz presented her group's learning objectives, which consisted of "ROC" (i.e., rate of change) and "Graphical Interpretation."

Excerpt 66, Class #1

Liz: Objectives...obviously you have to somehow approach rate of change, and you're going to do that...and graphical interpretation, whether it's taking a graph and being able to interpret what you see, or taking information and making a graph to represent that...

Liz provided no indication of a meaning for rate of change that she believed the quiz assessed, nor any articulation of the ideas, ways of thinking, or ways of reasoning that might be involved in interpreting a graph. Furthermore, after she had finished speaking, none of the teachers challenged Liz to make her meanings explicit.

Next, Annie presented her group's ideas on the types of instruction that her group believed would prepare students to take the quiz, which consisted of: "Have students define rate of change and give examples," "Students discuss and share relationships between two variables," "Students discuss and share ways to represent this relationship," and "Independent practice."

Excerpt 67, Class #1

Annie: So, we would have students define rate of change and give some examples...we would have our students discuss and share, we would want to make sure that they share with the groups the relationships between two variables...and ways to represent this relationship, and then do some independent work.

As with her group partner, Annie spoke of rate of change as a concept that her group would promote or meanings that they anticipated students might possess, but she did not say what those meanings would be. In addition, Annie spoke of specific relationships or representations “between two variables” that she had in mind (or even the variables that were involved) for students to “discuss” and “share.” We know from prior research the great difficulty with which students master concepts of constant rate of change, so it is significant that Annie took “discussing” and “sharing” to be, evidently, unproblematic. Finally, Annie did not articulate how her group’s instruction would promote the development of the learning goals. As with Liz, once Annie had finished speaking, none of the teachers challenged her to say what she meant.

Although the teachers exhibited a general disinclination to make their own meanings explicit, how they handled each other’s unstated meanings depended on whether the discussion centered around mathematical understandings and ways of thinking or on instructional methodologies. If the discussion focused on the mathematical understandings and ways of thinking, then the teachers were

disinclined to challenge one another to make their meanings explicit. In these instances, only the course instructor (Pat Thompson) pushed the teachers to explicate their meanings. If the discussion focused on instruction, then there were times when the teachers did challenge one another to make their meanings explicit.

For example, in the earlier excerpt where Alyce introduced the idea of to “teach with meaning” (Excerpt #64), Gwen did attempt to clarify her interpretation of Alyce’s statement. Gwen’s attempt to clarify Alyce’s comment was one of only five instances, amongst 35 hours of class video, in which one teacher challenged another for clarification. Four of these instances involved clarification to what I would describe as instructional methodology (e.g., “Any ideas on your contextual questions or discovery activities?” and “What did you envision doing data collection on?”). In these instances it appeared as if teachers were looking for new ways to present the mathematics, without questioning the mathematics itself.

Teachers’ mathematical understandings. Further investigation into the teachers’ mathematical meanings provided a more explicit image of the teachers’ understandings of the mathematics that they teach. During the group presentations of the learning goals and instruction, one of the teachers (Marcy) commented on the conciseness of another group’s learning goals (Group #4 had listed only rate of change and graphical interpretation as learning goals).

Excerpt 68, Class #1

Marcy: I'm not saying that ours are learning objectives either...but they should really be things that kids are able to do...and we know what rate of change means, but what's the objective?"

Although Marcy asserted that, "We know what rate of change means," none of the groups said what they meant by rate of change. Marcy's comment also provides an indication of the nature of the mathematics that she intended for students to learn. Specifically, that mathematics involves "things that kids are able to do."

Several of the groups identified objectives or sub-objectives that involved students "doing" things—performing procedures or skills or recalling facts. Marcy and her partner identified several sub-objectives under the larger objective of "rate of change": create graphs, interpret from the graph a real-life application, difference between distance from start and total distance travelled. These sub-objectives involved demonstrable actions on the part of the students, not ways of reasoning or ways of thinking. Other groups identified learning goals, such as: "know the relationship between distance, rate, and time" (which I interpret as the ability to recall and employ the formula $r = d/t$), and "understand when you read a graph input is on the horizontal, output is on the vertical" (which I interpret as the ability to recall conventions).

All six of the groups' learning goals can be partitioned into the following two categories: 1) un-stated mathematical meanings, and 2) things that students can do

(including conventions students can recall). In Table 4, I have shaded what I identify as un-stated mathematical meanings, with the non-shaded cells being identified as skill or facts. As Table 4 illustrates, a majority of the groups listed rate of change (5 of the 6 groups), interpreting graphs (5 of the 6 groups), and covariation (4 of the 6 groups) as learning goals, and they did so without articulating the meanings that they had in mind for these terms or phrases.

Table 4

Identified Learning Goals for Interpreting/Creating Graphs Quiz

Group #1	Group #2	Group #3
Rate of change	Rate of change over various intervals	Meanings of variables
Graphic interpretation	Create graphs	Relationships of two variables
Graphic creation	Accumulation of distance over time	Interpreting graphs
Covariation of different units	Similar rates that are “signed” indicate direction traveled	Covariation
	How to interpret from a graph a real-life application	Multiple representation of relationships
	Difference between “distance from start” and “total distance traveled” Slope of a piecewise function indicates the rate of change in each interval How total distance vs. time, velocity vs. time, and distance from a point vs. time are different	
Group #4	Group #5	Group #6
Rate of change	Know how to read a graph’s covariation	Graphical interpretation
Graphical interpretation	Know the difference between distance, rate, and time	Contextual interpretation
	Understand when you read a graph input is the horizontal axis, output is the vertical axis	Covariation
	Understand rate of change	Rate of change
	Be able to look at variables as they change in small increments and relate that to the overall picture	

The teachers general disinclination to articulate their meanings for terms like rate of change, or to challenge one another to make their meanings explicit, suggests that the teachers viewed these terms as being unproblematic—that they were all talking about “the” meaning for rate of change, although none of the teachers

actually stated what “the” meaning might be. Such a viewpoint is consistent with both a traditional perspective and a *perception-based perspective* of mathematics (Simon, Tzur, Heinz, Kinzel, & Smith, 2000; Tzur, Simon, Heinz, & Kinzel, 2001), in which there is “an implicit assumption that mathematical relationships exist as part of an external world (i.e., are independent of human activity)...and that what is perceived is the same for each person” (Simon, et al., 2000, p. 592).

One of the project’s main objectives was to provide opportunities for teachers to experience coherent mathematical ideas as students, and to design, implement, and assess instruction that focused on building meanings and developing significant mathematical ideas in the minds of their students. Throughout their involvement with the project, including the earlier Functions courses, the teachers had struggled to conceptualize the term “idea” as put forward by the research team. The teachers had been provided with a definition for the term idea (e.g., a factor, a graph) as something that has “context and meaning, is generalizable, and has the potential of being foundational for other ideas and ways of thinking; ideas are ways of thinking that can generate or motivate the procedures that have traditionally been the center of instruction” (P. W. Thompson, 2009). As such, procedures and skills emerge naturally from proficient reasoning about the ideas, rather than through proficient remembering.

During the seventh meeting of the Functions 3 course, the teachers engaged in another activity (in groups) designed for them to articulate and reflect on their conceptions of learning goals, tasks and activities, and teaching—the *Didactic Triad*.

Specifically, the teachers engaged in an activity requiring that they define what they mean by a learning goal, and discuss the relationships amongst learning goals, tasks and activities, and teaching. Table 5 exhibits the five definitions for learning goals provided by the groups.

Table 5

Group Definitions for “Learning Goals”

Group #1	Group #2	Group #3	Group #4	Group #5
What I want my students to have an understanding of (the big ideas)	Specific goals to focus students’ understanding as they represent and relate concepts within a big idea	The big idea the students should understand and could apply	The understanding you want students to leave a lesson with	Anything you want them to learn (both the big ideas and smaller ideas)

As indicated in Table 5, four of the five groups identified “big ideas” as their meaning for learning goals. During the group presentations, Group #1 (Liz, Adriana, and Annie) indicated that covariation would be an example of a learning goal, a big idea; Group #5 (Bernardo, Faye, and Story) indicated that covariation would be an example of a big idea, with linear rate of change and constant rate of change as smaller ideas.

Between the first and seventh class meeting (fourteen calendar days), the teachers had engaged in activities designed to perturb, and perhaps modify, their conceptions of constant speed, average speed, and increasing and decreasing speed.

During these activities the teachers were pushed (by the course instructor, Pat) to build meanings for these ideas, but neither learning goals, nor the notion of an “idea” were discussed during these activities. Therefore, how the teachers understood learning goals during the seventh class meeting should be similar to that from the first class meeting (see p. 58; Piaget, 1968). This suggests that those unarticulated mathematical terms (e.g., covariation, rate of change, interpreting graphs) listed in Table 4 were how the teachers conceptualized an idea—as an undefined mathematical topic.

As an out-of-class assignment, prior to the second class meeting, the teachers were requested to create a transcript that would describe a lesson that they would teach to one of the learning goals that they had identified on the “Interpreting/Creating Graphs” Quiz (Appendix E). These transcripts provide further indication for how the teachers conceptualized the mathematics that they teach.

The following excerpt comes from Faye’s transcript, which centers around the problem displayed in Figure 13. Faye indicated that the transcript was teaching to the learning goal, “Interpreting Distance Versus Time Graphs.” Note that T denotes the questions and utterances that Faye envisioned making, and S1, S2, etc. denote how she imagined her student responding.

T.J. left his house and rode his skateboard to school for five minutes at a constant speed (has there been a discussion of a meaning for speed?). He waited one minute, at Knox and 44th ST, for his friend Kristina. After he met up with Kristina, T.J. walked the rest of the way to school (at a constant rate?) for six minutes. He walked at a slower pace (pace = speed?) because he was chatting with his friend. Draw a distance from home versus time graph to represent the T.J.'s trip to school.

Figure 13. Problem from Faye's transcript.

Excerpt 69, Faye's Transcript

- T: What variable did you label on the x-axis and why?
- S1: I labeled time because it is the independent variable.
- T: What unit was the time measured in, and what was your scale?
- S1: Minutes, because it was given in the problem, and I put 12 scale marks to represent each minute.
- T: What variable is labeled on the vertical axis, and why?
- S2: Distance because it's the dependent variable.
- T: How long was T.J. riding his skateboard to school?
- S1: For five minutes.
- T: Does your graph show that T.J. rode his skateboard for 5 minutes?
- S1: Oh, I guess the first piece of my graph should have stopped at 5 minutes instead of 8 minutes.
- T: All right, do you want to go ahead and redraw that part of your graph? S1, how long did T.J. wait for Kristina?
- S1: 1 minute.

- T: Ok, does your graph show that he waited for 1 minute?
- S1: Oh sorry. The horizontal piece should be shorter since he waits only one minute. I'll fix that part too.
- T: What are you noticing about your work?
- S1: I need to pay more attention to the scale on the x-axis and make sure the length of the pieces of my graph match the description in the problem.

The majority of Faye's questions require specific answers to facts (e.g., "What variable is labeled on the vertical axis?"), and focus students' attention on the products of their reasoning (i.e., outcomes, solutions), rather than on their reasoning itself. This suggests that Faye has conceptualized "Interpreting Graphs" as an undefined mathematical topic that consists of facts the student should be able to recall and apply (i.e., a "skill set" associated with the topic), rather than as a scheme of meanings. Throughout the excerpt, Faye focused student attention on the variables individually, where each variable represented a number, rather than a quantity with magnitude; such a focus de-emphasizes covariation, an idea four of the six groups (including Faye's) had indicated as being an important learning goal for the quiz. Furthermore, Faye's transcript suggests that (for Faye) to prepare students to "interpret graphs" on a quiz requires that students practice "interpreting graphs" (repeat a behavior), rather than repeat an activity (i.e., repeat a way of reasoning).

In order to increase the likelihood that the teachers would express their meanings, the course instructor (Pat) took an active role in pushing for meanings to

be made explicit—by “holding teachers’ feet to the fire.” Pat did so during the seventh class meeting after each of the groups had presented their ideas pertaining to the relationships amongst learning goals, tasks and activities, and teaching.

Excerpt 70, Class #7

Pat: Suppose somebody said, okay the learning goal is that...they know how to solve a quadratic equation...that’s the learning goal...

Bernardo: Okay [Quiet for six seconds]

Pat: I’m asking...would you form that as a learning goal...could you accept that as an example of a learning goal? [...]

Janine: It could be one of them, but I think my goal would be...for students to understand quadratic, that’s the big goal [...]

Gwen: When I first heard you say that I thought, well which way, because there are so many different ways and I think the main goal for me when I teach quadratics is that I want kids to understand, you know, when you’re solving...okay, so you find and x and an x , sometimes you find one x , sometimes you don’t find any x ’s, but what does that mean, and that’s my main goal is maybe teaching that. I mean because we teach it lot’s of different times without actually ever saying solve the quadratic or you’re finding the two places on the x -axis where it crosses, you know, I mean, we use all sorts of different language that doesn’t have any meaning to them anyway, but...

For Gwen, solving quadratic equations appeared to be a legitimate learning goal (an idea, a mathematical topic), with a variety of methods (members of a skill set) for solving for x (e.g., quadratic formula, factoring, solving graphically). In addition, Gwen expressed that what gives “meaning” to the “solution,” is the idea that the root(s) are the place(s) where the graph crosses the x -axis. This could indicate that for Gwen the focus is on “understanding” the solution, that is, what it “means” (graphically, with the focus solely on one variable, x) to be a solution to a quadratic. In addition, although Gwen asserts that, “We use all sorts of different language that does not have any meaning to [the students] anyway,” she does not indicate that she views this as problematic, only that it is the norm. In the following (subsequent) exchange, Pat attempted to push the teachers to give meaning to quadratic.

Excerpt 71, Class #7

Pat: Suppose that you’re learning goal is “I want them to learn how to solve quadratic equations”...versus “I want them to learn...what...to learn what,” turn that one into learning an idea.

Story: Well Janine said understand quadratic equations.

Pat: But that’s not saying what they’re supposed to understand yet...what are you supposed to understand when you understand quadratic equations?

Janine: A whole lot of things...recognize them, use them to solve problems...

Pat: But the hallmark of an idea is how it's related, is its meaning and how it's related to other ideas...my question was...what does it...mean to understand quadratic equations? [...]

Pat: So...if they're going to understand quadratic equations...don't they have to have a meaning for quadratic equations...what meaning do you want them to have? And we can't just say "the meaning", because that doesn't answer, that doesn't make anything ... so what meaning do you want them to have? [...]

Janine: Well, you have a particular relationship between variables where...

Story: It could be two of them...that work.

Janine: There's a particular...

Story: That relate with one...

Janine: There's a particular relationship between those two variables where squaring is happening...

Although Janine initially expressed part of her skill set for understanding the topic of quadratic equations (e.g., recognize them, use them to solve problems), she was unable to, as were the other teachers, come up with a meaning for quadratic equation. This illustration was indicative of the teachers' difficulties in conceptualizing the mathematics that they teach as being composed of ideas that entail meanings, rather than topics that entail associated procedures, skills, and facts. No teacher offered anything that related "quadratic equation" to

- the idea of polynomial (a quadratic being a polynomial of a particular form), to the idea of a function (an invariant relationship between two quantities values as they vary),
- the idea of equation (any statement about a function that requests the values in the function's domain that produce values in its range of a particular characteristic), or
- what it means to solve an equation (determine the set of values in the functions domain that produces those values of interest in its range).

I take the teachers' disinclination to make their meanings explicit, and their disinclination to relate what they state as an idea to other ideas, as an expression of the teachers being insufficiently mindful of their own meanings so as to operationalize them. This disinclination to take their meanings, their reasoning, as objects of thought appears to be a consequence of how they understand the mathematics they teach.

I hypothesize that many of the teachers conceptualized an "idea" as a mathematical topic (e.g., "rate of change", "quadratic"), and "understanding an idea" as becoming proficient with the skill set (procedures, skills, and facts) associated with that topic. As such, several of the participating teacher's identified instructional goals consisted of getting students to become proficient with a given topic's skill set, where the focus was on doing (or demonstrating) something (where the something is very visible), not on developing ideas and building meanings. Something is an idea, not because it is a broad mathematical topic, constituted by procedures, skills,

and facts. Rather something is an idea, because it has context and meaning, and is foundational for and connected to other ideas.

The claim that the teachers' mathematical understandings were deficient of meanings is significant, because in the Silverman and Thompson (2008) framework for the development of mathematical knowledge for teaching, a teacher transforms her ways of operating via reflective abstraction. Such a transformation requires the teacher to take her reasoning, her meanings, as objects of thought. The vague and superficial meanings that teachers did operate with obstructed their ability to think about meanings their students should construct.

Mindful of Others' Mathematics

In the preceding section, I attempted to develop the claim that the teachers were disinclined to make their meanings explicit and that the meanings they did operate with were vague and superficial. In addition, I attempted to demonstrate the robustness of an epistemic way of operating—a general disinclination on the part of the teachers to focus on meanings (i.e., meanings were not on the teachers' radar).

In addition to a disinclination to make their meanings explicit, the teachers demonstrated a disinclination to think about how others might interpret, or are interpreting, their actions. That is an indication of a second epistemic way of operating. Specifically, the teachers demonstrated a general disinclination to be mindful of others' (i.e., students', colleagues') mathematics. The groundwork for this claim was partially laid in the previous section, through illustrations of the teachers' disinclination to challenge one another to make their meanings explicit.

The Functions 3 course was designed to provide opportunities for the teachers to engage in mathematical tasks and conversations about those tasks with colleagues (other teachers in the project), the course instructor (Pat), and with their own students. In addition, the teachers engaged in activities in which they observed classroom video of a teacher and her students (as part of the reformed-Algebra 1 case study component of the project) engaged in mathematical tasks and conversations about those tasks. Therefore, several of the activities in the Functions 3 course were explicitly designed to provide the teachers with opportunities to be mindful of others' mathematics (e.g., to think about how others might understand, to be sensitive to others' thinking, to be oriented toward others' thinking, to listen for cues to others' thinking).

During the group discussion regarding the "Interpreting/Creating Graphs" Quiz (Class #1), Gwen described her solution to problem #1 part d (describe a person's biking behavior given a graph of his distance from start relative to time; Appendix E).

Excerpt 72, Class #1

Gwen: Clown blew a tire and had to walk. When did this happen? How do you know?

Gwen: It happened right there... [Points to a place on the graph]

Although the video did not allow for me to see exactly where Gwen pointed to on the graph, her subsequent comments lead me to believe that she indicated the

area that I have circled in Figure 14. Note that the original graph did not include the labels a, b, c, and d, which I added here for purposes of discussion.

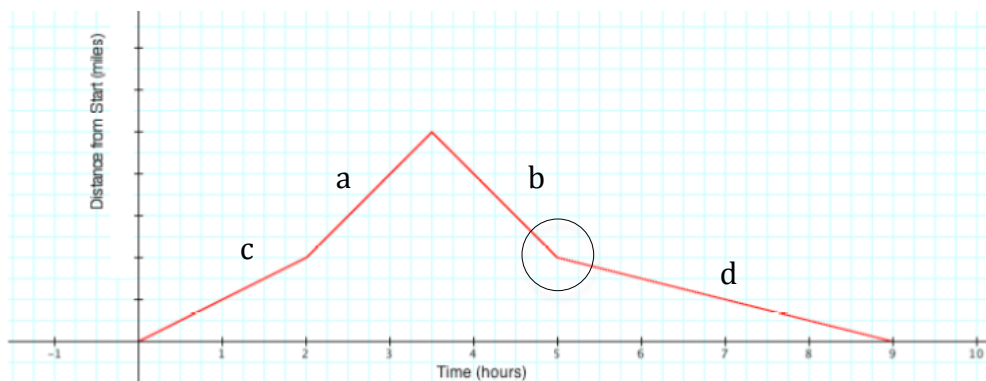


Figure 14. Gwen's solution to problem #1d.

Gwen: Because this is the same as that, so if he's heading back toward...then this should be the same as that, but it's not

Gwen: Which they would not...I don't know if kids would be able to see that either...

Gwen: So, what's the learning goal?

Alyce: If your test reflects what you're teaching, then they would be able to do it.

Gwen: That's true.

In justifying her reasons for choosing the place where Clown blew a tire, Gwen reverted to shape thinking—where, the shape of the graph has meaning, rather than the relationships between the quantities. I interpret Gwen's comment as follows (Figure 14), "Because this shape [line b] is the same as that shape [line a], so if he's heading back toward start, then this shape [line d] should be the same as that

shape [line c], but it's not." Where by "shape," Gwen could mean the slope, or "angle" of the line.

Gwen commented that "this is the same as that" and "this should be the same as that," where her "this" and "that" were different in each assertion. Gwen's use of pronouns and her disinclination to ascertain whether the other teachers' images (how Alyce and Eve had interpreted her comments) were aligned with her own is an expression of her disinclination to think about others' mathematics.

In addition, since neither teacher challenged Gwen to make her meanings explicit, this is an indication that neither Alyce nor Eve believed that Gwen could have meant anything different from how they had interpreted her—indicating that they too were disinclined to think about others' mathematics. A disinclination to think about others' mathematics is indicative of a disinclination to decenter—a disinclination to attempt to see the mathematical understanding from the perspective of another.

During the initial class meeting, prior to the teachers' engagement with the "Interpreting/Creating Graphs" Quiz (Appendix E), Pat asked the teachers to share their thoughts regarding their participation in the project, "From the point of view of ...who you are now relative to who you were then." Janine provided the following description of her instructional design methodology.

Excerpt 73, Class #1

Janine: I have this really strong desire...to present...to make sure that I really understand the material, and that I can present the most clear, concise lesson, but for me it's always been about me...

Janine's response is very indicative of an image of instruction that is based on transmission; one that is grounded in the belief that students learn by being shown—characterized as a traditional approach to mathematics teaching (Tzur, et al., 2001). Janine's response indicates a lack of decentering, both in her instructional design and in her teaching. Janine's assertions suggests that her image of lesson design and teaching is such that she makes the material and the presentation clear to her, not necessarily to her students—that she is not mindful of students' mathematics (i.e., that she is not thinking about a mathematics for her students to construct).

As indicated earlier (p. 224), prior to the second class meeting, the teachers were requested to create transcripts that would describe a lesson that the teachers would teach to one of the learning goals that they had identified on the “Interpreting/Creating Graphs” Quiz. The following excerpt comes from a discussion of the transcripts that occurred during the second meeting of the Functions 3 course.

Excerpt 74, Class #2

Pat: Anything else someone wants to share about the experience of writing a transcript of a lesson to teach the ideas on that quiz.

Faye: It was kind of hard for me to create the situation in my head...you know it's easier when you're in the classroom, but it's hard to...I should maybe have brain stormed a little bit of where I wanted it to go...it was just hard for me instead of just having kids' thought process going, I had to think of maybe what they would do and I don't know if I did it well, because it was hard without an actual kid thinking and responding.

Faye indicated that she found it difficult to think about how students might know an idea, and then create a possible trajectory for the conversation that she could have with students to increase the likelihood that students develop the intended understandings. Rather, she stated her preference to responding to students in a moment of instruction. This suggests that Faye did not have an image of how students might understand the idea, understandings that the students would need to support the development of the intended idea, or an image of how the idea could develop in the minds of her students—in the minds of someone other than herself.

In addition, Faye implied that it would have been easier for her to think about what students were understanding had there been “an actual kid thinking and responding;” that she needed to have a “kids' thought process going” in classroom. This way of thinking was illustrated later in the Functions 3 course, during the fifth class meeting, where the teachers were discussing an assignment that they had been given which required that they view several of the case study videos and discuss

their feelings (e.g., hesitancy, eagerness) with regards to discussing videotapes of their or another teacher's teaching.

Excerpt 75, Class #5

Pat: One of the things I noticed in all of the comments about watching videos, there wasn't anything about getting insight into kids learning...it was all about watching the teaching [...]

Pat: Your observations were about the teaching. I was the one who pushed issues of, "What did this student understand? What did that student do? Was that significant?" Why? I'm really curious.

Gwen: I think it's hard for us to look at video of a class we haven't been in and we don't know the kids, and it's really hard to assume what the kids are thinking [...]

Pat: My comment is that actually this is better practice for you, because one of the worst things that you can do is presume you know what a kid is thinking. Just because you know the kid doesn't mean you know what the kid is thinking.

Gwen: But, we also have a better guess of how certain kids are thinking...Last year I had one kid who anytime he went like this [raises right hand and waves fingers], I knew he had some off the wall question to ask. It was just the way that he raised his hand, you knew that he had an off the wall question that was not only off the wall, but would also bring in things mathematically that we had never thought of before [...]

Faye: When I watched it, I was definitely just looking at the teaching...I wasn't even looking at the kids, I was focusing on the teaching.

Janine: I think there's a reason for that...everyday you are in your class with 100-150 students, you have such an opportunity day after day after day to try and figure out what the kids are thinking. But we almost never have a chance to watch somebody else teach. It's so rare, and somebody else teaching well that you're just like mesmerized by that person, you don't even look at the kids. You almost don't even want to go to what they're thinking, you're just trying to look at her and say. "What is she doing that I can bring back to my classroom, so that I can figure out what my kids are thinking?" So I think that it's a matter of practicality, it's just such a big thing to be looking at the teacher...

Gwen's response suggests that she had a non-cognitive meaning for "assume what the kid's are thinking"—that you needed to know something about students' personalities or know the history of their behavior in order to draw inferences about their thinking. Faye and Janine expressed clearly that students were unimportant to them as contributors to the instruction they viewed in the video. This is indeed important, because the teacher they watched was highly attentive to student thinking and was conditioning many of her actions on the thinking she discerned in students' remarks. Both Faye and Janine's comments suggest that they are thinking of teaching in isolation of student learning, not as a means to influence students' mathematics.

Several of the course activities and discussions centered around the finger tool, a didactic tool designed to: (a) help students (and the teachers) conceptualize covarying quantities; (b) promote a conception of graphs as emergent as aspects of covariation; and, (c) provide a way to think about functions dynamically instead of statically (P. W. Thompson, 2002). The teachers engaged in the finger tool activities during the second and third class periods (Appendix A). These initial activities involved the teachers progressively tracking the values of quantities in dynamic situations, first tracking the value of one quantity with one finger (horizontally), then tracking the value of the other quantity with the other finger (vertically), then with both fingers simultaneously, and finally with both fingers, keeping the vertical finger “on top of” the horizontal finger. During the fourth class period, Pat asserted that he had not seen much “real” finger tooling going on throughout the activities.

Excerpt 76, Class #4

Pat: You were not really engaging with the tasks as teachers...looking at [the tasks] and attempting to describe the mathematics in them and also, at the same time, thinking about how the students might understand them, but you’re looking at it from the point of view of the teacher, and...not really engaging with [the tasks] as a student in the your class, where you turn off what you know [...]

Pat: Do you think that that’s an accurate characterization...Janine?

Janine: I guess I’m acting from my own perspective I don’t find it helpful, but maybe I just need to pretend like I’m a student and been told to use the

finger tool, and try to use it...I'm not sold on the finger tool is my problem...am I the only one?

Story: No

Liz: I think the finger tool helps you visualize it, and as teachers we already visualize it, so you skip that step where you wouldn't as a student. But then, we're not also analyzing things as to what they would do for my students...we're just saying, "O.K. we get that, let's move on."

Janine asserted that she does not find it helpful, from "her perspective," that the finger tool is not helpful for her, suggesting that she is not mindful of how the finger tool might be helpful to her students. In addition, it is not clear whether if Janine were to pretend to be a student and use the finger tool, she would do so by attempting to decenter, to engage in the task with the mathematical understandings of her students (how she imagines her students understanding).

Liz's comment that the finger tool "helps you visualize it" is unclear whether she means that the activity is helpful in visualizing the graph of an unfamiliar function or she means that it is helpful to imagine the graph of a function that you already know. We know from prior research that teachers who fail to see the benefit of training students to use the finger tool are generally those who see the finger tool as simply the act of tracing an already-familiar graph with one's finger (P. W. Thompson, 2009).

Pat next attempted to clarify that the finger tool was not simply "waving fingers," but tied to the idea of covariation.

Excerpt 77, Class #4

Pat: The finger tool is not necessarily this [moves two fingers simultaneously, right hand moves to the right; left hand moves to the right and up and down]. The finger tool is conceptually tracking two quantities' magnitudes as they vary simultaneously. So if I asked what the behavior was of cosine of x squared, what are you going to do?

Janine: Well, then I was using the finger tool.

Pat: Say more.

Janine: Well, you just said the finger tool is not necessarily this [gestures with fingers], it's visualizing two quantities changing simultaneously, then I was using it.

Pat: Umh...when?

Janine: In the last class, when...you said that people weren't really using the finger tool, I was using it in my head...I just wasn't doing this [moves fingers], because to me that's not...

Janine's comment provides further indication that she is not considering how the finger tool might be helpful for her students. Janine asserted that the finger tool was not necessary for her, because she can visualize two quantities changing simultaneously in her head—she is focused on her understanding, not the understandings of her students; she is not thinking about her students constructing their own mathematical realities.

The subsequent discussion suggests that Pat interpreted Janine's comments as indicating that she (and perhaps other teachers) did not have a theory of how concrete activity results in mathematical learning.

Excerpt 78, Class #4

Pat: Okay, that's useful. Do you think that your 9th graders can think that way? How are you going to get them to think that way?

Several: No

Johnny: By doing the finger tool [moves fingers]

Pat: There's a theory of abstraction built in here. The way people develop concepts is that actions are embodied first; you do things. Whether it is drawing something or enacting something like a finger tool, or going through a calculation in a very laborious way so that you do every step. After a while you start doing it mentally. Almost every theory of learning has that kind of structure built in. You have to engage in the activity in all of its complexity and concretely before you can envision it just in thought, before you can do a thought experiment. There is no theory of learning that does not contain that.

Pat: So that's what the finger tool is. It's helping students become like Janine. Because you can't say "Watch what's going on in my head. This is what I'm envisioning."

Janine: Does it always have to be the finger tool? [...]

Pat: It's a finger tool as long as you are insisting that they pay attention to both quantities simultaneously. We're going to look at videos that will show how easy it is to think that you're doing it and you're not. That's the point of the finger tool, is to make sure that both quantities get paid attention to.

The claim that the teachers' were not mindful of others' mathematics is significant, because in the Silverman and Thompson (2008) framework, developing MKT involves transforming (via reflective abstraction) a teachers' personal understanding of a particular mathematical concept to an understanding of: (1) how this understanding could empower their students' learning of related ideas; and, (2) actions a teacher might take to support students' development of it and reasons why those actions might work. Such a transformation requires decentering.

Empirical Orientation Toward Learning and Teaching Mathematics

In the preceding sections of this chapter, I attempted to develop and demonstrate the robustness of two epistemic way of operating—a disinclination in the teachers to make their meanings explicit and a disinclination to be mindful of others' mathematics. Regarding mathematical and instructional meanings, the teachers stated technical terms (e.g., rate of change, quadratic functions, discovery learning) without suggesting their meanings, and never discussed meanings unless prompted by instructor. Furthermore, the teachers' mathematical meanings consisted of action sequences—things students “can do” (e.g., determine the x-intercepts of a quadratic function)—where the focus is on producing outcomes (the

products of reasoning). Regarding being mindful of others' mathematics, the teachers rarely ascertained the compatibility of their meanings with those of others, and rarely considered how different ways of thinking will lead to different interpretations; the teachers appeared to be projecting their own understandings and ways of thinking onto others. Chapter 5 illustrated how these understandings and ways of thinking were problematic for productive reflection; that is, the two identified epistemic ways of operating constrained the teachers' capacity to reflect.

As I asserted earlier, I am describing the way the teachers operated (i.e., the way they were), so that I can talk about how it is that the way they operated got in the way of other things (e.g., reflection). In order for such a discussion to be productive to future research in mathematics teacher education, requires that I describe an orientation toward learning and teaching mathematics that explains how the teachers operated—an *empirical orientation toward learning and teaching mathematics*. Furthermore, such a description must operationalize the orientation, so that others can productively employ the description.

Simon, Tzur, and their colleagues (Simon, et al., 2000; Tzur, et al., 2001) have identified three cognitive structures with regards to teachers' perspectives on learning and teaching mathematics, work that does much to highlight and illustrate the cognitive structures that teachers use to make sense of teacher development opportunities. Simon, Tzur et al. (2000; 2001) describe two perspectives that work well to characterize the schemes of the teachers in the current study, a traditional perspective and a perception-based perspective. According to Tzur et al. (2001),

both perspectives include a platonic view of knowing, in which mathematical knowledge is viewed as existing external to the individual; mathematics “exists as part of an objective reality accessible to all” (p. 246)—a general perspective expressed by the teachers in the Functions 3 course. According to Simon et al. (2000) the main distinction in these two perspectives is that the “traditional perspective may or may not include an emphasis on mathematical understanding as connections, and it does not emphasize direct personal perception” (p. 593).

Although these two perspectives might provide characterizations of the teachers in the Functions 3 course, the data corpus is such that a detailed analysis of any individual teacher was not possible. Furthermore, I believe that the teachers’ pedagogical ways of operating are not captured exclusively by either a traditional perspective or a perception-based perspective. Finally, accounts of teachers’ practice (Simon, et al., 2000; Tzur, et al., 2001) do not address a teacher’s own learning of mathematics. Specifically, the characterizations of teachers’ perspectives approach does not account for teachers’ ways of operating during engagements designed to transform teachers’ personal mathematical understandings and ways of thinking. Therefore, I will attempt to describe an orientation to learning and teaching mathematics that accounts for both the teachers’ (in the study) personal and pedagogical ways of operating—an orientation characterized by a platonic ontological view of mathematics, and egocentric (p. 80) and empirical in nature. By empirical, I mean an orientation that is not only guided by perception, but focuses on the “visible”—that which can be perceived.

I shall attempt to develop this perspective by initially focusing on a particular way of looking at the role of a mathematics teacher from a platonic view of knowing: The role of a mathematics teacher is to teach “the” mathematics (indicated by the curriculum, standards, textbook, etc.) to students, and to assess the level at which the students understand “this” mathematics. Regardless of whether the teacher believes that “students passively receive mathematical knowledge by listening to and watching others, usually mathematics teachers, and by reading about mathematics (in textbooks)” (Simon, et al., 2000, p. 593), or that students come to “see mathematical ideas and relationships through their own experience of the mathematics” (Tzur, et al., 2001, p. 247), the orientation that I will attempt to characterize focuses on the teaching and learning of the visible (the perceivable).

Cobb, Yackel, and Wood’s (1992) description of a representational perspective to learning is consistent with what I have in mind regarding an empirical orientation toward learning and teaching mathematics, where learning is viewed as:

A process of acquiring accurate mental representations of fixed mathematical structures, relationships, and the like that exist independently of individual and collective activity. Within this frame of reference, the teacher is in a privileged position because he or she has apprehended these external mathematical structures, whereas the students are yet to do so. [As such], our instructional options are relatively limited as we attempt to make the mathematical structures apprehensible to students. We can of course

develop external representations to make the mathematical structures as transparent as possible. (p. 27)

This attempt to make things “transparent” to students is how I describe a teacher attempting to make things visible to her students. The idea of focusing on the visible did not originate with me; Ball and Bass (2000) describe a teacher’s ability to decompress—to “deconstruct one’s own mathematical knowledge into less polished and formal form, where elemental components are accessible and visible” (p. 98). Ball, Thames, and Phelps (2008) describe teaching as “making features of particular content visible to and learnable by students” (p. 400), and illustrate how a teacher would need to choose an appropriate choice of numbers to make visible the conceptual structure of the algorithm for subtracting multi-digit numbers (Ball, Thames, et al., 2008). In differentiating their framework for MKT from that of others, Silverman and Thompson (2008) assert that, “Ball and her colleagues have focused on special ways teachers must know the mathematics, which are visible during instruction...[and] focus...on the ways teachers treat this visible mathematics, which are sensitive to students’ understanding of it” (p. 500).

I will attempt to define what I mean by an orientation toward the visible, which is more in line with that indicated by Ball and her colleagues (e.g., Ball & Bass, 2000; Ball, Thames, et al., 2008) than with the “visible” mathematics of the workplace (e.g., Noss, Hoyles, & Pozzi, 2000). By a focus on the visible, I do not mean to simply imply a focus on mathematics that can be visually perceived (e.g., written equations, graphs), although that is part of it. I mean an orientation toward objects

that can be perceived in some manner, such as, but not limited to: visually (e.g., through words and symbols written on a page or a white-board, a graph), and orally (spoken words, phrases or symbols)—an orientation that allows for the teacher to (quickly) assess for “correctness” or “level” of understanding via visible, or perceivable, indicators.

One way that an orientation toward the visible is expressed is through instruction that focuses students’ attention on the visible—a focus on action sequences, but action sequences involving procedures, skills, and facts to arrive at an outcome (rather than a focus on reasoning). The teachers in the Functions 3 course expressed such an orientation. During the initial class meeting of the Functions 3 course, as the teachers were describing their involvement with the project, Eve indicated a perspective involving an orientation toward the visible, that is a mathematics consisting of procedures and skills.

Excerpt 79, Class #1

Eve: When I was student teach...I had a very good mentor teacher and he used the ELL code, like every step in color, teaching step by step, I was sitting there thinking, “Wow, he’s such a great teacher, and he teach step by step so clearly that students actually learn and learn so well.” And when I try, the first year teaching and it actually worked for me and I think, “Maybe I will use this way all the time when I’m teaching”...

Although Eve did not articulate what she meant by “students actually learn and lean so well,” the “it” that actually worked for her, or what she meant by

“worked”, she is describing an orientation that focuses their students’ attention on objects that are very visible.

Boyd (1992) characterizes one mathematics teacher’s (Mae) mathematics subject matter knowledge in terms of “objects and actions that she envisioned being applied to those objects” (p. 57). Boyd describes how Mae and her students employed the Blocks Microworld program to explore division as sharing. The Blocks Microworld program enabled Mae’s students to explore division as sharing in the context of reasoning about quantities or blocks (Boyd, 1992). In addition, the program made “explicit the connection between changes in the blocks representation of a quantity with changes in the numerical representation of that quantity” (Boyd, 1992, pp. 20-21).

After working with Blocks Microworld for several lessons, the students’ tasks changed from sharing a given number of blocks among a specified number of containers (in Blocks Microworld) to word problems worked with paper and pencil, a change intended to help the students generalize their understanding of division as sharing (Boyd, 1992). According to Boyd (1992), the pattern of the word problems was to share a given number of objects (e.g., apples, chairs, donut holes) among a specified number of recipients (e.g., baskets, rooms or children).

During a discussion of sharing chairs, “Mae had asked her students: ‘When we use computers we share blocks. These aren’t blocks, these are what?’ Students suggested ‘numbers’ and ‘chairs’. Mae continued: ‘These are chairs’” (Boyd, 1992, p. 58). According to Boyd (1992):

Chairs were objects in the problem Mae had presented while blocks in Blocks Microworld were intended to always be portrayed as representing numerical value. Mae's connection between the two indicated that she thought of her activities with Blocks Microworld as being about blocks as objects and not about blocks as representations of numerical value. (p. 58)

Boyd (1992) asserts that, "The word problems Mae chose to help students extend and generalize their understanding of division as sharing reflected her tendency to think of objects and patterns of action which employ objects" (p. 58). Furthermore, these problems fit a common theme: to share a given number of objects (e.g., chairs or pieces of paper) among a specified number of recipients (e.g., children), and were "easy to visualize as physical sharing" (Boyd, 1992, p. 58), but "did not address the abstract idea of division as mental partitioning, constrained by base-ten numeration, of some quantity that had a numerical value" (Boyd, 1992, p. 58).

The notion of a focus on objects and the actions being applied to those objects is compatible with my characterization of the Functions 3 teachers' conception of mathematics as composed of concepts and associated skill sets (p. 230). For example, the teachers indicated that quadratic functions were "objects" of the form $y = ax^2 + bx + c$. These objects had associated "actions" attributed to them (as envisioned by the teachers), actions that you perform on this object, such as determining the x-intercepts, or the vertex.

An empirical orientation toward learning and teaching mathematics (as illustrated by the Functions 3 teachers) places an emphasis on providing students with opportunities to “understand” the objects by engaging the students in tasks that involve performing actions on the objects, although the focus is not on actions (i.e., reasoning). Rather, the focus is on performing a sequence of steps to arrive (and perhaps justify why certain steps are “allowed”) at an outcome. The focus is on the product of reasoning, where the reasoning itself stays “hidden.”

The types of action sequences not commonly expressed by the teachers in Functions 3, involved getting students to abstract, generalize, relate, and interpret. The following excerpt illustrates the only time these types of action sequences were mentioned by someone other than the Functions 3 instructor (Pat), and occurred during the presentations of the group-identified learning goals and instruction for the Algebra 1 quiz (Class #1).

Excerpt 80, Class #1

Marcy: I think yours was two big ideas, but not necessarily learning objectives...they should really be things that kids are able to do, and so, I think, obviously you were short and sweet and we know what rate of change means, but what’s the objective?

Adriana: What should they be able to do?

Marcy: Learning objective needs to be somehow related to that big idea, but what about it? [...]

Bernardo: I think you need to say, "This is what I mean by a learning objective. A learning objective is something the students can actually perform"...one and one is two or whatever, or "A learning objective is a big idea and we're just calling it 'learning objective'," so...

Marcy: So, it's not really clear to us what that means...learning objective.

Bernardo: I think that it will vary from group to group...

Marcy: Sure, sure. [...]

Pat: So, what work should a learning objective do? [Quiet for 9 seconds]

Gwen: I think that maybe everything that you're doing in the unit or the lesson, or whatever, should bring the kids back to that learning objective...should be in some way getting the kids to the ending result, or whatever it is...

Sheila: I would say this is in one way I've changed...I was pretty old school...a learning objective was, "Kids will be able to do this," where I've moved more through this whole process is, "Students will understand," versus "be able to do," and I think that's where some of the differences in what we're writing comes from as well...being "able to do" versus "being able to understand, and interpret, and relate, and generalize, and abstract," as you said, so...there's a whole bunch of semantics going on from the first few boards to the next few boards...I think we're all saying very similar things, but in perhaps slightly different ways...

Although Shelia brought up the idea that the teachers were talking about learning goals using different action sequences for students, she asserts that the

differences are really only in word usage. I argue that the differences should be in the mathematics, and that students “being able to do” involves a focus on the visible, whereas getting students to abstract, generalize, and interpret involves a mathematics that is very difficult to make visible.

Hill et al. (2008) describe how one teacher, Lauren, questioned a particular student (Josh) about his solution to what numeric equivalent he would assign for the label “unlikely” on the “likelihood line” (Figure 15).

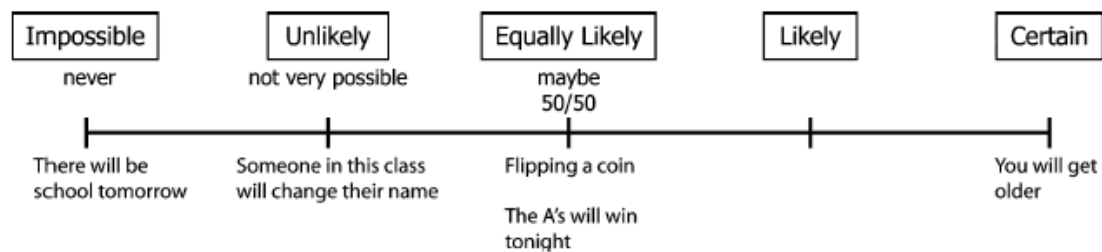


Figure 15. Likelihood line (Hill, Ball, Blunk, et al., 2008).

In the example, the class had earlier associated 0% with an event that was “impossible,” and 100% with an event that was “certain.” Josh had provided 25% to the label “unlikely.” Hill et al. (2008) provide the following excerpt, in which they assert the teacher (Lauren) “draws” the student out “to make his reasoning visible to the class” (p. 455):

Excerpt 81, Hill, Ball, Blunk, et al. (2008)

Josh: I know another way too. You just subtract twenty-five off of all of them. So likely it's gonna be, I'm not gonna say the answer though.

Lauren: No go ahead. Tell me what you mean by subtracting twenty-five percent. Can you clarify that for us?

Josh: So, one hundred minus twenty-five is seventy-five. Seventy-five minus twenty-five is fifty. And fifty minus twenty-five is twenty-five. And twenty-five minus twenty-five is zero.

Lauren: All right. How many of you agree with what he's saying. [Class indicates yes.] Okay.

I would argue that Josh did not make his reasoning visible, he simply made visible the chain of steps that he employed to arrive at the solution; without identifying what it was he was finding and how he believed his “steps” would help him arrive at a solution. What Lauren moved Josh to do, was to make the steps he employed in the solving the problem visible. Rather than the mathematics being about meaning, abstracting, generalizing, or justifying, it was about making the procedure visible—what Thompson et al. (1994) classified as a calculational orientation to mathematics teaching.

Simon et al. (2000) describe a teacher's (Ivy) expectation that that her students would “see” the connection between solving division problems with base-ten blocks and with paper-and-pencil using the division algorithm, because she oriented her students' perceptions toward the connection. Rather than designing

the activity with the intent that the students abstract a way of reasoning, the teacher (Ivy) designed the activity with the intent that she make “the connection” (between steps of their base-ten block solutions and steps in the algorithm) visible to her students. Simon et al. (2000) assert that Ivy’s perspective was such that she believed the mathematical relationships that could be perceived (by her students) as “properties of the objects being considered (e.g., base-ten blocks), not a function of the knowledge of the perceiver” (p. 592) and that she could make the connection visible (i.e., perceivable) to her students. This perspective is consistent with an empirical orientation toward learning and teaching mathematics, except that a teacher with an empirical orientation would attempt to make her own understandings (how she has perceived the mathematics) visible to her students, and not focus on how others (i.e., her students) might perceive, or interpret instruction.

Although the previous two examples illustrate what I mean by a focus on the visible, I would like to attempt to further clarify the notion of an empirical orientation toward learning and teaching mathematics by focusing on the teachers in the Functions 3 course. Specifically, I would like to elaborate on the orientation by focusing on two Functions 3 teachers as they interacted with their students.

Prior to the 13th class meeting, the teachers were asked to submit a video of themselves in their classroom, interacting with their students. More specifically, Pat asked the teachers to share a video of their teaching, in which the teachers attempted to hold a conversation with their students that had the attributes of being

conceptual. As described in chapter 3 (p. 92), a conceptual conversation is one that has “a diminished emphasis on technique and procedure, and an increased emphasis on images, ideas, reasons, goals, and relationships” (A. G. Thompson, et al., 1994; A. G. Thompson & Thompson, 1996). The teachers were also asked to submit a document outlining the context of their lesson and to provide feedback to five of their fellow teachers regarding their lessons.

Janine submitted a lesson involving rational functions. Janine asserted that she intended for the lesson to help students to discover why different rational functions have asymptotes where they do. Janine stated that the students investigated a variety of rational functions, and that she hoped that students could reason why an asymptote was occurring where it was, rather than have to memorize rules. The following excerpt is from Janine’s lesson (video).

Excerpt 82, Janine’s Classroom Video

Janine: You’re going to come up with some definitions for these different pieces of vocabulary that I have on the board. Vertical asymptotes, horizontal asymptotes, and we’re, you’re going to figure out how you can go about finding all of these things. We’ll start out by graphing $f(x)$ up there, which is a rational function and it is $\frac{x^2}{x^2 - 1}$, so go ahead and graph that and we’ll start analyzing the graph. [Janine graphs function on calculator and displays graph on the overhead projector]

Janine did not attempt to ascertain the students' images of rational functions or asymptotic behavior prior to graphing the given function. She did not know whether any student meant by "asymptote" anything more than that the graph appeared to go straight up, down, or sideways. In addition, she did not indicate whether she expected students to provide meaning to the terms or provide a means to determine where the asymptotes occur.

Excerpt 83, Janine's Classroom Video

Judy: Oh, is this the one with the asymptotes, and then you have...where they can't go past on the curve? [Students graph function on graphing calculators]

Amy: Mine doesn't look like that.

Judy: Mine does.

Janine: Yours may or may not look like this depending how far out your window is and also on what type of calculator you have [...]

Janine: Is everybody seeing some version of this on their calculator, and if you're seeing something different what are you noticing? [...] So if you don't have the vertical lines, put your calculator up in the air [seven student hold calculator in the air]

Janine did not acknowledge Judy's comment regarding what she imagined the graph to look like. In addition, Janine is not clear what she means by "this," what is the "this" that the students are supposed to be focusing on? Janine neither

clarified her statement, nor attempted to ascertain where students were focusing their attention.

Excerpt 84, Janine's Classroom Video

Janine: What do you notice? What kind of calculators do you all have? [...]

George: TI-84 silver edition.

Janine: So, the 84s are showing something a little bit different than the 83s.

Briana: Mine is the same, it's showing that...exactly what you have on the board
[Janine checks calculator]

Janine: [pointing to vertical asymptote at $x = -1$] Some of you aren't seeing this piece, does anybody know what that is?

Several: Asymptote.

Judy: Vertical asymptote.

Janine: A vertical asymptote, what is a vertical asymptote? We need a definition.

Janine has indicated that "this piece" of the graph on the overhead, which she has asserted is calculator-dependent, is a vertical asymptote. She did not attempt to ascertain where the students were focusing their attention, or how they were interpreting her assertions.

Excerpt 85, Janine's Classroom Video

Judy: Where the graph approaches, but never quite [...] crosses it.

Janine: You're right about that, it won't ever cross it...

Judy: It's like a restriction, it can't go outside of it [...]

Seth: Mine doesn't look like that.

Janine: Let's come up with a definition for a vertical asymptote and then we'll figure out why we're seeing them in the places that we're seeing them. So, what do you think a good definition would be for a vertical asymptote?

Janine moved directly to a definition without attempting to ascertain what the students understood. In addition, she made no attempt to get students to think about what Judy might have meant. Janine appeared to have a set definition in her mind and was attempting to get the students to "connect" the definition that she had yet to reveal with the dotted line.

Excerpt 86, Janine's Classroom Video

Judy: A restriction...on where...

Janine: But what type of a line is it?

Several: Vertical [...]

Judy: It's a vertical line that...

Janine: We'll say x equals anything is a vertical asymptote if what?

Amy: If the graph, if $f(x)$ does not cross the...

Janine: But what is $f(x)$ doing if it's not crossing it...

Judy: It's approaching it, but it never quite...

Janine: But what is $f(x)$ doing as it approaches that vertical line?

Janine made no attempt to ascertain what students might have meant or how students might have interpreted the comments that $f(x)$ "approaches the vertical line," that $f(x)$ "does not cross" or even "what is $f(x)$ doing?" Janine appeared to be

focused on how she interpreted the statements or the meanings that she attributed to the comments.

Excerpt 87, Janine's Classroom Video

Judy: It curves.

Janine: What's it doing, where's it going?

Amy: Down.

Janine: Down [...] it's going to negative infinity here isn't it? And what's it doing over there [points to the vertical asymptote at $x = 1$] [...] it's also going to negative infinity, and we'll see some graphs where it's gonna shoot up...

Amy: Yeah, because up there, it's going up...

Janine: Right, on this side it's going to positive infinity, so what's happening around this vertical line at x is equal to negative one?

Judy: It's causing it to go...

Janine: So, that is a vertical asymptote because...

Taylor: It approaches, but never touches...

Janine: The function value is approaching positive and negative infinity as it approaches that value [writes this definition below vertical asymptotes]

Seth: Is it?

George: Yeah.

Janine provided the definition to the students, even though she had yet to give meaning to the term or ascertain what images the students might have attributed to the term. Janine had made the term vertical asymptote visible to her

students by “connecting” the definition with a graphical representation—although the connecting was done via perception, rather than through reasoning and reflection on the behavior of a function near some value in its domain.³

A second teacher, Johnny provided video of a lesson that he taught, in which his students were to employ information that they had previously been taught regarding central angles, intercepted arcs, inscribed angles, and regular polygons to answer what Johnny described as novel problems. In the video, Johnny stated his intent was to use information that the class “knows” involving the area of a circle to find the area of a polygon.

Excerpt 88, Johnny’s Classroom Video

Johnny: [draws a circle on the white-board] We can already figure out how to find the area of a circle, right? How do I find the area of a circle, what are the things I have to know?

Linda: The radius or the diameter...

Johnny: Okay, either the radius or the diameter, so I at least have to know that...and if I know the diameter, how does that help me, Nancy, to find the area of a circle? [...]

Nancy: Because knowing the diameter of the circle, then you can find the area of the circle, or find the circumference?

³ I should note that the visual definition toward which Janine strived is totally inadequate for understanding the asymptotic behavior of $y = \sin\left(\frac{1}{x}\right)$, which has the line $(x, 0)$ as a horizontal asymptote even though its graph crosses $(x, 0)$ an infinite number of times.

Johnny: I want to find the area of the circle...so how does knowing the diameter help me?

Nancy: I don't know...

Johnny: If I know the diameter, then what do I know?

Eva: Where the circle is halved.

Johnny: I know the radius, don't I? [...]

Johnny: If I know the diameter, I automatically know the radius...because the radius is always what?

Several: Half...

Johnny: Half the diameter [...]

Johnny: So Tim, what is the area of a circle?

Tim: It's just $2\pi r$...

Johnny: Ah, ah...area, area...

Fred: Length times width... [Johnny puts head down]

Johnny: Length times width is what guys?

James: Area of a rectangle.

Johnny: That's the area of a rectangle...the area [of circle] is what?

Jenny: $\pi \cdot r^2$

Johnny: $\pi \cdot r^2$

In the above excerpt, Nancy and Eva's comments suggest that they were interpreting the information differently from the images that Johnny had intended.

Johnny appeared to desire that the students not only recall the formula $A = \pi \cdot r^2$, but that the students understand (how he understood) that the problem required that they make use of the formula. Nancy and Eva appeared to be attempting to answer the question of how the diameter relates to the circle. Rather than attempting to get at the understandings that Nancy, Eva, and other students might have been developing (i.e., how they might have been interpreting the situation), Johnny moved the conversation forward to get to the desired formula (a focus on the outcome). Johnny focused the students' attention on Johnny's conception of the product of "proficient" reasoning (i.e., the formula, something visible), rather than getting students understandings and reasoning on the table, as objects of discussion.

In contrast to preceding two excerpts, which highlight an empirical orientation, consider the following excerpt from the reformed-Algebra 1 case study component of the project. In the excerpt, the teacher (Jessica) asked the students how, if given the equation for a quadratic function, they could determine the vertex.

Excerpt 89, Reformed-Algebra 1 Case Study

[Jessica has written on the board, Original: $y = ax^2 + bx + c$]

Jessica: Get me started...I have a quadratic, I want to find its vertex, I don't know what to do, somebody help me...

Fred: Take the rate of change function

Jessica: Okay...why, Fred, would I want the rate of change function to help me find a vertex?

Fred: So, you can get the “x”

Jessica: Okay, okay...So, how about this? I'll make one up really quick [Jessica writes $y = x^2 - 10x + 5$ on the white board]

Jessica: Sure, I made that up...So, Fred says that I need the rate of change equation, so what's the rate of change equation for this?

Fred: $2x - 10$

Jessica: [Writes $ROC = 2x - 10$]

Jessica: Okay...all right, so I found the rate of change equation, who cares? [...]

Jessica: Somebody other than Fred...somebody other than Wallace too.

Tammy: Make it equal zero

Jessica: [Writes $0 = 2x - 10$]

Jessica: Why in the world would I do that? [...]

Tammy: So you can solve to find where the y value is zero

Jessica: Okay, so why is it...that my vertex is gonna happen when of rate equals zero? What's going on with y-values being zero on the rate of change graph to where that's going to lead me toward vertex? I mean I can do it, but why? Ryan?

Steve: It's where it crosses the x-axis

Jessica: That's where it crosses the x-axis, hence the y-value being zero, but why is that the location of vertex? Tabitha?

Tabitha: The turning point where it's the positive and negative, or an increase or decrease

Jessica: Okay...tell me more about it being the turning point, and positives and negatives, just elaborate.

Tabitha: Umh, once it hits the x-axis, it can either...

Jessica: So, are you picturing a line in your head?

Tabitha: Yeah...and like on the x-axis, like right where it hits the x-axis [Jessica draws graph depicting the situation that Tabitha is describing, Figure 16], it was a negative increase before that point and it would be positive after that.



Figure 16. Jessica's depiction of Tabitha's description of her image of the function's graph.

Jessica: Okay, and so what does that mean for the curve, the fact that prior to this x-intercept it's negative and after that x-intercept it's positive?

Tabitha: That's where the curve's turning point is going to be...

Jessica: And, in this case what is it turning from, a what to a what?

Tabitha: A....gosh...an increase, it looks like a "U"... it's a negative increase and then it goes to a positive increase...

Jessica: So, you draw a "U," so tell me again [draws parabola on the graph of the rate of change function, Figure 17] [...]



Figure 17. Jessica's depiction of Tabitha's description of her image of the graph of the rate of change function.

Jessica: So, you read a graph like a book, so it changes from what to what?

Tabitha: Negative to positive

Jessica: Okay, the rate of change changes from a negative to a positive, so what's the curve going to do...change from...

Tabitha: Negative to positive?

Jessica: Okay, but what does negative mean on the curve? Negative rate of change...

Tabitha: Decrease

Jessica: Decrease, good, very good...all right...ladies and gents, setting equal to zero [points to the rate of change function] it's this whole idea of x-intercepts, yes...but that is when you're rate of change changes...in this case Tabitha said it's a negative rate of change prior to that spot and then it's a positive rate of change after that spot, that's when you're curve turns, in this case from decrease to increase, very good...so, setting it equal to zero that's your rate of change being zero...turning point, exactly...so, I can solve this in my head and I get x equals 5...

In the preceding excerpt Jessica attempted to manage the conversation so that the students coordinated and reflected on their meanings regarding covariation, rate of change, x-intercept, graph, vertex—the students reflected on their scheme of meanings to arrive at a solution. Although Jessica summarized the “sequence” of reasoning, the mathematical reasoning involved during the discussion is not visible in the sense that I have been using the term in the previous examples. This last statement is misleading, because Jessica had a preconceived idea of how the conversation might unfold and managed the conversation to increase the likelihood that students would reason in the manner that she had intended—the reasoning was visible to Jessica and she attempted to make the reasoning visible to

her students. But the action sequences did not consist of procedures, facts, and skills; rather, the actions sequences consisted of reasoning, meanings, and ideas.

The previous excerpt not only serves to provide an example that does not involve a focus on the visible, it will also serve to demonstrate how the teachers' (in the Functions 3 course) orientation made it not possible for them to "see" what had in fact occurred in the excerpt. During the first Functions 3 course of the fall semester (Class #8, August 20th, 2007), the teachers watched the preceding video. After watching the clip that I transcribed above, Pat asked what the teachers made of what they had seen.

Excerpt 90, Class #8

Pat: So...what strikes you about this video?

Tami: She's not teaching the formula, the negative $b/2a$.

Pat: Okay [...]

Tami: She's approaching it from a very different perspective.

Sheila: She's teaching calculus to freshman [several teachers laugh]

Pat: Actually, is she teaching this to freshman?

Alyce: No, they discovered it on their own.

Pat: Did they discover it on their own?

Alyce: Well, I'm sure there was plenty of guiding, but...they came up with that.

Pat: Right, but why are they even talking about it? Jessica chose to talk about it, right?

- Tami: She's relating the increase and decrease and covariation to vertex, so she's pulling it one level deeper...
- Pat: Okay, but she's not...see that's why, where Alyce is right...she [Jessica] is not doing that relating.
- Tami: The students are.
- Pat: The students are...but now contrary to what Alyce said about them discovering it, they're not discovering this, so...Jessica's not teaching it and they're not discovering it, so what's happening? [...]

The teachers' comments suggest that they cannot "see" how Jessica managed the conversation so that the students reasoning became objects of thought, objects for reflection. It appeared that the reasoning, and Jessica's role in getting students to reason, was not visible to the teachers.

Excerpt 91, Class #8

- Bernardo: Well, they're having a conversation
- Pat: Go ahead and say more
- Bernardo: Jessica's talking with them and they're talking with Jessica to figure it out together.
- Pat: Jessica posed the problem right?
- Bernardo: Yeah, Jessica posed the problem.
- Pat: Yeah, and that in itself is very structuring—"Okay kids whatever you want to talk about doesn't matter, we're going to talk about this". So that really set's the boundaries.

Bernardo: Right.

Pat: So, that's no small thing right there, just selecting what you want to talk about...okay now say more about the conversation. [Quiet for 9 seconds]

Bernardo: Jessica posed the problem, and basically threw up her hands and said what do I do, and then they came up with...whoever it was, Fred I think, came up with rate of change function, and then Jessica started asking why.

Alyce: She probed.

Pat: Yeah, he suggested find the rate of change function.

Bernardo: And then Jessica said, "Well why, why would I want to do that? What will that get me?"

The teachers' comments suggest that they are missing Jessica's purposeful actions (it is not visible to them); that Jessica anticipated, designed, and pushed the conversation so that it went in a particular direction. Just as importantly, the teachers seemed not to consider that the conversation Jessica had with her students was enabled by their prior learning that had happened also by her design. Although the teachers discussed what Jessica did (what they perceived), they did not appear to conceive of Jessica having anticipated and managed the conversation.

Excerpt 92, Class #8

Pat: Now, was that just a random question?

Several: No.

Pat: What might have been Jessica's motive for asking that? [Quiet for 4 seconds]

Annie: Well one to kind of check for understanding, that he's not just throwing something out there.

Pat: All right.

Annie: That he had an idea of what he was talking about.

Annie's comment suggests that she is not seeing how Jessica's questions moved students to start thinking about the rate of change function (i.e., to reason); rather, Annie's comment indicates that she conceived Jessica's questions as a means for checking for understanding.

An empirical orientation toward learning and teaching mathematics, with its egocentric nature and focus on the visible, places an emphasis on getting students to attain "the" understanding to concepts (i.e., how the teacher has conceptualized the mathematics). Such an emphasis makes the role of interpretation (the idiosyncratic understandings of others), and an informed interventionist model of teaching, "hidden" to the teacher. A discussion that occurred during the 10th class meeting will serve to illustrate this point. This particular conversation occurred as the teachers were discussing the video of Bernardo's geometry lesson (prior to reflective episode #2). In particular, the teachers were discussing the portion of the video where the students (in the video) were presenting their solutions to problem #6 of the homework (Appendix C), as described in chapter 5 (p. 162).

Excerpt 93, Class #10

Liz : Okay, these kids are drawing these little lines and saying...that they are taking these perpendicular bisectors of a segment. Do you ever see a segment that they're bisecting?

Marcy: No.

Liz: And if you'd drawn it in there, there's no way his are perpendicular, so how are the kids that have an understanding...that's perpendicular to that. To what?

Bernardo: To what?

Annie: And that's one thing that was brought up in the conversation was, "What is a perpendicular bisector?" Kids have a lot of different meanings for what a perpendicular bisector was, they weren't consistent with what they thought.

Bernardo: No they weren't [...] I thought they knew what it was, but they didn't.

Johnny: Maybe that's just important to find out.

Bernardo: Oh, yeah.

Marcy: How can you have a conversation about it, if the vocabulary you're talking about doesn't have the same...

Annie's comment suggests that she interpreted "a lot of different meanings" as meaning a lot of incorrect understandings—that the students do not have the "correct" understanding (as conceptualized by the teacher). In addition, Bernardo's comment suggests that he was focused on the idea that the students did not know

“the” meaning for a perpendicular bisector was, where “the” meaning was how he defined it himself (which went unarticulated in this discussion). Although Johnny clearly expressed the notion that it might be important for teachers, in general, to be mindful and attempt to ascertain how students are understanding the mathematics during a lesson, his assertion appeared to be ignored—the notion was not on the teachers’ radar.

Next, Bernardo stated that he had recently given the students a quiz involving bisecting line segments, and many of the student had thought that bisector and intersect were synonyms.

Excerpt 94, Class #10

Alyce: I heard a girl later in the discussion say that they were perpendicular and you said, “What’s perpendicular,” and she goes, “They just have to cross.”

Marcy: You see and I have that happen in my class every day, they didn’t get what perpendicular meant.

Bernardo: What’s the solution by the way? [Quiet for five seconds]

Bernardo: What’s the solution...to the problem?

Story: The point of concurrency of the perpendicular bisectors, is it called the orthocenter?

Rather than discussing the ideas of interpretation and being mindful of others’ mathematics, the teachers appeared unconcerned by the discussion, and moved on to the solution to the problem (the product of their reasoning, which Story asserted was simply the “orthocenter”). In fact, Marcy asserted that every day

in her class, students “didn’t get what perpendicular meant,” a fact that she did not appear particularly upset by.

A second discussion will serve to further illustrate my point. During the fourth and fifth class meetings (June 25 and June 27, 2007), the teachers were asked to review several classroom videos from the reformed-Algebra 1 case study. The assignment required that the teachers watch the videos and be prepared to discuss what they had observed in the videos, placing a particular emphasis on the students and the teaching. The videos consisted of the Algebra 1-2 teacher (Jessica) leading her students through the finger tool activities (Appendix A), and engagements designed to build a scheme of meanings involving: quantity, variation, covariation, proportionality, constant rate of change, average rate of change, quantitative relationships, and graphs as emergent aspects of covariation. Early in the videos, Jessica had allowed for meanings and being mindful of students’ mathematics to somewhat fall to the background. This resulted in several student misconceptions as the lessons progressed (e.g., several students focused on the shapes of graphs, rather than on the relationships between the variables).

During the eighth (and final summer) class meeting (July 2nd, 2007), Pat asked the teachers to characterize this sequence of videos that they had watched.

Excerpt 95, Class #8

Alyce: Sometimes despite our best planning, students interpret things differently and we have to go back and try to undo it.

Pat: Umh, okay, and if somebody said, “So, what do you mean by that?”

Alyce: That it appeared that students were able to use the finger tool, and understand the finger tool in the beginning, but they weren't really...their right hand was just tracking time, and then whenever they had to put together both distances, it became a little more complicated, and then they had to rethink the finger tool, and then whenever you include the bungee you had to slow back down and look at the finger tool slowly again, because....sometimes we start with things that are easy, but then they miss the point

Pat: Okay, and you started off, Alyce, by saying despite our best intentions, umh, can you elaborate on that?

Alyce: So, I think that we plan these great lessons that in our mind make sense, but they skip steps that students can't see [...] I'm sure that as you were planning those lessons...the progression made sense and looked like "Oh, well they'll be able to handle this and they'll be able to handle this, and then we'll try this," [...] and then it was growing...but sometimes it doesn't work the same way we think it's going to work.

Alyce's comments suggest that (for Alyce) "students interpret things differently" amounts to students attaining incorrect understandings, that the teacher must then attempt to "undo." In addition, Alyce characterizes her model of lesson design as one sequenced to make sense to the teacher, one that focuses on how the teacher understands the material. Although Alyce did indicate that such

instructional design often misses things that students need to “see,” she does not indicate an awareness of how such knowledge could impact instruction.

A focus on the visible, as I have described here, emphasizes knowledge of the empirical, that is, knowledge that is figural (Piaget, 1976a). Figural knowledge is not operative, and “yields a series of static, centration-afflicted snapshots” (Campbell, 2001, p. 22). A focus on objects (i.e., the things that are perceived or imagined), rather than actions (i.e., reasoning) limits abstractions to those based on the objects themselves. It is worth noting that figural knowledge is not the same as perceptual knowledge. Its main feature, as the quote from Campbell attests, is that it is *static*. Being static, the actions producing figural knowledge are not easily visible to the person having it.

I must take care in my description, because I do not want to claim that the teacher’s were limited in their capacity to abstract, that is, limited to empirical abstractions. There is ample evidence of the teachers’ focus on the products of their reasoning (i.e., pseudo-empirical abstraction), but an empirical orientation as I have described also entails an egocentric focus (a disinclination to decenter). According to Piaget (1962):

A [person] whose perspective is determined by his action has no reason for becoming aware of anything except its results; decentering, on the other hand, i.e., shifting one’s focus and comparing one action with other possible ones, particularly with the actions of other people, leads to an awareness of “how” and to true operations” (p. 9).

Therefore, an empirical orientation toward learning and teaching mathematics, as I have described it, would account for the teachers' focus on their own perceptions, for a lack of focus on meanings and others' mathematics, and for the difficulties that the teachers exhibited in their attempts to imagine instruction designed to make meanings and reasoning explicit, as objects of thought. Such an orientation does not admit reflection, because the teacher's own or someone else's actions (i.e., reasoning) are not visible—reasoning is not on their radar screen. In the next chapter I will attempt to illustrate how such an orientation accounts for the teachers' actions as they attempted to reflect on two distinct activities which the teachers had engaged in as students.

CHAPTER 7

TEACHERS REFLECTING ON ASPECTS OF THEIR PRACTICE (PART II)

Having illustrated the difficulties the teachers displayed in their capacity to reflect during the earlier reflective episodes (chapter 5), identified two epistemic ways of operating that were suggested through my analysis of the data corpus (chapter 6), and proposed an orientation toward learning and teaching mathematics to account for the teachers' ways of operating (chapter 6), the next stage of my analysis involves demonstrating how my proposed empirical orientation exhibited itself during the remaining two reflective episodes.

In this chapter I, will describe and provide my analysis of two reflective episodes, each of which involved the teachers first experiencing a lesson as students and then attempting to re-construct the lesson that they had experienced as instructional designers. I will also attempt to demonstrate how an "empirical orientation" accounts for the teachers' actions in the two reflective episodes.

The chapter will also provide a means to discuss how a teacher's personal understandings of the mathematics they teach influences their capacity to reflect on their pedagogical conceptualizations of that mathematics. More specifically, I will discuss how even with more developed personal understandings (regarding ideas related to speed), the Function 3 teachers' empirical orientation still constrained their capacity to reflect productively.

Reflective Episode #3: Ideas Involving the Concept of Speed

The first reflective episode that I will discuss in this chapter, which occurred during Class #12, was designed to have the teachers act, over the class period, with two hats. The first was with the hat of a student of a lesson involving ideas related to the concept of speed (i.e., an object whose speed is always increasing), identified as Phase 1. The second was with the hat of an instructional designer creating the lesson that they just experienced, identified as Phase 2.

Throughout the earlier (summer) portion of the Functions 3 course, the teachers had engaged in several activities designed to transform their conceptions of constant speed, average speed, and increasing/decreasing speed (Classes #3 and #4). Therefore, although there is insufficient data to determine whether any of the teachers had developed a scheme of meanings (i.e., a KDU) involving ideas related to speed, the teachers (in general) had more developed understandings regarding speed than they did for the second reflective episode (reflective episode #4) that I will describe and analyze in this chapter (which involves data analysis). This fact will serve as a point of contrast between the two episodes and as a means to explore the role that robust personal mathematical understandings (i.e., KDUs) play in the development of powerful pedagogical schemes of meanings.

The reflective episode had two main phases with ten sub-phases in the first and nine sub-phases in the second. A chronological overview of the lesson's phases (Class #12, November 5th, 2007) is given in Table 6.

Table 6

Chronological Overview the Reflective Episode from Class #12

Class Meeting (Date)	Phase	Duration (minutes)
	Phase 1 – Teachers as Students of the Lesson	
Class #12 (11/5/07)	Phase 1.1 – Orienting the Discussion: What Do You Notice?	1
Class #12 (11/5/07)	Phase 1.2 – Bringing the Focus on the Idea of Constant Speed	1.5
Class #12 (11/5/07)	Phase 1.3 – Getting Meanings for Constant Speed on the Table	4.5
Class #12 (11/5/07)	Phase 1.4 – Taking Measurements	6
Class #12 (11/5/07)	Phase 1.5 – Was the Speed Constant?	2
Class #12 (11/5/07)	Phase 1.6 – Orienting the Discussion: What Shall We Mean by “Speeding Up”?	11.5
Class #12 (11/5/07)	Phase 1.7 – Developing a Meaning for “Always Speeding Up”	29.6
Class #12 (11/5/07)	Phase 1.8 – Building Meaning to the Idea of Average Speed	3.7
Class #12 (11/5/07)	Phase 1.9 – Construct a Graph of the Situation	16.5
Class #12 (11/5/07)	Phase 1.10 – Discussing the Teachers’ Methodologies in Constructing the Graphs	24
	Phase 2 – Teachers as Instructional Designers of the Lesson	
Class #12 (11/5/07)	Phase 2.1 – Individual Lesson Re-Construction	13.8
Class #12 (11/5/07)	Phase 2.2 – Group Re-Construction: Clarifying the Task	4.3
Class #12 (11/5/07)	Phase 2.3 – Attempting to Identify a Reason for an Instructional Action	1.5
Class #12 (11/5/07)	Phase 2.4 – A Continued Focus on the Steps	1.8
Class #12 (11/5/07)	Phase 2.5 – Emphasizing the Need to Manage the Conversation	13.5
Class #12 (11/5/07)	Phase 2.6 – Losing Track of the Sequence of Steps	7
Class #12 (11/5/07)	Phase 2.7 – Constructing a Definition for “Always Speeding Up”	7.5
Class #12 (11/5/07)	Phase 2.8 – Constructing and Discussing the Graphical Representations	6
Class #12 (11/5/07)	Phase 2.9 – Lesson Logic Completion and Retrospection	5

Phase 1 – Teachers as students of the lesson. Phase 1 of the lesson

involved the teachers attempting to make sense of a dynamic situation involving an

invariant relationship (i.e., always speeding up) between two covarying quantities (i.e., distance traveled and elapsed time). Throughout Phase 1, the course instructor attempted to motivate the teachers to take their meanings, and those of their fellow teachers, as objects of thought. Specifically, Pat's intent was for the teachers develop a conception for non-constant speed (i.e., a rate of change that changed linearly) that was coherent with their meaning for constant speed.

Phase 1.1 – Orienting the discussion: What do you notice? Pat initiated Phase 1 of the lesson by standing on a chair, dropping a screwdriver, and asking the teachers, "What did you see?"

Excerpt 96, Class #12

Tami: It fall and hit the ground

Pat: It fell and hit the ground, OK. Can you hand it to me? [Tami hands screwdriver back to Pat]

Pat: All right...now watch it again...can you see it hit the ground?

Several: Yes

Pat: So, now tell me again what did you see? [Drops screwdriver]

Tami: It fell, it hit the ground, it bounced.

Pat: It fell, it hit the ground, and it bounced, Okay.

Rachel: And it flipped over.

The teachers initially provide a very concrete observation of what they saw. Pat next moved the teachers to focus on what they observed as the screwdriver "fell." This action anticipated a discussion of the objects speed by focusing the

teachers' observations on the screwdriver "as" it fell. Although Pat did not overtly ask the teachers to comment on the screwdriver's speed, he did manage the conversation in a manner that would motivate such a discussion.

Excerpt 97, Class #12

Pat: What about its' falling?

Rachel: It increased...the speed increased as it was falling.

Johnny: How do you know that?

Story: No.

Pat: Did you see that...are you giving me theoretical knowledge or is that what you saw?

Rachel: No, it did look like is going a little faster.

Johnny: How do you know that?

Rachel: I don't, he said what did you see...it looked like it was...

Johnny: It's falling straight down, it's not taking a curved path

Pat questioned whether Rachel was commenting on what she had seen, or employing understandings regarding falling objects that she already possessed.

Johnny's question indicated that he was not certain whether Rachel was only commenting on what she had observed.

Phase 1.2 – Bringing the focus on the idea of constant speed. Pat next moved the teachers to focus on what they observed regarding the speed of the screwdriver as it fell; specifically, whether the screwdriver had fallen at a constant speed. This instructional action anticipated a discussion of what it would mean for

an object not to be moving at a constant speed. The teachers had previously provided the following working definition for constant speed: the change in distance compare to the change in time stays in the same proportion for any interval chosen (Class #4).

Excerpt 98, Class #12

Pat: Let's look at the speed, all right? [Places screwdriver above head]. So, watch it and tell me what you see in regard to its speed [drops screwdriver].

Pat: Did it look like it fell at a constant speed?

Johnny: Mhm.

Several: Yeah.

Pat: Johnny thinks it looked like it fell at a constant speed.

Johnny: It seems to me that it's constant...but then, again, I'm not making any measurements or anything like that, so it's really hard to say that yes that's what its doing.

Story: It's either negative 32 feet per second, or it's 9.4....

Pat: I asked you what you saw [Story laughs]

Pat: No, no...honest...

Story: No, I don't know cause I've always struggled with that and I've asked people in calculus stuff about this...

Pat's question was intended to move the teachers to respond in a manner consistent with their conception of the situation. Although several of the teachers

based their responses on what they had perceived, Story (and perhaps Rachel) brought in their prior conceptions regarding falling objects. In addition, Story's comment suggests that she had classified the situation (the problem) as being of a specific type, one whose solution she anticipated would involve the acceleration due to gravity of a falling body.

Pat next prepared to drop the screwdriver again and informed the teachers to "look carefully at it in regard to its speed as it falls." After he had dropped the screwdriver, Pat moved the teachers to make explicit their images of what it would mean for the screwdriver to have fallen at a constant speed.

Excerpt 99, Class #12

Pat: All right now, before I ask you did it fall at a constant speed, what would it look like if were it to fall at a constant speed?

Tami: The amount that it falls would be the same for every increment of time [moves her hand slowly down], so if you split a second into ten parts, the amount that it fell would be exactly the same in each increment.

Pat: Okay, all right [Tami hands Pat the screwdriver]

Tami's response suggests that she may have been thinking of constant speed as an object whose component parts (distance traveled and elapsed time) stay in constant proportion. Pat did not push other teachers to provide their meanings for constant speed, but next moved the conversation toward determining a way to test for constant speed.

Phase 1.3 – Getting meanings for constant speed on the table. Pat next moved the teachers to reason about how they could answer the question of whether the screwdriver fell at a constant speed, experimentally, rather than relying on what they “knew” to be true (accelerating due to gravity) or what they visually perceived. When the teachers responded with ideas such as CBR, CBL, and Logger Pro, Pat asserted that he was looking for a means to test for constant speed “right now”. Johnny asserted that they would need a way to time the fall, and Rachel stated that they would need some form of measuring device.

Excerpt 100, Class #12

Pat: I’m talking about right now, how could we do it right now?

Rachel : If you had a tape measure...uh, some measuring device, and looked at a certain thing [spreads hands above and below] and see how long it falls there, and use that same increment [moves hand downward, but with same spread]...you would have to drop it several different times to see how long...how far it went for this distance [hands apart]

Tami: So, we could have somebody like... where you are, and somebody [points arm to lower]...

Pat: So if we had...how many of you have stop...or watches that are...stopwatches, or a calculator that’s a stopwatch? Or a phone that has a stopwatch? [...]

Both Rachel and Tami proposed that the teachers should calculate the time that the screwdriver fell for two distinct distances, divide distance traveled by

elapsed time for each pair of data (to obtain “speeds”), and then compare these two calculated values—these proposals suggest that each teacher was thinking about speed as the object obtained once you have divided distance traveled by elapsed time. This conception differs from how Tami had described constant speed in the previous excerpt (excerpt 99, class #12), suggesting that Tami (and possibly other teachers) might have been reflecting on their conception of constant speed.

Pat next restated Tami’s original explanation for what it would mean were the screwdriver to fall at a constant speed, and asked again how they could test this definition. This action moved the teachers to think about their meanings for constant speed and to construct images of how the screwdriver’s fall could be tested employing this meaning (i.e., what their meaning would look like in practice).

Excerpt 101, Class #12

Rachel: Could you start like with an increment here [hand out], time that. Double that [moves hand up] time it again to see if... then take that up another, whatever that increment is...

Story: Oh, that’s a good idea.

Pat: Ahh, [holds screwdriver above head] so if it falls at a constant rate then ...however much time it takes to fall from this height, it should take half that amount if we made it half as high.

Rachel’s comment suggests that she may have been thinking of constant speed as a multiplicative relationship between distance traveled and the time taken to travel that distance. Such a conception is different from how Rachel had

expressed her understanding in the previous excerpt (excerpt 100, class #12), suggesting that she might have been reflecting. Pat next stated that they were going to take several measurements from two heights, 260 cm and 130 cm (measured with tape measure), that they would then look at the distributions of time, and then make estimates of the speeds.

Phase 1.4 – Taking measurements. The teachers recorded the elapsed times that the screwdriver took to fall from two heights, 260 cm and 130 cm for several trials. The data from these trials is displayed in Table 7.

Table 7

Recorded Times and Heights for the Screwdriver's Fall

Height (cm)	Trial	Rachel's Times (sec)	Liz's Times (sec)	Alyce's Times (sec)
260	1	.7	.6	
260	2	.7	.58	.7
260	3	.7	.6	
260	4	.7	.69	.7
130	1	.5	.39	.5
130	2	.5		.4

After the data had been recorded for the fourth trial at 260 cm, Pat stated that the time looked like it was about .7 seconds. After the data had been recorded from the second trial at 130 cm, Pat stated, "So, even if it's .5 or .4, is it half?"

Excerpt 102, Class #12

Johnny: Nope.

Pat: Is it close to half?

- Tami: It's accelerating
- Johnny: Nuh-uh.
- Pat: Well, what would be the explanation? [...]
- Liz: That it's not traveling at a constant speed [...] It's greater than .35, so
anyway it's not traveling at a constant speed
- Pat: Okay, but how could we argue from this that it's accelerating?
- Rachel: Because it has to cover...
- Story: Because the first 130 centimeters is 0.5 or 0.4 and then it's hitting at .7, so
the last 130 should be ...
- Liz: Half that time.

The teachers' comments suggest that they might be thinking of constant speed as a multiplicative relationship between distance traveled and the time taken to travel that distance.

Phase 1.5 – Was the speed constant? Pat next asserted that it was the acceleration that explained why it had taken longer to cover the first 130 cm than it had taken to cover the second 130 cm.

Excerpt 103, Class #12

- Pat: So, if it takes, if it takes longer to cover the same distance, what does that
tell you?
- Tami: It's going slower

Pat: It's going slower. So, it's going slow in the first hundred...now is it...Okay, now, let me come back to this. It's going slower...when we say it's going slower in the first half, than it is in the second half...

Sheila asked Pat how it was that from that data they could say that in the first 130 centimeters the screwdriver was traveling slower than in the second half, because they had not measured the top half. Pat asserted that the measurements that they had taken from 130 cm, in which they obtained an average time of about .5 second, was actually measuring the top half of the 260 cm fall, which had an average time of .7 seconds (Figure 18).



Figure 18. Diagram displaying top and lower halves of screwdriver's fall.

Pat next moved the conversation back to whether the speed was constant, and stated that the top half of the fall had taken .5 seconds, and the bottom half of

the screwdriver's fall had taken .2 seconds; and that, therefore, based on this information, the screwdriver was going faster in the second half of its fall.

Phase 1.6 – Orienting the discussion: What shall we mean by “speeding up”? Pat next asked the teachers to determine the screwdriver's average speed for the fall of 260 cm, then noticed that he was moving ahead of himself (skipping a part of the lesson's logic). Pat subsequently moved the conversation toward developing a meaning for “speeding up.”

Excerpt 104, Class #12

Pat: All right, what does that mean, that it was speeding up?

Rachel: At the same increment...it would...it covered.

Faye: In the same amount of time it covered more distance.

Rachel: Greater distance in a shorter period of time.

Pat: I'm sorry, say that again Rachel.

Rachel: For each increment of the same size, it covered that distance at a ...in less time.

Both Rachel and Faye's comment suggest that they are not thinking about a direct proportional relationship between distance traveled and amount of time required to travel that distance (i.e., scaling one trip into the other). Rachel and Tami, in particular, had exhibited conflicting conceptions of speed throughout the episode, suggesting that they had not developed a robust understanding of constant speed as a direct proportional relationship between distance traveled and amount of time required to travel that distance. In addition, these comments suggest that the

teachers were focused on demonstrating that it did speed up (obtaining the outcome), rather than thinking about a meaning for “speeding up.” Pat continued to move the teachers to make their meaning for “the object was always speeding up” explicit.

Excerpt 105, Class #12

Pat: Okay...so, it went a constant speed in each increment?

Rachel: No.

Tami: But for each increment of time, it covered more distance.

Pat: Okay...so it went at a constant speed during each increment of time?

Tami: No, it's accelerating.

Pat: Okay, but say we look at one-tenths ...so, does that mean it's going a constant speed in each tenth of a second?

Several: No.

Liz: But the average of that.

Tami: It's going more distance, the more time that passes. Each increment of time, each, is umh...

Tami's comments indicated a need to compare intervals of time—she had allowed for time to fall into the background, she was not covarying. Pat attempted to focus the teachers' attention on whether this implied that the speed was constant in each interval; a point, which perturbed the teachers' conceptions of “always speeding up,” as they attempted to coordinate their conceptions of constant speed with their images of an object that whose speed was increasing. Liz indicated that

they would be comparing average speeds for two different intervals (although she provided no meaning for average speed), but Tami appeared to focus exclusively on comparing distances along the trip. In addition, not one of the teachers indicated that they recalled working on a meaning for “faster,” or even constant speed, during the summer portion of the Functions 3 course (Classes #3 and #4).

Pat next requested that the teachers develop and present (in groups) a meaning for “it was always speeding up.” The groups consisted of: Story with Adriana; Sheila with Johnny; Faye, with Liz, and Marian; Janine and Alyce; and Tami and Rachel. Prior to working in their groups, Johnny asked whether they were only to use the information that they had obtained from the experiment, or if they could use their prior knowledge regarding falling objects. Pat asserted that they were to only use their “intuition that it always sped up.”

The teachers worked in groups for the next 5.5 minutes. During this time it was not possible to hear the comments of any specific group.

Phase 1.7 – Developing a meaning for “always speeding up.” After the groups had finished developing their statements, Pat asked that a representative from each group write their group’s meaning on the whiteboard. The group statements are listed in Table 8.

Table 8

Group Elaborations on the Phrase “It was Always Speeding Up”

Group	Statement
Story and Adriana	For equal increments of time, the distance traveled is greater for each consecutive increment
Sheila with Johnny Faye, Liz, and Marian	In traveling equal increments of distance, less time is required For every increment of time, the Rate of Change between the distance covered related to time elapsed is increasing
Janine and Alyce	For every increment of time, the distance traveled in that increment of time is greater than the distance traveled in the previous increment of time
Tami and Rachel	As time elapsed the distance covered increased at an increasing rate

As illustrated in Table 8, each group either focused their attention on either comparing distances (for equal increments of time elapsed) or comparing times (for equal increments of distances traveled)—the groups were not focused on covarying quantities (they had allowed one quantity to fall into the background). Such conceptions focus on the comparison and on obtaining a result of the comparison, a focus on the objects (the two “speeds”), the action (“comparing”) on those objects, and the result of the comparison (the product of reasoning). In addition, none of the groups based their definition on a meaning for speed (that is a meaning for “always speeding up” that cohered with their meaning for speed) and that allowed for comparisons of any increment of distance traveled and any increment of time elapsed.

After each group had written their statement on the board, Pat move the conversation toward comparing and contrasting the statements, and on whether the statements were missing anything.

Excerpt 106, Class #12

Pat: All right, now [...] look at these...do they all say the same thing?

Sheila: No.

Liz: I don't think so.

Story: They intended to, but...

Story: They need that equal increments included in the definition.

Pat: Story?

Story: Well, it has to have equal increments either of the distance or the time

Pat: For every increment of time [pointing to the definition given by Faye, Liz, and Marian], so .3 seconds and .7 seconds are two increments of time...

Story: But they are...

Pat: Okay

Story: So that's a necessary part [...]

Pat: So, we need to say something about that all the increments are the same length, right? [Writes "of a given size" to the statements provided by Faye, Liz, and Marian, and Janine and Alyce, which now read "For every increment of time of a given size,..."] [...]

Pat: We need to be talking about equal increments if we are going to make comparisons [...]

Story asserted that in order for the teachers to make comparisons between the speeds, that they needed to be comparing intervals of equal distance or of equal time (indicating that she was not covarying and that she was focused on making the comparison). This suggests that Story was not thinking about scaling one trip into the other; she was not thinking about a direct proportional relationship between distance traveled and amount of time required to travel that distance. Pat allowed for Story's assertion to move the direction of the conversation.

Excerpt 107, Class #12

Pat: Let's see, there's one up here that doesn't talk about equal increments of time. The third one.

Johnny: That's us

Alyce: Equal increments of distance

Pat: Equal increments of distance

Johnny: We broke the experiment up such that the distances were the same...then we referred to, well, ...then, how would the time be? [...]

Pat: Okay, yeah, umh...We were just asking what do you mean by "it was always speeding up". So the...the question is, is that adequate to capture what it means to be always be speeding up?

Story: Yeah, mhm [...]

Alyce: Well, I think that's good, because ours started...Janine and I started out with a definition similar to the very last one, but then we decided that we

needed to specify more about the distance, because of course the distance traveled is going to be greater because more time has passed.

Janine: Right.

Alyce: So, you need to talk about the distance traveled within the increment...so that's how ours got more wordy.

Pat: Yours is this one Alyce? [Points to statement on whiteboard]

Alyce: The second one, yeah.

Johnny and Sheila focused their comparison on needing equal intervals of distance and comparing the time elapsed in each interval. Pat next moved the discussion to challenge the teachers' conceptions of "equal increments" and how he believed that they were imagining the time (or distance) as being partitioned into equal increments.

Excerpt 108, Class #12

Pat: Okay, ...now...It looks like everyone is imagining that if we were to talk about this being [draws line] from 0 to 0.7 seconds, that we're talking about increments of time like this [draws tick marks], say that are each say... point zero seven... would that be if they were all cut up into each size of seven-hundredths of a second, then we cut it up into ten slices, okay (Figure 19)? [...]

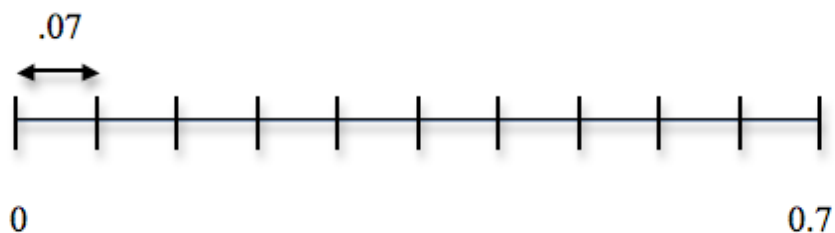


Figure 19. Time-line partitioned into ten equal segments.

Pat: Now, what about from [draws second line segment on board], what about from .009 to .104? From nine one-thousandths to 10.4 one-thousandths? Do any of those....can we talk about intervals of time like that (Figure 20)?

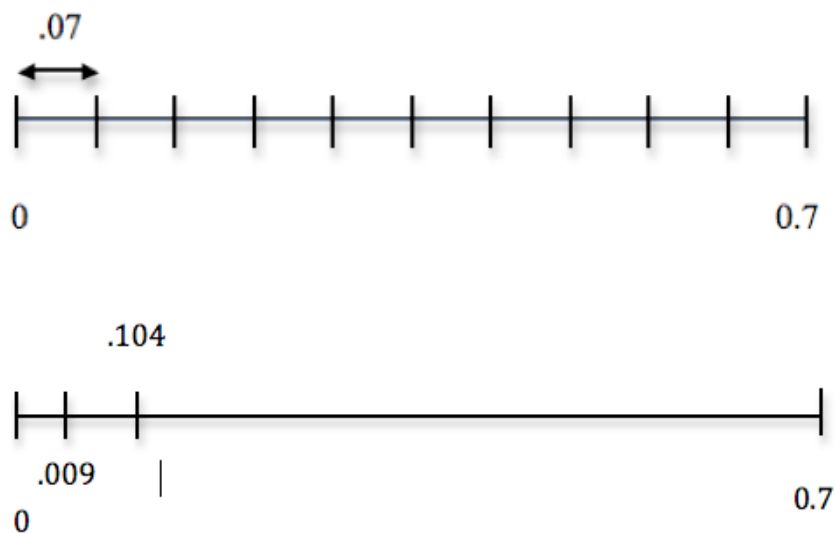


Figure 20. Time-segment overlapping equal partitions.

Alyce: It depends on what's being represented on your...is your scale all time?

Pat: Mhm.

Alyce: So, would that be representative of the distance?

- Pat: I'm just asking...
- Liz: Could we break it up like that?
- Pat: Suppose that you broke it up into ...increments of seven one-hundredths of a second, then what does that...does that tell you anything about this interval here that's five one-thousandths of a second?
- Alyce: That could be the amount of time that it takes an object that is slowing down to travel an equal increment of distance, and it takes a total of .7 seconds to do the whole trip.
- Pat: Okay...what issue do you hear me raising when I bring, when I ask about this, this funny interval that...
- Tami: For any interval, for any random interval of time.
- Pat: If we, okay, for every increment of time for a given size, for every increment of time for a given size [pointing to board]
- Rachel: We can take, any increment we can take it and we can make it smaller and smaller...
- Pat: Right.
- Rachel: So, we're looking at what happens in that increment...
- Story: So, it's the ratios of those, is that what you're saying? [...]

The teachers appeared to be unclear what point Pat was trying to bring up. Although Pat attempted to get the teachers to focus on a meaning that would work for any increment of time (similar to the discussion that they had during Classes #3

and #4) and a means to determine in which of two intervals the speed would be greater, the teachers appeared uncertain as to the issue that Pat was raising.

Excerpt 109, Class #12

Pat: So... suppose that we're...all right, let me come up here. Suppose that we say that the increment of time is .07, so we go...here are the two that we want to compare. The first one is 0.07 and the second one is...it's from 0 to 0.07, the second interval goes like this from... 0.01 to 0.08, in other words, those two intervals overlap...can we compare those intervals by any of those meanings?

Liz: I think so [...]

Rachel: But if you want to say that they need to be sort of...because you don't want overlapping intervals.

Pat: Well, suppose they are overlapping.

Alyce: That's okay, because you're still talking about the same equal increment...same equal increment of time.

Pat: In which one should more distance be traveled?

Alyce: The .01 to 0.08.

Alyce's comments suggests that she understood the issue that Pat was raising as needing to be able to state which of two intervals would have the greater speed; she is focused on answering the question, not on thinking about a way to describe which interval to choose in general. Pat next pushed the teachers to think of a way to describe meaningfully which of two intervals would have the greater speed.

Excerpt 110, Class #12

Pat: Why?

Alyce: Because it is further on in the trip.

Pat: Okay, so of ...okay, so for every increment of time of a given size...

Liz: And that's where theirs has the previous time.

Pat: Okay, let's go with this one [gestures to Janine and Alyce's statement] that just talks about distance and time, so that we're not using rate of change to describe, to define what we mean by rate of change. Um, for every increment of time of a given size, the distance traveled in that increment of time is greater than the distance traveled in the previous...so, the previous increment of time [underlines statement], is there a previous increment of time here between these two?

Several: No.

Liz: Distance traveled in the ...

Rachel: Previously...just put.... instead of increment, put previously

Pat: But, how was it previously if they overlap?

Liz: It's not previous it began at...

Rachel: But it is previous, ... the increment may not be, but the start point...

Pat: Oh yeah, but you know what? Two-tenths...two one-hundredths of a second into that they're both moving.

Rachel: Right.

Pat: So, how is that previous?

- Liz: It would have to begin ...
- Pat: See, previous means before, right?
- Pat: So, can we fix this, Can we... by...we have equal increments of time.
- Liz: It would have to be on the interval of time that began first.
- Pat: So, the one with the later starting time...
- Marian: Has a greater distance.
- Pat: Will have a greater distance, okay?
- Pat: So, for every increment [moves to board, points to words while stating] of time of a given size the distance traveled in that increment of time, umh...is greater than the distance traveled in ...
- Liz: The interval of time that began first [...]
- Pat: Any increment, any same sized increment [crosses out "the distance traveled in the previous increment of time"]
- Story: Beginning sooner?
- Pat: That has an earlier start time.
- Story: Ah, okay.

Pat wrote the new definition on the whiteboard: For every increment of time of a given size, the distance traveled in that increment of time is greater than any same sized increment that has an earlier start time.

Throughout the preceding discussion, Pat made a point to emphasize that in order to compare speeds (and give meaning to the idea that the screwdriver was always speeding up) that the teachers: should look at intervals of time "of a given

size”; needed to think about whether the screwdriver’s speed would be constant within any given interval of time; and needed a meaningful way to determine in which of two intervals the speed would be greater.

Pat next began to challenge the idea of whether their definition worked only for time increments of a given size, or for increments of distance of a given length. Pat asked how they could say their definition more mathematically, to which the teachers responded with uncertainty to what Pat meant by “more mathematically.” Pat asserted that he was looking for the teachers to state more precisely what they meant by the quantity that they called a speed, anticipating a need to cohere their meaning for “always speeding up” with their meaning for speed. Pat, having identified the tenuous and fragile nature of the teachers’ conception of speed (as a direct proportional relationship between distance traveled and amount of time required to travel that distance), decided to move the discussion to one involving the teachers’ conceptions of average speed, and stated that they would come back to this issue of constructing a “more mathematical” definition later.

Phase 1.8 – Building meaning to the idea of average speed. Pat next moved the teachers to reflect on the coherence between their conceptions of average speed and the fact that the screwdriver was always speeding up. From the information that the screwdriver fell 260 cm in 0.7 second, the teachers determined that the average speed of the screwdriver was 371.4 cm/s.

Excerpt 111, Class #12

Pat: So, does that mean that we traveled 371 centimeters?

Johnny: Nope.

Pat: Does that mean that we traveled for a second?

Johnny: Nope, that just means that...

Rachel: Within a second, that's how much time...that's how far you'd go.

Rachel comment suggests that she was not thinking about the fact that the screwdriver was always speeding up; rather, she was thinking that the average speed represent the screwdriver's actual speed (that it would have fallen 371 cm in one second). Pat attempted to get the teachers to coordinate their meaning for average speed with the fact that the screwdriver was always speeding up.

Excerpt 112, Class #12

Pat: Within a second...in a second...so, if we had let that thing drop for a second, then it would have gone 371 centimeters?

Johnny: Nope.

Tami: No, it would have gone farther.

Liz: No, if you're dealing with average.

Rachel: That's just an average speed, if it went at a constant speed...

Pat: Okay, if it went at a constant speed...what?

Sheila: If our object traveled at a constant speed of 371.4 centimeters per second for point 7 seconds it would have traveled the 260 centimeters.

Pat next moved the teachers to reflect on their meaning for average speed, by coordinating images of the fall of the actual screwdriver, with that of a fictitious

screwdriver falling at a constant speed of 371.4 cm/s (the calculated average speed of the screwdriver).

Excerpt 113, Class #12

Pat: Ah, okay, so...it would have been like if we had two things dropping, one was the screwdriver, the other was some ball. So, if the ball dropped at ... a constant rate at that [points to board] speed, then what would we have seen happening?

Story: They would have hit the ground at the same time.

Pat: Yeah, but what would we have seen in between?

Story: They'd be level the whole way down.

Liz: This one, would fall faster than that one

Story's comment suggests that she was imagining that the screwdriver (with a speed that was always increasing) would have been level with the ball (traveling at a constant speed) all the way down—that the average speed was the screwdriver's actual speed throughout the fall (this suggests that she was having difficulty coordinating her meanings for average speed and constant speed with her image of "always speeding up").

Excerpt 114, Class #12

Pat: Story said they'd be even the whole way down

Several: No

Rachel: That's not true

Liz: The one falling at the average speed would start out as being ahead and the other one would eventually catch it, until they hit at the same time

Pat: Ah, okay, so...here's the constant speed one it would be going down like this [moves arm smoothly down] at a constant speed. This one would start out slower, but then [moves arm down in increasing manner] would then...

Liz: Would catch it when they hit.

Pat: And they would then hit simultaneously, okay.

As illustrated in the preceding excerpt, the teachers concluded that the fictitious screwdriver would start out ahead of the actual screwdriver, but that the actual screwdriver would gradually catch up to the fictitious screwdriver, until they hit at the same time. Pat next moved to the next activity, which involved the teachers (individually) constructing graphs of the situation. This action anticipated a discussion of non-linear functions (e.g., quadratic function).

Phase 1.9 – Construct a graph of the situation. Pat asserted that the screwdriver had dropped 260 cm in 0.7 of a second, and was always speeding up. He then requested that the teachers construct a graph, given that information. Pat distributed graph paper to each of the tables and stated that the graph should display the screwdriver's distance from the ground in relation to the elapsed time since the screwdriver had been dropped.

As the teachers began to construct their graphs, Pat re-stated the specific task. Specifically, that the teachers were to sketch a graph of the screwdriver's

distance from the ground in relation to the elapsed time since it was dropped so that their graph satisfied three constraints: (1) that the screwdriver's initial height was 260 cm, (2) that it took 0.7 second to drop that height, and (3) that the screwdriver was always speeding up.

As the teachers worked on constructing their graphs, Pat circulated around the room, clarifying the task as needed. After four minutes, Pat stated that one table (Liz, Faye, and Marian) had simply sketched a curve, and that they did not have the curve pass through any actual points—suggesting that the teachers had anticipated the task as one involving the construction of a parabola (they were focused on the outcome, rather than their meanings and reasoning).

Pat next reiterated to the class, that they could use the data points that they had gathered in the experiment. After 9.5 minutes, Pat asserted that the majority of the teachers were not looking at any increments of time; that they were thinking about a start time, an end time, that the graph must be curved, and simply drawing a curve (i.e., a parabola) that passed through the points (0, 260) and (130, 0). Pat asserted that the teachers were not making their graph reflect the information in their statement for “it was always speeding up”—this again suggests a focus on the anticipated outcome of the task. This suggests that the teachers had assimilated the task as one involving the concept (i.e., object) “quadratic function,” and an associated “skill set” (i.e., action) performed on that object, where the action involved constructing a graph of a quadratic function representing the contextual situation.

Pat stated that he believed the teachers had put on their “teacher hat,” and had drawn the graph that they knew to be true. This suggests that Pat believed that the teachers were focused on the outcome of the activity (obtaining the graph), and were not reasoning about the situation—they were not coordinating their actions (i.e., meanings) to construct the graph.

Pat distributed additional graph paper to the teachers and requested that they start over and to construct the graph using the given information (i.e., to look at increments of time and distance traveled in it). Pat discussed Alyce’s method with her and stated that he had noticed that Alyce was one of the few teachers who had actually tried to use the given information to construct the graph.

After all of the teachers had completed their graphs, Pat collected the graphs, and while the teachers went on a short break, scanned the graphs so that they could be used as objects for discussion.

Phase 1.10 – Discussing the teachers’ methodologies in constructing the graphs. After the break, Pat next displayed several of the graphs on the overhead (see Figure 21) and asked each teacher to discuss their reasoning during the graph’s construction. This provided an opportunity for the teachers to reflect on their reasoning and to think about each other’s reasoning.

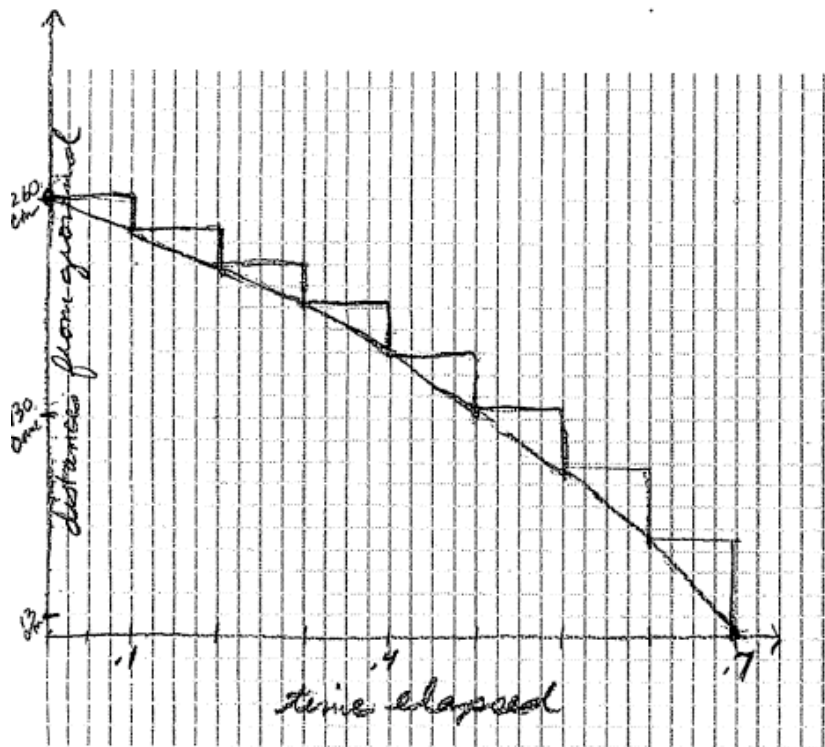


Figure 21. Sheila's distance traveled versus elapsed time graph.

Pat asked what it would mean for a graph to reasonably depict the situation. The teachers concluded that the graph needed to start 260 centimeters off of the ground, that it needed to hit the ground at seven-tenths of a second, and that at five-tenths of a second it needed to be 130 cm off of the ground. Pat then asked whether there were any other constraints, an action that he intended to move the teachers to make explicit how they had reasoned about the statement that, "For every increment of time of a given size, the distance traveled in that increment of time is greater than any same sized increment that has an earlier start time."

Excerpt 115, Class #12

Pat: Is there anything else?

Rachel: Maybe to check to see that within each...depending on how many...however many squares your going over...however many increments you're increasing each time...that you're decreasing [...] So, for every three squares that they're increasing, how many squares down they're going each time.

Rachel's comment suggests that she was thinking about the need to account for the screwdriver's increasing speed, but that she was uncertain as to how to achieve this (either by increasing or decreasing the distance traveled) given that she needed to use set increments of time—this suggests that she was having difficulty coordinating her meanings, or at least that she was having difficulty thinking about a way to articulate those coordinations. During the discussion of the teacher's graphs, Pat made a point to emphasize the fact that several of the teacher's were naturally inclined to make the increases increase linearly (i.e., made the rate of change increase linearly). The following excerpt illustrates this point.

Excerpt 116, Class #12

Pat: Sheila, tell us a little bit about yours (Figure 21).

Sheila: I did it similarly to Janine...umh, I didn't measure it out, I didn't make a conscious choice of how much greater each time, it was just...I just made sure that my vertical distance, that's shown there, in each of those triangles was a little bit longer each time.

- Pat: So, do you think that you changed yours linearly as well?
- Sheila: Based on previous knowledge, probably I'm pretty close to, yes
- Pat: Umh, I mean was that...was that part of your thinking?
- Sheila: No, I eyeballed it
- Pat: Okay, you eyeballed it...but it would be interesting to see if, in eyeballing it, that this linear increase in the rate of change is somehow natural.

Later, as they discussed Liz's graph, Pat continued to promote the idea of making the rate of change linear as a way to systematically keep track of the situation.

Excerpt 117, Class #12

- Pat: It looks like your increments are pretty close to each other...so, this business of increasing so that the increase changes linearly, seems to be pretty natural doesn't it?
- Johnny: Mhm [...]
- Pat: It certainly allows you to systematize...you know, you make an initial approximation and then you know how to change it after that [...] you had to make a decision somehow, and so the question was how...it looks like what the first three did just intuitively was to make the increases increase linearly.

Next, Johnny questioned Pat as to whether they were making the assumption that the acceleration was constant. Pat asserted that they had not stated that the acceleration was constant, but that the big idea was that they had provided an

intuitive solution to a differential equation. Pat asserted that they were looking for a function with the following boundary conditions: it had a value of 260, when the argument was zero, it had a value of zero, when the argument was point 7, and it had a rate of change that was always speeding up. Pat stated that although there were several functions that satisfied those constraints, the teachers had intuitively chosen to make the rate of change increase linearly.

Pat concluded Phase 1 by emphasizing the idea that the activity focused the teachers' attention on the two covarying quantities (distance traveled and elapsed time) and the invariant relationship between them (always speeding up and a rate of change that changed linearly). In addition, the activity was designed such that meanings emerged from the discussions of the situation, and allowed for a natural introduction to functions with non-linear rates of change (i.e., a quadratic function), rather than emphasizing a conception of a quadratic function as one whose graph is a parabola.

Phase 1 – Summary. Throughout Phase 1, the teachers demonstrated the fragile nature of the meanings that they had developed during the earlier Functions 3 class meetings (i.e., Classes #4 and #4). The teachers appeared constrained in their capacity to recall the meanings and understandings that they had worked to transform—they gave no indication that they recalled developing meanings for increasing speed and constant speed in the earlier class meetings.

In addition, throughout Phase 1, the teachers consistently focused on the products of their reasoning, rather than their reasoning—their reasoning was

hidden in their products. Finally, the teachers were focused on their perceptions, their experiences in the lesson (demonstrative of an egocentric orientation to learning), rather than their meanings and the need to have meanings make sense (that ideas and meanings needed to develop and cohere)—they were disinclined to take their meanings, their reasoning, as objects of thought.

Phase 2 – Teachers as instructional designers of the lesson. The second Phase of the lesson had the teachers attempt to reconstruct the logic of the lesson. Neither the individual nor the group lesson reconstructions were in a Lesson Logic format (p. 499); rather, Pat requested that the teachers create an outline of the conversation that unfolded and the ideas that ended up coming out and being discussed.

Phase 2.1 – Individual lesson re-construction. During the first part of the lesson re-construction, the teachers were asked to individually answer the question, “How did we get from standing on a chair dropping a screwdriver, to discussing these graphs” (i.e., reconstruct the logic of the lesson). Pat requested that the teachers write out their re-constructions and submit these to him when they were completed.

The teacher’s individual lesson logics varied in detail from listing very specific concrete actions (Figure 22) to describing the instructional actions at length (Appendix F), but without indicating a logic behind the actions.

- Chain - Visualize
- Measurable : how (is it speeding up, how can you tell?)
- ~~measurements~~ Suggestions on how to measure
- measurement
- Discuss Increasing speed, what does it mean, how can you tell?
- Make graph using what we know
- Discuss how students make graphs
- Different methods, different units

Figure 22. Liz's logic for "Always Speeding Up" lesson.

All of the teachers' lesson logics consisted of outlining the actions that the teacher would take, which more often than not entailed getting the students to "do" something. Out of the nine lesson logics that were submitted, only four of these contained any reasons for the instructional actions that the teacher would be make; these reasons did not focus on the understandings or ways of thinking that they intended for students to develop (e.g., "perfect what it means to say 'it was always speeding up'"), nor did they suggest that the teachers were thinking about a logic for the lesson. In general, the logics focused on outcomes (e.g., have students answer the question or make a graph). In addition, the teachers' descriptions of instructional actions (and reasons) appeared to be very empirically based, focusing on what they had perceived, not on how they envisioned someone else coming to develop similar ways of thinking.

Phase 2.2 – Group re-construction: Clarifying the task. After the teachers developed their individual lesson logic reconstructions, they created a group reconstruction. Faye volunteered to be the group’s recorder and typed the teachers’ logic into a Word document that was projected on the overhead display for everyone to see.

Faye asked for Marian to provide the first instructional action; Marian stated, “I go with standing on a chair dropping a screwdriver.” Faye next asked Pat if they needed to add the justification for each action, to which Pat responded that it was “not so much justifications as explanations.” That if the teachers believed that somebody (other than themselves) would need an explanation in order get a sense for why they were standing on a chair and dropping a screwdriver, that they should then include a reason. This comment indicated the need for the teachers to focus on the lesson’s logic.

Excerpt 118, Class #12

Adriana: Okay, stand on a chair and drop a screwdriver and ask students [...]
asking students, umh, what they see happening

Teacher: Or what do you observe? [...]

Johnny: As the screwdriver dropped [...]

Teacher: The way it drops, or...?

Faye: Okay...anything else for step 1?

Rachel: I don’t know if you need to put it, but he did it several times and just asking us to continue to look and see...I mean [...] when we say, “Well,

when this happens,” like “are you sure you saw that or is that prior knowledge?” I don’t know how you want to put that?

Johnny: Multiple observations.

Faye: Flesh out all possible...umh...

Rachel: But having us go back and seeing...

Faye: To what Marian? [...]

Liz: Can you see it? [...]

Faye: Questioning students’ observations or something? Carefully questioning...

Faye has typed:

Step 1

- Stand on a chair and drop a screwdriver asking students what they observe as the screwdriver drops, the way it drops

The teachers’ comments suggest a focus on their participation in the lesson, not on the motivations behind the instructional actions—not on the how the teachers’ actions (including their questions to students) intended for students to reason in specific ways.

Phase 2.3 – Attempting to identify a reason for an instructional action.

Next, Sheila moved the discussion toward an attempt to identify a reason for the instructional action of “asking students what they observe.”

Excerpt 119, Class #12

Sheila: Would you have to keep asking until you got out the idea that you wanted? We talked about the actual dropping, yeah, it hit the ground and it bounced, but you're looking...to start a discussion on speed, speed has to come out of that observation somehow.

Faye added the action: "Repeat several times and carefully questioning students observations, keep asking until you get the discussion on speed that you want."

Excerpt 120, Class #12

Faye: Okay?

Liz: I don't think you need to put the response that you want.

Faye: Okay [deletes "that you want"]

Adriana: So, whether or not it was constant [Faye adds: "on speed, whether or not speed was constant"]

Faye has typed:

Step 1

- Stand on a chair and drop a screwdriver
 - asking students what they observe as the screwdriver drops, the way it drops
 - repeat several times and carefully questioning students observations, keep asking until you get the discussion on speed, whether or not speed was constant

Although the modification of the phrase “keep asking until you get the discussion on speed that you want” to the phrase “keep asking until you get the discussion on speed, whether or not speed was constant” does highlight the focus of the conversation, it does not make explicit how the teachers imagine this focus being of benefit to the students.

Phase 2.4 – A continued focus on the steps. As the teachers continued with the lesson re-construction, they maintained their focus solely on the steps that had occurred in the lesson (the steps that occurred as they participated in the lesson), rather than attempting to think about the lesson’s logic. This focus on what they had experienced constrained their capacity to think about how they intended for students to think (were they to lead the lesson). In the preceding and subsequent excerpt, the teachers moved from getting students to focus on constant speed to asking them how they could prove the speed was constant, without questioning whether that action made sense (in terms of the lesson’s logic) with the ideas and meanings that they intended the lesson to develop.

Excerpt 121, Class #12

Adriana: Ask them how we could prove...what would be a good method if we wanted to prove that it was falling at a constant rate [...] or just prove that it was falling at a constant rate

Sheila: And how to prove and measure that.

Liz: I think that brought out measurements and that led to the discussion of how could we measure finding that.

- Faye: Should we start a new...do you want to keep running this under...the same action...Okay?
- Faye: So, what was that Liz?
- Liz: Well, I think, that led to...like how we were saying could we find measurements came out...well, how could we measure to show that?
- Sheila: I would suggest changing the word measure to determine, or something along the lines, cause nothing...nothing was led as far as let's measure that, that came from us.
- Liz: But with the discussion, I said we talked about and we said let's measure...yeah, could we prove it, "well if we took the time or recorded the time and things...and so we brought it up, but that...
- Johnny: Yeah, but I think that determine.
- Sheila: So, what are we writing? What we said, what we did, or what... how the lesson progressed?
- Faye: Should we put it like in parentheses? We...we know...bring it up...
- Liz: But with the discussion, I said we talked about and we said let's measure...yeah, could we prove it, "well if we took the time or recorded the time and things...and so we brought it up, but that...

The preceding excerpt highlights the teachers' apparent unawareness of the role that managing a conversation played in the lesson they experienced as students. By "managing the conversation," I mean specifically the way that Pat repeatedly inserted the issue of prior meanings of constant speed that they had

developed and the bearing that those meanings had for deciding whether the screwdriver dropped at a constant speed. By “managing the conversation,” I also mean the way that Pat explicitly forced teachers to take “what they saw” seriously and to test whether what they saw jibed with what they supposedly knew.

Regarding the act of “measuring,” Sheila asserted that, “Nothing was led as far as let’s measure that, that came from us,” and Liz stated that, “We brought it up.” This suggests that a focus on their participation in the lesson constrained their capacity to “see” how Pat managed the conversation, to promote specific ways of reasoning—the focus was on getting to the measurements (the visible), rather than the reasoning.

At this stage in the lesson re-construction, Faye had typed:

Step 1

- Stand on a chair and drop a screwdriver
 - asking students what they observe as the screwdriver drops, the way it drops
 - repeat several times and carefully questioning students observations, keep asking until you get the discussion on speed, whether or not speed was constant
 - asking how we could prove what would be a good method if we wanted to prove it was falling at a constant rate
 - how could we determine the situation to show that the screwdriver was not falling at a constant rate

As illustrated in the teachers' logic (Step 1), and the accompanying discussion to this point, the teachers had not considered the lesson's logic in their re-construction, but had focused exclusively on outcomes and their own perceptions.

Phase 2.5 – Emphasizing the need to manage the conversation. At this point in the lesson re-construction, Pat intervened to point out that (as the teacher) they needed to discuss how to manage the conversation in such a way that the ideas that they intend to come forward (for the benefit of the student) actually come forward.

Throughout the Functions courses, the instructor (Pat) served as a model of what it means to manage conversations so that the ideas are developed as the students (i.e., the participating teachers) engaged with instruction; whereby the students (i.e., the participating teachers) were encouraged to take their mental operations, their reasoning, as objects of thought. Throughout the lesson logic reconstruction, the teachers appeared to be unaware of how Pat managed the conversation so that the teachers were prompted to reason (or even that they were participants in a managed conversation), as illustrated in the following excerpt.

Excerpt 122, Class #12

Pat: You're not remembering how the conversation was managed so that those things happened [...] My point was that...if you want this...if that's important to you, that you manage the conversation in such a way that they come up with "well you need to know how far you're dropping it,

you need to know how much time it's taking," then just make that as a note [...] that you want to manage the conversation so that those ideas come forward.

Adriana: So, just stating you want to manage the conversation?

Pat: Yeah.

Adriana: Okay [...] Encourage the ideas... [Faye adds "want to manage the conversation to encourage students bring up measurements"]

Pat: Now, how did...do you recall...do you recall how that came about? When someone said, "Well, measure how high it is, measure distance and time, and then measure it halfway."

Adriana: And Liz said she could run out and get her tape measure, but you already had one.

Pat: Right, so what...but there was a question that drove that, do you remember the question?

Several: How...

Adriana: Was it a constant speed? [...]

Pat: How would you test rather or not it's dropping at a constant speed?

Liz: Well I think that's...when we came up with "how can we determine"
[Faye types "ask students how to"]

As the preceding excerpt illustrates, the teachers continued to focus on their participation in the lesson; they were disinclined to think about how they (as teachers) could manage the conversation so that specific issues would be raised. In

addition, the teachers added the phrase “manage the conversation” without any discussion of what that meant or would look like were they to lead the lesson.

Although the teachers modified some of the wording in their logic (e.g., change “how can we determine” to “how could we test”) taking into account Pat’s suggestion, they moved back to re-constructing a lesson that was devoid of logic.

In addition, although the teachers were proficient at identifying how one outcome (i.e., product of their reasoning) led to the next outcome, they were unable to re-construct how ideas were developed or meanings were built—they were unable to re-construct the lesson’s logic. The following excerpt, which occurred after the teachers (with much prodding from Pat) had stated that they would question their students until the discussion moved to whether or not the speed was constant, illustrates this point.

Excerpt 123, Class #12

Liz: How could we prove or how could we test for the same.

Adriana: How could we prove or disprove.

Liz: Yeah...

Adriana: That the speed is constant [...]

Rachel: And then we actually did it, we actually measured...the second step where we started actually measuring...

Faye: Step 2? [...]

Pat continued to attempt to focus the teachers’ attention on imagining how they would manage the conversation so that ideas would come forward.

Excerpt 124, Class #12

Pat: Right now “how can you test” is just one of those bunch of things that you ask, right? No more important than anything else.

Pat: But “how can you test?” whether...

Liz: Just write Step 2 in there.

Pat: But “how can you test?” ends up being the main idea...that, you know, managing the conversation will bring up measurement [...]

As the teachers continued to modify the logic, their focus stayed on what they had done as students—the teachers appeared to be disinclined to think about the reasons for the actions that they would make as teachers of the lesson. After Faye had added, “ask students measured and recorded the data for a given height” to their logic, the discussion turned to the wording of the statement.

Excerpt 125, Class #12

Liz: Not “for” a given height, but “at”

Faye: Record data for, “at” a given height?

Liz: At given heights

Rachel: How about at a set height? Because it’s not given, we decided how high it was [...] Do we need to say that we need...we’re looking at it at one height and then half that height?

Story: We came up with that though...

Rachel: But that’s how, look, if you’re just doing it at one height and just timing, that’s not going to do us a whole lot of good...

Story: No, of course not...

Rachel: In comparing those

Story: How about “set heights”

Rachel: Oh, okay [...]

In the preceding excerpt, Rachel indicated that they had taken measurements from specific heights (“one height and then half that height”), suggesting that she was thinking about there being a logic to the two heights under consideration. Story’s comment that, “We came up with that,” suggests that she was unaware that Pat had intended that the conversation move toward the teachers obtaining measurements at heights that would allow for the discussion to focus on whether the speed was constant—that the lesson had a logic to it and Pat had managed the conversation with regards to that logic.

Excerpt 126, Class #12

Story: Meaningful heights, I guess, would be good. It’s got to be a meaningful height unless it’s not going to help us.

Faye: Meaningful heights?

Johnny: Specific.

Faye: Do you want set height?

Pat: But how would a reader of this know what a meaningful height is?

Alyce: I did the total distance and half distance, that’s what I wrote in my step.

Johnny: I did the same thing.

Faye: Okay, total distance and half distance.

At this point the teachers indicated that they were going to move on without indentifying why it would be important for the students to take measurements at the total height and half the height, or how they (as teachers) might manage the conversation so that the students would be motivated to do so. This suggests that the teachers were focused on obtaining the measurements, that the measurements themselves (as objects) were important, not the logic behind taking the measurements—taking measurements was just another step in the sequence of steps in the lesson. The teachers appeared not to be thinking about why taking measurements (and which measurements) would be a natural thing to do given what they'd already done and where they were trying to head. Pat interjected to get the teachers to focus on these issues.

Excerpt 127, Class #12

Story: Repeat several times.

Rachel: Do we want to say that?

Pat: Do you remember...remember why?

Story: For average speed.

Pat: No, not repeating several times [...] Why at the initial height and half the height?

Tami: To determine whether or not it was a constant...umh...increase, or a...

Adriana: It would be half the time if it was half the distance.

Pat: We were test...we were testing the hypothesis that...

Johnny: That it was a constant.

Pat: That it was dropping at a constant rate of change, and then the hypothesis then translates into what? [...] If the screwdriver's dropping at a constant rate of change then...

Rachel: Half the distance would be half the time.

Pat: Half the distance would be half the time. That's what we were actually, that's why we were recording the data to see if in fact it took half the time [...]

Pat next indicated that the teachers needed to help Faye decide on what needed to be added to the group's logic.

Excerpt 128, Class #12

Tami: We discussed if it was an average, or what the average would be...

Adriana: And we want students to come to the conclusion that it's not [Faye types "discussed what the average speed would be" and "want students to come to the conclusion that it's not a constant rate"]

Sheila: Did we do average speed there or was it after we wrote the statement?

Marian: I think it was after.

Adriana: But we want students to come to the conclusion that it's not a constant rate...using that, those measurements.

Sheila: But did we talk about average speed there?

Faye: Oh, we did?

Johnny: No, I think...

Sheila: I thought we did the statement, and we clarified the statement before we got to average speed.

As the preceding excerpt illustrates, as Pat pushed the teachers to focus on explicating the reasons why instructional actions were taken, and describing how they as teachers would motivate student reasoning, the teachers' capacity to recall the sequence of events (i.e., the sequence of outcomes) began to become problematic. The teachers appeared constrained to reflect on their own reasoning in a manner that would allow for them to think about someone other than themselves developing similar understandings.

Phase 2.6 – Losing track of the sequence of steps. Next, Rachel asserted that they needed to clarify that when they dropped half the distance that they actually talking about the top half versus the bottom half fall. This comment contributed further confusion to the group's sequence of steps, as illustrated in the following excerpt.

Excerpt 129, Class #12

Liz: But did that happen, but I think Step 2 is really our whole thing about getting how to measure...

Rachel: Right.

Liz: Did that discussion come up in...

Marian: Step 3.

Liz: Step 2?

Liz: I mean if your saying, “Oh, now go to 130 and measure there, or take the times from there,” does it come up in that part, where we were talking about what that 130 meant?

Rachel: Probably right after that.

Liz: Maybe we...

Faye: So, where should we start Step 3 guys?

Sheila: After the testing.

The teachers spent the next 6 minutes (including some time spent with “computer issues”) discussing the progression of outcomes, the steps and the ordering of the steps. During this discussion, the teachers gave no indication that they were thinking about the lesson’s logic or how it made sense (logically, developmentally) for one step to follow the next. In addition, this discussion focused only on the teachers’ perceptions of the lesson they had experienced and on outcomes (the products of the teachers’ reasoning), not on student reasoning or managing the conversation. The teachers’ logic at this point was as follows:

Step 1 – Observation

- Stand on a chair and drop a screwdriver
 - asking students what they observe as the screwdriver drops, the way it drops
 - repeat several times and carefully questioning students observations, keep asking until you get the discussion on speed, whether or not speed was constant

Step 2 – Testing

- how could we determine the situation to show that the screwdriver was not falling at a constant rate
- want to manage the conversation to encourage students to bring up measurements
- ask students to measure and record the data at set heights” to “ask students to measure and record the data at meaningful heights—total distance and half the distance
- test hypothesis that it was dropping at a constant of change (if the screwdriver is dropping at a constant rate, half the distance would take half the time)

Step 3 – Discussion

- discussed what the average speed would be when dropping from half the distance we are talking about the top half of the graph
- want students to come to the conclusion that it’s not a constant rate

Phase 2.7 – Constructing a definition for “always speeding up.” The teachers next discussed how they had developed a definition for “always speeding up” (and attempted to determine where to place this event in their logic).

Excerpt 130, Class #12

Faye: Step 4...

Liz: What is meant by increase in speed?

Teacher: What does it mean...right

Rachel: What does it mean to always speed up, that's what the question that was posed at that point

Adriana: To always be speeding up

Rachel: Within groups we had...

Faye: So then...then, we got into groups, right here, so...

Rachel: Yeah, we had to work within groups...to discuss that

Sheila: Groups discuss and write...a...

Liz: Definition...

Teacher: Precise

Faye: Concept definition...didn't you say that...or did I just throw that in?

Story: You probably thought conceptual [...]

Faye: Conceptual?

Rachel: Conceptual...statement instead of definition, cause he really wanted a definition.

Faye: Statement.

Story: Yeah, statement is better.

Story: What was it a definition of?

Rachel: To answer the question

Faye: Of the first point there, "What does it mean to always be speeding up?"

Story: That's the definition, "always be speeding up"?

Adriana: And then share

Faye: And then share ...share with class.

Faye: And then maybe modify?

Rachel: Compare...share, compare, and modify [...]

The teachers provided no indication of the actual meaning that they intended or students to develop, why they believed such a definition would be important to student learning, or how they would manage the conversation so that students would be motivated to develop the intended meaning—they appeared to be simply recalling what they had done in the lesson that had engaged in as students.

The conversation next focused on the idea of using intervals of the same size, how the intervals did not have to be consecutive intervals, and how to determine “which one travels farther” if the intervals did overlap. This discussion again focused on what the teachers had done (their perceptions), not on how they imagined leading the lesson with their own students.

After a brief discussion and some modification of the order of the events, including a determination of where to place the discussion of average speed (which again provided no indication that the teachers were thinking about the lesson’s logic), the teachers agreed on the following logic for Step 4.

Step 4 –

- What does it mean to always be speeding up
- groups discuss and write a precise conceptual statements
- share and compare and modified
- discuss average speed and its meaning
- intervals of equal size

- discuss how intervals do not have to be consecutive they can be overlapping
- with equal intervals of time of a given size, even if they overlap, which one travels further

Phase 2.8 – Constructing and discussing the graphical representations.

The teachers next moved on to Step 5, which they identified as involving the construction of the graphs (the screwdriver's distance versus elapsed time). During the conversation, there was some disagreement as to whether to include the constraints that the teachers were given as part of the logic, as illustrated in the following excerpt.

Excerpt 131, Class #12

Marian: Meeting the constraints.

Faye: Should we have constraints as a different...

Sheila: Of the experiment

Pat: I think that it might be really useful to actually say what they are.

Faye: Yeah, should we have them as separate...

Rachel: But that's not something you have in your lesson, because you do a different...if we're doing it our heights going to be a different height...

Liz: But you can say the constraints of the time it took to fall that height.

Rachel: Oh, okay, yeah.

Liz: Not those specific...

Pat: Yeah, testing that...

Faye: Initial height?

Rachel: Start time, and start height and time...

Pat: Yeah, that it fell from a certain height, that it took a certain amount of time, and it was always speeding up.

Teacher: Right.

Rachel: And...I don't know if we need this...to emphasize that we're looking at graphing the increments and not just as we all did, what we thought it would look like...but focus on graphing it incrementally.

Sheila: Well, if we're talking about the students, they're not going to have our previous knowledge... I don't know if it's...

In the preceding discussion, Rachel asserted that the students should be forced to construct their graphs using “the increments,” but does not articulate what she means by “the increments” or why she believed that to be important. This suggests that Rachel had perceived the use of “the increments” to be important to Pat, and something that they should include in their logic. In addition, the teachers’ comments suggests that they viewed the focus of the lesson to be about obtaining the graph of a quadratic function—that the lesson was about obtaining a parabola, not about developing a meaning for non-constant rate of change, and naturally and coherently introducing the idea of a function with a rate of change that changed linearly.

The ensuing discussion focused on issues of student’s prior knowledge, and whether all students, some students, or just honors’ students would know to expect

the graph to be parabolic—which again placed the focus on the graph, rather than the logic.

Excerpt 132, Class #12

Sheila: Yeah, I think you should be stressing using the statement you created.

Liz: Making your definitions work for you.

Faye: Well, let's put that in [laughs]

Story: Wow...wow.

Although the teachers indicated that they liked the term “making your definitions work for you,” they did not state how they imagined the definition working for either themselves or for their students—they simply appeared to be latching on to what they perceived as a TPC² catch phrase.

The teachers next moved to discussing what occurred after the graphs had been constructed, but not in terms of the ways of thinking or ways of reasoning that such a discussion could promote. Pat asserted the need for the teachers to imagine how they would represent the graphs, so that they could be used as objects for discussion.

Excerpt 133, Class #12

Sheila: Well, once we did it, we just examined multiple representations and
looked for commonalities in methods in creation

Adriana: We shared and discussed the examples of the graphs

Pat: How are you going to do that? [...]

Liz: We could have each student or group have their own white board and do it and then hang the white boards up.

Pat: Okay.

Faye: But it's good to have the graph paper...yeah.

Sheila: Either that or you could give them their own overhead sheet...or a piece of butcher block.

Pat: So, the point is that you want them to...you want it to be created in some sort of displayable [...] You would have to think through how you would make that happen [...]

The teachers' comments regarding the graphs, that they were "shared and discussed," suggests that the teachers viewed this action as yet another step in the sequence of steps that made up the lesson. Rather than as an instructional action that intended for students to reflect on their own and one another's reasoning (i.e., that there was a logic to the instructional action). The teachers concluded their discussion of the graphs (Step 5) with the following logic:

Step 5 – Graph

- Graph the screwdrivers distance from the ground as a function of time elapsed, meeting the constraints of the experiment
- Constraints of initial height, initial time, and it was always speeding up
- Stress your conceptual statement, making your definitions work for you!

- Examined multiple representations, sharing students ideas, camera/elmo/overheads/whiteboards looking for commonalities in the methods of creation

Phase 2.9 – Lesson logic completion and retrospection. After the teachers had completed Step 5 in their logic, Sheila stated that she believed they should add a Step 6 that involved having students (depending on their level) “write a mathematical representation of their earlier conceptual statement.” Pat asserted that the teachers would in effect need to say, “For every epsilon greater than zero, *et cetera*.”

Excerpt 134, Class #12

Adriana: But we want the students to think about rate of change
changing...linearly...

Pat: Systematically.

Faye: Should we go back to Step 5 for that?

Marian: Yep.

Faye: [typing] And then want students to think about rate of change
systematically?

Adriana: Changing systematically.

Faye added another bullet to the end of Step 5: “Want students to think about rate of change changing systematically.” Although the teachers brought up the idea of getting students to think about rate of change “changing systematically,” they did not articulate how they had interpreted Pat’s use of the word “systematically” or

how they intended to manage the conversation to promote students' to think in such a manner. In addition, the teachers did not discuss the logic behind adding this sub-step. The teachers concluded the lesson logic re-construction with Step 6.

Step 6 – Extension

- Depending on student level, have them create or write a mathematical representation of their earlier mathematical statement, for all epsilon greater than 0...

The teachers' completed group lesson logic is displayed in Appendix G. Pat concluded Phase 2 by providing a recap of the reflective episode. Pat asserted that the teachers first engaged in a lesson as students, a lesson that introduced the idea of quadratic functions, which required Pat to keep the teachers from using too much of what they already knew. Pat then described how he had then asked the teachers to re-create the lesson, by having them think about what made the lesson happen; specifically, how (as a teacher) could they get from standing on a chair dropping a screwdriver to discussing intuitive differential equations. Pat asserted that as the teachers think about creating lessons, that they should stand back and think about, "Okay, so how would I systematically get from here to there."

Phase 2 – Summary. Although the teachers were able, for the most part, to re-construct the lesson involving concepts of speed, they did so by simply re-constructing the sequence of events. What the teachers were constrained to re-construct was the lesson's logic. Throughout the lesson re-construction, there was little discussion of the actual meanings that would either guide their instructional

actions or that would be desired learning outcomes for students. Rather than re-constructing the logic of the lesson, the meanings that the lesson intended to build, the ideas that the lesson intended to develop, and the teacher's orchestration of the lesson to promote these intended understandings and ways of thinking in their students, the teachers focused their attention on re-constructing the sequence of successive outcomes (the step by step products of the teachers' reasoning).

Reflective Episode #3 – Summary

As illustrated in the preceding reflective episode, the teachers' empirical orientation toward learning and teaching mathematics focused the teachers' attention on their own perceptions of the lesson that they had engaged with as students, on objects and actions that they perceived as being applied to those objects, and on the products of their reasoning. Such a focus constrained the teachers' capacity to take their meanings as objects of thought and to take the point of view of others. Furthermore, such an orientation focused the teachers' attention on the visible (e.g., comparing speeds, the graph); thus, making the idea of creating and implementing a lesson possessing a logic, one that systematically develops and builds meanings and ideas, hidden to the teachers.

Reflective Episode #4: Data Analysis

The second reflective episode that I will describe in this chapter was similar to the reflective episode involving ideas related to the concept of speed (Class #12) in that the teachers first engaged in an activity as students of mathematics, and were then asked to re-construct the lesson as an instructional designer creating the

lesson that they had just experienced. The episode involved the teachers experiencing and re-constructing a lesson involving data analysis, and comprised the 13th and 14th class meetings (November 19th and December 3rd, 2007), with an out-of-class assignment due prior to Class #14. The reflective episode had four main phases with 6 sub-phases in the first phase, eight sub-phases in the second phase, and six sub-phases in the fourth phase. A chronological overview of the lesson's phases (Class #13 and #14) is given in Table 9.

Table 9

Chronological Overview of Reflective Episode from Classes #13 and #14

Class Meeting (Date)	Phase	Duration (minutes)
	Phase 1 – Teachers as Students of the Lesson	
Class #13 (11/19/07)	Phase 1.1 – Unorganized Data is Not Very Useful	3.3
Class #13 (11/19/07)	Phase 1.2 – Giving Meaning to the Role that Organization Plays in Data Analysis and Generating Hypotheses	19
Class #13 (11/19/07)	Phase 1.3 – Reinforcing the Role that Organization Plays in Analyzing Data	11.8
Class #13 (11/19/07)	Phase 1.4 – Introducing TinkerPlots: Reinforcing the Role that Organization Plays in Analyzing Data	14.3
Class #13 (11/19/07)	Phase 1.5 – Constructing Images of Data Organization that are Propitious for Solving Problems (Group Work)	16.9
Class #13 (11/19/07)	Phase 1.6 – Addressing Issues Related to Data Analysis and Instructional Design	28.8
	Phase 2 – Teachers as Instructional Designers of the Lesson	
Class #13 (11/19/07)	Phase 2.1 – Lesson Logic Review	4.5
Class #13 (11/19/07)	Phase 2.2 – The Teachers' Images of Pat's Initial Instructional Action and the Reason for that Action	6.7
Class #13 (11/19/07)	Phase 2.3 – Trying to Identify Ideas and Meanings	6.6
Class #13 (11/19/07)	Phase 2.4 – The Type of Organization Matters	5.4
Class #13 (11/19/07)	Phase 2.5 – Diverging from the Lesson that was Experienced	3.3
Class #13 (11/19/07)	Phase 2.6 – Attempting to Return to the Lesson that was Experienced	3.7
Class #13 (11/19/07)	Phase 2.7 – Uncertainty in Recalling the Lesson that was	4.7

Class #13 (11/19/07)	Experienced Phase 2.8 – The Lesson that was Experienced was Not the Logic of the Lesson	3.8
	Phase 3 – Constructing the Logic of an Observed Lesson and Comparing the Two Logics	
Out-of-class assignment	• Individual Lesson Logic Construction of the Lesson Experienced in Class	n/a
Out-of-class assignment	• Individual Viewing of Video of Similar Data Analysis Lesson (9 th Grade Reformed-Algebra 1 Case Study))	n/a
Out-of-class assignment	• Individual Lesson Logic Construction of Similar Lesson from Video	n/a
Out-of-class assignment	• Comparison and Discussion (Individual) of the Two Lesson Logics	n/a
Class #14 (12/3/07)	Phase 4 – Whole-Class Discussion of Phases 2 and 3	
Class #14 (12/3/07)	Phase 4.1 – Orienting the Discussion: A Comparison of Understandings	4.5
Class #14 (12/3/07)	Phase 4.2 – Separating Teacher Mode from Student Mode	1.2
Class #14 (12/3/07)	Phase 4.3 – Discussing Phase 2: Difficulty Re-Constructing the Lesson	2.7
Class #14 (12/3/07)	Phase 4.4 – Being Mindful of Students' Understandings	8
Class #14 (12/3/07)	Phase 4.5 – Orienting the Discussion Toward Lesson Structure	11.5
Class #14 (12/3/07)	Phase 4.6 – Making the Teachers' Conceptions of Lesson Structure Explicit	7

Prior to initiating the reflective episode, Pat informed the teachers that they were going to engage in an activity similar to that which they had engaged in during the precious class meeting (Class #12). Specifically, that the teachers were first going to engage in a lesson as students of mathematics, and that they were then going to talk about the logic of the lesson. Pat also stated that the lesson that they were going to experience was a lesson on statistics that had been taught to 9th graders, and that teachers were going to be asked to review a video of this 9th grade lesson and compare the two lessons' logics.

Phase 1 – Teachers as students of the lesson. The first phase of the reflective episode was designed to develop and give meaning to the role that organization plays in data analysis. Throughout Phase 1, the course instructor (Pat) attempted to focus the teachers' attention on the role that organization played in generating and testing hypotheses, and consistently moved the teachers to construct images of productive data display.

Phase 1.1 – Unorganized data is not very useful. Pat initiated Phase 1 by stating that a recent study had been conducted at a high school where students were given an anxiety assessment just before taking a mathematics test. Pat then distributed (to the teachers) a two-sided sheet containing unorganized data, explaining that it came from this study. The sheet contained four columns listing test score on one side and four columns listing anxiety level score on the reverse side (Figure 23). He asked the teachers what information they could make from the data. This action enabled Pat to lead the discussion to the conclusion that little could be made of unorganized data.

Test score	Test score	Test score	Test score	Anxiety level	Anxiety level	Anxiety level	Anxiety level
64	79	56	68	84	37	18	46
46	85	75	74	17	32	21	81
50	49	70	68	8	99	12	18
83	70	68	94	34	38	25	50
78	98	88	75	35	46	47	13
82	55	60	68	31	7	10	50
90	72	60	69	34	89	23	45
73	58	81	42	43	19	19	16
62	75	72	59	35	28	55	96
90	80	59	74	35	42	33	42
82	90	67	89	52	36	42	33
63	88	66	67	85	41	74	70
57	63	100	68	94	64	45	80
70	87	88	82	18	41	40	44

Figure 23. Unorganized student test scores. First four columns were on one side of the page; second four columns were on the other side of the page.

The teachers said that they could not make sense of data. Though one teacher (Johnny) suggested that there might be some relationship between the variables, Pat steered the conversation toward the need for a better organization of the data by asking if they could be sure that the n^{th} score on one side of the sheet corresponded with the n^{th} score on the other. This instructional action anticipated future discourse regarding the role that organization could play in identifying relationships, and subsequently in generating hypotheses, based on the data.

Phase 1.2 – Giving meaning to the role that organization plays in data analysis and generating hypotheses. Pat next distributed a second data sheet, which showed anxiety level scores together with test scores (Figure 24). Pat asked the teachers whether they felt that this organizational display was better than the

first. This action focused the teachers' attention on identifying the benefits that organized data provided and constructing images of more propitious displays. In addition, the conversation anticipated the notion of statistical case, by emphasizing the necessity for the teachers' to assume each row of data went with one and only one student.

Anxiety level	Test score	Anxiety level	Test score	Anxiety level	Test score	Anxiety level	Test score
84	64	32	100	55	76	59	68
17	46	46	68	20	59	82	58
8	50	81	74	68	69	48	94
34	83	18	68	48	89	29	74
35	78	50	94	48	69	71	71
31	82	13	75	26	69	32	86
34	90	50	68	17	79	91	56
43	73	45	69	36	65	83	70
35	62	16	42	45	84	31	86
35	90	96	59	62	65	45	84
52	82	42	74	45	76	38	78
85	63	33	89	95	58	39	79
94	57	70	67	69	65	49	85
19	79	80	68	63	64	50	83
75	65	44	82	37	68	46	69

Figure 24. Student anxiety level scores and their corresponding test score.

Several of teachers impeded their reasoning by focusing their thinking on the development of plausible explanations to relationships within the data, based on the quantities under consideration (e.g., test scores and test anxiety levels), rather than maintaining their focus on developing hypotheses based on the numbers themselves. This is illustrated in the following excerpt.

Excerpt 135, Class #13

Liz: Is a high anxiety score good or bad?

Pat: On that first one, is it possible to pair the data [...] do you know that the first test score goes with the first anxiety score?

Several: No [...]

Pat: On this one, what are some of the things that you can tell?

Annie: Still not a lot because the data's not organized.

Liz: You still don't know if high anxiety...this number's a good thing or a bad thing [...]

Johnny: Does a high anxiety score mean that you're anxious or not anxious, so we don't know the directionality of the score.

Liz (and Johnny) indicated that in order to develop a hypothesis, they first needed to be able to interpret the anxiety level score (i.e., was a high anxiety level score good or bad?). This suggests that they might have intended to develop their hypotheses based on what made sense (e.g., high test scores go with low anxiety scores). Rather than focusing their attention on the need for a better way to organize the data, these teachers focused their attention on obtaining an outcome; that is, on how they had interpreted the task, as one involving the development and verification of hypotheses.

Pat next encouraged the teachers to generate hypotheses pertaining to the data, but insisted that they base their hypotheses on the numbers.

Excerpt 136, Class #13

Pat: What are some of the things that you can tell? [...]

Rachel: It seems...a lot of them...the ones with a higher [test] scores, a lot of those, have...not all of them... seem to have the score for the anxiety level is a little bit lower [...]

Pat: So...

Annie: I don't think I agree with that.

Rachel: Not all of them, but some of them...your 100's your 90's, the ones where...your scores are in the 100's, in 90's, the anxiety level is quite a bit lower.

Pat: Okay, we hear some disagreement...so, what's your general hypothesis?

Tami: The lower the anxiety level, the higher the test score.

Rachel: Or you could go the other way, the higher the test score, the lower the anxiety.

Tami: Well if you think you're prepared for the test, then you don't have anxiety over the test.

Pat: Okay, that's an explanation for the hypothesis, but not a hypothesis [...]

Rachel: I'm saying not on all of them...

Pat: So why is it hard to make that argument?

Annie: But then you have a couple of kids that don't care that have seven and eights there.

Johnny: You could have outliers.

Alyce: It could be that they're clueless, and don't know what they are doing.

Rachel: And they don't care.

Several: (Laugh)

Johnny: But see that's the thing is that because you're clueless and you don't really care, that doesn't mean that you're gonna ... so you have that as an issue too...

Pat: Johnny, stay to the numbers...what is keeping you from moving forward?

As illustrated in the preceding excerpt, several of the teachers attempted to justify Rachel's claim that many of the higher test scores were associated with lower anxiety level scores by focusing on plausible non-cognitive aspects of the students (e.g., student preparation for test, or whether the student was "clueless" or "did not care"), rather than the numbers themselves. These teachers' attempts to explain why Rachel's hypothesis might be true, constrained their capacity to build meaning for the role that data's organization plays in making interpretations of it. Pat insisted that the teachers generate hypotheses based on the numbers, not construct explanations for a hypothesis. In addition, Pat attempted to get the teachers to think about how the organization (or lack of organization) played a role in their ability to generate hypotheses, and how a different organization could possibly help them to make and justify claims (i.e., hypotheses).

Pat continued to question the teachers as to why they believed they were experiencing difficulty in generating claims. This action led to the suggestion (by Eve) that they order the test scores from highest to lowest, and look for a

relationship between the variables. Pat used Eve's suggestion to move the conversation toward discussing what the teachers' envisioned such a display (ordering test scores from highest to lowest) would provide. This conversation led to the notion that particular types of organization offer particular types of information and again highlighted the importance of assuming that each row represented data from a unique individual. In addition, this conversation explicitly highlighted the idea of a statistical case (as the fundamental unit of analysis in statistics).

Pat next moved the conversation back to generating hypotheses. The teachers constructed the following two hypotheses: a) Higher anxiety scores tend to go with lower test scores, and b) Higher test scores tend to go with lower anxiety score. Pat pushed the teachers to imagine organizations of the data that would enable them to judge the hypotheses' veracity. This action emphasized the role that data organization plays in generating and testing hypotheses, thus giving meaning to the idea that how a display's organization of the data matters greatly in data analysis.

Phase 1.3 – Reinforcing the role that organization plays in analyzing data. Pat next introduced of a third variable (each student's teacher) to the data set. After each of the teachers had received the third data sheet (Figure 25), Pat asked what hypotheses they could make with the addition of the third variable. This instructional action required the teachers to coordinate their notion of statistical case with their images of organizational displays to construct hypotheses based on

the data. In addition, this action served to reinforce the idea that particular organizations could provide particular kinds of information.

Teacher	Anxiety level	Test score
Bloom	84	64
Bloom	17	46
Bloom	8	50
Carlson	34	83
Carlson	35	78
Carlson	31	82
Carlson	34	90
Hobart	43	73
Hobart	35	62
Bloom	35	90
Bloom	52	82
Bloom	85	63
Carlson	94	57
Carlson	19	79
Carlson	75	65
Hobart	37	79
Hobart	32	85

Teacher	Anxiety level	Test score
Carlson	32	100
Carlson	46	68
Carlson	81	74
Hobart	18	68
Jimenez	50	94
Jimenez	13	75
Bloom	50	68
Bloom	45	69
Carlson	16	42
Hobart	96	59
Jimenez	42	74
Jimenez	33	89
Jimenez	70	67
Jimenez	80	68
Jimenez	44	82
Bloom	60	70
Bloom	16	78

Teacher	Anxiety level	Test score
Jimenez	55	76
Bloom	20	59
Bloom	68	69
Bloom	48	89
Bloom	48	69
Carlson	26	69
Hobart	17	79
Hobart	36	65
Jimenez	45	84
Jimenez	62	65
Jimenez	45	76
Bloom	95	58
Bloom	69	65
Bloom	63	64
Carlson	37	68
Hobart	14	80
Hobart	67	68

Teacher	Anxiety level	Test score
Hobart	59	68
Hobart	82	58
Hobart	48	94
Hobart	29	74
Bloom	71	71
Bloom	32	86
Carlson	91	56
Carlson	83	70
Carlson	31	86
Hobart	45	84
Hobart	38	78
Hobart	39	79
Hobart	49	85
Hobart	50	83
Hobart	46	69
Jimenez	77	76
Jimenez	54	79

Figure 25. Teacher, anxiety level score, and test score by student.

As with the earlier two data sheets, the teachers focused their attention on creating and explaining hypotheses based on the quantities (the three variables under consideration), rather than on the number themselves.

Excerpt 137, Class #13

Liz: Now you can see how test anxiety relates to a certain teacher...or if test relate to a different teacher.

Bernardo: Oh, yeah, if test scores...could relate test scores to a different teacher
[...]

Pat: Now what, you have one more piece of information. It's the same data that you already had...it's just one more piece of information. Is it more interesting now? [...]

Johnny: Now, does the teacher control the anxiety? Do certain teachers make more anxiety...

Pat: How can we say that...without any value statement? [...]

Annie: What effect do the teachers have on the anxiety level? [...]

Pat: That sounds like teachers are causing anxiety...how about, just, do different teachers have students with different anxiety levels?

Faye: Within the teacher, or across teachers?

Pat: That's across teachers, but you would be grouping by teacher [...]

Pat: Do different teachers tend to have students who have higher or lower [...]

Liz: If you knew certain teachers, students go through and pick a teacher that they know could help them. So, I know maybe I always have high anxiety, so I pick this particular teacher, as well as all my friends, so they didn't cause it, I just naturally have that...

Pat: Okay, you're trying to explain a hypothesis before we've formulated it.

Pat next sorted the data, as requested by the teachers, using an Excel spreadsheet with the data embedded. The sorted data was displayed on a computer projector so that the teachers could see the results of their requested organization and attempt to generate hypotheses. This visual display again highlighted the importance of the idea of a statistical case and highlighted a need for a better way to

organize the data, thus reinforcing the role that organization plays in analyzing data (i.e., in generating and testing hypotheses).

Phase 1.4 – Introducing TinkerPlots: Reinforcing the role that organization plays in analyzing data. Pat next introduced *TinkerPlots*, an exploratory data analysis computer program, by constructing a variety of organizational displays of the data provided in the third data set. In addition, Pat focused the conversation on how each of the various displays addressed specific questions regarding the data (Figure 26).

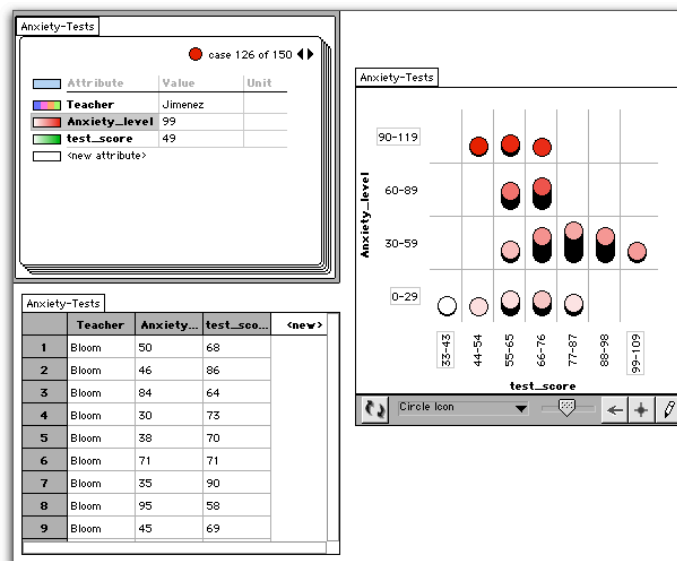


Figure 26. Tinkerplots display.

These actions served to provide the teachers with a sense of the program's capabilities and anticipated the necessity to construct displays that addressed different questions related to the data. This activity, and the activity that followed,

also demonstrated how to use a computer program to organize students' activities even though the students themselves do not have access to the program.

Phase 1.5 – Constructing images of data organization that are propitious for solving problems (group work). After concluding the *TinkerPlots* overview, Pat next partitioned the teachers into four groups and distributed (to the teachers) a sheet with questions that pertained to the data (Figure 27). The questions were designed to reinforce the teachers' images of the role that organization plays in data analysis and to provide a natural way in which the idea of conditional probability could arise in the context of instruction.

Anxiety-Test Data (In Groups)

- Consider these questions one at a time, focusing first on the one you chose.
 - Think about how the data could be displayed so that you can answer the question just by looking at the display.
 - Sketch how you want the display to be organized. Don't try to guess how the data would look, just say how you want it organized so that you can look at it and answer the question.
 - Come up to my desk and show me your sketch. I'll make the display you want and then print it out for your group.
1. Is test anxiety a severe problem among Math 1 test takers?
 2. Is test anxiety more severe among some classes than others?
 3. Did some classes do better than other classes on this test?
 4. Suppose you talk to a middle test scorer. How anxious were they likely to be when taking the test?
 5. Suppose you talk to a highly anxious student? How likely are they to have a low test score?
 6. Suppose you talk to a highly anxious test taker. To which class are they most likely to belong?

Figure 27. Anxiety-test questions.

Each group selected a question from the sheet, with no two groups having the same question, and worked to develop a sketch of the organizational display that they believed would allow them to address their question. A representative of each group showed a sketch of their desired organizational display to Pat, who generated the requested display using *TinkerPlots*. The displays appeared on a projector screen as Pat generated them. This action required that the teachers imagine organizing the cases in a manner that they believed was most propitious for addressing their chosen question.

Phase 1.6 – Addressing issues related to data analysis and instructional

design. During the last part of Phase 1, each group presented their designed display and discussed how and why they believed their display (a) allowed them, in principle, to answer their question and (b) supported their specific answer to their question. This allowed Pat to address specific issues related to both data analysis and instructional design, including: a focus on analyzing distributions, interpreting the attributes of a case, attending to how others (i.e., students) might interpret a question, attending to how others (i.e., students) might interpret a display, and a focus on how certain understandings or ways of thinking might be propitious or deleterious to future learning.

During the group presentations, the teachers focused on their own perceptions about their own participation in answering their respective questions. Missing from these presentations, other than when Pat pushed the issues, were discussions of the teachers' images of their students' engagements with the activity and their role (as teachers) in managing the activity with their students. In addition, throughout the group presentation, Pat consistently pushed the teachers to think about how others might be interpreting their statements (e.g., stop using pronouns), to articulate their own meanings and reasoning, and to push others to do the same.

Bernardo presented his group's results. The group (Annie, Bernardo, and Faye) had chosen Question #3: Did some classes do better than other classes on this test?

Bernardo asserted that since his group was to compare classes, they needed to group the test scores by teacher. Bernardo's organizational representation (Figure 27) displayed teacher along the horizontal axis and the distribution of test scores along the vertical axis (the count of the scores in each interval). Bernardo stated that to answer the question, his group had decided to look at each teacher's percentage of scores that were greater than 66%. Bernardo stated that his group had determined the number of students in each teacher's class and the number of these students who scored 66 or above on the test. They then calculated the percent of students, for each teacher, who scored 66 or above on the test, and compared the results. From the results, the group concluded that there was a difference in test scores between the classes.

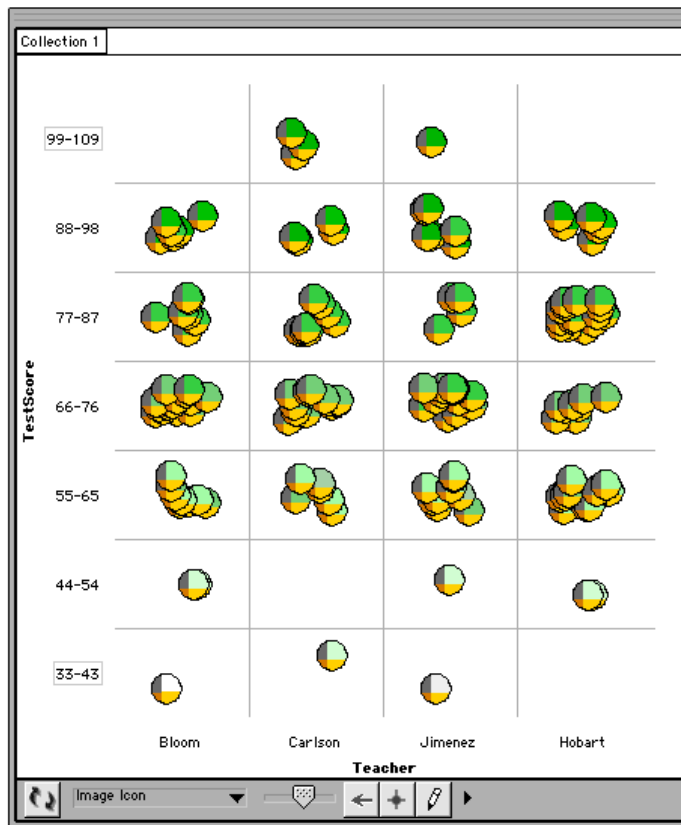


Figure 28. Display from Bernardo's presentation.

After Bernardo concluded his presentation, he asked for comments or questions. Story asked Bernardo to clarify his calculations. After re-stating his conclusions, it was quiet for 16 seconds, before Pat moved the focus of the conversation toward making the teachers' reasoning explicit and a topic of discussion. In addition, Pat moved the teachers to be mindful of epistemic student ways of thinking regarding summary statistics and the advantages (to student learning) of focusing students' attention on analyzing distributions.

Excerpt 138, Class #13

Pat : Well, maybe it's occurred to you and you think it's so trivial, so you're not going to say it...they answered that question without calculating a mean.

[Quiet for nine seconds]

Story: The mean score...

Pat: The mean score of each class.

Bernardo: Mhm.

Story: That's a good idea actually.

Pat: You didn't notice that?

Story: No [...]

Pat: Just look at the question, how would you normally have answered that question?

Liz: The average, by taking...

Pat: By taking the average, right?

Liz: Yeah.

Johnny: I want to know what the spread of the data is, because then we can talk about the variability [...]

Pat: They provided an analysis and responded to the question according to...a distributional characteristic...the percent of the class that scored above a certain cut-off level, that's a distributional characteristic. Did it seem like it was unnatural picking that?

Bernardo: No, not at all

Pat: Would calculating a mean have felt more natural?

Annie: No.

Bernardo: It didn't even occur to us [...]

Johnny: I'm assuming you picked the 60%...because you wanted to know, what's the percentage that passed.

Faye: Pretty much, yeah [...]

Rachel: If he'd given us these questions and asked us to answer them prior to us seeing this software, we would have gone for the mean probably.

Pat: One of the things that is often lost in [...] student understanding of summary statistics is that they are supposed to give some information about a distribution. Now, unfortunately, by themselves, a mean gives no information whatsoever about a distribution, none. You can have wildly looking distributions all have the same mean.

Annie: That's true [...]

Pat: Bringing them closer to the data, almost leads naturally, without even thinking about it, to analyzing the distribution. Which is fundamentally much more statistical than calculating a mean. Distributions are far more powerful. The idea of distributions and distributional analysis is far more powerful than calculating summary statistics.

As illustrated in the previous excerpt, the teachers appeared to be disinclined to take their meanings, their reasoning as objects of thought. In addition, the teachers appeared to be disinclined to think about student engagements with the

activity. Specifically, how promoting certain ways of thinking (as teachers), such as a focus on distributions, would be productive to student learning.

The last two groups to present involved Question #5: Suppose that you talk to a highly anxious student, how likely do they have a low test score? (Eve, Marian, and Story), and Question #6: Suppose you talk to a highly anxious test taker, to which class are they most likely to belong to? (Janine, Johnny, and Rachel). These two questions involved conditional probability, a concept that had not been discussed during the lesson. After these groups had presented their respective solutions, Pat moved the conversation toward a discussion of how the lesson's logic was such that the notion of conditional probability arose naturally in the context of the data and the activity.

Phase 1 – Summary. Although the lesson involved developing coherence among ideas related to data analysis (i.e., how organizational display influences what can be analyzed, the statistical case, constructing hypotheses, distributional analysis, and conditional probability), the development of these ideas was of secondary importance to the lesson. Of foremost importance was the emphasis on the instructional design of the lesson, which focused on meanings and developing significant ideas. Thompson (2008a) asserts that students (and teachers) in the United States, “rarely experience a significant mathematical idea and certainly rarely experience reasoning with ideas (p. 46), and “seldom experience coherence or generativity in mathematics” (Silverman & Thompson, 2008, p. 501). The data

analysis lesson provided the teachers with an experience of a lesson that that did focus on meanings and developing coherent mathematical ideas.

Throughout Phase 1, Pat placed an emphasis on the meanings and ideas that the teachers were intended to develop, specifically: the role that organization plays or can play in data analysis, the idea of a statistical case, and the role and nature of hypotheses in statistical reasoning. In addition, Pat consistently highlighted the importance of constructing images of student understandings as they engage with instruction.

The lesson promoted an image of instruction that focuses on the instructional actions that a teacher might take to support student development of mathematical ideas and the reasons why those actions might work. More specifically, the lesson promoted an image of sequencing a lesson in terms of the ideas that the teachers want students to develop, the way in which these ideas could be developed, and reflective discourse around which this development could occur. It should also be noted that throughout Phase 1, Pat modeled instruction designed to generate discussions around the intended meanings and ideas; and, although Pat did not draw attention to this fact, the instruction served to model this form of instructional design.

Throughout Phase 1, several of the teachers exhibited actions consistent with aspects of an empirical orientation to learning mathematics. For example, rather than focusing their attention on meanings, on their reasoning (e.g., the role that organization plays or can play in data analysis, the idea of a statistical case, the role

and nature of hypotheses in statistical reasoning), several teachers focused on the products of their reasoning (e.g., established statistical case, established hypotheses, sketched displays). In addition, as they engaged with the lesson, several teachers focused on their participation in the lesson to the exclusion of thinking that the lesson might have a logic. Finally, throughout the group presentations, the teachers' focus on their own perceptions, such as the decisions that they made (e.g., the values they attributed to "high anxiety"), hindered their capacity to think about someone other than themselves (i.e., their students) struggling with needing to make similar decisions, or their role (as teachers) in managing the conversation so that their students focus their attention on meanings and reasoning.

Phase 2 – Teachers as instructional designers of the lesson. The second phase of the lesson provoked reflection on the part of the teachers as they attempted to reconstruct the logic of the very lesson in which they had participated as students (e.g., create a *Lesson Logic*, the logic of the lesson). Specifically, this phase was designed to provoke reflection on the instructional actions that the teachers might take to support student development of the intended meanings and ideas, and the reasons why those actions might work. These images also include the tasks and classroom discourse that the teacher would employ.

The lesson logic reconstruction involved the teachers creating a *Lesson Logic* of the lesson that they had experienced. Since the *Lesson Logic* plays a significant role in this phase, I will offer a brief description of a Lesson Logic and its creation in order to give a sense of what the teachers were being asked to construct. Pat

Thompson developed the Lesson Logic as a way for a teacher to outline how a lesson's main ideas will develop; it is not a plan for a day, but a plan for something that the teacher is going to teach, and can therefore cover several encounters with students. A Lesson Logic "lays out what the teacher will say (probably), and the students will think (probably), and how what the teacher says and the students think are steps along a logical path that leads to comprehending the lesson's big ideas" ("The Logic", 2008).

A Lesson Logic template (Appendix H) requires the listing of major ideas of the lesson, listed in a way that summarizes the logic. In addition, the Lesson Logic should list meanings that students must understand at the outset of the lesson if they are to be productive participants in the lesson. Finally, the Lesson Logic details "each action to take and the reason for taking it, [where] actions include statements and questions," and the reason "explains why [the teacher] believes each action will likely advance the students' understanding one more step toward grasping the lesson's main ideas" ("The Logic of the Lesson Logic," 2008, January/February).

The construction of the Lesson Logic gives the "teacher a chance to anticipate obstacles [that] students may encounter as they step through the logical sequence of the lesson" ("The Logic of the Lesson Logic," 2008, January/February). Therefore, the Lesson Logic only "anticipates what the teacher will probably say and the students will probably think at each step—it is always possible that discussion or a question will pull the group in a new direction" ("The Logic of the Lesson Logic," 2008, January/February).

Phase 2.1 – Lesson logic review. Pat initiated the second phase of the lesson by giving a brief review of a Lesson Logic. Lesson Logics were not new to the teachers, since they had developed Lesson Logics on several other occasions (i.e., in Functions 1 and in their PLCs). In addition, the teachers had constructed a lesson logic in Class #12, although one that was much less formal than what was expected in Class #13.

Throughout this review, Pat emphasized several key ideas and component parts to the Lesson Logic. Pat described the three main parts to a Lesson Logic: the major ideas of the lesson; meanings that students need to have before the lesson and meanings that they must have so they that they can participate meaningfully in the lesson; and, the actions that the teacher plans to take and the reason why they plan to take those actions (Appendix H). Pat made a point to distinguish between skills that students must possess (e.g., students must know how to divide fractions), which are not included in a Lesson Logic, and meanings that students need to have before the lesson so that they can participate meaningfully in the lesson (e.g., students must have a meaning for division that encompasses both sharing and partitioning), which are central to a Lesson Logic.

Pat next elaborated on how a Lesson Logic is created. Pat emphasized the fact that the way that a Lesson Logic looks at the end is not the way that they are created—that in creating a Lesson Logic the teacher will jump all over the place. Pat concluded the first sub-phase by emphasizing the need for the teachers to think

about and to construct an image of the understandings that they want their students to develop.

Phase 2.2 – The teachers’ images of Pat’s initial instructional action and the reason for that action. One of the teachers, Rachel, was chosen to act as recorder and instructed to record the groups’ responses as they attempted to reconstruct the logic of the lesson. Rachel recorded the groups’ responses on a blank Lesson Logic form that was projected onto the classroom projector screen. After the recorder (Rachel) asked where the group wanted to begin, there were seven seconds of silence before Pat interjected in an attempt to get the teachers moving. Pat emphasized the fact that the teachers were not being asked to create a Lesson Logic from scratch, but rather to describe the lesson that they had just experienced.

Bernardo commented that Pat handed out raw data (an instructional action) and that Pat did so in order to establish a need to analyze the data (the reason for the action). Rachel typed Bernardo’s suggestions onto the Lesson Logic form (Figure 29).

Steps in the Lesson Logic

<u>Step</u>	<u>Action</u>	<u>Reason</u>
1.	Hand out raw data	Establish need for analyzing the data
2.		
3		

Figure 29. The initial action and reason of the Lesson Logic.

When it became apparent that the teachers intended to move to the next action (in step 2) without discussing what Bernardo had offered, Pat interjected and asked whether everyone agreed with what Bernardo had suggested.

Excerpt 139, Class #13

Pat: Is everybody square with that? [...]

Bernardo: Hey, that's my contribution, you know.

Pat: No, I'm just saying that just because somebody said it doesn't mean...

Faye: Should it be more like for categorizing the data or organizing?

Bernardo: That's probably a better word, actually.

Story: For what?

Bernardo: Organizing the data, yeah [...]

Bernardo: The first thing was he had to establish why are we doing any of this?

Here's raw data, why do any of it? We had to have a need established for organizing the information [...]

The preceding discussion centered on questioning whether the reason for the instructional action should be to establish "a need to analyze" or "a need to organize." Although this brings the notion of "organization" into the discussion, there failed to be any conversation regarding the thinking or ideas that the action intended to promote; rather, that some "need to organize" would be established (an outcome). In addition, the teachers failed to indicate that instruction first needed to give a sense as to the unhelpfulness of unorganized data.

In the preceding exchange Bernardo's comment that, "We had to have a need established for organizing the information," gives the sense that organizing the data was just something that needed to be done, an action on an object (data), rather than an idea that needed to be developed. Rather than an image of organization as a way of thinking that can generate or motivate procedures, the teachers' image of organization appeared to be that of an action being performed on an object.

Next, Johnny disagreed stating that he believed that Bernardo's action ("hand out raw data") did not support the reason ("establish a need for organizing the data"), because Johnny felt that their reason needed to answer the question as to why Pat handed out a data set.

Excerpt 140, Class #13

Johnny: I'm just saying that we handed out a data set...

Bernardo: Yeah, we handed out a bunch of data.

Johnny: What's the reasoning for handing out a data set?

Bernardo: Well, they need to see that they can't get anything from it.

Rachel: But is that, because...

Bernardo: Is that an action or not?

Rachel: When he handed it out were we looking at analyzing it, or were you looking at...the way that he handed it out was it to organize it?

Story: He said did it tell you anything?

Bernardo: Yeah, did it tell us anything?

Story: Can you determine anything from the data?

Bernardo: So, can we find out...maybe a better action then would be, "Can we find out anything from unorganized data?"

Johnny: Yeah, see that's what I'm just saying, cause by saying handed out data...

Rachel: So, what do you want?

Johnny: I'd rather fix that because your reason doesn't support that action [...]

Liz: How is your reason a question?

Rachel: What do you want to say?

Bernardo: What did I say? Umh...

Faye: Can we organize the data so that we could analyze it?

Rachel: Are we talking about action or reason?

Faye: Reason [...]

In the preceding exchange, the focus continued to be on actions—the teachers appeared disinclined to focus on the thinking that the actions might engender in students. Johnny's comment suggests that he believed that the action needed to be better connected to the reason and that "handed out data" was too vague—as in, action: hand out unorganized data, reason: establish a need to organize data. Rachel and Faye indicated that they were unclear which they were discussing action or reason. If the teachers' image of an instructional action and the reason for that action both involve "doing" things, albeit with different actors (teacher for the action, student for the reason), then distinguishing between the two would be problematic.

The comments of Bernardo and Rachel continued to focus on the instructional actions as being undertaken with the intent that the students do something, rather than to think in some manner. There was a lack of focus on ideas being developed, or of even ideas as being part of the lesson. Organization appeared to be part of an action sequence, something to do, rather than an idea that had been reasoned with as a means to assist in solving a problem. In fact, there seemed to be a general lack of discussion regarding ways of reasoning and ways of thinking.

In addition, the teachers focused their attention on attempting to recall what they had perceived, rather than thinking about the logic of the lesson. Rather than asking, “What would have been a natural thing to do given what we’ve done and where we’re trying to head,” the teachers focused exclusively on actions that they had performed and in the sequence that they could recall.

Furthermore, there appeared to be general confusion amongst the teacher as to what they were actually discussing (an action or a reason). Liz asked how their reason could be a question and Faye suggested that they include the question as part of the action. The teachers agreed to change the first action to: “Hand out raw data and ask ‘Can we analyze this data?’” Rachel next asked whether the group wanted to change the initial reason and the teachers agreed to change the reason to: “Establish need to organize data” (Figure 30).

Steps in the Lesson Logic

<u>Step</u>	<u>Action</u>	<u>Reason</u>
1.	Hand out raw data and ask, "Can we analyze this data?"	Establish need to organize data
2.		

Figure 30. Modifying the initial action and reason.

At this point the teachers had spent almost seven minutes attempting to reconstruct the initial action and reason for the lesson that they just experienced as students.

Phase 2.3 – Trying to identify ideas and meanings. Rachel asked what the group wanted to say next and Story stated that Pat had given them another sheet. The ensuing discussion centered on the teachers' attempts to simply recall the lesson that they experienced, what transpired and in what order, rather than on focusing on the logic of the lesson. The focus appeared to be on recreating the specific actions that Pat undertook or the questions that Pat asked causing the teachers to act or react, rather than on developing ideas—the focus was on the teachers' activities as learners, not as managers of others' learning the ideas.

Excerpt 141, Class #13

Rachel: Now what?

Story: Then he gave us another set that had more information on it [...]

Johnny: Wasn't there a couple of more things that we did with this one, wasn't there?

Annie: It's the same thing.

Rachel: Couldn't you put those together, because they were sort of asking the same question too.

Story: Yeah, that's what I'm saying.

Annie: You don't need to say, then he handed out another set of data.

Faye: No. [...]

Bernardo: That's getting more into the lesson plan part perhaps.

Johnny: Yeah, right.

In the preceding exchange, Johnny stated that there were a couple of more things that were “done” with the first sheet, but gives no indication that any ideas were being promoted, or that there might have been a logic to how Pat had distributed the data sheets. In addition, Rachel stated that the two data sheets were “sort of asking the same question,” or asking the student to “do” the same thing; which makes unclear what she felt the difference was—although, again, the focus was not on meanings or ideas, but on “doing” things (i.e., on actions). If the teachers conceived of organization as something that you do (an action performed on an object), then the two data sheets would have tended to reinforce the importance of “organizing” (doing the act of organizing), without giving any meaning to how or why organization is important.

As the teachers continued to attempt to construct the Lesson Logic, they struggled to make explicit both the ideas that instruction intended to promote and how the lesson developed them. Specifically, teachers struggled to re-construct the

logic of the lesson. As indicated earlier, to overcome this barrier the teachers appeared to have simply attempted to recall the order that things were done.

The teachers' focus on the products of their reasoning, throughout the lesson they had experienced as students (Phase 1), constrained their capacity to reflect on their reasoning—their reasoning was hidden in the products. In addition, throughout the lesson the teachers had engaged in as students (Phase 1), the teachers focused on their participation in the lesson to the exclusion of thinking that the lesson might have a logic. Therefore, the teachers' egocentricity and inattention to meaning during the lesson (as students) constrained their capacity, later, to imagine the lesson in a way that they might lead it.

Excerpt 142, Class #13

Marian: Can we add then to the reason, cause I think then if we're going to include all the data that he gave us, it wasn't just to organize data but also to establish a...

Story: Relationship.

Marian: Case.

Story: Oh.

Annie: Establish a hypothesis.

Rachel: Case, hypothesis?

Faye: A hypothesis, right? Because a case is just one...

Marian: Yea, but see that was the whole thing between the first and the second and the third, yeah.

Faye: Okay, yeah.

Rachel: So case?

Marian: Maybe both?

Faye: I think a hypothesis.

Bernardo: A case then a hypothesis

In the preceding exchange, Marian indicated some sense of a logic between hypothesis and case with her statement “whole thing between the first and the second and the third.” At this point perhaps, some of the teachers, in reflecting on the lesson, were beginning to construct an image of the lesson that involved the development of ideas. In fact Bernardo stated that a case was established first, then a hypothesis, but did not indicate how one promoted the other—suggesting that he (and possibly others) were having difficulty coordinating their meanings for statistical case, relationship, and hypothesis.

In addition, the way the terms “case” and “hypothesis” were mentioned and ultimately included in the reason (Figure 31), which seems to imply that “hypothesis” and “case” are nearly equivalent, suggests that these terms were merely facts being established (outcomes) not coherent meanings and ideas that were being discussed amongst participants in the lesson.

Steps in the Lesson Logic

<u>Step</u>	<u>Action</u>	<u>Reason</u>
1.	Hand out raw data and ask, "Can we analyze this data?"	Establish need to organize data and establish a hypothesis/case
2.		

Figure 31. Adding hypothesis and case to the reason in step 1.

As the teachers were discussing whether to include case or hypothesis (or both) as a reason, Pat asked whether the teachers believed that any ideas or meanings had come up in their discussion that needed to be addressed. Prior to Pat's question, the conversation had remained strictly focused on attempting to recall the events of the lesson; there had been no conversations amongst the teachers regarding meanings or ideas.

Rachel moved the cursor to the "ideas" section and asked whether they wanted to add anything.

Excerpt 143, Class #13

Rachel: Okay, do we want anything in there? [...]

Rachel: Pat was asking do we want to put anything in those, so I was...

Story: On the top, Okay [...]

Johnny: One of the ideas was the fact that, uh, they have attributes. That data has attributes. And I'm thinking that's part of that when we work on that, when we have them analyze data, is that they have to have some

understanding about the attributes and how everything works within that.

Johnny: Because then we added, OK, like we looked at the teacher, so now each student there is three different attributes to that student...

Story: Or levels of the variable.

Johnny: Now, well the variable [...] Yeah, but you have a teacher, you have an anxiety score, and you have a test score, those are the attributes for each student. And there's this idea, I'm thinking to myself, that's part of what they have to understand. And I'm not sure if that's in the Lesson Logic, but I think it's one of the major ideas.

Rachel: Ideas are the meaning students must have, does that follow from the meanings? That they have to realize that those three things all correspond to one student?

Although Johnny asserted that “data has attributes” was an idea, he failed to give any meaning to what this idea entailed. In fact, he seemed unclear as to what meaning he had regarding attributes. In the lesson the teachers had engaged in as students, Pat stated that the fundamental unit of analysis in statistics was the case, which has attributes. Johnny's comment suggests a focus on the attributes. In addition, although Pat had pointed out that discussing important ideas is critical to constructing a Lesson Logic, Johnny questioned whether the “major idea” of data having attributes should be included in the Lesson Logic.

Excerpt 144, Class #13

Rachel: So, when we talked about when we move the numbers, if we move one, all those values have to move, you can't just change that one column.

Johnny: Yeah, that's what I'm saying, is that the attributes all linked to each individual student. That's the idea of a relational...

Rachel: So, is that a meaning? That they need to understand before they have this lesson?

Bernardo: That will unfold in the lesson.

Several: Yeah [...]

Faye: It will unfold, yeah.

Johnny: So, is that...

Faye: Important.

Johnny: It's important, I mean, yeah.

Rachel: So, do you want to just go back to the lesson logic? The steps, and see?

Rachel: Where do we want to be? Do we want to be in this area here [ideas], or do we want to be back in the steps right now?

Bernardo: Have we decided if that...if what Johnny was talking about is a major idea?

Marian: The attributes.

Bernardo: The attributes, is that something important?

Faye: I think so.

Rachel and Johnny's initial focus was on the action of moving one column (one variable); they are "doing" something (an action) and must understand that

what they “can do” is limited by the connected attributes. The emphasis appeared to be on how the connection of the attributes (to each specific case) limits what can be done, rather than a focus on the cases (the students) as the unit of analysis, and that these cases have attributes. This suggests a focus on actions being performed on objects—indicative of an empirical orientation toward learning and teaching mathematics.

Excerpt 145, Class #13

Johnny: Well, the fact is that when you start doing the sorting, everything has got to stay, you know, you gotta stay connected.

Marian: Yep.

Bernardo: Then, if I think that the answer to that is yes, then we put it up there...

Rachel: Where? Which one?

Bernardo: Under the, uh... major ideas.

Johnny: Major ideas, Yeah, see I’m thinking it’s a major idea.

Bernardo: The following lesson logic provides a structure in which the surrounding conversation unfolds these ideas.

Tami: Sorting the data.

Johnny: Data has attributes...relational attributes [...]

Story: The attributes stay linked to the student.

Faye: Yeah.

Liz: Data has specific cases...

Faye: i.e. teacher, test score,...

Bernardo: Data has cases and each case can have several attributes.

Several: Yeah.

Rachel: So, what do you want to say? [...]

Bernardo: Data has cases and each case can have several attributes [...]

Although the teachers listed Bernardo's statement as an idea, they concluded the discussion regarding case and attribute without delving into the development of these ideas, or discussing how this idea is connected to or helped to develop other ideas. Furthermore, although the teachers discussed what they wanted students to understand regarding this idea, specifically that if they were to move one of the variables, then all of the corresponding variables must stay together (or remain linked) with their specific case, they did not include this as part of their description of the idea (Figure 32).

The following lesson logic provides a structure in which the surrounding conversation unfolds these ideas:

1. Data has cases and each case has several attributes.
2. (Major ideas of the lesson, listed in a way that summarizes the logic)
- 3.

Figure 32. Listing the idea that data has cases with attributes.

Rachel next focused the discussion on meanings that students might need prior to the data analysis lesson.

Excerpt 146, Class #13

Rachel: Do they need any meanings before the lesson? So far? [...]

Tami: They should know what is meant by a case and certainly what is meant by a hypothesis.

Bernardo: I think just hypothesis, because case is something that will develop in the lesson [...]

Liz: Like can I sort? No, cause those have to go with it.

Rachel types in "What is meant by a hypothesis?" under the meanings section of the Lesson Logic.

Excerpt 147, Class #13

Bernardo: Hypothesis definitely.

Tami: Or even attribute. Students need to know what is an attribute.

Johnny: But that comes with the case though, the whole idea [...]

Rachel: Back to steps.

Although the teachers subsequently listed hypothesis as an understanding that was required for students to participate productively in the lesson (Figure 33), they did not discuss what meaning for hypothesis they had in mind.

Meanings students must have before the lesson:

- 1) What is meant by a hypothesis?
- 2) (Things students must understand at the outset if they are to participate productively in the lesson. This is not the same as things they must be able to do.)
- 3)
- 4)

Figure 33. Listing the necessity that the students have a meaning for hypothesis.

In addition, although they had earlier agreed to place “case” in the ideas sections of the Lesson Logic, no one questioned Tami’s suggestion that case was a meaning that students must understand. In addition, the teachers failed to address how a meaning for hypotheses was necessary for students to participate in the lesson or its role in developing any ideas. Not only did the teachers fail to make explicit the meanings that they believed were needed to participate productively in the lesson, they did not give meaning to the ideas that they viewed instruction as having addressed (e.g., case). The teachers next agreed to move back to the steps of the Lesson Logic.

Phase 2.4 – The type of organization matters. Rachel moved the cursor back to the action for step 2 in the Lesson Logic and 24 seconds quietly passed (there was some side conversation between Liz and Marian) before Bernardo asked when they wanted to mention that the kind of organization mattered. Liz stated that organization matters is what Pat had gone to next (during the lesson they had engaged in as students) and pointed to the white board where Pat had written (in Phase 1):

“Particular organizations can provide particular kinds of information,” such as:

- Ordering test scores from highest to lowest

Rachel asked what the action for this reason should be and Faye suggested that they write discuss organization of data. Pat warned that if the Lesson Logic is too vague in its creation, then people will not remember what to do, or that other

people who may attempt to use the Logic might be unclear as to what to do. Liz suggested that their action should be stated as: “Discuss why organizing data in certain ways matters.” The teachers agreed with Liz’s suggestion to use what Pat had written in Phase 1 (including the example) as their reason (Figure 34), without any discussion of how they understood this as fitting into the lesson’s logic.

Steps in the Lesson Logic

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<u>Step</u>	<u>Action</u>	<u>Reason</u>
1.	Hand out raw data and ask, “Can we analyze this data?”	Establish need to organize data
2.	Discuss why organization of data matters.	Particular organizations can provide particular kinds of information (e.g., ordering test scores from highest to lowest)
3.		

Figure 34. The action and reason for step 2.

After she had typed the agreed upon action and reason, Rachel next moved the cursor to the action in step 3.

Phase 2.5 – Diverging from the lesson that was experienced. Faye stated that they had next developed a hypothesis. Bernardo stated that at some point they had talked about the case, and asked whether that was when they had discussed hypothesis. Rachel stated that they had discussed case just before hypothesis, when Pat had asked if they could just change one variable as the teachers had attempted to organize the data. In the conversation that ensued, the teachers agreed that case was a big idea and that they did need to address it in the steps. Johnny commented

that when they had previously talked about case as a big idea, they had stated that it would get fleshed out in the Lesson Logic, so he felt that they really we needed to work on how they would accomplish that.

Tami stated that she liked the question that Rachel had earlier posed, that if you attempted to separate one variable, what would happen? Rachel asked whether this question should be included in step 2 and the group agreed to move back to step 2. Johnny stated that if there were going to organize data, then they needed to make sure that everything stayed connected, because each case had several attributes and that they needed to make certain that all of the information for one case stayed with that case.

Excerpt 148, Class #13

Johnny: If we are going to be organizing data, then we have to make sure that everything stays connected. Because, if you do this or that to make sure that when you do the sorting [...] each student has certain several attributes and you want to make certain that all the student's information stays with that student.

Bernardo: So, is that a separate step or is it like 2a, or does it matter?

Rachel: You could almost take the things and show the thing using a spreadsheet that has the data and just move one piece of information and have them see what's going on and see if they catch, "Oh yeah, I can't just focus on this one..."

Bernardo: Well, I think we...

Rachel: Well, that could be that where they're actually seeing...and see if they'll come up and say, "No you have to look at them together. You can't just let them look at that one column and move it around and have everything else stay the same."

Rachel: Because, some of them may not be able to pick that up if they don't see that...

Bernardo: Actually see that happen, yeah.

Johnny: Yeah, I think that that is a good idea.

Rachel: So, how should I say that?

Bernardo: Well, how about just say what you did, I mean, cause, I mean at this point we had our spreadsheet and...

Story: Well you're doing is making certain the attributes stay with the case, I guess is what you're doing...

Bernardo: Right, but we want them to see that.

Story: Yeah [...]

Rachel: Use software to move a single column of data [...]

Marian: I think software's going to come in after the fact, don't we want them to come up with an idea before we start showing them? [...]

Rachel: No [...] we're just trying to show that just even take a set of data and move, change one and show that it doesn't move everything else unless you keep them all together.

Rachel: But I don't know.

Annie: That would come later I think.

Story: That was just on the regular spreadsheet, though, you didn't really need
[...]

Marian: Oh, I see what you're saying...

Annie: That would come later.

Rachel: Well, that came about because we have that understanding, but kids
aren't going to have that understanding.

The activity that the teachers discuss is designed so that the students "see" that they cannot move [sort] a single variable (column of data in the spreadsheet) without moving all of the other variables that are connected to each specific case. This suggests that the teachers were thinking about a way to make their meaning of case visible to their students. Specifically, that everything associated with each case must stay connected. This was the first, and only, occurrence where the teachers appeared to be concerned with promoting an explicitly discussed and detailed understanding and how that understanding might be promoted.

This occurrence is also an indication of the teachers' decentering—attempting to see the mathematical understanding from the perspective of another (their students). Rachel asserted that "kids aren't going to have that understanding" in regards to the idea of case that they were intending to develop. This was the only occurrence in any of the reflective episodes where the teachers indicated that they were mindful of others' ways of thinking.

Phase 2.6 – Attempting to return to the lesson that was experienced. Liz

pointed out that if they changed the reason for step 2 that they would need to change the reason in step 1, because they then would not have discussed hypothesis and case until step 3, suggesting that Liz was thinking about the lesson's logic. Rachel moved "establish a hypothesis/case" from the reason in step 1 to the reason for step 3, and asked what the corresponding action should be for step 3?

Faye suggested that the action be to get kids to use software, and Rachel began to type: "Use software to move a ..." as the action for step 3. Some of the teachers expressed uncertainty with what Rachel was typing, which caused Rachel to stop mid-sentence. Marian stated that she felt that they were getting ahead of the game and that they had not touched software yet (in the lesson they engaged in as students). Faye stated that she believed that what they were discussing was new, not what they had done, but something that they could do at that point (with their students). Rachel asserted that this new step came about because although the teachers understood that the variables' values for each case needed to stay together, that their students would not have that understanding.

During this discussion Annie had stated that she thought that what they were considering would come later. Pat picked up on Annie's comment and had her elaborate on why she believed that they were getting ahead of themselves. Annie replied they had first constructed their hypothesis based on the original data and that they had not used any software at that point; Liz added that they had not even brought in the third variable yet. Each of these comments referred to what they

could re-call occurring, rather than whether it made sense regarding the lesson's logic.

Phase 2.7 – Uncertainty in recalling the lesson that was experienced. Liz suggested that the action in step 3 should be that Pat asked whether they could create a hypothesis based on the data that they had and in the form that the data was in. Rachel typed: "Create a hypothesis based on the data we have" and asked if that was what the group wanted (Figure 35).

Steps in the Lesson Logic

<u>Step</u>	<u>Action</u>	<u>Reason</u>
1.	Hand out raw data and ask, "Can we analyze this data?"	Establish need to organize data
2.	Discuss why organization of data matters.	Particular organizations can provide particular kinds of information (e.g., ordering test scores from highest to lowest)
3.	Create a hypothesis based on the data we have,	Establish a hypothesis/case

Figure 35. The action and reason for step 3.

Excerpt 149, Class #13

Liz: Yeah, I think #3, he started asking can we create a hypothesis based on the data that we have? [...]

Liz: So, the data that we have in this form, or this thing, and that's kind of where we came up with our different ones, to establish these hypotheses, these cases, that was addressed more from his questioning. [...]

Rachel: Is that what you want? [Quiet again for 17 seconds, although Marian and Liz again softly discuss something]

Johnny: I'm just wondering, when we...got the data with the teacher addition part to it, had we already...started doing with the hypotheses on those.

Several: Yeah

Liz: We did that...

Marian: We just had two variables in the first one...

Annie: We just did two variables...

Liz: And then after those came up we got the teacher one.

Johnny: And then the question was is there anything else we can see?

Liz: Yeah, so...We did this, create the hypotheses, umh...

Liz: I just think you're asking to create hypotheses, the reason is to establish one but I think there was more of a reason than that.

Rachel: But I think before we went there, they have to see the connectedness between the numbers.

Liz: And that's well, the hypotheses establish a case, was brought out in here, and then a hypotheses of what do we see, that's more...

Story: Of the relationship. So you have to look for relationships first. Did we talk about that [...]

Liz: I think that's kind of where we're at is to make a relationship, but part of them creating that brought out the idea of case. When we started talking about that was when, "Well if I was to try to show this," that's where the,

well we can't just sort things out we need to worry about, the case was brought out with this idea of coming up with hypotheses that we saw.

As illustrated in the preceding exchange, the teachers displayed uncertainty recalling the events they experienced in the order that they unfolded (one variable, two variable, hypotheses, case, relationships).

The exchange between Rachel, Liz, and Story suggests that they might have been thinking about lesson's logic, "Before we went there, they have to see the connectedness," and "the hypotheses establish a case," and "you have to look for relationships first." Although these statements, taken out of context, could give a sense that the teachers were considering developing ideas, taken in context, they continued to demonstrate that the teachers were simply attempting to reconstruct the order of the events—"We did this, create," and "before we went there," and "when we started talking about that was when."

The following exchange between Johnny and Liz exhibits Johnny's difficulty in coordinating his meanings regarding organization, statistical case, and hypotheses as he attempted to recall the order of the events. This suggests that Johnny, and perhaps other teachers, had developed isolated meanings, rather than a scheme of meanings (i.e., a KDU).

Excerpt 150, Class #13

Rachel: So, is this what we want and where do we go from here?

Marian: Maybe provide a better reason than just establish a hypothesis.

Marian: I think we need to establish a case right here, by the way.

Liz: Yeah, right, to establish case [Quiet for 10 seconds, with Liz and Marian continuing to communicate to one another]

Johnny: I'm just wondering though, I don't want to keep going back over but, we were given the data set like this and we were asked some questions about, you know, what kinds of things we could see and all that. And then all of the sudden, we get it organized in such a way, and unfortunately for most of us we already making connections, well more than likely, this data points connected over here, and it's more in an organized manner. And I think maybe some of us were already making connections, but we already know you got to keep them connected, because they're the student.

Liz: Yeah, but we knew that.

Johnny: Yeah, and see that's what I'm saying...But when we got it organized in this way, weren't we also getting to that point right away with the connection with the case? ...With each case being connected to the attributes...

Liz: But it wasn't brought out yet...

Johnny: Yea, and that's...

Liz: It wasn't anything you had addressed, it might have been something we thought about but we didn't address it or bring it out to part of our logic yet.

Johnny: I'm just trying to remember when...

Liz: Yeah.

- Johnny: We did this what were we putting this form first for?
- Liz: We just said that we couldn't see anything based on this, so we passed out another one. Based on this one...
- Annie: What could you see...
- Liz: What can you tell from this? And that kind of brought out well we need to organize things. We hadn't talked about how to organize it, which would then have brought out a case, until when well we started talking about making a hypotheses and how would we organize it, which then needs to worry about cases.
- Johnny: So I guess, OK, so I guess #2 is OK, because really that answers...that's why we were doing this for #2 then, discuss...because when you are organizing the data, that's when you could say, "OK, we'd had a rather unorganized data set with an organized data set," just in that manner. OK, ...I'm just trying to remember where we were [...]
- Liz : He asked us to establish a case and to start seeing a relationship between the data. Like, that's why you ask what hypothesis do you see, so that we can make relationships between the data that we see.

Phase 2.8 – The lesson that was experienced was not the logic of the

lesson. Prior to the group moving on to step 4, Pat interrupted and stated that at the rate they were moving, the teachers were going to be there a long time. Pat reinforced the notion that they were constructing the logic of a lesson that they had experienced, but that their ability to reconstruct that logic was not a straightforward

activity for them. Pat stated that what the teachers' felt that they had gotten from that lesson was not the logic of the lesson.

Pat asserted that constructing a logic of a lesson required the structuring of ideas, not the structuring of activities; and required the teachers to structure how they anticipated these ideas to unfold. Pat asserted that although constructing a lesson logic was difficult it should not be avoided, because working through a lesson's logic is where you identify those ideas that are key and that you can actually build from.

Phase 2 – Summary. From a global perspective, the teachers were unable to re-construct the lesson that they had just experienced as students. Although teachers understood every idea discussed in the lesson without difficulty, they could not reconstruct the lesson's logic. As the teachers began to reconstruct the lesson, they began to disagree about steps in the lesson and the logic of successive steps. Rather than reflecting on their reasoning, the teachers attempted to recall the products of their reasoning. As the lesson re-construction progressed, the teachers began to disagree about the sequencing of the outcomes (e.g., "Did we first establish a case or a hypothesis?"). Further on in their attempt to re-construct the lesson, the teachers began to introduce actions that had not occurred in the actual lesson, actions pertaining to how they would teach the lesson (e.g., focusing on using software to make the idea of statistical case visible to students). By the end of the lesson (as time ran out), the teachers had segued into constructing a lesson that they themselves would have taught.

Even though Pat had emphasized that the construction of the *Lesson Logic* (see Appendix I for the teachers' constructed *Lesson Logic*) required that the teachers think about and construct images of the understandings that they want their students to develop, the teachers did not discuss what it was that students needed to understand in order to participate productively in the lesson or the understandings that they developed as the lesson progressed. Throughout Phase 2, the teachers appeared disinclined to reflect on the meanings they held, the meanings they were building, or the ideas that were being developed—the logic of the lesson. Rather, the teachers simply attempted to recall the order of the activities (those things that Pat's actions required of them) that they had experienced.

In order to reconstruct the logic of the lesson they had experienced, the teachers would have needed to do more than simply recall the lesson. They would instead need to summarize the meanings and ideas that they understood as being promoted, and the activities and conversations which were designed to promote the development of these meanings and ideas. In order to construct the Lesson Logic, by reflecting on the lesson they had experienced as students, it would have been necessary for the teachers to take their thinking and meanings as objects of thought. The only times the teachers gave any indication that they were even thinking about meanings or ideas was when Pat pushed them to confront these issues, or when they diverged from reconstructing the lesson they had experienced to creating a modified lesson.

Phase 3 – Constructing the logic of an observed lesson and comparing the two logics. Phase 3 consisted of the following out-of-class assignment, due prior to the 14th (and final) class meeting:

- Complete (individually) a Lesson Logic for the lesson that they experienced in class (during Class #13).
- Watch a video (on CD) of a similar lesson that Pat taught to 9th grade algebra students (part of the reformed-Algebra 1 case study).
- Create (individually) a Lesson Logic for the lesson observed on the videotape (the similar lesson taught to 9th grade Algebra 1 students).
- Reflect on the sources of difference between the two lesson logics. Were they because the lessons differed, because your recollections differed, or because your understandings of the lessons differed?

Seven of the eleven teachers that engaged in the lesson during Class #13, completed the out-of-class assignment. In general, the logic re-constructions of the observed lesson (the reformed-Algebra 1 case study) were a dramatic improvement to the one's that the teachers had re-constructed of their own experience. The logic re-construction of the observed lesson provided a much clearer indication that the teachers were thinking about someone other than themselves (i.e., students) progressing through the sequence of steps, and their role (as the teacher) in moving students to progress through these steps.

As illustrated in Faye's logic (Figure 36), the capacity to re-watch the video of the observed lesson (the reformed-Algebra 1 case study) allowed for the logic re-constructions (i.e., the steps) to be much more detailed and explicit.

2.	• Teacher shows students data on computer software Tinker Plots. • How can we organize/display the data to answer a question? • Stack data on software • Why do we get less in the group when we make a smaller range? • Can you say how anxious group is when the data is scattered all together? • Show different displays • What questions can you answer by looking at the display, or changes to it? • Where are the most anxiety levels? • Suppose you ask if anxiety levels are evenly distributed...looking at what we have now, can you answer the question clearly? (With scattered data) But if you look at it the way we did (separated into ranges) now can you answer question? • Can you tell if any teacher has greater anxiety problem than any other? If can't can reorganize data.	• To show how you can organize data any way you want • Show each circle represents one student • To get a sense for quantities in a given range • Big Idea: can answer question by organizing data • Can have a question about a situation and gain insight into the question by changing the way the data is displayed.

Figure 36. Portion of Faye's lesson logic from observed 9th grade Algebra 1 class.

The majority of the observed logic re-constructions focused on making things visible to the students—indicative of an empirical orientation toward teaching mathematics. As illustrated in Liz's logic (Figure 37), rather than managing the

lesson so that students focused on their own meanings, and reasoning, and the meanings and reasoning of other students (and made their own connections), the teacher “forced or allowed students to see” or “made students think.”

6.	Have students come up with questions that might have been asked that would have caused us to create this graph.	Make students think of reasons as to why we would organize data.
7.	Looking at the data, have students come up with questions that cannot be answered using the data as it is organized.	Force students to see that just because we can organize the data in a specific manner does not mean it will help us answer all possible questions about the situation.
8.	Have students start sketching how it might be organized to answer a question that the students come up with “Do different classes have different anxiety levels?”	Give students an idea of what they will be trying to show. Help them see the importance of different representations of the same information.
9.	Create the graph as described; discuss if the graph help answer the question or not.	Allow students to see if the need to reorganize is present and to begin discourse of the
10.	Assign groups and pass out questions for groups to work on, allow groups to pick a question they want to answer	Students feel more involved when they are allowed to choose what they work on. Plus they usually will pick something that interests them.
11.	Students sketch the display that should help them answer the question, then have instructor create the graph.	Give students the opportunity to decide how things should look in order to answer the questions they chose.

Figure 37. Portion of Liz’s lesson logic from observed 9th grade Algebra 1 class.

In addition, Rachel’s logic re-construction (Figure 38) illustrates that, although the logic re-constructions of the observed lesson provided a clearer logical reason, in terms of student learning, for each instructional action (each step), these steps were not consistently logically connected. Rachel’s logic also suggests that a

major focus of the teacher's instructional actions were to gauge students' levels of understanding, not necessarily for students to reflect on their own conceptions.

3.	Using Tinker Plots organize data by separating anxiety levels into 3 intervals.	Allows students to see how data is grouped in each range.
4.	Make intervals smaller (Why do we get less in groups as the intervals are made smaller?) while increasing the number of groups.	Shows how the data will go on either side of the dividers
5.	Compare scattered data plots to a display where data is arranged neatly in intervals	The neatly arranged data shows that most of the data falls in the middle intervals rather than the edges
6.	Stacked the data	Gives a clearer image of how many students are in each interval; like a histogram
7.	Asked "What question can you get from looking at the data?"	Give the teacher an idea of what students are thinking as they look at the data
8.	Organize data in a graph so that the question posed can be answered	Students can see from the data display that anxiety scores are not evenly distributed
9.	Asked the students to come up with a question that could be represented by the data, but isn't currently shown, and made sure they all had think time	See if students are relating info given and how the change of the display could represent the situation

Figure 38. Portion of Rachel's lesson logic from observed 9th grade Algebra 1 class.

Also absent from these seven logics were indications that the teachers were reflecting on the meanings that they believed that students needed prior to the lesson, and the ideas that they intended for students to develop. From the seven logics that were submitted, Liz failed to list both the meanings that she believed students would need to productively participate in the lesson, and the ideas that the

surrounding conversation would unfold; Faye and Janine listed only the ideas that the surrounding conversation would unfold.

Sample ideas that the teachers listed as having unfolded in the 9th-grade Algebra 1 class were as follows:

Faye:

1. If you have a question about a large data set, you can answer it by organizing the data.
2. Can have a question about a situation and gain insight into the question by changing the way the data is displayed.

Rachel:

1. How to organize data so that you can answer questions.
2. Calculate things as summaries of what is seen in displays.

Sheila:

1. How to take data given in raw form, organize the data so questions can be answered and conclusions can be drawn.
2. How to make mold data to address specific questions.

As illustrated, these ideas involve performing actions on objects (i.e., data) and focus on physically manipulating the data to produce some outcome—indicative of an empirical orientation toward teaching mathematics.

Sample meanings that the teachers listed as being important for students to possess prior to the lesson (from the 9th-grade Algebra 1 class) were as follows:

Alyce:

- 1) How to read the horizontal and vertical scale of a graph
- 2) How two values covary

Marian:

- 1) Variables
- 2) Histograms

These “meanings” are not well articulated. Marian, for instance, listed concepts (i.e., objects) without providing any indication of she meant by “variables,” and did not indicate in the steps of her lesson logic where or how these understandings were important for student learning. In Alyce’s logic, reading the scale of a graph is simply an action (reading scale) performed on an object (graph). This action does not identify meanings or reasoning, nor how it relates to the lesson’s logic. In addition, Alyce did not articulate what, regarding covariation, she believed was important for students to Although the teachers appeared more capable of taking the point of view of a student engaged in the lesson (as the student moved through the sequence of steps) in the observed lesson logic re-construction, and how they (as the teachers) could move students through the steps, the focus was still on the teachers doling out the mathematics, and making things visible to the students. In addition, the teachers were still constrained to take their meanings, their reasoning as objects of thought.

Only three of the teachers completed the fourth part of the out-of-class assignment: Reflect on the sources of difference between the two lesson logics. Were they because the lessons differed, because your recollections differed, or

because your understandings of the lessons differed? Three additional teachers were unable to complete the assignment due to having missed all or parts of Phase 1.

Marian asserted that the main difference in the lesson logics were due to her perspective, one as an observer of a lesson, the other as one engaged in experiencing the lesson. Faye stated that the main difference in her logics was that she “streamlined the main ideas” in the beginning of the lesson logic of the observed lesson, although she did not articulate what she meant by “streamlined the main ideas.” Rachel asserted that the differences in her logics were due to minor differences in the lessons themselves, such as in the observed lesson the students “started off with the data that contained teachers, test scores, and anxiety score and then very quickly moved into Tinker Plots and they spent more time asking different questions and rearranging the data to fit that situation.”

Phase 3 – Summary. The improvements in the teachers’ lesson logic re-constructions, from the re-construction of the lesson they experienced as students to the lesson they observed, consisted mainly in an improved capacity to decenter. The teachers’ logics were less egocentric, less based on their own perceptions, and more focused on moving students through the sequence of steps. The reasons for this could be that the teachers were basing the re-construction on what they observed, not attempting to recollect what they perceived. Alternatively, the fact that the teachers could rewind the video (of the 9th grade Algebra 1 course) allowed for them to not have to base their logic re-construction on what they re-called

perceiving and allowed for the focus to be more on what they observed—what questions Pat asked, what instructional action did Pat make, and how did students respond or react.

Phase 4 – Whole-class discussion of phases 2 and 3. Phase 4 of the reflective episode was designed to move the teachers to reflect on their conceptions of the two lessons and the two lesson re-constructions. Specifically, the teachers were provoked to compare and contrast their understandings of the two lessons and their re-constructions of the two logics. In addition, Pat attempted to move the teachers to articulate their conceptions of lesson structuring. Phase 4 of reflective episode #4 consisted of the first 34.9 minutes of Class #14 (December 3rd, 2007).

Phase 4.1 – Orienting the discussion: A comparison of understandings.

Pat initiated Phase 4 by stating that he had been fascinated by the differences in the teachers' understandings of the lesson that they had participated in and their understandings of the lesson that they had watched.

Excerpt 151, Class #14

Pat: If we could talk about that for a little bit, the difference between experiencing a lesson and observing a lesson [...] I don't mean that in general, but in particular...reflecting on the lesson that you participated in in comparison to reflecting on the lesson that you watched. Is there a difference, and if there is, what is it? [...]

Tami: I think it's much easier as a teacher to participate in something, because then you have an idea of exactly what your students are

experiencing...and you have a better idea of what to look for, what questions to ask, umh, what procedures to go through, maybe some nuances to share with the students versus just watching and basically critiquing somebody else.

Pat: Anybody else?

Liz: It think it was easier to be more detailed on your lesson you observed, like to pick out the little things that are trying to be brought over than what you're trying to learn as it goes on. It's hard to see the little details of what's being asked or what you're thinking about while you're doing it, versus watching a group of people go through it, it's easier to see the little details, "Oh, this question was asked probably to bring this out, it was easier to notice that." [...]

Tami's comment suggests that she believed that experiencing the lesson prior to teaching it provides the teacher with insight into what to expect, the questions that you (as teacher) might want to ask, and "nuances" to share with the students. What Tami did not indicate was whether this insight would emphasize her perception of the lesson or how she imagined a student engaging with the lesson.

Liz's comment indicates that she believed that observing the lesson allowed for her to better understand why Pat asked the questions that he did (each individual question), but she did not provide any indication that this helped her to better understand the lesson's logic.

Next, Rachel asserted that the video allowed her to re-wind sections that she was not clear about, which enabled her to focus on what questions Pat had asked and how the students had responded. Faye agreed with Rachel's comment and asserted that she was able to catch Pat's "questioning technique" by watching the video, although she did not articulate what she believed this "technique" to be.

Phase 4.2 – Separating teacher mode from student mode. Faye continued by indicating that she experienced the lessons in different modes, student versus teacher mode, and found that this allowed her to better understand the lesson.

Excerpt 152, Class #14

Faye: I was kind of going through the experience at the same time in student mode, not thinking about teacher mode in a sense, so it was nice to sit back and watch...and I liked watching that after we had gone through it, I think it would be different if we had just watched something, but after experiencing it, I could get a better understanding for, uh, the steps and maybe the reasoning.

Pat: It sounds like you're saying that after having gone through it and then watching it, then you can think more clearly about both sides of it, about the student experience and the teacher experience. [...]

Johnny: While we're watching I'm thinking, "Oh, what was I thinking myself when we were looking through that or some of the questions that were being asked." And so, that kind of colored how I viewed it, too...so I reflected back on how I experienced it, so...I have an idea of what the students

going through, but also you do start picking up on some of the nuances of the questioning [...] after we've gone through and then watching, now you go, "Oh, now...I can get a clearer idea where you're going with it."

Faye's comment suggests that she believed that the sequencing of the phases (first experience the less as a student, then observe the lesson as a teacher) to have benefited her understanding for the steps and "maybe the reasoning." This suggests that such sequencing might have helped her to better conceptualize the lesson's logic. Johnny's comment suggests that his focus in observing the lesson was on attempting to recall how "he" understood the lesson. Although Johnny asserted that the sequence of the phases allowed him to get a better idea of where the lesson was going, his comment suggests that the logic that was clearer (to him) was the logic of how he understood the ideas, rather than how a student might come to understand the ideas.

Phase 4.3 – Discussing phase 2: Difficulty re-constructing the lesson. Pat next asserted that one of the things that had struck him, during the teachers group re-construction was the difficulty that the teachers had in re-constructing the lesson that they had just experienced. Pat asked the teachers to compare the re-construction process of the two lessons. Rachel asserted that the observed lesson was easier to re-construct, because Pat had been much clearer in the video where the lesson was going—that Pat had explicitly stated in the video what they were going to do and where the lesson was headed. Rachel's comment suggests that she was not thinking about how she conceived of the lesson's logic, but rather required

Pat to detail the logic (although it is unclear whether Rachel was focused on the steps or the logic).

Phase 4.4 – Being mindful of students’ understandings. Pat next moved the teachers to think about what they believed the students in the video had experienced in the lesson.

Excerpt 153, Class #14

Pat: Let’s try to characterize your best take on the students’ experience in the videotaped lesson that you watched [...] How did the lesson play out, do you think, for the students who were in there? What was their experience? [...]

Faye: I think they learned a lot about different ways to display the data [...] and it really hit home that one circle represented one person [...]

Pat: It made it very clear that this was data collected on cases. [...]

Faye: I think the kids learned from it, you could tell that they were able to come up with an idea to answer a question, or at least to test it [...]

Pat: From your perspective, what do you think was taught?

Eve: I remember at the end, you gave us a piece of paper with questions, and we had to come up with the graph, so we can use that graph to answer the questions. But we need to be able to put the right object into the right axis, that’s what I get from that lesson...I think kids probably do the same thing [...]

As the preceding excerpt illustrates, although the teachers were better able to re-construct the lesson that they had observed, they still exhibited difficulty in identifying the meanings and ideas that the lesson developed. In addition, in commenting on what the students in the video were taught, Eve focused on a specific action (on an object, the graph) that *she* had engaged in during the lesson that *she* experienced as a student and translated that experience (i.e., what she found as being significant in her experience) to the students—her egocentrism constrained her capacity to think about what the students in the video might have experienced.

The next three minutes were spent discussing a particular student in the video who had asked for a specific type of graph (which could not be constructed) and how Pat had made it clear to the students (in the video) that they could ask for a different organizational display if they were not satisfied. Pat next moved the conversation back to Eve's remark and his question regarding what was taught in the lesson. Rachel responded that, "The way your data is represented will help you answer various questions." The conversation concluded with Pat asserting that if the students were practicing anything, "they were practicing making decisions about how to display their data."

Phase 4.5 – Orienting the discussion toward lesson structure. Next, Pat moved the teachers to articulate their conceptions regarding a lesson's structure.

Excerpt 154, Class #14

Pat: So, did the lesson have a structure to it? [Quiet for 11 seconds]

Faye: Not strictly, I mean you knew what you wanted to accomplish and you had certain tasks do that, but there was room within the structure for students to be creative [...] so, I think you had it structured in terms of what you wanted to accomplish, but not necessarily, “This is what we’re going to do first, that’s second, and then third, and then move on” [...]

Faye’s comment suggests that the tasks accomplished what Pat had envisioned the lesson accomplishing regarding student learning. In addition, although Faye indicated that the lesson had a logic, but not a rigid sequence of events, she did not articulate what the logic was or how the logic unfolded (i.e., how Pat managed the lesson).

Pat next stated that the lesson was designed to give the students room to be creative, that the lesson provided the students with “some ideas to work with, a hint at ways of thinking, and some...ideas of capabilities, things to do, and then a context in which they could employ them in a natural way.” Pat next asked where the teachers believed the teaching was, which resulted in Alyce asserting that Pat had relied on the students’ prior knowledge with statistics. Specifically, Alyce asserted that the students’ prior experiences with graphing and charts had allowed the students to progress successfully through the activity. Pat stated that he had not consciously counted on the students being experienced at constructing scatter plots.

Excerpt 155, Class #14

Pat: Where was the teaching?

Sheila: Student to student [...]

- Pat: Sheila said student to student. [Quiet for 15 seconds]
- Tami: But also a little with you manipulating the computer and showing them things that were possible...
- Pat: In terms of the demonstration that I did?
- Tami: Yeah [...]
- Eve: You showed them that a graph can help them see [...] they were talking about it, having discussion between them [...] so, they are teaching each other [...]

Pat asserted that there had been a lot of student-to-student interactions, that the students appeared to be very much on task, and the interactions were productive (i.e., the students made each other think).

Excerpt 156, Class #14

- Pat: But in a standard sense of teaching, where was the teaching? [...]
- Sheila: In the standard sense there wasn't any [...] here's an example, you do an example with me, now you do twelve of them, if that's what you mean by the 'standard sense' [...]
- Pat: So, in that sense, there was no teaching.
- Sheila: Right. [...]
- Janine: You asked them quite a few questions in the beginning, because you first put the data up there as a big mess. And then you asked them if they could answer any question with this. And then you organized the data and asked them, "Now what questions can I answer." You kind of led them

through the questioning in the beginning, you had some questions that drew them into what you wanted them to see, that you want your data to be useful, you have to organize it to fit your needs. And you have to organize it differently if you want to answer different questions. And you kind of demonstrated that for them in the beginning before you set them free. But it was what you did before you set them free that gave them what they needed to do what they...what you wanted them to do.

In the preceding excerpt, Sheila asserted that she was aware that Pat had not taught the lesson in a traditional sense. In addition, Janine's comment suggests that she was aware of how Pat managed the lesson so that he initially set the stage. Specifically, how Pat had moved the students to focus on what he believed was important for them to focus on, the issues that Pat believed the students needed to be cognizant, and the capacities of the computer software, prior to letting them go off into their groups to work on the problems.

Pat next described how he had structured the lesson, which anticipated a discussion on how the teachers structure their lessons.

Excerpt 157, Class #14

Pat: The beginning was highly structured, I was in control, which student responses I would pick up on. There was a very definite end point, where students start thinking about data, organizing it to answer questions, and thinking creatively about displays. Then, these questions that I handed out added another level of structure, but now much looser [...] I did not

ask the students, “Okay, go investigate interesting questions, ask a question and investigate it.” [...] Either they wouldn’t know what to do, they wouldn’t have a question to ask, or if they asked a question it would be very trivial. And so, the questions, then, that I made up were a second level of structure.

Phase 4.6 – Making the teachers’ conceptions of lesson structure explicit.

Pat used his description of how he had structured the data analysis lesson with the 9th grade Algebra 1 students to move the discussion toward how the teachers structure their own lessons.

Excerpt 158, Class #14

Pat: So, How do you think about structuring lessons? [Quiet for 15 seconds]

Sheila: For me it varies, because I teach freshmen and I teach calculus, so it’s not the same. And even within freshmen, I think, the structure of my lesson is different depending on the topic that I’m discussing.

Pat: Go ahead and say more. [...]

Sheila: What have we done...we’ve explored systems of equations, and just solving them by substitution, nothing further [...] I’ve tried to work to get them to tie it back to, what is a line, you know, we’re finding these two numbers what are they? [...] with re-writing formulas, a much different structure, it’s just a skill they need to have [...]

Liz: It depends on what level or what class I’m teaching [...] my off-track classes, I tend to be a lot more structured, providing them more guided

things or more examples and things like that...maybe my freshmen class or my higher-level student classes, I tend to, umh, be more of the open discussion type things, "So, what do we think?" On some concepts, complex numbers, how to simplify those things, I don't think I necessarily truly understand everything about complex numbers, so I tend to fall back on a very structured...me standing at the board teaching, "You need to know this" [...] Some of it depends on my familiarity with things, too, as to how I teach [...]

Both Sheila and Liz asserted that the way they structure a lesson is dependent upon the topic and their conceptions of their students. For Liz, her familiarity and comfort level with the topic, and whether she conceived of her students as being of a higher-level or not, determined whether she employed direct instruction (e.g., the teachers "standing at the board") or more open discussions. For Sheila, her conception of lesson structure appeared to be more dependent on the topic and, to a lesser extent, her students' levels of mathematical experience (e.g., freshmen versus calculus students). Neither teacher has indicated that they consider both the students' prior knowledge and their future learning, when considering a lesson's structure. Although Sheila did indicate that she attempted to make connection among topic whenever possible, her comment suggests that she does not think about the coherence of the entire curriculum.

The conversation next moved to whether having taught a class more than once, and having taught the next class in a sequence of course, changed the way the teachers structured their lessons.

Excerpt 159, Class #14

Sheila: Teaching the next course...it does completely change your conception of what is it that you're trying to get them to learn, because now you know where they need to go.

Faye: Exactly, yep.

Sheila: So, if you teach nothing but freshmen and never teach that sophomore level you don't have the same...

Faye: Backwards-planning almost, where they need to go, and what they need to know...

Sheila: Yeah, vision of why are you doing that for them [...]

The preceding excerpt suggests that at least some of the teachers had considered their students learning in more than just isolated courses. Although this suggests that the idea of making the curriculum coherent was at least part of their thinking, what was not clear was whether the teachers they were thinking about coherence in terms of their own understandings or the understandings of their students.

Phase 4 – Summary. Although the logic re-constructions that the teachers created of the lesson that they observed suggested an increased capacity to decenter, the conversations throughout Phase 4 suggested a general egocentric

orientation toward both lessons. Several of the teachers commented that as they observed the 9th grade Algebra 1 lesson, they attempted to recall their experiences, how they had understood the lesson that they engaged in as students. Several of the teachers commented that the ability to rewind and replay segments of the video allowed them to get a better understanding of Pat's instructional actions (particularly his questions) and the reasons for those actions. Although the capacity to replay the video appeared to assist the teachers' in conceptualizing the reason for any individual step, it did not appear to benefit their understanding of how the lesson was sequenced to build meaning and develop ideas (the overall logic of the lesson).

Reflective Episode #4 – Summary

There are several possible reasons for the difficulties that the teachers exhibited in their attempts to re-construct the lesson. One reason involved the teachers' own mathematical understandings—the teachers had not developed a KDU for the role that organization plays in data analysis (i.e., they had not transformed their own mathematics understanding). Rather than having developed a scheme of meanings (e.g., how organizational display influences what can be analyzed, the statistical case, constructing hypotheses, distributional analysis, and conditional probability), the teachers had developed isolated meanings.

In addition, throughout the re-construction process the teachers were disinclined to take their meanings as objects of thought—indicative of an empirical orientation toward learning and teaching mathematics. This constrained their

capacity to reflect on their reasoning as they attempted to re-construct the lesson. Rather than reflecting on their coordinations of meanings (e.g., the role that organization plays or can play in data analysis, the idea of a statistical case, the role and nature of hypotheses in statistical reasoning), the teachers focused on the products of their reasoning (e.g., established statistical case, established hypotheses, organization matters). Their focus on the products of their reasoning constrained their capacity to reflect on their reasoning—their reasoning was hidden in the products, and impeded their capacity to develop thematic images of that reasoning.

Furthermore, during Phase 1 of the reflective episode the teachers focused on their participation in the lesson to the exclusion of thinking that the lesson might have a logic. As a consequence, the teachers were focused on their activities as learners, not as managers of others learning these ideas. Therefore, the teachers' egocentricity and inattention to meaning during the lesson constrained their capacity, later, to imagine the lesson in a way that they might lead it.

Finally, Phases 2-4 suggests the possible benefits to having teachers watch video of themselves engaged in a lesson as students. This would allow for discussions to focus on meanings, ideas, and lesson structuring, without needing to rely on the teachers' capacity to recall the sequencing of events. In addition, as suggested at the end of Phase 4, allowing teachers to experience a sequence of courses (either as a teacher or through video), might assist teachers in focusing their attention on making ideas coherent throughout the sequence—although the emphasis must be placed on making the ideas coherent to the students.

Summary of Reflective Episodes #3 and #4

As illustrated in the preceding sections, the teachers' empirical orientation toward learning and teaching mathematics focused the teachers' attention on their own perceptions of the lesson that they had engaged with as students, on objects and actions that they perceived as being applied to those objects, and on the products of their reasoning. Such a focus constrained their capacity to attempt to take the point of view of others and to take their meanings as objects of thought.

Although the teachers were able, for the most part, to re-construct the lesson involving concepts of speed (or at least more effectively than the data analysis lesson), they did so by being more proficient at re-constructing the sequence of events—where, these events were again the products of their reasoning. In addition, although the teachers were more proficient at identifying how one outcome (i.e., product of their reasoning), led to the next outcome, they were unable to re-construct how ideas were developed or meanings were built (i.e., the lesson's logic).

If the teachers had more sophisticated understandings to work with in the reflective episode involving concepts of speed than they did in the data analysis reflective episode, and these understandings played a role in supporting reflection, then what impeded their capacity to reflect productively? I hypothesize that an empirical orientation to learning and teaching mathematics provides an explanation. Although the teachers might have developed robust understandings and ways of thinking involving concepts of speed, the teachers were oriented

toward a focus on the products of their reasoning, rather than on taking their meanings as objects of thought.

In addition, as with data analysis lesson, the teachers focused on their participation in the lesson (during Phase 1) to the exclusion of thinking that the lesson might have a logic; and, as a consequence, focused on their activities as learners, not as managers of others learning these ideas. Furthermore, this reflective episode highlighted the teachers' inability to "see" a managed conversation. Not only did the teachers express difficulty in imagining instruction designed to promote the articulation of meanings and reflective discourse, they appeared to be unaware that they had been participants in a managed conversation.

CHAPTER 8

DISCUSSION AND IMPLICATIONS

This concluding chapter highlights the study's key findings as well as its contributions, implications and limitations. Specifically, I will discuss how an empirical orientation toward learning and teaching mathematics constrains a teacher's capacity to reflect on their practice, as illustrated by the participating teachers. I will describe teacher education and professional development instruction that this study suggests might prove productive at transforming such an orientation. In addition, I will detail the limitations of the study. Finally, I will conclude the chapter by returning to the perspectives of Silverman and Thompson and of Ball, Hill, and colleagues, to describe this study's implications for any framework's characterizations of the development of mathematical knowledge for teaching.

Discussion

A salient finding of this study is that the participating teachers consistently demonstrated a tendency not to reflect on their practice. I could find very few instances that suggested any kind of reflection on the part of the teachers though they had many opportunities, from my perspective, where it not only would have been natural to do so, but instruction was designed so that they would. I saw few indications of teachers questioning their own or their colleagues' meanings, questioning their assumptions, questioning the coherence of their meanings, and questioning whether their meanings were aligned or even compatible with those of

their students or colleagues. The few instances that did suggest reflection included the teachers' discussions involving speed at an instant (p. 149-153) and constant speed (p. 281-283).

In order to help me to understand why teachers were disinclined to reflect, I asked how teachers were thinking so that it would be *natural* for them not to reflect. I explored the data corpus in an attempt to identify how the teachers were operating. From my analysis of the data corpus, I identified two particular epistemic ways of operating: (1) a disinclination to focus on meanings and (2) a disinclination to be mindful of others' mathematics. These ways of operating that teachers' possessed constrained their capacity to reflect productively.

These epistemic ways of operating, both characterized as a "disinclination" to act in specific manners, merit further comment. As a construct, a "disinclination to act" might imply unwillingness or an aversion to doing something, which is not what I have in mind. Rather, by "disinclination," I mean that the teachers' orientations were such that it was natural for them not to focus on meanings, and not to be mindful of others' mathematics. Therefore, I use "disinclined" to summarize a persistent pattern of the teachers not doing something that included those occasions where they were urged to do it, but still did not. For example, a "disinclination to make meanings explicit" points to many occasions where they did not make meanings explicit, and to occasions where they were urged to say what they meant but still did not.

One might ask, “There were many things that the teachers did not do besides reflect. For example, they did not focus on the quantum nature of light. Why is it then, that I am not concerned that they did not focus on the quantum nature of light?” The answer lies in the fact that there were no occasions where it would have been either natural or appropriate for them to do so. Conversely, the Functions 3 course was explicitly designed to provide the teachers with opportunities to reflect; yet, they consistently demonstrated a disinclination to do so. Therefore, my concern with the teachers’ tendency not to reflect is not only understandable, but warranted.

I proposed and attempted to operationalize an orientation toward learning and teaching of mathematics (an empirical orientation) as a potential way of thinking that would account for the identified ways of operating and help to explain the teachers’ reticence to reflect. Such an orientation is characterized by a platonic ontological view of mathematics, and is egocentric and empirical in nature. An empirical orientation focuses one’s attention on the visible and constrains one’s capacity to assume the point of view of others.

A teacher with an empirical orientation toward learning and teaching mathematics focuses on his personal perceptions, and conceptualizes mathematics as consisting of objects and actions that the teacher envisions being applied to those objects. In addition, such an orientation places an emphasis on providing students with opportunities to “understand” objects by engaging students in tasks that involve performing actions on those objects, although the focus is not on actions (i.e., reasoning). Rather, the focus is on performing action sequences involving

procedures, skills, and facts to arrive at an outcome. The focus is on the product of reasoning (the visible), where the reasoning itself stays hidden.

The identified epistemic ways of operating and empirical orientation assisted me as I attempted to answer the three research questions.

Summary of Findings

Regarding the specific research questions that this study attempted to answer:

Research Question #1: In what ways do teachers' mathematical understandings and ways of thinking support or constrain their capacity to reflect on their practice?

An empirical orientation to learning and teaching mathematics, as exhibited by the teachers in this study, is one that does not focus on meanings, but rather emphasizes a person's perceptions. Teachers' focused their attention on objects and on actions applied to those objects rather than on their meanings or their reasoning. This focus hindered the types of abstraction available to the teachers, in essence making reflection unattainable. In addition, the teachers' disinclination to decenter, placed their focus on the products of their reasoning; in effect their reasoning was hidden in the products. This preparatory step to reflection impeded their capacity to develop thematic images of that reasoning.

Research Question #2: In what ways do teachers' images of their students' mathematics support or constrain their capacity to reflect on their practice?

An empirical orientation to learning and teaching mathematics, as exhibited by the teachers in this study, is one that is egocentric in nature—one that is not

mindful of others' mathematics. Rather than focusing their attention on how someone (other than themselves) might come to construct mathematical understandings and ways of thinking, the teachers operated unproblematically from the basis of their own understandings. This disinclination to decenter constrained their capacity to reflect on how their actions (as teachers) could help promote the development of understandings and ways of thinking in their students. In addition, the teachers' focus on their own perceptions constrained their capacity to think about the lessons that they had engaged in as students as having a logical structure. This constrained the teachers from imagining their management of conversations around that logic were they to engage their own students in similar lessons. In effect, teachers took their own mathematics, by default, as their students' mathematics.

Research Question #3: In what ways do teachers' images of the tasks they will employ in their teaching, in relation to their mathematical understandings and their understandings of their students' thinking, support or constrain their capacity to reflect on their practice?

An empirical orientation to learning and teaching mathematics, as exhibited by the teachers in this study, is egocentric in nature, focuses on one's own perceptions, and focuses on objects and on actions being applied to those objects. Rather than focusing their attention on their role (as the teacher) in providing motivation for and opportunities to reason, abstract, generalize, and justify, the teachers focused on getting students to come to understand objects and the

important actions (as identified by the teachers) on those objects. Such an orientation obscured the pedagogical notion of managing a conversation and appeared to provide unwarranted agency to tasks, thus limiting the teachers' role to that of making the subject matter visible to the students and assessing the students' levels of correctness. Although my descriptions of the teachers' images of the tasks that they employed in their teaching are speculative at best (due to the nature of the data corpus), the very small roles that meanings and reasoning play in an empirical orientation toward the learning and teaching of mathematics warrants further investigation.

In addition to the three research questions, my analysis suggests that, for some of the teachers in the study, their personal mathematical understandings were intertwined with their conceptions of their students and their teaching. Simon and Tzur (1999) describe similar "complex interrelationships among different aspects of teachers' knowledge and their relationships to teaching" (p. 263), including the teacher's personal mathematical knowledge, their understanding of students' thinking, and their conceptions of their role as teacher.

For some of the teachers in the current study, their capacity to reflect on their conceptions of their students (to construct a mathematics of students) was hindered by their egocentric orientation. This was exhibited during instruction designed for the teachers to re-conceive their teaching and re-conceptualize their students. For these teachers, Pat's intent to provoke reflection (on the part of the teachers) regarding the teachers' implementation of instruction (with their own

students) similar to that which the teachers themselves engaged in as students (to develop similar ideas and ways of thinking) was constrained by the teachers' capacity to decenter. For example, Janine's egocentrism, and her belief that the finger tool did not work for her (because she could visualize two quantities changing simultaneously in her head), did not allow her to conceive of how a focus on covarying quantities and graphs as emergent aspects of covariation might benefit her students who could not reason as she did—she could not get past her own ways of thinking to imagine her students reasoning without it (p. 238-240).

For other teachers, their capacity to reflect on their own personal mathematical understandings was constrained by their conceptions of their students. For example, Story appeared unwilling to reflect on her own conceptions regarding speed, because of how she viewed her students' reaction to a conception involving "fractions" (p. 130-131). Specifically, that her students would not be open to the idea of a proportional relationship between distance traveled and amount of time required to travel that distance, because the relationship "involves fractions." Pat's intent to provoke reflection regarding Story's own mathematical understandings was constrained by her view of students. Furthermore, these conceptions were based on her beliefs about affective aspects of her students, such as her belief that students would be resistant to, and therefore would not be capable of, developing certain ideas or ways of thinking.

Implications

In explicating their notion of “mathematics worth teaching,” Romberg and Kaput (1999) describe curricular activities that involve students in problem solving that encourage mathematization. Such tasks include situations involving quantification, variation, and context. Such tasks also require students to express ideas and meanings, to reason, to compute, to abstract, to generalize, to justify, and to formalize. This envisioned curriculum places its focus, not on the visible, but on that which remained hidden to the teachers in this study—hidden to teachers with an empirical orientation toward learning and teaching mathematics. If an empirical orientation toward learning and teaching mathematics, as exhibited by the teachers of this study, is common among practicing secondary mathematics teachers, then the goal of immersing students in meaningful mathematical experiences seems a daunting task indeed.

The two lesson re-construction reflective episodes are particularly relevant to the idea of “model lessons” in teacher education and professional development settings. Mathematics teacher educators often think that if you want to help teachers improve their teaching, then you should: teach teachers what you think they should teach and model how you believe teachers should teach it, utilizing the materials that you believe they should employ. The reflective episodes involving data analysis and acceleration were two such “model lessons” and the participating teachers could not re-construct the logic of either lesson. Although the teachers were able to re-construct (for the most part) the sequence of actions in the lessons

(more so for the speed lesson than the data lesson), they could not re-construct the lesson's logic—they appeared constrained in their capacity to conceive of the lesson as *having* a logic. Though we were unable to observe teachers attempting to replicate either lesson, the data provides little reason to suspect that they could. Rather, the data suggests that the lessons they would teach would be compatible with their empirical orientation—lessons that focused on objects and on performing a sequence of actions on those objects.

This study's finding suggest that in order for such model lessons to have the possibility of achieving their desired effect, or for classrooms such as those envisioned by Romberg and Kaput (1999) to have a chance at becoming a reality, requires more than providing teachers with experiences that focus on either their personal or pedagogical mathematical understandings in isolation. Rather, the intermingling of a teacher's personal mathematical understandings with their conceptions of their students and their teaching may require that teachers' experience significant mathematical ideas in concert with developing a mathematics of students.

Throughout the reflective episode that centered on lesson re-constructions (chapter 7) the teachers consistently moved away from discussions of the ideas that were developed or the meanings that were built, as if the issue of knowledge development was not a focus of their ways of thinking. Furthermore, throughout the Functions 3 course the teachers appeared unaware of how understandings (either their own or their students) could develop, or how instruction might be designed

and implemented in order to promote the development of intended understandings and ways of thinking—even when these issues were the topic of conversation.

Although the teachers did not openly express a theory of knowledge development, several of the teachers did exhibit an epistemology in practice, one of a transmission of procedures, skills, and facts—a transmission of “skill sets” for given mathematical topics (a focus on understanding objects and actions applied to those objects). Such a conception can be described as a traditional perspective on mathematics learning (Simon, et al., 2000) which is consistent with what I have characterized as an empirical orientation toward learning and teaching mathematics. Such an orientation, one that focuses on the visible, makes the notion of interpretation and the construction of one’s own mathematics hidden. Understanding, to teachers with an empirical orientation, then consist of levels of “correctness”—where correctness consists of the teacher’s perception of the topic.

Simon et al. (2000), in describing their accounts of mathematics teachers’ practice, differentiate among three perspectives toward mathematics and mathematics learning: a traditional school-mathematics perspective, a perception-based perspective, and a conception-based perspective. According to Simon et al. (2000), both traditional and perception-based perspectives are grounded in a platonic view of knowledge, where “mathematics exists independently from human experience” (p. 593). For Tzur et al. (2001), a traditional perspective “can be characterized by teachers’ attempts to transmit particular mathematical ideas to students” (p. 247). In contrast, teaching based on a perception-based perspective

“emphasizes students coming to see mathematical ideas and relationships through their own experience of the mathematics” (Tzur, et al., 2001, p. 247). Therefore, rather than directly transmitting the intended understandings to their students, a teacher with a perception-based perspective “must orchestrate conditions that engage students in actively seeing and connecting those ideas” (Tzur, et al., 2001, p. 247). According to Simon et al. (2000), whereas a perception-based perspective views mathematics as “an interconnected and understandable body of knowledge” (p. 593), one requiring “firsthand experience in perceiving (discovering) the mathematics” (Simon, et al., 2000, p. 593), a traditional perspective, “may or may not include an emphasis on mathematical understanding as connections, and it does not emphasize direct personal perception” (Simon, et al., 2000, p. 593).

I argue that there exists a perspective between these two, one that differentiates between a view of mathematical learning as creating experiences designed for students to understand the mathematics as the teacher perceives it, from one that views mathematical learning as creating experiences designed for students to understand the mathematics as the teacher imagines the students might come to understand it (mathematics of students)—which is how I interpret Simon et al.’s (2000) perception-based perspective.

According to Simon et al. (2000), for a teacher with a conception-based perspective, “perception is understood as an interaction situated in one’s experiential reality and thus structured by one’s conceptions” (p. 593). Furthermore, mathematics learning is conceived as “a process in which

mathematical objects and relationships are constructed by the learner on the basis of her current knowledge and experiences; that is, the objects and relationships are reinvented, not discovered” (Simon, et al., 2000, p. 593).

The work of Simon, Tzur and their colleagues suggests a spectrum of perspectives of teachers in transition, based, in part on a spectrum of epistemologies. The teachers in the current study exhibited epistemologies that ranged from direct transmission of knowledge, to those involving some level of attention to student perceptions—all in an attempt to make things visible to their students. In addition, these teachers described lesson structure, how they design instruction to promote the development of intended ideas in their students, as dependent upon how they viewed their students and how they viewed their own understanding of the mathematics. Missing from the teachers’ descriptions were discussions of their epistemology—how they viewed the role that their actions played in their students’ construction of knowledge.

This suggests that before a teacher with an empirical orientation can become “aware” of students’ mathematics, they must first confront their own epistemology. That, in order for teachers to seek to develop a coherent curriculum for their students to learn (i.e., a mathematics of students), that they must first transcend their own egocentric nature. Thompson’s (2010) characterization of the development of a KPU (described in chapter 2, p. 32) suggests the possible phases in such an epistemological transformation:

- 1) Phase 1 – the teacher has developed a rich understanding of an idea (i.e., a KDU); at this stage, student thinking is not an issue.
- 2) Phase 2 – the teacher is oriented to student thinking, but tacitly assumes that information is all that students need; the teacher projects herself (i.e., attributes her thinking onto her students) by default.
- 3) Phase 3 – the teacher becomes aware that students think differently than she anticipates they do, but the teacher is overwhelmed by a seeming cacophony of student thinking. Note that throughout Phases 1-3, the students are constructing an image of the teacher's thinking and expectations.
- 4) Phase 4 – the teacher begins to imagine different epistemic ways of thinking among students; these ways of thinking are grounded in the teacher's ways of thinking.
- 5) Phase 5 – the teacher begins to imagine how different ways of thinking among students will lead to different interpretations of what she (i.e., the teacher) says and does; the teacher begins to develop a mini-learning of actions that might help students think the way she intends.
- 6) Phase 6 – the teacher adjusts both her understanding of the mathematical idea as she adjusts her image of the ways students' think about that idea, and her understanding of how students might think about the mathematical idea as she adjusts her understanding of it.

Future Research

Future research should address two issues. First, we need to know if there are levels of empirical orientations and the distribution of these levels amongst secondary mathematics teachers. Second, future research must explore the role that making teachers' epistemological perspectives objects of thought and discussion plays in a teacher's transformation toward more conception-based perspectives (Simon, et al., 2000; Tzur, 2008; Tzur, et al., 2001). Such research could involve teacher education in either academic or professional development settings, in which teachers would engage in critical epistemological discussions, in concert with analyses of how students interpret instruction (and the teacher's role in that interpretation).

Although this study focused on the capacity of practicing mathematics teachers to reflect, the implications derived from my analyses do not limit themselves to in-service teacher education. The difficulties expressed by the teachers in this study to move away from their own perceptions, their own understanding of the material, suggests pre-service mathematics teacher education that focuses on constructing images of teaching and a mathematics of students, concurrent with engagements designed to develop powerful personal mathematical understandings in student teachers. In addition, results from this study suggest pre-service mathematics teacher education that focuses student teachers' attention on theories of knowledge construction, provoking them to take their epistemology as objects of thought and discussion. Furthermore, such discussion must also focus the

student teachers' attention on their (as teachers) role in what it is that their (future) students will inevitably learn.

Limitations

This study drew on four principle sources of data: 1) whole-group class discussions, 2) small-group discussions, 3) small-group written work, and 3) individual teacher's written work. The nature and quality of this data corpus both supported and constrained my analyses.

The classroom discussions in the Functions 3 course were generally designed and managed with the aim of having teachers confront and reflect on ideas addressed in the instructional activities. These discussions turned out to contain considerable evidence of the teachers' understandings and ways of thinking. On the other hand, these discussions were often messy, disorderly, and non-recurrent. Saldanha (2004) describes his experience in analyzing similar "messy" data, analyses which,

- precipitated the need for verbatim transcripts;
- emphasized an analytical focus on the "emergent and dynamic interplay" (p. 1) amongst instructional activities, instructional interactions, and student engagements; and
- necessitated that certain assumptions and imputations be made "concerning students' foci of attention and objects of discourse" (Saldanha, 2004, p. 270).

Saldanha's (2004) descriptions and analyses served to orient and guide my own analysis.

Both the teacher-instructor and teacher-teacher interactions contained within these discussions were often complex and frequently developed out of unforeseen issues and events. In addition, the majority of the small-group discussions were difficult both to track and to hear (due to audio issues). Furthermore, there did not always arise opportunities in either the whole-group or small-group discussions for all teachers to contribute or to extensively articulate and elaborate their ideas and ways of thinking. Similarly, it was not always possible for the course instructor (Pat) to consistently seek elaboration on individual teachers' expressions of understandings and ways of thinking (e.g., their assertions, comments, or statements).

A consequence of these features was that teachers' understandings and ways of thinking often emerged in incomplete, tentative, or halting snippets. In addition, the verbal participation in these discussions of any individual teacher was neither consistent nor recurrent, and many possible visual expressions of ways of thinking were not caught on camera. Consistent with Saldanha (2004), my analyses sometimes necessitated that I make certain assumptions and imputations concerning what I interpreted as the teachers' objects of discourse and foci of attention. Consequently, due to the nature of the data corpus, analyses of some of the teachers' understandings and ways of thinking are conjectural and laden with interpretation. Therefore, following Saldanha (2004), the analyses detailed in

chapters 5 through 7 “should be taken as *viable* rather than *hard* claims about [teachers’] understandings and underlying images and conceptual operations” [author’s italics] (p. 270).

Although the Functions 3 course generated a rich data corpus, I was unable to conduct individual teacher interviews. This prohibited me from making substantive assertions about any individual teacher’s development. Collecting such interview data at the end of each reflective episode might have provided a basis for making systematic comparisons of individual teacher’s longitudinal thinking, and might have also allowed for further elaboration of insights developed from my analyses of the classroom discussions and the teachers’ written work.

Furthering the Discussion of Mathematical Knowledge for Teaching

This study made exclusive use of the Silverman and Thompson perspective on the development of Mathematical Knowledge for Teaching. As such, the framework of Ball, Hill et al. was limited to discussions of comparison and contrast with that of Silverman and Thompson in chapter 2. I would like to close by bringing the framework of Ball, Hill et al. back into the discussion to describe this study’s possible impact on *any* framework for mathematical knowledge for teaching.

Although the framework of Ball, Hill et al. can be used in conjunction with observation instruments that are aligned with it to gauge teachers’ progress in improving their mathematical knowledge and practice at the elementary level (i.e., grades K-6), the mathematical ideas at the secondary level (i.e., grades 7-12) are more complex and sophisticated than those addressed by this framework because of

the cumulative nature of mathematical knowledge. One response to this observation might be to expand Ball, Hill et al.'s framework to include knowledge that is more germane to teaching secondary mathematics. I suspect that this approach will not work simply because there would be too many "things" that a teacher would need to know. Furthermore, those "things", if kept in the spirit of the Ball, Hill et al. framework, would be highly dependent upon mathematics curricula viewed as unproblematic (i.e., a taken-for-granted curricula).

Taking the underlying mathematical ideas as unproblematic—that all teachers are teaching the same mathematics, perhaps some better than others, but the targeted mathematics is unproblematic—works somewhat for the K-6 curriculum, but it does not work at all for the 7-12 curriculum. For example, Thompson (2008a) describes the difficulties that middle school and secondary school students in the United States have with trigonometry, resulting from an "incoherence of foundational meanings developed in grades 5 through 10" (p. 48). Specifically, middle school and secondary mathematics textbooks develop two unrelated trigonometries: the trigonometry of triangles and the trigonometry of periodic functions. Rather than developing a meaning of angle measure that supports a single trigonometry that encompasses both triangle similarity and periodic behavior, typical curricula develop them separately and unrelatedly. Therefore, if we theorize about teachers' mathematical knowledge for teaching trigonometry without taking into consideration the underlying incoherence of the middle school and secondary mathematics curriculum, our theories will be, at their

root, about how teachers cope with incoherence instead of how they might support high quality student understanding.

Finally, the data from this study suggests that for teachers to transform their practice to focus more on students' mathematics, they must achieve three benchmarks: They must create personal mathematical meanings, they must become aware of them, and they must reflect on them in terms of the ways they cohere with instruction and learning.

The results of this study have implications for the Silverman and Thompson framework for the development of MKT. Although Silverman and Thompson's framework tacitly frames the evolution of *sound* MKT, it does not have a means to capture the MKT that teachers actually have. While this study suggests that the conditions for developing powerful KPUs (key pedagogical understandings) indeed requires teachers to reflect on their personal meanings and to project those reflections to others, it also highlights a shortcoming of the Silverman and Thompson framework. Specifically, that the Silverman and Thompson framework does not address the important problem of *how* to help teachers construct powerful personal meanings (KDUs) for mathematics they teach when they do not already possess them, it does not address *how* to support teachers reflections on those meanings, and it does not address *how* to support teachers' projections of those meanings to a mathematics of students (KPUs). Therefore, while the Silverman and Thompson framework does seem to provide an image of what needs to happen,

future research is needed to understand how to do it. I suspect that in researching the *how* of this, the framework itself will undergo significant refinements.

The Ball, Hill et al. framework addresses an issue unaddressed by that of Silverman and Thompson: That elementary teachers do indeed seem to have *categories* of knowledge. While the idea of categories of knowledge might seem incompatible with the *schemes of meanings* approach taken by Silverman and Thompson, it need not be. First, schemes, in effect, categorize a person's world into things and situations that the schemes assimilate and things and situations that they do not. Therefore, a focus on *products* of assimilation and of scheme formation might produce a framework that has many features of the Ball, Hill et al. framework, but to focus on categories at this moment of theory formation would be premature.

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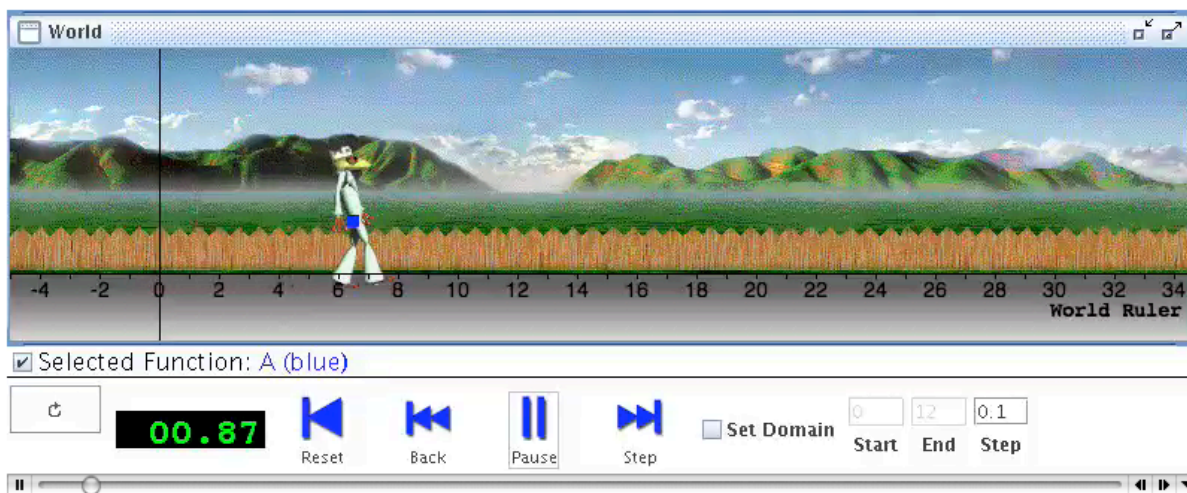
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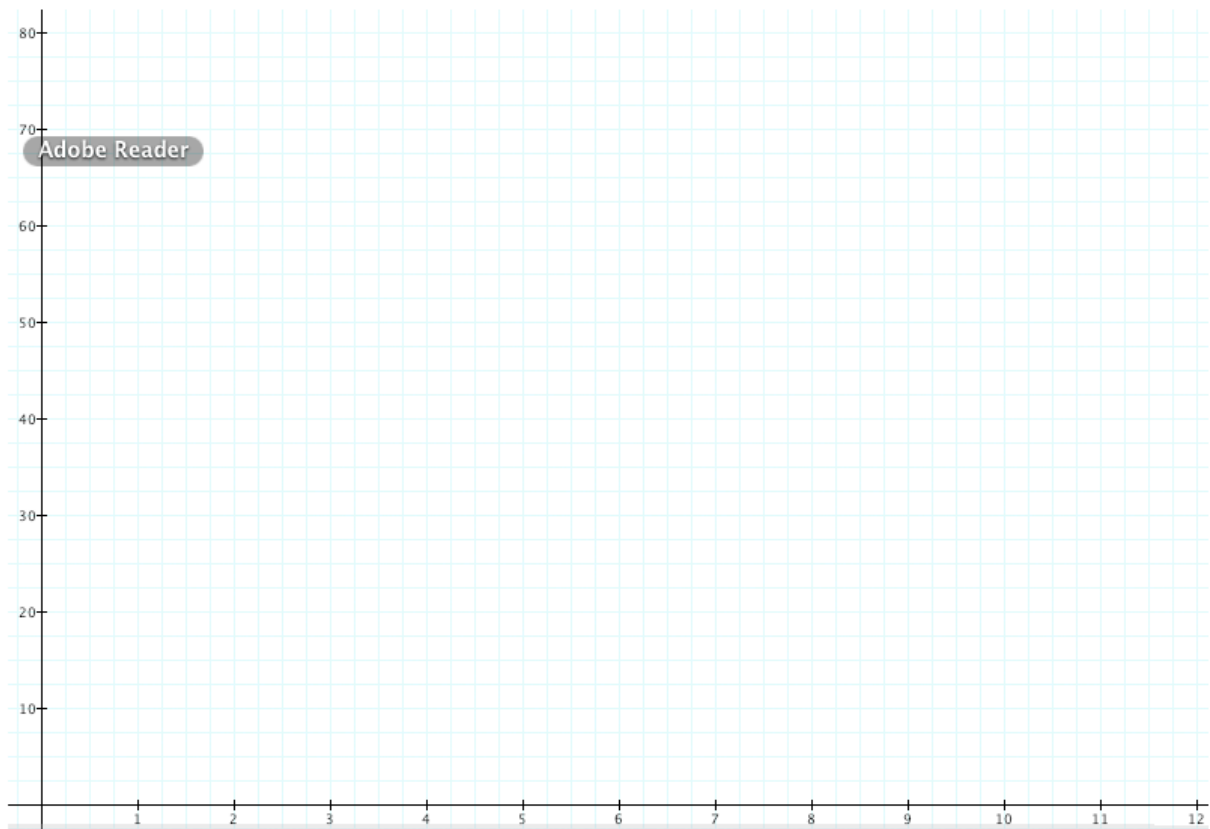
APPENDIX A
FINGER TOOL ACTIVITIES

Activity 1: The Finger Tool, Part 1

Play the movie of Dude taking a walk. Use a printout of the coordinate system at the end of this page as you practice doing these activities. Read all seven steps before beginning!

1. Track the timer along the horizontal axis using your right index finger. Do this all by itself. Go to Step 2 only after you are comfortable doing this.
2. Track Dude's total distance along the vertical axis using your left index finger. Do this all by itself. Go to Step 3 only after you are comfortable doing this.
3. Do steps 1 and 2 together. Keep your distance finger along the vertical axis and your time finger along the horizontal axis. Go to Step 4 only after you are comfortable doing this.
4. Do steps 1 and 2 together. Keep your distance finger directly above your time finger.
5. Pretend that you've dipped your distance finger in pixie dust. Do step 4 again.
6. Now use your pencil to show the pixie dust trail your finger left behind.
7. Pick three particles of pixie dust. How much time had elapsed and how far had Dude walked when you dropped each particle of pixie dust?

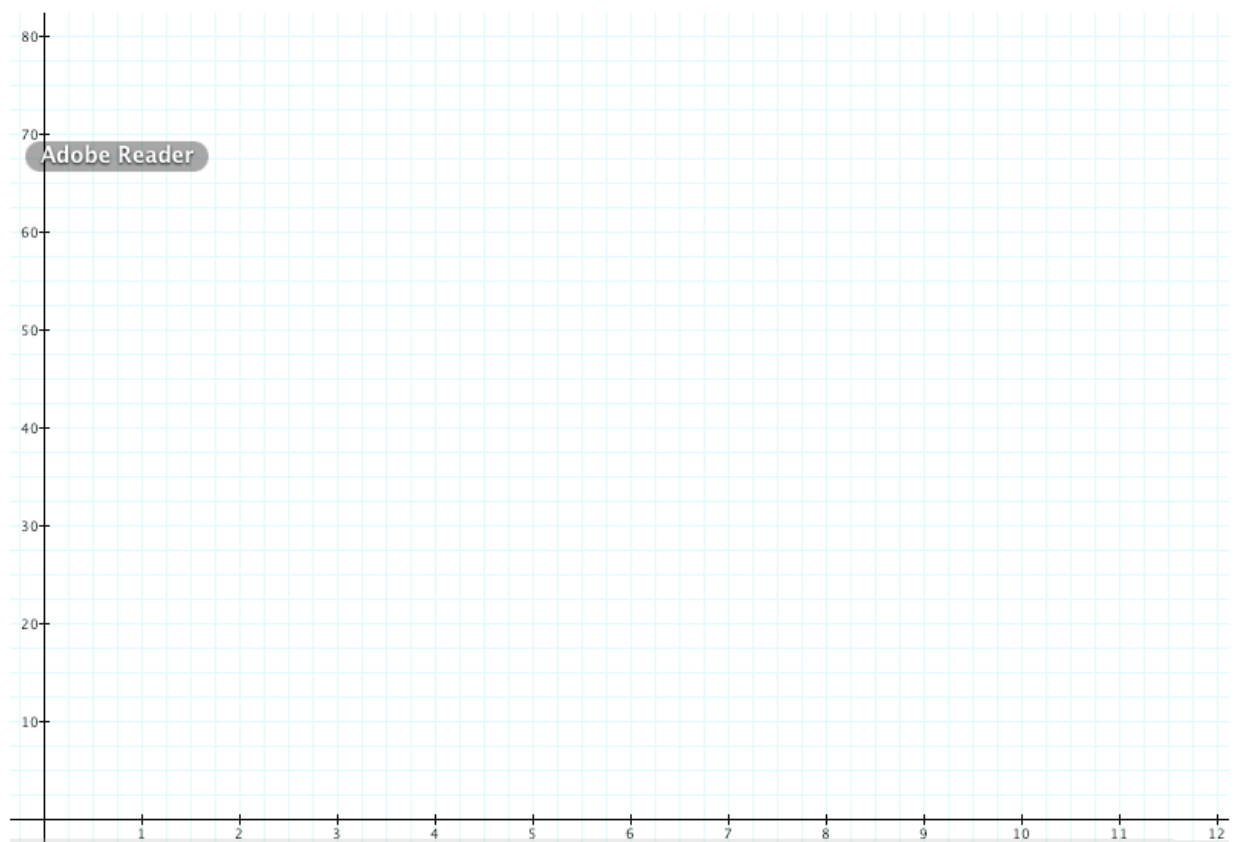
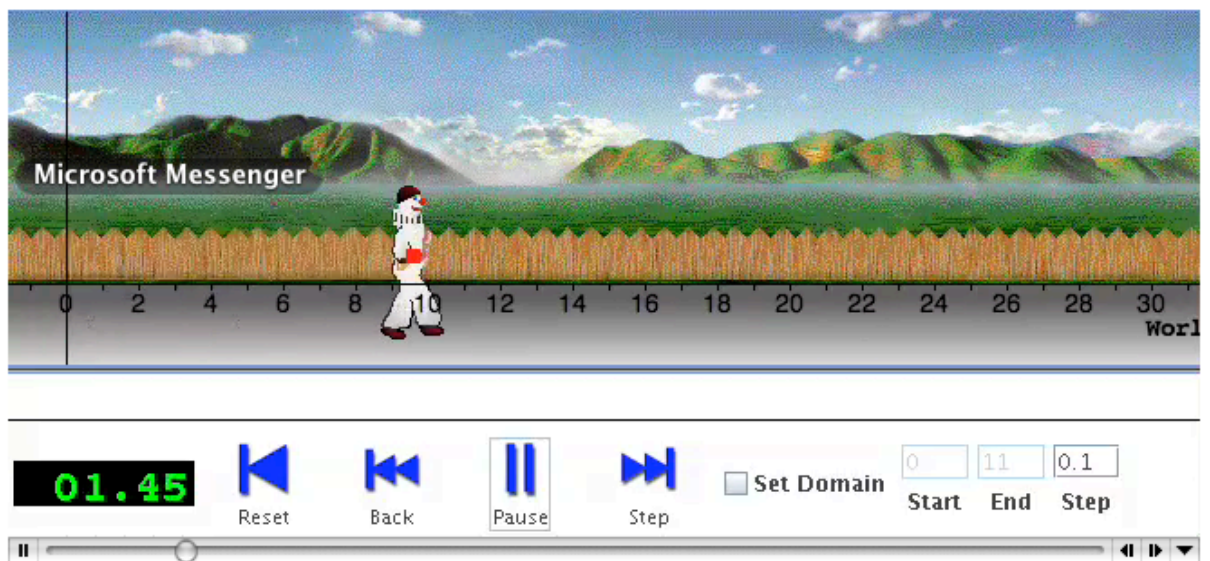




Activity 2: The Finger Tool, Part 2

Play the movie of Clown taking a walk. Use a printout of the coordinate system at the end of this page as you practice doing these activities. Read all seven steps before beginning!

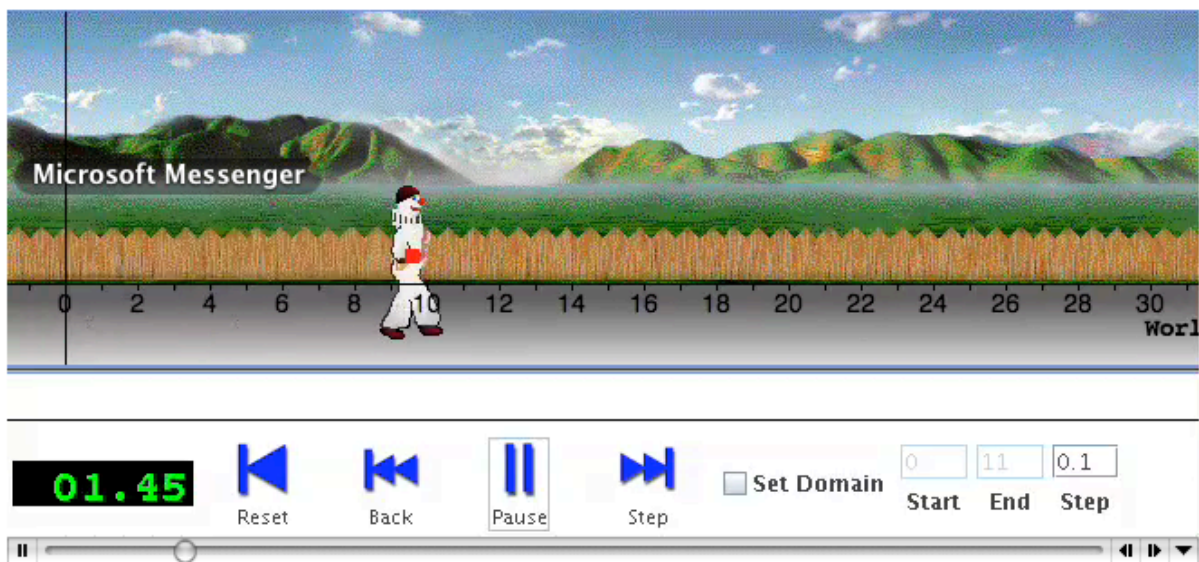
1. Track the timer along the horizontal axis using your right index finger. Do this all by itself. Go to Step 2 only after you are comfortable doing this.
2. Track Clown's total distance along the vertical axis using your left index finger. Do this all by itself. Go to Step 3 only after you are comfortable doing this.
3. Do steps 1 and 2 together. Keep your distance finger along the vertical axis and your time finger along the horizontal axis. Go to Step 4 only after you are comfortable doing this.
4. Do steps 1 and 2 together. Keep your distance finger directly above your time finger.
5. Pretend that you've dipped your distance finger in pixie dust. Do step 4 again.
6. Now use your pencil to show the pixie dust trail your finger left behind.
7. Pick three particles of pixie dust. How much time had elapsed and how far had Clown walked when you dropped each particle of pixie dust?



Activity 3: The Finger Tool, Part 3

Play the movie of Clown taking a walk. You may pause the movie and then use the left and right arrow keys to move Clown little bits at a time. Use a printout of the coordinate system at the end of this page as you practice doing these activities. Read all steps before beginning! (Note: You may want to use your fairy-dust graphs from the earlier activities.)

1. Track Clown's total distance along the horizontal axis using your right index finger. Do this all by itself. Go to Step 2 only after you are comfortable doing this.
2. Track Clown's distance from the start along the vertical axis using your left index finger. Do this all by itself. Go to Step 3 only after you are comfortable doing this.
3. Do steps 1 and 2 together. Keep your left finger along the vertical axis and your right finger along the horizontal axis. You and your partners may need to talk about a strategy for Step 3. Go to Step 4 only after you are comfortable doing this.
4. Do steps 1 and 2 together. Keep your left finger directly above your right finger.
5. Pretend that you've dipped your distance finger in pixie dust. Do step 4 again.
6. Now use your pencil to show the pixie dust trail your finger left behind.
7. Pick three particles of pixie dust. What do each of these particles represent?



APPENDIX B

BERNARDO'S PRE-LESSON LEARNING GOALS

Goals for 070824 Lesson: Homework Problem Review

Some of the overarching goals of the class are to develop the skills students need in order to solve new types of problems using previously learned information and to take ownership of their own learning through classroom discussions.

I wanted to spend the majority of the lesson discussing a problem from our first quiz. If time allowed, we would discuss another challenging problem from the quiz. Essentially, I had asked them to construct a point equidistant given three non-linear points. For the students, the problem was placed in the context of three friends wanting to start an Elvis club, but no friend wanted to walk farther than the other two. Prior to this problem, students had constructed segment bisectors, and we had discussed some suspected properties of points on the bisector in relation to the original segment, and they had constructed points equidistant from two other points.

I had two specific goals for this lesson:

- Students would take ownership of the given problem, and through classroom discussions develop a method to solve the problem focusing on explaining their reasons for doing things. I would stay out of the picture as much as possible, and only serve as a 'traffic cop' to redirect/rephrase questions back to the class. I would use very direct methods to achieve this goal. For example, "Amber did you hear what Zach mentioned? No? Then it is your responsibility to ask Zach to repeat himself, so ask..."
- For the students to see how they can use seemingly unrelated information to use to solve new types of problems. In this case, they didn't immediately know how to solve the problem, but they knew they could find points equidistant from two points.

APPENDIX C

REFORMED-GEOMETRY CASE STUDY (MATH 3-4) ASSIGNMENT

3. Explain the difference between a constructed geometric figure and a drawn geometric figure?



4. Given points A and B (below):

- a) Construct two points (label them H and J) that are equidistant from points A and B (below). Explain how you can be sure that the points you've constructed are equidistant from A and B.

Construction:



Explanation:

- b) Construct segment HJ. What do you suspect is true about *all* the points on segment HJ?

1st Period Math 3-4

Homework

Name: _____

5. Given points E and F (below), *explain a strategy* for constructing a point X so that X is 3 times as far from F as it is from E. Then, construct point X. Explain how you know that the point you have constructed really has is 3 times as far from F as it is from E.

Strategy:

Construction:



Explanation:

1st Period Math 3-4

Homework

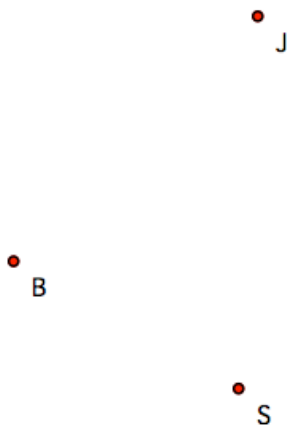
Name: _____

6. Ben, Juan, and Suzy lived on three different ranches in a sparsely settled area of the Mojave desert. They wanted to form an Elvis fan club, but they needed to settle on a meeting place. Now, each of the three is highly self-centered and wants to have the club headquarters at his or her house. But none of the three want to be any farther from club headquarters than anybody else. They agreed that they would set up a tent at a spot equidistant from all three of their homes. They got out a map, but each time they picked a spot someone would insist that they had been slighted by having to travel farther than someone else.

Give a strategy for locating a place on the map so that it is the same distance from each of Ben's, Juan's, and Suzy's house. Then construct a point Y that represents that point. Finally, explain how you know that your construction locates a point on the map that everyone should be happy with.

Strategy:

Construction:



Explanation:

APPENDIX D

ANNOTATED TRANSCRIPT OF REFORMED-GEOMETRY LESSON

The video began with Bernardo drawing three non-collinear points on the whiteboard, labeled J, B, and S, as in problem #6 of the homework (Figure 39, or Appendix C). After providing a brief review of what had been discussed during the previous class meeting, Bernardo initiated the lesson.

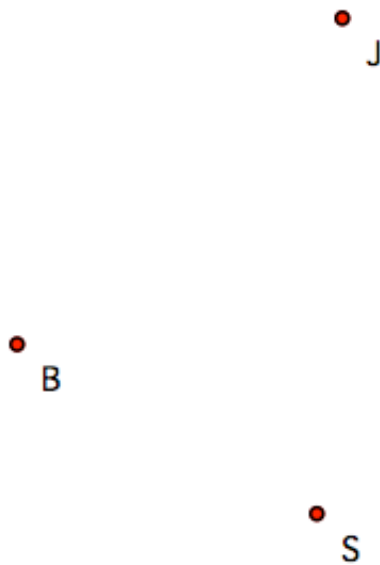


Figure 39. Determine a point equidistant to three non-collinear points.

1. Bernardo: So, we're going to start off with number six in our homework and work backwards, okay? Again six and five were the hardest, I thought anyway.
2. Bernardo: Now, before we start, do you know how to find a point equidistant from two points?
3. Several: Yeah.

4. Bernardo: Yeah. And now, all we're asking you to do is do it with three, okay?

Sounds easy, huh? Well, start off with what you know, okay? Umh, any volunteers?

Tina volunteered to display her solution. As Tina began to draw her diagram, Bernardo asserted that the students needed to "take ownership," that it was their responsibility to ask questions. Bernardo asserted that the students needed to ask one another questions if they did not understand.

In drawing her diagram, Tina first drew a circle through point J; it was not clear what center this circle had. She then drew a second circle that passed through points B and S. Tina next drew a dotted line that connected points J and S. Next, Tina placed a point on the circle that passed through point J and drew a dotted line that connected point B to this new point. After she had drawn this second dotted line, Tina stated, "It's not really straight, but it's kind of the concept." Finally, Tina drew a point where the two dotted lines intersected, and labeled it A. When Tina had completed her diagram (Figure 40), she returned to her seat.

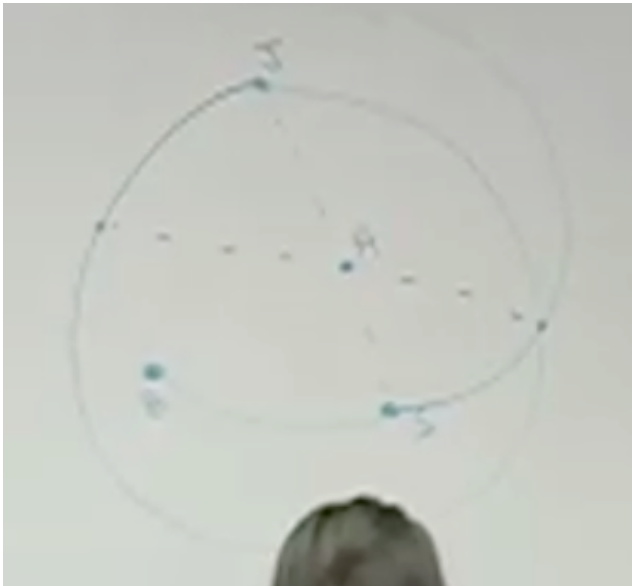


Figure 40. Diagram displaying Tina's solution for problem #6.

5. Bernardo: Ted, do you know what Tina's doing? [...]
6. Ted: She's getting the point that's right in the middle of them.
7. Bernardo: The point that's right in the middle of what?
8. Ted: Umh...S and J [...]
9. Bernardo: So, how do you know she's getting the point in the middle of S and J?
10. Ted: Because she drew the circle with radius B and S, and then she did the one with J and B, and then she put a dotted line between them, and where they crossed she put a dot labeled A.
11. Bernardo: Okay [...]
12. Bernardo: Tina, is that what you did?
13. Tina: Yeah.
14. Bernardo: Yeah?

15. Tina: It's really bad, like on here [gestures to her homework paper] it's more straight (Figure 41).

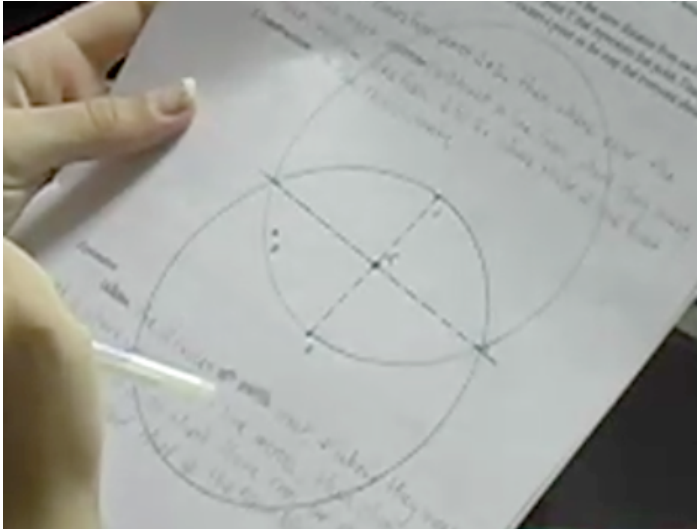


Figure 41. Tina's diagram for problem #6 from her homework.

16. Bernardo: Oh, okay.
17. Tina: So, it doesn't really kind of like curve or anything, but...
18. Bernardo: Okay, 'cause I'm kind of confused too, so what did you construct from points J and S?
19. Tina: Uhh...a dotted line.
20. Bernardo: So, you constructed like segment JS, that's how I'm interpreting it...
21. Tina: Okay.
22. Bernardo: So, you've got a line...that point A there, what is that point?
23. Tina: That is where the middle of J...S and B...it's all equidistant.
24. Bernardo: Oh, okay...how do you know?
25. Tina: Because I...constructed it.

26. Bernardo: Okay, would you mind kind of stepping back up there again please for us? Thank you.

Tina moved back to the whiteboard.

27. Bernardo: Okay, you're now saying that point A is the same distance from point B as it is from J and S?

28. Tina: Yes, on my paper it is.

29. Bernardo: Okay, let's talk about this then. Umh...how do you know that A is the same distance from all three?

30. Tina: Umh, because...Oh, because, umh, you kind of find the distance between J and S with this line [gestures to dotted line segment JS] and the distance between B and the rest with this line [gestures to second dotted line]

31. Bernardo: Okay, I don't [...] Amber do you see how Tina's finding the distance between B and S?

32. Amber: Not really, I'm kind of confused.

33. Tina: If you saw my paper you wouldn't be confused, this is so crooked.

34. Bernardo: Well, if you're confused guys you need to ask a question.

35. Amber: Well, I know how she's trying to find the middle of J and S, like the center point of that, but I don't know how she found that it's equal to B also.

36. Bernardo: Okay, how would you find the center point of B and S?

37. Amber: Umh, draw another segment...

38. Bernardo: Okay, why don't you try it. Thank you Tina. Everybody give Tina a hand. It's not easy being up there folks.

Tina moved back to her seat, as the class clapped for her.

39. Bernardo: So, Amber would you mind, for us? [...] Thank you Amber.

Amber moved to the whiteboard and drew a dotted line from point B to point A (Figure 42) on the diagram that Tina had previously drawn, and stated, "I don't know if that would work."



Figure 42. Amber's addition to Tina's drawing.

40. Bernardo: Okay, so...tell us why, tell us why you think that would work? You might be right, you might be wrong, I don't know. [Quiet for 13 seconds]

41. Amber: Because it meets up with the center?

42. Bernardo: Okay, again, how do you know that it meets up with the center? Now by center, are you referring to point A?
43. Amber: Yeah.
44. Bernardo: Okay, so how do you know that that dotted line then meets up with point A? [Quiet for 11 seconds]
45. Amber: Because it's the same distance.
46. Bernardo: If it's the same distance, I agree with you 100% on that, I guess, how do you know it's the same distance? [...]
47. Deb: I have a question.
48. Bernardo: Guys, if you've got a question just speak up, holler out.
49. Deb: Okay, I have a question. I don't know what you were doing with the dotted line, like...
50. Bernardo: Okay now, be specific...
51. Deb: I know you're trying to find the center, but why is there one coming off of B? Like where is it going, does it just stop? [...]
52. Bernardo: Larry...is that a question?
53. Larry: Okay, I don't think it's done yet, because you have to draw the circle around B [...]
54. Bernardo: I'm kind of lost here, Peter...could you kind of get us caught up here? What do you think we're trying to do, 'cause everybody had some good comments there.
55. Peter: They're trying to find the distance from B and S [...]

56. Bernardo: What distance from B and S?

57. Peter: Rephrase that? [...]

58. Bernardo: I heard you say you're trying to find the distance from B and S...what distance from B and S?

Tina raised her hand.

59. Bernardo: Does anybody have like a picture of what we're trying to find? Okay [...]

Larry. That's right I promised that you could go up next. Amber, thank you very much. Larry why don't you go up there and...

As Amber moved back to her seat, the class clapped, and Larry moved to the whiteboard.

60. Larry: Can I redraw this?

61. Bernardo: Okay, go right ahead [...]

Larry proceeded to erase Tina's (and Amber's) drawing. Larry first marked a center point and drew a circle; then marked a second center point and drew a second circle that intersected the first circle. He then marked a third center point and drew a third circle. Next, Larry marked points at the two points of intersection of the second and third circles. At this point he stated, "Wait," and went back to his seat to get his homework. Larry then marked points at the two points of intersection of the first and third circles. Next, Larry marked points at the two points of intersection of the first and second circles. Finally, Larry then drew line segments connecting the three pairs intersection points (Figure 43). When he had finished, Larry stated, "Okay, like that."

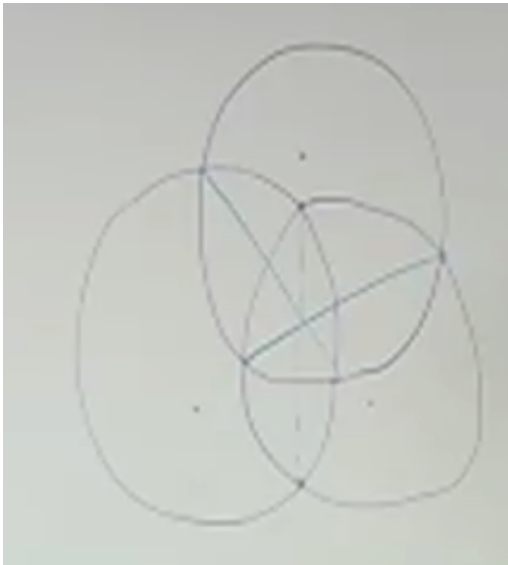


Figure 43. Larry's diagram displaying his solution to problem #6.

62. Bernardo: Okay.

63. Larry: When you add the other circle, they all cross right there [gestures to the place where the three line segments intersect].

64. Bernardo: Now, what is...what is the point where all three of those blue lines cross?

65. Larry: Where it's the equal distance from all the points?

66. Bernardo: And did we have to label that Y? Yeah, okay, could you label your points please?

Larry proceeded to label the points Y, S, B, and J (Figure 44).

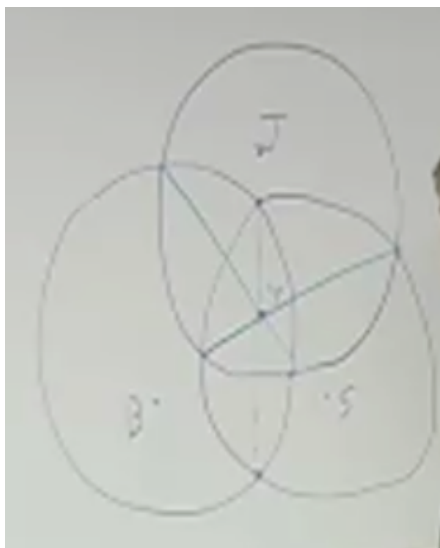


Figure 44. Larry's labeled diagram.

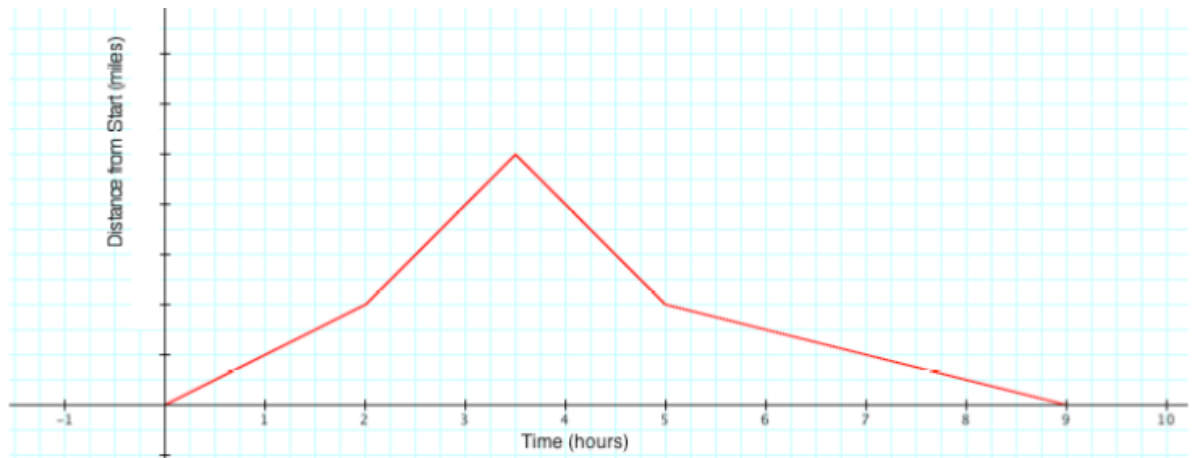
67. Bernardo: Okay, so we have J, B, S, and Y. Now, Larry said that Y was equal distance from J, B, and S. Okay Larry, you did the hard work up there, uh...who wants to explain...hold on [...] Fred, can you tell us about, what you suspect, uh, Larry's trying to do?
68. Fred: I'm guessing that the three blue lines are the perpendicular bisectors of each of the segments of the triangle, and where they meet would end up pointing out the middle of the triangle.
69. Bernardo: Okay...

APPENDIX E

REFORMED-ALGEBRA “INTERPRETING/CREATING GRAPHS QUIZ”

Interpreting/Creating Graphs Quiz

1. Clown went on a bike trip. The following graph shows the number of miles that Clown was from the start relative to the number of hours since he started.

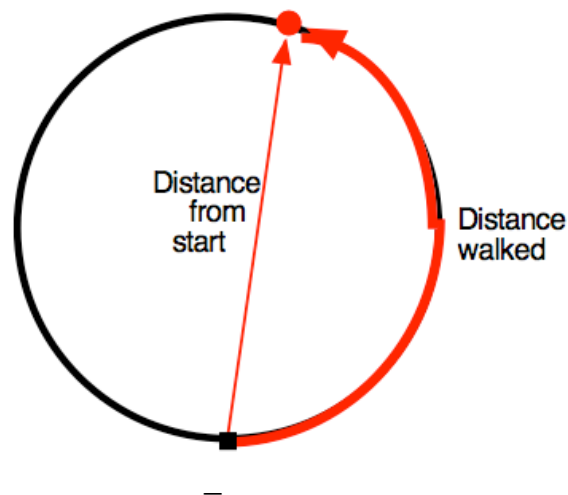


- a) In the first part of his trip Clown traveled at a speed of 5 mi/hr. Put numbers on the vertical axis so that the graph is accurate.
- b) How many miles did clown travel in the third part of his trip? How do you know?
- c) How fast did Clown travel in the fourth part of his trip? How do you know?
- d) In one part of his trip, Clown blew a tire and had to walk. When did this happen? How do you know?

Quiz

Page 2

2. Matt walked around a circular track (see diagram). The black square shows where he began. The arrows show that we are tracking Matt's distance from his current location to the starting point and the distance he had walked to get to his current location.
- a) Imagine Matt walking around the track once. Use your finger tool to sketch a graph of Matt's walk. Keep track of how far he had walked with your *horizontal* finger and his distance from the start with your *vertical* finger. (Hint: Think of playing Matt's walk a frame at a time to keep track of both.)



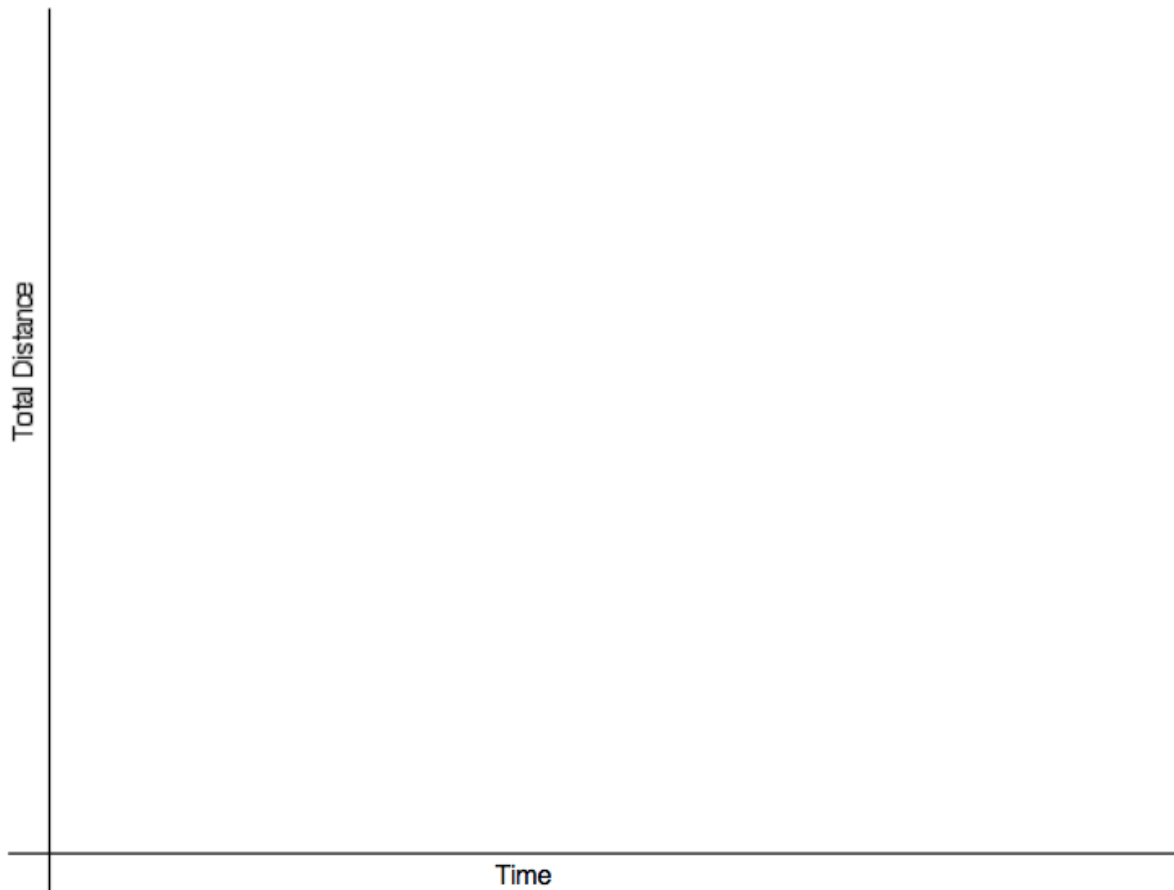
Distance from Start

Total distance

Quiz

Page 3

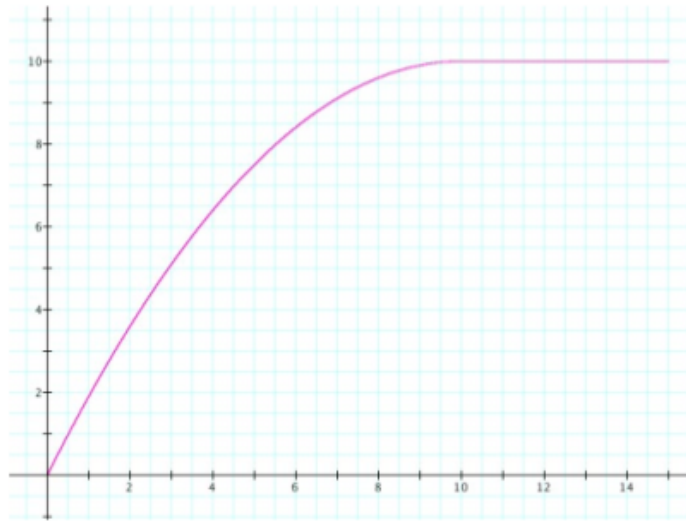
3. Miss Coombs' new years resolution is to run more. Sketch a graph of her TOTAL distance relative to the number of minutes she's been jogging while she was out on the following run:
- a) She began at a steady jog for 5 minutes (warming up, you know).
 - b) Then, she picked up the pace to a run, running top speed for 3 minutes.
 - c) She got a huge cramp and had to slow way down. She went at this slower speed for 4 minutes.
 - d) The cramp would not go away so she stopped for one minute
 - e) After that, she decided to toughen up and she sprinted all the way back home



Quiz

Page 4

4. The following graph tracks an object's SPEED relative to the number of seconds it had been moving:



- a) Describe this object's motion over the 15 seconds shown in the above graph.
- b) Sketch a graph (don't worry about total accuracy) of the object's distance from start in relation to the number of seconds it has been moving. Explain your graph (use the back of this sheet if necessary).

APPENDIX F

TAMI'S LESSON LOGIC FOR REFLECTIVE EPISODE #3

outline lesson logic of ideas on speed

- 1) Dropping a screw driver trials from chair
- 2) look at speed of screw driver
- 3) Discuss the speed at which the screw driver is dropping. Is it a constant speed? How can you tell?
- 4) measure height of drop
Take several measures of time that it takes to drop the given height.
- 5) Measure $\frac{1}{2}$ of the drop height, take several measures of time for that new distance.
- 6) Discuss how you can tell if the screw driver is traveling at a constant speed or if it is accelerating.
- 7) Discuss what is meant by an increased amount of speed over given increments of time.
- 8) Explore what this looks like on a graph by modelling the data collected

- 9) Discuss how you determine the distance traveled relative to the given increment of time elapsed.
- 10) Is the difference in the rate of change linear as you go to successive increments of time? How do you know?
- 11) Draw a graph of the path of the object relative to the distance from the ground as time elapsed,

APPENDIX G

GROUP CONSTRUCTED LESSON LOGIC FOR REFELCTIVE EPISODE #3

How did we get from there to here

How did we get from

Standing on a chair dropping a screwdriver

to

Discussing graphs of distance from the ground in relation to elapsed time?

Step 1- Observations

- a) stand on chair and drop screwdriver
- b) asking students what they observe as the screwdriver drops, the way it drops
- c) repeat several times and carefully questioning students observations, keep asking questions until you get the discussion on speed, whether or not speed was constant.

Step 2- Testing

- a) how could we test the situation to show that the screwdriver was not falling at a constant rate
- b) want to manage the conversation to encourage students to bring up measurements
- c) ask students to measure and record the data at meaningful set heights- total distance and half the distance
- d) test hypothesis that it was dropping at a constant rate of change (if the screwdriver is dropping at a constant rate, half the distance would take half the time)

Step 3- Discussion

- a) want students to come to the conclusion that it's not a constant rate when dropping from half the distance we are talking about the first half of the graph

Step 4-

- a) What does it mean to always be speeding up
- b) groups discuss and write a precise conceptual statement
- c) share and compare and modified
- d) discuss average speed and its meaning
- e) intervals of equal size

- f) discuss how intervals do not have to be consecutive they can be overlapping
- g) with equal intervals of time of a given size, even if they overlap, which one travels further

Step 5- Graph

- a) Graph the screwdrivers distance from the ground as a function of time elapsed, meeting the constraints of the experiment
- b) constraints of initial height, initial time, and it was always speeding up
- c) stress your conceptual statement, making your definitions work for you!
- d) examined multiple representations, sharing students ideas, camera/elmo/overhead/whiteboards looking for commonalities in the method of creation
- e) want students think about rate of change changing systematically

Step 6- Extension

Depending on student level have them create or write a mathematical representation of their earlier mathematical statement, for all epsilon greater than 0...

APPENDIX H
LESSON LOGIC TEMPLATE

(Lesson Name):
Logic of the Lesson

(Author)

The following is a *lesson logic* for teaching (the major idea or ideas) in (course, topic, or grade level).

A lesson logic is the outline of how you will develop the lesson's main ideas. It does not pay attention to time, meaning that the "lesson" may transcend several class periods. It does not give the level of detail that a lesson plan gives, meaning it might not say how you will organize the classroom, how you will transition from one activity to another, etc. Instead, it focuses on the ideas you will develop, the way you develop them, and why you take the approach you take.

The following lesson logic provides a structure in which the surrounding conversation unfolds these ideas:¹

1. (Major ideas of the lesson, listed in a way that summarizes the logic)
- 2.

Meanings students must have before the lesson:²

- 1) (Things students must understand at the outset if they are to participate productively in the lesson. This is not the same as things they must be able to do.)
- 2)

Steps in the Lesson Logic

<u>Step</u>	<u>Action</u>	<u>Reason</u>
1.		
2.		
3.		

¹ Your description of the major ideas that this lesson addresses should evolve *from* your attempts to create the lesson logic for teaching them. In other words, someone reading your lesson logic will read your description of the big ideas *before* reading your steps for teaching them. But, you will have created your description of big ideas *after* having created the steps for teaching them.

² Your list of "meanings students must have" should evolve *while* you write the steps in your lesson logic. Write them as they occur to you, and be alert to when you are plan a step that presumes students have a meaning that is essential for them to participate.

APPENDIX I

GROUP CONSTRUCTED LESSON LOGIC FOR REFLECTIVE EPISODE #4

(Lesson Name):
Logic of the Lesson

(Author)

The following is a lesson logic for teaching (the major idea or ideas) in (course, topic, or grade level).

A lesson logic is the outline of how you will develop the lesson's main ideas. It does not pay attention to time, meaning that the "lesson" may transcend several class periods. It does not give the level of detail that a lesson plan gives, meaning it might not say how you will organize the classroom, how you will transition from one activity to another, etc. Instead, it focuses on the ideas you will develop, the way you develop them, and why you take the approach you take.

The following lesson logic provides a structure in which the surrounding conversation unfolds these ideas:¹

1. Data has cases and each case has several attributes.
2. (Major ideas of the lesson, listed in a way that summarizes the logic)

Meanings students must have before the lesson:²

- 1) What is meant by a hypothesis.
- 2) (Things students must understand at the outset if they are to participate productively in the lesson. This is not the same as things they must be able to do.

Steps in the Lesson Logic

<u>Step</u>	<u>Action</u>	<u>Reason</u>
1.	Hand out raw data and ask "Can we analyze this data?"	Establish need to organize data
2.	Discuss why organization of data matters.	Particular organization can provide particular kinds of information. (i.e. ordering test scores from high to low)
3.	Create a hypothesis based on the data we have.	Establish a case and see a relationship between the data
4.		

¹ Your description of the major ideas that this lesson addresses should evolve *from* your attempts to create the lesson logic for teaching them. In other words, someone reading your lesson logic will read your description of the big ideas *before* reading your steps for teaching them. But, you will have created your description of big ideas *after* having created the steps for teaching them.

² Your list of "meanings students must have" should evolve *while* you write the steps in your lesson logic. Write them as they occur to you, and be alert to when you are plan a step that presumes students have a meaning that is essential for them to participate.

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