Secondary Teachers' and Calculus Students' Meanings for Fraction, Measure and Rate of Change

by

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A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Approved July 2016 by the
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ARIZONA STATE UNIVERSITY

August 2016

ABSTRACT

This dissertation reports three studies of students' and teachers' meanings for quotient, fraction, measure, rate, and rate of change functions. Each study investigated individual's schemes (or meanings) for foundational mathematical ideas. Conceptual analysis of what constitutes strong meanings for fraction, measure, and rate of change is critical for each study. In particular, each study distinguishes additive and multiplicative meanings for fraction and rate of change.

The first paper reports an investigation of 251 high school mathematics teachers' meanings for slope, measurement, and rate of change. Most teachers conveyed primarily additive and formulaic meanings for slope and rate of change on written items. Few teachers conveyed that a rate of change compares the relative sizes of changes in two quantities. Teachers' weak measurement schemes were associated with limited meanings for rate of change. Overall, the data suggests that rate of change should be a topics of targeted professional development.

The second paper reports the quantitative part of a mixed method study of 153 calculus students at a large public university. The majority of calculus students not only have weak meanings for fraction, measure, and constant rates but that having weak meanings is predictive of lower scores on a test about rate of change functions. Regression is used to determine the variation in student success on questions about rate of change functions (derivatives) associated with variation in success on fraction, measure, rate, and covariation items.

The third paper investigates the implications of two students' fraction schemes for their understanding of rate of change functions. Students' weak measurement schemes

obstructed their ability to construct a rate of change function given the graph of an original function. The two students did not coordinate three levels of units, and struggled to relate partitioning and iterating in a way that would help them reason about fractions, rate of change, and rate of change functions.

Taken as a whole the studies show that the majority of secondary teachers and calculus students' studied have weak meanings for foundational ideas and that these weaknesses cause them problems in making sense of more applications of rate of change.

Dedicated to my mom Norah O'Neill and my dear friend Ioana Elise Hociota.

My mom, Norah O'Neill, taught me that it is possible to struggle greatly in this world, have everything fall apart, and then to put the pieces back together. Seeing you face chemo and terminal cancer for the past three years with a smile on your face helped me see that I could write a dissertation with a smile as well. Math is easy compared to chemo! Thanks to your pioneering career as the first woman pilot at Flying Tigers I never felt uncomfortable in male dominated mathematics classrooms, or that I couldn't do something just because not many other women were around. I'm going to miss you forever.

Ioana, I will always remember you as the most inspiring, crazy, and wonderful person I met in graduate school. I'd do all of my Abstract Algebra homework over again if I could do it at your house with you. I will always cherish the grand canyon obsessed, outdoor loving family I became a part of after you died. Andrew, Etsuko, Matthias, Anna, Monika, Iulia, Rich, Chris, and Kari. Your passion for life and adventure reminds me so much of her and kept me excited and passionate and pushing though the hard times. Most of the words of my dissertation were written in your favorite coffee shop, thinking of your grand plans for graduate school. Thank you Ioana, for bringing Tina into my life. Tina helped me put my life back together after you left in so many ways and I will always cherish her friendship. Betsy Elizabeth-I don't even know how many times you've been there for me when I was sad and for coming with me on so many amazing adventures as I tried to explore as much has Ioana did!

ACKNOWLEDGMENTS

Thank you to my advisor Pat Thompson. You pushed my thinking more than anyone I've ever known, and I'm a more careful and precise thinker as a result of years of being pressed by you in research meetings. Thank you for writing grants-having the NSF pay my tuition and research stipend made possible some amazing research experiences with you and the Aspire team. Thank you also for being a wonderful and understanding mentor when Ioana died, and my mom was diagnosed with cancer. Your intellectual curiosity and kindness helped lead me out of despair and back to my dissertation.

Thank you to Marilyn Carlson for being a wonderful mentor with lots of ideas and enthusiasm for your students. Thank you for Jim Middleton for always having a wealth of interesting ideas whenever I come to talk to you. Thank you to Luis Saldanha for being a wonderful Research in Undergraduate Mathematics Education professor and one of the most careful paper editors I've known. Thank you to Allen McNamara for showing me what it meant to model scientifically and hundreds of places that calculus is useful in science I didn't know about before.

Thank you to all of the amazing people who supported me on this long road. Writing the papers and doing the math was hard, but not the hardest part. The hardest part was to keep trying through all the sad parts of life along the way.

My dad, Scott Byerley, taught me many useful lessons that served me well throughout graduate school. Knowing that buying fancy things is not necessary for happiness made it easy to live on a grad student budget for years with no stress. Thank you for always loving me, and being there for me no matter what. I'm so grateful you helped bring Bren, Linda, Emma, Piper, Mason and Brandon into my life.

All of the graduate students I met along the way have played a part in making ASU the wonderful place I loved. Thank you Yoon Hyunkyoung for being an amazing office mate and leaving me little "you can do it notes" and always believing in me. Thank you Surani Joshua-with me from beginning to end and so nice even when I was stressed and grumpy. Thank you Neil Hatfield for helping me with the very first projects that led to my dissertation and always being good for tech help, chocolate, and stats consultation! I can't imagine Aspire or grad school without you. Thank you Stacy Musgrave for being an amazing friend who I connected with the very first day I met you. Thank you Michael Tallman-I'm so thankful I spent five years with a fellow student with such high intellectual integrity and curiosity. You have also been my role model for organizing my office! I'm thankful for great conversations with Matt Weber, Emily Kuper, Morgan Sellers, Alison Mirin, Gabriel, Kristin Frank, Bethany Fowler Parsons, Katie Underwood, Frank, Owen Davis, Cody, Kristen Marie Whitney, Krysten Pampel, Kevin Moore, Carlos Castillo-Garsow, Aviva Halani, Eric Weber, Erika David, Ben Whitmire, Grant Sander, Jeff Chapin, Linda Ag, Mike McCallister.... I hope I'm not forgetting someone! And of course the amazing Caren Burgermeister, who is not technically a grad student, but takes care of all the ASU grad students and even officiated my wedding!

And, of course, to my husband Andy Darling, who loved me and supported me through the past five and a half years of graduate school. Our adventures sustained me. Your love and willingness to just hold me has made it possible to keep going at putting my ideas out into the world. Your interest in science has expanded my horizons and taught me so many new things. Thank you. Thank you. Thank you.

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INTRODUCTION

Mathematics is a powerful tool for describing relationships between quantities and variables. In fields as varied as economics, medicine, physics, geology, chemistry, biology and politics it is of central interest to understand how quantities change together. Comparing quantities' measures multiplicatively often is essential when exploring and characterizing relationships and trends in these contexts (P. W. Thompson, Carlson, Byerley, & Hatfield, 2014). Multiplicative comparisons of relative size are at the heart of many topics in secondary mathematics. From a mathematical perspective it seems critical to understand fractions and quotient before making sense of rate and rate of change functions. In addition, understanding radian measure, trigonometric functions, rational functions, and linearity entails imagining comparing two quantities multiplicatively (e.g. (Moore, 2013).

However, we do not know as a research community how calculus students struggle with measure, fraction and rate. We also do not know how this impacts their success in calculus. I hypothesize that students' issues with foundational mathematics may be one reason for the lack of success of the calculus reform movement.

Despite the millions of dollars spent on reforming calculus instruction there has been shockingly little evidence of the existence of classes of calculus students where the majority of the students learn the major concepts of calculus (See paper two for details). This study intends to contribute to improving calculus instruction by documenting an often overlooked source of difficulty that partially explains these discouraging results: calculus students' meanings for fractions and measure. I hypothesize that so many well-intentioned interventions have not worked as well as hoped, in part, because the students' did not have meanings for the foundational topics they needed to make sense of conceptual calculus instruction on derivatives.

The first study of secondary teachers' meanings for rate of change could also shed light on why calculus reforms might have failed. The high school teachers might not have the meanings that high school students will need to succeed when they take calculus. Thus the high school students do not learn the ideas they need before coming to the university. By the time they are at the university and enrolled in calculus, it is typically assumed that prior experiences in middle school and high school provided the students with useful understanding of concepts related to fraction and measure. The studies of teachers' and students' meanings taken together could help us understand one source of students' problematic mathematical meanings in calculus.

My fundamental research questions that span all three papers are:

- What meanings do students and teachers hold for fraction, measure, and rate of change?
- Are the problematic meanings documented in small qualitative studies present in larger samples?
- In the case of calculus students, how do their fundamental meanings for fraction, measure, quotient and rate impact their success in learning about rate of change functions?

All these studies shed light into our mathematical culture surrounding multiplicative comparisons. Study one made use of prior qualitative research on teachers' meanings for rate of change to design items intended to categorize teachers' meanings (P. W. Thompson, 2015). Earlier research noted that small groups of teachers had unproductive meanings for measure and rate of change (Coe, 2007). Study one contributes to the field in an important way by showing that the problematic meanings observed in small groups of teachers are present in a majority of the 251 secondary mathematics teachers we sampled.

Study two was similar, in the sense that it built on small sample studies of calculus students' meanings for quotient and fraction to investigate the meanings of a larger sample of students. Study two showed that the majority of calculus students in two sections of Calculus 1 at a large public university have weak meanings for fraction, measure, and constant rate. Because we have larger samples of both calculus students and secondary teachers, I can make stronger claims about the lack of preparation students receive in high school for success at the university. Study two used statistical methods to investigate the relationship between success on fraction, measure, and constant rate items and success on a test on rate of change functions. The statistical study does not explain how students' fundamental meanings are related to their construction of rate of change functions. Making the connection between trends in population statics and why what is actually occurring in student thinking requires qualitative study, and thus an additional interview-based study was conducted in the same class for study three.

The qualitative methodology of study three investigates how weak meanings for foundational topics negatively impact students' abilities to construct rate of change functions. The qualitative study also paints a picture of how a student who earned a particular score on the pre-test is likely to reason. The third study showed that for many students who had low scores on the fraction and measure items, their difficulties were deep-seated, hard to resolve, and caused them many problems in calculus.

PAPER ONE: SECONDARY TEACHERS' MEANINGS FOR RATE OF CHANGE

It is critical to understand teachers' meanings for rate of change to see if the teachers' meanings will support students in making sense of a variety of important quantitative situations. Prior studies of secondary teachers' meanings for slope, rate of change, and quotient typically focused on small numbers of teachers to model their meanings or to characterize their proficiency with these ideas (Ball, 1990; Coe, 2007; Fisher, 1988; McDiarmid & Wilson, 1991; Stump, 1999, 2001a; P. W. Thompson, 1994b; P. W. Thompson & Thompson, 1994). Large scale investigations of mathematical knowledge for secondary teaching, such as the TEDS-M study of mathematical knowledge and pedagogical content knowledge did not release any items related to quotient, rate of change, fraction or measurement (Tatto et al., 2012). Studies of elementary teachers have demonstrated that teachers' scores on assessments of Mathematical Knowledge for Teachers are related to improvement in their students' performance (Hill, Ball, Blunk, Goffney, & Rowan, 2007). Developing a valid instrument to measure teachers' knowledge was critical to understanding the link between teachers' knowledge and their students' opportunities to learn.

Since little is known about secondary teachers' meanings for the content they teach, we developed a diagnostic instrument, *Mathematical Meanings for Teaching secondary mathematics* (MMTsm), to help researchers and professional development leaders diagnose secondary teachers' mathematical meanings (P. W. Thompson, 2015). Our aim was to help professional development leaders to design interventions that would address weaknesses in teachers' meanings. At the same time, while the aim of the

MMTsm is diagnostic, using it on a large scale allows us to examine the prevalence of particular meanings in larger populations of teachers.

Our development of the rate of change items on the MMTsm was guided by qualitative work that characterized students' and teachers' thinking about rate of change, quotient, and slope (Coe, 2007; Lobato & Thanheiser, 2002; Martínez-Planell, Gaisman, & McGee, 2015; Nagle, Moore-Russo, Viglietti, & Martin, 2013b; Planinic, Milin-Sipus, Katic, Susac, & Ivanjek, 2012; Stump, 1999, 2001a; P. W. Thompson, 1994a, 1994b; P. W. Thompson et al., 2014; P. W. Thompson & Saldanha, 2003; P. W. Thompson & Thompson, 1994; Walter & Gerson, 2007; Zaslavsky, Sela, & Leron, 2002). Construction of quality items and rubrics required articulating productive meanings for rate of change that are useful in many contexts such as calculus, science and economics. We also needed specific descriptions of common unproductive meanings for rate of change. In addition, other studies provided evidence that secondary teachers' meanings for quotient is weak and suggested that investigation of teachers' meanings for "elementary" ideas is important (Ball, 1990; McDiarmid & Wilson, 1991).

This paper reports 251 teachers' responses to MMTsm items that focused on high school teachers' meanings for slope and rate of change. We were interested in whether teachers' meanings for rate of change were additive or multiplicative or both. We recorded if teachers were able to differentiate between situations best modeled with subtraction versus division.

We also were interested in the extent to which teachers' meanings for rate of change appeared to be connected to meanings for quotient as a measure of relative size.

LITERATURE REVIEW

A. Thompson and P. Thompson (1996) used the phrase Mathematical Knowledge for Teaching (MKT*)* to describe teachers' schemes for ideas they teach and which they hold at a reflected level. They described teachers' reflected schemes as guides for teachers' interactions with students whom they hope will develop the meanings and ways of thinking that the teacher intends. Silverman and Thompson (2008) expanded this scheme-based meaning of MKT by examining how teachers might create what they called Key Pedagogical Understandings from a basis of their personal, well-formed schemes-schemes which Simon (2006) called Key Developmental Understandings. A Key Pedagogical Understanding is a mini-theory a teacher holds regarding how to help students create the schemes that the teacher intends. In other words, A. Thompson, P. Thompson, and Silverman used "knowledge" in the sense of Piaget and von Glasersfeld—as schemes and ways of coordinating them that enable people to function adaptively in light of their goals and experienced situations. We see teachers' schemes as more than a set of declarative facts that the teachers learned about students and mathematics. An example of a declarative fact is "when students square a binomial they often distribute the exponents and forget the middle term." We want to model teachers' more general scheme for rate of change that would allow us to predict how teachers' might respond in a large variety of situations and not just in a specific context such as teaching the procedure to square binomials.

Ball, Hill and colleagues (Ball & Bass, 2002; Hill et al., 2007; Hill, Schilling, & Ball, 2004) used the phrase MKT quite differently than did A. Thompson, P. Thompson, and Silverman. Although Ball, Hill et al. have not said what they mean by "knowledge", they explained their motivation for the phrase “mathematical knowledge for teaching” as being rooted in their quest to understand the mathematical knowledge that supports teachers in their work of teaching mathematics.

> We began a close examination of the actual work of teaching elementary school mathematics, noting all of the challenges in this work that draw on mathematical resources, and then we analyzed the nature of such mathematical knowledge and skills and how they are held and used in the work of teaching. From this we derived a practice-based portrait of what we call “mathematical knowledge for teaching”—a kind of professional knowledge of mathematics different from that demanded by other mathematically intensive occupations, such as engineering, physics, accounting, or carpentry. (Ball, Hill, & Bass, 2005, pp. 16-17)

Although A. Thompson, P. Thompson, and Silverman agreed with Hill et al. that teaching a mathematical idea requires more mathematical sophistication than a basic mastery of that idea, differences in underlying theoretical perspectives led to different methods of studying teachers’ thinking (Byerley et al., 2015). In 2013, P. Thompson began to use the phrase mathematical *meanings* for teaching instead of the phrase mathematical *knowledge* for teaching, for three reasons: (1) readers often failed to understand that he was using “knowledge” in the sense of Piaget and Glasersfeld and not in the sense of Ball and colleagues (P. W. Thompson, 2013, p. 85); (2) to Piaget, knowledge and meaning were largely synonymous and both were imbued with the idea of scheme (Montangero & Maurice-Naville, 1997), and (3) readers understood easily that “meaning” connotes something personal and that a person’s meanings are intertwined, whereas “knowledge” seemed less personal and more declarative, standing apart from the knower. We used the term meanings instead of knowledge because we did not want readers to think our diagnostic instrument was designed to measure whether or not teachers had mastered a set of declarative facts related to teaching mathematics.

In light of the above, we named our instrument *Mathematical Meanings for Teaching secondary mathematics* (MMTsm) because we wanted to measure the meanings a teacher holds for the mathematics they teach. We hasten to say that we intend mathematical meanings for teaching to be understood as a teacher-centric construct, not a normative construct. Put another way, every teacher has meanings for the mathematics they teach. An individual teacher's meanings might be incoherent, superficial, coherent, or productive, but they are her meanings for the mathematics she teaches. We define "productive mathematical meanings" to be meanings that a teacher holds which would be productive for students' long-term mathematical learning were they to hold them also. We acknowledge immediately two concerns: (1) whether a meaning is actually productive for students' learning depends on schemes available to the students at the moment of instruction, and (2) determining whether or not a meaning is productive for students requires collecting empirical evidence from students. A particular meaning may seem more productive from the perspective of a more advanced knower, but there may be unforeseen consequences when attempting to teach this meaning to someone first experiencing the idea.

Schemes and Meaning

We do not have space to delve deeply into our meaning for scheme. See (Derry, 1996; P. W. Thompson, 2013; P. W. Thompson et al., 2014) for explanations of the idea of scheme and its relationship to meaning, understanding, assimilation, and accommodation. For our purposes here we will use the definition offered by Thompson et al. (2014):

> We define a scheme as an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity. (p. 11)

For example, Thompson and Saldanha (2003) described students' well-formed fraction schemes as emerging through a progressive coordination and integration of schemes for measurement, multiplication, division, and relationships of relative size. Similarly, Thompson and colleagues described students' well-formed rate of change schemes as emerging through the progressive coordination and integration of schemes for quantity, variation, covariation, change, accumulation, and proportionality (Silverman & Thompson, 2008; A. G. Thompson & Thompson, 1996; P. W. Thompson, 1994b; P. W. Thompson, Byerley, & Hatfield, 2013; P. W. Thompson & Thompson, 1994). Unlike "knowledge" which people often understand to mean knowing something that is agreed upon to be true, a teachers' scheme can be productive or unproductive. For example, a pre-service teacher inappropriately applied her part-whole scheme for fractions in a variety of situations. She drew a stick of length four inches and cut it into two equal pieces of length two inches. She described each piece as "one fourth" because she had one "part" out of the "whole" of four inches (Byerley & Hatfield, 2013).

The MMTsm includes several items that address contexts that we see as involving slope or rate of change. Our meanings for rate of change and slope, discussed shortly, served as a basis for our analysis of teachers' meanings. Our theory of meanings predicts that, for a teacher who has disconnected meanings for slope and rate of change, different contexts that involve slope or rate would trigger different schemes. Our theory also predicts that, for a teacher who has a coherent system of meanings for slope and rate of change, these same contexts would trigger different aspects of one scheme. For example, a teacher who understands the slope of a graph as a depiction of the relative size of changes in two quantities could connect a graph of rate of change of a car and a verbal description of a car's distance traveled, time elapsed, and the values on the car's speedometer. In contrast, some teachers understood slope to only describe real-world situations with horizontal and vertical components such as the steepness of a roof. These teachers do not connect the slope of a graph to more general rate of change situations such as the speedometer of a car even after instruction (Stump, 2001a).

Some teachers' conveyed dramatically different meanings in different contexts that we (and our consultants and advisory board) took as involving the same central idea. For example, some teachers computed average rate of change using an arithmetic mean of rates in one item and by using the formula $\Delta y/\Delta x$ in another item. We inferred that each of these teachers had more than one meaning for average rate of change. Each of these teachers' responses suggested that they had at least two schemes associated with the phrase "average rate of change", one that was sensitive to "average" as referring to a discrete collection of numbers and one that was sensitive to what they took as involving two continuous changes. Other teachers appeared to have a more comprehensive and coherent meaning for average rate of change because they employed the same meaning for average rate of change across contexts. Their responses conveyed that the average rate of change of a function on an interval of the independent quantity is a hypothetical constant rate of change of one quantity with respect to another that would produce the same change in the dependent quantity on that interval as was achieved by the function.

Our theory of meanings has strong implications for issues of reliability and validity of assessments and their items. If teachers conveyed similar meanings on multiple items intended to measure meanings for rate of change, the set of items would be considered to have internal consistency reliability. An assessment that validly assesses subjects' meanings with regard to ideas for which the population of subjects has a wide variety of disconnected meanings will likely have low internal consistency reliability. Subjects will respond differently to items that the assessment's writers see as tapping the same idea. This is exactly the case reported by Carlson, Oehrtman, and Engelke (2010). They conducted hundreds of interviews to establish the validity of their instrument's items with regard to their interpretations of students' answers, but most students answered some items that targeted a key idea correctly and other items targeting the same idea incorrectly—they had disconnected, and sometimes unproductive, ways of thinking about different contexts that (to Carlson et al., 2010) involved the same idea. To understand teachers' meanings it is critical to look at their responses to multiple rate of change items because we predict that their responses to one item will not be a strong predictor of their responses to other similar items unless they have strongly connected meanings for slope and rate of change.

Meanings for Quotient, Fraction, Magnitude, Variation and Covariation, and Rate of Change

In line with our stance that all meanings are personal, we attempt to convey the meanings of quotient, fraction, magnitude, covariation, and rate of change that we used in this study. We also explain why these particular meanings are productive and coherent for student learning and why it is important to understand high school mathematics teachers' meanings regarding quotient, fraction, magnitude, covariation, and rate of change. The meanings that we summarize here are described in greater detail in (P. W. Thompson, 1994a, 1994b; P. W. Thompson & Carlson, 2016; P. W. Thompson et al., 2014; P. W. Thompson & Saldanha, 2003).

Quotient. A multiplicative comparison of two quantities is the mental operation of comparing them with the intention of determining their relative size. Determining the relative size of two quantities means thinking of and expressing the magnitude of one quantity in terms of a multiple of the magnitude of another. For example, by using the height of one bean sprout to measure another bean sprout's height we can compare their relative heights. After measuring we could say "this bean sprout is one and three fourths times as tall as that bean sprout." A quantitative quotient is the value of a multiplicative comparison of quantities. The numerical value of a quotient need not be the result of division. When the value of a rate is provided as information about a situation, a person who understands the rate's value as a quotient understands that it gives the relative size of changes in one quantity with changes in another. Many students and teachers understand quotient (without knowing the word) only as the numerical result of division, without having an affiliated sense that they have determined a relative size. Other students and teachers understand the word quotient only as the name of a figural configuration that involves a vinculum—a horizontal division bar. Finally, many studies show that school students', future teachers', and teachers' meanings of division are non-quantitative and have little to do with ideas of relative size (Ball, 1990; Byerley & Hatfield, 2013; Byerley, Hatfield, & Thompson, 2012; McDiarmid & Wilson, 1991; M. Simon, 1993).

Fraction. In line with Thompson and Saldanha (2003), our meaning of fraction is that it is a scheme that emerges from the coordination of schemes for measurement, multiplication, division, and relative size. To say that A is 3/4 times as large as B means that A is three times as large as 1/4 of B, and that a part of A being 1/4 times as large as B means that B is four times as large as that part of A. Also, to say that A is 3/4 times as large as B implies the reciprocal relationship that B is 4/3 times as large as A. Moreover, the meaning of fraction is not necessarily bound to specific ways of inscribing one. To us, the statement "A is 3/4 as large as B" points to the same scheme as "A is 75 percent as large as B." The two statements indeed entail inscriptional schemes, but, in the end, they do not change the meaning of fraction. It should be clear that this meaning of fraction plays a large role in understanding quotient as a measure of relative size. The literature on students' and teachers' understanding of fractions is that, to many of them, a fraction connotes that if A is some fraction of B, that A is a subset of B. For a discussion of research on the limitations on students' part-whole meanings of fraction see Norton and Hackenberg (2010). Other students and teachers think that the mathematical meaning of a fraction is entirely bound to the context in which it is used, such as "3/4" being the slope of a line means "up 3 and over 4" (Stump, 2001a).

Magnitude. The common idea of magnitude is about size. We follow Thompson et al. (2014) by distinguishing among six meanings of magnitude: gross perception of size, size as measure being a count of a specific unit, size as measure relative to a specific unit (*Steffe Magnitude*), size as independent of specific units (*Wildi Magnitude*), relative size of measures in specific units, and relative size independent of units (*Relative Magnitude*). The last four meanings of magnitude (*Steffe Magnitude*, *Wildi Magnitude*, and two forms of *Relative Magnitude*) are all based in multiplicative comparisons of quantities' measures. The last three meanings involve the understanding that the size of the measured quantity is invariant across changes in unit. We claim that school instruction emphasizes the first three meanings of magnitude, whereas it is *Wildi Magnitude* and *Relative Magnitude* schemes that are foundational for mature understandings of rate of change and applying rate of change in science.

Variational and Covariational Reasoning. Both Confrey and colleagues and Thompson and colleagues have written extensively about the covariation construct (Confrey & Smith, 1995; Saldanha & Thompson, 1998). In this study, we use Thompson's meaning of covariation, expanded as in Thompson and Carlson's (P. W. Thompson & Carlson, 2016) framework for describing different levels of variational and covariational reasoning. This expanded framework includes a distinction introduced by Castillo-Garsow (C. Castillo-Garsow, Johnson, & Moore, 2013; 2012) between what he called "chunky continuous reasoning" and "smooth reasoning". A person reasons about a quantity or variable varying in "continuous chunks" by thinking that it attains a next value, that intermediate values exist, but without thinking that the quantity or variable actually attained any of those values. Thinking with smooth continuous variation is defined as,

> The person thinks of variation of a quantity's or variable's value as increasing or decreasing by intervals while anticipating that within each interval the variable's value varies smoothly and continuously.(P. W. Thompson & Carlson, 2016)

Thompson and Carlson (in press) then defined smooth continuous covariation as a person conceptualizing the values of two quantities varying simultaneously, while also having conceived of the quantities values varying smoothly and continuously.

The distinction between chunky and smooth continuous reasoning helps us distinguish between meanings that students and teachers have for slope and rate of change. A person who thinks in terms of chunky continuous variation is likely to think of a slope of ¾ as "up three and over four." Given one point on a line, some teachers find another point on a line with a slope of ¾ by moving over three and up four. However, they have difficulty reasoning about the values of the points in between the two points at either end of the "chunk." P. W. Thompson (2013) gave an example of a teacher who could not find the values of points on a line in between the two points she produced by moving up and over in chunks on a graph (p. 81). For this teacher, "division did not produce a quotient that has the meaning that the dividend is so many times as large as the divisor—3/4 as a slope was not a number that gave a rate of change. It gave a 'slantiness'" (P. W. Thompson, 2013, p. 81). A person who thinks in terms of smooth continuous variation is more like to think of slope as "changes in *y* are 3/4 times as large as changes in *x*." This meaning allows them to anticipate that the line takes on infinitely many points between any two points and allows them to find coordinates for these points if they need to. We believe that it is important to note if teachers or students meanings for covariation are chunky or smooth to understand differences in their meanings of rate of change.

Rate of Change. A mature meaning for rate of change involves imagining covariation of quantities as well as a relative size or relative magnitude scheme. This is consistent with Thompson and Carlson's (P. W. Thompson & Carlson, 2016) argument that "for students to conceptualize rates of change requires that they reason covariationally, but it also requires conceptualizations that go beyond covariational reasoning, such as conceptualizations of ratio, quotient, accumulation, and proportionality". Norton and Hackenberg (2010) provided evidence collected in teaching experiments with students that the development of productive meanings for rate of change and proportion are dependent on strong meanings for fractions. Understanding constant rate of change entails imagining two quantities covarying such that an accumulation of changes in one quantity is proportional to the associated accumulation of changes in the other quantity (P. W. Thompson, 1994b). A person's meaning for *proportional* may vary from a cue to cross multiply to a complex scheme built on images of comparing the measures of two quantities (Lobato, Orrill, Druken, & Jacobson, 2011). A student might use a *Steffe Magnitude* scheme to understand that the associated changes in the two quantities maintain a constant measure of relative size when the measure of one quantity is measured in terms of the other. A more advanced understanding of rate of change entails understanding a rate of change with a *Relative Magnitude* scheme. This would entail understanding that the relative magnitude of associated changes in the two quantities' values remains invariant even when one changes the unit in which either quantity is measured.

Qualitative Studies With Examples of Teachers' Meanings for Slope, Proportion, Rate of Change, and Quotient

Prior studies give examples of a variety of ways of thinking about slope, rate of change, and quotient that prove useful for interpreting teachers written responses to MMTsm items. The ways of thinking we will highlight from this literature are additive meanings, thinking about slope as an index of slantiness, the disconnect between slope and division, and difficulty in creating a quantitative image of division by a fraction.

Teachers' meanings for proportion. Fisher (1988) discussed secondary teachers' ability to help students transition between additive and multiplicative meanings for proportion. She described common informal additive methods for proportional situations that rely on halving or doubling. For example, if two batches of cookies takes 1.5 cups of sugar then four batches of cookies takes three cups of sugar. The number of batches of cookies increases in discrete amounts instead of continuously. Using Castillo-Garsow's (2012) language, we would say that the students imagine the batches of cookies varying in chunks of size two. Reasoning about quantities varying in chunks does not require the multiplicative understanding that the number of batches of cookies that can be made is always (2/1.5) times as large as the number of cups of sugar. Fisher (1988) found that the twenty teachers in her study were not likely to help students transition from additive to multiplicative ways of thinking about quantities changing proportionally. The teachers in her study did not talk about the connections between additive and multiplicative meanings for proportionality, and instead discussed procedures associated with proportions such as cross multiplication. In their solutions for four proportion problems, they were unlikely to use multiplicative proportional reasoning (Problem 1: 1 of 20 responses involve proportional reasoning, Problem 2: 1 of 20, Problem 3: 2 of 20, Problem 4: 4 of 20). Fisher's (1988) results show that teachers commonly use algebra to deal with situations that involve multiplicative comparisons from our point of view, but the teachers do not speak explicitly about these relationships.

Chunky meanings for slope. Coe (2007) and Stump (2001) interviewed in-service and pre-service secondary teachers who conveyed a chunky meaning for slope. While teaching a lesson on slope, Joe defined slope as, "'vertical change/horizontal change,' and presented a graph of the line passing through the points (0,0) and (3,2). He emphasized that the slope as a fraction, 2/3, up 2, over 3" (Stump, 2001, p. 216). Joe conveyed a chunky, non-multiplicative meaning for slope and never said for any size change in x the change in y is 2/3 as large. We believe his language in interviews and teaching would convey to a student that the division bar (vinculum) serves to separate numbers that tell us how to move in horizontal and vertical directions. This meaning is limited to Cartesian coordinate systems and can not be applied to polar coordinate systems. One consequence of the meaning for slope Joe conveyed was that a student in his class did not understand that "the two fractions 5/-6 and -5/6 could both represent the same slope" (Stump, 2001, p. 216). Joe noted in a post-teaching interview, "'They think you are describing a movement as opposed to you describing a number, a measurement'" (Stump, 2001, p. 216).

The three experienced secondary mathematics teachers whom Coe (2007) studied also conveyed chunky meanings for slope. For example, when Becky interpreted the slope -98/5 "she saw it as (-98) for every (5)" as opposed to that the change in y is -98/5 times as large as an associated change in x (Coe, 2007, p. 104). Peggy was asked "why do we use division to calculate slope?" and she replied that she didn't know because "she never really thought of it as the division operation" (Coe, 2007, p. 207). Peggy understood slope as directions on how to move up and over on a graph and did not imagine comparing the relative size of numerator and denominator.

Slope as an index of steepness. We believe most teachers' meanings for the idea of slope are multi-faceted and vary based on the situation they encounter. A chunky meaning for slope is useful for plotting lines. Coe (2007) and Stump (2001) found that teachers also employed a meaning for slope as an *index of slantiness* to various degrees of success. A teachers' sense of slantiness does not have to involve comparing changes in x and changes in y, but simply associating particular numerical values with particular graphs based on repeated exposure to graphs of lines. One limitation of remembering what a particular slope "looks like" is the dependence on the graphs being displayed in a rectangular coordinate system whose axes are in the same scale. Other limitations will become apparent in discussions of data that we present later.

Two pre-service teachers in Stump's (2001) study described slope as an index of slantiness. Natalie said, "Slope is a term used to associate the incline of a line with a numerical value" (Stump, 2001, p. 217) and Tracie said "assigning a number to a 'slant' is something that students just learning about slope are not accustomed to" (Stump, 2001, p. 217). Despite the methods instructors' emphasis on the connection between slope and real world comparisons of changing quantities such as distance and time Natalie choose to focus on steepness, inclined plane examples and developing "rise over run" as a measure of steepness (Stump, 2001, p. 221). Natalie was "resistant to including the notion of slope as a measure of rate of change in her work" (Stump, 2001, p. 221). The physical situations Natalie used in instruction included real-world examples such as inclined planes and ski slopes where steepness was visually apparent in the situation. She did not help students understand that slope could be thought of as the rate of change of any two quantities.

Coe's (2007) study complements Stump's work by showing the constraints of operating with a primary meaning for slope as a measure of steepness. Mary, described slope as "how steep a line is" (Coe, 2007, p. 115). Mary was unable to use her meaning for slope as a measure of steepness to make sense of multiple questions involving basic applications of slope. Coe asked Mary to answer the question in

Figure 1.

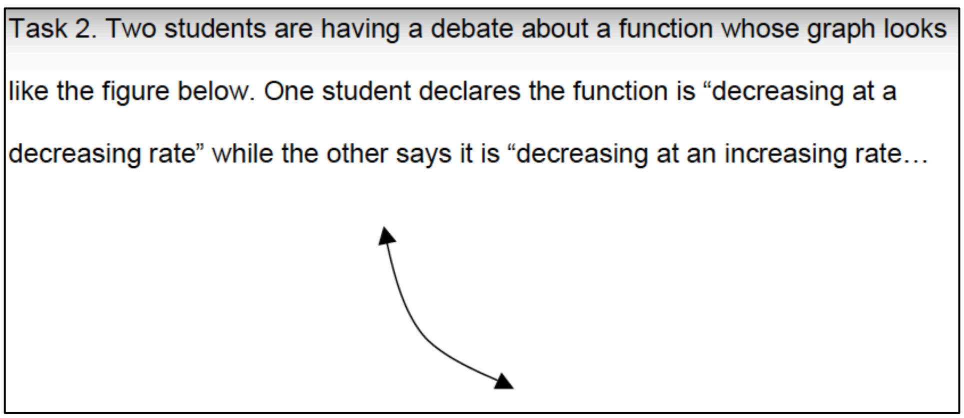


Figure 1. Task from Coe (2007).

Mary said she "Did this on the test the other day… Okay this is decreasing at a decreasing rate [pointing to given question] because as you come down here [right end of curve], tangent line has a smaller slope than this one [pointing to tangent sketched on left end of curve]" (Coe, 2007, p. 129). Mary graded student tests incorrectly because of her meaning of slope as an index of steepness. She did not consider that the changes in y in the graph are negative, and simply looked at how steep the graph appeared, as if it were a hill. A slope of negative two is smaller than a slope of negative one because $-2 < -1$. However, a slope of negative two appears steeper from the perspective of thinking about the slantiness of a hill.

Teachers' meanings for quotient. All three studies we found that investigated secondary teachers' meanings for quotient showed that teachers had significant difficulties with the idea (Ball, 1990; Byerley & Hatfield, 2013; McDiarmid & Wilson, 1991). We hypothesize that teachers with weak meanings for quotient are less likely to think of rate of change and slope multiplicatively, and more likely to resort to chunky or 'index of steepness' meanings.

McDiarmid and Wilson (1991) presented 55 alternatively certified secondary teachers with four story problems that prompted them to choose which story problem could be solved by dividing by ½. Only 33% were able to identify a quantitative situation that involved division by a fraction. In interviews, some teachers in their study could see no real world application for division by fractions.

> We can't really relate it to what does this mean to divide something by a half. What does that really mean? I know what it means to divide something by two. You're dividing it into two equal portions and then we start thinking about that becomes a half. But to divide something by a half, that's very abstract-you have to really push. What does that really mean and do I ever really use something like that? I mean, am I ever doing something like that? (McDiarmid & Wilson, 1991, p.99).

This teacher quoted above did not display a meaning for quotient as a relationship of relative size. Instead, he displayed a meaning that division means to cut something into a number of pieces. One of many situations the teacher could have used to explain division by fractions is in cooking. Suppose one has 1 ¾ cups of flour and each recipe called for ½ cup of flour. How many recipes could he make?

Ball (1989) asked prospective teachers "to develop a representation—a story, a model, a picture, a real-world situation—of the division statement $1\frac{3}{4} \div \frac{1}{2}$" (p. 21). Five out of nine prospective secondary teachers and zero out of nine elementary teachers responded appropriately (p. 22). Byerley and Hatfield (2013) asked 17 pre-service secondary teachers who were taking an upper division teaching methods course to draw a picture representing a division problem (See Figure 2). The results supported the hypothesis that many secondary teachers do not have strong quantitative meanings of quotient as a measure of relative size. Two said “I don’t know.” Seven out of 17 gave computational explanations without drawing a picture. For example, “20.15 times .39 is 7.86”. Two pre-service teachers drew a picture to remind them of computations (See Figure 2).

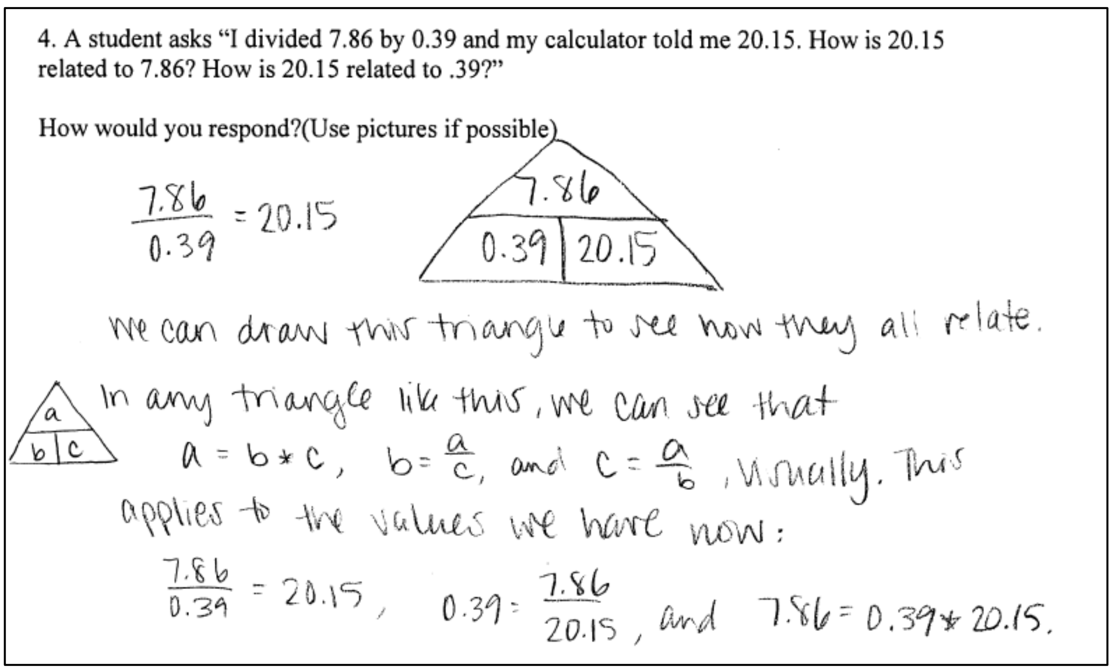


Figure 2. Pre-service Mathematics Teacher Drew Picture of Computations. Item Adapted from Byerley, Hatfield & Thompson (2012).

Six out of 17 represented the relative size of 7.86 and .39 in an image to explain the meaning of a quotient (Byerley & Hatfield, 2013). See Figure 3 for an example.

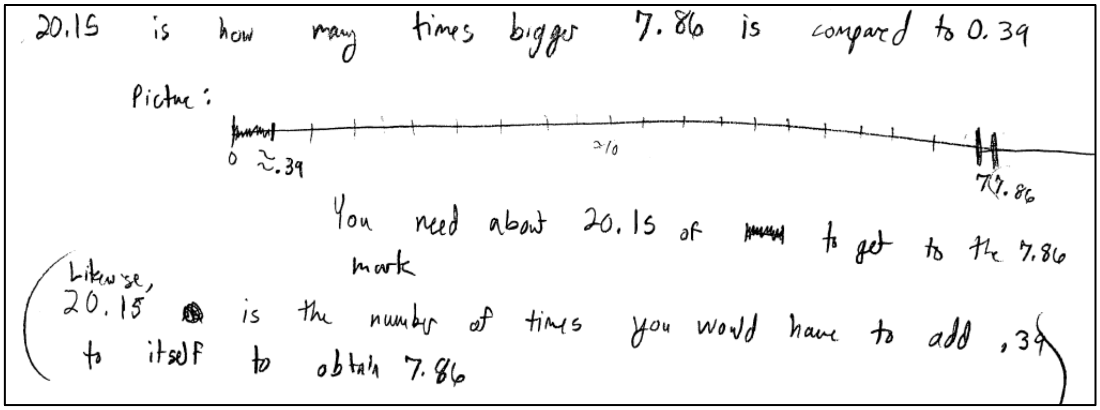


Figure 3. Diagram Depicting Relationship Between Numbers in Division Problem.

Only one of the seventeen pre-service teachers in the study was able to explain why division was used to calculate slope. We hypothesized that without an image of quotient as a measure of relative size, and an image that values of *y* and *x* varied together, it was hard for them to employ a meaning for slope as a measure of the relative size of a change in *x* and the corresponding change in *y.* We should note that it is possible for a teacher to understand quotient as a measure of relative size and have a chunky or indexical meaning for slope. The teacher must not only have a strong image of quotient, but must also have reflected on why division is used in the slope formula.

Teachers' meanings for measure. Two previously reported MMTsm items provide information about teachers' meanings for measurement (Byerley & Thompson, 2014; P. W. Thompson et al., 2014). One item asked teachers to convert between liters and gallons given a conversion factor.

A container has a volume of *m* liters. One gallon is $\frac{189}{50}$ times as large as one liter. What is the container's volume in gallons? Explain.

Figure 4. MMTsm Item *Gallons to Liters*. © 2014 Arizona Board of Regents. Used with Permission.

The second item asked teachers to convert between measures in the imaginary units "Nerds" and "Raps" given a conversion factor (Figure 5).

In Nerdland they measure lengths in Nerds. The highlighted arc measured in Nerds is 12 Nerds. In Rapland they measure lengths in Raps. One Rap is 3/4 the length of one Nerd. What is the measure of the highlighted arc in Raps?

Figure 5. Item *Nerds and Raps*. © 2014 Arizona Board of Regents. Used with Permission.

As of 2014 we had collected 100 secondary teachers' responses during the MMTsm's development phase. Only 24% of secondary mathematics teachers were able to convert between liters and gallons (Byerley & Thompson, 2014). Fifty out of 100 teachers were able to convert between the imaginary units *Nerds and Raps.* Only 17% of the 100 teachers solved both measurement problems correctly. Both items are similar to middle school level Common Core measurement items and the problematic responses demonstrate that teachers lacked strong measurement schemes. In particular, they did not imagine that since a gallon is larger than a liter, the number of gallons in a container must be smaller than the number of liters in the same container. Quantitative measurement meanings are critical for developing the ability to conceive of the change in one quantity measured in terms of the change in the other, so the weak performance on measurement items suggests that the teachers may also lack meanings for slope and rate based on measuring the change in y in terms of the change in x. Teachers' responses also suggest that many do not reason about the quantities in the situation. Other researchers have also found that students' and teachers' difficulties with multiplicative situations are due to a lack of orientation to reason about the measures of the quantities in the problem (Lobato & Siebert, 2002; M. Simon & Blume, 1994a, 1994b).

The responses in Figure 6 and Figure 7 demonstrate that two teachers' difficulties were not just due to an oversight or answering the problem too quickly. These responses also demonstrate the teachers' difficulties with reasoning about the measures of quantities.

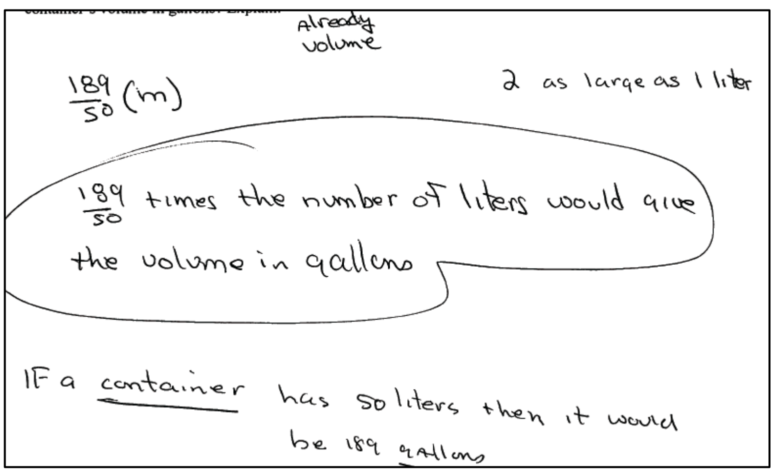


Figure 6. Example Where Teacher Wrote Meaning of *m* Explicitly.

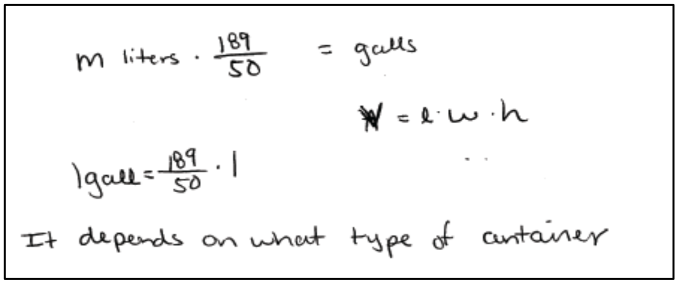


Figure 7. Teacher Needed to Know Shape of Container to Use Appropriate Volume Formula.

The teachers in Figures 8 and 9 wrote in complete sentences and appeared to fully consider the problem and yet were unable to compare a container's measure in two different units. The response in Figure 7 shows that the teacher's meaning for a measure of volume was dominated by the formulas he had learned. His reasoning was not constrained by an understanding that converting one measure of volume to another does not require a volume formula for a particular shape.

Teachers' meanings for rate of change. At least four studies have investigated secondary teachers' meanings for rate of change (Bowers & Doerr, 2001; Coe, 2007; Person, Berenson, & Greenspon, 2004; P. W. Thompson, 1994b). Data from each study supported the claim that many secondary teachers have meanings for rate that are chunky and additive. However, teachers' chunky, additive rate of change schemes actually work for them in much of school mathematics.

Person, Berenson and Greenspon (2004) investigated a secondary preservice teachers' "lesson plans on rate of change and right triangle trigonometry in light of his beliefs of ratios and fractions" (p. 17). Brian described rate of change as "the amount something changes in a given time" (Person et al., 2004, p. 21). Using Castillo-Garsow's language we notice Brian's description conveys an image of competed chunks of change. Brian conveys that a rate of change is an *amount* of change. He doesn't convey comparing a measure of an amount of change with a measure of amount of time. Time does pass in the background, but Brian's meaning for rate of change was not about multiplicative comparisons of two amounts. Brian also describes rate of change as "something per something" as in miles per hour (Person et al., 2004, p. 21). This suggests that the word "per" cued Brian to think of rate of change. At first Brian's descriptions of a graph of a function that had a constant rate of change did not make use of the idea of proportionality, ratio, or comparisons of relative size. Brian described a constant speed with the idea of cruise control. Building on observations made by Stroud (2010), we notice that Brian conveyed the idea of speed as the number to which a speedometer points instead of as a relationship of relative size between number of miles traveled and number of hours elapsed.

Bowers and Doerr (2001) investigated 26 secondary teachers' thinking about the "mathematics of change" in two university technology based mathematics classes. They designed the first two instructional sequences to help the participants understand the Fundamental Theorem of Calculus by exploring relationships between linked velocity and position graphs (Bowers & Doerr, 2001, p. 120). Given a non-constant velocity versus time graph, over half of the fifteen teachers at the first university found the total distance traveled by simply multiplying time elapsed by the velocity at the end of the time interval by inappropriately applying the formula $d=rt$ (Bowers & Doerr, 2001, p. 124).

Thompson (1994b) reported a classroom teaching experiment designed to help nineteen senior and graduate mathematics education students understand connections between rate of change functions and accumulation functions. Thompson (1994b) reported that students had difficulties understanding the rate of change of a cone's volume with respect to its height partially because they confused "changing" with "rate of change" and "amount and change in amount" (1994b, p. 257). Thompson wanted students to understand why the function giving rate of change of the cone's volume with respect to its height was identical to the function of cross sectional area of the top of the cone in terms of the height of the cone. One student, Adam, struggled to explain the relationship because he identified the idea of "rate" with the idea of "change" (P. W. Thompson, 1994b, p. 261). Adam understood that the volume of the cone got larger as the surface area of the cross section at the top of the cone increased. He did not connect the amount of change in volume to the amount of change of cross sectional area (P. W. Thompson, 1994b, p. 261).

Connections between literature review and our methodology

The literature review reported multiple conceptual analyses of productive meanings for rate of change, fraction, quotient, and slope. We used prior conceptual analyses of productive mathematical meanings to design items and identify high level responses for the MMTsm. While designing the diagnostic instrument we focused on teachers' meanings because the word meaning conveys that we are interested in more than what a teacher can do or what facts they recall (P. W. Thompson, 2015). We want to build models of teachers' schemes. In our methods section, we will describe how we used rubrics to categorize teachers' responses according to the meanings the response might convey to a student instead of scoring a response as correct or incorrect.

Our conceptual analysis of productive meanings for rate of change focuses not only on making multiplicative comparisons of changes in quantities, but also on how the quantities covary. We noted that quantities can be imagined to covary smoothly or in chunks (C. W. Castillo-Garsow, 2012). We stayed attuned to this distinction when analyzing teachers' responses and attempted to create items that would give us insight into the teachers' images of covariation.

The teachers' meanings for slope and rate of change described in the qualitative studies in the literature review helped us categorize teachers' written responses on the MMTsm. A summary of the most important constructs is in Table 1

Table 1. Examples of Evidence of Various Types of Reasoning about Slope, Quotient, and Rate of Change.

Construct:	Example of evidence of this type of reasoning
Chunky covariational reasoning	A slope of three means that every time *x* changes by 1, *y* changes by 3.
Smooth continuous covariational reasoning	A slope of three means that as *x* and *y* covary, for any sized change in *x* the associated change in *y* is three times as large.
Slope is an index of steepness/rate is an index of fastness	Slope is a number we assign to a slant to describe how steep it looks.
Formulaic meaning for slope/rate of change	$\Delta y/\Delta x$, rise/run, d=rt, etc.
Quotient is the result of division	When I use long division and follow the steps I get a quotient.
Quotient is a measure of relative size	I can estimate the quotient (A/B) by comparing the measure of quantity A to the measure of quantity B.

In prior qualitative studies teachers often spoke additively about rate of change and slope. For example in both rate of change and slope contexts teachers spoke about changes occurring in chunks and did not talk about comparing the relative magnitude of the measures of changes (Coe, 2007; Stump, 2001a). Teachers' also spoke about slope as an indication of how slanty a line is without a focus on the underlying comparison of changes in *x* and changes in *y*. Teachers' tendency to speak of slope and rate of change without a focus on multiplicative comparisons may be because their weak meanings for quotient noted in multiple studies (Ball, 1990; McDiarmid & Wilson, 1991) do not support more productive multiplicative reasoning. Related to their additive meanings for slope, were issues such as confusing rate of change with amount of change (P. W. Thompson, 1994b). Teachers were also reported to use rate of change formulas inappropriately because of a lack of focus on the meanings for constant rate of change (Bowers & Doerr, 2001). The majority of these qualitative studies relied on interviews or teaching experiments in which the researchers used multiple sources of evidence to infer a teachers' meaning for slope or rate of change. We drew upon these descriptions of teacher thinking, in addition to our own teacher interviews, to help us infer a teachers' potential meanings based on their sometimes terse written responses.

METHOD: THE DEVELOPMENT AND ADMINISTRATION OF THE MMTSM

This section describes the methods used during the four-year project Aspire to design and validate the MMTsm diagnostic instrument. We also explain how our instrument development built upon findings of prior qualitative studies and our theory of meanings. The motivation for Project Aspire was to develop a diagnostic instrument that would identify weaknesses and strengths in teachers' meanings in a way that would be useful for designing and evaluating professional development. The instrument is designed to be useful to give information about the common ways of thinking about mathematical ideas in a group of teachers and to determine if professional development had a positive impact on a group of teachers' mathematical meanings. It is not intended to evaluate teachers for purposes of employment or do create diagnostic models of individuals. Because of the purpose of the instrument we will not report all psychometric properties that would be necessary to establish reliability and validity for an instrument designed to diagnose individuals.

Item Development

One of the primary goals for the items was to give the teacher the opportunity to convey the sense they made of an item, which then gave us grounds to discern meanings they employed in making that sense. The relationship between our theory of meanings and the methods are discussed in greater detail in Thompson (2015). It was important that teachers could interpret a question in their own way, and that the question would prompt them to display their meanings explicitly enough that we could interpret and categorize them confidently. We also had to create items that prompted teachers to use higher-level meanings if they were able to do so. For example, if we wanted to determine whether a teacher could think about slope as the relative size of the change in y and the change in x, we could not ask a question that could be solved by routinely using the slope formula.

Thompson (2015) summarized the process of creating items and rubrics for the MMTsm. We followed typical instrument construction guidelines such as making many revisions to items based on interviews with teachers and pilot administration of the instrument. Feedback from our advisory board and other experts was also essential to item development. Once we created rubrics for items, we made more edits to the items to improve the ease of scoring.

Rubric Development

In Summer 2012 administered the draft versions of the slope and rate of change items discussed in the results sections to secondary mathematics teachers voluntarily participating in Math Science Partnership professional development programs. We categorized the thinking revealed in 144 teachers' responses to items using a modified grounded-theory approach (Corbin & Strauss, 2007). The modification was that we began our data analysis with the conceptual analysis magnitudes and rates of change described in the literature review, as well as multiple descriptions of teachers' meanings from prior qualitative studies.

We developed rubrics by grouping grounded codes into levels based on the quality of the mathematical meanings expressed. By reading a teacher's written response to an item we do not believe it is possible to model their meanings with the same level of assurance as if we interviewed the teacher. However, we did interview a subset of teachers to check whether the meanings we attributed to their written comments were consistent with the meanings they expressed later to us. Further, we hypothesize that teachers' use the same meanings and mathematical resources to respond to written items about teaching as they use while teaching. This is not to say that their written responses will exactly reflect what they say in classroom, only that their written and spoken descriptions of mathematics will be based on their meanings in either context.

When scoring responses we did not attempt to determine the depth of the teachers' understanding of mathematics that they left unarticulated. Instead, we read the teacher's response literally and asked, "If this is what they said to a class, what meanings for the mathematical idea might students' construct?" During team discussions of rubrics and responses, we continually asked ourselves. "How productive would this response be for a student if this is what the teacher said while teaching?" We are aware that some people who describe slope as "rise over run" might have more advanced and productive meanings, but we scored their responses as written and did not attribute additional unexpressed understandings to teachers without evidence. We relied heavily on prior research on student thinking to make determinations about the ways of thinking about particular ideas that would be more or less productive for students. For example, we pointed to Coe (2007) to warrant our concern with responses that conveyed a primary meaning for slope as an index of steepness.

At multiple stages in rubric construction and refinement we asked researchers internal and external to the project to score randomly selected subsets of teacher responses after participating in sessions focused on using the rubrics. Scorers had a variety of areas of expertise and included one or more statisticians, mathematicians, high school math teachers, and math education researchers. The scoring team included researchers from various institutions who learned secondary mathematics in a variety of foreign countries.

Sample and Scoring

We administered the MMTsm to 251 high school teachers in two different states in Summer 2013 and 2014. The teachers were participating in Math Science Partnership professional development programs (NSF or state funded). The sample had 63 teachers with at least a mathematics BA, 81 teachers with at least a mathematics education BA, and 107 with a BA in another subject. Many of these teachers also had masters degrees in a variety of fields. The number of years they had taught high school math varied from one year to over fifteen years.

The Aspire project team, which includes the first and second author, scored all teacher responses. To estimate interrater reliability (IRR) an outside collaborator scored 50 overlapping responses for each item and an Aspire team member scored all of the responses separately. Note this was the final round of many rounds of IRR for Aspire team members but it was the first time the outside collaborator had used the rubrics. Non-perfect agreement was scored as disagreement. Items with complex responses had lower IRR than items with simple or numerical responses. For example, when teachers explained how they would teach slope they often did not use complete sentences and used pronouns with unclear referents, so it often was difficult to determine whether or not a student could make sense of the teacher's explanation. The following table summarizes the interrater reliability scores for all items reported in the results section.

Table 2. Interrater Reliability Scores for MMTsm Items.

Item Name	Number of responses scored by two scorers	Percent Agreement	Cohen's Kappa
Gallons to Liters	50	.94	.917
Nerds to Raps	50	.94	.905
Meaning of Slope Part A	50	.84	.773
Meaning of Slope Part B	50	.72	.621

Meaning of Over Part A	50	.86	.814
Meaning of Over Part B	50	.9	.849
Slope from Blank Graph Part A	31	.968	.957
Slope from Blank Graph Part B	31	.903	.847
Increasing or Decreasing from Rate Part B	49	.959	.926

Table 2 shows that most items have a high level of agreement in each rater's score. The interrater reliability is strong enough to support inferences about what types of meanings we could expect to find in a group of teachers. Percent agreement scores of less than 90 are insufficient to use the item to diagnose the meanings of a particular teacher. The MMTsm is not designed to evaluate a particular teacher's meanings for any high stakes reason, such as employment or teacher evaluation. Rather, the MMTsm is designed to give researchers and professional development leaders information about groups of teachers that will be useful for planning professional development projects.

RESULTS

We present each item, the scoring rubric, and the distribution of teachers' responses on that item. After presenting all of the items and results we examine what the responses to the set of items convey about teachers meanings for slope and rate of change.

The Item *Meaning of Slope*

We designed the item in Figure 8 to reveal teachers' meanings for slope. We designed Part B to reveal teachers' ability to use constant rate of change to determine the change in dependent variable for any change in the independent quantity. We wanted to see if teachers could move beyond thinking of slope in terms of chunky, one-unit changes in *x*.

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.

Convey to Mrs. Samber's students what 3.04 means.

(next page)

Part B.

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.

A student explained the meaning of 3.04 by saying, "It means that every time *x* changes by 1, *y* changes by 3.04." Mrs. Samber asked, "What would 3.04 mean if *x* changes by something other than 1?"

What would be a good answer to Mrs. Samber's question?

Figure 8. The Item *Meaning of Slope* was Designed to Reveal Meanings for Slope. ©2014 Arizona Board of Regents. Used with Permission.

Rubric and results Part A on *Meaning of Slope*. The summary rubric for *Meaning of Slope* Part A is given in Table 3.

Table 3. Rubric for Part A of *Meaning of Slope*. © 2014 Arizona Board of Regents. Used with Permission.

Level A3 Response:	The teacher conveyed that *x* can change by any amount and that *y* changes by 3.04 times the change in *x*.
Level A2a Response:	***Any*** of following: – The teacher wrote that for every change of 1 in *x*, there is a change of 3.04 in *y*. – The teacher wrote that for every change of 2.7 in *x*, there is a change of 8.2 in *y*. – The teacher wrote that a difference in *x* values is compared to a difference in *y* values.
Level A2b Response:	The teacher conveyed in words or graphically that the slope gives information about how to move horizontally and vertically. For example: – If *x* moves to the right 1 space, *y* moves up by 3.04. – If *x* runs by 2.7, *y* rises by 8.2. – The slope tells us to move horizontally by one and vertically by

	3.04.
Level A1 Response:	***Any*** of following: − The teacher conveyed that 3.04 is the result of a calculation. − The teacher used a phrase such as "average rate of change", "constant rate of change" or "slantiness" without addressing the question of how 3.04 relates changes in *x* and changes in *y*. − The teacher simply stated the idiom "rise over run" without describing the changes.

Level A3 responses convey a multiplicative meaning for slope. A multiplicative meaning for slope builds on the meaning for quotient as a measure of relative size. Level A2a and level A2b responses convey an additive or chunky meaning for slope. Level A2a responses are considered slightly more productive for students than A2b responses because the meaning of slope in Level A2a responses is not constrained to horizontal and vertical motion on a Cartesian graph, but could be used productively in concrete situations. Level A1 responses on our rubric represented more than one possible meaning for slope, but each of these meanings are similar in the sense that they convey that the meaning of slope is something to memorize. While scoring we kept track of teachers' mathematical errors separately from their meaning of slope, so some responses scored at A1 have incorrect formulas such as $\Delta x/\Delta y$ or y/x. We scored responses that did not fit any other category at level A0. In cases where one teacher responded with multiple meanings for slope we categorized the responses by focusing on the highest level meaning.

The most common meaning conveyed in our sample was a chunky, additive meaning for slope (See Table 4).

Table 4. Responses to Part A *Meaning of Slope*.

Response	Math Majors	Math Ed Majors	Other Majors	Total

A3-relative size	3	3	4	10
A2a-chunky	27	18	37	82
A2b-chunky graphical	24	47	41	112
A1-memorized	7	12	19	38
A0-other/ IDK	1	1	2	4
No response	1	0	3	4
Total	63	81	107	250

* We included 250 teachers instead of 251 because one teacher did not state his major.

Only ten teachers out of 250 conveyed a multiplicative meaning for quotient in explanations of slope in Part A. Approximately 78% of teachers showed a chunky or additive meaning for slope. About 81% of teachers who majored in mathematics and 80% of teachers who majored in mathematics education conveyed a chunky, additive meaning. There is no evidence that there is a statistically significant relationship between degree type and the teachers' description of slope ($\chi^2(6,n=242)=10.71, p=0.097$). The eight teachers who did not give a response or gave an other response were excluded from the chi squared analysis. In all tables the designation "IDK" means that the teacher stated that they did not know how to answer the problem.

Rubrics and results Part B on *Meaning of Slope*. Part B gave teachers an additional opportunity to convey a multiplicative meaning for slope. In Part B, it was still uncommon for teachers to express the multiplicative relationship between corresponding changes.

Figure 9. Multiplicative Response to *Meaning of Slope*.

Figure 9 displays an example of a response that conveys a strong multiplicative meaning in Part B.

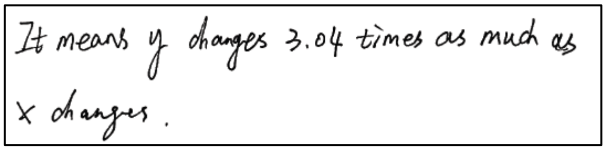


Figure 9. Multiplicative Response to *Meaning of Slope*.

A response is consistent with additive reasoning if the teacher imagines adding one so many times and adding 3.04 the same number of times. Figure 10 shows an example of an additive response to Part B.

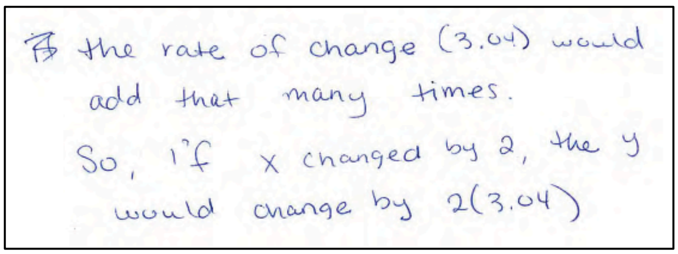


Figure 10. Part B Response to *Meaning of Slope* that Conveys Additive Reasoning.

The teacher appears to use a meaning of multiplication as repeated addition to compute the change in *y* given the change in *x* of 2. We did not take use of the operation of multiplication in the response to necessarily convey multiplicative reasoning to a student. The response in Figure 10 is consistent with imagining x varying in chunks of size one and *y* varying in chunks of size three. In our rubric we did not ask scorers to determine whether or not a response conveyed multiplicative or additive reasoning because this is difficult for a number of reasons. Instead we listed statements teachers actually made in their responses in the rubric and then drew inferences about whether or not these statements were consistent with additive or multiplicative reasoning.

The Aspire team categorized the responses to Part B using the rubric in Table 5.

Table 5. Rubric for *Meaning of Slope Part B*. © 2014 Arizona Board of Regents. Used with Permission.

Level B4 Response:	***All*** of following: + The response answered the question "what does 3.04 mean?" and did more than explain how to find the change in *y* given an arbitrary change in *x*. + The teacher conveyed the meaning of 3.04 as that the change in *y* will be 3.04 times as large as *any* change in *x*.
Level B3 Response	***All*** of following: + The teacher gave a mathematically reasonable explanation of what 3.04 means (e.g. constant of proportionality, ratio of changes, ratio, multiplication factor, multiplier, etc.) + The teacher did ***not*** convey that 3.04 means that the change in *y* will be 3.04 times as large as the change in *x*.
Level B2a Response	***All*** of following: + The teacher gave clear and explicit instructions describing how to find the change in *y* given an arbitrary change in *x*. + The response conveys clearly that to find the change in *y*, the change in *x* should be multiplied by 3.04. + The response did ***not*** convey what 3.04 means beyond a number that is used to compute a change in *y*.
Level B2b Response:	***All*** of following: + The teacher's explanation, in essence, amounts to giving one or more specific examples of how to find the change in *y* given a new change in *x*. + The teacher did ***not*** state a general relationship between changes in *y* and changes in *x*.
Level B1 Response:	The response appears to answer the question "how do you find the change in *y*?" but does so without explicitly mentioning the change in *y*. For example: − The teacher might say "multiply it by 3.04" without mentioning the change in *y*. − The teacher might say the change in *y* is proportional to the change in *x* without also saying what this means.

The data for *Meaning of Slope* suggests that it was hard for many teachers to make sense of Part B as intended without a strong multiplicative meaning for slope. Rubric levels B4 and B3 correspond to responses that answered the question "what does 3.04 mean?" and levels B1, B2a and B2b correspond to responses that answered the question "how do you find the change in *y* if *x* changes by something other than 1?"

Table 6 only has 155 teachers because while all 251 teachers saw the same Part A, only the 155 teachers from Summer 2014 answered a version of Part B that was improved after Summer 2013 pilot testing.

Table 6. Responses to Parts A and B of *Meaning of Slope*.

Response	B4 "Δ*y* 3.04 times as large as Δ*x*"	B3 Multiplier (vague)	B2a Explain how to find Δ*y*	B2b Example of finding Δ*y*	B1 "multiply it by 3.04"	B0	NR/ IDK	Total
A3-multiplicative	1	0	1	1	0	0	0	3
A2a-chunky	6	7	5	6	7	10	0	41
A2b-chunky	4	13	13	7	11	31	0	79
A1-memorized	0	2	4	4	6	8	4	28
A0-other	1	1	0	0	1	0	1	4
No response	0	0	0	0	0	0	0	0
Total	12	23	23	18	25	49	5	155

Ten (8.3%) of 120 teachers who gave chunky (level 2a/2b) responses to part A conveyed multiplicative meanings for slope in Part B. Table 6 suggests that it was much more common for teachers who conveyed chunky meanings on Part A to continue to convey chunky computational meanings on Part B. Also, 41 (34.2%) of the 120 teachers who conveyed a chunky meaning on Part A gave a level 0 response to Part B. Although level 0 responses are widely varied, they all failed to deal with the Part B prompt coherently. This suggests that having a chunky meaning for slope is insufficient to deal meaningfully with situations where the input variable changes by something other than one.

We emphasize that we ignored mathematical errors when we categorized responses to both parts. We attempted to categorize responses by the overall meaning conveyed. For example, on Part B many teachers confounded the change in *y* with *y* and wrote $y = 3.04x$. Fifty-nine (59; 23.5%) of the 251 responses to Part A contained a mathematical error of some kind, and 73 (47.1%) of 155 Part B responses contained an error of some kind. We suspect that errors were more common on Part B because teachers were asked to cope with a situation that, from their perspective, was unusual, while on Part A they expressed what they would say customarily.

The Item *Relative Rates*

Although additive meanings for slope and rate are productive in certain settings, these meanings can lead to invalid models of physical situations. The response to the item *Relative Rates* (Figure 11) shows one consequence of thinking of a rate of change additively. We first discussed this item in Byerley and Thompson (2014). This paper includes responses from 150 more teachers than the first report.

Every second, Julie travels j meters on her bike and Stewart travels s meters by walking, where $j > s$. In *any* given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a. Julie will travel $j - s$ meters more than Stewart.
b. Julie will travel $j \cdot s$ meters more than Stewart.
c. Julie will travel j / s meters more than Stewart.
d. Julie will travel $j \cdot s$ times as many meters as Stewart.
e. Julie will travel j / s times as many meters as Stewart.

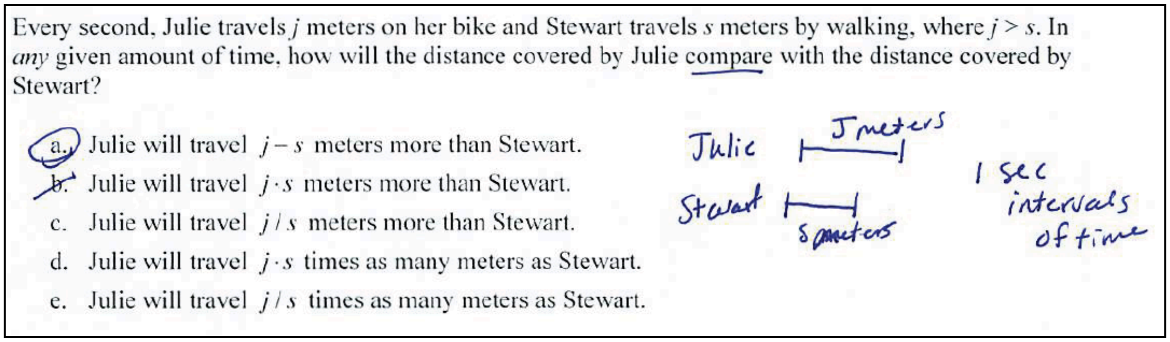
Figure 11. Item Called *Relative Rates.* © 2014 Arizona Board of Regents. Used with Permission.

We suspect that thinking of rate of change additively makes it more difficult to identify situations that are modeled with quotients. The responses to the item *Relative Rates* demonstrate that many teachers used an additive interpretation in a rate of change situation when a multiplicative interpretation was appropriate. We conducted interviews to understand the reasoning behind the choice of the highest level answer, *j/s* times. One teacher responded *j/s* and noted that the quantity *j-s* tells us how much father Julie travels in one second instead of in *any* amount of time. His explanation was:

> I selected (e) because we have proportional quantities here. In four seconds she travels 4*j* and he travels 4*s*. I assume she is traveling farther but it doesn't actually matter. She will always travel, since they are both traveling the same distances each second she will always travel some constant value some value that is a constant times greater than his, like *k* times greater than his distance traveled.

Some teachers selected (c) or (e) after giving *j* and *s* values such as 4 and 3, and then making a table of values to determine the answer.

Choice *j-s* corresponds with thinking about constant rate of change additively. In one interview the teacher highlighted the word "any" while reading the statement. She explained the choice of *j-s* by saying she cares "about distance traveled so time doesn't make any difference." There are multiple ways of thinking about the problem that result in choice (a), so there is no way to determine precisely what type of thinking a teacher engaged in to pick (a). Despite the variety of potential solution paths the interview data suggests that teachers who picked (a) were thinking about the situation additively. For example, some teachers thought of "*j*" as a changing quantity that represents Julie's distance for any given amount of time. With *j* representing a changing quantity instead of the value of a fixed unknown rate, the additive response *j-s* made sense to these teachers. Another teacher drew a velocity versus time graph and thought of the total distance traveled as the area under the curve. They named the areas "*j*" and "*s*" so that the distance between them was equal to “*j-s*”. Some teachers appear to solve the problem for one second intervals of time (see Figure 12).


Every second, Julie travels j meters on her bike and Stewart travels s meters by walking, where $j > s$. In *any* given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a. Julie will travel $j - s$ meters more than Stewart.
b. Julie will travel $j \cdot s$ meters more than Stewart.
c. Julie will travel j / s meters more than Stewart.
d. Julie will travel $j \cdot s$ times as many meters as Stewart.
e. Julie will travel j / s times as many meters as Stewart.

Figure 12. A Teacher's Response to *Relative Rates*. © 2014 Arizona Board of Regents. Used with Permission.

Figure 12 shows one consequence of having an additive meaning for rate of change (“1 unit of distance for each 1 unit of time”). For those with an additive meaning, speed is the distance travelled in a 1-unit interval (i.e. chunk) of time as opposed to the relative size of the measure of distance travelled and the measure of elapsed time to travel that distance. In teacher responses to other items, interviews, and in the literature, we also noticed teachers using the formula $d = rt$ without considering quantitative relationships that this formula entails, and therefore expected to see a product as part of the answer (Bowers & Doerr, 2001). Table 7 Presents the Results for *Relative Rates* by Major.

Table 7. Responses to Item *Relative Rates.*

Response	Math Majors	Math Ed Majors	Other Majors	Total
j/s times (e)	18	21	30	69
j/s more (c)	4	8	8	20
j-s (a)	36	44	56	136
*j*s* (b or d)	2	6	9	17
Other	2	2	3	7
No response	1	0	0	1
Total	63	81	107	250

The majority of the 250 teachers (54%) inappropriately use an additive model (*j*-*s*) of a situation that requires a multiplicative comparison. Moreover, only 27% of teachers gave the highest-level response that used multiplicative language as well as a multiplicative comparison. Teachers with math or math education degrees were not more likely to give a high level response. There is no evidence of a relationship between response and degree type ($\chi^2(8, n = 249) = 2.63, p = 0.955$) . The teacher without a response to *Relative Rates* was excluded from the Chi-square analysis.

The Item *Meaning of Over*

One way of coping with an underdeveloped meaning of rate of change is to make use of key words. The word "over" often refers to division and cues the use of slope or rate of change formulas. These patterns in usage allow teachers to choose the operation of division and solve many problems correctly without conceptualizing rate of change as a multiplicative relationship between changes in two quantities. The responses to the item *Meaning of Over* reveal teachers' tendencies to be cued by the word "over" to inappropriately model an additive situation with division instead of subtraction (See Figure 13).

A college science textbook contains this statement about a function *f* that gives a bacterial culture's mass at moments in time.

> The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this statement?

Part B. Express the textbook's statement in mathematical notation.

Figure 13. The Item *Meaning of Over*. © 2014 Arizona Board of Regents. Used with Permission.

Thompson (2015) first presented results to this item to illustrate our methodology for writing items that reveal teachers' meanings. Here, we use it to demonstrate that most teachers' meanings for rate did not allow them to write an additive model instead of multiplicative model for the statement. This article reports additional teacher responses not presented in Thompson (2015), additional interview data, scores based on a refined rubric, and new examples of teacher work.

Rubric Part A for *Meaning of Over*. The Aspire team scored responses using a rubric designed to distinguish between the meanings of "over" as during and as divide (See Table 8). The highest level response to Part A was "during" or a response that referred to the passage of time. We scored responses that conveyed the meaning of over as "elapsed time" or "amount of time" at level A2. Responses at level A1 conveyed that "over" meant division. Responses conveyed this meaning in a variety of ways including words such as ratio or using mathematical symbols for division. We scored some responses at Level A0 because the teacher did not write a meaning for the word "over."

Table 8. Rubric for Part A of *Meaning of Over*. © 2014 Arizona Board of Regents. Used with Permission.

Level A3 Response:	The response conveys that "over" means "during," or otherwise refers to the passage of time while the culture's mass is changing.
Level A2 Response:	The response conveys the meaning of "over" as the equivalent of "elapsed time" or "amount of time".
Level A1 Response:	The response conveys that "over" means division, i.e. "*a* over *b*" means "divide *a* by *b*".

Rubric Part B for *Meaning of Over*. Part B asked teachers to rewrite the sentence in mathematical notation. The scoring rubric focused on whether the teacher wrote a difference or a quotient (See Table 9).

Table 9. Rubric for Part B *Meaning of Over*. © 2014 Arizona Board of Regents. Used with Permission.

Level B3 Response:	***Any*** of the following: – The teacher represented the difference of 4 grams in the culture's mass at beginning and end of a time period. If the response contains a variable other than *m* or *y* to stand for mass, it *must* be defined. – The teacher presented a graph whose symbolic equivalent would fit the first bullet.
Level B2 Response:	***Any*** of the following: – The teacher represented a change in the culture's mass, but does not refer to the passage of time. – The teacher wrote a quotient that is equivalent to representing a change in mass (e.g., $\Delta m / \Delta x = 4/\Delta x$ or $m/\Delta t = 4/\Delta t$).
Level B1 Response:	The response does not fit Level B2 and contains a quotient or an algebraically equivalent statement (e.g., $m/\Delta x = 4$, $m = 4\Delta x$).

The highest-level responses represented a difference and took into account the passage of time in some way (See Figure 14).

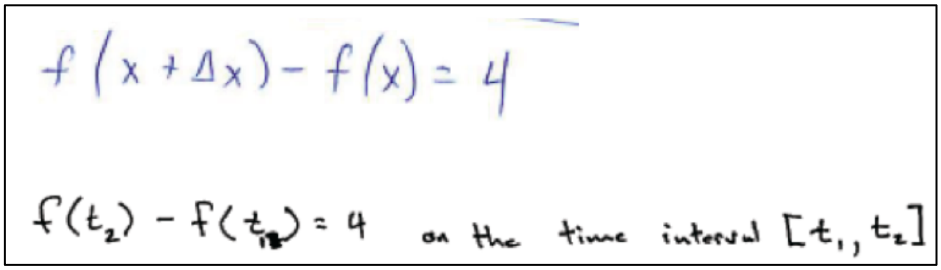


Figure 14. Two High-level Responses to *Meaning of Over*.

Some teachers did not incorporate the passage of time into their responses and gave answers like Δm = *4 grams*. These responses were scored at level B2 and considered to be mathematically acceptable, but less productive for a student than level B3 responses.

We scored the response $\frac{\Delta mass}{\Delta x} = 4$ at level B1. There were many variations of this response that included additional mathematical errors such as responding "mass/time" The response in Figure 15 was considered B1 with a mathematical error because the response confounded mass with change in mass and used function notation inappropriately.

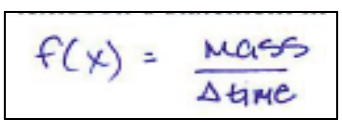


Figure 15. Part B Response to *Meaning of Over* Scored at B1.

Some responses included division as well as a level B2 answer (See Figure 16).

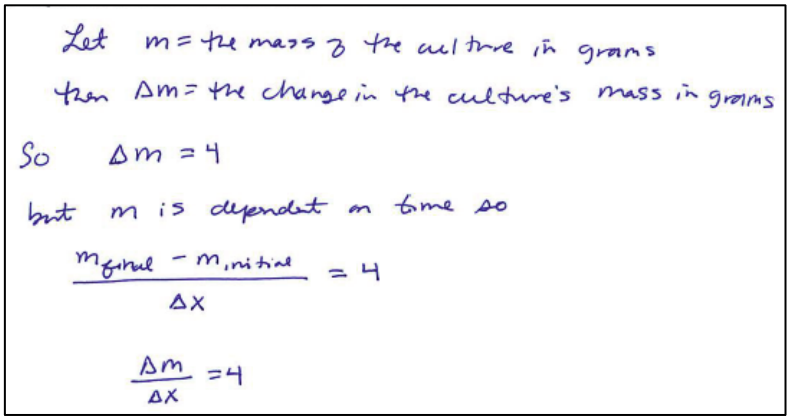


Figure 16. Example of Part B Response to *Meaning of Over* that Fits Both Levels B1 and B2.

We scored responses that fit multiple levels at the lower level because students would be confused if the teacher represented the same statement with and without division. The teacher who gave the response in Figure 16 did not appear to be perturbed by the contradictory equations. We scored responses such as $f(x) = 4$ at Level B0 because they were neither mathematically acceptable nor described by another level.

Results for *Meaning of Over*.

Table 10 shows the distribution of responses to Part A and Part B. One hundred thirteen (113) out of 251 teachers (45%) gave the high-level response of "during" or equivalent.

Table 10. Responses to Part A and Part B for *Meaning of Over* [1]

Response	B3 "subtraction"	*B2* $\Delta m=4$	B1 "divide"	B0 "Other"	NR/IDK	Total
A3 "during"	12	6	46	40	9	113
A2	1	1	14	15	2	33
A1 "divide"	0	4	67	0	0	71
A0	1	1	6	5		13

NR/IDK	1	0	0	0	20	21
Total	15	12	133	60	31	251

Table 10 also shows that 71 of 251 teachers (28.3%) said that “over” means division. This is not surprising, because “over” frequently means division in textbooks. However, it is surprising that only 18 of 113 teachers (15.9%) who said that “over” means during also represented a change in mass, and that 46 of 113 teachers (40.7%) who said that “over” means during used division to represent the statement symbolically. The latter teachers’ meanings for quotient did not contradict their notion of duration even though one concept is multiplicative and the other is not. We interpret this to mean that these teachers’ meaning for quotient is not multiplicative, but instead is a symbol used to separate two numbers that happen in connection with each other. For them, the concept of “duration” did not conflict with their non-multiplicative meaning for quotient. Furthermore, 40 out of 113 teachers (35.3%) who appropriately described over as meaning “during” gave a mathematically unacceptable, level zero response when they attempted to represent the sentence symbolically.

Many responses to the item *Meaning of Over* conveyed that the vinculum in r = d/t was loosely connected to the idea of comparing the relative size of two quantities in the teachers’ mind. The response from the teacher Naneh in Figure 17 conveyed that the word “over” was part of the definition of slope, “change in y over change in x,” and thus meant division. Naneh’s work suggests that she translated each aspect of the sentence to mathematical symbols using key words. For example she seems to have written “$\Delta x=4$” as a direct translation of the last four words of the sentence. This translation does not take into account that, in the context of the complete sentence, Δx refers to an unspecified interval of time.

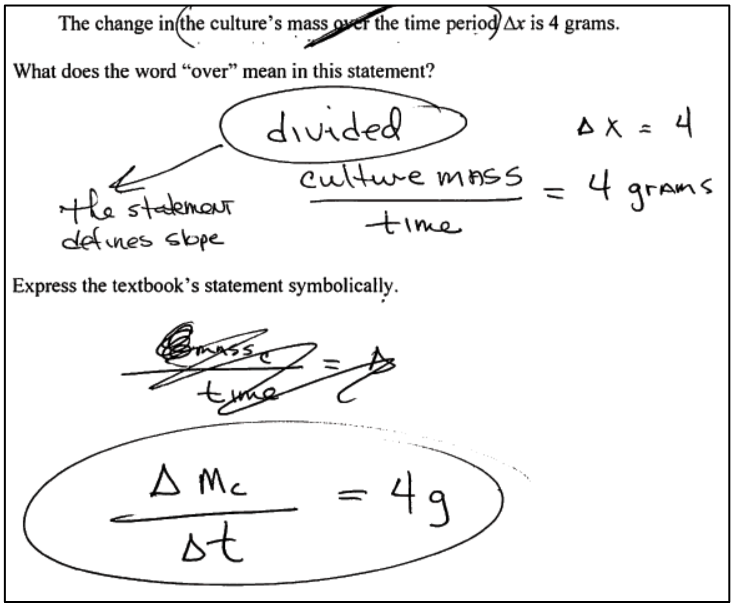


Figure 17. Response to *Meaning of Over* which Conveys "Over" is a Key Word Indicating Division.

The marks Naneh made suggest strongly that she parsed the statement as (The change in) (the culture's mass over the time period Δx) (is 4 grams). Other teachers parsed it as (The change in the culture's mass) divided by (the time period Δx) is 4 grams. The only way for a teacher to avoid either reading is to constrain himself by the fact that the result is four grams, and not four grams per time unit.

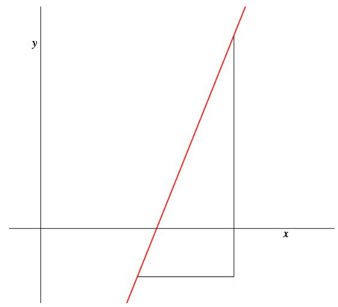
Notice that in Part B of Figure 17, the teacher's symbolic expression equates a quotient of two extensive quantities with four grams. We interviewed more than one teacher who had mismatched units and some were bothered by the mismatch and some were not. When a teacher's meaning for slope is primarily focused on the change in y, it allows room for him to understand the statement "Δmass / Δtime = 4 grams" unproblematically. To teachers like this, the vinculum does not mean a measure of relative size of changes. Rather, the vinculum means that mass changed and time changed.

There is no evidence that having a math or math education degree made it significantly more likely that teachers represented the textbook's statement appropriately $(\chi^2(8, n = 250) = 9.756, p = .282)$.

The Item *Slope from Blank Graph*

It is possible to know the formula for slope and be unable to estimate a slope from a blank graph with same-scaled axes. If a student or teachers' meaning for slope is the numerical change in *y* in relation to a change of one in *x*, they will need numbered axes to determine the changes in *y* and *x*. The Aspire team designed the item, *Slope from Blank Graph* in Figure 18 to see whether teachers could use a meaning of slope as a relative size of changes in *y* and changes in *x* to estimate a numerical value of slope given a graph without labeled axis. We believe that if a teacher "understands the quotient $\Delta y/\Delta x$ as the measure of Δy in units of Δx, then [he or she] can decide to estimate the numerical value of *m* simply by physically measuring Δy using Δx as a unit" (P. W. Thompson, 2015, p. 443).

Part A. There are two quantities *P* and *Q* whose values vary. The measure of *P* is *y* and the measure of *Q* is *x*. *y* and *x* are related so that $y = mx + b$. The graph of their relationship is given below, with *x* and *y* in the same scale. What is the numerical value of *m*?



Part B. What would be the numerical value of *m* if the *y*-axis were stretched so that the distance between 0 and 1 is 2 times as large as the original?

Figure 18. Item Named *Slope from Blank Graph.* Diagram is Larger in Actual Item. © 2014 Arizona Board of Regents. Used with Permission.

Thompson (2015) designed Part B to reveal teachers who compared the relative lengths of the sides of a triangle without thinking about the quantities that those lengths represent. Rescaling the *y*-axis would not change the relationship between the measures of the quantities *P* and *Q* and so the slope would remain unchanged even though the vertical leg on the triangle would be longer. The relative magnitude of the legs of the triangle would be unchanged because when the distance between 0 and 1 on the *y* axis is twice as large, the graphical unit of measure in the *y*-direction becomes twice as large as well, so the actual measure of the change in *y* remains constant.

Thompson (2015) reported 96 teacher responses to this item from a pilot sample. 52% of the teachers in his sample gave an approximation of slope between two and three. Thompson noted that 90% of the teachers who gave a reasonable estimate in Part A of the relative size of Δy and Δx answered “half” or “double” in Part B. He concluded that “this suggests that though they understood slope to be about relative size, they compared side-lengths of a triangle and not what those lengths represented” (Thompson, 2015, p. 444).

This paper provides additional teacher responses as well as a discussion of a similar item administered to calculus students.

Rubric and results Parts A and B *Slope from Blank Graph*. The Aspire team scored the responses to Part A using the rubric in Table 11.

Table 11. Rubric for Part A *Slope from Blank Graph.* © 2014 Arizona Board of Regents. Used with Permission.

Level A3 Response:	***Any*** of the following: – The teacher estimated a value between 2 and 3 (i.e. gave a number in the closed interval [2, 3]). – The teacher gave an explanation that includes a meaning of slope beyond a formula, such as "the change in y is 2.5 times as large as the change in x."
Level A2 Response:	***Any*** of the following: – The teacher solved for m in $y = mx + b$, getting $m = (y-b)/x$. – The teacher wrote a mathematically valid formula that could be used to determine the value of the graph's slope and did not estimate the slope's value.
Level A1 Response:	The teacher gave an estimate for the slope smaller than 2 or larger than 3.

If a teacher said the slope was y/x instead of $\Delta y/\Delta x$ the response was marked as "confounding y with Δy" in an additional scoring column and the mistake was ignored in the assignment of rubric level for Part A. We considered the distinction between y and Δy important because Thompson (1994b) observed that his students who failed to make this distinction struggled to understand the relationship between accumulation and rate of change. Teachers who gave estimates outside of the range two to three provided a variety of reasons; their reasons were generally mathematically invalid. We scored responses to Part B using the rubric in Table 12.

Table 12. Rubric for Part B of *Slope from Blank Graph*. © 2014 Arizona Board of Regents. Used with Permission.

Level B3 Response:	***Any*** of following: – The teacher wrote that the value of m is the same as in Part A without explanation.

	– The teacher's response suggests that he or she thought that the relative magnitude of changes in *y* and changes in *x* does not change.
Level B2 Response:	The teacher did not know whether the question meant that the *y*-axis only is rescaled or the *y*-axis and the triangle are both rescaled.
Level B1 Response:	***Any*** of the following: – The teacher wrote that the slope would be half of the value in part A. – The teacher wrote that the slope would be double the value in part A. – The response explains that the triangle, as a geometric object, did not change.

Note that is possible to give a high level response to Part B and a low level response to Part A. For example, a teacher might say that the slope is "*m*" in Part A and say the slope is still "*m*" in Part B. This teacher is unlikely to understand that the relative magnitude of changes in *y* and changes in *x* is invariant when the graph is stretched. Thus, responses to Part A and Part B must be considered together to identify teachers' meanings (See Table 13).

Table 13. Results on Parts A and B of *Slope from Blank Graph*.

Response	B3 "same"	B2	B1 "half/double"	B0/IDK	No Response	Total
A3 "2 to 3"	6	2	24	1	0	33
A2 "formula"	10	1	47	13	0	71
A1	0	0	4	1	0	5
B0/ IDK	2	0	4	9	0	15
No Response	0	0	0	0	32	32
Total	18	3	79	12	32	158

Thirty-three out of 158 the high school teachers estimated a value of the slope between two and three. We only have 158 responses to this version because we modified the item to clarify Part B after Summer 2015 pilot testing. Consistent with findings reported by Coe (2007) and Stump (2001) is not surprising that almost half (71 out of

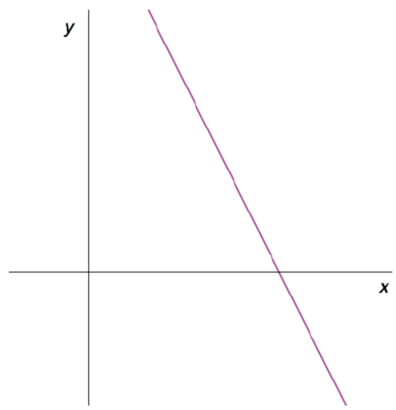
158) of the teachers provided a symbolic formula for slope. In addition to categorizing responses with the rubric we also noted whether the teachers' formula on Part A was of the form $\Delta y/\Delta x$ or y/x. We found that 39 out of 158 high school mathematics teachers confounded the change in a quantity's value with the quantity's value on Part A. The prevalence of this mistake across items suggests that many teachers' quantitative meanings for slope do not involve the comparison of changes.

Seventy-two percent (72%) of teachers who indicated a value of slope between two and three said that the slope would be either halved or doubled when the graph was resized. These teachers were able to compare the relative sizes of the legs of the triangle in Part A. In line with Thompson's (2015b) hypothesis, teachers' Part B responses suggest they were not thinking about the quantities those lengths represented and how a change in the graph would (or would not) affect the measures of those quantities

Item validation for *Slope from Blank Graph*. It is possible that some of the 71 teachers who gave a formulaic response (Level A2) to Part A would have been able to estimate the slope had they been pressed to do so by an interviewer. However, even when asked for a *numerical value* for slope their tendency was to give a formula. We considered rewriting the item to make it clearer we were asking for a numerical value, but we wanted teachers to be free to express the meanings they might convey in teaching. In six item interviews we asked teachers and calculus students what "numerical value" meant and everyone we asked responded that we meant a number, not a formula. For example, one calculus student said the question asked her "To actually find a number, but I don't know how I'm going to do that right now."

As further evidence that estimating slope is truly problematic, calculus students were asked to "estimate a numerical value of slope" on a course test, after their instructors discussed in class how to estimate slope given a graph with blank axes scaled identically.

15. The values of y and x are related so that $y = mx + b$. The graph of their relationship is given below. The scale used on the x and y axis is the same. Estimate the numerical value of m.



a) It is not possible to estimate m without numbers on the axis, but m is decreasing.
b) $m = 2$ c) $m = 0$
d) $m = -1/2$ e) $m = -2$

Figure 19. Item Given in a 2014 Calculus Test. © *2014 Arizona Board of* Regents. Used with Permission.

Only 96 of 170 students answered the item correctly after explicit instruction on the topic. Fifty-nine picked choice (a). In this item on the calculus test, the answer choices clearly show what a numerical value means. The student version of the item used the word "estimate" explicitly which suggests students truly do not think slope can be estimated without numerical values. Our data on *Slope from Blank Graph* leads us to suspect that these students' secondary teachers had a propensity to convey formulaic meanings for slope instead of measurement meanings, which then contribute to the calculus students' difficulties with their version of this item.

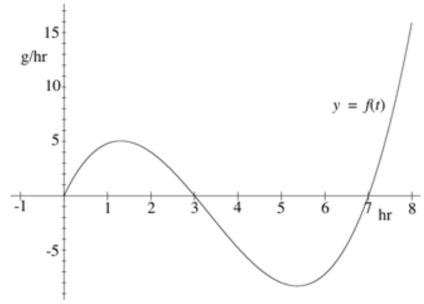
The calculus students' most common incorrect response was (a). Choice (a) said that "it is impossible to estimate the slope without numbers, and that *m* is decreasing." The statement "*m* is decreasing" is mathematically incorrect. We added this statement to the distractor "it is not possible to determine the slope" so that students with a productive meaning for slope would not be tempted to pick choice (a) because it is not possible to precisely estimate slope from an unlabeled graph. The statement "*m* is decreasing" confounds the values of *y* decreasing with the slope decreasing. Not noticing the problem with "*m* is decreasing" is consistent with an additive meaning for slope that is related to the change in *y* values.

The Item *Increasing or Decreasing from Rate*

The Aspire team designed the item *Increasing or Decreasing from Rate* to require teachers to differentiate between the idea of an increasing rate of change and an increasing mass (See Figure 20).

The values of function f give the rate of change (in grams/hr) of a bacterial culture's mass t hours after measurements began.

Over what intervals within the first 8 hours is the culture's mass increasing? Explain.



a) $0 < t \leq 1.4$ and $5.5 < t \leq 8$

b) $0 < t < 8$

c) $0 < t < 3$ and $7 < t \leq 8$

d) None of the above. My answer is __________________________

e) I don't know

(Next Page)
Part B. The graph from the prior page is repeated below.
Highlight the point (2.5, 2.25) on the graph of f. What does this point represent?

Part C.
Would you like to change your answer to the question on the prior page? Make the appropriate selection.

a) $0 < t \leq 1.4$ and $5.5 < t \leq 8$

b) $0 < t < 8$

c) $0 < t < 3$ and $7 < t \leq 8$

d) None of the above. My answer is __________________________

e) I don't know

f) I do not want to change my answer.

Figure 20. Item Named *Increasing or Decreasing from Rate.* © 2014 Arizona Board of Regents. Used with Permission.

The purpose of Parts B and C, displayed on the page after Part A , were to minimize the possibility that teachers picked the incorrect choice (a) just because they did not realize the graph was a rate of change function, or fell into the trap of thinking that an increasing graph corresponded to an increasing mass. For example, some pre-service teachers we interviewed initially answered (a) then switched their answers fairly quickly after seeing part (b). Although the concrete context of mass and time was quite similar to the item *Meaning of Over* the use of a graphical context led to different types of problematic responses. One obvious mistake that we expected some teachers to make was to treat an increasing rate of change of mass with respect to time as the same idea as an increasing mass.

In interviews some teachers showed a persistent difficulty with the item due to their meanings for rate of change. If someone considers slope to be the change in *y*, it is more difficult to differentiate between an increasing bacteria count and an increasing rate of change of bacteria. In some teachers' mind the rate of change of bacteria and the amount of bacteria are both extensive quantities directly tied to the number of bacteria. The secondary teacher Annie identified the function as a rate of change function but still lapsed into thinking about the graph as if it were a mass versus time graph in the middle of the explanation. (See Excerpt 1).

Excerpt 1. Annie's Explanation of *Increasing or Decreasing from Rate.*

> [The teacher reads problem aloud, emphasizes *grams/hour.*]
> We interpret increasing .. umm…let's see the function gives the rate of change in grams per hour… and so umm…what we are going to look at I would look at the rate of change being positive or negative, if we have a positive rate of change the grams per hour the mass is increasing per hour, is getting larger, so I look at where I have a positive rate of change, and I try to identify where I have no rate of change [*highlights maximum where the rate of change is a approximately positive 5, but the acceleration is zero*], this is telling me where the mass is staying the same, and then I have a negative slope so mass is getting small down to a zero rate of change so I'm not getting any smaller or larger…
> [Teacher determines the intervals from choice (a)]
>
> Interviewer: So just a little bit ago you said on the interval from 0 to 1.25 you said the change in the rate of change was positive.
>
> Annie: Positive, right.
>
> I: So the change in the rate of change of the bacterial culture's mass was increasing, so that meant that the mass was increasing as well?
>
> A: Right, right.

I: What is this a graph of again?

A: This is a graph of the rate of change of the culture's mass as time progresses.

I: So what would, so here we have a clear point. [teacher interrupts interviewer]

A: So think calculus here… my rate of change is positive…oh geez, I can't believe I did that, you are making me think here. The change in the rate of change is positive, because again what I'm seeing is grams per hour, so from 0 to 3 [mutters, geez, I can't believe I did that] my rate of change is positive, even though it looks like it is decreasing, well, it is decreasing but it is still positive. From 0 to 3 I have a positive grams per hour, from 3 to 7 I have a negative grams per hour, and then I have a positive grams per hour again.

Annie was not the only teacher who brought in the notion of the change of the rate of change into the discussion and ended up having difficulties interpreting the graph.

Table 14 shows the results on *Increasing or Decreasing from Rate*.

Table 14. Results for Part A and Part C on *Increasing or Decreasing from Rate*.

		Response to Part A					
		Chose (c) on Part A	Chose (a) on Part A	Chose (b) on Part A	Other/ IDK	Blank	Total
Response to Part C	Chose (c) on Part C	86	35	0	3	0	124
	Chose (a) on Part C	1	77	0	1	0	79
	Chose (b) on Part C	0	3	11	1	0	15
	Other/IDK	0	2	0	17	1	18
	Blank	0	0	0	0	1	1
	Total	87	117	11	22	2	239

Eighty-seven out of 239 (36.4%) teachers chose the highest level answer, (c) on Part A. After being prompted to examine the meaning of a point on the graph approximately half (51.8%) of teachers chose the highest level answer.

Table 15 shows that there were 25 out of 239 (10.4%) teachers who were able to correctly interpret the meaning of a point on the rate of change graph who did not select (c) for a final answer. This suggests that although they were aware of the axes labels their meaning for graphs and rate of change was insufficient to select (c). Additionally, 29 teachers struggled to explain the meaning of a point, but were still able to choose (c).

Table 15. Responses to Part B and Part C on *Increasing or Decreasing from Rate*.

		Response to Part B			
		Correct Meaning of Point	Incorrect Meaning of Point	Blank /IDK	Total
Response to Part C	Chose (c) on Part C	95	29	0	124
	Chose (a) on Part C	16	60	3	79
	Chose (b) on Part C	5	10	0	15
	Other/ IDK	4	13	3	20
	Blank	0	0	1	1
	Total	120	112	7	239

Descriptions of teachers' thinking from qualitative studies helps us hypothesize reasons for the teachers' difficulties (Coe, 2007; P. W. Thompson, 1994b). Teachers who consider slope as an index of slantiness and not as a comparison of two changes could relate the word "increasing" with a graph that is slanted up. Understanding the distinction between the rate of change of bacteria and the amount of bacteria makes this problem sensible. The tendency to speak of slope additively makes it more difficult to distinguish between a rate of change and an amount of change. Confounding an amount of change of a quantity with an amount of the quantity was common among the teachers in our study. Taken together, these fuzzy ways of thinking about rate as an amount of change, and of an amount of change as an amount, makes it difficult to understand the relationship between a rate of change graph and an amount graph. The graphical context of the item also could lead many teachers to consider the shape of the graph instead of the quantities that covaried to make it (Moore & Thompson, 2015).

LOOKING ACROSS ITEMS

In this section we look across items for consistencies and inconsistencies in teachers' meanings and ways of thinking about slope and rate of change. We used tables of values, interviews, and teachers' written work to understand their multi-faceted meanings for rate of change. We remind the reader that a teachers' responses might convey a stable meaning that they use across items, or, different meanings if they have multiple ways of thinking about (what we take as) related items. The high level meanings on each rubric are related to an image of rate of change as multiplicative comparison of associated changes in two quantities. The lower levels of the rubric arose from grounded coding of teachers responses and are not necessarily consistent across rubrics. For example, a teacher might make sense of one item about a bacteria culture's mass by using a key word and another item about a bacterial culture's mass by focusing on the shape of the graph.

Relationship between Measurement Responses and Rate of Change Responses

Coe's (2007) interviews suggested that teachers' meanings for rate of change are not tightly linked to their measurement schemes. This section investigates the hypothesis that teachers with weak measurement schemes will have difficulty constructing multiplicative meanings for slope and rate—meanings that are based on each as a measure of the relative size of a change in one quantity and a change in another.

We use teachers' responses to *Gallons to Liters* and *Nerds to Raps* (Figure 4 and Figure 5) as a way to quantify the strength of their measurement schemes. These items required teachers to convert between two units given a conversion factor. Our earlier analyses separated low level responses into multiple levels to attribute meanings to various types of response. In this analysis we will group all low level responses to *Gallons to Liters* and *Nerds and Raps* into one category to simplify our tables. Ten of the 251 teachers left *Gallons to Liters* blank but answered *Nerds to Raps.* We replaced their score of "No response" with a zero (low-level) because teachers had ample time to complete the test so missing data was rarely related to time constraints. We know this because teachers were instructed to write "no time" on their test if they ran out of time for a question.

***Relative Rates* versus measurement responses**. Table 16 shows the relationships between teachers' success on two measurement items and the item *Relative Rates.* The responses to *Relative Rates* are ordered so that the highest level meaning is at the top of the column.

Table 16. Response to *Relative Rates* Versus Number of Measurement Items Correct.

	Number of Correct Responses to Two Measure Items			
	2 Correct Responses	1 Correct Response	0 Correct Responses	Total
j/s times (e)	20	24	25	69
j/s more (c)	2	7	11	20
j-s (a)	29	55	53	137
Other	2	2	20	24
No response	0	0	1	1
Total	53	88	110	251

Thirty-eight percent (20/53) of teachers who answered two measurement items correctly score at the highest level on *Relative Rates*. Twenty-three percent of teachers who answered zero measurement items correctly score at the highest level of *Relative Rates*. There is a statistically significant association between the number of correct answers to measurement problems and responses to *Relative Rates* (Spearman's Rho = 0.16). Responses to these two simple measurement questions are more strongly associated with response to the rate of change item than with whether or not the person has a math degree or a random degree.

***Slope from Blank Graph* versus measurement responses.** Table 17 compares teachers' tendency to measure a change in y in terms of a change in x to their success on two measurement items.

Table 17. Responses to *Slope from Blank Graph* Part A Compared to Number of Measurement Items Answered Correctly.

	Number of Correct Responses to Two Measure Items			
	2 Correct Responses	1 Correct Response	0 Correct Responses	Total
Estimate "2 to 3"	14	11	8	33
A2 "formula"	16	22	33	71
A1/A0/IDK	2	4	16	22
No Response	7	12	13	32
Total	39	49	70	158

Scanning the columns of Table 17, we see that 35% (14/39) of teachers who answered two measurement items correctly estimated an appropriate numerical value for slope between two and three. Only 5% (7/39) of teachers who answered two measurement items correctly said something incoherent on *Slope from Blank Graph* (Levels A1/A0/IDK). In contrast, only 11.4% (8/70) of teachers who answered both measurement questions incorrectly estimated the slope of a line by measuring Δy in units of Δx. Although some teachers with two incorrect responses on the measure items provided acceptable formulas for slope without a numerical estimate, 22.9% (16/70) said something mathematically incoherent about the slope of the line. The association between responses to *Slope from Blank Graph* and responses to the two measurement items is statistically significant (Spearman's Rho = 0.19). This result is consistent with the hypothesis that being able to imagine a measurement process was a key idea for estimating slope from a graph with blank axis.

***Increasing and Decreasing from Rate* versus measurement responses**. Table 18 compares teachers' success on two measurement items with the strength of their interpretations of a rate of change function. There are only 239 teachers in this table because one group took a shorter version of the MMTsm that did not include *Increasing or Decreasing from Rate.*

Table 18. Comparison of Response to Part A of *Increasing and Decreasing from Rate* to Number of Measurement Items Answered Correctly.

	Number of Correct Response to Two Measure Items			
	2 Correct Responses	1 Correct Response	0 Correct Responses	Total
Choice (c) on Part A	30	34	23	87
Choice (a) on Part A	19	42	56	117
Choice (b)/Other/IDK	4	9	21	34
Blank	0	0	0	0
Total	53	85	101	239

Fifty-seven percent of teachers who answered two measurement items correctly interpreted the rate of change function and its relationship to the amount function on the first try (choice c). Twenty-three percent of teachers who answered zero measurement items correctly gave the highest-level answer on their first try. There is a statistically significant association between number of measurement items correct and responses to *Increasing or Decreasing from Rate* (Spearman's Rho = 0.28)

Meaning of Over **versus measurement responses**. The hypothesis that strong measurement meanings are associated with differentiating between additive and multiplicative situations is consistent with the results of *Meaning of Over*. Table 19 shows the relationship between responses that model the textbooks' statement as a difference and the number of correct responses to measurement items.

Table 19. Meaning of Over Responses Versus Number of Correct Responses to Measure Items.

	Number of Correct Response to Two Measure Items			
	2 Correct Responses	1 Correct Response	0 Correct Responses	Total
B3: difference	7	5	3	15
B2: $\Delta m = 4$	2	5	5	12
B1: divide	32	49	52	133
B0/IDK	9	23	39	71
No response	3	6	11	20
Total	53	88	110	251

Thirteen percent of teachers with two correct measurement responses interpreted the textbook's statement appropriately using a difference. Only three percent of teachers with zero correct measure responses appropriately interpreted the textbook's statement.

There is a statistically significant association between number of measurement items correct and high-level responses to *Meaning of Over* (Spearman's Rho = 0.19).

Responses that suggest teachers did not imagine measures of quantities, are associated with responses that indicate incoherence in their rate of change schemes. Understanding rate of change as a measure of the relative magnitudes of two quantities can be used in many contexts. Other meanings for rate of change are productive in limited contexts. Teachers who do not apply foundational measurement meanings to understand rate of change are more likely to address varied rate of change situations with various unproductive meanings.

Conclusions about measure and rate of change items. By comparing teachers' responses to measurement items and rate of change and slope items we determined that stronger responses to measurement items are correlated with stronger responses to slope and rate of change items. Correlations support the hypothesis that meanings for measurement ideas are foundational for productive rate of change meanings. It should be noted that an observational study cannot show that teachers developed more productive rate of change meanings because they had stronger measurement schemes. It is interesting that teachers' responses to two middle school items was more predictive of their rate of change meanings than holding a degree in mathematics or mathematics education.

Limitations of Chunky, Additive Meanings for Slope and Rate of Change

Interviews and in-depth analysis of individual teachers' responses illuminated ways in which chunky, additive meanings for rate and slope proved problematic for them. For example, only 25% of teachers who conveyed chunky meanings for slope on

Meaning of Slope estimated the relative size of changes on *Slope from Blank Graph*. Three of the 28 teachers who gave a memorized meaning for slope on *Meaning of Slope* estimated a numerical value for slope. Even though computational and chunky meanings for slope can work in many settings that demand no greater understandings, these meanings disabled most teachers from estimating a numerical value of slope on a blank graph.

Slope is the change in *y*. Teachers often expressed similar ways of thinking on more than one item. On *Meaning of Slope* some teachers confounded Δ*y* with slope. Kristen's Part A response in Figure 21 is consistent with thinking that slope gives information about how to move vertically and horizontally on a graph. Kristen conveyed a chunky meaning for slope because the changes occur in chunks of one and 3.04.

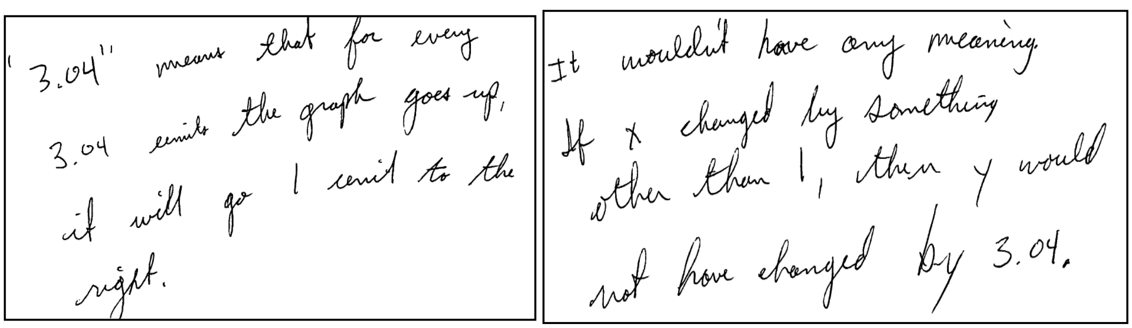


Figure 21. Kristen's Chunky Response to Part A (Left) and Part B (Right) of *Meaning of Slope*.

The Part B response in Figure 21 provides confirmation that, for Kristen, 3.04 is more strongly associated with the change in *y* than with a comparison of the relative size of corresponding changes in *y* and *x*. The response in Figure 21 foregrounded the change in *y* and kept the change in *x* in the background. We suspect that some chunky thinkers understood the difference between Δy and Δy/Δx, while some confounded the two concepts.

Martha’s Part A response in Figure 22 was scored at level A1 because it is formulaic and conveys slope as an index of steepness. Martha’s Part B response conveys that the slope is strongly associated with the change in *y* instead of the relative size of associated changes—Martha focused on how *y* would change, not on the meaning of 3.04. This is similar to Castillo-Garsow’s (2012) observations about a student who focused on the change in output while time passed in the background.

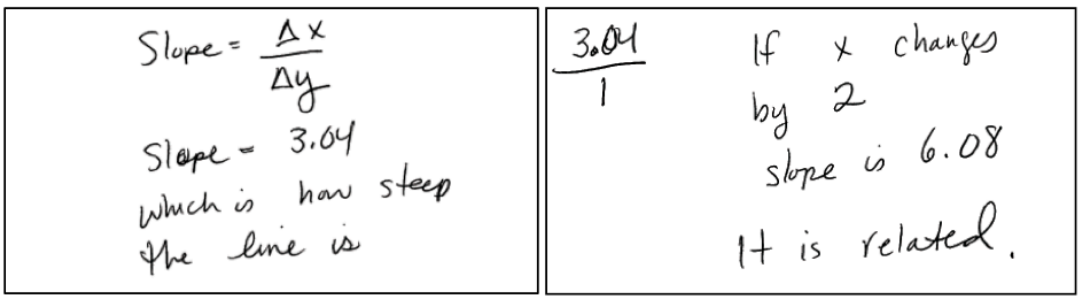


Figure 22. Martha’s Response to Part A and Associated Chunky Part B Response to *Meaning of Slope.*

Bren’s response (Figure 23) provides yet another illustration that thinking of slope as the change in *y* with respect to a change of one in *x* makes it difficult to estimate slope numerically. Labeling the change in *y* as *m* conveys that slope is not a comparison of two changes, but rather the change in *y* that is associated with a one unit change in *x*.

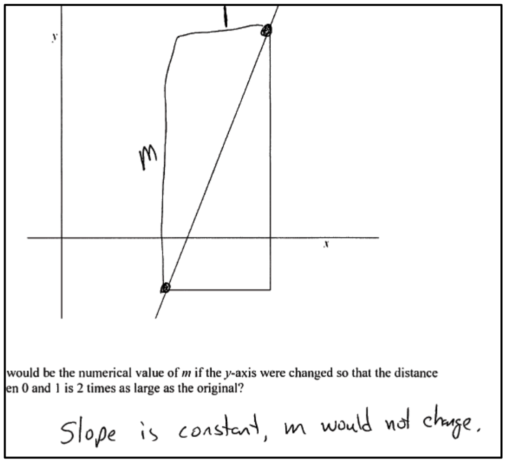


Figure 23. Bren's Response that Conveys that Slope is the Change in *y*.

On Part B Bren did not appear to answer the question. He appeared to associate the straight line with a constant slope. It appears that Bren understood that for any change of *x* of 1, the length of *m* (Δy) stayed the same. The slope of the line does not change as *x* varies. However, "slope is constant" does not make sense as an explanation for why *m* does not change when the axis are stretched.

Inability to estimate slope is one of the potential consequences of teaching the additive or formulaic meanings for slope that were frequently conveyed in the item *Meaning of Slope*. Estimating slope is extremely useful in a variety of situations such as checking answers or drawing graphs of a derivative given the graph of the original function.

Slope is the distance between two points. Chunky thinking leads to a variety of problems beyond confounding Δy with $\Delta y/\Delta x$. Nari's chunky responses conveyed that the only points on the line that "mattered" were the points obtained by the process of moving over and up in fixed chunks (see Figure 24). Nari's response is inconsistent with imagining that between any two points on the line there are infinitely many points.

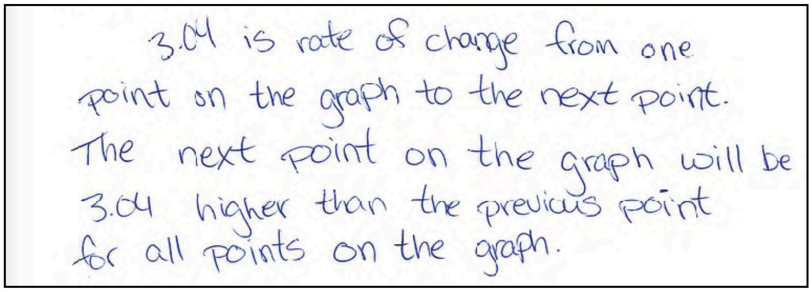


Figure 24. Nari's Chunky Response to *Meaning of Slope* Conveys that the Points on the Line Only Occur at Fixed Intervals.

There are a variety of consequences of thinking that points on the line only occur at fixed intervals. If points only occur at fixed intervals it is possible to conceptualize slope as the distance between two points on a line. Andy explicitly said that the slope is a distance between two points (see Figure 25).

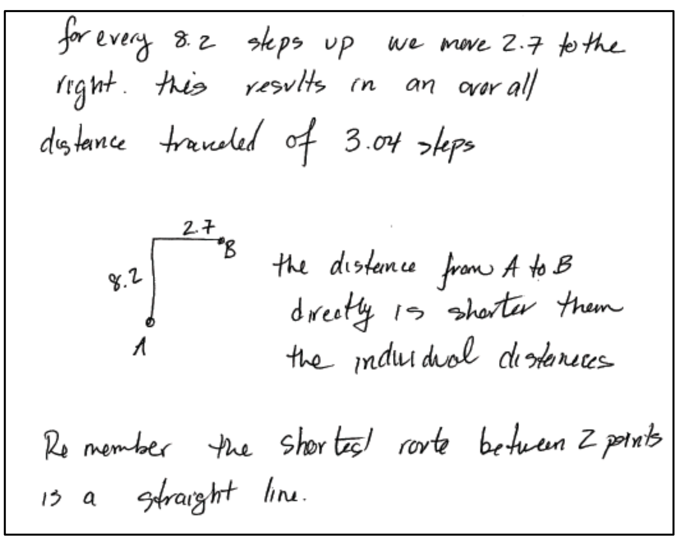


Figure 25. Andy's Additive Response to *Meaning of Slope* Conveys that Slope is a Distance Between Two Points.

The response in Figure 25 conveys that slope gives directions on how to get from one point to the next and that 3.04 is a distance. Unfortunately, Andy's meaning for slope as distance overpowered the visually obvious fact that the hypotenuse of the triangle is longer than either leg. We suspect that Andy's meaning for slope as distance is stable because he had taught algebra 1 five times, algebra 2 twice, and geometry three times. Andy had explained slope hundreds of times without realizing the inconsistency in his meanings.

The tendency to disconnect the meaning of slope from a quantitative meaning for division, reported by Coe (2007), is a potential explanation for viewing slope as a distance. For example, Daniel, a university calculus student we interviewed on *Meaning of Slope* as part of its rubric's validation, explained his meaning for quotient using a diagram that shows two segments whose lengths were labeled A and B. He explained that "A divided by B" means "the amount of A's that would fit into B" and "B divided by A" means "the amount of B's that would fit into A". Daniel then responded to Part A of *Meaning of Slope* by writing $\left(y_2 - y_1\right)/\left(x_2 - x_1\right)$. Excerpt 2 contains Daniel's explanation of the meaning of 3.04.

Excerpt 2. Daniel Explained his Meaning for Slope of 3.04.

> Daniel: So the 3.04 is the slope between these two. So that is basically the change between the two.
>
> Interviewer: When you say the change between the two, what are the two things you are talking about?
>
> D: The two different points. The points. [Daniel and interviewer clarify that the student computing the slope had two points, but that they are not given in the problem but we can imagine them.]
>
> I: When you say the change between the two are you talking about the length between the two points on this piece of paper?
>
> D: Yeah. It would be the length between these two [Daniel highlights hypotenuse.]

I: So the slope is the length between the two points.

D: Right.

I: Okay. So why do you divide the change in *y* and the change in *x* to get a length?
D: Because, it's… you've got the one *x* here and the other one here and so you are trying to find the way which they both get to each other basically. That's…

I: Okay. [Daniel laughs] Is that at all related to seeing how many B's fit into A or is that like a separate thing in your brain?

D: If you are doing the slope it's different, I guess, I'm seeing it different in my brain, I guess it is because of the word slope gave this a different meaning.

I: What does the bar in between them mean to you?

D: I just… divide [laughs]

I: Alright. It's just that you were not using the how many times B fits into A language at all when describing the slope so that is why I was asking.

D: Yeah. No, not with slope.

Daniel realized that length is not a good description of slope when the interviewer drew multiple triangles of different sizes on the same line. The slope of the hypotenuse of each triangle was the same, but the lengths of the hypotenuses differed. Daniel quickly connected slope and division after subsequent instruction, but the meaning he carried from secondary school into calculus was that the meaning of division in the slope formula differed from the meaning of division he learned in school. A teacher who conveys an additive, chunky meaning for slope allows students to assimilate the teacher's instruction without connecting the idea of slope of a line to the idea of a quotient of changes. They do not see "m/n" as a multiplicative comparison of numerator and denominator. Instead, they think that the numerator gives the number of steps to take in one direction and the denominator gives the number of steps to take in another directions, and the vinculum merely separates them.

Teachers' slope and rate meanings are tied weakly to quotient. Other interviews confirm that some teachers did not connect their meaning of the vinculum in formulas for slope or average rate of change with their meaning of division. We asked Ross, a high school math teacher, "Why do you divide to calculate the slope of the line?" Ross first repeated the definition of slope to himself and then explained:

Excerpt 3. Ross's Explanation of the use of Division in the Slope Formula.

> The division can be used because we talk about the slope being the average rate of change and "average" is the total [taps fingers] uhh… the total of the observations divided by the number of observations. And then of course we have to make a distinction of what is… what we interpret the total observations and the number of observations, so if I want to talk about… i.e., for example, total distance traveled by some total time during the travel so umm…total distance traveled would be delta distance over delta time but we can also see this also as a ratio of the two differences.

Ross thought of arithmetic mean and distance divided by time as the same idea. He made the common mistake of thinking that average rate of change is computed using an arithmetic mean (Yoon, Byerley, & Thompson, 2015). Ross made other inappropriate connections in a variety of situations that involve division, connecting topics superficially based on the appearance of the division operation. The interviewer pressed Ross to explain his meaning for ratio to better understand how he connected the idea of arithmetic mean and his inappropriate formula "total distance traveled would be delta distance over delta time."

Excerpt 4. Ross's Explanation of his Meaning for the Word "Ratio."

> The problem I'm (pause) unfortunately from the different stuff I've been looking at now, we can unfortunately use the word ratio unfortunately to mean both a

> comparison of two different units to each other but also the terminology as a fraction a part to whole, there is this way we use ratio and fraction together, but the idea is you know, I'm just trying to get at the idea of what slope is, it is this ratio, for every change in this I do the change in this. The reason I don't want to use the word fraction for this is because again we think of a fraction having the same units. You know, one fourth of a pie, two thirds of a gallon. But when I talk of a ratio of distance to time those are two different units. And so it's for every change of this element I have a change in the other element. For every one hour, change of one hour I drive down the road I go an additional 65 miles down the road, so it is not a fractional concept like we normally think of it, it's part to whole, because unfortunately when we see division you can also interpret that as fraction. But the fact is because we don't have the same two units, it doesn't fit as neatly with some of those notions we've been taught about the differences between fractions and ratios.

Ross had been taught common meanings for ratio such as "comparison of two quantities with different units" and fraction such as "part out of whole." Ross's meanings seemed to hinder Ross from being able to make sense of the use of division in the slope formula as producing a quotient—a measure of the relative size of changes. It is possible to compare the associated changes in two quantities measured in different units by comparing the measure of one quantity and the measure of another quantity. Ross did not express an overarching meaning for fraction and ratio as a comparison of the relative size of two quantities, or the value of one quantity measured in terms of another. His meaning for ratio in the context of driving was chunky: "For every one hour, change of one hour I drive down the road I go an additional 65 miles down the road." He talked about two changes happening in tandem but did not compare the relative sizes of the measures. The interviewer gave Ross another chance to revise his thinking about the problematic connection between arithmetic mean and his "delta distance over delta time" formula. Ross was unable to clarify his thinking and concluded, "So maybe it is kinda a mish-mash. Maybe I put things together that might confuse the kids at times." Although Ross seemed aware of the incoherence of his responses, he did not know how to revise them. We believe that his foundational meanings for ratio and quotient were focused on tangential issues such as the comparison of two quantities with unlike units and his meanings did not allow him to build more complex multiplicative ideas on foundational ideas of measure.

Both Ross and Daniel struggled to connect their meanings for quotient to their idea of slope. Daniel thought of slope as “the change between two points” and that the vinculum in a slope formula served to separate vertical and horizontal changes and did not cue his meanings for division. Some teachers’ responses to *Meaning of Over* illustrated how Daniel and Ross might have used slope hundreds of times without connecting it to their meaning for quotient. The response in Figure 26 conveys that the vinculum in a rate of change formula merely separates the numbers that tell the teacher how to move on a graph.

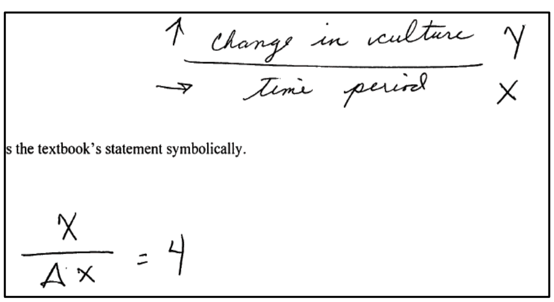


Figure 26. Response to *Meaning of Over* Where the Vinculum Merely Separates Two Numbers that Tell how Far to Move Horizontally and Vertically.

Some responses conveyed that “over” meant “above” or that the word “over” referred to a spatial arrangement of symbols (See Figure 27).

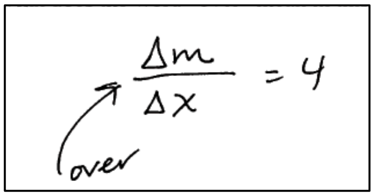


Figure 27. Response to *Meaning of Over* that Conveys "Over" Means "Above."

Teachers' and students' tendency to use non-multiplicative meanings for slope that downplayed the role of the quotient contributed to a variety of mathematical inconsistencies in their mathematical models.

Use of Key Words for "Over"

One problem with the disassociation of the idea of slope and quotient is that it allowed teachers to comfortably model one situation in two inconsistent ways. Many teachers understood that the word "over" could mean duration in some contexts but still believed the word over should always be converted to over in symbolic contexts. Figure 28 shows a teacher who recognized that "over" meant "during" in Part A, but who then apparently read "over" as meaning division when representing the statement symbolically.

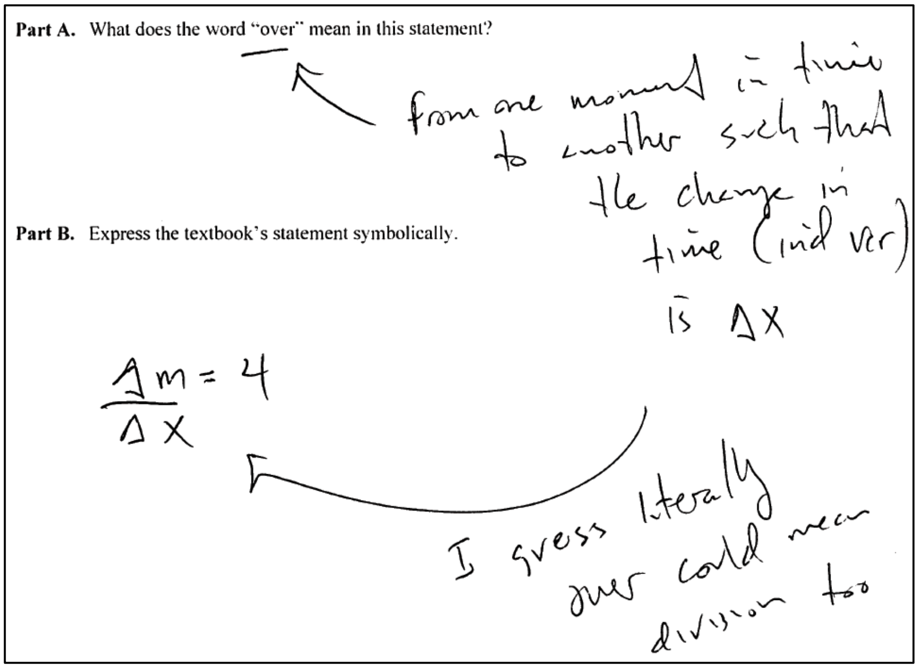
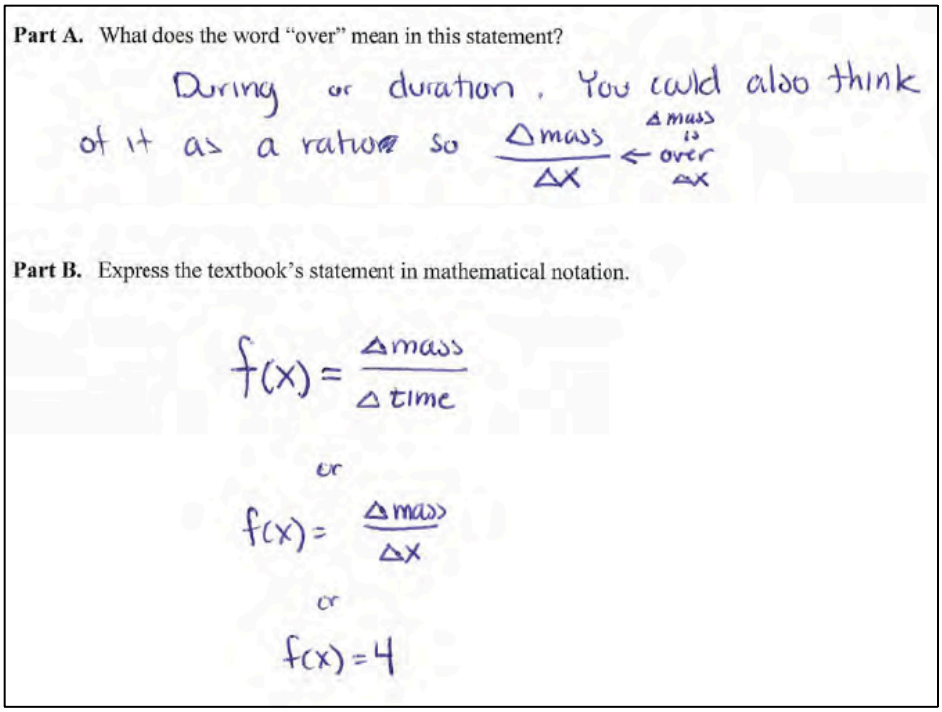


Figure 28. Response to *Meaning of Over* that Conveys "Over" Means both Division and During.

We interviewed, James, who taught algebra II, geometry, and precalculus two times each to better understand why someone would say over meant both during and divide in the same statement. First the interviewer asked James to respond to a blank version of the item and then asked James to analyze his initial response (See Figure 29).



Part A. What does the word "over" mean in this statement?

During or duration. You could also think of it as a ratio so $\frac{\Delta mass}{\Delta X}$ ← Δ mass is over ΔX

Part B. Express the textbook's statement in mathematical notation.

$$f(x) = \frac{\Delta mass}{\Delta time}$$

or

$$f(x) = \frac{\Delta mass}{\Delta x}$$

or

$$f(x) = 4$$

Figure 29. James' Initial Response to *Meaning of Over.*

The Aspire team conducted the interview shown in Excerpt 5 six months after James took the MMTsm.

Excerpt 5. James Discussed his Responses to *Meaning of Over.*

> James: [Reads question carefully aloud.] [Over means] during or duration. You could also think of it as a ratio, so change in mass over, yeah so during or duration, so in your math class when they say "something over something," they always mean a divide sign so a ratio.
>
> I: Do you think they are both saying the same thing?
>
> J: Well, yeah, I think that. Well yeah, they are saying. I think the during or duration is more saying conceptually what is going on, and the divided by or over I see the reason behind that, I think I'm more pointing out mathematically what we mean when we say over with no explanations as to why, it is just the way it is.
>
> I: So is the mass, the change in mass divided by the change in time, is that how you write the idea of duration?
>
> J: Can you repeat the question?

I: Is the "delta mass divided by delta *x"* a mathematical way of saying duration?

J: I want to say the change in x is the way of saying duration. I want to say the change in x is representing duration. But maybe we could include the division sign. So no, I would not say that "delta mass over delta *x"* is a way of saying duration. So this is funny.

I: Okay, I think it is funny because "over" sometimes means divide and sometimes not.

J: [James read Part B. James looked at his old answers from when he took the MMTsm six months earlier, found them all problematic and crossed them out and explained he used function notation incorrectly.]

I: What would you say today?

J: I like the idea of the function. I would keep the function.

I: You can work it out on a paper.

J: Yeah, just give me a second. The change in the culture's mass…[pause]

J: Change in mass over [divided by] change in x equals 4. That would be my new thing.

I: Four what?

J: Four grams. [James showed no discomfort with a quotient being equal to 4 grams.]

James accepted that "over" could mean both divide and during in the same situation by saying that "during" is the conceptual meaning and "divide" is the mathematical meaning. Even though James realized that divide and duration are not expressed in the same way mathematically, and the interviewer confirmed that "over" only sometimes means divide he still used division in his symbolic representation of the statement. James did not say that a quotient being equal to four grams was problematic, even when the interviewer called attention to the units. James's understanding that "over" always means divide (because “that is just the way it is”) was so prominent he kept the division sign despite the inconsistencies he noticed and discussed with the interviewer. A combination of James’ strong association of over and division, and his lack of multiplicative meaning for quotient allowed him to inappropriately model an additive situation with division. Despite the interviewer’s interventions to help him notice the problems with his representation of the statement he remained confident with his response.

Another teacher, Samuel, recognized that his units in Part B of *Meaning of Over* did not make sense during his interview. Samuel wanted to change the textbook’s statement to be about rate of change by changing the units on four grams to four grams per unit of time.

Excerpt 6. Samuel's Response to Mismatched Units in his Answer to *Meaning of Over.*

> The change in culture’s mass is 4 over the time period delta x is four grams. So I’m a little bothered by this right now. Yeah unless it is a mistake. [Sighs.] Umm… [pause] Yeah I don’t think that… I’m going to pick on the college science textbook, but I would say four grams per time frame is what I would claim that to be.

Samuel concluded that either he was confused or the textbook’s statement was incorrect. Although dimensional analysis did perturb Samuel enough to rethink the problem, his conclusion was the conclusion the textbook’s statement did not make sense. He felt that over must represent divide and that therefore the units must be changed.

Taken as a whole, these teachers’ responses to multiple rate of change questions show that the commonly presented meanings for slope and rate of change can contribute to serious inconsistencies in their applications of ideas. We’ll now use tables to understand inconsistencies in responses across our entire sample of teachers.

Responses Across Rate of Change Items

Because most teachers approached different rate of change contexts with different unproductive meanings there were no strong associations between one teacher's response to different rate of change items. For example Table 20 shows that many of the teachers who appropriately distinguished between additive and multiplicative models in *Meaning of Over* did not do so in *Relative Rates* and vice versa.

Table 20. Responses to *Meaning of Over* Compared to Responses to *Relative Rates*.

	Response to *Relative Rates*			
	High level (j/s)	Additive (j-s)	Other	Total
B3: difference	5	10	0	15
B2: Δm = 4	4	7	1	12
B1: divide	45	75	13	133
B0/IDK	26	36	9	71
No response	9	9	2	20
Total	89	137	25	251

Forty-five of the 133 teachers (34%) who were overpowered by the word "over" in *Meaning of Over*, chose a multiplicative comparison appropriately in *Relative Rates*. It may be that the key word "over" was so strongly associated with division that they did not make use of the scheme that allowed them to make sense of *Relative Rates.* Only nine out of 27 (33%) of the teachers who appropriately modeled the situation in *Meaning of Over* chose the appropriate multiplicative response in *Relative Rates.*.

Even though the items *Meaning of Over* and *Increasing and Decreasing from Rate* both involved interpretation of a situation involving the growth of a bacterial culture's mass, differences the two items resulted in widely different responses from the same teacher. *Meaning of Over* had a key word commonly associated with division that strongly included responses. *Relative Rates* had a distractor that was attractive to chunky thinkers. We view the responses across items show the complexity of teachers' thinking and how it varies by context. The group of rate of change items does not measure their understanding of one aspect of rate of change, but models how they use their meanings in a variety of rate of change contexts.

Table 21. Response to *Increasing and Decreasing from Rate* Compared to Response to *Meaning of Over*.

	Response to *Increasing or Decreasing from Rate*			
	High level Part A (c)	Part A, Choice (a)	Other	Total
B3: difference	8	5	2	15
B2: Δm = 4	6	5	0	11
B1: divide	54	57	17	128
B0/IDK	12	40	13	65
No response	7	10	3	20
Total	87	117	35	239

The uppermost left table entry shows that 53% of teachers who appropriately represented the statement in *Meaning of Over* properly interpreted the rate of change graph in *Increasing or Decreasing from Rate.* Similarly 42% of teachers who inappropriately translated "over" into division properly interpreted the rate of change graph. This shows that despite having difficulties determining when the key word "over" does and does not mean division in a situation about bacterial growth, they did properly interpret rate of change graphs.

CONCLUSION

Taken as a whole, the responses to the rate of change items in this paper are alarming. On *Meaning of Slope* four percent of teachers described slope as involving multiplicative comparisons between two changes. While the many teachers who gave chunky (77.6%) and formulaic (15.2%) descriptions of slope were not incorrect, we found substantial qualitative and quantitative evidence in our study that these additive and computational meanings were typically not productive in a variety of situations. Furthermore we found that some students and teachers with chunky and or formulaic meanings for slope also developed problematic meanings for slope such as that slope is the distance between two points on line.

The data demonstrates that many teachers have substantial difficulty in coping with rate of change and slope situations that are highly related to the secondary mathematics standards they are teaching. Only 20.8% of teachers estimated the slope of a blank graph with equally spaced axis when asked to find a numerical value of slope. Approximately half of teachers determined where the amount of bacteria was increasing given a rate of change of bacteria with respect to time graph. Ten percent of teachers used subtraction to model a change in mass. There is strong evidence that the teachers who used division instead of subtraction to model a difference were distracted by the use of the word over to mean during. Twenty-one percent of secondary mathematics teachers correctly converted between units of measure on both measurement problems. Furthermore there was evidence that weak understanding of measurement was correlated with problematic responses on all of the rate of change items. Much of the qualitative work done by us and other researchers suggests that secondary teachers have weakly developed meanings for quotient and that their meanings for rate of change and slope are not strongly tied to their meanings for quotient. Given the utility of comparing the relative size of two quantities multiplicatively in mathematics, science, medicine and business, we find this evidence alarming.

Our sample of 251 high school mathematics teachers was a convenience sample. It was not selected randomly from their states or from the United States. We therefore cannot claim that it is representative of any specific population. However, the sample is large for an educational study and the teachers in it were from many regions in their states. We therefore feel that there is a distinct possibility that the meanings we found are common within the United States, and therefore that it is probable that an alarmingly high percentage of high school mathematics teachers, even teachers with mathematics and mathematics education degrees, are likely to convey problematic meanings for slope and rate of change that will hinder their students' efforts to learn science and mathematics. We suspect that, like their teachers, students will learn to model additive situations with quotients and multiplicative situations with differences. We believe many students will develop problematic meanings such as that slope is the same idea as the change in *y*. We acknowledge immediately that these suspicions beg future research.

The results of our study, if they apply broadly, demonstrate a serious problem with U.S. high school mathematics teachers' meanings for slope and rate of change. Results regarding other areas assessed by the MMTsm show that the problem is much broader than meanings for slope and rate of change (Musgrave & Thompson, 2014; P. W. Thompson et al., 2015; P. W. Thompson, Hatfield, Byerley, & Carlson, 2013; Yoon et al., 2015). However, we urge readers not to view these results as a condemnation of teachers' capabilities, but rather as pointing to a systemic, cultural problem within U. S. mathematics education. Our cautionary note is in line with Stigler and Hiebert (1999), who saw the results of their video study as providing insights into cultures of teaching rather than as a critique of individual teachers.

We interacted with many of the teachers who responded to our instrument. They all were highly motivated to improve their mathematics teaching, which was their reason for participating voluntarily in NSF Math/Science Partnership professional development programs (neither designed nor conducted by us). They wanted to talk about mathematics, and many teachers were disturbed that the questions we asked made them aware that their meanings were not sufficient to provide satisfactory (to them) answers. They also agreed that the meanings emphasized in the MMTsm are important for their students to develop. We therefore do not see our results as pointing to teachers' individual failings.

The reason we believe that our results point to a systemic, cultural problem in U.S. mathematics is that the meanings we probed are not taught in undergraduate mathematics programs. Rather, they are meanings that teachers developed as school students and became reinforced by their experiences in teaching from mathematics textbooks that support, directly or inadvertently, the same meanings as the textbooks they used as students. Put another way, we believe our results are related to what Lortie (1975) described as the cultural regeneration of schools. Lortie claimed that school students who identified positively with their schooling and with their teachers were most likely to enter teaching, thus regenerating for future students the schooling experiences they internalized. While our data says nothing about why teachers enter teaching, it gives another perspective on the issue of cultural regeneration. It seems quite plausible to us that a process like the following regenerates the problem of mathematical meaning in U.S. school mathematics:

- Many students leave high school with poorly formed meanings for ideas of the middle- and secondary-school mathematics curriculum.

- Students take mathematics courses in college that are designed with the presumption that students have basic mathematical meanings they in fact do not have.
- Instructors of these college mathematics courses presume, or do not care about whether, students have basic mathematical meanings they in fact do not have.
- Students apply coping mechanisms (e.g., memorization) in college mathematics that allowed them to succeed in high school.
- Students return to high schools to teach ideas they understood poorly, rarely revisited, and for which they still have poorly-formed meanings.

The process of cultural regeneration that we described is somewhat reminiscent of Felix Klein's (1932) description of the "double discontinuity" experience by students who become teachers.

> The young university student found himself, at the outset, confronted with problems that did not suggest, in any particular [way], the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honored way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching. (Quoted in Buchholtz et al., 2012, p. 107)

The difference between cultural regeneration as we described it and Klein's double discontinuity is that Klein presumed that the "young university student" of which he spoke had unproblematic mathematical meanings for secondary mathematics. Klein was mostly concerned with helping students connect those (presumably well-formed) meanings to university-level mathematics—to relearn elementary mathematics in terms of higher mathematics (e.g., different geometries being characterized by different groups of symmetries). Klein believed that his program's success depended upon educating future teachers regarding the connections between school and higher mathematics.

It is beyond the scope of this paper to address the systemic problem we've described. Thompson (2013) outlines one research and political agenda that addresses this issue partially. However, we must say here that we are not advocating a return to Klein's program of *Elementary Mathematics from an Advanced Standpoint.* Rather, our argument is that the cultural regeneration cycle can only be broken with sustained, intensive professional development for current mathematics teachers to support them in developing productive meanings for students' mathematical learning, a parallel effort in the redesign of high school mathematics pre-service teacher preparation programs. The professional development effort, in our opinion, must also focus on helping teachers select curriculum materials that cohere with their effort to re-conceptualize their mathematics in terms of supporting students' construction of coherent mathematical meanings.

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PAPER TWO: CALCULUS STUDENTS' UNDERSTANDINGS OF FRACTION, CONSTANT RATE, AND MEASUREMENT AND THE ASSOCIATION WITH THEIR UNDERSTANDINGS OF RATE OF CHANGE FUNCTIONS

While many studies have focused on calculus students' understandings of rate of change and derivative (Asiala, Dubinsky, Cottrill, & Schwingendorf, 1997; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Habre & Abboud, 2006; Orton, 1983), few studies have investigated calculus students' understanding of the mathematical ideas foundational to rate of change and derivative. Strong meanings for quotient, fraction, and measure could support students' meanings for rate of change and derivatives. Conversely, weak meanings for fraction, quotient, and measure might hinder students' understandings. Although multiple researchers have noted the importance of understanding rate and ratio for learning calculus (Orton, 1983; P. W. Thompson, 1994b; Zandieh, 2000), we know little about how calculus students' foundational meanings for fraction, quotient, and measure inhibit or enable them to develop a meaning for rate of change that is useful in calculus. The small-sample studies that have been conducted revealed that calculus students' meanings for fraction and quotient are not sufficient to help them make sense of rate of change and derivatives (Byerley & Hatfield, 2013; Byerley et al., 2012).

This study uses primarily quantitative methods to investigate 153 calculus students' success on fraction, measurement, and rate of change items. If their meanings for elementary ideas are extremely weak it would provide one explanation for why so many interventions designed to teach calculus conceptually have not been successful. The study also investigates the statistical association between students' results on a Pretest and their success on a test on rate of change functions in a conceptual calculus course.

THEORETICAL FOUNDATION

This section describes the meanings for rate of change function (derivative function), constant rate of change, and fraction that were foundational for this study. The value of a rate of change function at any value in its domain gives the rate of change of another function with respect to the other function's input at the same value in its domain. The rate of change of a function *f* at a certain value can be imagined in multiple ways. Here is one way: Suppose we want to know the rate of change of a function at $x = a$. First imagine a tiny change in x from the value $x = a$ to $x = a + \Delta x$. Then measure the associated change in *f(x)* in terms of the tiny change in x. For sufficiently small values of Δx this average rate of change, $\frac{f(a+\Delta x)-f(a)}{a}$, is approximately equal to the rate of change of the function at $x = a$ (P. W. Thompson, Byerley, et al., 2013).

This meaning for rate of change function is consistent with Zandieh's (2000) conceptual analysis of the concept of derivative that focused on ratio, limit, and function as the major organizing ideas (p. 107). The meaning for function used in this description of rate of change function is a relationship between two covarying quantities (P. W. Thompson & Carlson, 2016). Imagining two quantities varying together is critical for developing a productive meaning for rate of change. Imagining a quantity involves imagining some attribute of an object and a means to measure that attribute (P. W. Thompson, 2011). Conceiving of a rate of change of one quantity with respect to another involves comparing the measures of two changes in quantities (P. W. Thompson, 1994a).

For example, making a multiplicative comparison of a measure of distance traveled to a measure of time elapsed produces the new quantity speed.

Different people hold different meanings for fractions, quotients, and measure, but not all are equally useful. A person's meaning for an idea is that person's scheme for that idea. I use Thompson's definition of scheme:

> "We define a scheme *as an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity* (Thompson et al., 2014, p. 11).

This definition for scheme allows for complicated schemes for fractions that entail many ideas and procedures that are triggered in a variety of situations. Mature fraction schemes allow someone to imagine a variety of different situations as being modeled by the same underlying idea of a multiplicative comparison of two quantities. A person with a weak fraction scheme might be triggered to use various fraction procedures in different situations without seeing the fundamental similarities across situations.

Conceptual analysis and empirical studies inform us that some meanings for fractions are more productive for proportional reasoning and making sense of rate of change (Nabors, 2003; Norton & Hackenberg, 2010; P. W. Thompson & Saldanha, 2003). For example, Thompson and Saldanha's (2003) meaning for fractions as reciprocal relationships of relative size is easily linked to a meaning for rate of change as a multiplicative comparison of two changes. This meaning for fractions entails imagining that A/B gives a measure of the relative size of A and B. By measuring a quantity of size A in terms of a quantity of size B, they show that A is A/B times as large as B.

Reciprocally, *B* is *B/A* times as large as *A* (Thompson & Saldanha, 2003, p. 32). To understand the prior statements requires coordinating meanings for multiplication, quotient, and measure to imagine measuring one quantity in terms of another. Being able to imagine comparing two quantities multiplicatively by both partitioning and iterating the two quantities is critical to their discussion.

Thompson and Saldanha (2003) list some of the many problems with only understanding *A/B* as *A* is a subset of *B*, or as *A* parts out of the whole that has B parts. (p. 2). In constant rate of change situations, it would not make sense to describe the fractions (10 miles / 2 hours) as ten miles out of two hours. One calculus student's understanding of fractions as parts of wholes caused her significant problems in many situations. For example, she drew a line of length four and cut it into two equal pieces while trying to explain her meaning for four divided by two. She described the size of the two pieces as one fourth because there were four parts in the "whole" and one piece in the "part" (Byerley & Hatfield, 2013).

LITERATURE REVIEW

This literature review first examines the many failed attempts at calculus reform. It then examines the little the field knows about calculus students' understanding of fraction and quotient. I conjecture that the limited evidence we have for calculus students' weak meanings for fraction and quotient is a partial explanation of the well documented difficulty calculus students have with rate of change and derivative. The research on calculus students' meanings for rate of change and derivative suggest their meanings do not entail envisioning multiplicative comparisons of relative sizes of changes. The review concludes with a few studies that have demonstrated that students meaning for fractions could explain their success or failure in understanding more advanced mathematical topics.

Studies of Calculus Reform

There have been many interventions in calculus class that promote active learning, use of technology, multiple representations of functions, etc. (Code, Piccolo, Kohler, & MacLean, 2014; Garner & Garner, 2001; Habre & Abboud, 2006; P. W. Thompson & Dreyfus, in press; White & Mitchelmore, 1996). While some of the interventions provided solid evidence that students learned more in the reformed class than a traditional class, I was unable to locate a report of an intervention that resulted in the majority of the class learning over 70% of the ideas the instructors intended to teach. In my attempt to locate evidence of a successful intervention I read multiple literature reviews on the state of calculus research, many individual studies on calculus interventions, and examined data collected from many courses using the *Calculus Concept Inventory*.

Larsen, Marrongelle, Bressoud, and Graham (in press), Rasmussen, Marrongelle, and Borba (2014), and Speer and Robert (2001) each conducted extensive reviews of the research on teaching and learning calculus. The three reviews did not give an example of a successful calculus class intervention. Speer and Robert cited 55 papers, Rassmussen et. al. cited 69 papers and Larsen et. al. cited 108 papers. The reviewers stressed the importance of doing applied research on reformed calculus classes so it seems likely that if they had known about a successful example of this research they would have reported it. They did mention cases where researchers helped students learn particular concepts successfully (Rasmussen et al., 2014, p. 510) and promising studies that had encouraging initial results (P. W. Thompson & Dreyfus, in press). Both Larsen et. al. (in press) and

Speer and Robert noted that the multiple calculus reform projects of the 1990's were not well-researched so the field did not fully understand how the reform projects impacted student learning. Although there were some comparative studies of student success in reform and traditional classes "the reported research provides very little information regarding the extent to which (and how) the innovations were informed by research, and very little information regarding how students and teachers engaged with the innovation" (Larsen et. al., in press, p. 28).

Many of the comparative studies found evidence that the interventions resulted in a statistically significant impact on student performance. Even though some students performed better in the reform classes, they still showed substantial difficulties on the major ideas their instructors wanted to teach. Here are a few examples. Garner and Garner (2001) studied courses taught using a popular reformed calculus textbook. They concluded that test scores were so low seven months after the students took calculus that it did not matter what type of course they taught: the students would not remember it. They wrote "although the comparisons between the two groups showed *statistical* significance, there was no evidence of *educational significance:* both reform and traditional students forgot most of what they supposedly had learned" (p. 108). Other studies are similarly pessimistic. Habre and Abboud (2006) designed a new calculus course that included use of technology and in-class discussion and contrasted student success with a course that mainly lectured and did not use any technology for visualization. Although some of the more advanced students liked the reformed class and learned more they found that, "in the end, the one thing that is most striking is the large percentage of dropouts (33 students out of 89) and failures (12 of the remaining 56 students) in the observed sections" (p. 67). They hypothesized that one of the reasons their approach failed was the prior knowledge of the students and decided not to try the experiment again. At the end of their intervention study White and Mitchelmore (1996) concluded "almost the only detectable result of 24 hours of instruction intended to make the concept of rate of change more meaningful was an increase of manipulation-focus errors in symbolizing a derivative" (p. 93). They did not think that a conceptual approach to calculus would work if students lacked foundational understandings of variables before arriving in class.

The final approach I used to locate an example of a successful calculus class was based on the data collected by Epstein (2013) as part of the development of the *Calculus Concept Inventory.* A panel of expert faculty with decades of experience drafted the first version of the CCI to measure a small set of basic constructs essential to calculus. In the piloting of the test at six institutions they realized that students' scores were at the "random-guess level" at the beginning of the semester and that there was "*no gain anywhere*" at any institution after a semester of calculus (Epstein, 2013, p. 1021). This was a shock to them and they decided the items were too hard. Their department chair agreed that the items should be easier and he said the items "needed to be at a level of 'point to your foot'" and the students will still struggle to answer correctly. After extensive piloting the item writers concluded that the "items needed to be at a level where most faculty would believe the items were utterly trivial" (Epstein, 2013, p. 1021). The easier version of the CCI has now been given in many U.S. universities and about a dozen other countries and the author gets a request to use it almost every week. Despite the huge number of instructors using the test the highest gain score he found was 0.44 in a reform course that used interactive engagement. This means that the students learned how to answer 44% of the "utterly trivial" items they could not answer at the beginning of the semester. Epstein reported that gains this high are very rare even in reform courses. Even famous mathematics educators such as Uri Treisman who are well known for their successful interventions (Fullilove & Triesman, 1990) had gain scores of 0.3. In contrast, gain scores of 0.7 are reported in the physics education literature on the Force Concept Inventory (Epstein, 2013, p. 1021).

Calculus Students' Understandings of Fractions and Quotients

Despite the vast research on students' understandings of fractions and quotients there is little research on how calculus students understand fractions or quotients. After extensive searching using numerous techniques I did not locate any articles focused on calculus students' understanding of fractions or quotients. In addition to key word searches I made use of other literature reviews. For example, Pinilla (2007) reviewed approximately 250 articles on learning fractions and none of her summaries or the titles of cited articles mentioned that the students' studied were in calculus. We do know that a small sample of calculus students displayed considerable confusion when trying to explain how the number 29.66 related to 0.236 when given the statement $7 \div 0.236 = 29.66$ (Byerley, Hatfield, & Thompson, 2012). In this study, Hatfield, Thompson and I (2012) found many calculus students were unable to give a situation in which one would divide by a fraction or explain why division was used in the slope formula. We found that the constructs such as "part-whole fraction scheme" used by Steffe and Olive (2010) to describe elementary students' thinking also described calculus students' thinking. Many calculus students had a primary meaning for fractions as "parts of wholes" and these meanings limited them in circumstances that required relative size meanings for fractions (Byerley, Hatfield, & Thompson, 2012; Byerley & Hatfield, 2013).

Some evidence suggests that there is a difference in the understanding of students correlated to their academic achievement. Schneider and Siegler (2010) found that 66 elite students with high SAT scores at Carnegie Mellon were able to compare the size of 29 different fractions to 3/5 with a mean error rate of 6% and a standard deviation of 5 (p. 1230). They found that the mean error rate (29%, SD = 20.8) on the same task was much higher at College of Allegheny County, a junior college with minimal entrance requirements. Even though fractions are an important part of the developmental mathematics curriculum at community colleges, "surprisingly, an extensive search of the literature revealed we know almost nothing about [community college students' mathematics knowledge and understanding]" (Stigler, Givvin, & Thompson, 2009, p. 5). They noted that even though placement tests are widely administered the results are rarely released. To fill this gap in knowledge Stigler et. al's (2009) study used 5830 Santa Barbara Community College placement test results to investigate what students' understood about the mathematics they had been taught in middle and high school. They found "several of the most common errors involved working with fractions" (Stigler et al., 2009, p. 13). For example, students simplified fractions incorrectly, added numerators and denominators to add fractions, and struggled to determine whether or not one fraction was larger than another (p. 11). This limited data suggests that college students' understandings of fractions range from incredibly weak to reasonably strong and is correlated to the admission requirements to the university. It is unknown how calculus students in particular might have fared on the tasks in these studies. Also, the placement tests' items that Stigler et al. surveyed were performance items. They were not designed to reveal what students understood about fractions.

We can infer some information about calculus students' fraction and measurement knowledge by examining the results of the National Assessment of Educational Progress (*National Assessment of Educational Progress, Mathematics Assessment*) of national sample of 12th graders. The following item from the 1996 test can be answered by comparing the relative size of cherry syrup and water in two situations and comparing the quotients:

> Luis mixed 6 ounces of cherry syrup with 53 ounces of water to make a cherry-flavored drink. Martin mixed 5 ounces of the same cherry syrup with 42 ounces of water. Who made the drink with the stronger cherry flavor?
>
> Give mathematical evidence to justify your answer.

In 1996 only 23% of 12th grade students gave a correct answer, and 26% gave a partially correct answer.

In 2009 62% of 12th grade students answered the following question about measurement correctly.

> Which of the following containers has the greatest liquid capacity?
> (1 gallon=4 quarts= 8 pints = 128 ounces)
> A. A 64-ounce orange juice container
> B. A 16-pint water jug
> C. A 5-quart bowl
> D. A 2-quart cola bottle
> E. A 1-gallon milk bottle.

In 1990, 46% of a nationally representative sample of 26,000 12th grade students showed "a consistent grasp of seventh grade material (decimals, percent, fractions, simple algebra)" (Mullis, Dossey, Owen, Phillips, 1991). We cannot assume that the population of students enrolled in calculus have the same results on the NAEP items as a random sample of 12^{th} graders, as students with more mathematical coursework tend to do better on division, fraction, and proportion tasks (Lawton, 1993; Mullis, Dossey, Owen, & Phillips, 1991; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2010). Taken as a whole, these studies suggest that although university calculus students may have stronger meanings for fractions and quotients than elementary and secondary students we should not assume that these ideas are unproblematic for them. Given the limited evidence suggesting that fractions and quotients could be a major source of difficulty for calculus students it is important to investigate the question explicitly. Many millions of dollars are allotted to improve calculus instruction and if students' middle school mathematics meanings are holding them back from success this issue should be addressed directly. Right now, the research does not exist to decide if fraction, measurement, and quotient interventions would be useful in calculus courses.

Calculus Students' Meanings for Rate of Change

A number of studies have documented calculus students' difficulties with the idea of rate of change (Carlson et al., 2002; Gravemeijer & Doorman, 1999; Hackworth, 1994; Herbert, 2013; Orton, 1983). Orton (1983) noticed that calculus students had unexpected difficulty with the basic calculation of a rate of change from a graph. In Orton's study, students struggled to apply the rule for dividing the difference in y by the difference in x to obtain a rate. He wrote, "it has been suggested already that one of the problems of learning about rate of change is that the ideas are basically concerned with ratio and proportion" (Orton, 1983, p. 243). Another possibility is that they had difficulty with the idea of change or with representing it.

Much of the research on secondary and calculus students' meanings for rates of change suggests that students' images of rate are not based on a multiplicative comparison of relative size, nor based on an image that entails a comparison of changes. The literature on secondary and calculus students' meanings for rate of change will be organized around the following points:

1. Many calculus students might be able to coordinate changes in two quantities without explicitly comparing the relative size of two changes.

2. Rate of change and slope are often considered to be one quantity that is a measure of steepness or fastness and not a comparison of two changes.

3. Some students' conceptions of rate of change are "chunky" meaning that they prefer to consider rate of change on integer-sized intervals of change.

4. Some students confound amount of change in an interval with a rate of change.

Point 1. Coordinate changes but not relative size of changes. Both Carlson et al. (2002) and Johnson (2010, 2012) investigated university and secondary students' abilities to coordinate the covariation of two quantities in a variety of contexts. For example, Carlson investigated student's understanding of the covariation of the amount of water and the height of water when water was poured into a bottle. Both found that students could coordinate the direction of change of one quantity with the direction of change of the other quantity. However, the majority of students in Carlson et al.'s (2002) and Johnson's (2010, 2012) studies did not compare the relative size of changes in two quantities even though this comparison would have helped them address the interviewers' various requests for descriptions of how the two quantities changed together.

Point 2. Rate of change and slope or single quantities. Despite students' ability to loosely coordinate changes in two quantities, many students convey a dominant meaning for slope and rate of change that does not involve comparing changes in two quantities. Instead students developed indexical meanings such as slope as an index of steepness or rate as an index of fastness. Stump (2001b) found advanced secondary students are more likely to think of slope as an angle measure. Nagle, Moore-Russo, Viglietti & Martin (2013a) found that 71% of 65 calculus students described slope as a *behavior indicator* in at least one of five questions about slope. To those students, steepness *indicates* how fast a function is increasing or decreasing. Although many students knew that slope formula involved changes in x and y, researchers found "little evidence that students actually considered a relationship between the variables x and y" (p. 1506). Thompson (1994b) noticed that students "who experienced difficulty [with interpreting a rate of change function] seemed to want to think of the difference quotient as 'the derivative' and interpret it as 'how fast it [the function] is changing,' without interpreting the details of the expression as an amount of change in one quantity in relation to a change in another" (p. 142). Nagel et. al's (2013) supports Thompson's observations. They reported, "in order to grasp the concept of the derivative, students need a conceptual understanding of slope beyond what was evidenced in the majority of their responses" (p. 1508).

Points 3 and 4. Change is chunky, and confuse amount with change in amount. It is possible to make sense of some rate of change situations without understanding fractions as reciprocal relationships of relative size. Rate of change can be considered the amount added to one quantity for a given change in another quantity. Many high school teachers employed this meaning when explaining a slope of three as "y changes by three when x changes by one (Byerley, Yoon, & Thompson, 2016). This meaning for slope conveys that the two quantities vary in chunks. A person with a chunky image of covariation considers two quantities varying in chunks as opposed to continuously (C. W. Castillo-Garsow, 2010, 2012).

A chunky meaning for rate of change is useful in some situations but can also lead to difficulties. Many teachers who expressed a slope of three as “up three and over one” struggled to imagine situations where the independent variable did not change by one. Some teachers with this chunky meaning also referred to the slope as the change in *y,* reasoning that the change in *y* and the slope were both three (Byerley et al., 2016).

Chunky meanings for slope do not depend on imagining multiplicative comparisons. I hypothesize that with this weak understanding it is hard to imagine that two quantities in a rate vary smoothly together. To do so would require the students to see a constant multiplicative relationship. To understand the definition of the derivative $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ it is important to imagine Δx becoming infinitely small while the change in the function also varies as Δx decreases. For those who think of rate as a multiplicative comparison of changes in two quantities it does not matter the size of Δx. Chunky meanings for slope and rate depend on the size of the “chunk” of change in *x* to remain constant and do not cohere well with images of Δx smoothly decreasing in size.

Calculus Students’ Understanding of the Derivative

Given the difficulties many calculus students have with the idea of rate of change, it is not surprising that research also finds that students struggle with the concept of derivative. Many students are unable to create graphical representations of the rate of change function from a given function (Tall, 1986; Ubuz, 2007). Students in the Ubuz study often focused on computing derivatives without considering a connection to the rate of change of the function. Unsurprisingly, they failed at constructing a derivative function by imagining the rates of change at a number of points in the domain (Ubuz, 2007, p. 611).

Meanings for One Topic Impact the Learning of Another Topic

Researchers have used both qualitative and quantitative methods to understand how students' prerequisite knowledge impacts their understandings of topics that build on that knowledge.

Nabors (2003) studied the relationship between seventh graders' meanings for fractions and their development of proportional reasoning. Students first worked on a variety of fraction items with Nabors so that she could build models of their fraction understandings. Nabors then asked students to solve challenging proportion and rate of change problems to understand how their meanings for fractions impacted their reasoning. Nabors found that students who demonstrated more advanced meanings for fractions in the first part of the experiment were better able to make sense of rate situations in the second. She used Steffe and Olive (2010) constructs to categorize the students' fraction schemes. The use of teaching experiment methodology, as described by Steffe and Thompson (2000), allowed Nabors to make explicit connections between students' fraction schemes and proportionality schemes. She found that the mental operations of partitioning, iterating, and thinking flexibly about units and relationships among units were essential to make sense of fractions and proportions maturely. This conclusion agrees with the findings of other researchers, as summarized by Norton and Hackenberg (2010, p. 343), that students with only a part-whole meaning for fractions have trouble extending their understanding to more advanced topics such as proportional reasoning.

Torbeyns, Schneider, Xin & Siegler (2014) found that "despite country-specific differences in absolute level of fraction knowledge, 6th and 8th graders' fraction magnitude understanding was positively related to their general mathematical achievement in all countries, and this relation remained significant after controlling for fraction arithmetic knowledge in almost all combinations of country and age group" (p. 1). In a similar study Siegler et al. (2012) found that fraction knowledge at age 10 was the strongest of five predictors of age 16 algebra knowledge and mathematics achievement (p. 693). Furthermore, the predictive relations for early knowledge of both fractions and division was stronger than other mathematical skills in the U.S. and the U.K. Siegler et al. (2012) concluded that mastery of fractions and division is needed if substantial improvements in understanding of algebra and other aspects of high school mathematics is to be achieved" (p. 696).

Statistical studies could not address how understanding one idea contributed to learning another idea. However, the statistical studies provided evidence that difficulty on fraction items predicted difficulty on other mathematical items. Together the qualitative and quantities studies suggest students' fractional reasoning impacts the sense they make of more advanced mathematics.

Relationship between Literature, Conceptual Analysis and Methods

The literature review and conceptual analysis of fraction and rate of change strongly informed the selection of items and the analysis of results. The conceptual analysis of rate of change functions suggested that additive or computational meanings for fractions and measure were likely insufficient to build productive meanings for rate of change functions. The models of student thinking produced in prior research were used to create items that would be difficult for students limited to unproductive meanings for a topic. For example, if a student's primary meaning for slope as a formula and an index of steepness, they would be likely to struggle to estimate slope from a graph with equally spaced but unlabeled axis. Students with a part whole meaning for fractions would likely struggle with the item *Fraction of Cloth,* discussed in methods (See Figure 30). A measurement item involving conversion from gallons to liters was included because a national study showed 12th graders struggle with measurement, and the conceptual analysis shows that measuring one change in terms of another is a productive way of making sense of rate of change. Torbeyns et al. (2014) motivated the inclusion of an item asking students to place fractions on a number line that included fractions larger than one.

METHOD

Participants

One hundred fifty-three (153) students taking a redesigned introductory calculus course in Spring, 2015 at a large Southwestern university consented to have their test scores used in this study. These students had a variety of backgrounds including prior coursework in conceptual precalculus or calculus at the same university, precalculus or calculus at a community college, and precalculus or calculus in high school. Twelve students did not give consent for their responses to be used in the study. The course, taught by two experienced instructors, met 150 minutes per week for one semester. The students also attended one recitation session a week taught by a graduate teaching assistant. To enroll in calculus students needed an adequate score on an un-proctored online placement test or credit for precalculus from a regionally accredited college or university.

Redesigned Calculus Course

Thompson designed a new approach to calculus that was built upon his conceptual analysis of the Fundamental Theorem of Calculus (P. W. Thompson, Byerley, et al., 2013). His goals for the course are "address two fundamental situations: (a) you know how fast a quantity is changing and you want to know how much of it there is, and (b) you know how much of a quantity there is and you want to know how fast it is changing" (1994b). In this class, derivative functions were usually called "rate of change functions" to remind students of a productive meaning for a derivative function.

I conducted the study in the context of the redesigned course because the instructors discussed quotient and constant rate in class and assigned homework and test problems on these topics. Furthermore, the instructors included a large number of conceptual items on rate of change questions on Test 3 that were difficult for students to solve if they had only learned to recall and apply procedures.

Item Selection and Improvement

The students answered questions about fractions and rate of change on a Pretest and on Test 1 that occurred after a month of instruction. They answered questions about rate of change functions on their Test 3 that occurred two weeks before the final exam. In Fall, 2014 I used qualitative and quantitative methods to develop and improve items for the three tests used in the Spring 2015 study. Using interview data, I clarified item stems that were unnecessarily confusing. I also added items about rate of change, magnitude, and function from Project Aspire to Test 1. Project Aspire was an NSF funded project to develop an instrument to diagnose teachers meanings for rate of change, function, magnitude and other topics. I used Aspire items because they were validated with interviews with content experts, calculus students, and secondary teachers (P. W. Thompson, 2015). Project Aspire items were designed to be sensitive to and detect reasoning abilities noted in the literature review to be critical for learning calculus. Some of these reasoning abilities and understandings are covariational and quantitative reasoning, measurement schemes, chunky meanings for rate of change, and additive versus multiplicative meanings for rate of change.

Pretest Items

Fraction and Quotient Items

Fraction of Cloth. Bradshaw, Izsák, Templin, and Jacobson (2014) designed *Fraction of Cloth* (Figure 30) to measure middle school teachers' mathematical knowledge for teaching fractions.

Ms. Roland gave her students the following problem to solve:

Candice has 4/5 of a meter of cloth. She uses 1/8 of a meter for a project. How much cloth does she have left after the project?

Ms. Roland had students use the number line so that they could draw lengths. Which of the following diagrams shows the solution? Assume all intervals are subdivided equally.

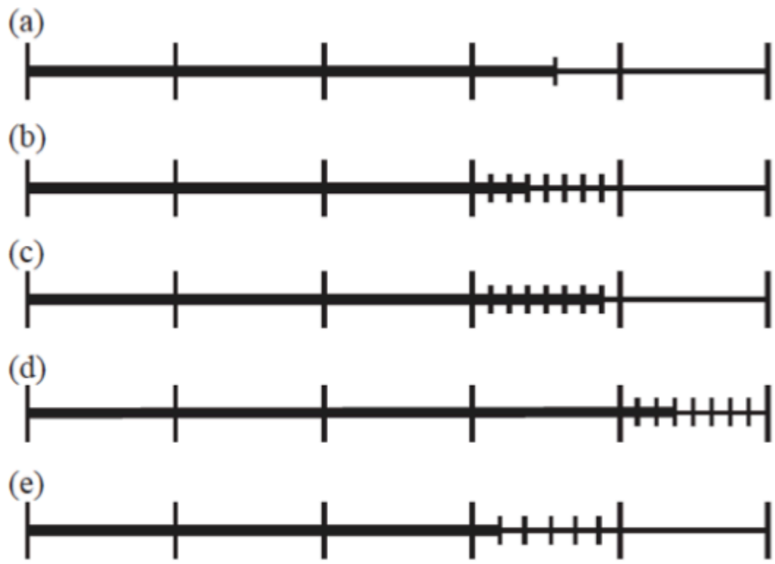


Figure 30. The Item *Fraction of Cloth* Designed by Izsák et.al (2014).

Each distractor reflects a particular problematic way of thinking. For example, choice (c), reflects subtracting 1/8th of 1/5th from 4/5^{ths} instead of subtracting 1/8th of one meter from 4/5ths.

Liters to Gallons. Project Aspire designed the item *Liters to Gallons*, shown in Figure 31, to see whether teachers would think about a unit conversion situation algorithmically or would take quantitative relationships of relative size into consideration (Byerley & Thompson, 2014; P. W. Thompson et al., 2014). When using a larger unit of measure (e.g. gallons instead of liters), the numerical value of the measure of the container's volume will decrease.

A container has a volume of **m** liters. One gallon is $\frac{189}{50}$ times as large as one liter. What is the container's volume in gallons? Explain.

Figure 31. The Item *Liters to Gallons*. © 2014 Arizona Board of Regents. Used with Permission.

I made the Aspire item multiple choice for the Pretest by adding the four choices,

$\frac{189}{50}m, \frac{189}{50}m^3, \frac{50}{189}m, \frac{50}{189}m^3$. These options were given as answers by many teachers on the Aspire assessment.

Fractions on Number Line. I designed the item in Figure 32 based on the studies that found relationships between students' ability to place fractions on number lines and their mathematical success (Torbeyns et al., 2014).

Place the following fractions as carefully as possible on the number line below. (The fraction ½ has been placed as an example.)

5/7 18/14 27/13 9/7

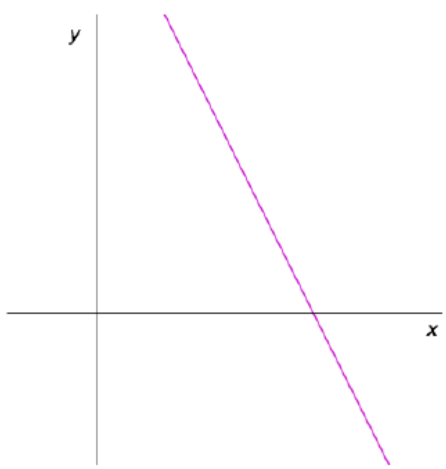
Figure 32. The Item *Fractions on a Number Line.*

The item *Fractions on a Number Line* was scored as correct if the student put the mark in the correct "bin." The number ½ would be scored as correct because it is in the "bin" between 2/5 and 3/5.

Slope and Constant Rate of Change Items. The Pretest included two items that Project Aspire designed to reveal the extent to which teachers' meanings for slope were based on multiplicative relationships.

***Slope from Blank Graph*.** The item *Slope from Blank Graph* determines if teachers use a meaning of slope as a relative size of changes in x and changes in y to estimate a value of slope given a graph (P. W. Thompson et al., 2014). The open-ended version was on the Pretest and a similar multiple choice version was on Test 1 (Figure 33).

15. The values of y and x are related so that $y = mx + b$. The graph of their relationship is given below. The scale used on the x and y axis is the same. Estimate the numerical value of m.



a) It is not possible to estimate m without numbers on the axis, but m is decreasing.
b) $m = 2$
c) $m = 0$
d) $m = -1/2$
e) $m = -2$

Figure 33. The Item *Slope from Blank Graph*. © Arizona Board of Regents 2015. Used with Permission.

In interviews, teachers and students with a primarily computational meaning for slope struggled to see how they could estimate a slope without being given points to put into the formula.

Meaning of Slope. The Pretest included a slightly modified version of the Aspire item, *Meaning of Slope,* shown in Figure 34.

> **Your friend is learning about the idea of slope. In his homework he divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.**
>
> **Explain to your friend what 3.04 means.**

Figure 34. Modified Aspire item *Meaning of Slope.* © 2014 Arizona Board of Regents. Used with Permission.

I scored the calculus students' responses with the rubric developed by the Project Aspire team (Byerley et al., 2016). Responses scored at the highest level conveyed that slope is a measure of the relative size of changes in input and changes in output. Mid-level responses conveyed a chunky meaning for slope. Lower-level responses conveyed a memorized or computational meaning for slope, such as slope formulas and the mnemonic rise/run. Responses that did not fit any of these three descriptions, were scored at level zero.

Constant Rate of Three. I designed the item *Constant Rate of Three* in Figure 35 in order to see if students understood the relationship between a change in R, a change in S, and the rate of change of R with respect to S.

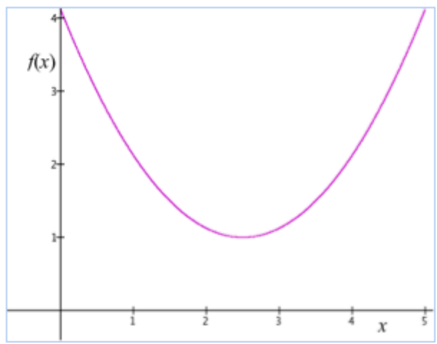
> **The constant rate of change of quantity R with respect to quantity S is 3.**
> **If S changes by 1.7 how much does R change by?**

Figure 35. The Item *Constant Rate of Three* Designed for Pretest.

Rate of Change on Graph Items. The Pretest included items that involved the ideas of rate of change depicted graphically. I wanted to know if items that involved rate of change or quotient embedded into contexts with functions, covariation, and graphs would be more predictive of student success than items that isolated the idea of fraction or constant rate.

Covariation and Change. The Aspire item *Covariation and Change* (Figure 36) involves covariation, graphical representations, frames of reference, and rate of change (Joshua, Musgrave, Hatfield, & Thompson, 2015).

The graph below is of a function *f* over the interval [0,5].



For small equal increases of the value of x starting at $x = 1$ and ending at $x = 2$, the corresponding **changes** in the value of *f* are....

a) positive and increasing
b) positive and decreasing
c) negative and increasing
d) negative and decreasing
e) I cannot tell

Figure 36. Item *Covariation and Change*. © Arizona Board of Regents 2014. Used with Permission.

Individuals who focus on the absolute value of changes often respond that the changes are negative and decreasing. Part B (not shown) asked “Is this sequence increasing or decreasing? -10, -9.5, -9, -8.5, …” to prompt people to focus on the idea that a sequence of negative numbers approaching zero are increasing. After this prompt the second page asks people if they want to change their answer. Answer choice (b) reflects thinking about the values of the function instead of the changes in values.

The Bottle Problem. The Precalculus Concept Assessment (PCA) is a validated multiple choice instrument that includes items related to function, covariation, and rate of change (Carlson, Oehrtman, & Engelke, 2010). The Pretest included the PCA item *The Bottle Problem.* This item that asks students to match the shape of a bottle to a graph showing the relationship between the volume of water in the bottle and the height of the water from the bottom of the bottle.

Test 1 Items

Fertilizer Function. The Test 1 item, *Fertilizer Function* (Figure 37), involved both the idea of relative size and function.

9. The instructions recommend using 64 oz. of a fertilizer for 75 ft^2 of garden space. Let A be the area of the garden in ft^2, and n be the number of oz. of fertilizer. Which of the following defines a function that determines the number of oz. of fertilizer needed as a function of the area of garden space?

a) $g(n)=\frac{64}{75}n$ b) $h(A)=\frac{64}{75}A$ c) $k(n)=\frac{75}{64}n$

d) $p(A)=\frac{75}{64}A$ e) None of these

Figure 37. The Item *Fertilizer Function*. This Item Taken From Test One From Precalculus Pathways (Carlson, Oehrtman, & Moore, 2010).

Students should notice that the number of square feet that can be fertilized is greater than the number of ounces of fertilizer. They also needs to interpret the quantities *A, h(A), n and k(n)* and understand a function as representing a relationship between varying quantities. There were other Test 1 items involving rate of change used in this study (See Appendix B). One item asked students to determine the distance a woman traveled given a graph of her rate of change versus time. Other items asked about the definition of constant rate of change. One item, *Relative Rates,* asked students to use information about rates of change to compare the distances two people traveled (Byerley & Thompson, 2014).

Test 3 Items

Student responses on Test 3 items were used as a measure of their understanding of rate of change functions. Test 3 occurred after a unit in which students created rate of change functions given initial accumulation functions. The instructors agreed to modify multiple-choice distractors on Test 3 that were not popular and replace them with distractors based on ways of thinking that had been noted in qualitative studies.

The graph in Figure 38 was the basis for six questions about rate of change.

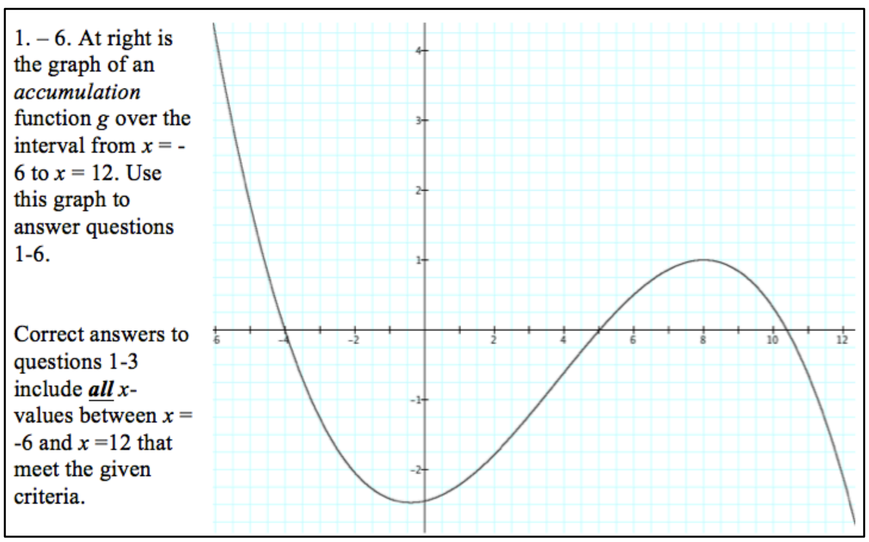


Figure 38. Graph Given on Test 3 for Questions One to Six.

The questions asked students to differentiate between an increasing amount and an increasing rate of change as well as between a positive rate of change and an increasing rate of change. Three of the questions asked about Figure 38 are shown in Figure 39.

4. At what value of x does the rate of change of g switch from positive to negative?

a) $x = -4$ b) $x = -3$ c) $x = -0.5$ d) $x = 4$ e) $x = 8$

5. Which of the following is true? The rate of change of g switches….

a) …from increasing to decreasing at $x = -4$
b) …from decreasing to increasing at $x = -0.5$
c) …from increasing to decreasing at $x = 4$
d) …from increasing to decreasing at $x = 8$
e) Two of the above are true

6. Which is closest to $g'(4)$? (*Use the scales on the axes in finding your estimate.*)

a) -1.5 b) 0 c) 0.5 d) 1 e) 2

Figure 39. Three Questions on Test 3 Associated with Graph in *Figure 38.*

Questions seven to twelve involved answering questions about an accumulation function given the graph of its rate of change function.

Other questions on Test 3 involved interpreting the meaning of rate of change expressions. For example, see Figure 40.

20. The radius of a spherical balloon is measured in meters. The function f gives the balloon's volume as a function of its radius. What does the expression $\frac{f(3+h)-f(3)}{h}$ represent?

a) The change in volume of the balloon as the radius changes from 3 to 3 + h
b) The approximate constant rate of change of the balloon's volume as the radius changes from 3 to 3 + h
c) The instantaneous rate of change of the balloon's volume when the radius is 3 meters
d) The slope of the tangent line to the curve $y = f(x)$ at $x = h$
e) The slope of the tangent line to the curve $y = f(x)$ at $x = 3$

Figure 40. Item on Test 3 about Rate of Change Functions.

The question shown in Figure 41 asked students to relate the rate of change of temperature to the temperature of an oven in an expression.

19. At 5 minutes after turning off the oven, the temperature inside the oven is $T(5)$ degrees Fahrenheit, and the rate of change of the temperature is $T'(5)$ degrees Fahrenheit per minute. Which expression gives the approximate temperature in the oven 6 seconds later, that is 5 minutes and 6 seconds after the oven is turned off?

a) $0.1 \cdot T'(5)$ b) $0.1 \cdot T(5)$ c) $T(5)+T'(5)$

d) $T(5)+0.1 \cdot T'(5)$ e) $T'(5)+T(5)/6$

Figure 41. Question on Test 3 Relating Rate of Change and Accumulation.

Five additional items used in the study were related rates or optimization problems.

Pretest Administration and Scoring

The calculus instructors required students to take the Pretest in the first week of class and gave students credit for taking it. Students' scratch work suggested that they took the Pretest seriously. The majority of Pretest responses were scored correct or incorrect. Minor computational errors were ignored if it was clear from a student's work that the student understood the question. I scored open-ended Aspire items using Aspire rubrics (Byerley et al., 2016; P. W. Thompson, 2015).

Multiple Linear Regression Model

Student answers to the Pretest and Test 1 items were used as predictor variables in a multiple linear regression model. The number of correct answers on the 21 Test 3 items about rate of change functions was used as the outcome variable in the regression model. Examinations of assumptions required to use multiple linear regression found no serious departures from the necessary assumptions. (See Appendix D.)

There were very few missing data points. No students skipped any questions on Test 1 or Test 3. Sixteen of 1495 student item responses on the Pretest were missing.

Because students had enough time to complete the Pretest, I hypothesized they skipped questions they did not understand and replaced the missing scores with zeros.

Validation of Items with Qualitative Interviews

Using data from the Pretest, I recruited six students with a range of low and medium understandings of quotient, measurement, and rate. I conducted two clinical interviews with each of the six students and asked them to explain their thinking test items. The interview data was analyzed in more detail as part of a qualitative study investigating the relationship between students' fraction meanings and their understanding of rate of change (Byerley & Thompson, submitted). However, some of interview data is reported here to help explain why students picked particular answers.

RESULTS: PRETEST AND TEST 1

Although these results do not generalize beyond the university in which they were collected, they do indicate that, at this university, students have substantial difficulty with ideas of quotient, fraction, constant rate of change, covariation, and graphically interpreting rates of change.

In the results, I report the scores of students who stayed in the class until at least Test 3 separately from the students who dropped the class before Test 3. I refer to the former students as *persisters* and the latter students as *droppers*. The study is primarily focused on *persisters* because those students showed substantial commitment to trying to learn calculus. See Figure 42 for a distribution of persisters' scores on the Pretest.

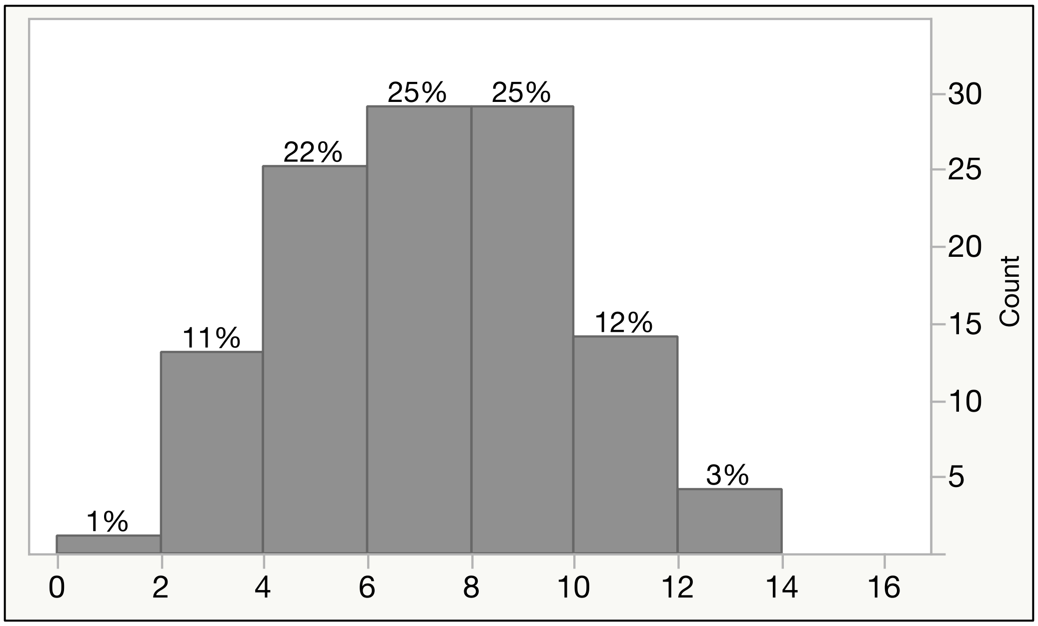


Figure 42. Distribution of 115 Persisters' Scores on the Pretest that had 15 Possible Points.

The six students interviewed in Spring 2015 had Pretest scores ranging from two to seven, making them fairly representative of the weaker 59% of class.

Fraction Item Results

Fraction of Cloth. Forty-one of 153 (26.7%) students who took the Pretest answered the question correctly. Thirty-three of 115 (28.6%) persisters and 21.5% of 38 droppers answered this item correctly. Student interviews on this item revealed substantial difficulties reasoning about fraction diagrams and understanding the meaning of statements like 1/5 of 1/8. Five of the six students interviewed on this item took over ten minutes to solve the problem even with numerous hints. During their attempts to solve the problem these five students made a number of statements indicating a lack of a basic understanding of fractions Table 22.

Interviews on Fraction of Cloth. All six students interviewed described fractions as parts of wholes. This meaning disabled the five students from identifying the size of a piece when the interviewer pointed to 1/40 of a meter on the diagram and asked them to identify its size with respect to a meter. The "baby piece" of size 1/40 of one meter is shown in Figure 43.

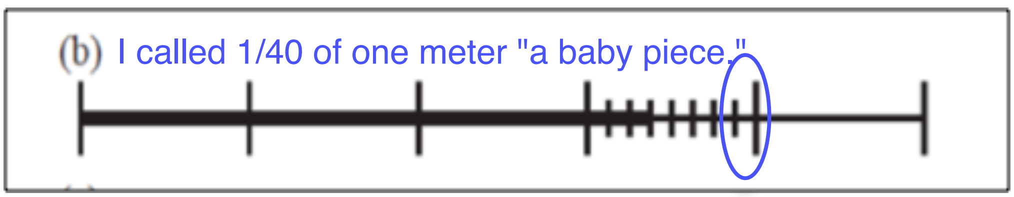


Figure 43. Correct Representation of 4/5-1/8 in *Fraction of Cloth.*

With the interviewer's help, these five students eventually identified the "baby piece" in answer choice (b) as 1/40 of a meter and this helped them find the correct representation for 27/40. The same five students initially inappropriately described 1/8 of 1/5 of one meter as 1/8 of one meter. They were not bothered that the line they labeled 1/5 of a meter was eight times as large as the line they labeled 1/8 of a meter. Table 22 summarizes the results of the six interviews.

Table 22. Summary of Student Interviews on *Fraction of Cloth.*

Time used to solve problem during interview	**Described fractions as parts of wholes?**	**Struggled to identify a piece sized 1/40 of a meter when asked directly?**	**Identified 1/40 of one meter as 1/8 of one meter?**	**Made major fraction mistake?**
Janet 12 min	Yes.	Yes.	Yes.	Yes. Many. Computed (1/5-1/8) to find 1/8 of 1/5. Could not find one half of seven.
Kristina 37 min.	Yes.	Yes.	Yes.	Yes. Many. Thought 1/5 was smaller than 1/8. Drew 1/40 of a meter by breaking 1/10 of a meter into two equal pieces. Struggled to divide line segment into 8 equal parts because she did not think of 1/8 as ½ of ½ of ½.
Daniel 16 min	Yes.	Yes.	Yes.	Yes. Identified 1/8 of one meter as drastically smaller than 1/5 of one meter and did not notice problem. Said one eighth of one fifth is one thirteenth because eight plus five is thirteen.
Alex 11.5 min	Yes.	Yes.	Yes.	Yes. Tried to identify 2.7 fifths on the diagram in effort to find 27/40.
Emma 4.5 min	Yes.	No.	No.	No. She missed the problem because she didn't think of identifying the size of the 1/40 piece to solve the problem.
Hannah 17.5 min	Yes.	Yes.	Yes.	Yes. She made true statement that 40 copies of a "baby piece" fits into one meter. She did not know if this meant the "baby piece" was 1/8 of a meter or 1/40 of a meter.

All six of the students eventually found a correct representation with interviewer prompting, but it took one student 37 minutes of focused conversation to make sense of the diagrams. During the conversations five of the six students made multiple incorrect statements about fractions some of which are shown in Table 22.

Excerpt 7 gives a representative example of the difficulties students experienced when trying to determine the size of a "baby piece". The interviewer used the word "baby piece" in the interview because she wanted to indicate a length of size one fortieth without giving away the name of the fraction. At first Janet thought that a "baby piece" is one eighth of one meter and then decided that this did not make sense. The interviewer helped her see that a "baby piece" is one eighth of one fifth of one meter but Janet did not know how to find one eighth of one fifth. Excerpt 7 shows a small portion of the long discussion about the size of the "baby piece."

Excerpt 7. Discussion with Janet about the Size of a "Baby Piece" on *Fraction of Cloth*.

Interviewer: This little piece is some fraction of a meter. Right?

Janet: Right.

I: But you don't know how to use one eighth of one fifth to figure out which fraction of a meter the little "baby piece" is?

J: Right. Wouldn't that be…it would be subtraction, right? So it is one eighth of one fifth. You have one fifth so you'd have to subtract one eighth from that.

Janet struggled to figure out that forty "baby pieces" fit into one meter and that this meant a baby piece was one fortieth of one meter. Seeing that forty "baby pieces" fit into one meter involves imaging each fifth cut up into eight equal pieces. Imagining a partition of a partition is one way to understand that eight-fortieths is equivalent to one-fifth. During our interactions there was numerous instances where Janet struggled with partitioning.

Byerley (submitted) includes evidence that Janet struggled to cut a circle into five equal pieces and coordinate the number of cuts with the number of pieces resulting. Excerpt 8 gives evidence that Janet did not imagine partitioning a partition to decide that 4/5 was equivalent to 28/35. Overall, five out of six students interviewed did not try to find the size of a "baby piece", and once prompted to do so struggled to coordinate multiple partitions of one meter to determine that it 1/40 of one meter.

Fractions on a number line. Students answered the fractions on a number line question without a calculator. Some responses had significant scratch work showing long division. Based on the interviews, it appears that many students' low scores on this item reflected difficulty coming up with a way to compare a number of sevenths to a number of fifths. Table 23 shows the responses of all students who took the Pretest, droppers and persisters.

Table 23. Number of Fractions Placed Correctly on the Number Line.

	Number of Fractions Placed in the Correct "Bin"				
	4	3	2	1	0
Persisters	62 (53.9%)	18 (15.6%)	20 (17.4%)	10(8.7%)	5(4.3%)
Droppers	18 (47.7%)	3 (7.9%)	8 (21.1%)	4 (10.5%)	5 (13.2%)
All students	80 (52.2%)	21 (13.7%)	28 (18.3%)	14 (9.2%)	10 (6.5%)

Half of the students who placed zero fractions correctly dropped the course by Test 3. In contrast, 22.5% of students who placed all four fractions correctly dropped the course by Test 3.

Although students did know procedures for finding common denominators, they did not always think to express two fractions with a common denominator to decide which was larger. Janet tried to draw pictures of 4/5 and 5/7 to determine which fraction was larger but her pictures were not accurate enough to help her. I suggested that she repartition the slices of "pie" in her picture and that was enough of a hint for her to find common denominators. She was unsure if the procedures she learned for adding and subtracting fractions would still work in the context of trying to decide if 4/5 or 5/7 was larger.

Excerpt 8. Janet Discussed if 4/5 or 5/7 is Larger.

> Janet: Oh, I see what you are doing, the common denominator… wouldn't it just be thirty-five? I guess I can't. I don't remember how to get a common denominator with a multiplication problem. Usually you do it with subtraction.
>
> Interviewer: This isn't a multiplication problem. We are trying to decide which of two fractions is bigger.
>
> J: so we …[writes down 28/35 and 25/35, two fractions equivalent to 4/5 and 5/7] Okay. So then four fifths is bigger. I multiplied them both by umm …what would give me the same denominator.
>
> I: Okay.
>
> J: So, four fifths I divided by seven (I'm sorry multiplied) and twenty five over… I multiplied that one by five. I guess that wouldn't make them the same thing because I didn't multiply them by the same thing. I guess that wouldn't work.
>
> I: You are worried that because you multiplied one fraction by five and the other fraction by seven you messed something up?
>
> J: Yeah.

There are a few important exchanges in Excerpt 8. First, Janet confuses the procedure for finding common denominators with the oft-heard maxim "do the same thing to both sides." Second, Janet, like other students interviewed, did not think to repartition the one fifth and one seventh intervals by cutting a fifth into sevenths and a seventh into fifths without the interviewer's suggestion. Eventually she was able to reorganize her thinking and come to her own conclusion as to why she can multiply any fraction by one (in this case 5/5 and 7/7).

Like Janet, the students who placed fractions on a number line incorrectly in their interview also revealed substantial and difficult-to-resolve issues with fractions.

Gallons to Liters. Responses to *Gallons to Liters* revealed major difficulties with fourth and fifth grade Common Core measurement standards (National Governors Association Center for Best Practices, 2010).

Table 24. Responses to *Gallons to Liters.*

	High-level answer	**189/50 m**	**Used cubic term (a or c)**	**Total**
Persisters	17 (15%)	62 (54%)	36 (31%)	115
Droppers	6 (18.7%)	14 (43.8%)	12 (37.5%)	32
All students	23 (15.6%)	76 (51%)	48 (32.6%)	147

Many students associated converting between two measures of volume with the appearance of a cubic term. This data is consistent with Dorko's research showing that elementary volume formulas were not well-understood by calculus students (Dorko, 2012; Dorko & Speer, 2014). The results suggest most students did not reflect on the relationship that the smaller the unit of measure used, the larger the measure of a given container.

Fertilizer Function. The *Fertilizer Function* item on Test 1 required a similar ability to relate the measure of two quantities-in that case the number of ounces of fertilizer and the number of square feet in the garden. Sixty-eight percent of the persisters answered *Fertilizer Function* correctly on Test 1.

Results of Rate of Change Items

Covariation and Change. Unsurprisingly, students struggled on a set of more complex items that involved the concepts of function, graphs, and rate of change. The results for the item *Covariation and Change* are shown in Table 25.

Table 25. Students' Final Answer Choice on *Covariation and Change.*

	Choice C: Correct description of changes	Choice D: Describes absolute value of changes	Choice A, B or E	Total
Persisters	23 (20%)	14 (12.2%)	78 (67.8%)	115
Droppers	6 (15.8%)	11 (28.9%)	21 (55.2%)	38
All students	29 (19%)	25 (16.3%)	99 (64.7%)	153

The calculus students' responses are similar to secondary teachers with mathematics or mathematics education degrees. Only 23 of 146 (15.8%) of secondary teachers with mathematics or mathematics education degrees gave a correct answer to this question (Joshua and Thompson, in preparation).

Bottle Problem. Carlson et al. (2002) reported that high performing calculus students struggled to coordinate changes in volume and height in a spherical bottle filling with water. The calculus students' responses from this study confirm her observations (1998; 2002) in a different sample of students. Only sixty-three of the 115 persisters (54.7%) answered the *Bottle Problem* correctly.

Constant Rate of Three. Only 62 of 115 persisters (54%) correctly answered *Constant Rate of Three* (Figure 35). In interviews, students displayed difficulty in explaining the relationship between changes, differentiating change in amount from amount, and interpreting the phrase "with respect to." One student, Daniel, said that whenever he encountered the phrase “with respect to” in a problem he knew he would probably answer incorrectly.

Estimating Slope from a Blank Graph. Students were asked to estimate the value of slope from a blank graph on the Pretest and Test 1. Results from the Test 1 version of *Slope from Blank Graph (*Figure 33) are reported because the instructors showed students that it was possible to estimate the slope on a blank graph in class. Sixty-four of 115 persisters (56%) answered the item in Figure 33 correctly on Test 1 after the instructors discussed this very issue in class.

A Comment on Generalizability

Despite the lack of statistical generalizability, this is the first study that studied this many calculus students’ meanings for fractions and measure, and strongly suggests that a nationally representative sample of calculus students would also struggle with the items. The proportions of correct answers on each item would be the same in a national sample as this isolated sample, but this study demonstrates that it is likely a random sample of calculus students would struggle with these middle school ideas and the area is worthy of future study.

Building a Regression Model to Predict Test 3 Responses

Multiple linear regression can be used to determine which items are predictive of student success on rate of change function items. Regression models can help build a case that students’ meanings for foundational topics are relevant to their understanding of a conceptual calculus class.

An ordinary least squares approach shows that three of the twenty-two items are statistically significant predictors of success on Test 3 items ($\alpha = 0.05$). The statistically significant items on the Pretest are *Gallons to Liters* ($\beta = 2.81$,t(113) = 2.49, p = 0.015), *The Bottle Problem* ($\beta = 2.12$, t(113) = 2.15, p = 0.034) , and *Covariation and Change* ($\beta = 1.29$,t(113) = 2.10, p = 0.038). Using a stepwise elimination approach the previous three items are still statistically significant. The two additional statistically significant predictors from Test 1 are *Fertilizer Function* ($\beta = 3.25$ F(118) = 11.96, p = 0.0007) and *Slope from Blank Graph* ($\beta = 2.22$, F(118) = 6.62, p = 0.01). All five items were individually significant predictors when used to predict variation in Test 3 rate of change item scores.

It is best to make a model using fewer predictors with greater degrees of freedom to avoid over-fitting. The model with the five statistically significant predictors explains 30.9% of the variation in the number of rate of change questions answered correctly on Test 3. The adjusted R squared, which takes into account that the R squared value increases monotonically with the number of predictors, is 27.7%.

Table 26 shows a relationship between students' scores on the five predictor items and their success on test three rate of change items. Two of the five predictor items were scored on a 0, 1, 2 scale so the total points possible on the five predictor items is seven.

Table 26. Score on Predictor Items Versus Performance on Test Three Rate of Change Items.

		Score on Five Predictor Items (Percent Students with Score)						
		0 (1%)	1 (12%)	2 (22%)	3 (29%)	4 (23%)	5 (9%)	6 (4%)
Test Score	4^{th} quartile	0%	14%	8%	21%	41%	80%	80%
	3^{rd} quartile	0%	7%	24%	39%	15%	10%	0%

	2nd quartile	100%	29%	44%	27%	33%	10%	20%
	1st quartile	0%	50%	24%	12%	11%	0%	0%

Table 26 shows that the majority of students (74%) scored two, three, or four on the predictor items. Students who scored four, five, or six on the predictor items are substantially more likely to be in the top quartile on Test 3. Half of the students with only one point on the predictor items are in the lowest quartile on Test 3. Table 26 is consistent with the regression model showing that variation in the responses to the five predictors is associated with variation in Test 3 scores.

Two of the fraction items, *Fractions on a Number Line,* and *Fraction of Cloth* were not statistically predictive of student success on Test 3. Interviews suggested that students' responses on these items were related to their meanings for fraction and that their meanings for fractions were associated with their ability to make sense of rate of change functions. There are multiple reasons why the qualitative and quantitative data are not consistent. One, many factors impacted student success on Test 3 so even if there is a relationship between responses to an item and Test 3 scores it might not be strong enough to be significant. Two, interviews suggested students were able to "guesstimate" the correct location of fractions without being able to explain the placement. Three, it may be that even if students who have relatively strong meanings for fraction, quotient, and measure, do not apply those meanings to making sense of calculus. Daniel's interview provided evidence for the third hypothesis.

Daniel's interview on *Meaning of Slope* showed that he had not connected his meanings of quotient to his meaning for the slope formula. He explained that "A divided by B" means "the amount of A's that would fit into B" and "B divided by A" means "the amount of B's that would fit into A". Daniel responded to Part A of *Meaning of Slope* by writing $(y_2 - y_1)/(x_2 - x_1)$. Excerpt 2 contains Daniel's explanation of the meaning of 3.04.

Excerpt 9. Daniel Explained his Meaning for Slope of 3.04.

> Daniel: So the 3.04 is the slope between these two. So that is basically the change between the two.
>
> [We clarify that he is imagining a line connecting two points.]
>
> Interviewer: When you say the change between the two are you talking about the length between the two points on this piece of paper?
>
> I: Yeah. It would be the length between these two [Daniel highlights hypotenuse.]
>
> I: So the slope is the length between the two points?
>
> D: Right.
>
> I: Okay. So why do you divide the change in *y* and the change in *x* to get a length?
>
> D: Because, it's… you've got the one *x* here and the other one here and so you are trying to find the way which they both get to each other basically. That's…
>
> I: Okay. [Daniel laughs.] Is that at all related to seeing how many B's fit into A or is that like a separate thing in your brain?
>
> D: If you are doing the slope it's different, I guess, I'm seeing it different in my brain, I guess it is because of the word slope gave this a different meaning.
>
> I: What does the bar in between them mean to you?
>
> D: I just… divide [laughs]
>
> I: Alright. It's just that you were not using the how many times B fits into A language at all when describing the slope so that is why I was asking.

D: Yeah. No, not with slope.

Daniel realized that length is not a good description of slope when the interviewer drew multiple triangles of different sizes on the same line. The slope of the hypotenuse of each triangle was the same, but the lengths of the hypotenuses differed. Daniel's meaning for quotient allowed him to construct a stronger meaning for slope, but only once the interviewer asked him to connect his meaning for quotient to his understanding of the slope formula.

DISCUSSION

Only 1 of the 115 persisters correctly placed four fractions on a number line, converted from gallons to liters, and represented 4/5 minus 1/8 with a diagram. Fifty-nine percent of persisters scored less than 50% on the Pretest. Five out of six students interviewed struggled to find 1/5 of 1/8 which explains why the success on *Fraction of Cloth* was barely above a random-guess level. Half of persisters placed four fractions in approximately correct places on a number line, but when interviewed some of these students could not explain their placements. Only 15% of students correctly converted from *Gallons to Liters* given a conversion factor and four multiple choice responses. As a whole the students approach to the problem was substantially less effective than random guesses between four choices. None of these fraction and measurement items on the Pretest involved ideas beyond middle school mathematics.

The results of individual fraction and measure items suggest that most students' meanings for fractions and measure are not strong enough to support their learning about rate of change functions. The evidence for this statement comes from three different sources. One, my conceptual analysis explains how a productive meaning for rate of change functions is built on productive meanings for measuring the change in *y* in terms of the change in *x*. Two, the six interviews, analyzed in Byerley & Thompson (submitted), showed numerous instances where weak meanings for fraction and measure caused significant difficulties as students tried to understand rate of change functions. Further, the students' responses on *Gallons to Liters* were statistically significant predictors of success on Test 3. This provides evidence that students' measurement schemes are related to what they can make sense of in calculus.

Students difficulties with fractions and measurement may explain the reason that so many studies documented that students' and teachers' meanings for rate of change are indexical, chunky, or computational and not based on the multiplicative comparison of two changes (C. W. Castillo-Garsow, 2012; Coe, 2007; Stump, 2001a; P. W. Thompson, 1994b). Students develop ways to cope with rate of change situations without having to rely strongly on their meanings for fraction.

The rate of change items show that students have many difficulties with interpretation of rate of change functions above and beyond their meanings for fractions. For example, most of the 115 persisters did not differentiate between the value of a function and a change in the value of a function (see Table 25). Forty-six percent struggled to appropriately use a graph to represent the covariation in two quantities (see *Bottle Problem* discussion). Forty-six percent struggled to find a change in *y* for a given constant rate of change and a change in *x* (see *Constant Rate of Three* discussion). This basic computation is critical for making approximations of an amount a quantity has accumulated over an interval. Forty-four percent of persisters were not able to estimate a slope given a graph without numerical values on the axis (see *Slope from Blank Graph)*. This skill is critical for drawing a rate of change function given an original function and deciding if answers to many questions are reasonable. If students struggle to imagine the changing slope of a function as x changes, they will struggle to understand the relationships between the function and the graph of its derivative.

Overall, this data suggests that there are many reasons it is difficult for students to understand conceptual instruction on rate of change functions, including but not limited to their understanding of measurement, variable, graph, function, fraction and covariation.

IMPLICATIONS AND CONCLUSIONS

The results of this study suggest that one of the reasons for the general lack of success of calculus reform is that students' foundational knowledge of measure, fraction and constant rate do not support students' construction of useful meanings for rate of change functions. This data also suggests an explanation for the many students who report disliking conceptual instruction (Habre & Abboud, 2006). For students to engage meaningfully in conceptually-oriented instruction, they must have schemes for ideas that ground classroom discussions that are sufficiently well-developed that they can think about implications of those ideas.

It is worth noting that students who lack the foundational meanings they need to understand a conceptual calculus lesson on rate of change functions can learn derivative rules without understanding them. Many students in the Spring 2015 class who had exceptionally weak fraction meanings scored above 90% on the course's derivatives mastery test.

My research is focused on students' understanding of fraction, quotient and measure instead of students' conception of variable, but there is a parallelism between this study and White and Mitchelmore (1996). They found that students' conception of variable was not strong enough for them to make sense of their conceptual calculus intervention. They determined that there was not enough time in the calculus curriculum to provide the remediation students needed. Adding to the difficulties already documented, my study shows that most of this sample of calculus students did not have the meanings for fraction, measure and quotient that would be useful to understand conceptual calculus instruction on rate of change functions. White and Mitchelmore (1996) suggested, "either entrance requirements for calculus courses should be more stringent in terms of variable understanding, or an appropriate precalculus course should be offered at the university level" (p. 93). Educators who are aware of this studies data have suggested that most of the students in this study should have been placed in precalculus. While it seems reasonable to expect calculus students to understand middle school mathematics, there are at least two issues with addressing calculus students' weak foundational knowledge by placing them into a lower class. First, it is unclear that a precalculus course will address students' weak meanings in measure, quotient, fraction, and rate of change that hinder students so severely in calculus. Second, many students in calculus have already passed college level precalculus and cannot be required to repeat it. Instead, we would need to turn back time so that students could learn these middle-school concepts when they should have and ensure that those who take precalculus are provided support in developing these meanings.

Even though many ideas from precalculus are critical for calculus, completing college precalculus with its current focus on developing procedural knowledge does not increase students' chances of eventually passing calculus (Sonnert & Sadler, 2014). Sonnert and Sadler (2014) pointed to a variety of studies showing that "traditional approaches that did not work in high school are simply repeated anew in college precalculus" (p. 1204). Precalculus curricula rarely focuses on the meanings of measurement, fraction, quotient and constant rate of change. Some curricula, such as Pathways Precalculus, are making adaptations in light of students' weak meanings for quotient and measure and incorporating these ideas explicitly into the student and teacher materials (Carlson, Oehrtman, & Moore, 2010). It would be interesting to research the impact of the research-based modifications to the curriculum on students understanding of these fundamental ideas. My experiences teaching pre-service teachers who learned from the Pathways curriculum indicate that at the very least they are more likely to notice the broad use of quotients in higher mathematics and they describe quotient as a measure of relative size frequently.

There is no reason to think that placing students to classes before Precalculus will help them earn a STEM degree. Very few students who take non-college-credit courses, such as developmental mathematics, complete a STEM degree (Stigler et al., 2009). Moreover, community college students often learn fractions in the same ineffective way they learned them in elementary and secondary school (Stigler et al., 2009). Even if community college teachers were well prepared to discuss fractions meaningfully, it may be that students would have a difficult time developing a new way of thinking about something they had already learned meaninglessly. Mathematics educators have tried to help teachers develop strong meanings for fractions and they found it "challenging at best and impossible at worst" to encourage teachers to think in more meaningful ways about computations they had been using for many years (Armstrong & Bezuk, 1995).

Placing students with weak meanings for fractions into precalculus or another remedial class is not even possible when the students have already passed precalculus at a university. Two of the students interviewed, Janet and Kristina, had very weak meanings for fractions and they had passed a university precalculus course with an A and a B respectively. They were able to pass prior classes because they both worked hard, completed all of their homework, and were fairly good at recalling and applying procedures for tests. Their former teachers cannot be "blamed" for passing them. Janet and Kristina both figured out ways to pass tests that the university math department intended to assess understanding, but which Janet and Kristina had learned to answer without the understanding that the writers presumed would underlie correct performance.

Calculus instructors must eventually teach calculus to students who arrive with meanings for fraction and measure that cannot support them in understanding calculus ideas. It is unreasonable, however, to expect calculus instructors to rescue students who are so woefully unprepared to learn calculus meaningfully. We must redouble efforts to support middle school and high school teachers in promoting strong meanings for measure, fraction, quotient, and constant rate.

In line with Larsen et. al.'s (in press) call for more applied research on the design of effective calculus courses, the field would benefit from taking seriously the long-term impediments to students later learning that are created by their failure to learn fundamental mathematical ideas in middle school. The field must also help mathematics departments change their thinking about what it means to succeed in college mathematics courses. We must consciously attend to the problem that our concept of success allows many students to “succeed” without learning much about what we think is being taught.

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PAPER THREE: TWO CALCULUS STUDENTS' FRACTION AND MEASURE SCHEMES AND THEIR UNDERSTANDING OF RATE OF CHANGE FUNCTIONS

This paper focuses on the mathematical meanings and ways of thinking that students bring with them to their study of university-level calculus. As Speer and Robert (2001) noted:

> It is accepted in much of the educational research community that students' understanding of one concept influences their learning and understanding of related concepts. Research has revealed that what may appear to be weaknesses in students' understanding of calculus concepts can really be just manifestations of their preexisting understandings of a related concept (Speer & Robert, 2001, p. 296).

The intervention study by White and Mitchelmore (1996) is one example of that research. They found that students' meanings for variable prevented them from learning from 24 hours of focused conceptual instruction on derivatives. White and Mitchelmore argued that while most existing research on calculus students' thinking is on topics such as function, tangent, and limits that are particular to calculus, "another aspect that needs to be considered is the question of what other concepts are involved in applying calculus knowledge" (1996, p. 79). Twenty years later research on students' calculus learning is still thin on how students' foundational knowledge impacts the way they can make sense of conceptual calculus curriculums.

There is evidence that many calculus students struggle with fraction, measure, constant rate of change, and interpreting a graph (Byerley, submitted). As a research community, we do not actually know what students are capable of understanding about conceptual calculus instruction when their middle school mathematics knowledge is constituted by unproductive and often inaccurate meanings for fundamental ideas.

To investigate how students' foundational knowledge impacts them in calculus, this paper discusses two students who were participating in a conceptual calculus class designed by P. W. Thompson, Byerley, et al. (2013). The research question guiding the study was:

> How do students' meanings for quotient, rate of change, fraction, and measure impact their understandings of rate of change functions in a redesigned calculus course?

It seems obvious that "the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience" (Glasersfeld, 1995a, p. 1). It is much less obvious how a student with problematic meanings for foundational mathematics understands conceptual calculus instruction.

LITERATURE REVIEW

There is recent research documenting that students having well-formed fraction and measure schemes is essential for their development of algebraic and quantitative reasoning (Nabors, 2003; Pekkan, 2008; Steffe, 2013; Torbeyns et al., 2014). In addition, Steffe and colleagues have made many arguments about how fraction and measure schemes develop from students' counting and unit-coordination schemes (Ellis, 2007; Hackenberg, 2007; Steffe, 2002, 2003, 2004; Steffe & Olive, 2010). Thus, long before students are ready for calculus they must develop the ability to coordinate multiple levels of units to support robust fraction and measure schemes. Steffe and colleagues' extensive teaching experiments show that students usually progress to a higher-level scheme by reorganizing a lower-level scheme. Much evidence suggests that students who struggle with basic fraction situations do not make sense of higher level fraction situations until they develop strong foundational schemes (Steffe & Olive, 2010). This literature review will focus on what we know about calculus students' foundational measure, fraction, and rate schemes.

Calculus Students' Understanding of Measure

Byerley (submitted) and Dorko and Speer (Dorko, 2012; Dorko & Speer, 2014) reported that many calculus students have difficulties with the idea of measure. Dorko (2012) investigated calculus students' abilities to answer basic questions about areas and volumes of a variety of shapes to better understand students' thinking when they try to use integration to find volumes of solids created by revolving a curve around an axis. She conjectured,

> It is plausible that the difficulty understanding these calculus concepts are, at least in part, due to a lack of understanding of more basic concepts. Though area and volume understanding has been investigated in elementary school students, it has not been studied in calculus students (Dorko, 2012, p. 4).

Dorko and Speer (2014) reported that 26.9% of 169 students who answered all tasks had correct units in all of their responses. Using interviews, Dorko (2012) "found that the student thinking behind length units used for other spatial computations appears to be that the units of the answer are the same units in which the shape was initially measured" (p. 5). Thus if the sides of a square were measured in centimeters, the students believed the area of that square should also be measured in centimeters (not centimeters squared). Dorko and Speer's study suggests many calculus students use computational instead of quantitative reasoning in measurement situations. By computational, I mean that students abstracted a rule from prior experience and use the rule without imagining how the quantities in the situation are measured. If the students had imagined the measures of area they would have been more likely to realize that centimeters are not appropriate measures of area.

Dorko and Speer's (2014) findings are consistent with the results of the National Assessment of Educational Progress, which tested a nationally representative sample of 12th graders (*National Assessment of Educational Progress, Mathematics Assessment*). Only 55% of twelfth graders in 1990 correctly answered "volume" in response to the item in Figure 44.

If a measurement of a rectangular box is given as 48 cubic inches, then the measurement represents the…

1. **distance around the top of the box**
2. **length of an edge of the box**
3. **surface area of the box**
4. **volume of the box**

Figure 44. 1990 NAEP Item Given to National Sample of 12th Graders.

The fact that almost half of high school seniors did not associate cubic inches with a measure of volume strongly suggests that they do not have a strong meaning for cubic inches. This could explain Dorko's (2012) finding that calculus students choose which unit of measure based on the units given in the problem and not based on a quantitative understanding of the situation. Additionally only 49% of a nationally representative sample of 12th graders solved the item in Figure 45 correctly. The students were told that one gallon equals four quarts and one quart equals two pints in the problem, so the difficulty was not due to forgetting the conversion factors.

The number of pints in 10 gallons could be determined by performing which of the following operations?

A. Multiplying 10 by 2
B. Dividing 10 by 6
C. Multiplying 10 by 6
D. Dividing 40 by 2
E. Multiplying 40 by 2

Figure 45. 1992 NAEP Item Given to 12th Graders.

Calculus students also struggle to convert between measures of volume, even when given a conversion factor. Byerley (submitted) reported that only 15% of 147 Calculus I students at a large public university correctly converted between gallons and liters given that a gallon is 189/50 times as large as a liter. Consistent with students' lack of meaning for the measures of volume, 37% of the students unnecessarily cubed the conversion factor between liters and gallon. This means they chose the answer that said *m* liters is equal to $(189/50)^3 m$ gallons. Students' success on converting gallons to liters was a statistically significant predictor of success on a test that assessed their understandings of rate of change functions. Calculus students' measurement schemes are a promising and little studied research topic that might shed light on their difficulties in calculus.

Calculus Students' Understanding of Fractions

Byerley and Hatfield (2013) and Byerley, Hatfield, and Thompson (2012) reported that many calculus students also had significant difficulty reasoning about basic fraction situations in interviews and teaching experiments. Students commonly displayed part-whole meanings for fractions that disabled them from making sense of situations requiring multiplicative fraction schemes. Byerley (submitted) reported that only 52% of

147 calculus students placed four given fractions appropriately on the number line. Only 27% of calculus students chose a correct representation of 4/5 – 1/8, given five options. Both items, *Fractions on a Number Line* and *Fraction of Cloth*, were used in interviews and are discussed later in this paper.

Calculus Students' Understanding of Rate of Change

There are a number of studies documenting calculus students' difficulties with the idea of rate of change (Carlson et al., 2002; Gravemeijer & Doorman, 1999; Hackworth, 1994; Herbert, 2013; 1983; P. W. Thompson, 1994b). For example Hackworth (1994) found, "most of the students currently entering first semester calculus have a weak understanding of rate of change, and this university's calculus classes do not improve it" (1994, p. 161). It might be that studying derivatives did not improve calculus students' understanding of rate of change because they did not have the foundational meanings they needed to make sense of instruction on derivatives. P. W. Thompson (1994b) found that students' images of rate substantially impeded their progress in making sense of the Fundamental Theorem of Calculus.

Rate of change as index of fastness. Many students have a dominant meaning for slope and rate of change that does not involve changes in two quantities. Stump (2001b) found advanced secondary students are more likely to think of slope as an angle measure rather than a comparison of sizes of the changes in two quantities. Stroud (2010) reported that students think of rate as the number a speedometer points to or an index of fastness. Given the difficulty many calculus students have with fraction and measure, it makes sense that they would develop a meaning for slope and rate of change that does not require them to compare quantities multiplicatively.

Rate of change as amount of change. Teachers frequently talk about a slope of *m* as meaning "as *x* changes by 1, *y* changes by *m*" (Byerley et al., 2016). The rate of change and amount of change are both *m* so some teachers lose track of the change of *x* of one and think of a rate of change as an amount of change (Byerley et al., 2016). Another consequence of these chunky meanings (C. W. Castillo-Garsow, 2010) for rate of change is students' difficulty dealing with changes in *x* that are not equal to one. Their meaning for rate of change does not entail a proportional correspondence between changes in *x* of any size and associated changes in *y* (C. W. Castillo-Garsow, 2012).

THEORETICAL FOUNDATIONS

I view learning as the process by which students make sense of new ideas by using and modifying their earlier schemes. When a student carries out an activity associated with a situation and cannot assimilate the result into their current expectation they will experience a perturbation. If the student is able to accommodate the unexpected result by modifying his or her schemes the perturbation will be eliminated (Glasersfeld, 1996, p. 68). In the language of Glasersfeld, this study investigates how two calculus students are able to modify their current schemes in their attempts to make sense of new calculus ideas.

In this study I use Thompson, Carlson, Byerley & Hatfield's (2014) definition for scheme:

> "We define a scheme *as an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity* (Thompson et al., 2014, p. 11).

This definition is broad enough to describe complex schemes, such as the one students' develop for rate of change functions. A well-formed scheme for rate of change function entails schemes for measure (and hence proportionality), function (and hence variable), graph, variation, and covariation. The conceptual analysis in this section details how students' images of rate of change functions could develop over many years from foundational measurement schemes.

Measurement Schemes are Critical for Quantitative Reasoning.

Thompson described the importance of researching students' quantitative reasoning (P. W. Thompson, 1993, 2011; P. W. Thompson et al., 2014). Following Thompson, many others have demonstrated the importance of reasoning quantitatively for making sense of algebra, precalculus, and calculus (Ellis, 2007; Johnson, 2010; Lobato & Siebert, 2002; Moore & Thompson, 2015). For an individual to quantify an object they must imagine a measure of the object:

> Quantification is the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bilinear, or multi-linear) with its unit (P. W. Thompson, 2011, p. 37).

Steffe (2013) investigated the foundational mental operations that are entailed in developing the images of measurement necessary to reason quantitatively. This conceptual analysis will focus on how students' ability to coordinate three levels of units is essential for the development of a multiplicative fraction scheme and a mature measurement scheme. Finally, it will discuss the importance of a measurement scheme for the construction of a rate of change function.

Development of Fraction and Measurement Schemes

Steffe & Olive (2010) analyzed thirty years of teaching experiments with elementary students to understand how students developed increasingly sophisticated fraction schemes. During these experiments they found extensive evidence that students developed their fraction schemes as modifications of their counting schemes. The ability to imagine number sequences as comprised of composite units, such as seeing ten as five copies of two, helped students develop the images of partitioning and iterating they needed to construct fraction schemes.

Counting schemes and unit coordination. When children are able to construct permanent objects-meaning they can imagine the object and its location even when they cannot see it, they become able to reflect on a sequence of identical objects. The awareness of a group of objects can motivate the desire to count the objects. Some students cannot count a group of objects that have been described to them but hidden from their view because "they need perceptual unit items in their visual field to carry out the activity" (Steffe, 2013, p. 18). As students develop mathematically they are able to imagine objects and processes more easily in their mind and reflect on these objects to develop more advanced reasoning. Steffe (2013) outlined a number of developmental steps that children progress through as they develop more advanced counting schemes.

Eventually children are not only able to hold one object in mind and operate on it, but to hold a collection of objects in mind and operate on that composite unit. For example, one student Johanna could think of nineteen as the sum of the composite units nine and ten and she operated with those composite units.

> Johanna was asked to take twelve blocks. The interviewer took some more and told Johanna that together they had nineteen and asked how many he took. After sitting silently for about 20 seconds, Johanna said "seven" and explained, "Well, ten plus nine is nineteen; and I take away two-I mean, ten plus two is twelve, and nine take away two is seven" (Steffe 1992, p. 291). Johanna disembedded ten and nine from nineteen and then operated on the two numbers until she transformed them into twelve and another number that when added to twelve, would make nineteen. (Steffe, 2013, p. 26)

Johanna's reasoning shows how students develop the ability to work with composite units in the service of counting goals.

The construction of a robust measurement scheme requires that students construct iterable composite units. Constructing an image of a composite unit made up of iterable units is critical for making sense of fractions:

> When it is a goal of a child who has constructed the partitive fraction scheme to mark off a unit part of a continuous whole, the child can disembed the part from the whole and iterate it to produce another partitioned continuous whole to compare with the original continuous whole in a test to find if the part is a fair part (Steffe, 2013, p. 36).

A student who has constructed a partitive fraction scheme can estimate one-tenth of a whole and then make ten copies of the one-tenth piece to determine if it is a fair share. The process of determining if a part is a fair share also produces a measure of the whole. It is in this sense "that children's fraction schemes can be used as measuring schemes in a way that is analogous to the claim that children' number sequences can be used as discrete quantitative measurement schemes" (Steffe, 2013, p. 35). To measure a continuous quantity students must project length units onto the quantity that they intend to measure (Steffe, 2013, p. 38). It is non-trivial to subdivide a quantity equally because the students must have conceptual structures in place to allow them to imagine a partition before acting. To develop more mature measurement schemes, the students must develop more than a partitive fraction scheme.

Multiplicative fraction schemes. Steffe (2013) explained why it is necessary to reason with three levels of units to construct a fraction as a multiplicative concept and why this is important for developing a mature measurement scheme (p. 37). Constructing an improper fraction as a multiple of its unit fraction involves thinking of a fraction as a multiplicative object.

Consider the fraction that is eight-sevenths of some original unit. First partition the original unit into seven equal pieces. The original unit and the one-seventh pieces are both units. Disembed one-seventh from the original unit and iterate it eight times to construct eight-sevenths. As Thompson & Saldanha (2003) elaborated, eight-sevenths, then, is eight one-sevenths, and one-seventh is one-seventh of 1 because 1 is 7 times as large as one-seventh of 1. In other words, to have a mature measurement scheme, one must understand fractions as entailing a reciprocal relationship of relative size between a unit fraction and the unit that one takes as 1. See Figure 46.

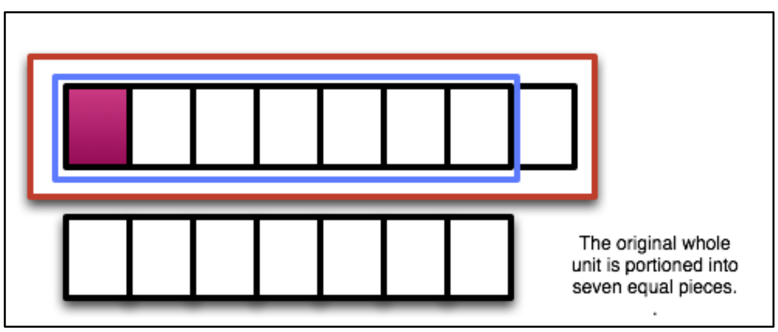


Figure 46. Seeing Eighth-sevenths as a Unit of Units of Units. Eight-sevenths is Eight One-sevenths, and One-seventh is in a Proportional Relationship with One.

It is common for students to interpret the resulting figure as eight-eighths, because they no longer hold in mind the original unit. The idea of a unit that is one, and remains one even after operating, is not in their thinking. To comprehend the new figure as eight-sevenths requires coordinating three levels of units-the original whole, eight-sevenths, and one-seventh. Figure 46 uses different colors to represent each of the three units that must be held in mind to understand the fraction eight-sevenths multiplicatively.

Reasoning with three levels of units and measurement schemes. Steffe argued that “a measurement of what an observer would consider as an extensive quantity is a multiplicative object in the same sense that a fractional number is a multiplicative object” (p. 37). By this, I understand Steffe to mean that the mental operations that are involved in constructing a multiplicative fraction scheme are the same as the mental operations involved in constructing a measurement scheme. In both cases a unit is iterated to make a whole; in measure, the iteration of a unit is what produces the measure.

It is important to be able to reason with three levels of units to understand the relationships of relative size in one system of units. Steffe (2013) gives three examples of why reasoning with three levels of units is important for the development of a mature measurement scheme. First, to understand 3/12 of one foot as equivalent to 1/4 of a foot “would entail establishing a composite unit containing four composite units each of which contains three units of 1/12 foot” (p. 39). The student would have to imagine a partition of a partition and keep in mind both partitions at once to establish 3/12 as equivalent to 1/4 of a foot. Second, to construct five feet as 60 iterations of 1/12 of a foot requires reasoning with three levels of units for the same reason imagining 8/7 requires reasoning with three levels of units. Finally, to imagine measuring one quantity measured by two units at the same time requires “constructing composite units as iterable units” (p. 39). To imagine measuring five feet as both 5/3 yards and 60 inches requires thinking of a yard as three feet and an inch as 1/12 of a foot.

Measurement and Calculus

Students’ ability to take three levels of units as given is essential for the development of a robust meaning for measurement and, in turn, conceptualizing a measure of a quantity is critical for quantitative reasoning. Imagining quantities and their measures is foundational for understanding the intended overarching ideas of Thompson’s redesigned calculus course: a course that examines how the measures of quantities co-vary. In particular, strong measurement schemes are critical for students’ construction of rate of change functions. I acknowledge that students’ images of variation and covariation are also critical for understanding rate of change functions, but will remain focused on measurement schemes in this paper. For more information on the relevance of students’ images of variation see White and Mitchelmore (1996), P. W. Thompson and Carlson (2016), and P. W. Thompson, Byerley, et al. (2013).

A differentiable function with a non-constant rate of change can be approximated to any degree of precision over a small interval with a linear function. The constant rate of change of the linear function is equivalent to the average rate of change of the original function on the small interval under consideration (P. W. Thompson, 1994a, p. 23). The constant rate of change of the linear function is a measure of the relative size of a change in output and a change in input.

Graphs are frequently used in conceptual calculus courses to convey the relationship between functions and their rate of change functions. For the sake of this analysis, let us assume that students understand a graph as a representation of the covariation of two quantities' measures and not as a static shape (Moore & Thompson, 2015). Constructing a graph of a rate of change function entails being able to imagine how a rate of change of the original function varies as the value of x varies. The student must be able to visualize a sequence of representations of changes in x and associated changes in y. They must imagine how the quotient $\Delta y/\Delta x$ varies by comparing the relative sizes of Δy and Δx. Being able to anticipate the measurement process is useful in deciding how small Δx intervals must be to produce linear approximations of the original function that have a desired accuracy. Holding a strong image of measure in mind also allows students to compare the relative sizes of two changes when they appear to them to be so small that it is not possible to carry out the measurement process physically. Thompson's (1994b) conceptual analysis of the Fundamental Theorem of Calculus was the inspiration for this analysis of what it means to understand a rate of change function.

Relative magnitude schemes. The measurement and fraction schemes described above are necessary but insufficient for making sense of calculus. P. W. Thompson et al. (2014) described five levels of reasoning with magnitudes that are critical for a mature understanding of rate of change. A person with a *relative magnitude* scheme can coordinate the comparison of two extensive quantities into an intensive quantity as well as the changes of units of measure for each extensive quantity (Thompson et al., 2014, p. 6). An important part of the relative magnitude scheme is that a person anticipates that the relationships between the two extensive quantities remains invariant even if the unit used to measure each quantity changes. In other words, 60 miles per hour represents the same relationship between distance and time as 88 feet per second. A student who understands the equivalence of 60 miles per hour and 88 feet per second understands that a mile is 5280 times as large as a foot and one hour as 3600 times as large as a second. Since one second is 1/3600th of an hour, an object will travel 1/3600th of 60 miles in one second. The student coordinated both the relative size of hours and seconds as well as the proportional relationship between distance traveled and time elapsed. Since one mile is 5280 times as large as one foot the number of feet traveled in a given amount of time is 5280 times as large as the number of miles traveled in that time period. If an object travels 1/3600th of 60 miles per second it travels $5280\left(1/3600 \text{ of } 60\right)$ feet in one second.

A relative magnitude scheme requires at a minimum coordinating three levels of units. Students must imagine one foot as the result of partitioning one mile and one second as the result of partitioning one hour. They must also understand that if they go some portion of the time they go the same portion of the distance. The two students discussed in this study did not have relative magnitude schemes, but it is important to show the schemes they operated with in relation to mature schemes I hypothesize would support well-formed understandings of rate of change functions.

Summary

It is critical for researchers to understand how students' foundational schemes impact new ideas they are able to understand. Students' measure schemes are foundational to their ability to reason quantitatively-a critical way of reasoning for making sense of the redesigned calculus course the students in the study were taking. Reasoning with three levels of units is critical for constructing multiplicative meanings for fractions and robust measurement schemes. Strong meanings for improper fractions and measure are critical in many areas of calculus including making sense of rate of change functions.

This theoretical perspective describes not only the advanced ways of reasoning about measurement but the development of these ways of thinking from elementary counting schemes. I decided to include descriptions of Steffe's research on the development of students' basic measurement schemes as modifications of their counting schemes after collecting the data on the two calculus students described in the results section. These two students did not reason with the multiplicative measurement schemes the conceptual analysis hypothesized were necessary for productive meanings for rate of change functions. Their reasoning was better explained by focusing on their basic images of partitioning and by using constructs describing earlier stages of development of measure schemes.

METHOD

Reform Calculus Class

Thompson designed a new approach to calculus that was built upon his conceptual analysis of the Fundamental Theorem of Calculus (P. W. Thompson, 1994b; P. W. Thompson, Byerley, et al., 2013; P. W. Thompson & Silverman, 2008). He designed the class to "address two fundamental situations: (a) you know how fast a quantity is changing and you want to know how much of it there is, and (b) you know how much of a quantity there is and you want to know how fast it is changing" (P. W. Thompson, Byerley, et al., 2013, p. 125). Janet, and Kristina, the two students described in this paper were enrolled in this redesigned course.

Overview of Mixed-Methods Calculus Study

The two students described in this paper were two of many students participating in a mixed-methods study designed to understand how students' fraction schemes impacted their understanding of calculus. In Fall 2014 I tutored five students weekly in individual sessions to develop a broad sense of how their understanding of fractions impacted their success in the course as well as to identify other relevant variables impacting success. In Spring 2015 I collected responses from 153 calculus students on a Pretest, Test One, and Test Three to understand how successful the students were on fraction and measure items and whether or not success on these items was predictive of success on rate of change function test questions (Byerley, submitted). In Spring, 2015 I also interviewed six students three times each to better understand their item responses and to understand how their fraction and measure meanings supported or hindered their understanding of the rate of change of a sine function.

I decided to focus on Janet and Kristina because they both worked hard in the class and completed all of the homework. Their difficulties could not be attributed to lack of attendance or effort unlike some of the other students I interviewed. Janet and Kristina earned an A and a B respectively from a college level precalculus course so under no typical placement system would they be denied entrance to calculus. It cannot be argued that they should simply be placed in a lower class to deal with the lack of foundational mathematical knowledge revealed in the study. In addition to these factors, I focused on Janet and Kristina because they both demonstrated substantial difficulties with basic fraction, measure and partitioning schemes.

Tutoring Kristina. The Fall, 2014 tutoring sessions with Kristina occurred once a week for the entire semester. I allowed her to ask for help with anything she was confused about in the class so I might develop a broad understanding of her experiences in the course. Because I paid her a small stipend for participation she also agreed to answer particular questions about fractions and rate of change functions and to be interviewed about her test responses. I tutored Kristina nine times in Fall, 2014 and recorded the important sessions to understand her experiences in calculus.

Interviewing Janet. The purpose of the first clinical interview in Spring 2015 was to model Janet's meanings for fraction, quotient, slope, measure, and rate of change. The first interview was based on her responses to the Pretest given in Spring 2015 (Byerley, submitted). The interviews analyzed in this paper concerned the items *Fraction of Cloth* and *Fractions on a Number Line* (described in results). Bradshaw et al. (2014) designed *Fraction of Cloth* as part of a national assessment of teachers' mathematical knowledge for teaching fractions. The item *Fractions on a Number Line* was inspired by Torbeyns et al. (2014). Torbeyns et al. (2014) found that student's ability to appropriately place fractions on number lines was highly predictive of their success in mathematics.

The second interview took place after the course instructor introduced the idea of creating a rate of change function given an accumulation function, but before Janet learned about derivative shortcuts. The goal of the exploratory teaching interview was to help students construct and understand a graph of the rate of change of the sine function. The discussion of the sine function was inspired by Precalculus Pathways curriculum (Carlson, Oehrtman, & Moore, 2010) and Moore's dissertation study (Moore, 2013). The output of the sine function is measure of a vertical length in terms of the radius. This mathematical context provided an additional opportunity to understand students' abilities to make multiplicative comparisons of varying quantities.

The third interview took place after Janet took her third exam (Test 3). Test 3 addressed students' understandings of rate of change functions. The interviews focused on the Test 3 problems that Janet answered incorrectly. Byerley (submitted) discussed Test 3 at length.

Exploratory teaching interviews. Both Janet and Kristina's sessions involved asking them questions and trying to help them understand a topic if they were confused. I spent more time helping Kristina with various issues because I was tutoring her, but in both cases I was open to helping either student construct new ideas. I wanted to "experience, firsthand, students' mathematical learning and reasoning" (Steffe & Thompson, 2000, p. 267) and not be constrained by a strict interview protocol if the student made an interesting comment or mistake. If students had difficulty with a particular task, I attempted to orientate them in a way that allowed them to continue their solution. The strength of this method is that it reveals the boundaries of students' understandings and the nature of those boundaries. A student who reaches an impasse in thinking about a problem, but can profit from a slight suggestion about another way to think about it has different understandings than a student who cannot overcome an impasse no matter the help given.

Analysis of Interviews

I made Pencast recordings of sessions with an Echo-Pen and transcribed them. I used open coding (Charmaz, 2006) to analyze the Pencasts. In the first pass of coding I identified instances that gave insight into students' meanings for fraction, quotient, measure, rate of change, or rate of change functions. I also identified places where their meanings for foundational topics impacted the sense they made of rate of change functions. This first pass of coding resulted in transcript of interviews that were highlighted in four colors. One color represented sections that gave insight into students' meanings for fraction, quotient, measure or rate. One color represented when their meanings supported them, and another represented when their meanings disabled them. A fourth color represented students meanings for rate of change functions. These color coded transcripts also included comments that described my initial hypotheses about how to describe the students' thinking. After this rough coding, I tried to connect the students' words to descriptions of student thinking provided in earlier studies. For example, I compared students' words to decide if their meaning for rate conveyed in an excerpt was consistent with thinking of rate as an index of fastness.

After the first pass of coding, I noticed that students' struggled to reason about a partition of a partition in the *Fractions of Cloth* item. As a result I looked for instances that shed light on students' abilities to coordinate three level of units and their use of measurement schemes. I reexamined the sections color-coded as about fractions to see if students appeared to be reasoning with two or three levels of units. I paid explicit attention to any circumstance where the students drew a partition or failed to use partitioning when it would have been useful. As a final pass of coding I used Tzur's (2014) summary of Steffe's research on levels of fractions schemes to organize the descriptions of Janet and Kristina. I looked for evidence that students did or did not reason at each level specified by Steffe (2010). See the discussion section for the table and evidence. The strength of this method is that by showing that students struggled to reason at stages hypothesized to precede multiplicative understanding of fraction and measure, I provided further support that they lacked mature fraction and measure schemes.

Other documents produced were summaries of students' performance on tests and quizzes and summaries of sessions with students. I often wrote the summaries of students' test responses before interviews and tutoring and the summaries included hypotheses about areas of weakness I might expect to see in a tutoring session or an interview.

RESULTS: THE CASE OF KRISTINA

This results section is the first of two. The first section is about Kristina. It gives a detailed example of what it means for a calculus student to fail to coordinate three levels of units. This is important because Steffe (2013) hypothesized that without being able to coordinate three levels of units the student would struggle to develop robust measure and rate schemes. At the end of Kristina's section I briefly describe the many ways her meanings for fractions caused Kristina problems in her study of calculus.

Who is Kristina?

Kristina was a hard working chemistry major, who earned a B in Pathways Precalculus from an experienced teacher at the same university. She had below average scores on three of the four tests in calculus and did not pass the class despite doing all of the homework and receiving extra tutoring at least once a week.

Two pizzas. As part of our tutoring sessions, I drew two pizzas as shown in Figure 47. I asked Kristina "what fraction of the two pizzas is shaded?"

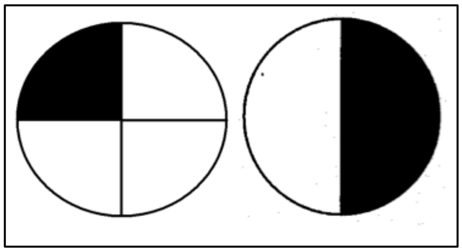


Figure 47. Kristina Tried to Determine what Fraction of the Two Pizzas is Shaded. Although she could identify that 1/4 of one pizza and 1/2 of another pizza was shaded, she could not determine that 3/8 of two pizzas was shaded. She did not see 1/4 of one pizza as the equivalent to 1/8 of two pizzas. Reasoning about 1/4 of 1/2 of the total amount of pizza was difficult for Kristina, which suggests that coordinating three levels of units was a challenge for her.

Fraction of Cloth Item. Kristina and I spent 47 minutes discussing the item *Fraction of Cloth* (Figure 48).

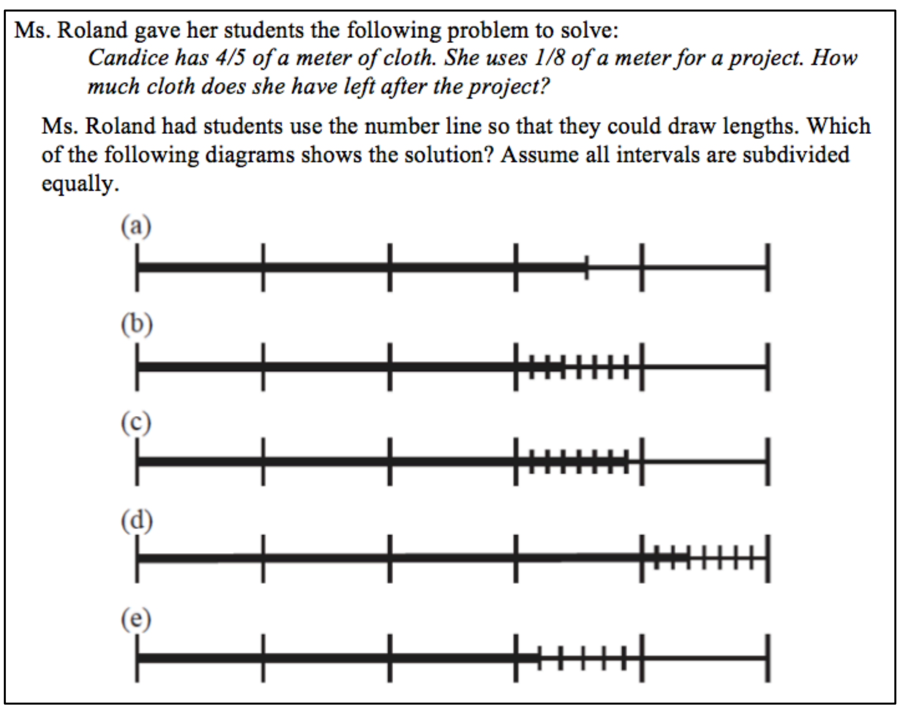


Figure 48. The Item *Fraction of Cloth* Designed by Bradshaw et al. (2014).

Initially we discussed that the horizontal lengths were partitioned into five equal pieces and why choice (d) did not make sense. Kristina saw the entire length as one meter and choice (d) showed more than 4/5 of one meter. Next Kristina drew a segment on the diagram that she called “1/8 meter” but her 1/8 meter was longer than 1/5 of the entire partitioned segment. I asked her to discuss her meaning for 1/8 to help her recognize her mistake. She explained 1/8 meant, “… how many times one-eighth goes into one meter. That doesn’t make sense. I really don’t know. I really don’t like fractions. This is really hard.” To help her visualize 1/8 I drew a line segment for Kristina and asked her to cut it up into eight equal pieces (Figure 49).

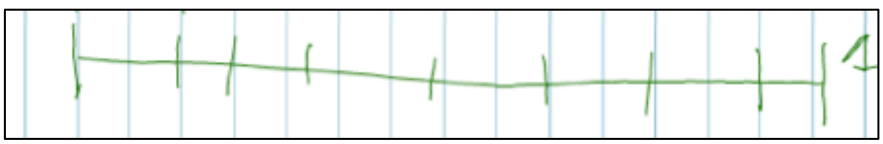


Figure 49. Kristina Struggled to Partition a Line into Eight Equal Pieces.

Instead of cutting the line into half, and then half again to help her find equal sized pieces Kristina created the segments one by one from left to right. I asked her to try again and to make the pieces all the same size and again she struggled and repeated the partition in the same way. Kristina knew the pieces should be equally sized, but did not have a good strategy to create equally sized pieces. I showed her how to make eight equal pieces by thinking of 1/8 as 1/2 of 1/2 of 1/2. She said, “that is a good way to think about it, but I never would of thought of it as more fractions of more fractions. But starting at one half and then reasoning half of that, half of that, that makes sense.” Although Kristina was able to cut segments into one half repeatedly when prompted, she did not anticipate how partitioning a partition could serve her goals of equi-partitioning a segment into eight pieces.

I repeatedly asked Kristina what 1/5 of 1/8 of a meter was. She was unable to answer even after numerous prompts. Eventually, I asked her to show me 1/40 on one of the diagrams. She immediately recognized that 40 was divisible by 5 and 8. I am including the majority of our lengthy discussion about 1/40 of one meter to convey how truly difficult it was for Kristina to coordinate her multiplication facts with her partitioning schemes and to reason with three levels of units.

Excerpt 10. Kristina Tried to Draw 1/40 of One Meter.

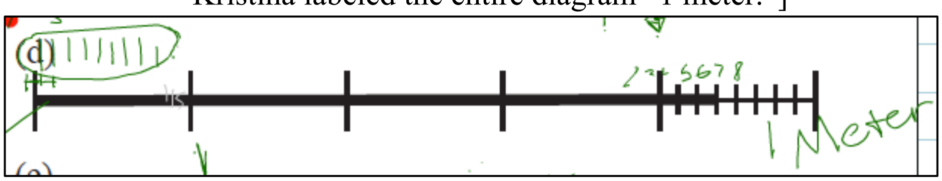
Interviewer: We are trying to figure out the size of a piece where 40 copies of it would make one meter.

Kristina: [long pause] Umm…I don't know. I don't know how to do that.

I: Could you draw a tenth of one meter?

K: A tenth of a meter.
I: So draw a piece so that ten copies of it would make one meter.

K: So you would just draw ten pieces.[Kristina draws 1/10 of 1/5 below. Note that Kristina labeled the entire diagram "1 meter."]



I: So that little piece you drew. If you made ten copies how much would you get.

K: Uhh. One fifth of ten.

I: Uhh… so this little piece. It is one tenth of something because you cut something into ten equal pieces. If you made ten copies of it like you did, what is it equal to, what is this whole?

K: One fifth.

I: It is one tenth of one fifth. I want you to draw one tenth of one meter. You drew one tenth of one fifth of one meter.

Excerpt 10 shows that Kristina was unsure about how to draw 1/40 of a meter, even though we had just had numerous conversations about the meaning of a fraction and drew 1/8 of a line segment and identified 1/5 of a meter. I asked her to draw 1/10 of a meter and she ignored the size of the whole and used a part-whole meaning to focus exclusively on the ten objects she cut up 1/5 into and not the size of the whole. Kristina's way of operating is consistent with her descriptions of one-tenth as one out of ten things. In Excerpt 10 I tried to help her construct a meaning of 1/10 of a meter as the size of a piece that when iterated 10 times makes a meter. Excerpt 11 continues the conversation from Excerpt 10.

Excerpt 11. Kristina Tried to Draw 1/10 of a Meter Again.

> Kristina: Okay. This is already five, so if I just break it down like that [cuts each fifth into two pieces]… is it in ten things now?
>
> Interviewer: Good job! How did you know how to break it up like that?
>
> K: Because it is already five. Because if I add… I don't know. I just knew.
>
> I: Does it have to do with five times two?
>
> K: Yeah.
>
> I: Like, if every fifth was in half you would end up with ten pieces overall. You are kind of using multiplication facts inside your brain. Like over here [I point to our attempts to draw 1/8], when I was thinking of drawing one eighth, an eighth is one fourth of one half. Four times two is eight. I was using my multiplication facts in my brain to make my pretty drawing over here.
>
> K: Yeah.
>
> I: Now this one is harder, but you can still use multiplication facts. We are kind of running out of things to draw on! [laugh] One fortieth of one meter. Try to draw one fortieth of one meter.
>
> K: [Cuts the tenths into halves to make 1/20 of a meter.]

I: You are breaking the tenths into halves.

K: Yeah.

I: So the little piece you just drew [I point to 1/20 of a meter.]. What fraction of the whole is that little piece?

K: One fortieth. Or….

I: If it was one fortieth and you counted them up how many would you find?

K: [She pauses to count the twenty pieces.] Oh, that is only twenty.

In Excerpt 11 Kristina figured out that if she partitioned 1/5 of one meter into two equal pieces she made 1/10 of a meter. She could not explain her reasoning so I tried to help her connect her partitioning activity to her knowledge of multiplication, namely, that ten is fives copies of size two. Additionally, I gave a second example of how I coordinated my multiplication facts with partitioning by reminding her to think of 1/8 as 1/2 of 1/2 of 1/2. Despite this, Kristina still believed that 1/2 of 1/10 is 1/40. She had to physically count 20 pieces to determine that 1/2 of 1/10 is 1/20. Her physical counting suggests she doesn't easily imagine cutting ten objects in half. I share the remainder of the conversation about drawing 1/40 of a meter in Excerpt 12 as evidence of how truly difficult it was for Kristina to coordinate her multiplication knowledge with her partitioning schemes.

Excerpt 12. Kristina Continued to Struggle to Draw 1/40 of One Meter.

Interviewer: So say you have ten cookies for a party and you cut them all in half. How many servings of cookies would you have now?

Kristina: Five. You have ten in half?

I: You have ten cookies. And then some more people come. More than ten. You want to have an equal amount for all of them so you decide to cut each cookie in half. How many pieces do you have?

K: Twenty.

I: So if you cut your ten equal pieces in half, how many pieces would you have?

K: Twenty.

I: And that is how many pieces you counted right?

K: Yeah.

I: So one twentieth is closer to one fortieth. But I still want you to show me what one fortieth of one meter looks like.

K: I don't know how to break it down any more. [pause] I could like cut this one in half.

I: If you cut each one of your twentieths in half, how many pieces would you end up with?

K: Forty?

I: If you took this little piece and made forty copies of it how much would you get?

K: One.

I: One meter.

Finally, Kristina, is able to recognize 1/40 of a meter on the diagram but she has yet to construct 1/40 as equivalent to 1/8 of 1/5. The conversation continues for some time with many more issues until we eventually decide how to model 4/5-1/8 on the diagram. Kristina's difficulties with fractions on this item, and in many other tutoring interactions, suggest that her partitioning schemes were weakly connected to her knowledge of multiplication and counting. Kristina showed no sign throughout our tutoring sessions of having developed the ability to reason with three levels of units despite my extensive attempts to help her do so.

Kristina's Experiences in Calculus

Kristina's difficulties with fraction, measure, and rate and her tendency to reason computationally instead of quantitatively substantially impacted her understanding of calculus. The following is an abbreviated list of rate of change situations that Kristina struggled to understand because of her fraction knowledge.

- She did not understand that if a car travels at a constant rate it travels 1/4 of its total distance in 1/4 of its total time. This meaning for constant rate of change requires imagining proportional partitions of two quantities.
- She thought that if A and B both approach zero the quotient A/B must approach zero. Her image for quotient did not entail comparisons of relative size and this caused problems when she considered tiny changes in x and y.
- Kristina's difficulty simplifying expressions made her much more likely to use the quotient rule when it was unnecessary (e.g. $f(x)=\frac{100x}{20}$), which led to even more complicated expressions. Her weak images of partitioning made simplification slow and difficult. For example, she needed a ruler to enact a physical measurement process to decide how many copies of two make five. It took her 13 seconds to decide that there are 5 copies of 20 in 100. Despite these difficulties she earned an A on a derivative rule mastery test that did not require students to simplify answers.

- Even though Kristina could easily recite procedures for optimization and related rates problems, it was difficult for Kristina to imagine how the quantities changed together which made it difficult for her to envision and model situations that involved rates of change or related rates. She tried to solve a “bucket problem” that involved imagining a bucket filling with water and imagining the relationship between the volume of water and the height of the water. On a bucket problem she could tell the height of water increased when the volume of water increased. However she could not imagine the values of the changes in volume or height. She struggled to use a table to help her understand the situation because of issues with finding the numerical values in the table. For example I told her that in 1/2 minute they would add half of 2.5 gallons to a bucket but she could not visualize 1/2 of 2.5 on a number line or produce the numerical value.
- She said repeatedly that if the amount of something is increasing, then the rate of change of that amount is increasing as well. I used examples of graphs and actual physical motion to help her see that an increasing amount did not imply an increasing rate of change. However, on tests and quizzes she reverted to her original incorrect assumption.
- When Kristina took derivatives of functions such as $f(x)=\pi^6$ she used the power rule to get $f'(x)=6\pi^5$. I graphed $f(x)=\pi^6$ and she said she could not find the rate of change of that constant function. When I asked her for her meaning of rate of change she wrote down “rate of change is distance something traveled in set amount of time” but this “distance something traveled” meaning

did not help her see the rate of change of a constant function is zero because she did not focus on the zero change in *f(x)* for any change in x.

- She tried to differentiate a function then integrate the result to check that she had carried out the procedures appropriately. She could not determine whether the following two expressions were equivalent: $\frac{3}{2}\frac{x^{1.5}}{1.5}+C=x^{3/2}+4$. She could not compute 3/2 divided by 1.5 or place 3/2 on a number line.

Taken together these instances demonstrate Kristina's weak fraction schemes were one of the many reasons she struggled to construct graphs of rate of change functions, apply derivatives to solves problems, and carry out procedures involving differentiation and integration. Kristina's procedural orientation to mathematics, including her procedural orientation to fractions was a significant obstacle in this conceptual calculus class. Further her understanding of graphs made it very hard for Kristina to even draw an appropriate representation of Δx and Δy to enact a measurement process to estimate the rate of change of a graph. The case of Kristina demonstrates that weak fraction schemes can prevent students from making sense of calculus, but also that weak fraction schemes probably also come along with a host of other problems that are also relevant.

RESULTS: THE CASE OF JANET

This results section focuses on Janet's fraction reasoning and its implications for Janet's difficulties in calculus as she attempted to make sense of the rate of change of a sine function. This section is important because it shows how a student who has a weak understanding of fractions struggles when trying to make sense of a conceptual lesson on rate of change.

Who is Janet?

Janet was a curious and motivated student. At the time of the interviews she wanted to triple major in anthropology, biology, and psychology. She took calculus because she had heard that calculus was crucial for being able to model the social and political world quantitatively. She said, “I wanted to get into math to model these large scale behaviors.” For example, she was interested in the relationship between the amount of parental investment in childrearing and the gender differential of people in power. Unlike some students who wanted to learn traditionally, Janet liked Dr. M’s conceptual approach to teaching calculus. She realized during the course that in prior math classes she often memorized procedures without understanding, and that calculus would not help her model the real world unless she understood the concepts.

Janet was the only student out of 153 who asked to come to my office and discuss her results on the Pretest to find out how to improve. In our first discussion we realized that her algorithm for long division was incorrect and she was astonished that her mathematics teachers had not noticed that she did not know how to divide. She also realized that she did not understand trigonometry even though she earned an A in precalculus and she agreed to work on areas of weakness by reviewing the Pathways Precalculus curriculum (Carlson, Oehrtman, & Moore, 2010).

One of the most notable aspects of Janet’s interviews was her ability to reflect on her own thinking. She was curious about how people learned and presented a poster entitled “The Role of Introspection in Children’s Theory of Mind Development” at a national Cognitive Development Conference. She wanted to know how Piaget’s ideas had been applied to mathematics education. She was confident in her thinking overall, but knew she had struggled more in mathematics classes than other subjects. Janet laughed about the anguish she felt when interviewed about what she described as second-grade math. “It is the worst! I’m doing well on the tests and I am doing well in the class but if you slightly lift up the veil it literally is terrible. How am I able to get A’s and really high B’s on these tests when I fundamentally don’t understand what a fraction is? [laughs] That blows my mind.” Her scores of 82% on both Test 1 and Test 2 were substantially above the class averages of 64% and 54% respectively and were curved to be an A and a high B.

Janet described her mathematical thinking as disastrous. The interviews helped her realize that her small mistakes actually were symptoms of profound difficulties with fundamental mathematical issues. She said, “I hate it when I do something and think it is [just] a stupid mistake, and then it unveils this whole hidden problem.” She came to realize that there were giant gaps in her elementary mathematical knowledge and that it was indicative of a major cultural problem that she had passed so many mathematics classes with no one noticing them.

Janet’s Meanings for Fractions

Janet’s feelings about fractions. Janet arrived in calculus knowing that fractions had caused her problems in prior math courses. She recalled “thinking that fractions were not a big deal, but then so many times I heard that fractions were scary and that we need to get rid of them, that I became scared.” I suspect her teachers “got rid of fractions” by either converting them to decimals or by multiplying both sides of an equation by the least common multiple of numbers in the fractions’ denominators. She did not feel that her high school mathematics experiences helped her improve her meanings for fractions because “the standard thing was that teachers gave us fractions and the first step was to get rid of them so we didn’t have to think about them.” She explained that when she looks at a fraction she “literally just see[s] those as symbols. It is literally not representing anything.”

Placing fractions on a number line. Janet’s original Pretest response to *Fractions on a Number Line* is shown in Figure 50. She discussed the item with me informally the week after the Pretest. She agreed to be part of the study and we recorded our second conversation about the Pretest a few weeks after the first. I was concerned that since we had already discussed her mistakes in depth, the second discussion would not capture her difficulties with fractions but this was not the case.

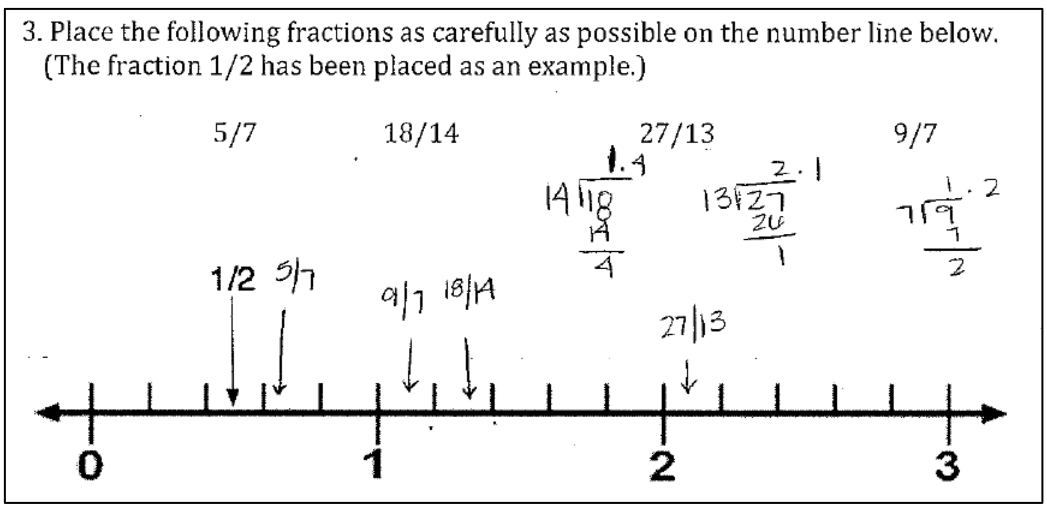


Figure 50. Janet's Pretest Response to *Fractions on a Number Line*. She Found that 9/7 was not Equal to 18/14 because of Misrepresenting the Remainder as a Decimal.

Janet noted that seven was bigger than five so 5/7 should be less than one but struggled to find a more precise estimate. With my help Janet saw she needed to decide whether 4/5 or 5/7 was larger to make a better estimate. Her intuition was that 5/7 is smaller than 4/5 but she did not know why. Janet decided to draw an image of 4/5 to help her justify her reasoning (Figure 51).

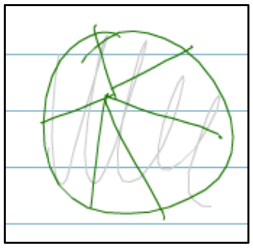


Figure 51. Janet's Pie to Show 4/5. (The Squiggle Marks Show that she Eventually Crossed Out her Image.)

Janet partitioned the pie into six pieces instead of five. Based on her repeated difficulties coordinating the number of cuts made and the number of pieces it was probably not a coincidence that her pie had the wrong number of pieces. Further, if Janet had had a strong image of five when she started drawing partitions it is unlikely that she would have ended up with six pieces. Her difficulty drawing equal sized pieces was not due to carelessness; she explained that she tried to draw the parts equally in Excerpt 13.

Excerpt 13. Discussing Janet's Picture of 4/5 of a Pie.

Interviewer: What is the problem with the pie? How come that isn't working for you?

Janet: Because I can't draw it equally. It is hard for me to visualize it. Cause it is like, you know what I mean?

I: You have a drawing limitation.

J: Right, not uh… yeah.

I: What you'd like is to draw four-fifths perfectly and five-sevenths perfectly.

J: And then be able to visualize.

I: But they are too close to each other to be able to tell from the imperfect drawing.

J: Yes. Right. That is exactly it.

Janet also did not think of repartitioning the pieces into thirty-fifths so that she could compare the sizes of the two fractions until I suggested this explicitly.

Excerpt 14. Janet Tried to Decide if 4/5 or 5/7 is Larger using a Common Denominator Approach.

Interviewer: [I suggested that she try multiplying the fractions by "one" to compare their sizes. I gave the example that 2/2 and 3/3 were equivalent to one.]

Janet: Oh, I see what you are doing, the common denominator… wouldn't it just be thirty five? I guess I can't. I don't remember how to get a common denominator with a multiplication problem. Usually you do it with subtraction.

I: This isn't a multiplication problem. We are trying to decide which of two fractions is bigger.

J: So we…[writes down 28/35 and 25/35.] Okay. So then four fifths is bigger. I multiplied them both by umm …what would give me the same denominator.

I: Okay.

J: So, four-fifths I divided by seven (I'm sorry multiplied) and twenty five over… I multiplied that one by five. I guess that wouldn't make them the same thing because I didn't multiply them by the same thing. I guess that wouldn't work.

I: You are worried that because you multiplied one fraction by five and the other fraction by seven you messed something up.

J: Yeah.

Had Janet had a mature partitioning scheme, it would have occurred to her to find common denominators by repartitioning the fractions. Further, when I suggested she multiply fractions by one, she did so, then doubted herself because she did not understand the connection between multiplying by 5/5 and partitioning a partition. She did not focus on the difference between multiplying by 5/5 and 7/7 rather than 5 and 7. She described multiplying by 5 instead of 5/5 and was not bothered when the interviewer intentionally mimicked her language multiple times. It is clear she did not understand the reasoning behind this procedure because she thought she did something wrong because she did not "do the same thing to both sides."

What is 1/8 of 1/5? The discussion of *Fraction of Cloth* showed Janet's difficulties with coordinating three levels of units in service of a goal. As part of the interview (Excerpt 15) I directed Janet's attention to the size of a "baby piece." A "baby piece" was 1/5 of 1/8 of a meter (see Figure 48).

Excerpt 15. Janet Tried to Identify the Size of a "Baby Piece" that is 1/40 of a Meter.

Janet: It would be one-eighth, right? One of these mini-pieces.

Interviewer: One-eighth of what?

J: [pause] One-eighth of one meter?… I guess?

I: You don't seem happy.

J: I don't…

I: You don't like a baby piece to be one-eighth of one meter.

J: No. Is that right? I guess I shouldn't ask if it is right.

I: It's all good.

I: So we are trying to decide on a name for this little guy. One possibility is one-eighth of one meter. What does one-eighth mean to you?

J: I don't think it is one-eighth because one-eighth of one meter that's saying this thing [points to 1/5 of one meter] is a meter and it is not.

J: Right.

I: But you don't know how to use one-eighth of one-fifth to figure out which fraction of a meter the little baby piece is?

J: Right. Wouldn't that be…it would be subtraction, right? So it is one-eighth of one-fifth. You have one-fifth so you'd have to subtract one-eighth from that.

Janet did not realize that there would be forty "baby pieces" in the entire diagram of choice (b), which would imply that the baby piece was 1/40 of one meter. Janet also struggled to find solutions to three problems of the form "find 1/A of 1/B" during our three interviews.

Janet's Difficulties with Measure.

In addition to difficulties with fractions, Janet did not distinguish a measure of length from a measure of volume. For example, one Test 3 problem would have been easy for her to answer had she noticed that cubic meters was a measure of volume. The question asked whether *dV/dt* or *dh/dt* was an appropriate representation of the rate of change of oil pouring into a tank at *M* cubic meters per second. *V* was defined as volume and *h* as height.

Excerpt 16. Janet Discussed her Meaning for Cubic Meters.

Janet: So what we wanted was the volume with respect to time? Is that correct? M was height. M represents height.

Interviewer: I don't think M represents height. It is a constant rate of M cubic meters per second.

J: I spent a long time on this one too, logically trying to sort through it.

I: what does cubic meters per second mean to you?

J: I don't know what that means. I know that is bad. I don't know what a cubic meter means versus meters per second.

I: So the phrase cubic meters per second did not help you at all.

J: No.

C: When it says oil pours into circular top of tank…

J: I knew it would have to do with volume because of what I've done in the past.

After I explained the difference between meters and cubic meters, Janet reported having no memory of learning the distinction between cubic units and linear units in school.

Janet's Construction of the Rate of Change of the Sine Function

Janet's difficulties with fractions and measure substantially impeded her ability to make sense of the sine function and the rate of change of the sine function. I report other mathematical issues that also caused problems briefly to bolster the point that many schemes come into play when students are trying to make sense of something as complicated as the rate of change of the sine function.

To begin the interview, I acted out the situation in Figure 52 with a ball on a string as well as showing Janet an animated version of the diagram.

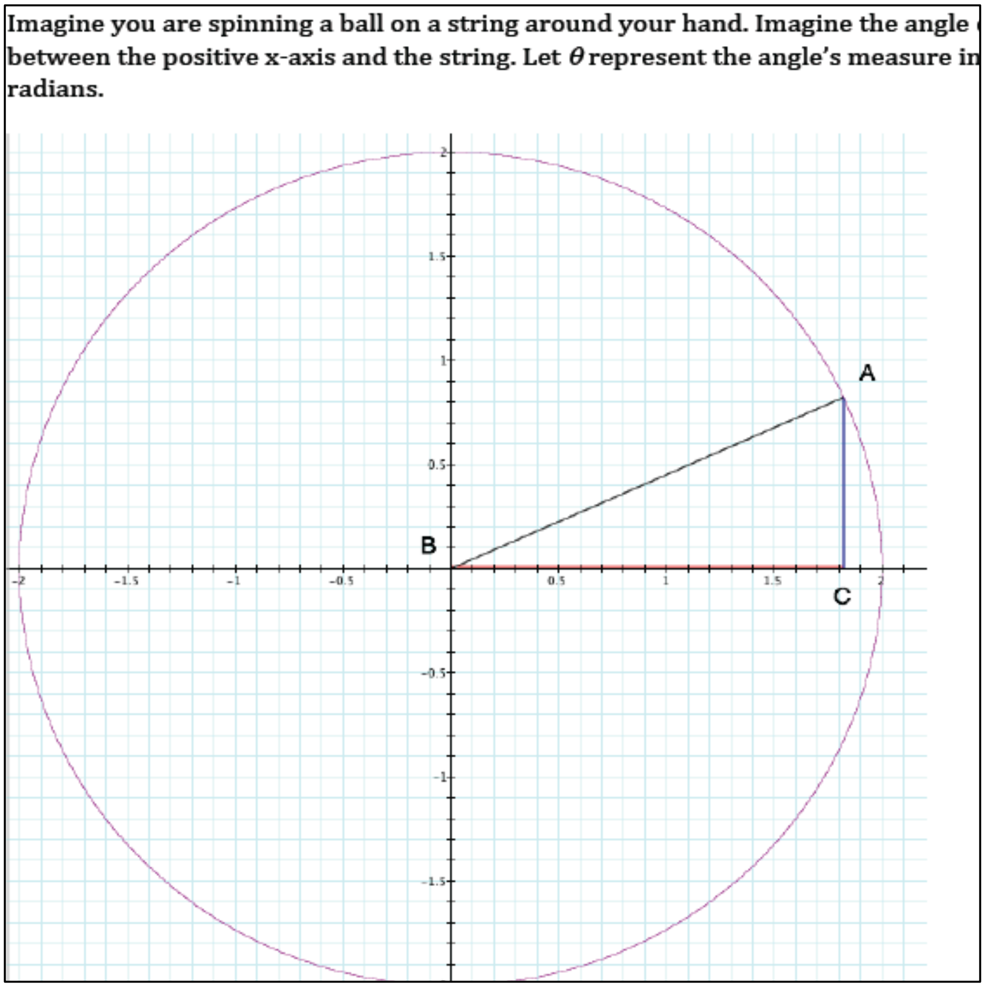


Figure 52. Image Given to Janet in Exploratory Teaching Interview.

Fraction meanings impede Janet's construction of sine function. Before we were able to discuss rate of change functions, Janet's meanings for fractions substantially impeded her progress in making sense of the sine function. As a first step Janet wanted to label the unit circle. Her struggle to label angles revealed the difficulties caused by her unit coordination schemes.

Excerpt 17. Janet Tried to Determine an Angle having a Measure of $3\pi/2$ Radians.

Interviewer: The entire circle is 2π radians. One-quarter of 2π is $\frac{\pi}{2}$.

Janet: This one is two-thirds π right? [Janet points to 3/4 of 2π.]

I: Three-fourths of 2π.

J: [Writes $\frac{3\pi}{4}$.]

I: No, three-fourths of 2π. So it's three-fourths times 2π.

J: So sixth-eighths?[pause] It wouldn't be times 2π right, cause… it would be three-eighths.

After 4 minutes and 40 seconds of discussion Janet determined that three-fourths of 2π is equal to $\frac{3\pi}{2}$. She did not know multiplication was an appropriate operation to find 3/4 of 2π. Once I told her that multiplication was appropriate she did not know how to simplify 6π/4 to 3π/2. Janet did not think of six-fourths as equivalent to three-halves. This suggests that she did not have an image of how both fractions referred to the same measure. One such image is in Figure 53.

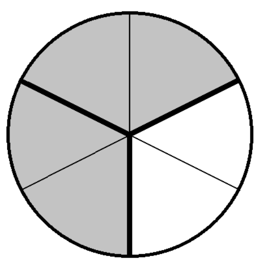


Figure 53. One way of Seeing Six-fourths as Equivalent to Two-thirds.

I suspect Janet's difficulty with seeing 3/2 and 6/4 as equivalent was that it involved coordinating three levels of units. On other problems she was able to reason with a partition of an object and seemed comfortable reasoning with two levels of units.

Next we discussed the covariation of the angle's measure of openness and the ratio AC/AB. The prompt was "as the angle's measure increases from 0 radians to π/2 radians, how does the quotient AC/AB change?" Janet's first instinct was to measure the longer segment in terms of the shorter segment and incorrectly estimated that AC/AB was greater than one. With redirection and substantial support she appropriately estimated AC/AB for several values of θ. Because she did not efficiently estimate ratios, Janet spent 19 minutes creating an image of how the ratio changed as θ varied from zero to $\frac{\pi}{2}$. Janet's difficulty reasoning with improper fractions and problems with unit coordination is consistent with Steffe's hypothesis that coordinating three levels of units is necessary for constructing an improper fraction scheme.

After a discussion about the use of negative signs, Janet described the ratio changing from zero to one to zero to negative one to zero as θ varied from zero to 2π. Her initial drawing of the changes in ratio as θ varied is in Figure 54.

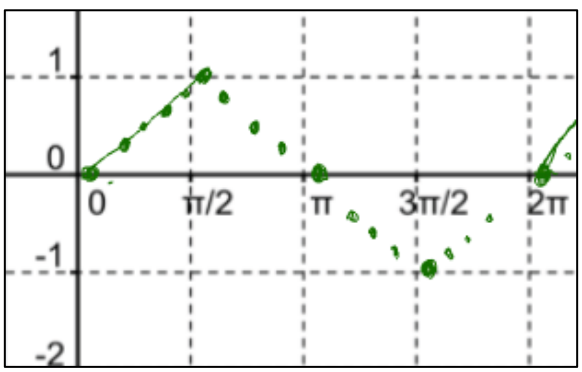


Figure 54. Janet's Initial Image of the Relationship Between the Ratio AC/AB and *θ*. Janet's initial coordination of the angle measure with ratio was insufficient to see that the rate of change of ratio with respect to angle measure was not constant.

Finding the average rate of change of sine. The next question in the protocol was "estimate the rate of change of the ratio AC/AB with respect to the angle measure θ for various values of θ. What could we type into graphing calculator to help us make our estimates?" At this point in the interview I told Janet the function was named sine so she could use Graphing Calculator to estimate values she wanted. Janet's first response focused on comparing the values of outputs of the sine function with subtraction.

> Janet: f(x) = sin (x). Use this to compare various θ ratio outputs to determine rate of change between different θ angles.

To find the rate of change of the ratio AC/AB with respect to θ she typed $\sin(\frac{\pi}{8})-\sin(0)$ into the graphing calculator. In Excerpt 18 I confirmed her meaning for a rate of change was amount of change.

Excerpt 18. Is the Average Rate of Change of the Sine Function $\sin(\frac{\pi}{8})-\sin(0)$?

> Interviewer: We found the difference, how much the ratio changed between 0 and π /8. We found out how much the ratio changed by. Is that what rate of change of ratio with respect to angle measure means to you?
>
> Janet: That is what I'm thinking.
>
> I: So the rate of change of the quantity is how much it changed.
>
> J: Right.

Janet's meaning for rate of change as amount of change is consistent with the hypothesis that students without multiplicative meanings for fractions cope by using additive meanings for rate of change. Janet nevertheless used her additive meaning to determine that the rate of change of sine was not constant, but she did not attend to the need to use equally sized intervals to draw this conclusion.

I told Janet she could not use differences in output to decide if the rate of change is constant because we also needed to consider the changes in input. Janet decided to make equally spaced intervals of angle measure.

Excerpt 19. Janet Tried to make Equally Spaced Fractions.

Janet: Right, after we go to $\frac{\pi}{8}$ we go to $\frac{\pi}{6}$, and then to $\frac{\pi}{4}$.
Interviewer: Are you trying to make the intervals the same?
J: Yeah.
I: [I write down the fractions in Figure 55] So the differences between all these fractions is the same?

J: Yeah.

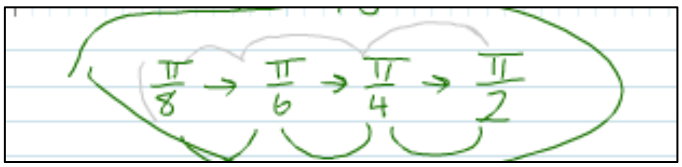


Figure 55. Janet's Attempt to Make Equal Sized Changes in Angle Measure.

After I drew two diagrams of fractions for Janet, she understood that her intervals were not equally sized. Later Janet explained that she was only focused on the denominator of the fraction and the pattern of adding two to the denominator. This data is consistent with the hypothesis that Janet did not think of a fraction as a multiplicative object and tended to reason additively in fraction situations. The combination of failing to make equally spaced intervals and using an "amount added" meaning for rate of change made it almost impossible for Janet to decide where the rate of change increased.

Estimating average rate of change. Next, I suggested to Janet that she divide the change in ratio by the change in angle measure so that she would not have to find equally spaced intervals. She made use of the lines we had already drawn together in Figure 56 to represent changes in the inputs and outputs of the function.

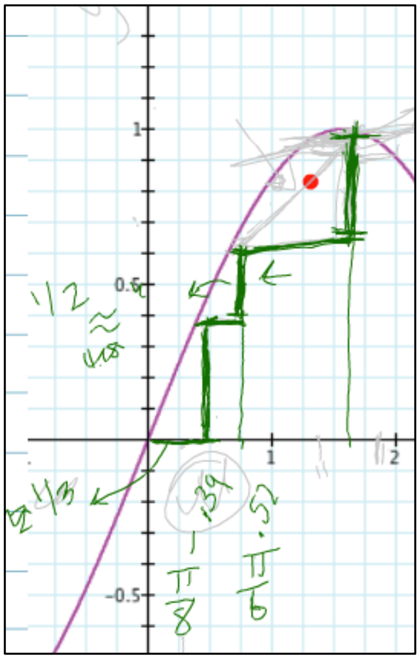


Figure 56. Janet Attempted to Estimate the Average Rate of Change of the Sine Function on Two Intervals.

At first Janet estimated $\Delta y/\Delta x$ by assuming that both axes had the same scales. She estimated the rate of change between 0 and $\frac{\pi}{8}$ to be three, and the rate of change between $\frac{\pi}{8}$ and $\frac{\pi}{6}$ to be two. These estimates were appropriate given the false assumption that the axes were equally scaled and could have help helped her see that the rate of change of the sine function decreased as θ varied from zero to $\frac{\pi}{2}$. However, for reasons that she did not explain, she crossed out two and three and measured Δx in terms of Δy to arrive at the estimates of one-third and one-half.

Excerpt 20. Janet Attempted to Estimate the Average Rate of Change of the Sine Function.

Janet: So the rate of change is more like one third, because *y*[2] is measured by *x*.

Interviewer: Okay.

J: This one is more like one half. And then his one would be like a little, I can't tell if they are the same or if the are… again just because we drew them on there.

Even though Janet had adopted the measurement meaning for quotient that her professor used, she was not able to estimate slope in a way that helped her understand how the rate of change varied. She measured Δx in terms of Δy, meaning she estimated Δx/Δy. She did not notice that the changes in *y* were smaller for subsequent equally sized changes in *x*. She appeared to rely on the accuracy of her drawing and was unable to mentally visualize the changes in *y* becoming smaller for equally sized changes in *x* on the concave down graph.

Later in the interview Janet again measured Δx in terms of Δy, meaning she estimated Δx/Δy when she wanted to estimate the quotient Δy/Δx. Janet knew she frequently measured the longer change in terms of the shorter change and disregarded that by convention Δy is in the numerator and Δx in the denominator (see Excerpt 21).

---

[2] As a sidenote on Excerpt 20, many students said *y* when Δy was appropriate and although I did not investigate the issue in this study, I believe students' reasoning about frames of reference as introduced by Joshua et al. (2015) is also relevant for their construction of rate of change functions.

Excerpt 21. Janet Reflected on Measuring Δx in Terms of Δy when Trying to Find Δy/Δx.

> Janet: I keep mixing up…[In context meaning she estimates Δx/Δy instead of Δy/Δx.]
>
> Interviewer: You mix up which direction you are going.
>
> J: I do.
>
> I: Usually you try to fit the smaller one into the bigger one.
>
> J: That is exactly what it is.

Whether or not Janet produced an estimate of Δy/Δx or Δx/Δy often depdended if Δy or Δx was larger because she strongly preferred measuring a larger change in terms of a smaller one. As in our discussion about the sine function outputs, Janet was more comfortable with fractions where the numerator was smaller than the denominator. Janet was so preoccupied with deciding how to measure that she struggled to make sense of how the measures changed as *x* increased.

Computing average rate of change. We tried to cope with Janet's estimation difficulties by using the graphing calculator to find the average rate of change over the interval 0 to $\pi/8$. Janet typed $f(\sin(\pi/8))/\sin(\pi/8)$ into the graphing calculator even though we had already defined *f(x) = sin(x)*. Difficulties with understanding "sin" as the name of a function prevented her from coping with her weak measurement schemes by using a calculator.

I pointed out her mistake and showed her how to compute the average rate of change. We discussed the meaning of each part of the rate of change function $r_f(x) = \dfrac{f(x+h) - f(x)}{h}$ where *f(x) = sin(x)*. Her instructor frequently used "*r*" to define a rate of change function to remind students of the meaning of derivative. Janet explained that h represented a change in x values. She saw $f(x+h)$ as the final value of y after the x values changed by h. She saw $f(x)$ as an initial value of y, and $f(x+h)-f(x)$ as the change in the value of y. Janet estimated values of the rate of change of the function on the graph and compared her estimates to the computed outputs of the rate of change function. We used the calculator to check many of Janet's estimates of average rate of change. She appeared to be making appropriate estimates of $r_f(x)$ at various values of x so we moved on to the next task: graphing the function y = $r_f(x)$.

Janet's dislike for small intervals. Even though Janet had just successfully estimated the rate of change of sine on a small interval around $\frac{\pi}{2}$ she forgot the estimate when she tried to graph $y = r_f(x)$. Her new estimate was $r_f(\frac{\pi}{2}) = 1$. It was not visually obvious to Janet that the value of $r_f(x)$ is essentially zero on a small interval around $x = \frac{\pi}{2}$.

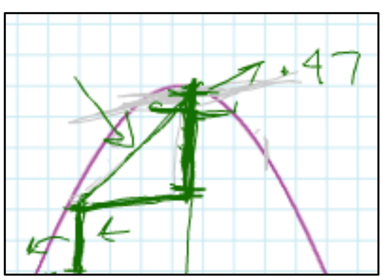


Figure 57. The Calculus Triangle Janet used to Estimate the Rate of Change of Sine at $\frac{\pi}{2}$.

Janet used a large interval to estimate the rate of change and again treated the *x* and *y* axis as if they were identically scaled.

Excerpt 22. Janet did not Like Small Intervals.

Interviewer: You said before you did not like small intervals.

Janet: Yeah. [We laugh.]

I: Is that stressing you out right now?

J: Uhh... I think it just makes it hard for me to conceptualize.

I: What is it about the small intervals that…? Can you articulate what you don't like about them? Maybe if you could articulate it you would like them!

J: [we laugh] It is that I… I can't … I can't visually create them. Because they are so small. By definition they shouldn't really be large enough that I can actually draw them. Does that make sense?

I: So…are the intervals we made right here for *h*, do those bug you? [I point to small, but visible intervals.]

J: No… but I don't know what they are I guess. They don't have values assigned, so I can't…so if you ask me to compute…We are saying that this is similar to the size of these intervals. I don't know what these are [points to Δx intervals], so I don't know the size of this [points to associated Δy intervals].

Janet was able to measure one length in terms of another if she could physically carry out the measurement process. Her images for measurement were not strong enough to support measurement when the lengths were too small to easily compare visually. I took this as evidence that her measurement scheme was not stable enough for her to carry it out in her mind without carrying out the activity of measurement. In Steffe's (2010) studies he also attributed more mature schemes to students who could carry out mathematical activities even if the objects they were counting were not in the visual field.

Impact of Janet's fraction schemes on her construction rate of change of sine. At the end of an hour and a half discussing the sine function, Janet had finally constructed a reasonable graph of the rate of change of sine and reflected on how the graph was related to the original rotating ball on a string situation. I suspect that without my frequent interventions to address her difficulties with fraction, measure, and rate, Janet would not have produced a graph of the rate of change of sine that day. To summarize, these are difficulties that Janet faced in envisioning the sine function and the rate of change of the sine function. I did not discuss evidence for points 4, 10, and 11 in this paper.

1. Finding 3/4 of 2π .
2. Visualizing how the sine ratio changed as θ increased.
3. Estimating a rate of change with a difference instead of a quotient.
4. Estimating rate of change by comparing two changes in y.
5. Trying and failing to make equally sized changes in x because she did not know how to write a sequence of equally spaced fractions.
6. Estimating a rate of change by finding Δx/Δy instead of Δy/Δx.
7. Struggling to see that a concave down function has a decreasing rate of change.
8. Inappropriate use of function notation to compute an average rate of change.
9. Estimating the rate of change of sine at π/2 as one because she drew too large of an interval.
10. Not differentiating between negative and positive changes.
11. Thinking that if two slopes appear to have the same steepness they are the same slope, even if one is negative and the other positive.

The evidence suggests the many foundational issues Janet faced en route to constructing the sine function made it much harder for her to focus on the meaning of a rate of change function. This is consistent with her poor performance on rate of change function items on Test 3.

DISCUSSION

I will use Tzur's (2014) discussion of Steffe's fraction constructs to organize the discussion of Janet's and Kristina's fraction schemes. Tzur (2014) summarized eight of the fraction schemes identified in Steffe's research and ordered them based on children's typical development. For example, a student learns to equally partition a line segment before they learn to partition a line and iterate the unit produced by the partition. Table 27 shows Tzur's (2014) summary of four of the eight levels of fraction schemes and the evidence that suggests Janet and Kristina struggled to operate at that level.

Table 27. Janet and Kristina's Fraction Reasoning and Steffe's Fraction Schemes.

Fraction Schemes (Tzur, 2014)	**Evidence concerning Janet and Kristina's reasoning with each fraction scheme.**
1. Equi-Partitioning: Using her concept of number as a template for a partitioning operation, a child can disembed a part and may anticipate that iterating that part would confirm if it is (or not) an equal share for N people (2-level unit coordination).	**Janet**: Struggled to cut a circle into five equal pieces and complained that it was difficult. Initially thought 1/8 of 1/5 of a meter was 1/8 of a meter, but was troubled by the fact that 8 copies of the piece did not make a meter. **Kristina:** Struggled to cut a line segment into eight equal pieces. Tried to make a piece such that ten copies of it would make one meter. She cut 1/5 of a meter into ten equal pieces producing 1/50 instead of 1/10

	and did not notice the problem.
4. **Iterative Fraction**: Iterable unit fraction (e.g. 1/7) resulting from partitioning is 'freed' from the whole; it can be disembedded and iterated as a 'thing'. When coupled with operations that produce three levels of unit coordination, the child can anticipate composing it with the whole (e.g. 7/7) to produce, say, 8/7 or 12/7 or 14/7 as two wholes. The child is aware the composed unit is also a potential result of iterating the unit fraction so many times (i.e., 8/7 = 8*1/7). For the child, then, any fraction m/n is an anticipated result of interating m units of 1/n. This 3 level unit coordination in regards to 1/n allows using the Iterative Fractions Scheme (IFS) to operate on an extensive quantitative unknown, but not yet on an extensive quantitative variable.	**Janet**: When estimating Δy/Δx Janet typically measured the larger quantity in terms of the smaller quantity, instead of measuring the numerator in terms of the denominator. She seemed to make mistakes while subconsciously avoiding improper fractions. **Kristina:** In the "Two Pizza" situation Kristina did not see two pizzas as 8 copies of 1/4 of one pizza. She did not see 1/4 of one pizza as 1/8 of two pizzas.
5. **Recursive Partitioning**: Operating mentally to partition a unit fraction (e.g. 1/4 of 1/7) allows the child to anticipate its result as if the second partitioning would have been applied (without/before enacting) to each and every part of the first partition (e.g., 1/4 of a single 1/7 would be as if each 1/7 is partitioned into 4/4 parts and thus could potentially result and thus could potentially result in partitioning the original 7/7 whole into 28/28 segments, making the 1/4 of a part (1/7) to be 1/28 of the whole (which marks this scheme as an anticipatory 3-level unit coordination that is multiplicative.	**Janet**: She computes 1/5-1/8 to find 1/8 of 1/5. **Kristina:** She did not anticipate that cutting a length in half three times produces 1/8 of the whole. She did not anticipate that cutting 1/5 into 8 equal pieces would produce 1/40 even though she knew 5 times 8 equals 40. She thought that 1/2 of 1/10 was 1/40. She had to physically cut all ten copies of 1/10 into two parts and then count the 20 parts to determine that 1/2 of 1/10 is 1/20.
8. **Any Fraction Composition**: Recursive partitioning (e.g., find 1/7 of …) can be applied, in anticipation, to a composed fractional unit (e.g. 5/9). The child's situation includes reversing the iterative	**Janet:** She doesn't see 6/8 as equivalent to 3/4. Understanding the equivalence would involve partitioning each 1/4 in 3/4 into two equal pieces to see 6/8 of the whole.

fraction scheme for composing 5*1/9 and disembedding these 5 pieces, then splitting each of them into the given number of mini-parts (e.g. 1/7 of each 1/9) and then composing the sought after result by integrating/iterating these (1/63rds) five times.	Janet did not think to repartition 5/7 and 4/5 into thirty-fifths to decide which fraction was bigger. **Kristina**: We did not discuss fraction situations this difficult because Kristina had not constructed this fraction scheme.

The evidence in Table 27 shows that it is unlikely either Janet or Kristina had developed multiplicative fraction or measure schemes or learned to reason with three levels of units by the time they enrolled in calculus. I initially hypothesized that students with primarily additive, part-whole meanings for fractions would struggle to understand rate of change functions. In particular, I hypothesized that constructing a rate of change function given the graph of an original function would be difficult without the ability to estimate rates of change from a graph. These hypotheses were confirmed in Janet and Kristina's sessions, but the discussions also demonstrated that their difficulties for fractions caused a wide variety of other problems that I had not anticipated. Kristina's problems were so deep-seated and hard to resolve that even with intensive effort on her part, on my part, and on her teacher's part she failed the redesigned calculus course. It is highly possible based on her A on the derivative shortcut test and ability to recall and apply procedures that she would have passed the majority of calculus courses in the country that primarily test students' ability to recall and apply procedures(Tallman, Carlson, Bressoud, & Pearson, 2016).

Janet's issues with fractions caused frustrated her greatly, but did not keep her from passing the redesigned calculus. One of the reasons Janet's grades on the first two tests were so far above the class averages was that she was good at remembering conceptual explanations for ideas. Even though she was struggling to deeply understand constant rate of change she read the course notes carefully and learned to appropriately use verbal descriptions of major concepts on tests that focused on both computations and verbal descriptions of the meanings of various mathematical concepts. Janet's construction of a sine function and its rate of change function was frustratingly slow because she was frequently held back by her limited ability to anticipate a measure of Δy in terms of Δx. She needed to see representations of Δy and Δx on a graph that were large enough for her to physically carrying out a measurement process. This led to difficulty coordinating multiple measures in the construction of a rate of change of sine function. She was restricted to operating on changes represented visually on the graph and it was non-trivial for her to draw appropriate representations of changes or draw appropriately-sized intervals. It appeared that her measurement schemes were not strong enough to imagine any two quantities, especially small quantities, being compared multiplicatively. The time it took her to estimate a ratio made it difficult for her to coordinate the values of a multitude of ratios with changes in the input variable. This prevented her from efficiently relating a function to its rate of change function.

Janet demonstrated many times that she could measure two lengths given two line segments, but that she did not always anticipate how a measure would be useful without prompting. Tzur (2014) argued that the strength of a student's scheme is partially determined by the degree to which the student can anticipate the need for the outcome of a scheme before acting. Tzur (2014) quoted Piaget: "success is not preceded by trial-and-error and is not a matter of luck but is assured by operational anticipation" (Piaget, Inhelder, & Szeminska, 1960, p. 319). Being able to anticipate the need to measure to solve a problem is a necessary to reason successfully about the relationship between a function and the graph of its rate of change function. Tzur (2014) noted first students learn to anticipate that certain effects will ensue after they carry about a goal-directed behavior. However for some time the student will not be able to predict the effect of carrying out the scheme without executing the activity (p. 176).

Tzur (2014) also pointed out that some students can carry out a scheme when prompted to by someone such as a teacher, but they do not independently anticipate the need to use a particular scheme (p. 178). For example, it did not occur to Janet to find a common denominator by repartitioning a fraction when trying to decide if 4/5 or 5/7 was larger. Kristina did not think of 1/8 as 1/2 of 1/2 of 1/2 but could see it that way after I prompted her. Both Kristina's and Janet's struggles to draw equally sized pieces suggests they lack a vision for a partition of a whole. I believe their difficulty imagining a partition without carrying it out made it less likely for them to spontaneously partition a partition in service of a goal.

Tzur (2014) said schemes can progress to a stable anticipatory stage, "in which a learner consistently calls upon and meaningfully uses the new scheme without prompting" (p. 177). Janet and Kristina did not "call upon and meaningfully use" their fraction and measure schemes to make sense of rate of change functions. I hypothesize that one of the reasons Janet passed and Kristina failed is that with extensive prompting Janet did make use of the measurement schemes she had available to make sense of rate of change functions. With my prompting and support, Janet was able to produce a collection of measures for different values of x and eventually construct a graph of the rate of change of sine. I believe that because Janet's measurement and fraction schemes did not allow her to anticipate how she would use them in service of a complex goal; she needed my support to make progress.

CONCLUSION

The results of the quantitative study of Janet's Spring 2015 calculus class (Byerley, submitted) strongly suggest that Janet's difficulties should not be dismissed as isolated issues only impacting a few calculus students. It is clear that many students' meanings for fraction, measure, and rate of change are weak enough to impede them from making sense of rate of change functions. Janet was one of the 153 students who took the Pretest for the quantitative study conducted in two Spring, 2015 calculus courses (Byerley, submitted). Her Pretest score was at the 34th percentile of the 115 students who stayed in calculus the entire semester. Her scores on the first two tests were above the class average and her scores on the last two were slightly below the class average.

Table 28. Janet and Kristina's Test Scores

	Pretest	**Test 1**	**Test 2**	**Test 3**	**Final**
Janet	33%	82%	82%	50%	65%
Kristina	Not given in Fall 2014	53%	74%	36%	44%
Spring 2015 Mean Score	45%	64%	54%	60%	59%

Kristina did not take the Pretest and her test scores suggest that she is a below average calculus student. However, with a B in college-level precalculus Kristina is significantly more mathematically successful than many college mathematics students and her extreme issues with coordinating three levels of units might reflect common difficulties in students who are struggling to pass remedial mathematics, precalculus, and calculus.

Byerley (submitted) reported results for 153 students on *Fraction of Cloth* that suggest many students struggled to reason about a partition of a partition. Forty-one of 153 (26.7%) students who took the Pretest answered the question *Fraction of Cloth* correctly. Five out of six students interviewed about this item struggled to find 1/5 of 1/8 (Byerley, submitted). Although one item cannot be used to assess students' ability to coordinate multiple levels of units reliably, the dismal scores suggest calculus students' unit coordination schemes are worthy of attention. If future research reveals that a substantial portion of calculus students struggle to coordinate three levels of units, it will have profound implications for what meanings they are prepared to construct when they enter calculus.

The calculus students in (Byerley, submitted) also showed weak measurement schemes in addition to problems with partitioning. Fifty-six percent of students who remained in calculus until the end correctly estimated the slope of a linear function from a blank graph with equally spaced axis. They were given four possible values of the slope and only one was remotely reasonable. Repeated failed attempts to teach Kristina to estimate slope shed some light on why the item was so difficult for students even though the instructors discussed this very issue in class. Estimating $\Delta y/\Delta x$ from a graph was a statistically significant predictor of success on tests about rate of change functions (Byerley, submitted). This is consistent with the interviews that showed that difficulty producing these estimates substantially impeded Janet and Kristina's understanding of rate of change functions. And as noted in the literature review, students' ability to convert gallons to liters given a conversion factor was shown to be a statistically significant predictor of success on rate of change function items (Byerley, submitted).

There is no simple solution to deal with calculus students' difficulties with coordinating three levels of units and weak measurement schemes. Steffe (2014) reported that the large number of fourth graders who do not coordinate three levels of units when they are asked to learn multiplicative fraction and measurement schemes poses an enormous unsolved problem in elementary mathematics education. He did not know how to efficiently help elementary students learn to coordinate three levels of units, and as a result, struggled to help many students construct multiplicative fraction schemes in his two to three year teaching experiments (Steffe, 2014, p. 38). After decades of study Steffe found no method that worked reliably to help another person imagine a partition of a partition and use this scheme to achieve a goal.

These observations about the deep-roots of calculus students' difficulties have implications for the move towards teaching calculus conceptually. Understanding calculus conceptually is incredibly useful for many students because they are able to see how mathematics is used to describe the world. However, the many students who dislike conceptual instruction may dislike it in part because to them it does not make calculus useful or sensible. Their roadblocks to understanding calculus are so deep-seated that they are unable to construct meaningful mathematical understandings even when their instructors design a lesson with this goal. Many students would rather memorize that the derivative of the three letters "sin" is the three letters "cos" than struggle through a frustrating hour and a half of analysis of a sine function graph like Janet did. I should be clear that I am not suggesting we teach calculus by rote. Although Kristina earned an A on her test of taking the derivatives of complex functions, there is overwhelming evidence that she had no idea what the computations meant or how to apply derivatives to model the world. I am suggesting that it is not enough to encourage teachers to use technology, multiple representations of functions and inquiry methods. Calculus teachers must also understand student thinking and how students' schemes impact the sense they make of instruction.

Further, to increase the chances of students making sense of rate of change functions in calculus, I believe it is critical to focus attention on their experiences in elementary and middle school. Tall's (in press) observations about students' difficulties with algebra might apply even more strongly to students' difficulties with calculus:

> Problematic ideas in algebra may have their origins in early arithmetic and accumulate though successive experiences over the years. If the problematic aspects remain unresolved, the spectrum of difficulties may become so complicated that it may no longer be easy to resolve problems arising in a particular topic because their origins are so deeply embedded in he subconscious mind of the learner (p. 4).

If Tall's observations are accurate then the best time to deal with calculus students' fraction and measure schemes is long before they arrive in calculus.

Elementary and secondary teachers have the best chance of helping students develop their fraction and measure schemes. Unfortunately, there is significant evidence that a nationally representative sample middle school teachers do not have the mature fraction and measurement schemes that we want students to develop (Bradshaw et al., 2014). Bradshaw et. al. investigated teachers' mastery of fraction ideas shown to be critical in many research studies: referent units, partitioning and iterating, multiplicative comparisons, and using multiplication and division appropriately. Only 31% had mastered referent units and 55-63% mastered the other three attributes (Bradshaw, et al., 2014, p. 9). Further, a majority of secondary mathematics teachers with mathematics or mathematics education degrees were unable to convert between gallons and liters given a conversion factor (Byerley & Thompson, 2014). Many of the teachers' striving to prepare students for college, do not have the foundational meanings the students' need to be successful.

One critical step in helping more students understand calculus is to help elementary and secondary teachers develop the mathematical meanings for fractions and measure we find to be critical for calculus. It is not enough to fix calculus by reducing class sizes, using group work, incorporating technology, changing curriculums, opening tutoring centers, or any other strategy for improvement. Although these strategies may help students, to truly make sense of calculus students need the opportunity to learn the ideas calculus is built on first.

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CONCLUSION: REFLECTING ON THE THREE STUDIES

In this conclusion I am not going to rehash the academic conclusions of the three papers. Each of the three papers clearly show that there are serious and hard to resolve issues with teachers' and students' meanings for ideas foundational for secondary and tertiary mathematics. Instead, I'm going to discuss what these results mean to me from the perspective of my experiences as a mathematics teacher. This conclusion has my opinions as a teacher-not scientific results from a study. I think is important to not only consider these results from a scientific perspective, but an emotional one. Teachers' and students' emotions are hugely influential in their decision making processes. It is not my goal here to study affect – only to tell a few stories to help my committee understand from a human perspective just how troubling my studies are to me as a teacher.

Every time I have been emotionally distressed because of the horrific things teachers and calculus students say about fractions, Pat gives me the same advice. To maintain my sanity he suggests I look at the interviews and item responses as "just data". While I wrote the three papers, I put on my researcher hat and calmly reported results like "this high school teacher thinks slope is a distance" or "the majority of math majors studied have not mastered 4^{th} grade measurement standards." With proper academic distance I stated the fact that only one in 115 calculus students answered four questions about fractions correctly. For people trying to teach calculus meaningfully and students wanting to pass calculus, the quantitative studies are depressing. Still, with those studies hope remained that if students reviewed fractions, measure, and rate in calculus it might help them pass.

It was the interviews I found to be the most discouraging. What was most disheartening was not that Kristina and Janet did not know how to reason with fractions; rather it was that even as a person writing a dissertation on fractional reasoning, I could not help them improve over many hours of tutoring. Their problems were too deep-seated and too many layers underneath the content they were currently focused on. From the prospective of a researcher, I am glad I identified that neither student appeared to reason with three levels of units. Steffe has done extensive work on the importance of a student's ability to coordinate three levels of units in order to construct multiplicative fraction and measure schemes. By noticing that Janet and Kristina did not coordinate three levels of units, it gave me a solid hypothesis to explain why repeated attempts at tutoring did not change their meanings for fraction and measure. From the prospective of a teacher, I was not at all happy to notice Janet and Kristina's issues with unit coordination. There is a fairly good chance that Steffe's conclusions about children's fraction schemes being critical to the development of their algebraic qualitative reasoning schemes are correct. I felt hopeless to help some students learn calculus as I understood it.

We need to consider how findings of these studies and others like them inform reform movements to teach calculus conceptually. There is overwhelming evidence that most students do not learn the major ideas of calculus in traditional classrooms. It is clear efforts to reform calculus must continue, however there is a real danger to making modifications to calculus without proper attention to student thinking. A hallmark of Thompson's course that distinguishes it from many calculus reforms is that the teachers explicitly discussed critical ideas of middle school mathematics and assigned related homework. Some students flourished in Thompson's redesigned calculus. They felt that math finally made sense and that they could understand it. However, the time available to reteach middle school math in calculus is scant and was insufficient for many students. Students who succeeded at the redesigned class were able to convey much stronger meanings for derivative and integral in Thompson's pilot study. However, failure and drop out rates were similar for the Thompson curriculum as traditional courses. Because the course was different than traditional calculus, some people found it was easier to incorrectly blame the redesign for those that did fail the course.

Given the central role of calculus for so many students, it is critical to think about the ethical dilemmas of asking so many students to understand ideas they may not be prepared to understand. The problem is not the clarity of ideas presented in the reform course – the issue is that 12 years of reasoning computationally means that not all students have the foundational schemes they need to make sense of the class. Through frequent discussions with students, I saw the high cost of failure. Some lost scholarships, others were at risk of being sent back to their war-torn home countries if their GPA was too low. Others had to take out thousands of dollars of additional student loans to spend a fifth year trying to graduate. Some needed a certain grade to study abroad, continue with their major of choice, or apply for a leadership role in their campus organization. The students who were aware of the high costs of failure worked hard, but they were rarely able to diagnose their difficulties s issues with foundational math and locate appropriate resources. It is clear that the solution to this issue must involve better support to elementary and secondary teachers.

It is also critical that reform calculus courses attend to issues of student thinking instead of just making changes to the format of instruction. I think that student-centered instructional techniques are useful. If you ask a student to respond to a clicker question , to use a whiteboard, or to discuss their ideas in a group, it increases the chances that the teacher notices that the student is misunderstanding the ideas. (Of course, this presupposes that the teacher understands the ideas in the first place.) However, if you ask a teacher to use student-centered instructional techniques as if the technique itself is the solution to the problem, these efforts are doomed to fail. Janet and Kristina were discussing animated visualizations, using hands-on representations, justifying their answers, drawing diagrams, yet still their foundational mathematical meanings held them back from success. Putting Janet and Kristina into discussion groups with whiteboards in calculus class is not going to solve their problems. Showing them representations of rates of change graphically, numerically and algebraically did not work in part because they didn't understand fractions and measure.

We also need to take seriously what we are asking high school math teachers to do from the perspective of the teachers' emotions. To help their students understand high school mathematics they also have to teach students foundational middle school math ideas-many of which the teachers themselves do not understand or are not prepared to teach. Teach for America told me that if I cared enough and worked hard enough and had enough innate teaching skill, my students could successfully learn grade level standards. As a new teacher I utterly failed to help my algebra students understand grade level standards, despite working so hard that I became sick from stress. To me it was an issue of my personal and moral worth to understand what it was possible for my students to do, given what they knew when they got to my class. Teach for America told us that if we set big goals and followed their vision of transformational teaching we could change the students' lives. They gave me almost no training on how students understand mathematics and in hindsight the lack of attention to student thinking made it impossible to set realistic and feasible goals.

My masters program in secondary mathematics education talked repeatedly about issues of execution: group work, technology, and think-pair-share. I have no recollection of discussing how students' conceptual understandings developed from foundational schemes and the state of the typical high school students' foundational schemes. At no point during my masters program did anyone check that I understood the mathematics I was teaching.

When we ask secondary teachers to teach mathematics conceptually and tell them that it will help their students understand more, we are putting them at financial and emotional risk if we do not also teach them about student thinking. Teachers need to understand the consequences of trying to help people make sense of ideas that they are not prepared to learn conceptually so they can actively address the issues that come up.

The gaps in my understanding of student thinking also caused me problems teaching calculus at my second high school. About half of the class was successful with my conceptual instruction because they had the foundational ideas they needed to make sense of it. In hindsight, I think that the other half of the class hated my conceptual instruction because they did not understand the basic ideas of rate of change, function, and algebra they needed to be successful in calculus.

When I tried to teach conceptual calculus at the prestigious private school, I made mistakes partly because of the conceptual calculus curriculum I used. Per instructions, I used the reform textbook that I had no reason not to trust: Hughes-Hallett Calculus. This textbook was filled with interesting problems, but never once mentioned that the teacher should check if the calculus students understood fractions or constant rate or try to investigate if that would hold them back from understanding calculus. In fact, the review chapter was filled with exponential and polynomial functions and other ideas that were important but much more advanced than what my students actually needed to review (or learn for the first time). Further, the second chapter included extensive coverage of epsilon-delta limit definition proofs. I still remember being observed struggling to teach a conceptually-based lesson on the definition of an epsilon-delta limit. The administrator's response to the lesson was that it looked like I tried hard but was not able to communicate with the students and I got in trouble. For teachers like me, who really cared and tried hard, it was heartbreaking to fail and be criticized over and over again. It is critical reform textbooks base the revisions on extensive research on student thinking and experience teaching. I believe one of the reasons teachers decide not to teach conceptually is because of the emotional suffering that frequently occurs when they try it and are not prepared to do it well.

Teachers are not omniscient creatures; they are a product of the educational system and have no way of knowing what they do not know if all of their experiences – including teacher preparation programs – fail to talk about mathematical ideas and only focus on the most outward and visible classroom activity techniques. This system of refusing to train the teacher and then blaming the teacher only means that the ones that really care burn out an leave, truly perplexed as to why they did everything they were asked to do and still failed at the "reasonable" goals that administrations and politicians set for them. Meanwhile, you are left with the teachers that either do not care, or learned

– nay, were taught to – not care, in order to survive and continue to provide financially for their families in the only way they know how.

If we don't want calculus to be a filter keeping students out of STEM fields, the issues in these three studies need to be addressed. We need to address the issues by incorporating research on student thinking into calculus curriculums and interventions as well as by providing more attention to the support of elementary and high school teachers.

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APPENDIX A

PRETEST ITEMS

2. The constant rate of change of quantity R with respect to quantity S is 3.
If S changes by 1.7 how much does R change by?

3. Place the following fractions as carefully as possible on the number line below. (The fraction ½ has been placed as an example.)

5/7 18/14 27/13 9/7

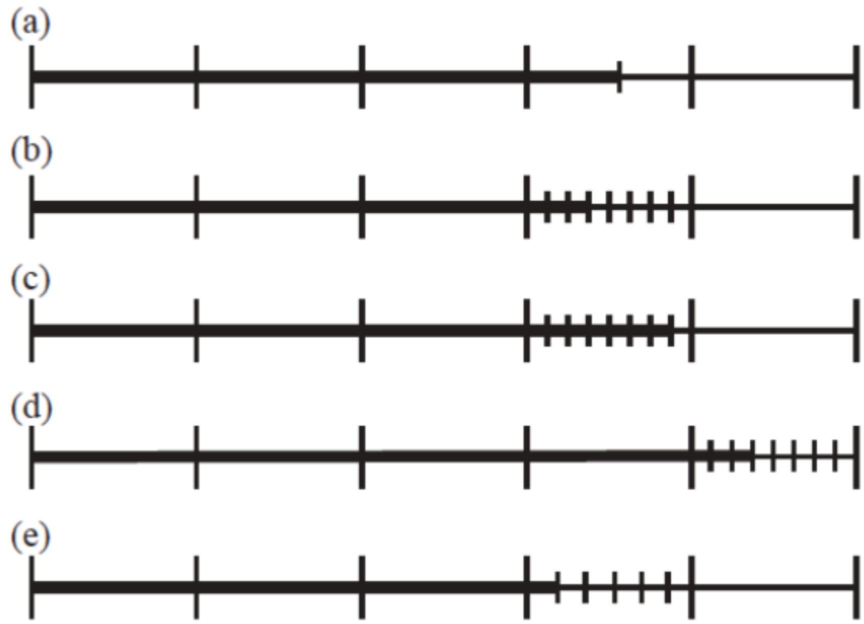
4. Your friend is learning about the idea of slope. In his homework he divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.

Explain to your friend what 3.04 means.

5. Ms. Roland gave her students the following problem to solve:

Candice has 4/5 of a meter of cloth. She uses 1/8 of a meter for a project. How much cloth does she have left after the project?

Ms. Roland had students use the number line so that they could draw lengths. Which of the following diagrams shows the solution? Assume all intervals are subdivided equally.

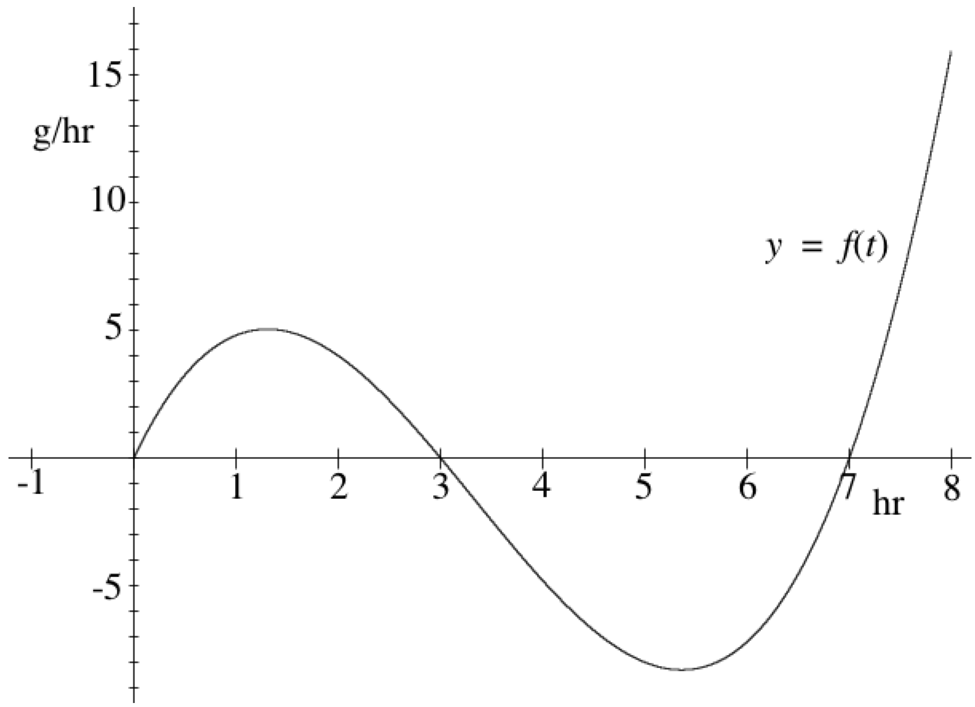


6. Some amount, call it B, is partitioned into *n* equal parts.

a) How large is B compared to the size of each part?

b.) How large is each part in relation to B?

7. The values of function *f* give the rate of change (in grams/hr) of a bacterial culture's mass *t* hours after measurements began.



Over what intervals within the first 8 hours is the culture's mass increasing? Explain.

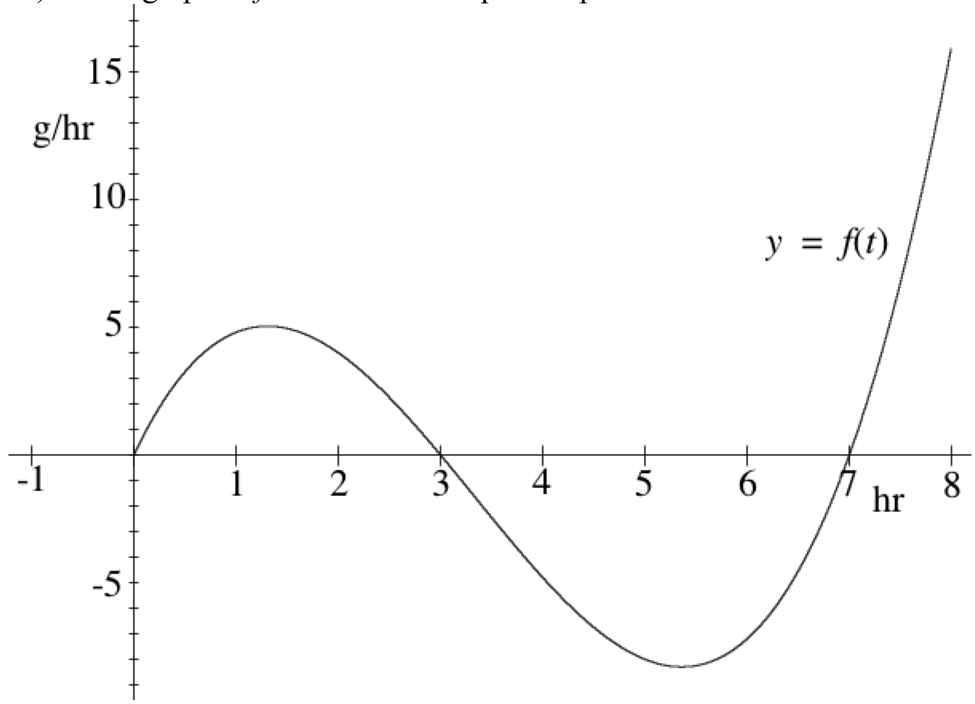
a) $0 < t \leq 1.5$ and $5.5 < t \leq 8$

b) $0 < t < 8$

c) $0 < t < 3$ and $7 < t \leq 8$

d) None of the above. My answer is ______________________________

Part B. The graph from the prior page is repeated below. Highlight the point (2.5, 2.25) on the graph of *f*. What does this point represent?



Part C.

Would you like to change your answer to the question on the prior page?

____

a) $0 < t \leq 1.5$ and $5.5 < t \leq 8$

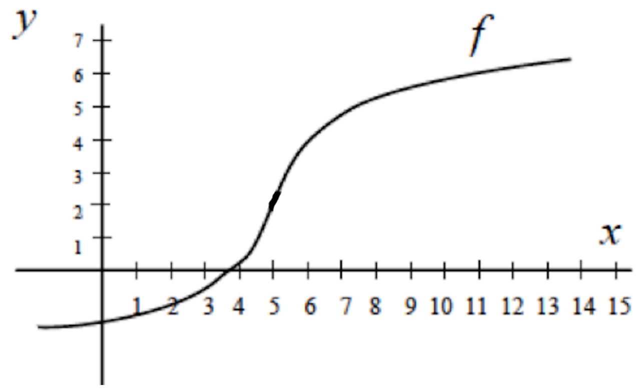
b) $0 < t < 8$

c) $0 < t < 3$ and $7 < t \leq 8$

d) None of the above. My answer is ______________________________

8. Using the graph below, explain the behavior of function *f* on the interval from

$x = 5$ to $x = 12$.



a) Increasing at an increasing rate.

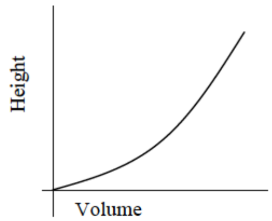
b) Increasing at a decreasing rate.

c) Increasing at a constant rate.

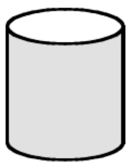
d) Decreasing at a decreasing rate.

e) Decreasing at an increasing rate.

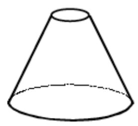
9. The following graph represents the height of water as a function of volume as water is poured into a container. Which container is represented by this graph?



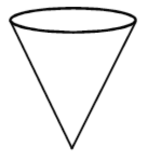
a)



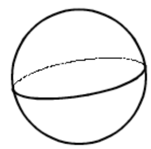
b)



c)

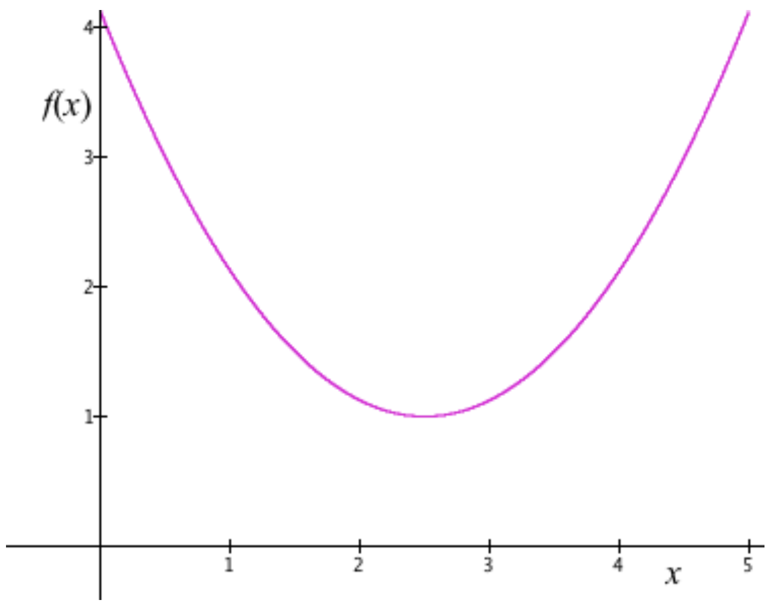


d)



e)

12. The graph below is of a function *f* over the interval [0,5].



For small equal increases of the value of x starting at $x = 1$ and ending at $x = 2$, the corresponding **changes** in the value of *f* are….

a) positive and increasing
b) positive and decreasing
c) negative and increasing
d) negative and decreasing
e) I cannot tell

Part B. Is this sequence increasing or decreasing? -10, -9.5, -9, -8.5, …

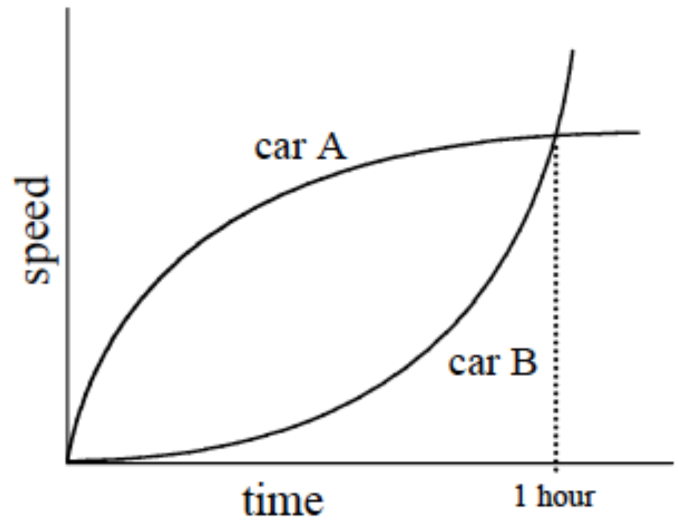
____

Would you like to change your answer to the question on the prior page?

____

a) positive and increasing
b) positive and decreasing
c) negative and increasing
d) negative and decreasing

13. The given graph represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.) Use this information and the graph below to answer item 8.



What is the relationship between the position of car A and car B at t = 1 hour?

a) Car A and car B are colliding.

b) Car A is ahead of car B.

c) Car B is ahead of car A.

d) Car B is passing car A.

e) The cars are at the same position.

14. A container has a volume of *m* liters. One gallon is $\frac{189}{50}$ times as large as one liter. What is the container's volume in gallons? Explain.

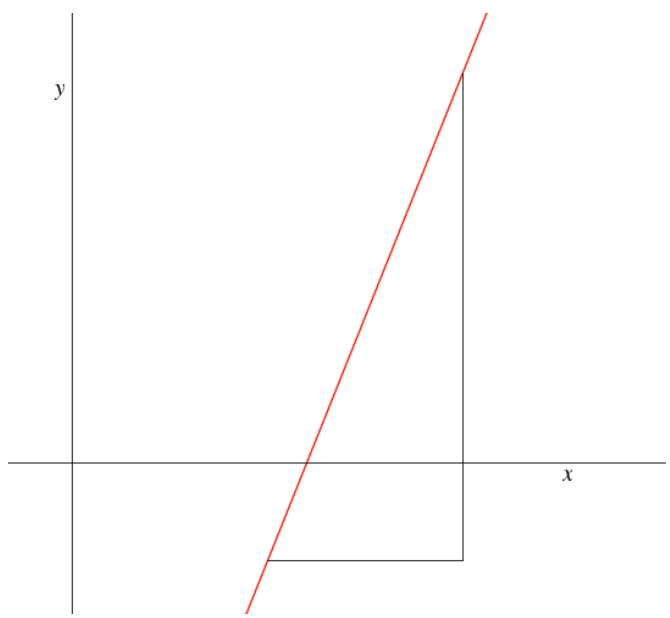
a. $\frac{189}{50}m$

b. $\frac{189}{50}m^3$

c. $\frac{50}{189}m$

d. $\frac{50}{189}m^3$

15. There are two quantities *P* and *Q* whose values vary. The measure of *P* is *y* and the measure of *Q* is *x*. *y* and *x* are related so that $y = mx + b$. The graph of their relationship is given below, with *x* and *y* in the same scale. What is the numerical value of *m*?



APPENDIX B

TEST 1 ITEMS

2. Which expression would calculate the change in cost of a Car A over a 2-year period starting at any time, i.e. starting at a years since 1985?

a) $f(a) - f(2)$ b) $f(a-2)$ c) $f(a+2) - f(a)$

d) $f(a+2) - f(2)$ e) $f(2) + f(a)$

5. Every minute, Julie travels j meters on her bike and Stewart travels s meters by walking. In *any* given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a) Julie will travel $j - s$ meters more than Stewart
b) Julie will travel $j\,s$ meters more than Stewart
c) Julie will travel $\frac{j}{s}$ meters more than Stewart
d) Julie will travel $j\,s$ times as far as Stewart
e) Julie will travel $\frac{j}{s}$ times as far as Stewart

6. The constant rate of change of quantity P with respect to quantity Q is 4. If Q changes by 1.3, how much does P change by?

a) 4/1.3 b) 1.3/4 c) $1.3 \cdot 4$

d) 4 e) None of the above

8. Which is an accurate interpretation of an object traveling 3.2 ft/sec?

a) Every 3.2 seconds corresponds to 1 foot traveled.
b) The number of seconds is always 3.2 times the number of feet traveled.
c) The number of feet traveled is always 3.2 times the number of seconds elapsed.
d) It is the distance traveled every 3.2 seconds.
e) Two of the above are true.

9. The instructions recommend using 64 oz. of a fertilizer for 75 ft^2 of garden space. Let *A* be the area of the garden in ft^2, and *n* be the number of oz. of fertilizer. Which of the following defines a function that determines the number of oz. of fertilizer needed as a function of the area of garden space?

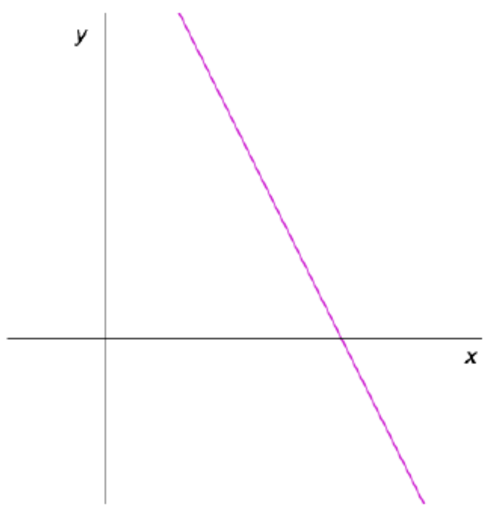
a) $g(n) = \frac{64}{75}n$ b) $h(A) = \frac{64}{75}A$ c) $k(n) = \frac{75}{64}n$

d) $p(A) = \frac{75}{64}A$ e) None of these

12. The value of *y* varies with constant rate of change with respect to *x*. Which statement must be true?

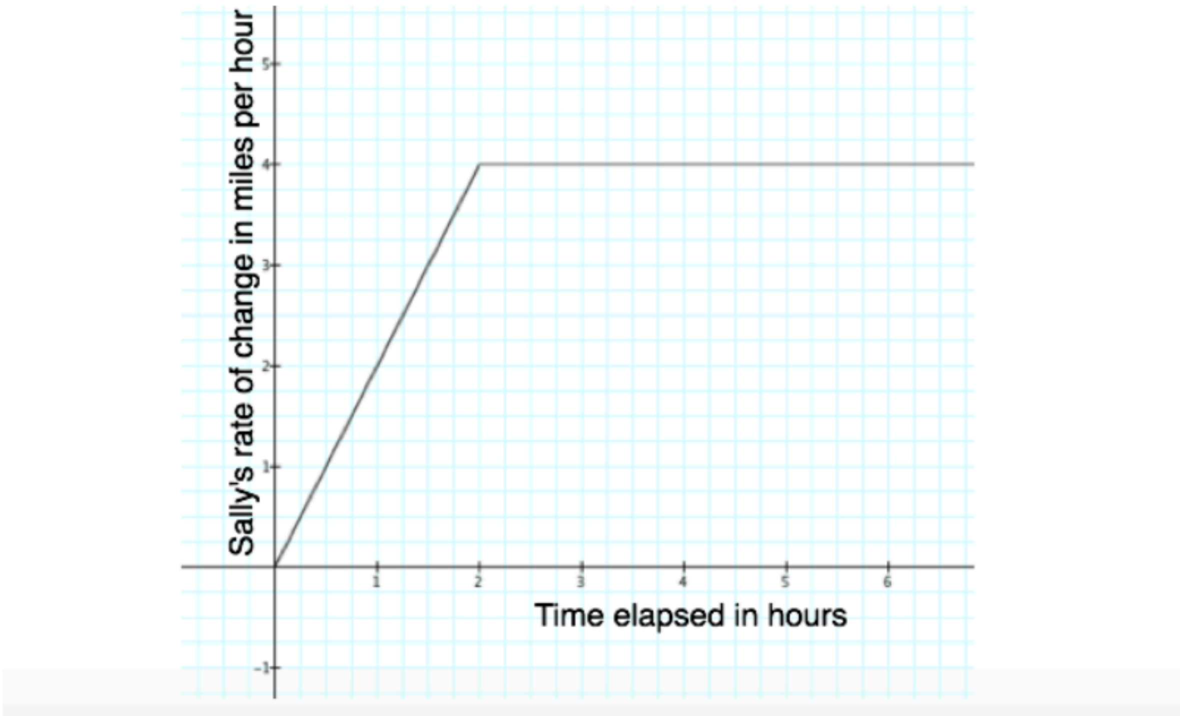
a) *y* is always a constant multiple of *x*
b) *y* and *x* each individually change at the same rate
c) the change in *y* is always equal to the change in *x*
d) the change in *y* is always the same value relative to the change in *x*
e) Two of the above are true

15. The values of *y* and *x* are related so that $y = mx + b$. The graph of their relationship is given below. The scale used on the *x* and *y* axis is the same. Estimate the numerical value of *m*.



a) It is not possible to estimate *m* without numbers on the axis, but *m* is decreasing.
b) $m = 2$ c) $m = 0$
d) $m = -1/2$ e) $m = -2$

For questions 16 and 17. The following graph shows the rate of change of Sally's distance from home measured in miles per hour with respect to time elapsed in hours.



16. How far did Sally's distance from home change between 3 hours and 4.5 hours?

  a) The distance did not change at all.
  b) The distance changed by (4.5)(4) miles.
  c) The distance changed by (1.5)(4) miles.
  d) The distance changed by (3)(4) miles.
  e) The distance changed by 3 miles.

17. Following standard conventions, the ordered pair (1.5, 3) is on the graph above. Which of the following is a valid interpretation of the meaning of this ordered pair?

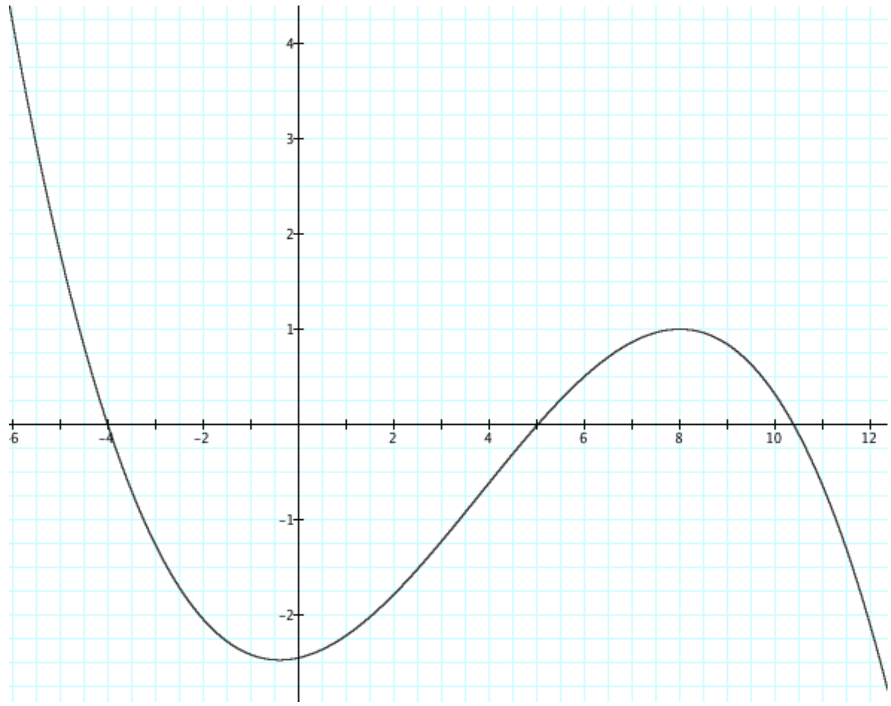
  a) Sally traveled 1.5 miles in 3 hours.
  b) Sally traveled 3 miles in 1.5 hours.
  c) Sally's speed is 1.5 miles per hour 3 hours into her trip.
  d) Sally's speed is 3 miles per hour 1.5 hours into her trip.
  e) None of the above.

APPENDIX C

TEST 3 ITEMS

1. – 6. At right is the graph of an *accumulation* function g over the interval from x = -6 to x = 12. Use this graph to answer questions 1-6.

Correct answers to questions 1-3 include ***all*** x-values between x = -6 and x =12 that meet the given criteria.



1. Determine all x-values where the accumulation function is increasing.
(The notation (a, b) means the interval $a < x < b$)
a) b) c) d) e)

2. Determine all x-values where the rate of change of g is negative.

a) b) c) d) e) a & d

3. Determine all x-values where the rate of change of g is increasing.

a) b) c) d) e)

4. At what value of x does the rate of change of g switch from positive to negative?
a) x = -4 b) x = -3 c) x = -0.5 d) x = 4 e) x = 8

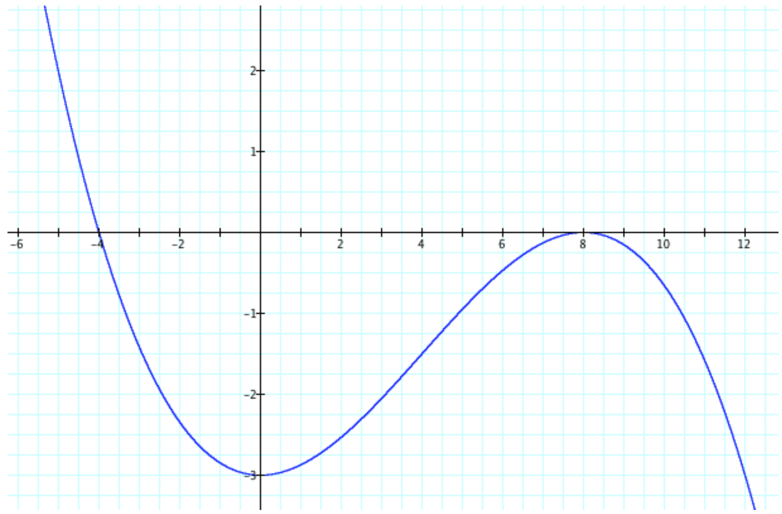
5. Which of the following is true? The rate of change of g switches….

a) …from increasing to decreasing at x = -4
b) …from decreasing to increasing at x = -0.5
c) …from increasing to decreasing at x = 4
d) …from increasing to decreasing at x = 8
e) Two of the above are true

6. Which is closest to ? (*Use the scales on the axes in finding your estimate.*)

a) -1.5 b) 0 c) 0.5 d) 1 e) 2

7. – 12. At right is the *rate of change* function r_h for an accumulation function h over the interval from x = -6 to x = 12.



Use this graph to answer questions 7-12.

7. Determine all x-values where is increasing.

a) b) c) d) e)

8. Which must be true about the accumulation function at x = 0? …

a) is increasing b) is decreasing c) has a negative value

d) pauses and has a relative minimum e) Two of the above must be true

9. Where does have a relative maximum?

a) x = -6 b) x = -4 c) x = 8

d) x= -4 and x = 8 e) Nowhere from x = -6 to 12

10. Which generally shows the graph of h for a small interval around x = 8?

a)  b)  c)  d)  e) 

11. Which is closest to the value of , i.e. the accumulation at x = -5?

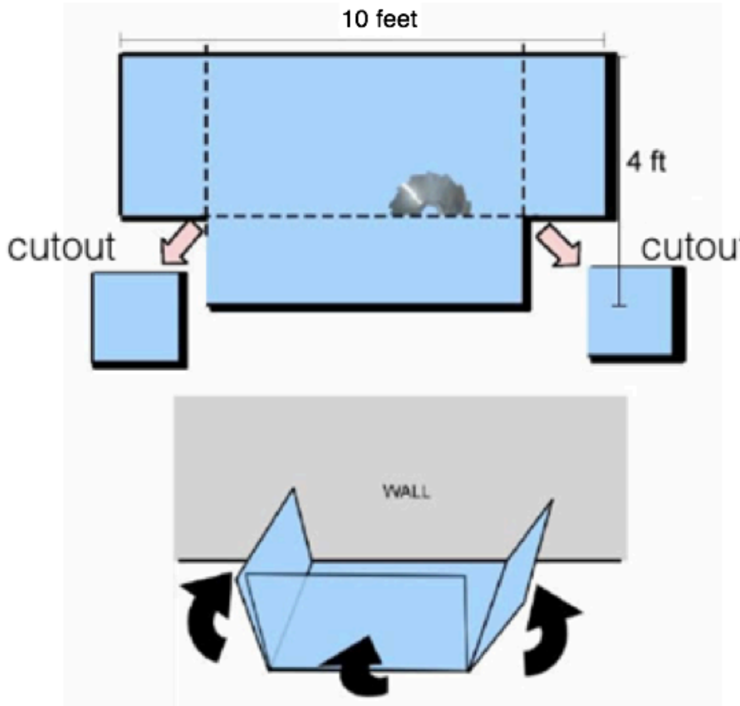
a) 2 b) 3 c) -3 d) -1/3 e) Not enough information

12. On what interval does the graph of h have this general shape: 

a) b) (0,4) c) (0,8)

d) Nowhere from x = -6 to 12 e) Not enough information

13. – 15. A 10 ft x 4 ft piece of wood is being used to build an open-top box. The box is formed by making equal-sized square cutouts from *two* corners of the wood at the ends of a 10-foot side. After these squares are discarded, three more cuts are made (at the dashed lines on figure) and these pieces are "folded up" and secured to create three of the four vertical sides. When the open side is placed against a wall, the open-top box is formed. Ignore the thickness of the wood when answering these questions.



13. Which quantities must you imagine as varying to determine which size cutout maximizes the volume of the box?

a) The length of the base of the box
b) The width of the original wood piece
c) The height of the box
d) The area of the base of the box
e) I imagine three of the above as varying quantities

14. If the variable x represents the length of a side of the square cutouts, what are the possible values of x that result in a box with positive volume?

a) $0 < x < 2$ b) $0 < x < 4$ c) $0 < x < 5$ d) $0 < x < 10$ e) None of these

15. Which of the following represents the volume of the box as a function of the length of a side of the square cutout?

a) $V(x) = (x - 10)(x - 4)(2x)$ b) $V(x) = (2x - 10)(2x - 4)(x)$ c) $V(x) = (10 - 2x)(4 - x)(x)$ d) $V(x) = (10 - x)(4 - 2x)(2x)$ e) $V(x) = (10 - 2x)(4 - 2x)(x)$

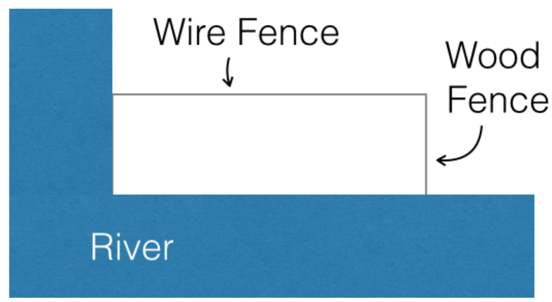
16. The number of traffic accidents in a year, A, is related to the population of a city, p, by the equation:

$$p^{3/2} - 500A = 2{,}500$$

If the population is growing at a rate of 500 people per year, find the rate at which traffic accidents will be rising, in accidents per year, when the population is 40,000 people. Choose the closest answer.

a) 50 b) 100 c) 200 d) 300 e) 500

17. A farmer needs a rectangular pasture of 3000 square meters in a place where a river makes a right-angle turn, such that the river will form the south and west barriers of the pasture. The north side will be a wire fence that costs \$15/meter and the east side will be a wood fence which costs \$30/meter. What is the minimum cost for creating the pasture? Choose the closest answer from the options provided.



a) \$1600 b) \$2400 c) \$3200 d) \$3800 e) \$4200

18. If f is an accumulation function, and $f'(c) = 0$, which must be true?

a) f has a relative maximum or minimum at $x = c$
b) f changes concavity at $x = c$
c) The accumulation has a value of 0 at $x = c$
d) For a small change in x from c, the change in $f(x)$ is essentially 0
e) Two of the above must be true

19. At 5 minutes after turning off the oven, the temperature inside the oven is $T(5)$ degrees Fahrenheit, and the rate of change of the temperature is $T'(5)$ degrees Fahrenheit per minute. Which expression gives the approximate temperature in the oven 6 seconds later, that is 5 minutes and 6 seconds after the oven is turned off?

a) $0.1 \cdot T'(5)$ b) $0.1 \cdot T(5)$ c) $T(5)+T'(5)$

d) $T(5)+0.1 \cdot T'(5)$ e) $T'(5)+T(5)/6$

---

20. The radius of a spherical balloon is measured in meters. The function *f* gives the balloon's volume as a function of its radius. What does the expression $\dfrac{f(3+h)-f(3)}{h}$ represent?

a) The change in volume of the balloon as the radius changes from 3 to 3 + *h*
b) The approximate constant rate of change of the balloon's volume as the radius changes from 3 to 3 + *h*
c) The instantaneous rate of change of the balloon's volume when the radius is 3 meters
d) The slope of the tangent line to the curve $y = f(x)$ at $x = h$
e) The slope of the tangent line to the curve $y = f(x)$ at $x = 3$

21. - 22. Oil pours into the circular top of a conical tank (pointing down) at a constant rate of *M* cubic meters per second. The tank is such that the diameter at top of the tank is half the total height of the tank. (Recall that for a cone: $V = \frac{1}{3}\pi r^2 h$)

21. What is an appropriate representation for the given value of *M*?

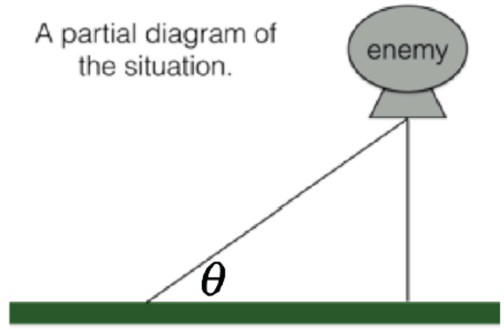
a) $\dfrac{dh}{dt}$ b) $\dfrac{dV}{dt}$ c) $\dfrac{dV}{dh}$ d) $\dfrac{dr}{dt}$ e) $\dfrac{dV}{dr}$

22. At any particular moment after the tank begins to fill, how fast is the surface of the oil rising in meters per second?

a) $\dfrac{16}{\pi h^2} \cdot M$ b) $\sqrt[3]{\dfrac{12V}{\pi}} \cdot M$ c) $\dfrac{4}{\pi h^2} \cdot M$ d) $\sqrt[3]{\dfrac{48V}{\pi}} \cdot M$ e) None of these

---

25. - 28. The enemy of James Bond is in a hot air balloon, which is tied to the ground with a 140-foot rope that creates a variable angle θ with the ground. The rope remains taut, and $\theta \le 90^\circ$. Bond is at the point where the rope is attached to the ground, and he wants to get to the balloon to fight the enemy. Bond can run along the ground at 20 ft/sec, and climb straight up a second rope to the balloon from directly below at 10 ft/sec. If the enemy can vary the angle θ, how should he position the balloon in order to keep Bond away from the balloon for the most time possible?



25. What is the general structure of function being optimized in this problem?

a) $Distance = D_1(t) + D_2(t)$ b) $Distance = D_1(t) - D_2(t)$

c) $Distance = D_1(\theta) + D_2(\theta)$ d) $Time = T_1(\theta) + T_2(\theta)$ e) $Time = T_1(\theta) - T_2(\theta)$

26. Which statement applies to creating the function in #25?

a) $Dist = \dfrac{Rate}{Time}$ b) $Dist = \dfrac{Time}{Rate}$ c) $Time = \dfrac{Rate}{Dist}$

d) $Time = \dfrac{Dist}{Rate}$ e) $\tan(\theta) = \dfrac{D_1}{D_2}$

27. Which two distances are involved in creating the function in #25?

a) $140\sin\theta,\ 140\cos\theta$ b) $\dfrac{140}{\sin\theta}, \dfrac{140}{\cos\theta}$ c) $\dfrac{\sin\theta}{140}, \dfrac{\cos\theta}{140}$

d) $140,\ 140\tan\theta$ e) $140, \dfrac{140}{\tan\theta}$

28. What is a critical step in finding the solution to the problem? ("the function" refers to the correct answer to #25.)

a) Determining where the function's output is equal to zero
b) Determining where the rate of change of the function is equal to zero
c) Determining where the rate of change of the function has a maximum value
d) Determining where the second derivative is equal to zero
e) None of these

APPENDIX D

CHECKING REGRESSION ASSUMPTIONS

Before doing hypothesis testing with a regression model it is important to make sure the assumptions are satisfied. This section shows that there are no major departures from the assumptions I made to make the regression model.

The following are departures from the regression model that can be studied with residuals.

1. The regression function is not linear.
2. The error terms do not have constant variance.
3. The error terms are not independent.
4. The model fits all but one or a few outlier observations.
5. The error terms are not normally distributed.
6. One or several important predictor variables have been omitted from the model. (Kutner, Nachtsheim, Neter, Li, 2005, p. 103).

I will address each of these issues with residual plots then address the issue of multicollinearity that is unique to multivariate models.

1. Regression function is linear

The residual versus predicted plot in *Figure 58* does not show any obvious patterns. The predictor variables are responses to all relevant Pretest and Test 1 question. The predicted variable is number of correct rate of change questions on Test 3.

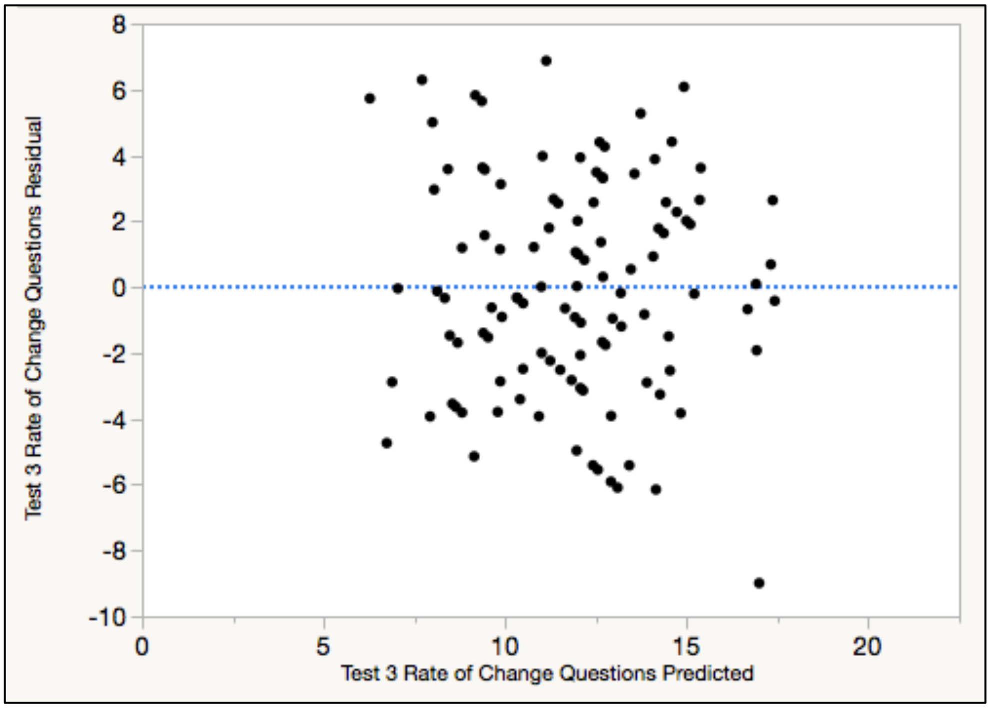


Figure 58. Residuals Versus Predicted Plot.

Students who are predicted to have high scores on the rate of change questions on Test 3 have similar residuals as students who are predicted to have low scores. This suggests the model doesn't systemically over or under predict for students with high or low scores. This suggests that a linear model is appropriate and that we do not need to transform our predictor variables. Figure 59 shows that the residuals for the model containing five predictors also looks like a randomly scattered cloud. This suggests a linear model is appropriate for the reduced model as well.

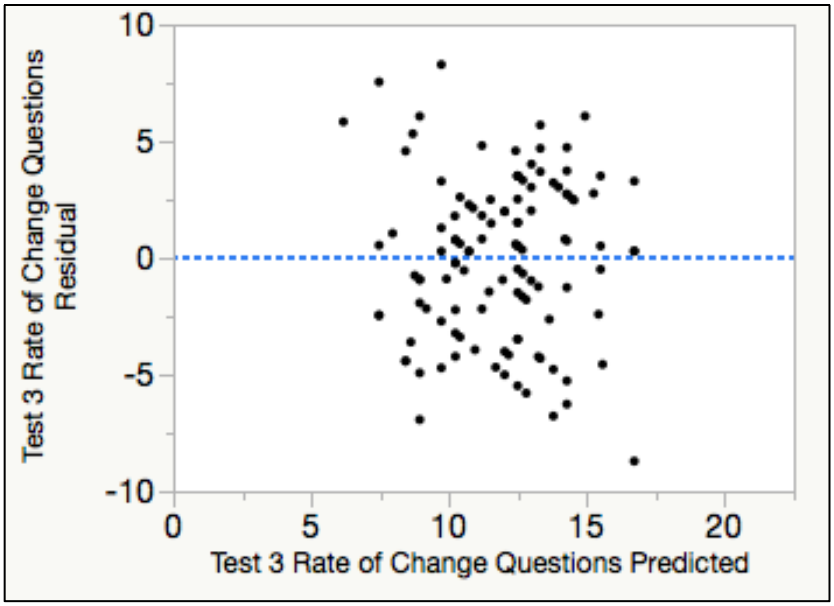


Figure 59. Residuals Versus Fitted Values for Model with Five Predictor Items.

2. Constant Variance of Error Terms

If there was not constant variance of error terms the residual versus fitted values plots in Figure 58 and Figure 59 might have a discernable shape, such as a "megaphone shape". These residuals do not show a pattern. The Brown-Forsythe test is used to formally test whether or not the "mean of the absolute deviations for one variable for one group differs significantly from the mean absolute deviation in the second group" (Kutner, et. al, 2005, p. 116). Using JMP we find, Brown Forsyth F ratio equals 0.19 ($df_n = 1$, $df_d = 113$, p=.66). We should not reject the null hypothesis that there is constant variance of error terms for different values of predictors.

3. Error terms are not independent

Error terms may not be independent if the data was collected over time. All data was collected in the first week of class in this study so we should not worry about time-series effects.

4. Presence of outliers

The box plot of residuals showed no obvious outliers. An outlier would be a student whose actual Test 3 score was extremely different from the score we predicted the student would receive. A Mahalanobis plot Figure 60 is a way to detect outliers in multivariate regression models. A point with a high Mahalanobis distance is an outlier on the model and should be checked. The plot does not show any outliers that may violate the assumptions of the regression model.

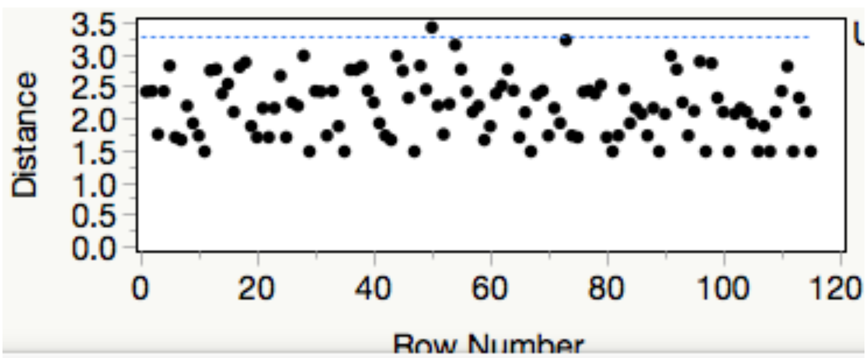


Figure 60. Mahalanobis Distances to Detect Outliers.

5. Error terms normally distributed

One way to tell if residuals are normally distributed is to plot the residuals and check that they form a bell shaped curve. Another way is to make a normal quantile plot and check to make sure it is approximately a straight line. If the normal quantile plot is not approximately linear it means there are many more large or small residuals than we would expect if we had a normal distribution. Figure 61 does not show any major departures from our assumption of normal distribution of residuals.

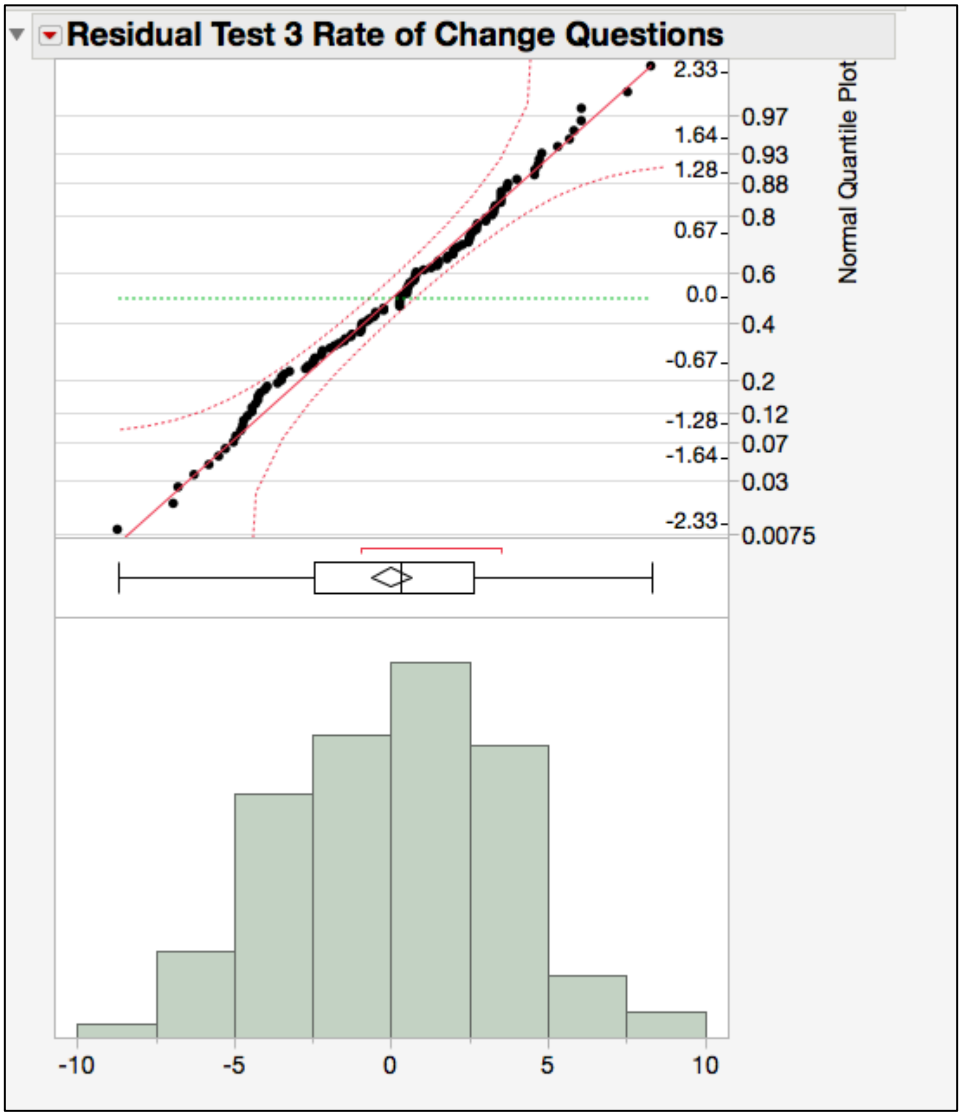


Figure 61. Plot of Residuals and Normal Probability Plot to Check that Residuals are Normally Distributed.

6. Predictor variables omitted from model

When creating the study I was aware that I was omitting predictor variables such as students' former GPA that should be predictive of success in calculus. However, I did include a variety of items concerning students understanding of measure, fraction, graph, covariation, and rate of change because I hypothesized that all of these issues could be relevant to instruction.

Check for Multicollinearity

If two or more of the predictor variables are highly correlated it can lead to artificial inflation of the r-squared value associated with the regression model. Considering that many of the items used were specifically designed to measure the same idea multicollinearity could be a major problem. Variance inflation factors are a widely used method of detecting the presence of multicollinearity (Kutner, et. al., 2015, p. 408). The variation inflation factor is a measure of how much the R squared value is inflated because of multicollinearity of predictors. Typically a value of 10 for the variation inflation factor means that the multicollinearity is severe and undulling inflating R squared. The values in Figure 62 or Figure 63 for the variation inflation factor (VIF) do not reveal problems with multicollinearity.

Term	Estimate	Std Error	t Ratio	Prob>|t|	VIF
Intercept	6.1699233	0.896563	6.88	<.0001*	.
P 12 Final Answer	1.2218683	0.431388	2.83	0.0055*	1.1097384
P9	1.7851898	0.684672	2.61	0.0104*	1.0661051
P14	1.2944181	0.506624	2.55	0.0120*	1.0216934
T1 Q9	2.76767	0.728989	3.80	0.0002*	1.0492121
T1 Q15	2.256356	0.666331	3.39	0.0010*	1.0060561

Figure 62. Test for Multicollinearity of Reduced Model.

Term	Estimate	Std Error	t Ratio	Prob>|t|	VIF
Intercept	5.6034386	2.42844	2.31	0.0233*	.
P2 0/1	0.6105658	0.952308	0.64	0.5230	1.2138183
P3 Number Correct	0.4617546	0.427078	1.08	0.2825	1.4009195
P4 Level	-0.530533	0.624924	-0.85	0.3981	1.2153107
P5	0.2949944	1.067042	0.28	0.7828	1.2634721
P7a 0/1 2	0.0921481	1.667653	0.06	0.9561	2.416487
P7b 0/1	-1.146989	1.12583	-1.02	0.3110	1.3260208
P7c	1.3647577	1.671638	0.82	0.4164	2.654265
P8 0/1	0.4537341	1.08604	0.42	0.6771	1.3318778
P9	2.1275018	0.99049	2.15	0.0344*	1.3086254
P 12 Final Answer	1.2908748	0.613904	2.10	0.0383*	1.327411
P13	0.3161496	1.310133	0.24	0.8099	1.3396537
P14	1.773953	0.712312	2.49	0.0146*	1.1804379
P15 Level	-0.072875	0.421323	-0.17	0.8631	1.1553072
T1 Q1A Q8M	-2.63195	1.443886	-1.82	0.0716	1.3571131
T1 Q5	0.7683615	0.711828	1.08	0.2833	1.1855631
T1 Q6	2.13385	1.574974	1.35	0.1788	1.3521202
T1 Q8A Q2M	1.6563415	1.335119	1.24	0.2179	1.3357368
T1 Q9	2.0996498	1.121708	1.87	0.0644	1.4667649
T1 Q12	1.1511221	0.953596	1.21	0.2305	1.18879
T1 Q15	1.8748639	0.998086	1.88	0.0635	1.3233958
T1 Q16	1.5917574	1.330695	1.20	0.2347	1.7268088

Figure 63. Multicollinearity Test for Full Model.

APPENDIX E

IRB APPROVAL

EXEMPTION
GRANTED

Patrick Thompson
Mathematics and Statistical Sciences, School of
480/965-2891
Pat.Thompson@asu.edu

Dear Patrick Thompson:

On 8/6/2014 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	Investigating the relationship between students' meanings for magnitude and their understanding of rate of change functions in Calculus.
Investigator:	Patrick Thompson
IRB ID:	STUDY00001334
Funding:	None
Grant Title:	None
Grant ID:	None
Documents Reviewed:	• Consent Form Study Part A, Fall 2014, Category: Consent Form; • Consent Form Study Part B, Spring 2015, Category: Consent Form; • Social Behavioral IRB application Thompson, Category: IRB Protocol; • Spring 2014 Sample Interview, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Spring 2014 Sample Tutoring Items, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Sample Test Form Spring 2015, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Sample Interview Items Spring 2015, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions);

	• Oral Recruitment-Byerley Spring 2015, Category: Recruitment Materials; • Oral Recruitment-Byerley-Fall 2014, Category: Recruitment Materials;

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (1) Educational settings on 8/6/2014.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-

103). Sincerely,

IRB Administrator

cc: Cameron Byerley Patrick Thompson Cameron Byerley Owen Davis Stacy Musgrave

APPENDIX F

PERMISSIONS

Pat Thompson and Marilyn Carlson gave permission for me to use items from the Mathematical Meanings for Teaching Secondary Mathematics diagnostic instrument in the Pre-Calculus Concept Assessment.

Pat Thompson, second author on paper one gave permission to use it as dissertation chapter.