Conceptualizing and Reasoning with Frames of Reference

in Three Studies

by

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ABSTRACT

This dissertation reports three studies about what it means for teachers and students to reason with frames of reference: to conceptualize a reference frame, to coordinate multiple frames of reference, and to combine multiple frames of reference. Each paper expands on the previous one to illustrate and utilize the construct of frame of reference. The first paper is a theory paper that introduces the mental actions involved in reasoning with frames of reference. The concept of frames of reference, though commonly used in mathematics and physics, is not described cognitively in any literature. The paper offers a theoretical model of mental actions involved in conceptualizing a frame of reference. Additionally, it posits mental actions that are necessary for a student to reason with multiple frames of reference. It also extends the theory of quantitative reasoning with the construct of a ‘framed quantity’. The second paper investigates how two introductory calculus students who participated in teaching experiments reasoned about changes (variations). The data was analyzed to see to what extent each student conceptualized the variations within a conceptualized frame of reference as described in the first paper. The study found that the extent to which each student conceptualized, coordinated, and combined reference frames significantly affected his ability to reason productively about variations and to make sense of his own answers. The paper ends by analyzing 123 calculus students’ written responses to one of the tasks to build hypotheses about how calculus students reason about variations within frames of reference. The third paper reports how U.S. and Korean secondary mathematics teachers reason with frame of reference on open-response items. An assessment with five frame of reference tasks was given to 539 teachers in the US and Korea, and the responses were coded with rubrics intended to categorize responses by the extent to which they demonstrated conceptualized and coordinated frames of reference. The results show that the theory in the first study is useful in analyzing teachers' reasoning with frames of reference, and that the items and rubrics function as useful tools in investigating teachers' meanings for quantities within a frame of reference.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

PAPER ONE: CONCEPTUALIZING AND REASONING WITH FRAMES OF REFERENCE: A THEORETICAL FRAMEWORK

The concept of frames of reference, though commonly used in mathematics and physics, is not described cognitively in any literature. The lack of a careful description of the mental actions involved in thinking within a frame of reference inhibits our ability to account for issues related to frames of reference in students' reasoning. In this paper we offer a theoretical model of mental actions involved in conceptualizing a frame of reference. Additionally, we posit mental actions that are necessary for a student to reason with multiple frames of reference. We also extend the theory of quantitative reasoning with the idea of a 'framed quantity'. This theoretical model provides an additional lens through which researchers can examine students' quantitative reasoning.

Consider the following problems that students encounter routinely in high school:

- Bobby is 3 years older than Lucy. When Bobby is x years old, how old will Lucy be?
- A particular engine can propel a boat at a maximum of 32 miles per hour. The boat travels 30 miles upstream from Port Adele to Port Chimney and then back, at maximum speed. The captain uses an anemometer before

starting to estimate the downstream current as 6 mph. Just considering travel time, how long will the round-trip take?

- Yolanda and Sydney ran in the same marathon. Sydney ran 5/3 times as fast as Yolanda. If Sydney finished the 26.2-mile race in 4 hours, what was Yolanda's average speed?

Students often struggle to manage the dual perspectives required in each task (Bowden et al., 1992; Monaghan & Clement, 1999; Panse, Ramadas, & Kumar, 1994); for instance, the first scenario provides a comparison of Bobby and Lucy's age relative to Lucy's age, then switches to describing Bobby's age from Bobby's perspective, and finally asks for Lucy's age relative to Bobby's. A student must similarly tease apart the ways in which the framing of information about quantities in a scenario switches between two frames in the other two examples. In our own work investigating teachers' meanings on similar tasks, we identified a type of quantitative reasoning that was involved in answering such tasks (Smith & Thompson, 2007; Thompson, 2011).

In line with physics terminology, we choose to describe the extra layer of complexity in the above problems as issues of "frames of reference". The purpose of a frame of reference is to facilitate one's measurement of quantities, and to be able to compare measurements taken in different situations like different speeds (Yolanda & Sydney) or from different starting points (Bobby & Lucy). Indeed, without a known frame of reference (explicitly stated or implicitly accepted by a group of people) any measurement is useless; for example, the phrases ‘Tokyo is 4,000 miles’ or ‘Pollution is 50% greater’ are meaningless with no mention of what location or previous pollution reading is referred to. Yet at least one study has shown that students almost never qualified their measurements or models with reference points (Marshall & Carrejo, 2008).

In this report, we introduce what we mean by *a conceptualized frame of reference* and *reasoning with frames of reference*, and explain why this is an area that deserves attention within and between the physics and mathematics education communities.

A definition of the noun phrase “frame of reference” would suggest that a frame of reference is an object external to the person reasoning with it. Such a perspective does not align with our goal of describing what it might mean for an individual to conceptualize a frame of reference. Therefore, we articulate the mental activity involved in conceptualizing and reasoning with frames of reference. While the *products* of the mental activity we describe align with the classical definition for frame of reference as a coordinate system or a system of measures, our emphasis is on the mental actions a student must employ to conceptualize a frame of reference. In particular, we use the phrase “frame of reference” to refer to a set of mental actions through which an individual might organize processes and products of quantitative reasoning – meanings that, if fully developed, would be a Piagetian scheme for frame of reference (Thompson, 2011). As such, conceptualizing frames of reference and quantitative reasoning are interrelated, with frames of reference providing an additional lens with which to look at quantitative reasoning. We therefore introduce an extension to the theory of quantitative reasoning with the idea of a *framed quantity*. Finally, we propose many problem contexts that our frames of reference constructs can be powerfully applied to besides relative motion.

Conceptualizing a Frame of Reference

An individual can think of a measure as merely reflecting the size of an object relative to a unit, or he can think of a measure within a system of potential measures and comparisons of measures. An individual conceives of measures as existing within a *conceptualized frame of reference* if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives meaning to a zero measure and all non-zero measures, and 3) committing to a directionality of measure comparison additively, multiplicatively, or both.

Committing to a Unit

As an example, a student can think about the measure “4.5 feet” in different ways. If the student focuses only on the value “4.5” and sees the unit as of secondary (or perhaps no) importance, there is no meaningful connection between the unit and the value for this student. In contrast, if the student sees a multiplicative relationship between the unit and the value, this provides a meaning for the measure. In this second case, “4.5 feet” is a length that is 4.5 times as long as the length of an object that is taken as a standard foot. A student who sees this relationship and the importance of unit in establishing meaning for each measure has taken the first crucial step towards conceptualizing a frame of reference.

Committing to a Reference Point

As a demonstration, consider the phrases “distance Ben walked” and “distance Ben is South of his house”. Both phrases describe quantities. The first phrase is vague and leaves a reader wondering if the quantity described is Ben’s distance walked today, Ben’s distance walked in his room, or the distance Ben walked since his birth. As such, the ambiguity in the phrase “distance Ben walked” creates ambiguity in the meaning of a measure. Saying the measure of “distance Ben walked” is m units fails to provide usable information for an individual trying to reason about the situation. Moreover, the vagueness of “distance Ben walked” would make it possible for an individual to inadvertently change his meaning for this phrase while reasoning within a complex situation. He might define formulas or expressions to model the situation without understanding that his inconsistent meanings for the quantity make his model incoherent. Another possibility is that two individuals can read a situation and internally ascribe reference points to the quantity without realizing that they have done so. They might then discuss a problem and never realize that they are talking past one another because they are operating and speaking within two incompatible conceptualized frames of reference.

The specificity of “the distance Ben walked from his house today” makes it a more useful description of a quantity. In particular, we can confidently say that if the measure of the quantity “distance Ben walked from his house today” is zero, then Ben hasn’t left his house today. Similarly, if the measure of that quantity is b units, with $b > 0$, then Ben walked b units outside of his house. The commitment to a reference point attributes a meaning to every measure of the quantity and avoids the problems associated with ambiguity described above.

Committing to a Directionality of Measure Comparison

Consider a student designing a study to investigate the relationship between people’s weight and Vitamin C consumption. The student plans to weigh each participant at the start and at the end of a two-month period, during which the participants will consume various amounts of Vitamin C daily. The student plans to examine the changes in the participants’ weights. This student could imagine these comparisons in two different ways. If the student is oriented to think always of positive changes, then the student would make the following kinds of statements: “Josh is 6 pounds heavier at the end of the study” and “Wanda is 6 pounds lighter at the end of the study”. In this case, the student has not thought of the comparison of measures within a frame of reference. Rather, the student adjusted his description so that a comparison always results in a positive number. Such adjustments constantly alter the directionality of comparison in order to think of the larger measure relative to the smaller. Should the student be asked what a participant’s change of 1.5 pounds means, he could not say definitively whether the participant gained or lost weight.

Alternatively, suppose that the student commits to a comparison of “pounds heavier at the end than at the beginning”. The additive comparison that the student has in mind is the post-weight minus the pre-weight. Here, the student would make statements like: “Josh is 6 pounds heavier” and “Wanda is – 6 pounds heavier.” In these statements, the student made use of the same direction in comparing the measures. Unlike the other case, the student now definitely interprets a change of 1.5 pounds as the individual weighed 1.5 pounds more at the end of the experiment than at the beginning.

We note that this commitment to the directionality is crucial when making multiple comparisons. For instance, most students can mentally shift between “heavier than” and “lighter than” when comparing two people’s weights. However, the activity of comparing three or more people’s weights proves much more difficult without committing to a directionality within a frame of reference.

An analogous commitment to a directionality when comparing measures holds for multiplicative comparisons. A student thinking within a frame of reference will be able to say “*x* is 3 times as large as *y*” and “*y* is one-third as large as *x*.” A student who avoids committing to a directionality of comparison will only be able to make the first statement, possibly because of a discomfort with non-integers. For example, he or she may end up making statements such as Hillary Crosley Coker’s headline ‘Women’s Soccer Team Paid 40 Times Less

Than Men' (Coker, 7/6/15), a remark that strictly means the women's team salary is not 1/40 but -39 times the men's salary.

As a final note, we emphasize that we are *not* suggesting people should commit to a single reference point or a single directionality of comparison for their entire engagement in a task. In fact, it is often the case that while solving problems, an individual must conceptualize more than one frame of reference. The commitments we refer to only occur *within* the act of conceptualizing one frame of reference; a student can choose to work with a different frame of reference for the same quantity within one context, but while working within one frame, he works consistently with the choices of reference point and directionality of comparison he made in order to conceptualize that frame of reference. The conceptualization of multiple frames of reference then requires further mental actions to bring information from multiple frames together, an activity we call *reasoning with multiple frames of reference*.

Reasoning with Multiple Frames of Reference

We identify two types of reasoning that a student might employ when engaging in a task that necessitates conceiving of multiple frames of reference. The first type is that a student *coordinates multiple frames of reference* when he finds the relationship between one or more quantities' measures in two frames, such that he can determine a measure given in one frame from a measure given in the other. A student who has coordinated two frames of reference could, given an event's representation in one frame, represent that event in another frame in order to compare similar quantities. The second type of reasoning is that of a student *combining multiple frames of reference* when he considers two different quantities simultaneously within their respective frames of reference. Below we discuss the mental actions that are associated with each type of reasoning.

Coordinating Multiple Frames of Reference

A student coordinates multiple frames of reference by carrying out three sets of mental actions. She must first recognize the need to transform the measures of quantities measured in different frames of reference into measures that have been measured in the same frame of reference. Second, a student must coordinate known *measures* of quantities in different frames in order to answer her question. Third, she must use those known measures to coordinate the *frames*.

We illustrate these mental actions in the context of the task presented in Figure 1.

Two children, Alice and Bob, walk together from school to home. Alice starts measuring the distance they have traveled by counting the sidewalk squares they have crossed since passing the tree. Bob starts counting the sidewalk squares they have crossed since passing the stop sign and noticed that there were 3 squares between the tree and the sign. Let u be the number of sidewalk squares Alice has counted. Write an expression that gives Bob's count of sidewalk squares.

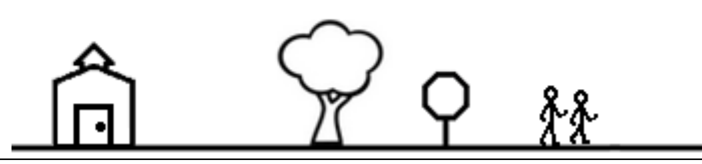


Figure 1. The Alice and Bob task.

Before beginning to coordinate multiple frames of reference, the student must first recognize that Alice and Bob each conceived of a comparable quantity within separate frames of reference. The student's recognition of this fact coincides with her envisioning what a distance of zero squares means to both Alice and Bob. The student must recognize that for Alice, "zero squares" means that the children are at the tree; likewise the student understands that "zero squares" to Bob means that the children are at the stop sign.

While the student could answer the prompt with a statement such as "Let *v* represent the number of squares that Bob has counted", she may feel the need to make use of the given definition for *u*. However, in attempting to use *u*, she imagines shifting from Alice's measurements (and frame of reference) to Bob's measurements (and frame of reference). The student anticipates that for the shift to work, she needs to find a commonality between the two frames of reference. The stem of the task in Figure 1 provides the student with a useful point of commonality between the frames. The student knows that Alice and Bob walk along the same path, counting the same sidewalk squares, with Alice starting to count at a tree and, three squares later, Bob starting to count at the stop sign. The stop sign serves as a point of commonality between the two frames of reference. The student knows that for Alice the stop sign is three squares from the tree. Likewise, she knows that Bob views the stop sign as zero squares from itself. Thus, a measure of three squares for Alice, 3_{Alice}, is the same point along the path as zero squares for Bob, 0_{Bob}. In establishing the link $3_{\text{Alice}} \equiv 0_{\text{Bob}}$, the student has coordinated known measures of comparable quantities from two different frames of reference. To fully coordinate the two frames of reference, the student must establish the relationship between the measure of a quantity in one frame of reference and the measure of the comparable quantity in other frame of reference. The student imagines that if Alice and Bob are at the stop sign and move forward one square, then both of Alice's and Bob's counts will increase by one; thus $4_{Alice} \equiv 1_{Bob}$. She anticipates that as they keep moving forward *any amount*, both Alice and Bob will increase their counts (e.g. they move forward another 0.5 squares, $4.5_{Alice} \equiv 1.5_{Bob}$). Likewise, she imagines that if Alice and Bob moved backward one square, their counts would increase by -1; thus $2_{Alice} \equiv -1_{Bob}$. In examining these connections based from the point of commonality, the student anticipates that Bob's count will always be 3 squares less than Alice's count. This supports the student in expressing Bob's count as $u - 3$ using Alice's frame of reference. Such a conclusion requires the student to keep track of both measurements while also thinking carefully about the ways in which both measurements change in relation to each other, or engage in covariational reasoning (Marilyn P. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002).

The task in Figure 1 involves coordinating two frames with different reference points. Other tasks may involve coordinating two frames with different units, such as coordinating lengths of an object in both centimeters and inches. Here our student would need to recognize that a point of commonality would be $0_{CM} \equiv 0_{IN}$. He would also need to see that as an object grows, the inches measure and centimeters measure increase and decrease together, and they do so at a fixed ratio so that $\Delta X_{IN} \equiv \Delta 2.4 X_{CM}$. Finally, yet other tasks involve coordinating two frames with both different reference points and measures, such as coordinating the measures of temperature in degrees Celsius and degrees Fahrenheit.

Coordinating multiple frames of reference is cognitively demanding. It requires that a student conceive each frame as valid, be aware of the need to coordinate quantities' measures within them, and carry out the mental process of finding a relation between the frames while keeping all relative quantities and information in mind.

Combining Multiple Frames of Reference

A student *combines* frames of reference when she considers multiple quantities that exist within separate frames of reference simultaneously. Combining frames of reference is a separate act from coordinating frames of reference. When combining frames of reference, the student does not have a goal of expressing measures of one or more quantities in terms of different frames. Rather, the student's goal is simply to hold quantities from multiple frames of reference in mind concurrently. In the above section, the student would have combined Alice's frame of reference with Bob's frame of reference had she stated "Alice and Bob's home is both *u* squares from the tree *and* *u* – 3 squares from the stop sign". As a further example, coordinating systems allow us (mathematicians, teachers, and students) to represent the measures of different quantities simultaneously when those measures stem from potentially different frames of reference. Figure 3 shows two examples of this; a coordinate system combining Alice's and Bob's frames of reference as well as a coordinate system for air temperature in Fahrenheit and Celsius. Students' acts of joining two or more number lines that represent measures of (one or more) quantities in different frames of reference, and anticipating that ordered pairs (or *n*-tuples) give information about the measures in relation to each other, is the heart of combining multiple frames of reference.

Relationships Between Our Constructs

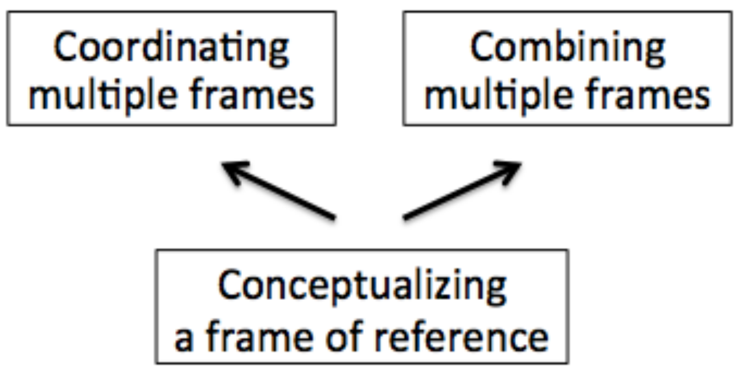


Figure 2: Relationships between constructs

In order to either coordinate or combine multiple frames, a person must first conceptualize each frame individually. The act of conceptualizing a frame is a necessary prerequisite as well as a part of both ways of reasoning with multiple frames. When a person engages in the mental actions of coordinating multiple frames he is attending first to measures taken in one frame of reference and then the other, as he takes a measure in one field and then draw conclusions about the corresponding measure in another frame. When a person engages in the act of combining multiple frames she is holding two measures in mind at once, essentially forming a multiplicative object (Saldanha & Thompson, 1998).

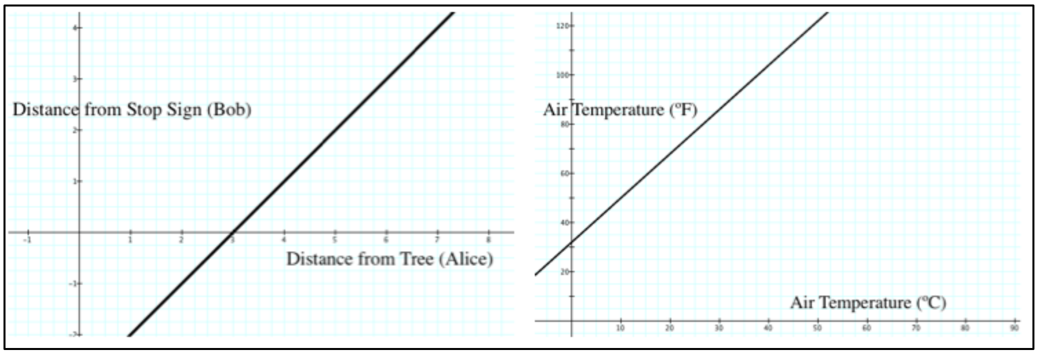


Figure 3. Examples of coordinate systems as combining multiple frames of reference.

We note that when the student imagines a point (an ordered pair) along either line in Figure 2 as representing the measures of quantities in different frames of reference, she has combined the frames. If, however, she sees the line not just as representing a set of coordinated measures of quantities, but as a transformational relation between values of the quantities, she sees the graph as representing a functional relationship between the quantities.

Placing Our Theoretical Perspective amongst Others

It has been widely reported that undergraduate students in mathematics have difficulty constructing meaningful formulas to represent how two quantities change together [support & cite]. Conceptualizing a quantity involves….An important aspect of defining a variable involves a precise statement of what quantity is being measured, including the reference point from which the measurement is being initiated and the units used to perform the measurement. A vaguely defined variable such as d = distance in feet that Jane has walked fails to establish a reference point from which the distance Jane has walked is being measured. This can lead to confusion, since establishing a frame of reference for a problem situation is important for facilitating the measurement of quantities. In fact it is not possible to conceptualize a quantity without making clear your framing of the problem context. Whenever a task involves multiple quantities defined within different reference frames, it is necessary for a person to have a clear understanding of those different frames in order to reason about the relation

Though both physicists and physics educators acknowledge that reference frames are a central concept in physics, they are never defined in terms of what an individual must conceptualize. Experts and teachers frequently speak of frames of reference in ways that can encourage problematic conceptions, eight of which I describe below. At the same time, though the concept has not been a major focus of study, it is widely acknowledged that students frequently struggle with problems involving coordination of multiple reference frames (Bowden et al., 1992; Monaghan & Clement, 1999, 2000). Finally, frames of reference as a concept is almost exclusively applied to relative motion contexts, and often reduced to the ability to make qualitative descriptions of motion of one object from the perspective of another.

Defining a Frame of Reference

'Frame of reference' is an idea frequently used in mathematics and the hard sciences, yet rarely defined to encompass all that must be understood in order to conceptualize a single frame or to reason with multiple frames. Typical physics textbook definitions range from "a coordinate system with a clock" (Young, Freedman, & Ford, 2011) to "a rigid system of 3 orthogonal rods welded together" (Carroll & Traschen, 2005) to "a set of observers at rest relative to each other"(de Hosson, Kermen, & Parizot, 2010) or "an arrangement of observers and measurement devices that determine the position and time of any event" (Scherr, 2007). Drivotin (Drivotin, 2014) took on the task of providing a formal mathematical definition of a reference frame using set theory that was motivated by finding a definition applicable to both inertial and non-inertial frames in both classical and relativistic mechanics, but did not include an argument for what his definition added to the conceptualization or use of reference frames. Frames of reference are also frequently defined implicitly, such as stating that "two observers… [are] in the same reference frame if they are at rest with respect to one another" (Scherr, Shaffer, & Vokos, 2001), or that each observer constitutes a distinct reference frame (de Hosson et al., 2010).

Perhaps the most extensive and detailed definition of a frame of reference, from an educational perspective, is the definition provided by Panse, Ramadas, & Kumar (Panse et al., 1994) in their study of what a fully matured idea of a frame of reference is *not*. Their study of undergraduate physics students revealed seven problematic alternate conceptions that students commonly hold about frames of reference, many of which may be promoted by teachers and experts using the definitions given above. These conceptions were so prevalent throughout the rest of the literature (both in problems that authors noted students had, and in the language that experts themselves used) that I find it useful to summarize them here for future reference.

- Alternative Conception 1 (AC1) is the belief that a frame of reference is a concrete object, in such a way that it could, for instance, experience friction.
- AC2 is the belief that a frame of reference is localized by the physical boundaries of the object that it is defined as being “attached” to, such that another object could exist “inside” or “outside” of the frame or that another object could be thrown from within the frame and exit the frame.
- AC3 is the belief that smaller objects on a larger object become part of the larger body’s frame, such that for a man walking on a ship (COMMA) the ship itself is at rest.
- AC4 is the belief that particular phenomena exist or take place within certain frames; for example, motion takes place within/with respect to one frame but not another.
- AC5 is the belief that motion is either “real” or “apparent” with the Earth’s surface frequently used as the definition of absolute rest; for example, a

train has “real” motion in the frame of the station, but the station has “apparent” motion in the frame of the train.

- AC6 is the belief that phenomena such as movement or existence only occur in a frame when it can be seen by an observer in the frame, such as a tree that only exists in a train station’s frame of reference if it is tall enough to be seen from the station platform.
- AC7 is the belief in pseudo-relativism, where the fact that descriptions of motion vary among observers is acknowledged and extended to the belief that a description of phenomena in a given frame of reference is not unique, but depends on “how it is viewed”.

Much of the existing work done on frames of reference, though valuable and contributing to educational efforts, actually promotes the problematic conceptions that Panse et al. found. For example, the definition of a frame of reference as rigidly welded rods (Carroll & Traschen, 2005) can easily promote a frame of reference as a concrete object (AC1) as well as the idea that a frame of reference has physical boundaries (AC2).

In addition to the seven alternative conceptions found by Panse et al., an eighth alternative conception has arisen out of my review of the literature:

- AC8, the belief that a frame of reference is useful primarily (or only) for an observer that remains at the origin of the frame’s coordinate system.

For example, the idea that every observer constitutes a different reference frame (de Hosson et al., 2010) can promote AC8.

Frames of Reference and Measurement

Despite the differing and sometimes limiting definitions of a frame of reference discussed above, they all agree in that the purpose of a frame of reference is to facilitate one's measurement of quantities. David Hestenes (Hestenes, 1996), developer of the Modeling Method paradigm for physics education, placed frames of reference at the center of an understanding of measurement when he defined a model as "a representation of structure in a physical system and/or its properties" and added that of the four aspects of structure, one was the representation of position and trajectory with respect to a reference frame. Indeed, every physical law we have, including ones as basic as Galileo's law for falling bodies $\Delta y = \frac{1}{2}gt^2$ is only true for an object in a specific reference frame (ibid). Yet students frequently struggle to internalize the idea that any measurement is meaningless without information about the frame of reference within which it was made, as in Marshall and Carrejo's study (Marshall & Carrejo, 2008) where almost no participants spontaneously qualified their measurements or models with reference points. Shen and Confrey (Shen & Confrey, 2010) studied how K-12 teachers understood that models were based on a reference frame by conducting teaching experiments on the issue of the technical equivalence of both the heliocentric and geocentric views. Though experts prefer the heliocentric view for the sake of elegance and parsimony, the teachers accepted the heliocentric view as "correct" and the geocentric view as “wrong” based on an argument to authority, not understanding that the views differed only by a change in the placement of the implicit reference frame’s origin.

At the same time, other studies on common physics student struggles show a lack of awareness even on the part of experts of how important a frame of reference is to imbue any measurement with meaning. In Goldberg and Anderson’s study of how students struggled with the idea of negative velocity (Goldberg & Anderson, 1989) frames of reference is never considered as a point of confusion. In Minstrell’s paper on helping students to understand the “at rest” condition in terms of Newton’s Second Law, it never makes mention of fact that an object is only “at rest” with respect to the particular reference frame of the Earth’s surface (Minstrell, 1982). Minstrell’s work is also of interest because, in studying student understanding of the “at rest” condition without mention of a frame of reference, he accidentally promotes a way of thinking about motion that leads to AC5 - the belief that motion is either “real” or “apparent” (Panse et al., 1994).

We also see students struggle to maintain a frame of reference in studies that focus on how students reason about rate of change, such as when they focus only on magnitude of average rate of change, or change the direction of comparison to growth or decay rate, in order to always have positive measurements to discuss (Ärlebäck et al. 2013).

Frames of Reference and Newton's Laws

Newton's laws of motion, though easy to state, are often difficult for students to grasp. Though typical physics textbook summaries of Newton's laws do not mention frames of reference (Young et al., 2011), an understanding of frames of reference is crucial for a student to be able to grasp Newton's laws. David Hestenes, a physics education reformer who developed the Modeling Method of Physics Instruction and the Force Concept Inventory, proposed that students be taught Newton's Zeroth Law, which is that every particle has a definite position x_i and a trajectory $x_i(t)$ with respect to a frame of reference, in order to prepare them for Newton's next three laws (Hestenes, 1992). Even more explicit in the literature is the idea that an understanding of frames of reference is central for students to understand the applicability of Newton's laws, since an inertial reference frame is frequently defined as one in which Newton's laws hold true (Hestenes, 1987; Ohanian, 2004) or, equivalently, a frame in which an observer sees no "inertial forces" (also referred to as "fictitious forces") (Arons, 1997; Bernstein, Fishbane, & Gasiorowicz, 2000).

Frames of Reference and the Principle of Relativity (Galilean)

The relativity principle, though most famous for its application to Einstein's special and general theories of relativity, is also fundamental to classical phyiscs. The principle of relativity was studied in detail by Bandyopadhyay et al. (Bandyopadhyay, 2009), when they identified three "apparently separate but in fact entirely equivalent [meanings] of the relativity principle… (i) the inadmissibility of the notion of 'a frame at absolute rest', (ii) the inability to determine the 'velocity of a frame' from measurements with respect to the frame itself and (iii) the absence of the 'velocity of frame' term in the pseudo forces for a non-inertial frame". I will not go further into their (iii) meaning as it seems to be based on an incomplete argument that only considers rotating non-inertial frames but not linearly accelerating non-inertial frames, and also appears to be more of a justification for meaning (ii) than a valid conceptual consequence in and of itself. However, meanings (i) and (ii) are ideas that can be used as markers to see if students truly grasp the principle of relativity and all of its consequences; in essence, they capture the idea that it is impossible to measure anything in terms of absolute space or motion.

The issue of whether one can determine absolute space and motion stretches back past Newton, who himself believed that absolute space and motion (and therefore absolute rest) existed, though his laws of motion do not require their existence. It was frequently thought that the luminiferous ether would provide a rest frame by which to measure absolute space, rest, and motion, and the search for such a rest frame only died with the search for the ether at the turn of the twentieth century (Zylbersztajn, 1994).

However, students continue to struggle with the idea that all measurement of space and motion is merely relative to a reference frame (or even relative to another object). They frequently believe AC5, that non-zero measurements of motion can be classified as "true" or "apparent" motion, implicitly defining the

Earth's surface as an arbiter of absolute rest. If the idea that the Earth itself is moving with respect to the sun is pointed out, many students revert to the idea that an at rest observer in space could determine absolute rest (McCloskey, 1982; Monaghan & Clement, 1999, 2000; Panse et al., 1994)

In non-physical contexts, or simply in the most general viewpoint of frames of reference as tools for measurement, I find the principle of relativity to be analogous to the idea that different frames can give different but equally valid measurements because they reference different frames for context and meaning. For example, a person on a bike can measure a deer's velocity at 25mph due North while another in a train can measure it at -15mph due North, and both measurements are equally valid (and, indeed, equally valid to the deer's velocity of 45mph due North with respect to the Earth's surface). To reach such an understanding would enable a student to fully use frames of reference as a conceptual tool.

Frames of Reference and the Principle of Relativity (Einsteinian)

No study of frames of reference can be complete without mentioning the centrality of reference frames to Einstein's theories. Indeed, Einstein developed his theory of special relativity by trying to reconcile the equal validity of all inertial frames of reference (the principle of relativity) with the natural conclusion of Maxwell's equations that light has the same speed when measured in any frame of reference (Carroll & Traschen, 2005). A study of freshman and senior physics majors in Brazil showed that while some students correctly predicted how physical laws would behave in reference frames moving at different speeds (including relativitistic speeds), they mostly justified these assertions with reference to common speeds or the authority of the scientific community instead of the principle of relativity. Other students believed that the theory of special relativity meant that strange and non-intuitive things happened *within* frames moving at constant relativistic velocities, though the principle of relativity states that an observer in such a frame would be incapable of measuring the frames velocity and therefore equally justified in claiming his frame is at rest (Pietrocola, 1999). These students do not see that to say that a frame is moving at a relativistic velocity only means that it is moving at a relativistic velocity *with respect to some other frame of interest*, and therefore measurements of the same quantity taken from each of these two frames will be different. Another study in Argentina of a lesson sequence developed to teach secondary students special relativity said that “that from an epistemological point of view, it is not possible for students to build the concepts of space and time in the special relativity theory unless they previously have the notions of ‘reference frame’, ‘observer’ and ‘simultaneity’ within the frame of classical Mechanics”. After carrying out their lesson sequence, they interviewed their students and considered one of the most significant results to be that more than half of the students admitted the need of a reference system to solve a problem involving the notion of movement and most students agreed on the need to redefine the concept of observer within the context of special relativity (Arriassecq & Greca, 2012). One of the consequences of special relativity is that simultaneity is relative; two events may be recorded as occurring simultaneously in one frame but not another, and both recordings are equally valid. Students frequently struggle with the idea that simultaneity is relative, and 2001 paper posited that this struggle results partially from an incorrect set of ideas that include the belief that each observer constitutes a distinct reference frame (Scherr et al., 2001).

Later in life, Einstein's formulation of the theory of general relativity was motivated by considering the principle of equivalence in mechanics which states that a uniformly accelerated non-inertial frame can be considered to be inertial if there is an equal but opposite uniform gravitational field in the opposite direction (Bandyopadhyay & Kumar, 2011). Recent studies have shown that students struggle to understand the principle of equivalence (Bandyopadhyay & Kumar, 2010a) as well as the principle of covariance; a physical law or principle is covariant with respect to some transformations of reference frames if their basic equations retain their form under said reference frame shifts (Bandyopadhyay & Kumar, 2010b).

Clearly consideration of frames of reference are not only central to understanding and using Einstein's theories (and those who built on his work), but were also central to the development of the theories of special and general relativity in the first place.

Frames of Reference and Visual Imagery of Relative Motion

Many approaches to student thinking about frames of reference in the literature focus on the utility of visual imagery, and of computer simulations as an instructional tool. Many microworlds, or programs designed to provide an immersive experience with ideas that students can freely explore and discover , have been developed to promote the use of visual imagery in reasoning about and coordinating frames of reference, such as Thinkertools, RelLab, BOXER, ScienceSpace, and NewtonWorld (Dede, Salzman, & Loftin, 1996; Monaghan & Clement, 2000). Monaghan and Clement (Monaghan & Clement, 1999) wrote that computer simulations helped students develop mental imagery and ability to switch between frames of reference (such as a driving car and a plane overhead). In this way students could eventually correctly answers questions about how, for example, a slow-moving car would appear to be moving backwards from the perspective of an observer in a faster vehicle, where both were moving in the same direction with respect to the Earth's surface. Similarly, a central focus of Dede et al.'s simulations was to allow students to "attach" themselves to different objects and observe the simulation from different perspectives. However, they made no effort to define or explain what they meant by frames of reference other than by referring to concrete objects that were at relative motion with respect to each other, and in doing so risked perpetuating AC1 (a frame as a concrete object), AC2 (a frame as being attached to an object) and AC6 (phenomena only exist within a frame when they can be visually seen from the origin of the frame). A central goal of Monaghan and Clement was to further the idea that for an observer in a frame, the frame itself as at rest. However, while it is useful to understand the idea of an object at rest with respect to a frame, an observer using a given frame does not have to be moving with that frame, and in fact the language of “in a frame” or “attached to a frame” can easily perpetuate AC2, the idea that a frame defines a space that an object can be inside or not (Bandyopadhyay, 2009; Panse et al., 1994). I would add that it also promotes AC8, the idea that a frame can only be used by an observer that remains at the origin.

Frames of Reference as a Problem-Solving Tool

There are comparatively few works that look at frames of reference as a problem-solving tool amongst others (such as vector addition, proportional reasoning, formula use, etc.), and while they tend to categorize a frames of reference approach as superior, they do not offer any justification for doing so. Bowden et al. (Bowden et al., 1992) looked at the different approaches students used to analyze problems that involved an object moving inside another moving object and concluded that few students focused on “distinguishing frames of reference” (p.263-264) as opposed to other problem-solving strategies. Kozhevnikov (Kozhevnikov, Hegarty, & Mayer, 1999) also studied how students solved problems but compared their strategies to their cognitive strengths and concluded that while students stronger in “spatial layout ability” were able to encode problems into concepts of frame of reference with only relevant information and move fluidly between frames, students stronger in "visual ability" envisioned all given information, relevant or not, and viewed information given in different reference frames as separate problems. Curiously enough, although both groups made intensive efforts to study student use of frames of reference, neither gave any definition of a frame of reference, though Bowden repeatedly refers to frames by the object to which they are "attached", language which could easily promote AC2.

Expanding the Theory of Quantitative Reasoning

As we showed above, whenever an author (usually of a textbook) did explicitly describe what he or she meant by a frame of reference, the description focused on a frame of reference as an object or objects (Carroll & Traschen, 2005; de Hosson et al., 2010; Young et al., 2011). Such definitions support a student in focusing on the object of a frame of reference itself. In contrast, a key moment in developing our theory was when we began framing the question as "How does a student think about measures within a frame of reference?" As we said earlier, we defined a fully conceptualized frame of reference by stating that "An individual conceives of *measures as existing within a frame of reference* if the act of measuring entails [three commitments]." In other words, the mental actions, behaviors, and skills that we traditionally associate with someone "understanding frames of reference" (whatever that means) have nothing to do with how one thinks about frames of reference and everything to do with how one thinks about quantities.

In his first article about quantitative reasoning, Thompson (Thompson, 1993) defined a quantity by saying that a “person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it”. For example, if a person looks at an object such as a chair and thinks about a measurable attribute of the chair such as height, surface area, or mass, as well as appropriate units (such as linear units of length for height) then that person is now thinking about a quantity. He also added in an unpublished paper available online that this includes implicitly or explicitly thinking of appropriate units (Thompson, 1990). We find this to be a useful definition that provides a place to start thinking and talking about quantities, especially with younger children. However, curricula that seek to emphasize quantitative reasoning have highlighted further aspects of quantities, such as measuring a quantity in relation to a reference point (Marilyn P. Carlson, Oehrtman, & Moore, 2013).

Therefore, we define the idea of a *framed quantity*, which refers to when a person thinks of a quantity with commitments to unit, reference point, and directionality of comparison. As an example, consider a person who thinks about measuring how far Yolie has traveled as she walks her dog, understanding that appropriate units would be linear units such as feet, meters, and miles. This person is thinking about a quantity. In contrast, a person thinking about measuring Yolie’s displacement to the east from her front door in meters is conceiving of a framed quantity. Not only does this person’s mental construction have all the aspects of a conceptualized quantity, but it also shows a commitment to a unit (meters), reference point (front door) and directionality of comparison (displacement to the east yields positive measures). In other words, the quantity is so well defined that any measure value contains all the necessary information to understand its meaning. If x = Yolie's displacement to the east from her front door (meters), then $x = 3$ means that Yolie is 3 meters to the east of her front door and $x = -5$ means that Yolie is -5 meters to the east of her front door (which could be interpreted as being 5 meters west of her front door if wanted, but also provides the same specific meaning without this reframing). No extra qualifiers are needed to make sense of the value, and there is a clear directionality of comparison: the value always says how much further in the eastern direction Yolie is than her front door.

In Thompson's 2011 paper he identified a number of dispositions that would aid students' construction of algebraic thinking from quantitative thinking, including a disposition to represent calculations in open form, propagate information, think with abstract units, and reason with magnitudes. To this list we can now add that a disposition to think about measures within a frame of reference, and specifically with a direction of comparison, aids students in algebraic thinking. In constructing formulas students are often perplexed as to how to choose between $a – b$ and $b – a$, or a/b and b/a. This confusion can now be explained by thinking about how students do or do not commit to a directionality of comparison. Let us think about a student that is comparing the heights of husbands and wives in a study of couples. If the student sometimes frames the results of the comparison as “the husband is 6 inches taller than the wife” and other times “the husband is 2 inches shorter than the wife” then he is internally switching between two quantitative operations, which have corresponding formulas of $h - w$ and $w - h$, where h represents the husband’s height and w represents the wife’s height, both in inches. Naturally such a student would have difficulty in developing a formula to compare heights. In contrast, another student may commit to a directionality of comparison by deciding the value of his measure will always describe ‘how much taller the husband is than the wife’. Since such a commitment entails always using the same quantitative operation, such a student will have far less obstacles to describing his process in symbolic form as $h - w$.

Since our theory was first presented in a conference paper (Joshua 2015), more work has been done on frames of reference based on our conceptual definitions. such as a pilot study on how students reason about changes within a frame of reference (Joshua 2016), how students reason about relationships about quantities and graphical representations within a frame of reference (Lee et al. 2019), and how teachers reason with frame of reference (Joshua 2019).

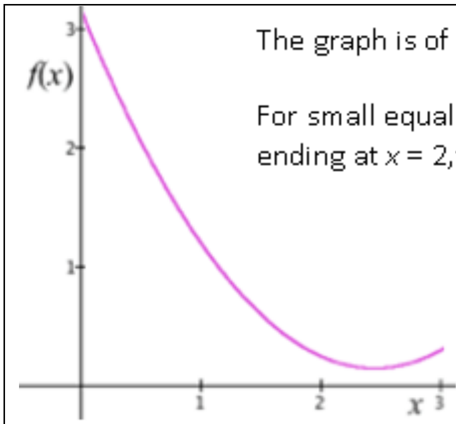
Future Directions

As can be seen from the above literature, there is still much work to be done on the concept of frames of reference. Much can be drawn from the work that has already been done, with modifications to avoid implicitly encouraging problematic alternate conceptions. We also find it significant that no work appears to have been done on the application of frames of reference to any contexts other than relative motion. Below we briefly discuss five topics, only two of which are relative motion, in which we see potential benefits when the constructs of a *conceptualized frame of reference*, *reasoning with multiple frames of reference*, and *framed quantities* are applied.

1) Personal experiences in teaching pre-calculus and calculus had shown us that students frequently conflate the value of a quantity and a change in that quantity, which leads to difficulties in understanding the ideas of change, slope, constant rate of change, and rate (derivative) functions. This confusion may be explained by a lack of attention to reference point for each measure; if a student does not commit to a reference point when measuring a quantity, there is little meaningful difference between the measure of the total quantity and a change in that quantity over a given interval. On the other hand, developing the idea that the total quantity is really a change from (a reference point of) zero provides parallel ideas with which to distinguish the two. Highlighting reference point commitment in teaching and discussion may help to alleviate this confusion.

2) In Project ASPIRE, an NSF-funded collaboration between Arizona State University and University of California at Berkeley, we found that student struggles when reasoning about multiple changes could be attributed to lack of commitment to a frame of reference.

Consider the task in Figure 3 that asks the reader to compare consecutive changes in the interval [1, 2].



The graph is of a function f over interval [0,3].

For small equal increases in the values of x starting at $x = 1$ and ending at $x = 2$, the corresponding changes in the value of f are...

a) positive and increasing
b) positive and decreasing
c) negative and increasing
d) negative and decreasing
e) I cannot tell

Figure 3. Comparing Changes Task. © 2014 Arizona Board of Regents. Used with permission.

This task proves challenging for people who do not think about changes within a frame of reference – specifically, people who do not maintain a directionality of comparison. Consider two hypothetical students: Dean who chooses option d) and Cathy who chooses option c). Assume both students understand the directionality of changes well enough to visualize changes as in Figure 4A.

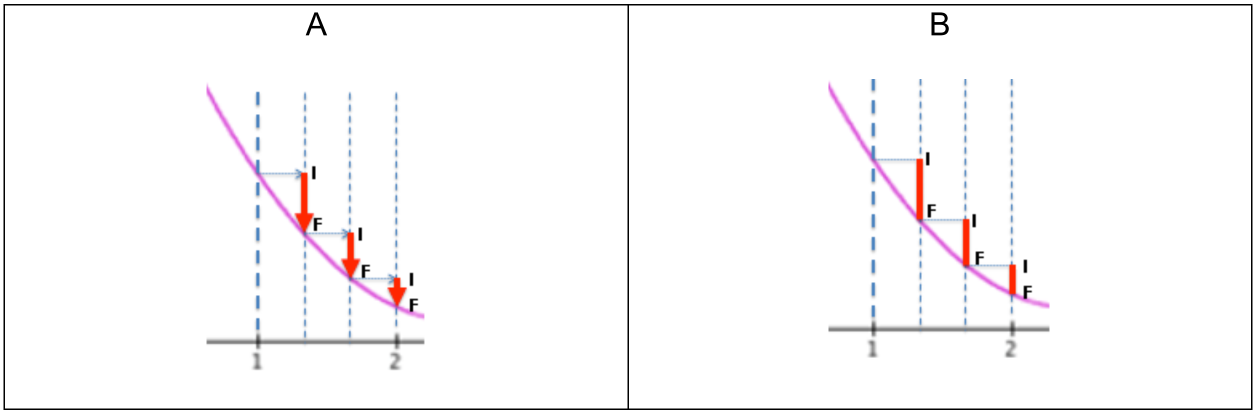


Figure 4. Different Visualizations of the Comparing Changes Task.

Dean says that the changes are negative and decreasing because he has inadvertently switched the direction of his comparison between deciding “the changes are negative” and “the changes are decreasing.” To determine that the changes are negative, he is engaging in a quantitative operation that we can formulize as [final *y*-value] – [initial *y*-value] and obtains a negative value for each. However, in deciding that the changes are decreasing, he is really only considering the magnitude of those changes, essentially switching his mental image to that shown in Figure 6B and engaging in a quantitative operation that we can formulize as [larger *y*-value] – [smaller *y*-value]. In comparison, Cathy says the changes are negative and increasing because she has maintained her directionality of comparison. For both her “changes are negative” and “changes are increasing” decisions, she engages in a quantitative operation that can be formulized as [final *y*-value] – [initial *y*-value]. We gain insight into individuals’ difficulties with this task by noticing a lack of commitment to directionality of comparisons.

3) Students frequently categorize all motion within a false dichotomy of “real motion” vs. “imagined motion”, where an object is only “really moving” if it is moving with respect to the surface of the Earth, and the measure of its speed or velocity is only “real” if measured with respect to the surface of the Earth (Panse et al., 1994). This hinders their ability to deal with relative motion tasks and has been a focus of study in physics education (Monaghan & Clement, 1999; Shen & Confrey, 2010). For example, students cannot accept that a bike moving 15mph towards a sign is also moving 5mph with respect to a walker and moving backwards with respect to a car. While Monaghan and Clement worked on developing their students’ visual imagery, we believe that teaching students about conceptualizing all quantities as measured with specific reference points, and comparing quantities with specific directionalities of comparison, may prove beneficial.

This common student struggle with “real” versus “imagined” motion stems from a lack of understanding of the fundamental physics principle of relativity (Bandyopadhyay, 2009) that states that there can be no way of verifying that any reference frame (or object) is at absolute rest, and therefore the entire notion of absolute rest should be abandoned. We believe emphasizing that a reference point and directionality of comparison are mandatory for any measure to be meaningful can provide a backdrop for students to also accept that what we talk about as motion measure in the real world always comes with its implicit assumption of a reference point (the surface of the Earth), and that if all reference points are arbitrary then the surface of the Earth is as well. Such an emphasis would also move students forward in understanding the principle of relativity which is fundamental for understanding both classical physics and special relativity.

4) One of the most common struggles students have in physics is in understanding the concepts of velocity and acceleration. For example, researchers have found it extremely difficult to change the student perception that a positive acceleration means an object must be speeding up (when in fact it may be going from -5mph to -2mph, meaning it is slowing down but increasing in velocity). We have found in personal conversations that even professors who are known for their work in physics education have been teaching students that an object going from -10mph to -20mph means that “the velocity is increasing in the negative direction,” probably to deal with these types of misunderstandings. But not only are such descriptions physically and mathematically inaccurate, they result in descriptions that are incompatible with observations about change and rate of change that can be derived from calculus. We believe that teaching students about a commitment to directionality of comparison is far more consistent and fruitful way to approach these concerns.

5) We are grateful to an audience member at our presentation of an earlier version of this paper at the RUME 18 conference, who offered the idea of electric potential as another concept we can reconceive through our constructs for frames of reference. It is true that students struggle with the idea of electric

potential, and our minds immediately went to the struggles that physics and engineering students have with Kirchoff's second laws for circuits. Briefly stated, Kirchoff's second law states that the sum of the changes in electric potential around any loop in a closed circuit is zero. Students often struggle with how to apply the rule because they feel a need to know where in the circuit the potential is "really zero" so that they can start their calculations there, not understanding that (like absolute rest) there is no such thing as absolute zero electric potential. These student difficulties may be alleviated by the same measures that help students to understand the principle of relativity in motion.

6) Early childhood and elementary school lessons on measurement tend to focus on measuring by iterating units and converting unit measurements as early as Grade 4 (Common_Core, 2015), but Project Aspire project found that even secondary teachers struggled to correctly convert a measure of m liters to an equivalent measure in gallons. We suspect that students will develop stronger measurement schemes if the idea of measurement itself was introduced with tasks designed to help students conceptualize a frame of reference, at age-appropriate levels. Such an endeavor would help students to develop the idea of magnitudes that exist independent of units, an essential part of developing what Thompson et al. called "a 'Wildi magnitude' sense of magnitudes" (Thompson, Carlson, Byerley, & Hatfield, 2014).

We believe that research on frames of reference and student thinking about frames of reference is warranted by the difficulties that students have with “typical” frames of reference problems. Moreover, we think that our cognitive definitions of a *conceptualized frame of reference*, *coordinating multiple frames*, *combining multiples frames*, and *framed quantities* offers a starting place from which to investigate not only student thinking in physics and relative motion problems but also many mathematical and scientific domains that were previously not thought of as related to frames of reference at all. From our review of the literature, of current problems in physics and math education research, and our own burgeoning research on frames of reference, we believe that the constructs proposed here offer new insight on student difficulties and contributes to a foundation for further research.

CHAPTER 2

PAPER TWO: CALCULUS STUDENTS' REASONING ABOUT CHANGES WITHIN A FRAME OF REFERENCE

Two introductory calculus students participated in multi-day teaching experiments designed to investigate how they reasoned about changes (variations). The data was analyzed to see to what extent each student conceptualized the variations within a frame of reference, coordinated variations in multiple frames, and combined frames in a coordinate system. We found that the extent to which each student conceptualized, coordinated, and combined reference frames significantly affected his ability to reason productively about variations and to make sense of his own answers. We end by analyzing 123 calculus students' written responses to one of our tasks to build hypotheses about how calculus students at our university reason about variations within frames of reference.

Keywords: Frame of Reference, Quantitative Reasoning, Difference, Variations

The report of the 5-year NSF-MAA calculus study (Bressoud, 2015) notes that despite many reform efforts over the past decades, college calculus still functions as a sieve that filters many students out of STEM careers, to the detriment of both the individuals and their fields of study. Higher education institutes in the U.S. are still struggling to improve their calculus pass and retention rates in calculus sequences and majors that require calculus. The 2010/2012 NSF-MAA study, often termed the “Big Calculus Study”, is a useful place to begin looking at the problem of college calculus success because it gathered extensive survey & interview data on 300,000 college Calculus 1 students. The study examined several aspects of calculus courses at colleges and universities throughout the country. These are common types of institutional variables that are often the focus of change. However, none of these variables address the quality of the content being taught. Institutions will likely fail to solve their retention problems if the content of calculus class is incoherent, disconnected, or meaningless to students learning it.

The Big Calculus Study’s “cognitive goals” section provides insight into our obstacles students face in making calculus ideas meaningful and coherent. A survey of 420 professors indicated that that their *goals* emphasized students being able to reason through problems on their own, but their *image* of students was that they memorize material the way it is presented (Bressoud, 2015). However, professors’ assignments and assessments tended to emphasize memorization, with a mean of 50% of problems focusing on computations and only 20% involving novel problems, proofs, or justifications (Tallman, 2016). We see a clear disconnect in this data between what professors want from their students and what they require students to do.

Many of the core ideas in calculus are founded in the idea that values of quantities vary.[1] Both instantaneous and average rates of change involve multiplicative comparisons of variations in quantities' values, which themselves are additive comparisons of quantities' values. Both net and total accumulations involve the summation of many small variations as an independent variable's value varies. A student therefore will have difficulty grasping any of these ideas if they cannot conceptualize variations in quantities' values as quantities in themselves and operate quantitatively on them.

To operate quantitatively on variations in quantities' values, a student must conceptualize a variation as a quantitative difference between two other quantities (P. W. Thompson, 1993). To conceive a quantitative difference, one must conceive the result of an additive comparison as a new quantity in and of itself, in relation to the quantities that make it. If students cannot think of a variation separately from the operation (canonically subtraction) that led to its value, they cannot keep track of the meaning of that variation or incorporate it into other information to solve a task.

[1] Thompson and Ashbrook (2016) explain that the word "change" is used ambiguously by many calculus instructors and calculus textbooks. At times instructors and texts use "change" to mean change in progress, at other times they use "change" to mean completed change, and at other times they use "change" to mean replace one thing (number, word, or phrase) by another. Thompson and Ashbrook also explain that the word "change" in "rate of change" always means change in progress. I follow Thompson and Ashbrook to use "vary" for change in progress, "variation" to mean completed change, and "change" to mean replace one thing by another.

We collected data at Arizona State University to investigate the degree to which Calculus 1 students differentiate the meaning of a variation in a quantity versus a total quantity (Musgrave, Hatfield, & Thompson, 2015). Musgrave et al. found that calculus students struggle with this idea, showing they have a weak base for building more complex calculus ideas. For example, students cannot comprehend rates of change if they cannot distinguish variations from totals. Accumulation cannot be understood if the relationship $dy=m \cdot dx$ is not seen as different from $y=m \cdot x$; the former can be used to find accumulation from any changing rate while the latter has extremely limited application.

We postulate that student difficulties in understanding and reasoning with variations stem at least partially from their inability to conceptualize a frame of reference within which to consider variations (Joshua, Musgrave, Hatfield, & Thompson, 2015), and that this is the source of many of their difficulties in calculus classes. For example, students commonly confuse a variation in a quantity with a total quantity, which can be attributed to a lack of commitment to reference point. If students have not conceptualized quantities as being measures from reference points, they have a weak basis from which to distinguish totals and variations.

To investigate how calculus students reason about calculus, we ran a multi-day teaching experiment with two students who recently finished Calculus 1, and also gave one of our tasks as a written item to 123 Calculus 1 students.

Our research questions were:

Research Question #1: To what extent do the two students combine multiple frames of reference?

Research Question #2: To what extent do the two students conceptualize a frame of reference by committing to reference point and direction of comparison?

Research Question #3: To what extent do the two students coordinate multiple frames of reference?

Research Question #4: To what extent do calculus students in our larger sample commit to direction of comparison?

Literature Review

Our interest in frames of reference and reasoning with frames of reference came about in an unexpected way. While analyzing teachers' responses to two items intended to target proportional thinking and rate of change, we found that teachers' responses to both items revealed struggles with coordinating quantities measured in what we came to realize were different frames of reference. Bowden et al. (Bowden et al., 1992) looked at the different approaches students used to analyze problems that involved an object moving inside another moving object (such as vector addition or proportional reasoning) and concluded that few students focused on "distinguishing frames of reference". Bowden et al. noted that they attempted to characterize students' meanings based on their entire transcripts; however, Bowden et al. did not explain what they meant by "frames of reference". Rather, they used "frame of reference" as the possession of some object, e.g. "the frame of reference of the boat," Likewise they did not explain what they meant by “students’ meanings.” Monaghan and Clement (Monaghan & Clement, 1999) wrote that computer simulations helped students develop mental imagery and ability to switch between frames of reference (e.g., as in a scenario involving a moving car and a plane flying overhead). However, they did not define or explain what they meant by frames of reference other than using pointers as Bowden et al. did. In further work they continued to use the construct of frames of reference without explicating what they meant by it (Monaghan & Clement, 2000). Panse et al. (Panse, Ramadas, & Kumar, 1994) investigated and identified “alternative [unproductive] conceptions” that students had about frames of reference, such as the idea that a frame of reference was a concrete object with boundaries or that a frame of reference is defined by the existence of a concrete object. While they did valuable work in describing alternative conceptions that hindered students’ ability to reason about physical situations, they did not describe their normative conception of frames of reference. In all literature focusing on the idea of frames of reference or student thinking thereof, the authors presume that they and their readers share a common understanding of what “frames of reference” entails.

The few times an author (usually of a textbook) did explicitly describe what he or she meant by a frame of reference, the description focused on a frame of reference as an object or objects that exist outside a person. Typical definitions range from “a coordinate system with a clock” (Young, Freedman, & Ford, 2011) to “a rigid system of 3 orthogonal rods welded together” (Carroll & Traschen,

2005) to “a set of observers at rest relative to each other” (de Hosson, Kermen, & Parizot, 2010), with no further discussion about how students must conceptualize a frame of reference in order to reason with them. Such definitions support a student in focusing on the object of a frame of reference itself. In contrast, a key moment in developing our theory was when we began framing the question as “How does a student think about measures within a frame of reference?” As we said earlier in this manuscript, we defined a fully conceptualized frame of reference by stating that “An individual conceives of *measures as existing within a frame of reference* if the act of measuring entails [three commitments].” In other words, the mental actions, behaviors, and skills that we traditionally associate with someone “understanding frames of reference” (whatever that means) have nothing to do with how one thinks about frames of reference and everything to do with how one thinks about quantities, or measurable attributes of objects (A. G. Thompson & Thompson, 1994; P. W. Thompson, 2011). Curricula that seek to emphasize quantitative reasoning have highlighted further aspects of quantities, such as measuring a quantity in relation to a reference point (Carlson, Oehrtman, & Moore, 2013).

Therefore, we define the idea of a *framed quantity*, which refers to when a person thinks of a quantity with commitments to unit, reference point, and directionality of comparison. As an example, consider a person who thinks about measuring how far Yolie has traveled as she walks her dog, understanding that appropriate units would be linear units such as feet, meters, and miles. This person is thinking about a quantity. In contrast, a person thinking about measuring Yolie's displacement to the east from her front door in meters is conceiving of a framed quantity. Not only does this person's mental construction have all the aspects of a conceptualized quantity, but it also shows a commitment to a unit (meters), reference point (front door) and directionality of comparison (displacement to the east yields positive measures). In other words, the quantity is so well defined that any measure value contains all the necessary information to understand its meaning. If x = Yolie's displacement to the east from her front door (meters), then $x = 3$ means that Yolie is 3 meters to the east of her front door and $x = -5$ means that Yolie is -5 meters to the east of her front door (which could be interpreted as being 5 meters west of her front door if wanted, but also provides the same specific meaning without this reframing). No extra qualifiers are needed to make sense of the value, and there is a clear directionality of comparison: the value always says how much further in the eastern direction Yolie is than her front door.

In Thompson's 2011 paper he identified a number of dispositions that would aid students' construction of algebraic thinking from quantitative thinking, including a disposition to represent calculations in open form, propagate information, think with abstract units, and reason with magnitudes. To this list we can now add that a disposition to think about measures within a frame of reference, and specifically with a direction of comparison, aids students in algebraic thinking. In constructing formulas students are often perplexed as to how to choose between *a* – *b* and *b* – *a*, or *a/b* and *b/a*. This confusion can now be explained by thinking about how students do or do not commit to a directionality of comparison. Let us think about a student that is comparing the heights of husbands and wives in a study of couples. If the student sometimes frames the results of the comparison as “the husband is 6 inches taller than the wife” and other times “the husband is 2 inches shorter than the wife” then he is internally switching between two quantitative operations, which have corresponding formulas of *h* – *w* and *w* – *h*, where *h* represents the husband’s height and *w* represents the wife’s height, both in inches. Naturally such a student would have difficulty in developing a formula to compare heights. In contrast, another student may commit to a directionality of comparison by deciding the value of his measure will always describe ‘how much taller the husband is than the wife’. Since such a commitment entails always using the same quantitative operation, such a student will have far less obstacles to describing his process in symbolic form as *h* – *w*.

Theoretical Framework

Conceptualizing a Frame of Reference

Our definition of a conceptualized frame of reference depends on the nature of how people reason about quantities, or measurable attributes of objects. Thompson defines quantitative reasoning as “the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships” (A. G. Thompson & Thompson, 1994), where a quantity is a person's conceptualization of an object and attribute of it so that it is measurable. "A quantity is in a mind. It is not in the world." (P. W. Thompson, 2011). Therefore, our definition of a frame of reference addresses the mental actions a person takes to think about quantities within a frame of reference rather than whether they understand an object called a 'frame of reference'. This clarification guided our eventual definition:

> An individual can think of a measure as merely reflecting the size of an object relative to a unit or he can think of a measure within a system of potential measures and comparisons of measures. An individual conceives of measures as existing within a *frame of reference*[2] if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives meaning to a zero measure and all non-zero measures, and 3) committing to a direction of measure comparison additively, multiplicatively, or both. Further discussion of these commitments can be found in (Joshua et al., 2015).

Coordinating Multiple Frames of Reference

Once a person has conceptualized quantities within a frame of reference, she can coordinate the measures of those quantities with measures taken in a different frame if she engages in several cognitive steps. The first step is to recognize that there is a need to coordinate two frames, because measures of

[2] A person can conceptualize a frame of reference additively or multiplicatively. Our uses of "frame of reference" in this paper will refer to additive frames.

quantities taken in different frames cannot be combined or compared meaningfully until the frames have been coordinated. The next step is for the person to coordinate individual measures between the two frames and to decide on the nature of the transformation that will express measures in one frame in terms of another frame. The final step is to apply that transformation to all measures in the first frame.

Combining Multiple Frames of Reference

Individuals combine two frames of reference they have conceptualized independently by considering two quantities within their respective frames simultaneously. A graph in a coordinate system is a representation of combining reference frames.

Not every person looking at or envisioning a graph is combining multiple frames. To conceptually combine multiple frames means that the person is engaging in the mental actions of holding two (or more) quantities in mind simultaneously. Thompson and colleagues (Saldanha & Thompson, 1998; Thompson & Saldanha, 2003; Thompson & Carlson, 2017; Thompson, Hatfield, Byerley, & Yoon, 2017) called the act of holding in mind two quantities values simultaneously “creating a multiplicative object” and demonstrated that it is a nontrivial act for students and teachers to accomplish.

Imagery

Thompson and colleagues (A. G. Thompson & Thompson, 1994; P. W. Thompson, 1996, 2013; P. W. Thompson, Carlson, Byerley, & Hatfield, 2014), in expanding on Piaget's construct of scheme, explained the role and importance of individuals' imagery in activating their schemes. Thompson (1994) explained that while images are fragments of remembered experience, they play different roles in a person's thinking depending on the level of a scheme's development (its operativity). A person's thinking might be at an early phase in regard to scheme development, meaning that images they form at a moment dominate their reasoning, or it might be at an advanced (third) phase in regard to scheme development, meaning that an image they form at a moment triggers a network of relationships and associated imagery. Piaget said of the first phase:

> The image is a pictorial anticipation of an action not yet performed, a reaching forward from what is presently perceived to what may be, but is not yet perceived. (Piaget, 1967, p. 294)

Piaget said of the third phase:

> [This is an image] that is dynamic and mobile in character … entirely concerned with the transformations of the object. … [The image] is no longer a necessary aid to thought, for the actions which it represents are henceforth independent of their physical realization and consist only of transformations grouped in free, transitive and reversible combination … In short, the image is now no more than a symbol of an operation, an imitative symbol like its precursors, but one which is constantly outpaced by the dynamics of the transformations. Its sole function is now to express certain momentary states occurring in the course of such transformations by way of references or symbolic allusions. (Piaget, 1967 p. 297)

The early imagery is not relevant for this study, as subjects in it used imagery beyond it. But Piaget described a second phase of imagery that is important for this study. He said this about a second phase of imagery:

> In place of merely representing the object itself, independently of its transformations, this image expresses either a phase or an outcome of the action performed on the object. … the most interesting feature of this type of image is its failure to anticipate the

> result of the transformation in a complete and accurate fashion. Whilst such images constitute an imitation of these very actions … they are barely able to keep abreast of the actions. In other words, the action cannot be adequately visualized all the way to its ultimate conclusion before it has actually been performed. (Piaget, 1967, p. 295)

We will see a subject who forms images that are in this second phase while working with frames of reference tasks. Images he forms suggest to him actions of thought with regard to measurements from a reference point, but he cannot coordinate multiple actions because his imagery does not imply the results of his actions and he therefore cannot engage in chains of reasoning regarding reference points. We will also see a subject whose imagery with regard to frames of reference is in the third phase.

The strength and applicability of a person's schemes entails a co-dependency between relationships among actions the person has formed and the imagery triggering them. If the imagery associated with a scheme is tied to surface features of tasks such as perceptual material, context, or sensorimotor experience, then the scheme will only be triggered by tasks similar to those that the she already experienced. However, if she has built imagery for the scheme that is related to the relationships between quantities that are invariant across these tasks, she is more equipped to apply a scheme to an unfamiliar context that still contains those kinds of relationships.

Researchers in physics education witnessed this phenomenon when comparing how novices and experts categorized textbook problems. Experts categorized them on the underlying physics principles they understood as involved in the problem situation. Novices categorized them based on the surface features of the problem (Chi, 1981). Piaget's idea of imagery connected to schemes gives us an explanatory framework for this dichotomy.

In brief, a person whose imagery is in the first or second phase in regard to contexts she witnesses or imagines by way of reading text will exhibit thinking that belies a greater dependence on context than a person whose imagery is in the third phase. We also take the reverse stance, that a person who exhibits thinking that seems to change among contexts *we* see as embodying the same ideas does so because of imagery that is at an early phase of development.

Methodology

DIRACC Calculus Class

All student participants took introductory calculus class in Fall 2018 at a large Southwestern university based on Thompson's Project DIRACC: Developing and Investigating a Rigorous Approach to Conceptual Calculus. Thompson designed the course around two fundamental questions:

(a) you know how fast a quantity varies at every moment; you want to know how much of it there is at every moment, and

(b) you know how much of a quantity there is at every moment; you want to know how fast it varies at every moment. (Thompson & Ashbrook, 2018).

The entire course was built around the goal of encouraging students to reason quantitatively, including reasoning quantitatively about variations. We chose to conduct this study only with students from this class because we knew that they had already had exposure to at least a semester of mathematics where they were asked to think and reason about variations as additive comparisons of quantities.

Participants

One hundred and twenty-three (123) students taking a redesigned introductory calculus course initially answered a single item (the Bank Account Task, in Figure _) as part of their class in October 2018. Those that were interested in participating in interviews were asked to provide their email, and the first fifteen (15) students who responded were given a pre-test and then audiotaped for a 2-3 minute interview about their thinking on the Bank Account Task they had answered in class.

As we reviewed the pre-tests and Bank Task interviews, we looked for how each student responded to my questions as they pertained to commitments to unit, reference point, and directionality of comparison. We wanted students who understood my questions and endeavored to answer them.

We selected two students, Loren and Gabriel (pseudonyms) who were far apart in their tendency to conceptualize variations within a frame of reference, but who demonstrated an ability to understand my questions and a willingness to talk at length about their thinking, to participate in multi-day teaching experiments. Both students were freshman in college and STEM majors (one chemistry, one biochemistry) who took DIRACC Calc 1 in their first college semester. Each student participated in a multi-day teaching experiment in Spring 2019.

Teaching Experiment

A teaching experiment is an “exploratory tool…aimed at exploring students’ mathematics” (Steffe & Thompson, 2000). Instead of seeking to understand a student’s current knowledge, a teaching experiment is designed to gently probe, support, and push students to the limits of their mathematical and reasoning capabilities in order to explore what those limits are. The experimenter is not an objective observer who stands outside the experiment, but an integral part of the experiment itself, acting as both treatment and observer.

This study with two participants took the form of a constructivist teaching experiment, where my goal was to gather data to help me to build models of the student’s way of thinking about variations in relation to reference point, directionality, and coordination and combination of frames. We also had larger theoretical goals which were to 1) test the power and utility of our frame of reference theory to explain student thinking, and 2) have a source of experimental data with which to refine this theory.

We developed our tasks with the expectation that most students would struggle in some way and the nature of those struggles would give me insight into how the student thought. Before the interviews started we built a sequence of tasks with different possible trajectories and questions for different ways that we anticipated the students might answer, which would give us insight into the extent to which they were conceptualizing the variations in the tasks within a frame of reference. However, unlike a clinical interview, we also left room to ask questions that arose in the moment in response to what the students said. We knew that because we were active participants in the experiment that we would need to be careful in our analysis when we made claims about what the students thought or believed and to find evidence for any such claims.

Data Collection

For all interviews we used a computer to display tasks on the left side of the screen, and displayed the screen of my iPad Pro on the right side of the screen using the Apple program. Students used an Apple Pencil to write or draw on the iPad, and used the computer mouse to gesture to parts of the problem on the left side of the screen. For some tasks the problem was embedded in an interactive program like Graphing Calculator on the left side of the screen, so they could also use the computer to change the parameters of the problem or to play videos. The entire screen (and audio) were video recorded. For the first day for each student I asked him to use the mouse to gesture so that his gestures are captured, and if he made movements away from the screen we added them to the data capture by commenting on them audibly (e.g. “I notice that you just put your index finger together and then moved them apart in opposite directions.”). For the subsequent days we added a camera facing the student so that I was also capturing the student’s face, body, and hands and arms. Surani Joshua conducted the interviews, Dr. Pat Thompson was the witness and we frequently had a second witness, Hyunkyoung Yoon, observing through a camera from another room. Immediately after each interview we all sat down to record initial impressions from the interview and to plan any modifications to the next day's plan. The data captured was therefore the students' written work, the camera recording of the student throughout the experiment, and the written notes of both witnesses.

The entire set of transcripts (6.5 hours total) was transcribed verbatim, with the transcription synced to the video via timestamps, using a professional service called rev.com. We then rewatched all the videos while reading the transcripts to check for accuracy, adding in notes about body language and fixing transcription errors.

Data Analysis

For analysis, we began with a pass of the data where we looked for episodes in which students' behavior or explanations seemed explicitly tied to frame of reference. We then analyzed those highlighted episodes in terms of whether they illuminated the student's ways of thinking about reference point, directionality of comparison, coordination of frames, and/or combination of frames. We then used these episodes to build models of how each student thought about measures and measure comparisons throughout the entire experiment. To build models we looked for indications of whether and to what extent the student committed to a reference point, committed to a directionality of comparison, or engaged in the activities of coordinating or combining a frame of reference, and to what extent these cognitive activities helped the student in reasoning about variations.

Two methodological issues we had not anticipated arose quickly. First, we saw that Loren had unproductive meanings for variation, covariation, quantity, and graphs that complicated our analysis of his data. Additionally, his thinking on frame of reference reasoning was very undeveloped; our theory did not have any explanatory power for a student's actions when he did not attend to quantities at all. We had to start thinking about the kinds of imagery that would be consistent with Loren's behavior in order to explain his decisions.

Second, it was impossible to separate analysis of a students' commitment to reference point or commitment to directionality of comparison with their coordination of multiple reference frames. We can clearly see a lack of either commitment only in the act of coordinating two or more reference frames. To say that a student has made an intellectual commitment means that they will give up that commitment in the context of a perturbation only if they see a need to do so and then do so intentionally. We were able to conclude whether Loren or Gabriel made these commitments only in settings where they needed to make a choice to either maintain a commitment or to compensate for it. Therefore, our analyses of commitment to reference point and directionality, and our analyses of coordination of reference frames, had to happen simultaneously and could not be separated from each other.

Results

Attention to Quantity & Combination of Frames

Our first task, tracking a clown's distance traveled in relation to elapsed time, was chosen to provide data on two questions about student reasoning. First, we wanted to tease out the ways in which each student attended to total quantities to have a basis for comparison when they reasoned with variations. Second, we wanted to see to what extent each student was able to combine two reference frames by considering two quantities simultaneously.

To do so, we gave them each an animation where a clown starts at x=0 and moves back and forth horizontally along a number line. The clown changes direction at different points and moves at different speeds (including some pauses) in each segment of the journey.

For the running clown, please sketch a graph of:

a) total distance traveled versus time since animation began

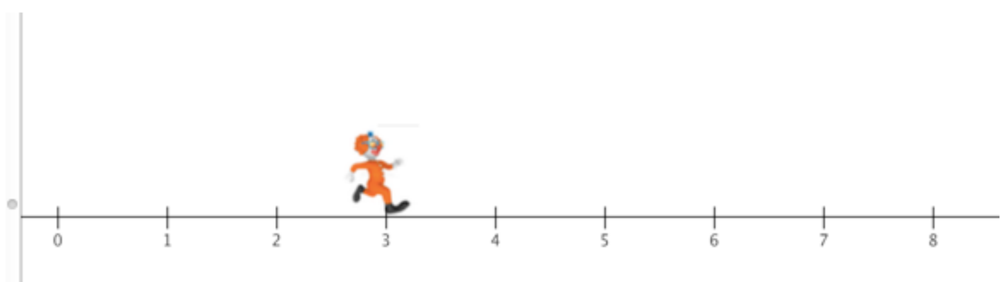


Figure 4: Clown Task

As both Loren and Gabriel worked their way through drawing the distance versus time graph and then commenting on their responses, we saw stark differences in how each student conceptualized quantities, and how each student was able to combine reference frames by attending to both quantities simultaneously.

Loren's Attention to Quantity & Combination of Frames

Our discussion of Loren's behaviors in the context of the Clown task is made difficult because of his fragile, often fleeting, connections among visual images, verbal phrases, and mathematical inscriptions. Much of the following illustrates this.

Loren struggled to find and represent quantities in his own graphs and to combine two frames by attending to both quantities. When he revised his answers to be more appropriate it was almost always in response to our probing questions – moreover, as we will see in later tasks, it became clear that these changes in his behavior were functional accommodations to our questions and the changes he made did not affect his meanings, since they were not carried into future tasks or sub-tasks.

<table>
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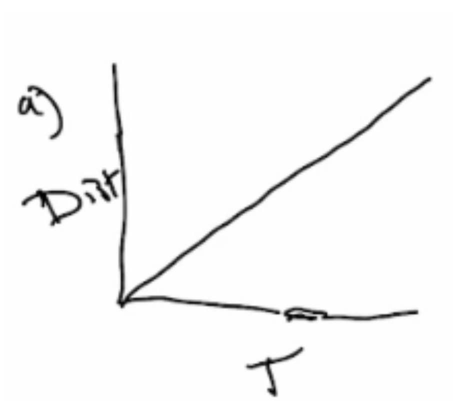
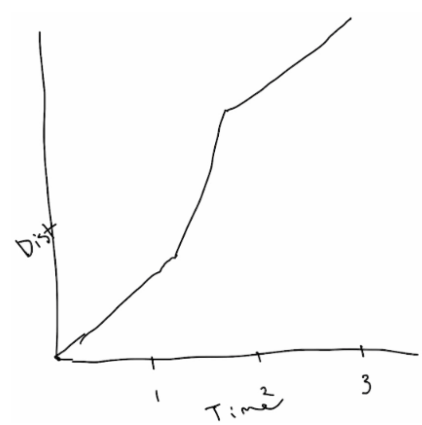

</td><td>b)

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<tr><td>c)</td><td>d)</td></tr>
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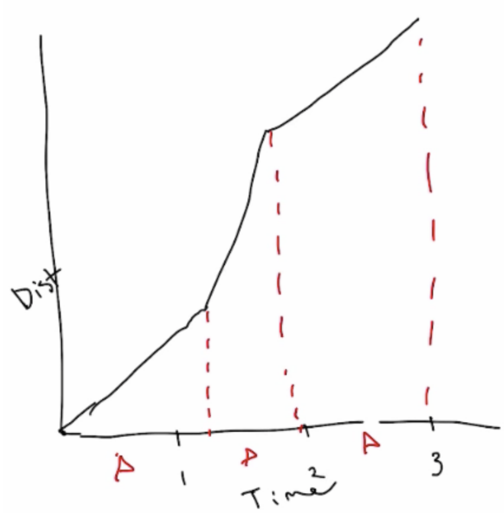
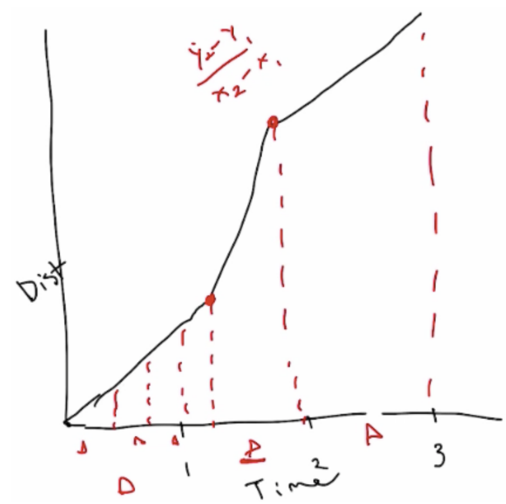
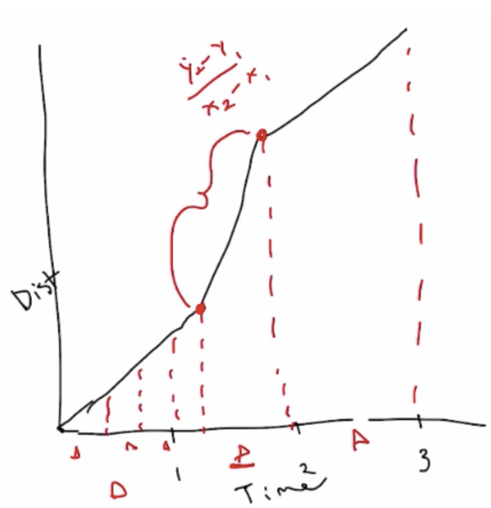


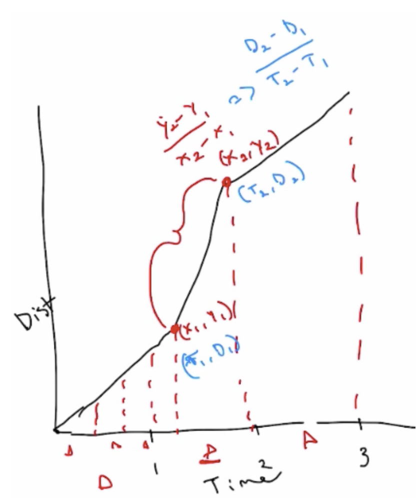
e)



f)



g)



h)

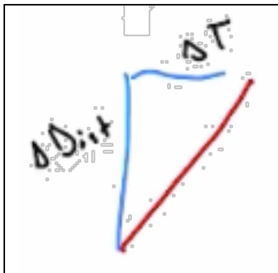
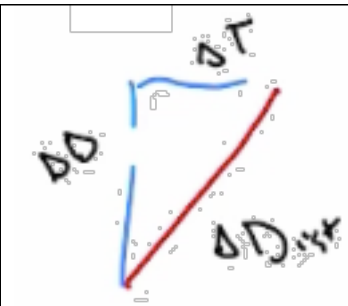
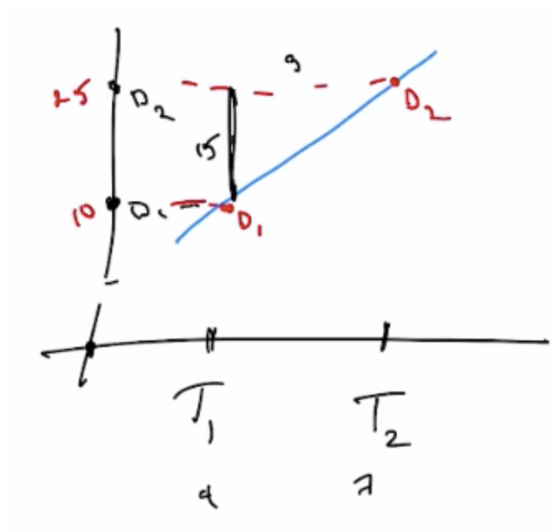
	
i) 	

Figure 5: Loren's drawings of distance vs time

In part a) of the clown task, Loren drew Figure 1a as a representation of the clown's distance walked versus time elapsed – a single linear graph. Loren explained, "Um, so as time increases the total distance traveled is increasing whether the person, the clown went forward or backward does not ... It's kind of like if you have a fit bit and you want to find out how many steps you had, if you move four or five feet, you're hoping that you would actually have total of, a larger number than what you began with." The mental imagery consistent with Loren's answer is that of imagining the display on a pedometer (step counter) ticking upwards as the animation plays, but without any kind of coordination to elapsed time. Loren attended to one quantity at a time.

Loren made no reference to magnitudes or measures in discussing his graph, nor did he coordinate Clown's elapsed time and distance walked. The vastly different speeds at which the clown walks in different segments (including some resting periods) are not reflected in Loren's graph or his explanations. He focused only on the fact that the pedometer display would keep increasing, and so drew a graph that "goes up" if one's eyes follow it from left to right.

During Loren's explanation of his initial distance graph (Figure 1a) his body language and voice suggested satisfaction with his response. In order to perturb Loren's thinking, we presented him with four alternative graphs that were non-linear but also monotonically increasing, and asked him to explain why his graph was more accurate than them. Loren responded by asking to redraw his graph, and drew the graph in Figure 1b for the first three seconds of the animation. Loren's new graph is consistent with coordinating measures of variations, and left open the possibility that he was truly combining frames by tracking the direction and the magnitude of variations in both distance and time. However, it was also possible that Loren was tracking perceptual information by engaging in shape thinking (Moore & Thompson, 2015, 2016) where he simply made the graph steeper when he saw the clown moving faster.

In order to discern which way Loren was thinking as he constructed his graph, we asked him to explain how far the clown had traveled in the third Δt-

interval marked in Figure 1c. What followed was 31 minutes of Loren trying several different tactics to make meaning out of his own graph. He first chose to write the formula (y2-y1)/(x2-x1) and when we asked what that would calculate, he said “To find your total distance covered at, ah, within this interval.” and marked the two endpoints of that part of the graph with red dots (Figure 1d). It is likely that Loren’s imagery for linear graphs was dominated by experiences of being asked to find the slope of a line. He did not anticipate that the result of the decision he had made was not what we asked him to find.

We next prompted Loren to think about rate of change of a linear graph so he might realize his formula produced a measure of speed and not of distance. However, when we asked how he would determine the clown’s speed in this interval of time, he said, “Then what you would do is take the derivative of this, uh, this the distance versus time graph. This distance versus time function,” highlighting a section of the graph with a curly bracket (Figure 1e). We decided to ignore Loren’s suggestion (there was no way to “take a derivative of the graph”). Instead, we took it as a sign of Loren’s loose connection between “rate of change” and “take a derivative”. We also noted that Loren apparently did not connect ideas of slope and rate of change, and had a loose connection between (what we call) the slope formula and the length of a triangle’s hypotenuse. Loren also seemed to conflate length of a segment of the graph with a variations in the clown’s total distance.

We then drew Loren's attention to the quantities in the clown context, that *x* represented Clown's elapsed time and *y* represented Clown's total distance traveled. asked him to rewrite the coordinates of the endpoints using "d" and "t" in place of "y" and "x" and then to rewrite his formula (Loren's work is in blue in Figure 1f). We asked Loren to circle the formula in blue formula explain what it calculated. He explained, "That is the, uh basically how fast you are going... it's your change in, um distance over time, which is ... equals distance over time so it's kind of, it can kind of be how fast you're going." When asked if he remembered what he had said earlier, Loren said "Ah, I said it represented the distance, but then I might have just misspoken." Loren had now identified the quantity whose measurement is given by the slope formula. We can see Loren's imagery in his meaning for graphs in his actions and decisions. Though he placed the labels "Dist" and "time" on the axes at my request, those labels did not provide quantitative meanings for the coordinate points in his mind when they were represented by the symbols x_1, y_1, x_2, and y_2. He was able to replace these symbols appropriately with d_1, d_2, t_1, and t_2 when asked to do so, but his capability for seeing that y_1 was the same as d_1 etc. was not enough to anticipate that the slope formula would also calculate speed. Loren had to carry out the actions of replacing the variables and rewriting the formula before he saw something that he identified as "change in distance over change in time".

After Loren identified the meaning of his chosen formula, we asked what would represent distance to see if Loren would circle or otherwise identify the numerator of his slope formula. Instead Loren said “Um, you can end up ... I don't know if that is over thinking or making it more complicated. But kind of making it a triangle to then set up the angle here or something like that.” and drew and labeled the image in Figure 1g and then said “Uh, I'm thinking, I'm thinking that the red line can represent, I'm actually just gonna ... I just wanna see something, um.” and started writing what he verbally identified as the Pythagorean formula (Figure 1h) as well as speculating about the meaning of the area of the triangle. Though Loren’s sketch in Figure 1g has a vertical line segment marked “∆Dist”, he does not connect his own annotations with the question of “How far did the clown walk in this interval?” His imagery for ‘distance’ led him to trying to assess the length of the segment of the graph. We asked him again what quantity he was looking for, and he said “The distance, is that it?” and remained focused on the length of the red line. It is important to note here that even when we clarified that the question was “How far he traveled in that interval, so the change in distance.” Loren remained focused on the red line though we were looking together at his own sketch where he had written “∆Dist” as a label for the vertical blue triangle leg. Instead of being recognizing that he had already labeled the quantity we were asking for, Loren went ahead and changed his drawing to Figure 1h where there is a ∆D and a ∆Dist and did not explain the difference between his two notations, or whether they were different to him at all.

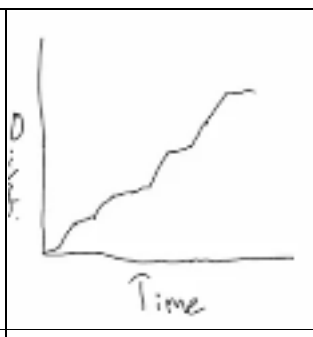
The remaining interview went for another 18 minutes much like we already described. Loren eventually identified the variation in distance as the vertical distance only when we provided sample numbers for him to use in calculations. He calculated the variation in distance and was able to identify a part of the graph with the same measurement (Figure 1i), but continued to focus on the length of the segment and the Pythagorean theorem. He could not decide whether the length of the segment or the slope of the segment represent the clown's velocity.

Loren's interview about a distance versus time graph, shows that he did not have clear meanings for the quantities in his own graphs, and that he struggled to combine two frames and consider two quantities simultaneously. He was spurred to make changes in his own answers by our probing questions, but frequently attended to the parts of the graphs which were already highlighted (by us or even by himself) and looked for places to use the formulas he was familiar with in relation to such shapes as straight lines or triangles. We hypothesize that Loren's imagery for linear equations included many experiences of finding slope with the slope formula or length with the Pythagorean formula, and so those are the things that Loren tried. His responses were less attuned to the quantities we were asking about, than the geometric figure he perceived in front of him. Each change in his answers was a functional accommodation to our questioning; we make this claim because, as we will see in future tasks, his general schemes for graphs and quantities were not impacted and so the conclusions that we reached in earlier items did not carry forth into future items.

Gabriel's Attention to Quantity & Combination of Frames

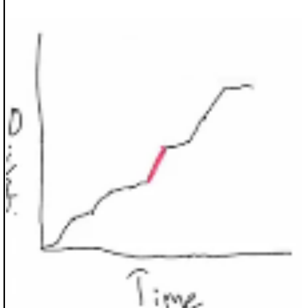
Gabriel displayed an attention to the quantities that he was representing on his graph, and demonstrated a conceptual combination of two frames by attending to both quantities simultaneously. When he did make mistakes he caught and fixed them spontaneously within a few seconds of making them on his own without our intervention. For example, a few times Gabriel referred to common procedures that were inappropriate for the task such as looking at the area under a graph, but then would quickly change his mind. His verbal statements indicate that he was able to do so because he kept returning his attention to the two quantities that he was working with both in the animation and that he represented simultaneously in his graph. Gabriel's imagery for his graph and the quantities represented in it allowed him to anticipate the results of actions he proposed, and to evaluate their appropriateness before he carried them out.

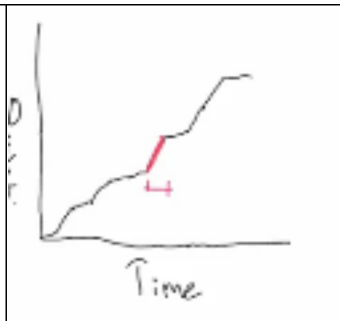
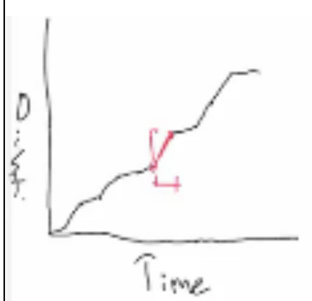
In part a) of the clown task, Gabriel drew the following graph of the clown's distance walked versus time since the animation began:

Gabriel:	
Surani:	What were you focusing on when you drew the graph?
Gabriel:	Uh, I was mostly focusing on the speed of the character and how long it was moving at an unspecific speed of ... it was more the absolute value

	of the velocity I guess. Because in total distance traveled it doesn't matter if it's moving to the right or to the left, only really matters its rate of change.
Surani:	Okay. And so, I see two elements of your graph. I see that you have segments that have different lengths and also different slopes. And can you tell me about how you decided on both of those?
Gabriel:	So for the ones ... for the different slopes, it's just the, the steeper slopes are for when the character's moving faster, and the shallower slopes are for when it's moving slower. The different lengths being just for how long it was moving at any one speed versus another.

Gabriel's graph and explanation, just like Loren's, left us with two possibilities. The first was that Gabriel was truly combining two frames by tracking both the direction and the magnitude of variations in both quantities together, and the second was that Gabriel was just tracking perceptual information (since his own words referred to making the slope of the graph steeper and shallower). We gauged how Gabriel was thinking about his graph by asking him the same question we asked Loren.

Surani:	For this [highlighted in red] part, where on the graph is represented how long he went? How long it took him to go there? 
Gabriel:	It would be from here to here [draws horizontal component of graph segment].

	
Surani:	Okay and then what about, how far did he go in that amount of time according to your graph?
Gabriel:	It would be this region right here [colors in area of triangle under segment]. Oh wait no, distance not rate of change, hold on. 
Gabriel:	Yeah, right. So then the distance traveled would be this bit right there [draws vertical component of graph segment. 
Surani:	Why did you change your answer?
Gabriel:	From the under the curve? It's just that was just a ... at this point I'm so used to looking at rate of change graphs that that's my like, first go to response. I didn't actually read ... put enough thought into it to correct myself.
Surani:	Okay, but this is not a rate of change graph to you?
Gabriel:	No. This is just distance versus time.

Gabriel showed that even though he used language of "steeper" and "shallower" commonly associated with shape-thinking, he still had an awareness of the extensive quantities involved in his graph. To Gabriel the height of a point above the x-axis represented the total distance a clown had traveled from its starting point, and the variation in height between two points represented a variation in distance. His imagery allowed him to make an inference and then consider its appropriateness in relationship to what he is working with and what he wanted to find. Subsequent questioning showed that he had similar meanings for the y-coordinates of points on his graph as representations of elapsed time by means of how far they were to the right of the y-axis. He engaged in combining the reference frames of both quantities as he both drew his graph and then extracted information from it later in response to our questions.

Commitment to Reference Point & Coordination of Frames

A commitment to a reference point gives a meaning to every measure of a quantity and avoids many problems of ambiguity. To discern the extent to which Loren and Gabriel committed to a reference point when speaking of quantities' measures, we gave them parts (b) and (c) of the clown task which are shown in Figure 2 below. In parts (b) and (c) the phrases "from x=0" and "from x=3" refer to physical locations on the number line the clown was running back and forth upon.

For the running clown, please sketch a graph of:
a) total distance traveled versus time since animation began
b) displacement from x=0 versus time since animation began
c) displacement from x=3 versus time since animation began

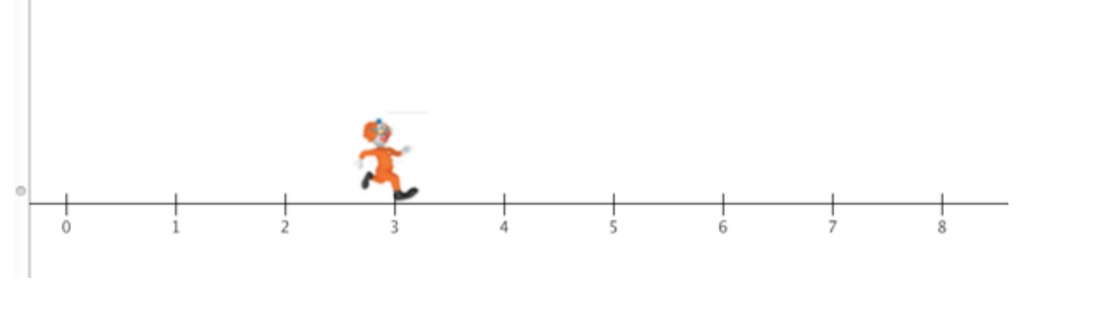


Figure 6: Clown Task Animation

Both students were asked about their meanings for "distance" and "displacement" before the clown task was even placed before them, and short discussions ensued to clarify the difference between the two.

Loren drew similar graphs for both parts (b) and (c) (see Figure 3), and was satisfied with these graphs until he was perturbed by our introduction of a perturbation about the initial values for the two graphs could not be the same. When presented with that perturbation he responded with a functional accommodation that resolved the conflict in his mind but did not include a coordination of two frames. Gabriel drew graphs that were vertical translations of one another and explained his reasoning in a way that is consistent with our theory of what it means to coordinate two frames.

Loren's Commitment to Reference Point & Coordination of Frames

In parts (b) and (c) of the clown task, Loren drew the following graphs of the clown's displacement:

b) displacement from x=0 versus time since animation began	c) displacement from x=3 versus time since animation began

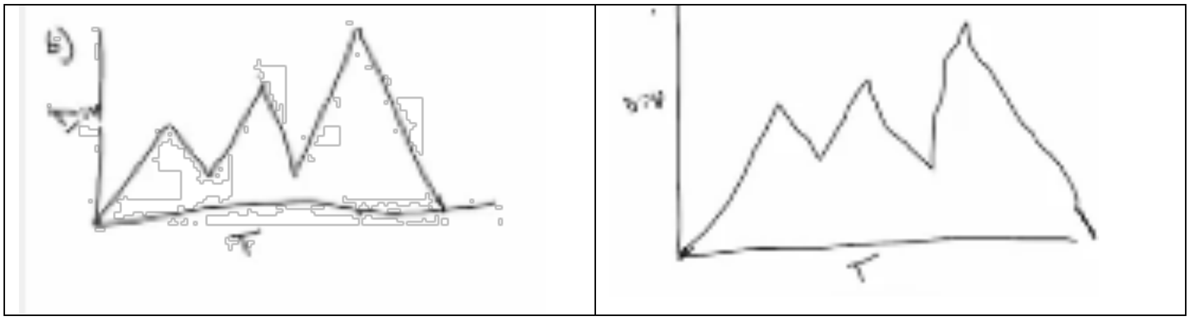


Figure 7: Loren's initial graphs

Loren’s graphs are inconsistent with thinking about the measurement of displacement from a given starting point. They *are* consistent with the thinking of a person whose dominant imagery was focused on location. Since the same animation is played for both parts (b) and (c), it seems that Loren was tracking the location of the clown in both parts, moving his pencil up when the clown moved to the right and moving his pencil down when the clown moved to the left (all the while moving his hand smoothly from left to right). Once again the differing speeds of the clown at different times, and the clown’s intervals of rest, are not represented on either graph. It seems that Loren’s imagery for a graph was to represent increases and decreases in the number on the x-axis underneath the clown’s feet, with no coordination with time.

After answering part c) of the clown task, we asked Loren about the starting value of displacement from x=3. His response shows that he had difficulty in distinguishing between a reference point for measuring and an initial point for motion. Thompson asked him to answer his question not by looking at his own graph, but by looking back at the original clown animation.

Dr. T:	Look up there [pointing to the clown animation]. What's your starting value?
Loren:	It's three, yeah three. My starting value-
Surani:	First let me ask in terms of location.
Loren:	Oh the starting value in this question?
Surani:	In part C yeah.
Loren:	Oh um ... his starting value ... if you're going to compare it, so displacement is how far you travel from your initial position
Dr. T:	How far you *are* from…?
Loren:	From your initial position.
Dr. T:	Not traveled from. How far you *are* from…?.
Loren:	Yeah yeah. Yeah not how far- [said at the same time as Surani's remark below]
Surani:	Not initial position though. remember he's not starting at position three, right? In the previous question we did displacement from position zero and can start from position zero. But this time he's starting here [points to position 0], but we're tracking displacement from position three.
Loren:	*Ooooh*. Ah, so I kind accidentally took it in a different where maybe, where I made three, technically could be your zero, but then um, you're ...
Dr. T:	So if 3 were, if you did that and three were your zero then where is he starting?
Loren:	Because you're comparing-
Dr. T:	Not why, where?
Loren:	Zero then.
Dr. T:	You said three is your zero.
Loren:	That is your initial position.
Surani:	So this is your zero? Oops [fast-forwards animation to point at which clown is standing at x=3] This time he's at position zero*, he's at zero displacement. Right?
Loren:	Yes.
Surani:	Okay. [rewinds the animation to the very beginning, when the clown is standing at x=0] Then where is he now? From this placement?
Loren:	Aaah, you want…negative three? He went , ah backwards.
Dr. T:	So where should you start? Where should your graph start?
Loren:	Then it [the initial point of the graph] should be yes in the negative. Yeah that's why I said that I took that as zero, but yes it should be at um in the negative section. So I mean you can make the graph lower.

*This was a mistake on my part. I should not have said the clown was at position zero when he is standing at position 3!

Loren's conversation kept referring back to the location of the clown, instead of to a measurement of a clown's displacement from a chosen reference point. His comment "I made 3, technically could be your zero" seems to suggest that we (the interviewers) were welcome to think of x=3 as position zero, but that it remained position three to him. His imagery was still about the clown's location, and to him the salient point about the beginning of the animation was that the clown was initially standing on position 0. He did not see the inconsistency in starting his graphs for both parts (b) and (c) with the initial point until we fast-forwarded the animation to a point where the clown was standing at position 3, and established that as a point where displacement from x=3 is 0. Only after he experienced that conversation did Loren recognize that the clown standing at x=0 must give a displacement value other than 0.

We then asked Loren if he would be able to fix his graph by "relabeling your graph right now and make it correct…. Don't redraw it just relabel it." Loren did not do so, but he eventually changed only the beginning of his graph (see Figure 4).

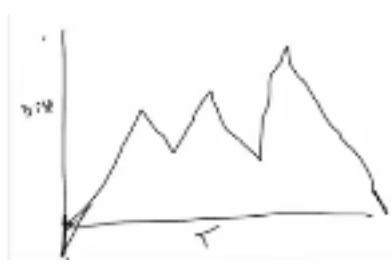


Figure 8: Loren's Final Answer to the Clown Task

Loren originally did not coordinate two frames because he did not see a difference in questions b) and c). From his perspective, both questions involved tracking the location of the same figure in the same animation, and so he drew the same graph for both tasks. His imagery for both distance and displacement graphs was related to location instead of measures of quantities. The setting in which he worked during part (c) was still the setting in which we asked him to think about part (b). That was the dominant experience, the imagery, that he had available to him going into part (c).

Our prompts to have him put the clown on x=3 and answer about displacement from there provided Loren to opportunity to see the need for coordinating a pair of measures: that the clown standing at x=0 would produce a measure of 0 in the part b) task but a negative) in the part c) task.

However, Loren did not coordinate the frames themselves. His final response coordinated only the initial measure in his graph but did not take into account the repercussions of that coordination for all the other measures of displacement in the rest of his graph. His final response is consistent with thinking that since the initial value for displacement is -3 he had to adjust the beginning of his graph, but that following that adjustment he could make the rest of the graph according to the clown's location. Loren's imagery for thinking about measures taken from different reference points did not allow him to keep track of the meaning of all the points on the graph simultaneously. He did not anticipate that the consequence of his initial point coordination was that all the other points on the graph should shift down 3 units as well.

Gabriel's Commitment to Reference Point & Coordination of Frames

In parts (b) and (c) of the clown task, Gabriel drew the following graphs of the clown's displacement:

b) displacement from *x*=0 versus time since animation began	c) displacement from *x*=3 versus time since animation began
	

Figure 9: Gabriel's Cown Task Responses

After Gabriel answered part c) of the clown task, we asked him to explain his answer to us "as if we cannot see the clown animation; we only can see your graph". His response shows that he repeatedly described the location of the clown in relation to the x=3 position that he was asked to use as his reference point.

Gabriel:	[Clown is at x=0] Um, so from these... or from this time [indicating the start of the animation], uh, the clown is traveling from the left of three towards three. Then it's traveling from three, from three away from three towards the left direction. Then it's traveling towards the right direction, crossing three and going above or to the right of three. Then it's once again traveling left, crossing three, uh leading to a spot left of three. Crossing three again going to the right, it's then approaching three and crossing three, heading back to the left. Could technically also make this value, negative three from three [indicating final function value].
Surani:	Is there anything different you paid attention to in this problem [displacement from x=3] than the last one [displacement from x=0]?

Gabriel:	Um, not really. The graph picture was the same, the only dis ... or the only difference being that what was considered zero was shifted up [sic] by three units and so the actual, I guess shape of the graph was just moved down three units.
Surani:	Okay. What if someone didn't see the questions and they just um, and they just saw your two graphs. And they said "Which one is right for displacement?" And they, again they only saw your graphs, they didn't see anything else about this task. And they said um, "Well they're both displacement graphs, which one's *right* [emphasized]? Like, which one's the *real* [emphasized] displacement?"
Gabriel:	I mean, it all depends on where you want to measure the displacement from. It's probably easier and more succinct to go from zero because you never would then go negative of zero, but at the end of the day they're both equally correct.

In both his graph and his answers to follow-up questions, Gabriel showed all the cognitive steps that we describe as constituting the act of coordinating two frames of reference. Gabriel recognized the need for different frames necessitated by the different quantities he was asked to represent in parts (b) and (c). He started by coordinating individual measures in each frame by paying explicit attention to the point from which he is measuring (we can see an example in his last comment where he said that the value when the clown stops at x=0 was "negative 3 from 3"), . Finally, he coordinated the frames themselves by concluding that a change in reference point affected *all* the points on the graph. Gabriel's imagery for coordinating quantities was centered on transformations, and allowed him to anticipate the consequence of his initial transformation for all the points without having to carry it out repeatedly. He was able to coordinate these frames because he had explicitly committed to a reference point when making measurements, as we see when he said "it all depends on where you want to measure the displacement from." Gabriel both committed to a reference point for his measurements, and used those explicit commitments in part b) and c) to coordinate the two different frames together.

Commitment to directionality & coordination of frames

To characterize each student's ability to commit to direction of comparison, we posed a task that required him to coordinate frames that required different directions of comparison. The task in Figure **10** below has two parts, each with the same company ledger: part (a) asks the student to represent how much money the company gained each month, and part (b) asks the student to represent how much money the company lost each month.

	BANK BALANCE		How much did my company gain this month? (in millions) ↓
	First Day (in millions)	Last Day (in millions)	
January	$38	$57	
February	$57	$69	
March	$69	$44	
April	$44	$83	
May	$83	$74	

	BANK BALANCE		How much did my company lose this month? (in millions) ↓
	First Day (in millions)	Last Day (in millions)	
January	$38	$57	
February	$57	$69	
March	$69	$44	
April	$44	$83	
May	$83	$74	

Figure 10: The Bank Task

In order to answer each question a student would need to commit to what positive values mean. For the gain task, a positive number means a net gain when answering the question, “How much did my company gain?” and a negative number means a negative gain, or net loss. In the second part, a net loss of money is a positive loss, and a net gain is a negative loss.

While Loren and Gabriel had similar answers to part (a), albeit with different levels of confidence, it is in their attempts to answer part (b) that we see the limits of their commitments to direction of comparison during the act of coordinating two frames.

Loren’s commitment to directionality & coordination of frames

Loren originally gave identical answers for parts (a) and (b) of the Bank Task, as shown in Figure 7 below.

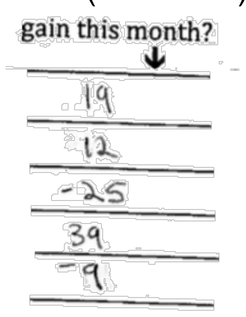
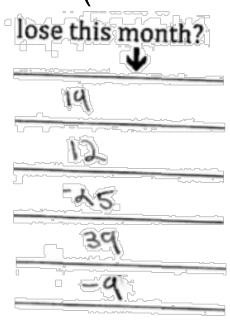
a) How much did my company gain each month (in millions)?	b) How much did my company lose each month (in millions)?
	

Figure 11: Loren's initial responses to the Bank Task

Loren’s answer to part (a) shows that he was able to commit to a direction of comparison and maintain that direction for all of part (a). He chose to represent the company’s gain each month by calculating to what extent the end of month balance exceeded the beginning of month balance. Unlike some Calculus 1 students, whose answers we will examine later in this section, Loren did not switch his direction of comparison in order to keep all of his answers positive. The excerpt below shows that he had imagery for what it meant to gain a negative amount of dollars.

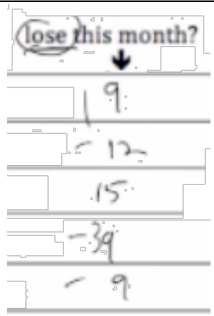
Loren:	In February, two, the company gained $12 million. And then, in March, the comp, wait, in March the company ended up gaining negative $15 million but then uh ended up, like meaning losing $15 million so gained 15 or lost $15 million. Um, in April they ended up, in April the company, my company ended up gaining 40. In May, my company ended up losing $9 million. Or gaining. Or yeah, yeah, losing $9 million and gaining uh, negative nine million dollars.
Surani:	Thank you. Um, and you ... I noticed that you said, "Gaining negative $15 million or losing $15 million," are those exactly the same thing? Or ...
Loren:	Um ...
Surani:	Or what if I, what if I'm the person that's like, that's not the same thing. Why don't you convince me of why you said that. You said a good thing.
Loren:	Mm-hmm (affirmative). Um, gaining and losing can become synonyms depending on what proceeds, um, that- that verb or verb um, but gaining a negative amount is, um ... sorry, what was the question again?
Surani:	Oh, just I'm curious why you wrote that because it's not actually the same thing that you wrote several months ago*, but I'm interested.
Loren:	Oh. So, I mean I sometimes I like to switch it up a bit I guess maybe that's why I just said lost $9 million rather than gain $9 million dol-, or gain a negative amount of money. Uh, I guess it's all colloquial, but then, colloquial for the- the whole world um, agreed upon convention if someone gains a negative amount of money you just say I ended up losing money.

*This statement was wrong; I had confused Loren with another student who did not participate in the teaching experiment. Both Loren and Gabriel gave the same initial answers to both parts in the teaching experiment as when they answered this task in their Calculus 1 class four months earlier.

When Loren moved to part (b) of the Bank Task, he struggled to coordinate the meaning of the frames “how much gain” and “how much loss”. He engaged in the same actions as in part (a) and ended up with the same responses. He did not initially show any discomfort with this, until we pointed out how a bank balance sheet could not show a simultaneous gain *and* loss of $19 million at the same time.

Loren:	Yeah. So, in January my company lost $9. $9. In February ... should I go on down the list? In February, my company lost $12. In March, my company lost -$15, or gained $15.
Surani:	Okay, and I want to pause you for a second because I want to compare with something you said on the previous page. When you said, "In January, the company gained $19 million," and here you say, "In January, the company lost $19 million."
Loren:	Mm-hmm (affirmative) In January the company lost 19.
Surani:	Are they both true?
Loren:	Uh, wait, say that again.
Surani:	Sure. Um, on this previous page you said, "In January, the company gained $19 million," right?
Loren:	Mm-hmm (affirmative).
Surani:	For this first line right here. And then, "In February ..." sorry and then on this page you said, "In January, the company lost $19 million." It's the same, it is however the same um, bank balance sheet.
Loren:	Mm-hmm (affirmative).
Surani:	So in January, did the company gain $19 million or lose $19 million?
Loren:	Yeah, they both, in both examples they gained $19 million, but it is all relative. Um, in this they're establishing um, with the… that gaining $19 million, having a positive um, uh, dollar amount is negative, or- or is- is um, means losing money and having a negative dollar amount would mean the opposite, uh, which is gaining money 'cause- 'cause of establishing a loss of money is positive and I don't know, that uh, a negative amount would be gaining.
Dr. T:	So, what's your plan when I ask you for January?
Loren:	So January, they, the company gains $19 million. They lost, they lost $19 million. Where'd it go? They lost, the company lost $19 million.
Surani:	Okay. In January, did- did they have more or less money in the end?
Loren:	They had less.
Surani:	Well, take a look at the first and the last day.
Loren:	Mm-hmm (affirmative). They ... oh. Oh, so, okay, I- I was thinking that is something else. I might not have read it.

Surani:	That's fine.
Loren:	I feel like I might have done this last time actually, but um ... I was just looking through it for a sec, but they ... okay, so they ... yeah, they ended up ... Oh, I'm not too sure... They gained ... I mean ... so yeah, okay, so I think losing a negative amount would mean that you would gain um, money. It- it could mean that.
Surani:	Okay.
Loren:	But they ended up making money. Yeah, they lost. They weren't, and in that way my company in January lost $19 million. Or gained, they lost -$19 million.
Surani:	Is that just as valid as saying gained $19 million?
Loren:	Yes, depending on the uh, what you're comparing it to.
Surani:	And in this problem, what are we always comparing to?
Loren:	If you are comparing to… [spoken at same time as comment below]
Surani:	Yeah, like when I ask you gained this month and lost this month, going back to our conversation about how all measurements are in respect to something. What are your ... this, like -19, what is that in respect to? What are you measuring?
Loren:	Mm-hmm. In respect to having um, less money in uh, in the beginning of the month.
Surani:	Okay, so you just pointed to the 38 [million initial dollars for January].
Loren:	Yeah.
Surani:	The 38 on the first one.
Loren:	Yeah.
Surani:	Okay. And then what would you say for the rest of your numbers?
Loren:	Um.
Surani:	Would they continue to be the same?
Loren:	Uh, no. Actually need to change them now. Um, so you have ... uh, in terms of losing $19 million, uh, they have more money at the end of February, for that you need, they lost a negative amount of money. Um, in March they ... my company ended up with, uh, ended the month with less money, so that would mean that they lost $15 million. So 15 positive meaning... Um, earn more money in April, my company, and that would mean that I lost a negative amount. And then my company had less money in May, so that would mean that I had, that my company lost a negative amount of, negative amount of money.

Loren seemed to have a strong image that when comparing measures additively we are always looking for ‘excess’ as the default. He was able to accommodate his meanings to handle negative excess. However, when the task asked about deficit, this didn’t fit his image at all. Having only imagery for the process of looking for excess, he proceeded to just look for excess again. Put another way, in both parts Loren seems to have imagined comparisons in terms of gains and losses over a month. He engaged in functional accommodations in Part B so that “gain” meant “loss” and “loss” meant “gain”, which became confusing for him.

While Loren was able to commit to a direction of comparison in part (a), his initial response to part (b) shows that this commitment was stable because it aligned with his imagery for what a comparison should look like. When asked to commit to a different direction of comparison, he did not do so on his own.

Loren’s ability to coordinate two directions of comparison is different from his ability to coordinate two reference points in a significant way. Just as in the clown task, Loren became perturbed by our asking him to explain his own statement that the company could have simultaneously gained and lost $19 million at the same time. Just as in the clown task, Loren used this contradiction to adjust his response and come up with an appropriate measure for loss in January – that "I think losing a negative amount would mean that you would gain um, money… they lost -$19 million". However, he also saw the consequences of this new meaning for "loss" for the rest of his responses and coordinated the new direction of comparison with the old, unlike in the clown task where his adjustment of his initial point provided no impetus to adjust the rest of his graph.

Even as Loren started to work with the idea that one could choose a direction of comparison other than looking for excess, we still saw him struggle to maintain that new commitment. His commitment to a direction of comparison for the gain task was clear, confident, and sustained. His commitment to a direction of comparison for the loss task was extremely tenuous; we see in the excerpt above how he repeatedly grasped and then lost his new idea of what it meant to measure with an eye to deficit. Loren seemed to start building some imagery of how the possibility of a different direction of comparison might come about when he said later in the experiment that "So I guess, so being a pessimistic boss or uh, supervisor, you say, you like to, you wanted the amount that was lost according to this question [circles the question of loss]". His words suggest that he was starting to think about how the nature of what one anticipates when asking a question (gain or loss) shapes the nature of the resulting measures.

Gabriel's commitment to directionality & coordination of frames

Gabriel’s answers for parts (a) and (b) of the bank task are shown in Figure 8 below:

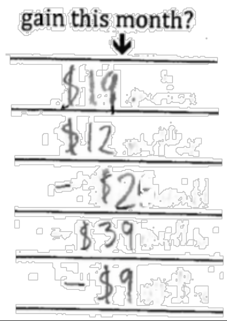
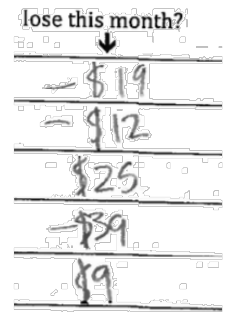
a) How much did my company gain each month (in millions)?	b) How much did my company lose each month (in millions)?
	

Figure 12: Gabriel's answers to the Bank Task

Gabriel’s answer to part (a) shows that he was confident in committing a direction of comparison and maintaining that direction throughout all of part (a).

Gabriel:	No. So, basically, how much the company gained would just be a last day minus first day, or I guess, delta dollars. I don't know. I'm not good at assigning variables. So, 19, uh, 12. Let's see. Negative if ... 25. Positive 39. And negative 9 [as he fills out the chart].
Surani:	And now, will you explain ... Is there anything that you would want to share about your thinking and how you reached those conclusions?
Gabriel:	Um ... None that immediately come to mind, other than the, the 19 representing a, a gain but in the opposite direction. So, it would be a loss in ... Yeah. I-it would be a loss.
Surani:	The 19?
Gabriel:	Oh, sorry. Not the 19. The, the, the 25 and the 9 [marks those two numbers].
Surani:	Oh okay.
Gabriel:	Those would be losses. So, if you're looking at the positive direction of gain, it would be negative.
Surani:	I had some other students that did what you did.
Gabriel:	Mm-hmm (affirmative).
Surani:	They got those numbers, but then they crossed out the two negatives, and they wrote "no gain," or they wrote, um, "zero," or they wrote, uh, "lost 25." And...it seems like you didn't feel the need to do that. You're okay with that negative 25. Well, what might you say to those students

	that were just not okay with putting a negative there?
Gabriel:	Uh, I wouldn't necessarily say that they're wrong. It just would give an incomplete picture. Because if you're just adding up ... If you're just adding up the gains that the company made, then by only putting a zero, for instance, you're only ... you're excluding any profits that would have been lost over a month. And by saying that it was instead a loss of 25, instead of just saying negative 25, you're then putting a positive where a negative should be. And so, if you were to add it all up later, if you forgot to switch that to a negative, then it would just appear like more profits, when in fact you've lost 25 million.

Though both Loren and Gabriel were able to maintain an appropriate direction of comparison in part (a), Gabriel's comment "if you're looking at the positive direction of gain" showed that he was justifying his answers with arguments parallel to those he would use in part (b). Gabriel showed an awareness that he had chosen to measure in a particular direction for part (a). He then continues to discuss the act of looking in a particular direction when he explained his responses to part (b).

Surani:	Okay. Wonderful. And now, will you answer, just like before, the different question of this pessimistic guy that wants to calculate how much they lost each month.
Gabriel:	All right. Just going off of a little bit of memory, a little bit of quick math. Basically, the, the same thing, but the signs are flipped, since you're instead trying to look at how much was lost instead of how much was gained.
Surani:	Okay. And then, what would you say to a student that looked at a negative 19, and they said, "But I gained the money. Why should I put a negative when I gained money, like gain. You know, *gaining*." Positive. Happy. Why did you put a negative?
Gabriel:	Mm-hmm. Um, um, because you're, you're losing negative money which is the same as gaining, a double negative being a positive. Or you could also look at it from like a more practical, I guess, standpoint where if you're losing expenses, you're gaining money, that type of way of looking at it, I guess.
	[…]

Surani:	You used the word direction at some point, a few seconds ago.
Gabriel:	Um, i-it's ... Not sure how to explain it. Basically, if you're only looking at the, the direction of how the money had increased, then this would then be, I guess, a, a negative direction. So, it's increasing in the negative direction. Well, it's not increasing, but it, it's continuing in the negative direction with respect to the month that it's in, versus the other ones are increasing. It's going in the positive direction in respect to the month that they're in, if your ... if you were to be graphing gain.
	[…]
Dr. T:	Um, when you, when you were talking about, um, direction-
Gabriel:	Mm-hmm (affirmative).
Dr. T:	... so that, um, positive direction on a loss would be the opposite of the negative direction on the loss.
Gabriel:	Mm-hmm (affirmative).
Dr. T:	But the negative direction in the loss would be like a gain?
Gabriel:	Correct.
Dr. T:	So, what were you thinking ... What were you thinking of *first* [emphasized]? I don't know if you can have access to that. But what were you thinking of *first* when you were thinking about, um... Amount of loss and whether it would be, whether you would represent that amount with a positive or negative number?
Gabriel:	Um, mostly, I was thinking about what exactly a positive number would mean. So, in, in the instances of loss, a positive number would mean amount lost versus a negative number being, I guess, amount not lost or amount gain.

Gabriel's comments show that his imagery for making additive comparisons included an explicit decision about direction of comparison as well as an act of anticipation. His comment about "if you're only looking at the, the direction of how the money had increased" shows that his imagery included the idea that one could look in either direction. We gained more insight into his imagery when he said that he made his decision about whether to use positive or negative numbers by "thinking about what exactly a positive number would mean". Gabriel started with the act of anticipating what a measure might mean within the system of relationships between measures that he had available to work with. When he referred to known and practiced procedures such as the fact that a double negative is positive, he justified the use of that knowledge with an analogy that gave meaning to that fact: “you're losing negative money which is the same as gaining, a double negative being a positive. Or you could also look at it from like a more practical, I guess, standpoint where if you're losing expenses, you're gaining money”. His commitment in both frames (gain and loss) was consistent and he showed confidence in both his decisions and the resulting responses.

In fact, Gabriel’s conceptualization and coordination of frames allowed him to speed through every task I gave him except the last one. I asked Gabriel to solve a complicated task involving relative motion, where a pitching machine lobbed “anti-gravity baseballs” at a catcher running towards the machine. He was asked to resolve two contradictory opinions, one which said the catcher received the balls at the same constant rate the machine threw them and the other which said the catcher kept getting the balls faster and faster because he was running toward them. With some thinking and no outside help, Gabriel determined that neither opinion was correct and that the catcher was receiving the balls at a constant rate greater than the rate the machine threw them. I next asked in what situation the pitching machine could throw the balls at the same rate that the running pitcher would receive them, and Gabriel finally reached the end of what his current ways of thinking would allow him to do. In short, I had to give him a task involving a situation where different people had different measurements for "one foot" and "one second" – a special relativity problem – in order for Gabriel's reasoning with frame of reference to break down.

Looking at a Larger Sample

Loren's and Gabriel's demographics were very similar– male freshman students at the same university in similar majors who both passed the same Calculus 1 class. Both had taken calculus and physics in high school as well. My teaching experiment with them allowed me to model their reasoning with variations within a frame of reference, but two students were not enough to gain insight about the general undergrad population's ability to reason about variations. To do so, I looked at a larger sample of students.

We gave the bank account task (from Figure **10** above) to 123 DIRACC Calculus 1 students in October of 2018 when they completed approximately 2/3 of the semester. They were given a maximum of 10 minutes to complete the task and turn it in. Because we were using this task to look only at students' ability to commit to a direction of comparison, we ignored the magnitude of their answers and looked only at whether a value was positive or negative. After examining all the responses and looking for commonalities, we used open coding to group students response to the gain question and developed the following rubric:

Level	Gain Task Response Description
Direction:	Student maintained an appropriate direction of comparison.
Direction with Discomfort:	Student maintained an appropriate direction of comparison, but also expressed discomfort with negative values by adding

	commentary to explain their result (e.g. “no gain” or “a loss”)
Excess Only:	Student described positive gains but not negative gains, choosing to reframe the negative gains in some way (e.g. “0”, “gained 0”, “lost”, “didn’t gain”, “none”, or left blank)
No Direction:	Student does not maintain a direction of comparison, describing all variations with the same sign.
Other:	Scorer cannot interpret response, it does not fit a higher level, or blank.

Figure 13: Bank Task rubric for part (a)

Examples of each level are shown below along, with how many responses fell into each category:

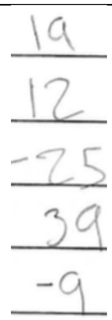
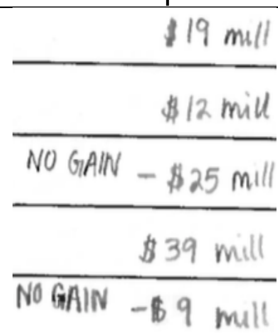
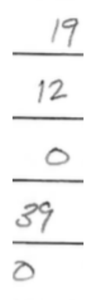
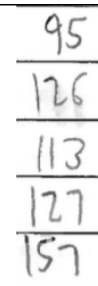
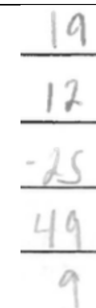
Direction Example	Direction with Discomfort Example	Excess Only Example	No Direction Example	Other Example
 19 12 -25 39 9	 $19 mill $12 mill NO GAIN -$25 mill $39 mill NO GAIN -$9 mill	 19 12 0 39 0	 95 125 113 127 157	 19 12 -25 49 9
58/123 (47.2%)	*10/123 (8.1%)*	*49/123 (39.8%)*	*4/123 (3.3%)*	*2/123 (1.6%)*

Figure 14: Examples of Bank Task part (a) rubric levels

We then repeated the process of open coding & categorizing for the loss task responses, and developed the following rubric:

Level	Loss Task Response Description
Direction:	Student maintained an appropriate direction of comparison.
Direction	Student maintained an appropriate direction of comparison, but also

with Discomfort:	expressed discomfort with negative values by adding commentary to explain their result (e.g. “no loss” or “a gain”).
Loss Only:	Student described positive losses but not negative losses, choosing to reframe the negative losses in some way (e.g. “0”, “lost 0”, “gained”, “didn’t lose”, “none”, or left blank).
Partial Gain Direction:	Student described positive losses as negatives, & reframed negative losses in some way (e.g. “0”, “lost 0”, “gained”, “didn’t lose”, “none”, or left blank).
Gain Direction:	Student gave Level G3 answers for loss question, keeping the same direction of comparison for both parts of the task.
No Direction:	Student does not maintain a direction of comparison, describing all variations with the same sign.
Other:	Scorer cannot interpret response, it does not fit a higher level, or blank.

Figure 15: Bank Task rubric for part (b)

Examples of each level are shown below along, with how many responses fell into each category:

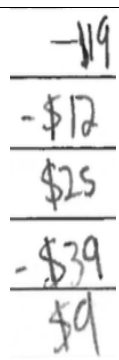
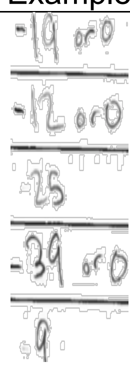
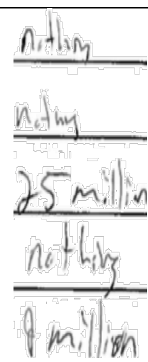
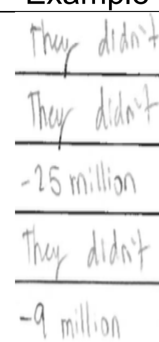
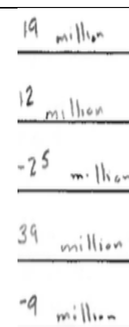
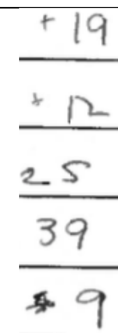
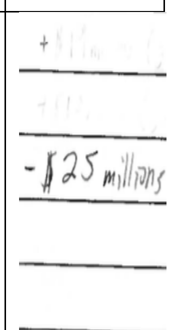
Direction Example	Direction with Discomfort Example	Loss Only Example	Partial Gain Direction Example	Gain Direction Example	No Direction Example	Other Example
						
-$19 -$12 $25 -$39 $9	-19 or 0 -12 or 0 25 -39 or 0 9	nothing nothing 25 million nothing 9 million	they didn’t they didn’t -25 million they didn’t -9 million	19 million 12 million -25 million 39 million -9 million	+19 +12 25 39 9	______ ______ -$25 millions ______ ______
39/123 (31.7%)	*1/123 (0.8%)*	*47/123 (38.2%)*	*7/123 (5.7%)*	*13/123 (10.6%)*	*9/123 (7.3%)*	*5/123 (4.1%)*

Figure 16: Example responses for the Bank Task rubric part (b)

All students' responses to both tasks are shown below in Table 1:

Table 1: Responses to Bank Task parts (a) versus (b)

Loss ↓ Gain→	Other:	No Direction:	Excess Only:	Direction with Discomfort:	Direction:	*Total:*
Other:	2	0	1	0	4	*7*
No Direction:	0	2	1	3	3	*9*
Gain Direction:	0	0	0	3	10	*13*
Partial Gain Direction:	0	0	7	0	0	*7*
Loss Only:	0	2	40	3	2	*47*
Direction with Discomfort:	0	0	0	1	0	*1*
Direction:	0	0	0	0	39	*39*
Total:	*2*	*4*	*49*	*10*	*58*	*123*

Our data shows that committing to a direction of comparison was not easy for most of the students in the sample. 44.7% of students (lowest three levels on part (a)) did not maintain a direction of comparison on the gain task even though negative values of money are commonly referenced in our culture, and a further 8.1% showed discomfort with simply letting a negative value for gain stand on its own. 67.5% of students (lowest three levels on part (b)) did not maintain a direction of comparison on the loss task.

The 10.6% of students that gave a Level L2a response showed that while they committed to direction for the gain task, they did not switch to a new direction for the loss task.

We found two results of note in our data. First, a student's performance on the gain task is highly predictive of their performance on the loss task, and vice versa with $p<.0001$; this is not surprising, but does tell us that even a simple task like the gain question can give us insight into our students' ability to commit to a direction of comparison. Our second result is even more interesting: we wanted to look only at the 68 students who gave the highest two scored responses to the gain task, and compared their responses to the loss task. The only difference between these students is that "Direction" responses had no explanation for negative values while students giving "Direction with Discomfort" responses felt some need to give additional explanations or interpretations to their negative values. Whether a student explained or didn't explain their negative gains was highly predictive of how they would do on the loss task with $p<.0001$. In other words, *simply adding an explanation to their negative values on the gain task* predicted that they struggled to answer the loss task.

Discussion

Our research questions for this study were:

1) To what extent do our two students combine multiple frames of reference?

2) To what extent do our two students conceptualize a frame of reference by committing to reference point and direction of comparison?

3) To what extent do our two students coordinate multiple frames of reference?

4) To what extent do calculus students commit to direction of comparison?

Loren and Gabriel provide us with two starkly different examples of reasoning with variations within a frame of reference. Gabriel combined reference frames, consistently committed to a reference point and direction of comparison for his quantities, and coordinated reference frames. His commitments within each conceptualized frame of reference helped him give appropriate responses to tasks about variations confidently and to justify his answers. Loren frequently did not commit to a reference point, a direction, or both. Along with his difficulties with variation and meanings for graphs, his lack of commitments within each frame caused him difficulties in making sense of variations. After analyzing these two students, we had evidence that conceptualizing a frame of reference affected student reasoning about variations.

Data from a larger sample of students suggests many calculus students struggle to commit to a direction of comparison and to coordinate frames that differ in terms of their directionality.

Interviews with Loren and Gabriel, and data from Calculus 1 students, suggests a wide range in Calculus 1 students' abilities to reason within a frame of reference, in terms of both commitment to reference point and commitment to direction of comparison. This is especially significant for calculus students for two reasons.

1) The expression $\left(f(x+h)-f(x)\right)$ in the difference quotient $\left(f(x+h)-f(x)\right)/h$ is supposed to mean a comparison of values of *f* within a frame of reference. Students' whose underlying imagery is to adjust

directions of comparison so comparisons always have a positive value cannot understand the difference quotient's numerator as evaluating a directed variations in *f*'s value. That understanding would conflict with their image of comparison, forcing them to understand the numerator unproductively. Students who reason effortlessly about $\left(f(x+h)-f(x)\right)$ as representing a framed comparison of values not only are positioned to understand rate of change as a framed quantity, they are positioned to develop a meaning of acceleration as a framed quantity.

2) The differential *dx* is fundamentally a difference. Students often presume unthinkingly that a value of *dx* always represents an increase in the value of *x*. But to understand the validity of the statement that accumulation from *a* to *x* equals the negative of accumulation from *x* to *a*, or

$$\int_a^x f(t)\,dt = -\int_x^a f(t)\,dt$$

they need to understand that *dt* in the right side of the equality has the opposite sign of *dt* in the left side.

The study reported here investigated calculus students' ability to reason within a frame of reference and where those abilities might break down, Statements (1) and (2), above, are conjectures to be researched in our next study.

Conclusion

As we analyzed our data and explored how Loren and Gabriel reasoned about variations, we found several places where our teaching experiment could have been improved.

One of the most glaring omissions was that we had no tasks where the students had to coordinate both a change in reference point and in directionality. Each task that involved coordination of two frames had a different reference point or a different direction of comparison, but not both. Future teaching experiments that investigate variations within a frame of reference will include new tasks such as “d) sketch a graph of the clown to the *left* of x=3 with respect to elapsed time”.

Our choice of Loren as a subject, who had such complicated and fragile connections between his imagery for graph, frame of reference, covariation and even quantity itself, make our analysis of our data more difficult than expected. We chose to write this paper about the first few tasks in the teaching experiment because the number of factors that were needed to explain Loren’s behavior on more complicated tasks became untenable. Additionally, we simply did not get to some of the most interesting tasks with Loren. Each student was originally scheduled for 3 hours, and Gabriel finished all our planned plus additional tasks in 2.5 hours, but Loren struggled to get through most of our asks even after we extended his interviews to a total of 4 hours.

The current experiment was limited to how students reason about variations within a frame of reference, but future work in this area can build upon these early results to study how students reason about rate of change and accumulation of infinitesimal variations as well. Our initial study included both of these ideas but it soon became clear, even before the start of interviewing, that the scope was too large for one teaching experiment.

We would also like to take more large-scale data about how students think about variations within a frame of reference. Our rubrics for the Bank Task item could be used with students at a variety of levels, from high school algebra student who need to think about variations in order to grasp constant rate of change, to pre-service teachers at all levels. Data on how in-service teachers reason with frame of reference, including one item focusing on variations, is forthcoming in a future paper.

CHAPTER 3

PAPER THREE: US AND KOREAN TEACHER'S REASONING WITH FRAMES OF REFERENCE

Our theory of what entails a conceptualized frame of reference is explained, along with items and rubrics designed to illuminate how teachers do or do not reason with frames of reference. We gave 539 teachers in the US and Korea frame of reference tasks, and coded the open responses with rubrics intended to rank responses by the extent to which they demonstrated conceptualized and coordinated frames of reference. Our results show that our theoretical framework is useful in analyzing teachers' reasoning with frames of reference, and that our items and rubrics function as useful tools in assessing teachers' meanings for quantities within a frame of reference.

Keywords: Frames of Reference, Mathematical Meanings, Secondary Teachers, Quantitative Reasoning

A frame of reference is an organizing tool most familiar in physics, yet it is also applicable to any mathematics task that involves quantities, or measurable attributes of objects (Thompson, 1994). Every time a person thinks about a quantity, its meaning is only fully understood within the frame of reference within which it was measured. To say a plane is flying at 35,000 feet only has meaning when we know height was measured in a frame where the reference point is sea level; to say a ball's free fall velocity close to Earth varies by -9.8m/s/s only has meaning when we know that acceleration was measured within a frame where distance measurements are always away from the surface of the Earth, as opposed to towards it.

If professional development programs and education researchers wish to address issues with how teachers help their students with the mathematics they teach, we first need more nuanced information about the teachers' own understandings of the mathematics. Many current assessments that focus on mathematical knowledge for teaching (Hill, 2005) categorize teachers' MKT by whether or not they can give normatively correct answers to tasks. In Project Aspire, we took an alternate approach by analyzing teachers' responses according to what those responses told us about the teacher's meanings at the moment of their response, and to compare different meanings by how productive, in our judgment, they would be for student learning were teachers' meanings to become students' meanings. Our work responds to critiques of the deficit model (Bak, 2001), in that we are interested in identifying mathematical understandings and meanings teachers *have* rather than understandings and meanings they do not have.

In this work, we draw on data from an assessment we developed to analyze teacher's mathematical meanings, drawing specifically from items meant to give insight into teachers meanings regarding frame of reference. We present five tasks designed to assess the ways in which teachers reason with frame of reference, the rubrics we wrote to score the responses in terms of the meanings displayed, sample teacher responses, and the results from giving these tasks to over 500 U.S. and South Korean teachers.

Our research questions for this study are:

1. How do the mathematics teachers in our samples reason with frames of reference?

2. Are there differences between the teachers in our United States sample and South Korean sample in how they reason with frames of reference?

Past Literature

Our first search of math education and physics education literature revealed no conceptual definitions of frame of reference (Joshua, Musgrave, Hatfield, & Thompson, 2015). By 'conceptual definition' we mean a definition of what mental actions a student must engage in in order to conceptualize a frame of reference. Instead, the definitions we found in both textbooks and academic articles referred to physical objects, such as "a set of rigidly welded rods" (Carroll & Traschen, 2005), "a set of observers" (de Hosson, Kermen, & Parizot, 2010), or "a coordinate system and a clock" (Young, Freedman, & Ford, 2011) among others. Several studies looked at ways in which students struggled with frame of reference tasks (Bowden et al., 1992; Trowbridge & McDermott, 1980) or reported results of interventions meant to improve performance on frame of reference tasks (Monaghan & Clement, 1999; Shen & Confrey, 2010) and one identified common student misconceptions about frames of reference (Panse, Ramadas, & Kumar, 1994). None gave a clear conceptual definition of frame of reference, which we concluded was needed to address the issue of how to help students think with frames of reference.

Theoretical Perspective

Quantity & Quantitative Reasoning

Our conceptual definitions for frame of reference reside within the larger theory of quantitative reasoning. Thompson defines quantitative reasoning as "the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships" (REF 1994, 2011), where a quantity is a person's conceptualization of an object and attribute of it so that it is measurable. "A quantity is in a mind. It is not in the world." (Thompson, 2011, p. 2).

Conceptualizing a Frame of Reference

When a student has fully conceptualized a frame of reference, the frame of reference itself is not the the student's primary object of consideration. Rather, the student is using one or more frames of reference as a systematic way to think about and organize the measures of quantities and their meanings, as well as the quantitative relationships between those quantities. This places our constructs of conceptualizing and coordinating frames squarely within the domain of quantitative reasoning (Thompson, 1993) and in fact is related to how students do or do not construct schemes for thinking with magnitudes (Thompson, Carlson, Byerley, & Hatfield, 2014). This clarification guided our eventual definition:

An individual can think of a measure as merely reflecting the size of an object relative to a unit or he can think of a measure within a system of potential measures and comparisons of measures. An individual conceives of measures as existing within an *additive frame of reference*[3] if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives meaning to a zero measure and all non-zero measures, and 3) committing to a direction of measure comparison additively, multiplicatively, or both.

In the items we present in this study, our analysis focuses on the latter two aspects of frame of reference reasoning: commitment to reference point, and commitment to direction of comparison.

A person commits to a reference point he has in mind the idea that a quantity's measurement is taken from some known chosen reference point, and that this reference point is important in giving meaning to the measurement. For example, consider the difference between thinking about "distance Robin drove" and thinking about "distance Robin drove since passing the café" taken from one of our tasks. The meaning of any measure of the quantity "distance Robin drove" has an ambiguity that can create difficulties when working on a complex task, especially when measurements from different reference points are involved.

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[3] One can conceptualize a frame of reference additively or multiplicatively. One thinks with an additive frame of reference when measures are rooted in additive change, such as measuring with a ruler. A person thinks with a multiplicative frame of reference when measures are rooted in additive and multiplicative change simultaneously, such as in exponential growth. Our subsequent uses of "frame of reference" will refer to additive frames. Multiplicative frames will be the subject of future publications.

A person commits to a direction of comparison when she conceives the comparison in general terms and not in terms of specific measures being compared. For example, suppose a teacher poses this scenario to her students: “Five students are outside our door. They will walk into our room holding a card stating their height in inches. Record each student’s height and answer the question *how much taller* each student is than the next student to walk through the door.”

1. Student 1 is ______ inches tall. Student 2 is _______ inches tall.
   Student 1 is ________ inches taller than Student 2.
2. Student 2 is ______ inches tall. Student 3 is _______ inches tall.
   Student 2 is ________ inches taller than Student 3.
3. Student 3 is ______ inches tall. Student 4 is _______ inches tall.
   Student 3 is ________ inches taller than Student 4.
4. Student 4 is ______ inches tall. Student 5 is _______ inches tall.
   Student 4 is ________ inches taller than Student 5.

The five students have heights of 56 inches, 49 inches, 52 inches, 55 inches, and 51 inches, in that order. The teacher has each student stand at the room’s front, holding his or her card. The teacher, anticipating some students will object to saying Student 2 is some number of inches taller than Student 3, has designed a scenario wherein she can manage a conversation about what it means to make a commitment to direction of comparison.

Coordinating Frames of Reference

Our use of the word ‘commitment’ is not a suggestion that people should commit to a single unit, reference point, or direction of comparison for any given task. Rather, it is important for a person to be aware of the frame of reference within which given measurements make sense, so that they can make explicit coordinations of those measures with corresponding measures in another frame should that be necessary or expedient for completing a goal. An individual is coordinating two frames of reference if she conceives each frame as a valid frame, is aware of the need to coordinate quantities’ measures within them, and carries out the mental process of finding a relation between measures within frames while keeping all relative quantities and information in mind (Joshua et al., 2015).

Schemes & Meanings

We follow Thompson et. al’s definition of a scheme as an “organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization’s activity.” (Thompson, 2013; Thompson, Carlson, Byerley & Hatfield, 2014). A person’s meaning is the space of implications of their understanding, and we call a person’s meaning stable if the person’s understanding in the moment results from an assimilation to a scheme. Unlike knowledge, which is often used to describe declarative facts that are either normatively correct or incorrect, a person’s meanings are revealed in the process of application: whether their meaning has been triggered, whether they are used consistently, and so on. When a person uses a meaning consistently, we say their meaning is a scheme. The items and rubrics used to score teachers' responses in this study focused on meanings because we wanted to explore the implications of teachers' understandings as revealed in tasks designed to trigger similar mental activity in a classroom.

Dependence on Context

Thompson and colleagues (A. G. Thompson & Thompson, 1994; P. W. Thompson, 1996, 2013; P. W. Thompson, Carlson, Byerley, & Hatfield, 2014) expanded on the Piagetian construct of scheme by explaining the role that individuals' imagery plays in activating their schemes. Images are fragments of remembered experience, and the extent to which they affect a person' thinking depends on the strength and operativity of that person's scheme (Piaget 1967). We wrote about how imagery can be used to analyze a person's scheme for frame of reference in (second dissertation paper):

> "The strength and applicability of a person's schemes entails a co-dependency between relationships among actions the person has formed and the imagery triggering them. If the imagery associated with a scheme is tied to surface features of tasks such as perceptual material, context, or sensorimotor experience, then the scheme will only be triggered by tasks similar to those that the she already experienced. However, if she has built imagery for the scheme that is related to the relationships between

quantities that are invariant across these tasks, she is more equipped to apply a scheme to an unfamiliar context that still contains those kinds of relationships.

In brief, a person whose imagery is in the first or second phase in regard to contexts she witnesses or imagines by way of reading text will exhibit thinking that belies a greater dependence on context than a person whose imagery is in the third phase. We also take the reverse stance, that a person who exhibits thinking that seems to change among contexts *we* see as embodying the same ideas does so because of imagery that is at an early phase of development."

Methodology

From 2012 to 2015, the Project Aspire team created the 46-item assessment *Mathematical Meanings for Teaching secondary math* (MMTsm; Thompson, 2016), to be used by professional development leaders to assess the effectiveness of their interventions. The project started with the intention of assessing teachers' meanings on six core constructs – function, magnitude, variation, covariation, structure, and rate of change. As the project continued we added an additional construct—frame of reference. The *Willie & Robin* item (described later) was originally designed to be a rate of change item, and the *Ivonne & Nicole* item (described later) was originally intended to assess teachers' meanings for proportionality by seeing if they would over-generalize their "proportion" schemes to include a situation that was fundamentally additive

(Greer, 1988; Iszak & Jacobson, 2017). However, teachers' responses in early trials of *Ivonne & Nicole* suggested the primary issue was teachers' coordination of frames of reference. Both items were re-categorized as frame of reference items. In the intervening time we identified frame of reference as a secondary construct in several other MMTsm items, one of which (*Subsequent Changes*, an item designed primarily to assess meanings of covariation) is discussed in this paper.

The first few rounds of creation included writing items and refining them using data from interviews with teachers, pilot testing with small and large groups of teachers, and input from experts (a panel of four PhD mathematicians and six PhD mathematics educators). The purpose of interviews and pilots was to see whether items elicited responses that gave the team insight into teachers' meanings they were designed to investigate. To do so, we created items so that teachers could not use practiced procedures or memorized definitions. Instead, we sought to create items that highlighted ways teachers understood the problem. With teacher data and feedback from the panel, items were then revised if warranted, and interviews repeated if the revisions were significant. In the process, we culled some items, refines others, and inspired by our experiences with the data, added a few more items.

The next stage involved writing rubrics with which to score teacher responses to items, most of which are open response. In doing so we reminded ourselves that we were investigating teachers' *mathematical meanings for* *teaching* not merely their mathematical meanings. In other words, because we were specifically assessing classroom teachers, the guiding philosophy of rubric creation was to capture the criterion “If a teacher shared the meanings we discern with a class, what meanings for the mathematical idea might students construct?” Moreover, we gave extra weight to meanings suggested by teachers’ initial responses because we presumed those were meanings most likely to come to teachers’ minds spontaneously in a classroom.

Since a major goal of Project Aspire is to provide information to professional development leaders, we also had to make the rubrics straightforward enough for people outside of our core research group to use with only minimal training (a few days). It was therefore necessary to write descriptions of rubric levels that would capture certain ways of thinking, without requiring that the scorer be familiar with the nuances of those ways of thinking. We used constructs in mathematics education research to make decisions about the rubrics, but then had to explain and illustrate the rubrics not with theory but with examples of responses where such meanings would play out. In the process of creating, testing, and refining rubrics we found that to make some items scorable it was necessary to significantly change them. Others needed to be discarded completely, and a few were refined to be multiple choice items. With the help of the BEAR team at UC Berkeley we ran several rounds of IRR and used the results to refine the items and rubrics.

The data from the U.S. discussed in this paper was collected from 253 high school teachers in 2014 and 2015 who volunteered for two professional development programs around the country and were scored by the Project Aspire team, with some overlapping scores with which to run IRR. In 2015 one of our Project Aspire team members, a native Korean speaker, translated the items to Korean, had them back-translated by another native speaker, and adjusted the translations with that feedback. In summer of 2015 this core research team member went to Korea and collected our Korean data from 366 South Korean teachers (264 high school teachers, 102 middle school teachers) that gathered in one place to undergo required teacher certification tests. She also interviewed 3 of the Korean teachers on each item, data that provides useful additional info on how those teachers were thinking. She then trained five Korean mathematics education Ph.D. students on the rubrics to score the entire Korean data set. The Korean member of the Aspire team then scored a subset of responses in order to run IRR. We have decided not to use our Korean data for the '*Subsequent Changes*' item because of issues with the translation that were revealed after the data was taken.

Table **2** summarizes the inter-rater reliability scores for all items reported. We have decided not to use our Korean data for the '*Subsequent Changes*' item because of issues with the translation that were revealed after the data was taken.

Table 2: IRR Scores for all MMTsm frame of reference items.

Item Name	Number of responses scored by two scorers	Percent Agreement (US / Korea)	Cohen's Kappa (US / Korea)
Willie Chases Robin Part A	100 (50/country)	92% / 97%	0.889 / 0.936
Willie Chases Robin Part B	100 (50/country)	68% / 93%	0.537 / 0.911
Willie Chases Robin Part C	100 (50/country)	92% / 69%	0.684 / 0.480
Ivonne & Nicole	100 (50/country)	96% / 97%	0.934 / 0.937
Subsequent Changes	50 (US only)	100% / N/A	1 / N/A

We cannot claim our U.S. or Korean data is representative of teachers in either country. U.S teachers were from two professional development programs in the American Midwest and Southwest and had taught a mean of 4.35 years (s.d. = 4.22), while Korean teachers were those taking a professional development program to get a certification required by the fifth year of teaching and had taught a mean of 3.99 years (s.d.=1.96). Neither set of teachers were chosen randomly, but instead were chosen from teachers willing and available to take our assessment at that time. All our claims are about *this* set of American teachers and *this* set of Korean teachers. However, we feel that our samples are large enough, and in some cases the national differences are stark enough, that our data should be considered of interest to the mathematics education community.

Results

In this section we will discuss the reasoning behind our five tasks and our scoring rubrics, analyze a few illustrative examples of teacher responses for each task, and look at the results of giving each task to large numbers of U.S. and Korean teachers. We discuss the US and Korean teacher's separately because the two populations differ greatly in their characteristics, both in background and in the nature and distribution of their responses to our items. We do not separate them for the purpose of comparing them and do not seek to analyze their differences in terms of which groups performed better or worse. Rather, we report them separately because that allows us to look within each group of teachers at relationships between their responses to items.

Four tasks involve relative motion. Three tasks are related to the same stem in the *Willie Chases Robin* scenario, and a fourth task is the *Ivonne & Nicole* task. The fifth task looks at teacher's reasoning about changes in the *Subsequent Changes* task. While the contexts and questions vary, all the items and rubrics are designed to characterize a teacher's ability to reason about quantities within a frame of reference and to coordinate multiple frames of reference.

Figure **17** shows the *Willie Chases Robin* context below, which includes our first three frame of reference tasks. *Willie Chases Robin* is a frame of reference context where an individual uses one clock to time two events that begin at different moments. Thus, a person must add 1/6 hour to Willie's elapsed time to represent Robin's elapsed time within Willie's frame of reference, or subtract 1/6 hour from Robin's elapsed time to represent Willie's elapsed time in Robin's frame of reference. Notice also the item's use of two different time units—speed measured in miles per hour and the difference in their elapsed times measured in minutes.

Robin Banks ran of a bank and jumped into his car, speeding away at a constant speed of 50 mi/hr. He passed a café in which officer Willie Katchim was eating a donut. Willie got an alert that Robin had robbed the bank, jumped into his patrol car, and chased Robin at a constant speed of 65 mi/hr. Willie started 10 minutes after Robin passed the café.
Part A. Let *u* represent the number of hours since Robin passed the café. Write an expression that represents the number of hours since Willie left the café.
Part B. Here are two functions. They each represent distances between Willie and Robin.

$$f(x) = 65x - 50\left(x + \frac{1}{6}\right), x \geq 0$$

$$g(x) = 65\left(x - \frac{1}{6}\right) - 50x, x \geq \frac{1}{6}$$

i) What does *x* represent in the definition of *f* ?
ii) What does *x* represent in the definition of *g*?
Part C. Functions *f* and *g* both give a distance between Willie and Robin after *x* hours. But *f*(1)=6.67 and *g*(1)=4.17. Why are *f*(1) and *g*(1) not the same number?

Figure 17: *Willie Chases Robin* MMTsm Item

Willie Chases Robin Part A: Item & Rubric

Robin Banks ran of a bank and jumped into his car, speeding away at a constant speed of 50 mi/hr. He passed a café in which officer Willie Katchim was eating a donut. Willie got an alert that Robin had robbed the bank, jumped into his patrol car, and chased Robin at a constant speed of 65 mi/hr. Willie started 10 minutes after Robin passed the café.
Part A. Let *u* represent the number of hours since Robin passed the café. Write an expression that represents the number of hours since Willie left the café.

Figure 18: *Willie Chases Robin* MMTsm item Part A

The *Willie Chases Robin* Part A item in Figure **18** above is designed to see if teachers would coordinate two different measurements of time for the two men, each using the reference point of when they left the café. For example, a teacher that thinks about two imaginary stopwatches, one for each man that starts when he leaves the café, might reason about the constantly changing measurements on those stopwatches and come to the conclusion that Willie's stopwatch will always display 1/6 of an hour less than Robin's at the same moment and therefore is represented by the quantity "*u*-1/6". We then categorized and ranked other responses in terms of how productive we thought they would be for students in the classroom. Figure **19** summarizes our rubric for Part A.

Coordinated directionality:	The teacher wrote *u*-1/6 or something consistent with *u*-1/6.
Reverse directionality:	The teacher wrote *u*+1/6 or something consistent with *u*+1/6.
Other:	The response doesn't fit a higher level, cannot be interpreted, has no clear answer, is off-topic, answered "I don't know" or was left blank.

Figure 19: '*Willie Chases Robin*' MMTsm Rubric for Part A

Willie Chases Robin Part A: Exploring the Responses

We now discuss what individual responses can tell us about the teacher's meanings for quantities within a frame of reference, at the time they took this assessment. Part A's rubric was written to capture only a teacher's commitment to directionality of comparison. Figure 20 shows three sample teacher responses to Part A that were scored at different levels.

Part A. Let *u* represent the number of hours since Robin passed the café. Write an expression that represents the number of hours since Willie left the café.		
a) $u - 10$	b) $u + \frac{1}{6}$	c) $U = 50x+10$ $50x+10$

Figure 20: Teacher responses to ‘*Willie Chases Robin*’ Part A

The responses in Figure 20a and Figure 20b were scored as such because a teacher that tries to reason with key words like the fact that Willie left “after” Robin might add 1/6 to *u*, because without committing to the reference point of café leaving she could confuse elapsed time with time of day. We therefore hypothesized that responses consistent with “*u*-1/6” display a higher level of frame of reference reasoning than “*u*+1/6”. While it is also important for a teacher to commit to a unit of hours, for the purposes of this rubric we accepted “*u*-10” as a highest level response because the direction of coordination is correct. The response “u-10” was rare, as most teachers did correctly coordinate the units, but we include it here to illustrate the range of our rubric categories. The response in Figure 20c was scored at the lowest level because it is not consistent with an ability to commit to a directionality of comparison.

Table **3** shows a breakdown of Part A responses from each country, further separated by what their university degree is in; 173 US and 366 Korean teachers saw Part A of the ‘*Willie Chases Robin*’ task.

Table 3. Responses to the ‘*Willie Chases Robin*’ Part A MMTsm item.

Directionality	United States	South Korea	*Total*

Coordinated	52 (30.1%)	226 (61.7%)	*278 (51.7%)*
Reversed	22 (12.7%)	71 (19.4%)	*92 (17.1%)*
Other	99 (57.2%)	69 (18.9%)	*168 (31.2)*
Total	173 (100.0%)	366 (100.0%)	*539 (100.0%)*

** Cells contain number of respondents and percent of column total.*

There is a statistically strong difference between US and Korean teachers' responses, (χ^2(df=2, N=539)=81.411, p<.0001). 60.7% of Korean teachers coordinated the directionality of comparison between Willie and Robin's elapsed time while only 30.1% of US teachers did so. However, Korean teachers were slightly more likely (19.4%) to reverse the directionality of comparison than US teachers (12.7%).

Willie Chases Robin Part B: Item and Rubric

Robin Banks ran of a bank and jumped into his car, speeding away at a constant speed of 50 mi/hr. He passed a café in which officer Willie Katchim was eating a donut. Willie got an alert that Robin had robbed the bank, jumped into his patrol car, and chased Robin at a constant speed of 65 mi/hr. Willie started 10 minutes after Robin passed the café.
Part B. Here are two functions. They each represent distances between Willie and Robin.

$$f(x) = 65x - 50\left(x - \tfrac{1}{6}\right), x \geq 0.$$

$$g(x) = 65\left(x - \tfrac{1}{6}\right) - 50x, x \geq 1/6.$$

What does x represent in the definition of f?
What does x represent in the definition of g?

Figure 21: '*Willie Chases Robin*' MMT item Part B

Willie & Robin Part B in Figure **21** is designed to see whether teachers would interpret the meaning of parts of function definitions by analyzing them quantitatively, with explicit reference to their domains. The highest level for this item is for responses where the teacher distinguished between both independent variables by the reference point of their values. The only way for two non-equivalent functions definitions to represent the same quantity (distance between the men) is for the independent variable in each to have different meanings, which is why responses that said both *x*'s have the same meaning were placed at fifth highest level. The intermediate levels were for responses that articulated the difference to some degree but did not specify the exact quantitative meaning of the *x*'s. Figure **22** summarizes our rubric for Part B, and the dotted line between the third and fourth rows indicates that when ranking we judged both of these types of responses to be consistent with approximately equal levels of productive meaning in the classroom.

Used reference points of quantities:	The teacher said *both* of the following things: - *x* in *f*(*x*) represents number of hours (or elapsed time) since Willie left café - *x* in *g*(*x*) represents number of hours (or elapsed time) since Robin left café
Used point of starting motion:	Matches highest response except that *x* in *g*(*x*) is since Robin left *bank*. Reference points are not chosen appropriately for quantity but rather are consistent with identifying the start of each man's motion.
Reversed reference points:	Matches highest response except teacher switched the reference points for *x* in *f*(*x*) and in *g*(*x*)
Only mentioned elapsed time:	Matches highest response except no reference points (café, bank) mentioned
Identical answers for i) & ii):	Teacher gave same meanings for *x* in *f*(*x*) as in *g*(*x*)
Other:	The response doesn't fit a higher level, cannot be interpreted, has no clear answer, is off-topic, answered "I don't know" or was left blank.

Figure 22: '*Willie Chases Robin*' MMTsm rubric for Part B

Willie Chases Robin Part B: Exploring the Responses

Part B elicited a wide range of responses, so we built a rubric that looked at all three of the commitments necessary to fully conceptualize a frame of reference: unit, reference point, and directionality of comparison. Figure **23** gives three sample responses to Part B scored at different levels.

Part B. Here are two functions. They each represent distances between Willie and Robin.		$f(x)=65x-50\left(x-\frac{1}{6}\right), x\geq 0.$ $g(x)=65\left(x-\frac{1}{6}\right)-50x, x\geq 1/6.$
(a) *i) What does x represent in the definition of f ?* x represent the number of hours since Willie left the cafe "x represents the number of hours since Willie left the café"	(b) *i) What does x represent in the definition of f ?* the number of hours Robin passed the café "the number of hours Robin passed the café"	(c) *i) What does x represent in the definition of f ?* Related to Robin given he is 1/6 hr ahead of Willie "Related to Robin given he is 1/6 hr ahead of Willie"
ii) What does x represent in the definition of g? x represents the number of hours since Robin passed the cafe "x represents the number of hours since Robin passed the café"	*ii) What does x represent in the definition of g?* the number of hours Katchim passed the café "the number of hours Katchim passed the café"	*ii) What does x represent in the definition of g?* Related to Willie given that he is 1/6 hr behind Robin "Related to Willie given he is 1/6 hr behind Robin"

Figure 23: Teacher responses to '*Willie Chases Robin*' Part B

The response in Figure **23**a was scored at the highest level because of three aspects we deemed importan. The teacher clearly specified "number of hours" so was identifying each *x* as representing a quantity; other responses merely referred to “time” which could apply equally to the passage of time or the time of day. The teacher also specified reference points and used the appropriate reference points (leaving the café for both men) to make sense of the function definitions. Without reference points for a quantity’s measurement, the relationship between a given measurement and the quantitative situation it represents is ambiguous. Finally, the teacher correctly identified that *f* gave the distance between the two men in terms of Willie’s time since leaving the café, where *g* is in terms of Robin’s time since leaving the café. In order to correctly identify each function’s independent value, the teacher had to reason about how one would adjust each man’s time in terms of the others to calculate his distance from the café, in terms of his speed times the number of hours he drove. Our model for how this teacher reasoned was that she conceptualized the quantity with an internal commitment to unit, reference point, and directionality of comparison.

The response in Figure **23**b was scored at the middle (third) level because it is identical to a highest level response *except* that the teacher reversed the meanings of the *x* in the definition of *f* and the *x* in the definition of *g*. The definitions he gave do not allow for *f* and *g* to represent the distance between Willie and Robin. The teacher’s response is consistent with using one directionality of comparison to define each measurement of elapsed time ([final time] – [initial time] to find the value of *x*) but the opposite directionality of comparison to define each man’s time in the other man’s frame of reference. Our model for how this teacher reasoned about Part B was that he conceptualized the quantity with an internal commitment to unit and reference point, but not directionality of comparison.

The response in Figure **23**c was scored at the lowest level because it did not fit any higher levels, and we can see why when we look at this response in terms of the commitments the teachers did and did not make. This teacher identified the difference in the *x*'s by a general indication that each one has something to do with one person in the context and referred to the difference of 1/6 hours in starting time between the two men. We can see that the teacher is hinting at something relating to the difference in reference points for each man's measurement of time, but she does not know how to interpret that difference by defining two quantities with different reference points. Our model for how this teacher reasoned is that she did not define either *x* in terms of any quantities (precise or vague) at all, so she made no commitments to unit, reference point, or directionality of comparison in this response.

Table **4** shows a breakdown of Part B responses, with 173 US and 366 Korean teachers. "RP" is shorthand for "reference points" in the level descriptions.

Table 4. Responses to the '*Willie Chases Robin*' Part B MMTsm item.

	United States	South Korea	*Total*
RPs of quantities	23 (13.3%)	144 (39.3%)	*167 (31.0%)*
Start points of motion	5 (2.9%)	8 (2.2%)	*13 (2.4%)*
Reversed RPs	35 (20.2%)	25 (6.8%)	*60 (11.1%)*
Only elapsed time	5 (2.9%)	72 (19.7%)	*77 (14.3%)*
Identical i) and ii)	27 (15.6%)	33 (9.0%)	*60 (11.1%)*
Other	78 (45.1%)	84 (23.0)	*162 (30.1%)*
Total	*173 (100.0%)*	*366 (100.0%)*	*539 (100.0%)*

** Cells contain number of respondents and percent of column total.*

There is a statistically strong difference between US and Korean teachers' responses, (χ^2(df=5, N=539)=91.815, p<.0001). Korean teachers were almost three times as likely as US teachers to identify both reference points of the quantities of elapsed time (39.3% SK; 13.3% US). However, if we ignore any reverse in directionality of comparison (adding the first and third levels), the two groups had much closer percentages: 46.1% of Korean teachers and 33.5% of US teachers used the reference points of elapsed time for each man. Our teacher responses to Part B show a much greater difference between the two populations in commitment to directionality of comparison than in commitment to reference point.

On the other end of the spectrum of responses, 15.6% of US and 9.0% of Korean teachers gave identical meanings for the quantities represented by x in $f(x)$ and $g(x)$ even though $f(x)\neq g(x)$, so they cannot logically represent the same quantity if x has the same meaning in each. These responses are consistent with only looking at algebraic formulas and not thinking about the quantities represented by each part of each function definition.

Willie Chases Robin Part C: Item and Rubric

Robin Banks ran of a bank and jumped into his car, speeding away at a constant speed of 50 mi/hr. He passed a café in which officer Willie Katchim was eating a donut. Willie got an alert that Robin had robbed the bank, jumped into his patrol car, and chased Robin at a constant speed of 65 mi/hr. Willie started 10 minutes after Robin passed the café. Here are two functions. They each represent distances between Willie and Robin. $f(x)=65x-50\left(x-\frac{1}{6}\right), x\geq 0.$ $g(x)=65\left(x-\frac{1}{6}\right)-50x, x\geq 1/6.$ Part C. Functions f and g both give a distance between Willie and Robin after *x* hours. But $f(1)=6.67$ and $g(1)=4.17$. Why are $f(1)$ and $g(1)$ not the same number?

Figure 24: '*Willie Chases Robin*' MMTsm item Part C

Willie & Robin Part C in Figure **24** is designed to see whether teachers could articulate how two functions could represent the same quantity yet have different values for the same value of their independent variables. The answer, as in Part B, is that *x* has different meanings in the two functions. For example, if Robin passed the café at 4:00pm, then the distance between the two men at 5:00pm is either *f*(1)=6.67 or *g*(1.16)=6.67. The value of a variables (or a quantity) has no meaning without a measure's reference point. Therefore we placed responses that explained the difference between the meanings of "1" in *f*(1) and "1" in *g*(1) at the highest level, and responses that identified a difference in meaning without elaboration at the middle level. Figure **25** summarizes our rubric for Part C.

Different reference point of	Teacher said *f*(1) and *g*(1) represent distance between men at two different moments in time, or made same statement for

quantities:	*x*=1 in *f*(*x*) and in *g*(*x*).
Unspecified or misattributed different meaning:	Teacher said *x*=1 has different meanings in both functions but a) did not elaborate on the meaning of *x*, b) described both *x*'s as representing distances, or c) described *f*(1) and *g*(1) as representing time passed; or, described *f*(1) and *g*(1) as representing distances but not specifically distances between men.
Other:	The response doesn't fit a higher level, cannot be interpreted, has no clear answer, is off-topic, answered "I don't know" or was left blank.

Figure 25: '*Willie Chases Robin*' MMTsm rubric for Part C

Willie Chases Robin Part C: Exploring the Responses

Part C was particularly difficult for teachers from both countries. We decided that the most valuable information from Part C responses was whether teachers committed to a reference point. Figure **26** shows three sample teacher responses to Part C scored at different levels.

Part C. Functions *f* and *g* both give a distance between Willie and Robin after *x* hours. But *f*(1)=6.67 and *g*(1)=4.17. Why are *f*(1) and *g*(1) not the same number?		
a) f(1)=distance between 2 men when officer has left cafe & traveled 1hr g(1)= distance between 2 men since Robin has travel 1hr beyond cafe-Cop is catching up	b) f(1) represent the distance Robin has gone in 1 hour. g(1) represent the distance the Officer has gone in 1 hour.	c) Because Willie started $\frac{1}{6}$ hrs at the café, the distance between two people should be represented by g(x). f(1) is fixed at café where Willie is.

Figure 26: Teacher responses to '*Willie Chases Robin*'

The response in Figure **26**a was scored at the highest level because this teacher described *f*(1) and *g*(1) as both representing the distance between the two men, but at different points in time because of the different meanings of *x* in each function. The prompt in Part C asks teachers to resolve a seeming contradiction. To do so, a teacher had to think about the quantitative meaning of the value x in each function, and realize that different reference points for the inputs necessarily implied different meanings for the dependent values as well. Our model for how this teacher reasoned is that he conceptualized all four quantities x [in $f(x)$], x [in $g(x)$], $f(x)$ and $g(x)$ with commitments to reference points.

The response in Figure **26**b was scored at the middle level because this teacher described $f(1)$ and $g(1)$ as representing distances at different points in time, but not specifically distances between men. To reach this conclusion, she had to keep her commitment to the definitions of each x, but not make the same conclusions about the dependent values as the teacher in Figure **26**a. Our model for how this teacher reasoned is that she conceptualized x [in $f(x)$] and x [in $g(x)$] with commitments to reference points, but did not conceptualize $f(x)$ or $g(x)$ with a commitment to reference points.

The response in Figure **26**c was scored at the lowest level because it did not fit any higher levels, and we can see why when we look at how this teacher was not able to resolve the seeming contradiction posed to him. We do not have enough information to speculate about how he conceptualized the quantities represented by x in each function, but we can conclude that he did not conceptualize the quantities $f(x)$ or $g(x)$ with commitments to reference points.

The breakdown of Part C responses appears in Table **5** with 173 US and 366 Korean teachers.

Table 5. Responses to the '*Willie Chases Robin*' Part C MMTsm item.

	United States	South Korea	*Total*
Different RP	12 (6.9%)	56 (15.3%)	*68 (12.6%)*
Different [other]	24 (13.9%)	139 (38.0%)	*163 (30.2%)*
Other	137 (79.2%)	171 (46.7%)	*308 (57.2%)*
Total	*173 (100.0%)*	*366 (100.0%)*	*539 (100.0%)*

** Cells contain number of respondents and percent of column total.*

There is a statistically strong difference between US and Korean teachers' responses, (χ^2(*df*=2, N=539)=50.759, *p*<.0001). South Korean teachers were more than twice as likely to identify the different reference point of elapsed time for each function as US teachers (15.3% SK; 6.9% US) and almost three times as likely to be able to identify a difference that had something to do with a quantitative meaning of a variable even if the explanation was either incomplete or misidentified certain quantities (53.3% SK; 20.5% US).

Ivonne & Nicole: Item and Rubric

Figure **27** shows the *Ivonne & Nicole* task. Two quantities' values, timed by a common clock, have different starting times. The item's intent is to have teachers represent a value of each quantity within the reference frame of the other quantity.

Ivonne and Nicole jog together at the local track because they run at the same speed. Ivonne arrived early and started running before Nicole arrived and started running. Ivonne had run A laps when Nicole had run B laps. Later, Ivonne thinks, "When I have run C laps, Nicole will have run ________ laps." Later yet, Nicole thinks, "When I have run D laps, Ivonne will have run ________ laps." Fill in each blank with an appropriate expression.

Figure 27: '*Ivonne & Nicole*' MMTsm item

The *Ivonne & Nicole* item in Figure **27** mentions two quantities' values, timed by a common clock, that have different starting times. The item's intent is to have teachers represent a value of each quantity within the reference frame of the other. There are several ways in which teachers need to conceptualize and coordinate frames of reference to answer this item. Initially, the teacher must identify the extent to which Ivonne's distance exceeds Nicole's, represented by the quantity (A-B) and recognize that this difference stays constant throughout the timeline of the entire problem. To find this quantity the teacher needed to either think about measuring the distance between them from the reference point of either woman, or measure both women's distance from the same point (most likely the start of their runs) and then compare them. The teacher must then coordinate the consequences of that difference for each woman's frame of reference. Figure **28** summarizes our rubric for Part C.

Coordinated RPs & directionality:	The teacher wrote C-(A-B) and D+(A-B) respectively, or equivalents.
Coordinated directionality only:	The response is of the form C-__ and D+__ where both blanks contain the same expression, but that expression is not A-B.
Inappropriate use	The teacher wrote a proportional relationship, such as BC/A

of proportionality:	and AD/B.
Other:	The response doesn't fit a higher level, cannot be interpreted, has no clear answer, is off-topic, answered "I don't know" or was left blank.

Figure 28: '*Ivonne & Nicole*' MMTsm rubric

Ivonne & Nicole: Exploring the Responses

The *Ivonne & Nicole* rubric looks at both the teacher's commitment to reference point and directionality of comparison. Figure **29** gives three sample teacher responses to *Ivonne & Nicole* scored at different levels.

Ivonne and Nicole jog together at the local track because they run at the same speed. Ivonne arrived early and started running before Nicole arrived and started running. Ivonne had run A laps when Nicole had run B laps. Later, Ivonne thinks, "When I have run C laps, Nicole will have run ________ laps." Later yet, Nicole thinks, "When I have run D laps, Nicole will have run ________ laps." Fill in each blank with an appropriate expression.

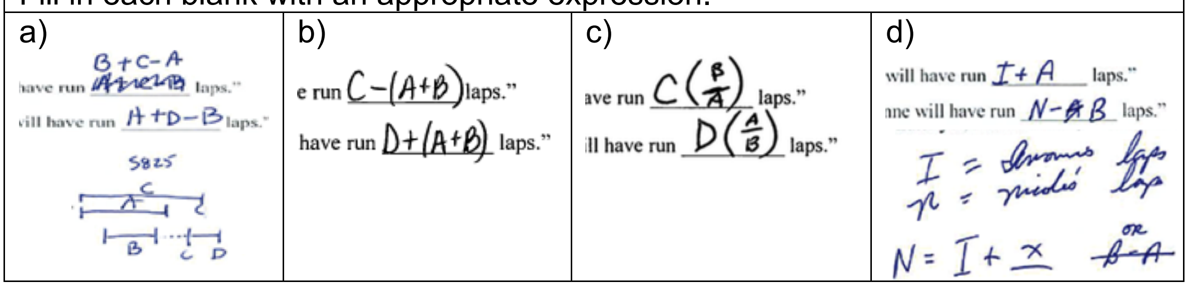


Figure 29: Teacher responses for '*Ivonne & Nicole*' MMTsm.

The response in Figure **29**a was scored at Level 3 because the response is equivalent to *C* – (*A*–*B*) and *D* + (*A*–*B*) when manipulated algebraically. We can see from the drawing that the teacher was thinking of two different reference frames and attempting to coordinate measures in both. The response in Figure

29b, while not correct, still gives us some insight into the teacher's meaning for coordinating measures from different perspectives. Though this response incorrectly represents the difference between Ivonne's and Nicole's lap counts (as A+B instead of A-B), it does show that the teacher recognizes that there must be a reciprocal additive relationship between Ivonne and Nicole's laps – that is, that there is some measure such that Ivonne's laps is that much more than Nicole's, and Nicole's laps is that same measure less than Ivonne's. This teacher correctly coordinated the directionality of comparison between Nicole and Ivonne's counts, but did not correctly coordinate the reference point between them. For that reason the response in Figure **29**b was placed at Level 2.

Because our rubrics were designed to capture ways of thinking about items, we frequently created categories to capture a kind of response that we saw frequently in the preliminary data. This happened with the '*Ivonne & Nicole*' rubric, where we saw several responses that replied to the item as if the different women's lap counts had a proportional relationship (we hypothesize that the presence of 4 quantities, 2 known and 2 unknown, triggered some teacher's scheme for proportional tasks). For that reason we categorized responses like the one in Figure **29**c in Level 1, to be able to keep track of how many teachers responded with proportional reasoning. The response in Figure **29**d was placed at Level 0 because it did not fit any higher levels.

Table **6** shows a breakdown of responses to this task, with 173 US and 366 Korean teachers.

Table 6. Responses to the 'Nicole Chases Ivonne' MMTsm item.

	United States	South Korea	*Total*
Coordinated RPs & directionality	57 (33.0%)	135 (36.9%)	*192 (35.6%)*
Coordinated directionality only	19 (11.0%)	18 (4.9%)	*37 (6.9%)*
Inappropriate use of proportionality	3 (1.7%)	122 (33.3%)	*125 (23.2%)*
Other	94 (54.3%)	91 (24.9%)	*185 (34.3%)*
Total	*173 (100.0%)*	*366 (100.0%)*	*539 (100.0%)*

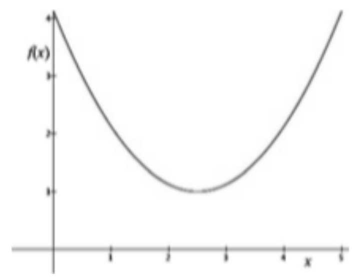
** Cells contain number of respondents and percent of column total.*

There is a statistically strong difference between US and Korean teachers' responses, (χ^2(*df*=3, N=539)=87.113, *p*<.0001). Approximately one-third of both groups (33.0% US; 36.9% SK) coordinated both the magnitude of the difference in reference point, and the directionality of comparison between the two women's laps. The largest difference between the two groups is that a third of Korean teachers (33.3%) inappropriately applied proportional reasoning to the relationship between Ivonne and Nicole's number of laps, which have a constant additive relationship and not a proportional one, while only 1.7% of US teachers did. This result is consistent with the original intent of *Ivonne & Nicole*, which was that it was a proportionality item in the MMTsm to be used in concert with items for which proportional reasoning was appropriate, to assess teachers' meanings for proportionality. Inappropriate use of proportionality or overgeneralization of linearity, has been well studied (Modestou & Gagatsis 2007; Greer 1997) and we hypothesize that the structure of an item with 4 quantities of interest led many teachers to assume that it was a standard missing-value proportion problem.

Subsequent Changes: Item & Scoring

Figure 30 shows the *Subsequent Changes* task. To determine whether the changes are increasing or decreasing in value, teachers must first consider changes in the function's value, and second, they must consider at least two Subsequent changes.

The graph below is of a function *f* over the interval [0,5].



For small equal increases of the value of *x* starting at *x* = 1 and ending at *x* = 2, the corresponding changes in the value of f are…

a) positive and increasing
b) positive and decreasing
c) negative and increasing
d) negative and decreasing
e) I don't know

(next page)

Part B. Is this sequence increasing or decreasing? -10, -9.5, 9, -8.4, …

____

Part C. Would you like to change your answer to the question on the prior page? Circle the appropriate selection.
a) positive and increasing
b) positive and decreasing
c) negative and increasing
d) negative and decreasing
e) I don't know
f) I do not want to change my answer.

Figure 30. The 'C03: *Subsequent Changes*' MMTsm Item.

If their reasoning is based solely on the function's values, they will likely select "positive and decreasing". If their reasoning is based on one change in *x*, then they are likely to choose "negative and decreasing. On the other hand, teachers could consider successive changes and compare them in magnitude, not in value, and also select negative and decreasing. We suspect these teachers will change their response after looking at Part B and seeing that the values of the changes are increasing. If a teacher says that the changes are "negative and decreasing," answers that the sequence in Part B is increasing, and leaves their response in Part A unchanged, then we believe that this teacher did not covary changes in the value of the function with changes in *x* in Part A (rather, they may have only considered one change in *x* and corresponding change in the function value).

Part B was designed to perturb teachers thinking after they responded to part A. We deliberate placed Parts B and C on the next page and left the header "Part A" off of the first part in the hopes that teachers would answer the first part before turning and looking at the rest of the item. In our sample almost all teachers had the same correct response for part B. Therefore the levels focus on the teachers' responses to Part A and C (the unlabeled follow-up question in Part B) with minor mention of part B" (MMTsm).

Figure **31** summarizes our rubric for the *Subsequent Changes* item, which depends on the teacher's responses to both Part A and Part C.

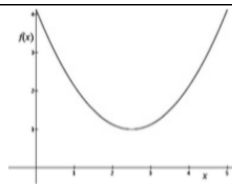
Attended to each change's RP & committed to directionality, in both parts:	Part A: (c) negative & increasing Part C: (c) or (f) kept same answer
Attended to each change's RP, originally switched directionality but committed to directionality after Part B:	Part A: (d) negative & decreasing Part C: (c) negative & increasing
Attended to inappropriate RP, after Part B attended to RP of change & committed to directionality.	Part A: (b) positive & decreasing Part C: (c) negative & increasing
Attended to RP of changes but switched directionality, in both parts:	Part A: (d) negative & decreasing Part C: (d) or (f) kept same answer
Attended to inappropriate RP originally, after Part B attended to RP of changes & switched directionality.	Part A: (b) positive & decreasing Part C: (d) negative & decreasing
All else.	The response doesn't fit a higher level, cannot be interpreted, has no clear answer, is off-topic, answered "I don't know" or was left blank.

Figure 31: *Subsequent Changes* MMTsm Rubric

Subsequent Changes: Exploring the Responses

There was substantial disagreement within the group of Korean consultants about the whether the Korean version of *Subsequent Changes* was equivalent to the English version. We therefore do not analyze Korean teachers' responses to *Subsequent Changes.*

This task was particularly difficult for U.S. teachers. Each answer asks teachers to commit to both a reference point and a directionality of comparison for the sequence of changes they are asked to consider. The first aspect of the multiple choice options, "[the changes are] positive" or "[the changes are] negative”, asks teachers to attend to the reference points of each change, which is different than the reference point of the functions’ total value. The second aspect of the options, “[the changes are] increasing” or “[the changes are] decreasing” necessitates comparing a change to a successive change while *maintaining the same commitment to directionality* as was used when deciding whether the changes were positive or negative. Figure **32** shows several answer possibilities for Part A.

<table>
<tr><td colspan="3">
Part A. For small equal increases of the value of x starting at x = 1 and ending at x = 2, the corresponding changes in the value of f are…

Part B. Is this sequence increasing or decreasing? -10, -9.5, 9, -8.4, …

Part C. Do you want to change your answer to Part A?</td></tr>
<tr><td>(a)

c) negative & increasing</td><td>(b)

b) negative & decreasing</td><td>(c)

b) positive & decreasing</td></tr>
</table>

Figure 32: Options to ‘Subsquent Changes’ Part A MMTsm item

The response in Figure **32**a is consistent with someone who committed to both appropriate reference points and directionality of comparison when answering Part A. In order to say that the Subsequent changes between *x*=1 and *x*=2 are negative, an individual would partition that part of the domain and then attend to measures of the changes in the function’s value. Figure 17 shows one possible example where a person might partition the interval [1,2] into three small, equal changes in the value of *x*, and identify the three corresponding changes in the value of *f* as negative (by taking their measure with respect to changing reference points, each one the value of *f* at the beginning of the interval). In order to answer that these changes are negative, this person has committed to the default directionality of comparison we use for real numbers, such that for example 10 > 2 > 0 > -2 > -10. In order to identify these changes as increasing, this person must *remain committed* to this directionality of comparison when comparing the changes to each other and acknowledge that changes which are negative and decreasing in magnitude are in fact changes whose measures are increasing (becoming less negative). This person would then easily identify the sequence in Part B as increasing, and remain satisfied with their answer in Part C.

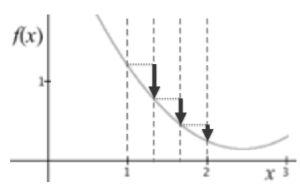
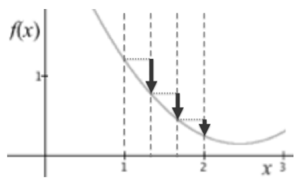


Figure 33: committing to a directionality

The response in Figure **32**b is consistent with someone who committed to appropriate reference points but not a directionality of comparison. This person might partition the interval [1,2] into a few equal small equal changes in x and identify the changes in the value of f as negative as in Figure **34**a. However, when comparing these changes, the directionality which was used to answer the positive/negative question is lost. This person might attend only to the magnitude of the changes as in Figure **34**b and conclude that the changes are decreasing.

However, the two answers are inconsistent. If the changes are negative then they are also increasing.

a)



b)

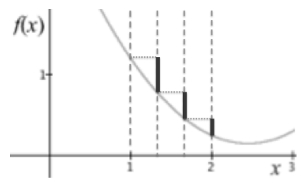


Figure 34: not committing to a directionality

The response in Figure **32**c is consistent with a person who looked at subsequent values of the function's value (as in Figure **35**a) and concluded that they were positive and decreasing.

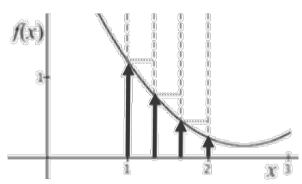


Figure 35: Function values to magnitude of changes

The responses in Figure **32**d are consistent with a person who interpreted the question as asking about subsequent values of the function instead of subsequent changes in the function's values, seeing Part B as irrelevant to the question at hand since the function's values are clearly in the first quadrant, and keeping the same answer for Part C.

Table **7** shows a breakdown of responses to Part A of this task; 253 US teachers responded to the *Subsequent Changes* task. "RP" stands for reference point and "D" stands for direction of comparison.

Table 7. Responses to *Subsequent Changes Part A*.

	United States
(c) Committed to RP and D	38 (*15.0%)*
(d) Committed only to RP	76 (*30.0%)*
(b) Looked at totals, not changes	129 (*51.0%)*
All Else	10 (*4.0%)*
Total	*253 (100.0%)*

** Cells contain number of respondents and percent of column total.*

15% of teachers made appropriate commitments to both the reference point and direction of comparison for changes in the function's values. A total of 45% of teachers (answering c or d) were able to commit to reference point for the changes. The majority of teachers did not commit to either.

Part A of the *Subsequent Changes* was on a separate page but not labeled "Part A" because we hoped to get the teacher's initial response before they flipped the page and looked at the next part. When they did turn the page, they found Part B which was designed to perturb teachers who struggled to maintain their commitment to direction of comparison. 235 of the 253 teachers (92.3%) identified the sequence "-10, -9.5, 9, -8.4, …" as an increasing sequence. We wished to see whether teachers would connect their knowledge about the decontextualized number sequence in Part A with the context of Subsequent changes in a function's value, and aske them if they wished to change their answers in Part C. Table **8** shows the relationship between each teacher's response to Part A and their answer to the same question in Part C.

Table 8. Part A vs. Part C Responses to *Subsequent Changes*.

Part A↓ Part C →	(c) Committed to RP & D	(d) Committed to RP	(b) Looked at totals	Other
(c) Committed to RP & D	38 (15.0%)^{6}	0 (0.0%)^{1}	0 (0.0%)^{1}	0 (0.0%)^{1}
(d) Committed to RP	24 (9.5%)^{5}	43 (17.0%)^{3}	4 (1.6%)^{1}	5 (2.0%)^{1}
(b) Looked at totals	12 (4.7%)^{4}	3 (1.2%)^{2}	103 (40.7%)^{1}	11 (4.3%)^{1}
Other	2 (0.8%)^{1}	0 (0.0%)^{1}	0 (0.0%)^{1}	8 (3.2%)^{1}

** Cells contain number of respondents and percent of total of all teachers.*
** Superscripts give rubric level that these responses are scored at.*

All 38 teachers who originally made two appropriate commitments maintained those commitments in their Part C response, and an additional 38 teachers (15% more of the total) were able to make those commitments when given Part B as a hint. However, most teachers continued to struggled with this item; 51.7% of all US teachers were scored at our lowest level where they did not demonstrate a commitment to either appropriate reference points or a directionality of comparison in either Part A or C.

Looking Across Items

Here we look across items to see if teachers used their meanings for frame of reference consistently. Since our rubric scores are ordinal, we ran the Jonckheere-Terpstra (JT) test for ordered alternatives. **Error! Reference source**

not found. displays the results for each pairwise test, treating the row item as the independent variable and the column item as the dependent variable. In each JT test our alternative hypothesis for each pairwise JT test was that the column item's scores increase as the row item's scores increase. We combined results from both countries for all tests.

Table 9: Pairwise tests for trend (Jonckheere-Terpstra)

	Willie Chases Robin Part B	*Willie Chases Robin* Part C	*Ivonne & Nicole*	*Subsequent Changes*
Willie Chases Robin Part A	JT = 71587.5, Z = 11.456, *p*<.0005	JT = 56349, Z = 10.105, *p*<.0005	JT = 59737, Z = 6.935, *p*<.0005	JT = 25552.5, Z = -.395, *p*=.693
Willie Chases Robin Part B		JT = 55645, Z = 9.213, *p*<.0005	JT = 57465, Z = 5.357, *p*<.0005	JT = 24848, Z = -.878, *p*=.380
Willie Chases Robin Part C			JT = 57584.5, Z = 5.857, *p*<.0005	JT = 25546, Z = -.412, *p*=.681
Ivonne & Nicole				JT = 29150, Z = 1.381, *p*=.167

The results in Table **9** show that there is a statistically significant relationship among teacher responses for *Willie Chases Robin* Parts A, B, and C, and *Ivonne & Nicole*, but not a significant relationship between any of those four items and the *Subsequent Changes* item. The *Willie Chases Robin* Parts A, B, and C, and *Ivonne & Nicole* are all problems involving the relative motion of two moving objects whereas the *Subsequent Changes* task involves only abstract quantities (x, $f(x)$, and change in x. This suggests that the teachers' imagery that triggered them to apply their schemes for frame of reference are context sensitive instead of relying on imagery entailing relationships among quantities generally.

A more powerful explanation for the difference between *Subsequent Changes* and the items *Willie Chases Robin* and *Ivonne & Nicole* is that teachers could reason about the latter by comparing values of total quantities, creating what we might call a first-order difference. In contrast, the *Subsequent Changes* task asks teachers to not only identify at least two subsequent changes in the values of $f(x)$ but also to compare these changes, creating what we might call a second-order difference—a difference of differences. In order to do compare differences, a person must think of each difference as not only a comparison of two other quantities but also a quantity in itself. To answer the *Subsequent Changes* task a person must conceptualize a new quantity within a newly conceptualized frame of reference where each difference is assumed to have the same reference point that is different from either differences reference point, while maintaining a commitment to the original direction of comparison. All this creates additional layers of complexity. Again, this gives us hypotheses for the imagery that the teachers associated with the process of both making meaning of measures (by conceptualizing a frame of reference) and coordinating measures taken within different frames.

Our five items and their associated rubrics all address ways of thinking about quantities within a frame of reference. Teachers who had strong meanings for how to conceptualize quantities within a frame of reference would have imagery triggering their schemes that were more closely related to the nature of relationships between quantities than the surface features of the tasks, and their schemes would have led them to perform similarly on all three tasks. Our results suggest that our teachers' meanings for frame of reference did not, as a group, entail imagery and operations that allowed them to apply their meaning for difference of quantities to difference of differences.

Discussion

Our research questions were:

1) How do the teachers in our samples reason with frames of reference in answering these five items?
2) Are there differences between the teachers in the United States and South Korean sample in how they reason with frames of reference?

Our answer to the first question is that a minority of teachers conceptualized the quantities involved in the tasks within a frame of reference and coordinated quantities measured in multiple frames without losing track of their reference point, direction of comparison, or both. Only in the task *Willie Chases Robin* Part A did a majority of teachers (51.7%) coordinate the directionality between the two quantities involved.

Our answer to the second question is that there are significant differences between the US and Korean teachers in our sample in how they reason with frames of reference. In general Korean teachers scored higher on our rubrics.

We say this even though a substantial minority of Korean teachers understood *Ivonne & Nicole* as being about a proportional relationship. We interpret their behavior as that their proportionality scheme was triggered by the text's surface features being similar to missing-value proportion problems with which they were intimately familiar. However, we cannot take these Korean teachers' behavior as evidence that they could not coordinate frames of reference.

The differences between the two country's samples suggest that teachers' difficulties are culturally conditioned by the education systems (and larger cultures) in which they developed their ways of thinking. Conceptualizing and coordinating frames of reference can be nurtured in educational reforms that emphasize attention to quantities instead of just objects, as well as exposure to a wider variety of tasks in school curriculum (and concurrent high-level discussions led by teachers) that will allow students to develop more generalized imagery to trigger their schemes for frame of reference. At the same time, the fact that so many teachers struggled with our tasks, from both countries, emphasizes that developing a stable meaning for frame of reference is a cognitively difficult task and any reforms targeting this way of thinking with students must be carried out carefully.

Limitations & Future Research

The first and most obvious limitation of our study is that our teachers were taken from convenience samples in both countries. Our conclusions can only be known to hold true for that sample. However, our Korean sample did encompass

95% of all South Koreans taking the first-class teacher exam in Summer 2015, and so is a more representative sample than our US teachers who were drawn from professional development summer opportunities.

Another limitation is that almost all of our tasks involve relative motion. One future direction is to develop more items with contexts other than relative motion within which to study teachers' reasoning with frame of reference, in order to see the nature and extent of their imagery for frame of reference reasoning. We have identified several more MMTsm items that, like the *Subsequent Changes* task, assess frame of reference as a secondary construct. Our next step to construct new rubrics that focus on what responses tell us about teacher's reasoning with frames of reference, and rescore the data.

We also came to see during our study that there were ways in which we could have improved our rubrics for the five items presented here. For example, we could have developed rubrics for different dimensions such as "committing to reference point", "committing to direction of comparison", or "coordinating quantities" and scored each response to a task along multiple dimensions. In the *Willie Chases Robin* task, we could create an extra dimension for "commitment to unit" to categorize those responses which use a "1/6" versus "10 minutes" in their response to Part A. Doing so would allow us to compare teachers' responses across items in more meaningful ways, and would give us more information about a teacher's thinking as a result. Such multi-dimensional rubrics would be especially useful for professional developers using our instrument.

Conclusion

Historically, the idea of frame of reference has been one associated with physics and problems typically seen in physics classes, and has been taught as if a frame of reference is an object external to a person. Our theory and items presented in this study show that the mental commitments and coordinations underlying a person's ability to reason with frame of reference are far more broadly applicable. The ability to conceptualize a frame of reference and to coordinate multiple frames of reference affects a person working on tasks spanning all contexts, both concrete and abstract.

Teachers in our sample struggled to reason with frame of reference and also did not show consistent reasoning across different contexts. The MMTsm assessment was designed to characterize teacher's mathematical meanings *for teaching*. Our tasks and rubrics were written to answer the question "If a teacher gave this answer in a classroom, how productive would that be for students?" Our results show that the teachers in our samples do not possess strong stable schemes for reasoning with frame of reference.

While our country comparisons show US teachers struggling more than Korean teachers, both samples struggled to maintain mental commitments and to coordinate measures of quantities taken in different frames. Moreover, the differences between the two samples suggest that teachers' difficulties with frame of reference are at least partly culturally conditioned, and that therefore their frame of reference reasoning can be improved with sustained educational effort.

We conclude with some hypotheses about the nature of educational reforms that might improve teachers' ability to reason with frame of reference and to teach their students to do the same. Just as our conceptual definitions of frame of reference do not treat a reference frame like an external object, so too must any professional development that seeks to help teachers build stable schemes for reasoning with frame of reference. Instruction about frames of reference is not what is necessary. Rather, teachers (and students) need multiple opportunities to work with quantities and the measures of quantities in complex tasks so that they can build experiences of giving meaning to measures (by means of making commitments to reference point and direction) and to coordinating measures taken in different frames. Repeated opportunities to reason in these ways, along with careful guidance in discussions and in choices made during tasks, can help teachers develop stable meanings for reasoning with frame of reference.

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