

Investigating the Advancement of Middle School Mathematics Teachers' Meanings for  
Partitive Division by Fractional Values of Quantities

by

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## ABSTRACT

Researchers have described two fundamental conceptualizations for division, known as partitive and quotitive division. Partitive division is the conceptualization of  $a \div b$  as the amount of something per copy such that  $b$  copies of this amount yield the amount  $a$ . Quotitive division is the conceptualization of  $a \div b$  as the number of copies of the amount  $b$  that yield the amount  $a$ . Researchers have identified many cognitive obstacles that have inhibited the development of robust meanings for division involving non-whole values, while other researchers have commented on the challenges related to such development. Regarding division with fractions, much research has been devoted to quotitive conceptualizations of division, or on symbolic manipulation of variables. Research and curricular activities have largely avoided the study and development of partitive conceptualizations involving fractions, as well as their connection to the invert-and-multiply algorithm. In this dissertation study, I investigated six middle school mathematics teachers' meanings related to partitive conceptualizations of division over the positive rational numbers. I also investigated the impact of an intervention that I designed with the intent of advancing one of these teachers' meanings. My findings suggested that the primary cognitive obstacles were difficulties with maintaining multiple levels of units, weak quantitative meanings for fractional multipliers, and an unawareness of (and confusion due to) the two quantitative conceptualizations of division. As a product of this study, I developed a framework for characterizing robust meanings for division, indicated directions for future research, and shared implications for curriculum and instruction.

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“ratios behave differently than fractions.” Pat – who had already addressed several fundamental problems in my work – responded by writing, “Now these things have behaviors? Matt, you need to do a lot of epistemological work so that your characterizations are coherent.” I will never forget this comment, which marked a turning point for me. Through the years Pat challenged me and molded me into a much more careful thinker and writer as I prepared for this dissertation. Additionally, I am indebted to Pat for introducing me to Piaget and constructivist theory, which has permeated nearly every aspect of my life.

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## CHAPTER 1

### INTRODUCTION AND STATEMENT OF THE PROBLEM

Researchers have identified two fundamental conceptualizations for division, known as *partitive* and *quotitive* division (Beckmann, 2011; Fischbein, Deri, Nello, & Marino, 1985; Gregg & Underwood Gregg, 2007; Harel, Behr, Lesh, & Post, 1994; Jansen & Hohensee, 2016; Kribs-Zaleta, 2008; Lo & Luo, 2012; Simon, 1993; Thompson & Saldanha, 2003). These conceptualizations are based on a quantitative, non-commutative model for multiplication whereby  $a \times b = c$  is taken to mean *a groups (or copies) of the amount b yield the amount c*. Partitive division (also called *sharing* division) is the operation that determines *b*, given that *a* and *c* are known. It is the conceptualization of  $c \div a$  as the amount of something per copy such that *a* copies of this amount yield the amount *c*. Quotitive division (also called *measuring* division) is the operation that determines *a*, given that *b* and *c* are known. It is the conceptualization of  $c \div b$  as the number of copies of the amount *b* that yield the amount *c*, or the measurement of *c* in terms of *b*.

Researchers have identified primitive intuitive rules regarding partitive and quotitive division that are rooted in operating with whole numbers, such as division makes smaller or that partitive division must involve a whole number of groups (Fischbein et al., 1985; Greer, 1994; Harel et al., 1994; Lamon, 2011; Rizvi & Lawson, 2007; Simon, 1993; Thompson & Saldanha, 2003; Tirosh, 2000). Some of these researchers have suggested that these primitive rules have contributed to students' and teachers' weak meanings for division involving non-whole values (Jansen & Hohensee, 2016; Rizvi & Lawson, 2007; Sharp & Adams, 2002; Tirosh, 2000), while others have

commented on the need for and the challenges related to helping students extend their meanings for division to accommodate non-whole numbers (Greer, 1994; Lamon, 2011).

Researchers have also studied the gaps in teachers' and students' thinking between conceptualizations of division and algorithms for numerical division (Borko et al., 1992; Kribs-Zaleta, 2008; Perlwitz, 2004, 2005; Rizvi & Lawson, 2007; Sharp & Adams, 2002; Simon, 1993; Tirosh, 2000). One such gap is the pervasive inability among teachers to explain the invert-and-multiply algorithm despite demonstrating some level of conceptual understanding of division (Rizvi & Lawson, 2007; Sharp & Adams, 2002; Tirosh, 2000). Another gap is teachers' and students' inability to resolve two apparently different answers, one obtained from numerical procedures for division and the other obtained through quantitatively modeling the operation (Borko et al., 1992; Kribs-Zaleta, 2008; Perlwitz, 2005; Simon, 1993). As an explanation for such gaps, The National Council of Teachers of Mathematics (NCTM) has suggested that in too many classrooms, teachers overemphasize procedural learning while neglecting conceptual development. To mitigate this issue, The NCTM has stated that "effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems" (National Council of Teachers of Mathematics, 2014, p.42).

In most of the studies mentioned above, the focus of the research has been on quotitive conceptualizations of division, or on symbolic manipulation of variables, as noted by Ott, Snook, and Gibson (1991) in the following remark:

Although the measurement (quotitive) meaning of division of fractions has received considerable attention, a review of the literature indicates that the partitive meaning for division of fractions has been almost totally ignored. The

partitive meaning of division of fractions has been very resistant to clear, concrete explanations. In fact, one writer . . . went so far as to say that partitive division of fractions does not make sense (Ott et al., 1991).

Despite the general tendency to focus on quotitive meanings, some researchers (Beckmann, 2011; Gregg & Underwood Gregg, 2007; Jansen & Hohensee, 2016; Kribs-Zaleta, 2008; Ott et al., 1991) have shared ways to conceptualize partitive division with rational numbers. Some of these researchers (Beckmann, 2011; Gregg & Underwood Gregg, 2007) have even shared justifications for the invert-and-multiply algorithm which are based on partitive conceptualizations. Very few researchers (Jansen & Hohensee, 2016) have focused more closely on partitive meanings with non-whole divisors, and I have found almost no insights into the development of these meanings.

My purpose for this dissertation study was to add to the field's understanding of partitive conceptualizations of division over the positive rational numbers, by focusing on the *advancement* of teachers' partitive meanings to accommodate fractional values. I am interested in this research for the following three main reasons: (1) partitive division over the positive rational numbers is detrimentally underemphasized in contemporary curriculum and standards for teaching mathematics, (2) partitive meanings form a conceptual foundation for other mathematical meanings, such as rates and proportional correspondence, and for the invert-and-multiply algorithm, and (3) limited research is available on this topic. For this study, I investigated and described the impact of an intervention that I designed with the intent of promoting the development of in-service middle school mathematics teachers' partitive meanings. I focused on the following two primary research questions, with accompanying secondary research questions.

**RQ1 (Primary Research Question 1):** What meanings, with their affordances and limitations, do in-service middle school mathematics teachers possess relative to partitive conceptualizations of division with non-whole divisors?

**RQ1.1:** What meanings do teachers reveal when they engage in tasks that I designed to elicit meanings for fractions as measures of relative size, with a focus on fractions as reciprocal measures of relative size?

**RQ1.2:** What meanings do teachers reveal when they describe and model symbolic (decontextualized) statements of multiplication, both general and specific, with a focus on fractional multipliers?

**RQ1.3:** What meanings do teachers reveal when they describe and model symbolic (decontextualized) statements of division, both general and specific, with a focus on fractional divisors?

**RQ1.4:** What meanings do teachers reveal when they engage in tasks that I designed to elicit partitive conceptualizations of division, with varying degrees of abstraction, and with a focus on fractional divisors?

**RQ1.5:** What justifications do teachers provide for the invert-and-multiply algorithm after working through the tasks mentioned in the previous research question?

**RQ1.6:** What cognitive obstacles do teachers further reveal as I actively attempt to promote the development of their meanings that are foundational to partitive division over the rational numbers?

**RQ2 (Primary Research Question 2):** How do these teachers' meanings change as a consequence of an instructional sequence that emphasized quantitative reasoning to aid in the advancement of these meanings?

**RQ2.1:** How do the teachers' post-intervention meanings compare to their pre-intervention meanings?

**RQ2.2:** What advancements to the teachers' schemes are evident and what challenges remain?

To answer these questions, I designed a qualitative study that allowed me to describe models of the teachers' meanings that explained their actions. The study included clinical interviews (Clement, 2000; Hunting, 1997) and a teaching experiment (Steffe & Thompson, 2000). To analyze the data, I used open and axial coding to generate and test grounded theory, as described by Corbin and Strauss (2008). As a result of this study, I was able to identify cognitive obstacles to the advancement of meanings for partitive division. I also described a framework for characterizing robust division in general. The findings from this study will address the research gap relative to understanding teachers' partitive meanings and will ultimately inform the development of curricular materials and future research efforts.

## CHAPTER 2

### LITERATURE REVIEW

As I briefly mentioned in the introduction, there are two fundamental conceptualizations for division that researchers have discussed, known as *partitive* and *quotitive* division (Beckmann, 2011; Fischbein et al., 1985; Gregg & Underwood Gregg, 2007; Harel et al., 1994; Kribs-Zaleta, 2008; Simon, 1993; Thompson & Saldanha, 2003). These conceptualizations are based on a quantitative, non-commutative model for multiplication whereby  $a \times b = c$  is taken to mean *a copies of the amount b per copy yield the amount c*. As discussed by Thompson and Saldanha (2003), I use the word *copies* instead of *groups* for a few reasons. The word *group* does not entail consistency in size, e.g., splitting students into groups doesn't necessarily mean the groups will have equal sizes. As such, thinking of a partial group is cognitively perturbing, because for some people, there exist only smaller groups and larger groups, but no partial groups. On the contrary, the idea of *copying* parallels meanings for iteration by implying consistency in the size of each copy. It is also easier to think of partial copies in general, although the quantity that is being copied can still affect whether partial copies are sensible. To facilitate discussions about multiplication, I use the following terms, placed in their respective positions of a statement of multiplication:

$$\text{multiplier} \times \text{multiplicand} = \text{product}.$$

I use the convention that the multiplier should precede the multiplicand in a multiplication statement, thus rendering the model conceptually non-commutative; 2 copies of 3 apples is not situationally equivalent to 3 copies of 2 apples, despite that there

are six apples in total in each situation. As for statements of division, I use the following terms, placed in their respective positions:

$$\text{dividend} \div \text{divisor} = \text{quotient}.$$

*Partitive* division, also known as *sharing* division, is the conceptualization of  $c \div a$  as the amount of something per copy such that  $a$  copies of this amount yield the amount  $c$ .

A second meaning for division is *quotitive*, or *measuring*, division, which is the conceptualization of  $c \div b$  as the number of copies of the amount  $b$  that yield the amount  $c$ . Or, characterized in another way,  $c \div b$  is the measurement of the amount  $c$  in units of size  $b$ .

A third meaning for division is that of determining *relative size* of one quantity as compared multiplicatively to the size of another quantity (Byerley, Hatfield, & Thompson, 2012; Thompson, Carlson, Byerley, & Hatfield, 2014). By *quantity* I mean the conceptualization of a measurable attribute of an object or phenomenon (Smith & Thompson, 2007)<sup>1</sup>. Since I consider quotient as relative size to be an abstraction of either partitive or quotitive division<sup>2</sup>, I will primarily focus my review of the literature on partitive and quotitive conceptualizations for division with fractions.

### **Primitive Models for Partitive and Quotitive Division**

Children are first exposed to meanings for division in the context of operating on whole numbers<sup>3</sup>. Several researchers have identified primitive models and subsequent

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<sup>1</sup> I say more about this in my theoretical framework chapter.

<sup>2</sup> I say more about this in my conceptual analysis chapter where I discuss ratios.

<sup>3</sup> According to the CCSSM, notions of partitioning begin in 1<sup>st</sup> grade and division with whole numbers is formally introduced in 3<sup>rd</sup> grade (National Governors Association, 2010).

intuitive rules for partitive and quotitive division that result from operating with whole numbers (Fischbein et al., 1985; Greer, 1994; Harel et al., 1994; Rizvi & Lawson, 2007; Simon, 1993; Thompson & Saldanha, 2003; Tirosh, 2000). Fischbein, Deri, Nello, and Marino (1985) noted that division with whole numbers leads children to develop misleading intuitive rules regarding division which are not generally true for division with rational numbers. Concerning partitive division, these intuitive rules include (1) the dividend must be larger than the divisor, (2) the divisor must be whole, and (3) the quotient must be no bigger than the dividend – i.e. “division makes smaller.” The third intuitive rule is a consequence of the second, and primitive models for partitive division that require at least the second and third rule are referred to as *fair-sharing* models (Gregg & Underwood Gregg, 2007; Rizvi & Lawson, 2007). Concerning quotitive division, the primitive model of *repeated subtraction* requires the divisor to be smaller than the dividend, in anticipation of a whole number quotient (Fischbein et al., 1985).

These primitive models, and their accompanying intuitive rules, for partitive and quotitive division are rooted in reasoning with whole numbers and research suggests that they exert an influence on the reasoning of students and teachers. For example, Harel, Behr, Lesh, and Post (1994) conducted a quantitative study involving more than 450 in-service and pre-service elementary teachers in which the researchers controlled for some of the factors – including primitive intuitions – that had been known to influence the success of students on division problems involving non-whole values. They observed a significant drop in the ability of the subjects to produce a correct answer when primitive intuitive rules were violated. In a later study, Rizvi and Lawson (2007) noticed that none of the 17 preservice teachers from their study could initially pose a word problem that

required division by a fractional value. Also, they observed that when they asked subjects to produce a word problem that required division by a whole number, the subjects produced problems involving only partitive division. Rizvi and Lawson attributed these limitations to the influence of the primitive fair-sharing and repeated subtraction models for division. They said the following about fair-sharing.

The fair-sharing, or partitive model is a traditional teaching model for division of whole numbers, but it can act as a barrier in the representation of division of fractions. For example,  $48 \div 1/4$  cannot be represented by the same model of fair-sharing because it is senseless to share 48 lollies among a quarter of a girl (Rizvi & Lawson, 2007, p.378).

In this quote, I do not believe that Rizvi and Lawson are suggesting that  $48 \div 1/4$  cannot be modeled by partitive division; but instead, I believe they are suggesting that thinking of partitive division as fair-sharing is limiting in its scope. In their study, Rizvi and Lawson observed that some of their subjects could successfully use the model of repeated subtraction (the primitive model for quotitive division) to justify why division by  $1/3$  is numerically equivalent to multiplying by 3. But they suggested that the requirement that the dividend must be larger than the divisor was an obstacle to conceptual development. They said the following about repeated subtraction.

This model (repeated subtraction) helps the learners to represent some division of fraction problems, but it also appears to be difficult for students to use this model to represent division situations when the divisor is bigger than the dividend. For example, in representation of  $1/3 \div 1/2$  it is confusing to ask how many times one half can be subtracted from one third, perplexing for the students of elementary classes who are not used to subtracting a bigger number from a smaller number. (Rizvi & Lawson, 2007, p.378).

Another consequence of primitive notions as described by researchers (Jansen & Hohensee, 2016; Simon, 1993) is the phenomenon of interpreting division by a unit fraction as a multiplication problem instead. For example, these researchers have observed that expressions such as  $6 \div \frac{1}{2}$  have a relatively high risk of being interpreted as one-half of six, yielding an answer of three, instead of the correct answer of 12. This can be attributed to the primitive notion that division must make smaller.

### **Extending Beyond Primitive Conceptualizations**

Primitive models for division are not conducive to productively thinking about division with non-whole values. Researchers have acknowledged the need to extend primitive meanings for division beyond the scope of whole numbers only. Cajori said the following:

That, in the historical development, multiplication and division should have been considered primarily in connection with integers, is natural . . . First come the easy but restricted meanings of multiplication and division, applicable to whole numbers. In due time the successful teacher causes students to see the necessity of modifying and broadening the meanings assigned to the terms (Cajori, 1897, p.183).

Researchers have since identified various cognitive barriers to the development of more robust meanings for division. Greer (1994) confirmed that persistent primitive intuitions regarding division were likely one such obstacle. He also alluded to research showing that children had difficulties creating word problems that elicited given acts of division with values that violate primitive intuitive rules, and that children had difficulty identifying an appropriate operation given a word problem when the numbers were less convenient. Greer went on to say that a “vertical extension of the meanings of the

operations is beset with cognitive obstacles; elucidating these and finding means to help children overcome them more effectively is a major challenge for research in this field” (Greer, 1994, p.81). Since Greer’s publication, researchers have shed more light on identifying and overcoming some of the cognitive obstacles to which Greer is referring. For example, Lamon (2011) highlighted several reasons why learning about fractions (including division with fractions) could be challenging. These reasons are summarized below.

1. Fractions are written using an unfamiliar notation; two numbers separated by a bar. As such, fractions can be perceived as a collection of two values, instead of as a single magnitude.
2. The language used to discuss fractions is unfamiliar.
3. The fraction symbol  $\frac{a}{b}$  can represent a variety of meanings.
4. Fractions can represent values of *intensive* quantities, which are multiplicative comparisons between the values of two quantities, e.g., dollars per mile, and which can be abstract and difficult to conceptualize.
5. Operations involving fractions are dependent on developing new ways to operate, e.g., a multiplier of 3 involves only iterating, but a multiplier of  $\frac{4}{3}$  involves partitioning and iterating.
6. Operating with fractions requires coordinating multiple levels of units, e.g.,  $\frac{2}{3}$  of one unit (a whole) can be thought of as 2 of another unit ( $\frac{1}{3}$  of a whole).
7. There can be interference from whole number intuitions, e.g., division makes smaller.

In regards to items (1) and (3) from the list above, Tzur and Hunt (2015) suggested some teaching activities to help build the meaning that a unit fraction represents a single amount that, when iterated a certain number of times, produces a whole amount. The activities involve students starting with a whole strip of paper and then creating a second

strip that estimates a unit fraction of the length. Next, the student iterates the estimation to determine its accuracy, and then makes and analyzes adjustments to the estimation until an appropriate length of the unit fraction is obtained. Tzur and Hunt said the following:

Through activities of iterating units, then, the child begins to understand unit fractions not just or mainly as shaded or folded pieces of a whole (e.g., one of five parts) but as a multiplicative relationship between a unit and the whole into which it fits a given number of times. In our example, the child comes to think of  $1/5$  as a unique quantity that, when repeated five times, exactly reproduces or fits inside of a referent whole (Tzur & Hunt, 2015).

In this quote, Tzur and Hunt point out that the described activities reinforce the notion of a unit fraction as a representation of a *single* magnitude. Conceiving of a fraction as a single value is an important development to make sense of division of fractions.

### **Relationship of Conceptualizations of Division to Constant Rate of Change**

In this section, I comment on literature regarding meanings for constant rate of change and how they relate to conceptualizations for division. Thompson, Carlson, Byerley, and Hatfield (2014, p.6) suggest that upon traveling 62 miles in 2.7 hours, quotitive meanings are inappropriate when dividing 62 miles by 2.7 hours. Instead they suggest that *proportional reasoning* (I could substitute here *partitive schemes*) be used to conclude that 22.96 miles corresponds to 1 hour of traveling, establishing a relative size between the value of distance traveled and the value of time elapsed. Further reflection on the situation leads one to conclude that if traveling at a constant speed, the number of miles traveled will always be 22.96 times as much as the number of hours elapsed.

In a teaching experiment with a 10 year-old fifth-grader called JJ, Thompson (1994b) investigated the development of the notion of speed as a rate. Initially, JJ's image of speed was a fixed length of distance, and not a ratio between distance traveled and time elapsed. Time was not an extensive<sup>4</sup> quantity in her thinking, only distance was. For her, time was a ratio between the distance traveled and one speed-length. As such JJ was initially only able to operate with distance-lengths and speed-lengths, dividing them and interpreting the quotient as the time elapsed. When Thompson asked JJ to find the rate required to go a certain distance in a certain non-whole amount of time, she resorted to guess and check strategies, guessing the speed-length and iterating it an appropriate number of times to determine if her guess was correct. She operated this way because she was restricted to operating only with speed-lengths and distance-lengths. It was only later in the teaching experiment that JJ could abstract time into an extensive quantity with which she could operate. She was ultimately able to conceive of speed as a ratio of total distance and total time, which she imagined as a rate, thus establishing a proportional correspondence between variable amounts of both distance and time.

JJ's conflation of speed, as a constant ratio between distance and time, with speed as distance was similarly observed by Person, Berenson, and Greenspon (2004). Their subject was a high school teacher who conflated rate of change (which relates *two* covarying quantities) with an amount of change (which is concerned with a *single* quantity). Thompson (1994a) made a similar observation when one of his subjects struggled due to thinking of a rate as a change in a single quantity.

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<sup>4</sup> An extensive quantity is a quantity that can be measured directly (J. Kaput & West, 1994; Post, Behr, & Lesh, 1988; Thompson, 1990).

## Subjects' Difficulties Connecting Procedures to Conceptualizations of Division

In the last few decades, researchers have been studying students' and teachers' connections between conceptualizations of division and procedures for numerical division. In a study of 30 preservice elementary teachers, Tirosh (2000) presented the subjects with four division tasks ( $1/4 \div 4$ ,  $1/4 \div 3/5$ ,  $4 \div 1/4$ ,  $320 \div 1/3$ ) and asked each of them to (1) calculate each result, (2) list common mistakes a seventh grader might make, and (3) describe potential sources for each of these mistakes. She noted that all 30 subjects suggested that a misapplication of the invert-and-multiply algorithm was a potential source of error. Only four of the subjects cited primitive intuitions regarding partitive division as a potential source of error. She also observed that all 30 subjects resorted to the invert-and-multiply algorithm to calculate each result, instead of relying on meanings for division, such as  $4 \div 1/4$  is 16 because there are 16 quarters in 4. Tirosh noted that the subjects could use the invert-and-multiply algorithm but could not explain why it worked. Similarly, Rizvi and Lawson observed that none of the 17 preservice subjects in their study could "explain the thinking that lies behind this (the invert-and-multiply) algorithm" (Rizvi & Lawson, 2007, p.382). To demonstrate formal, generalized justifications for the algorithm, Tirosh (2000) shared the following explanations.

1. Since division is the inverse of multiplication, division of fractions can be interpreted as a missing multiplicand problem. Understanding how to procedurally multiply fractions and reduce the terms allows one to arrive at the result of the invert-and-multiply algorithm.

$$\frac{a}{b} \div \frac{c}{d} = x \rightarrow \frac{c}{d} \cdot x = \frac{a}{b} \rightarrow x = \frac{ad}{bc}$$

2. Interpret division of fractions as a complex fraction and then put the complex fraction into a form with a unit denominator by appropriately scaling the numerator and denominator of the complex fraction.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a/b}{c/d} = \frac{a/b}{c/d} \cdot \frac{d/c}{d/c} = \frac{ad/bc}{1} = \frac{ad}{bc}$$

3. Get a common denominator then divide numerators and divide denominators.

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad \div bc}{bd \div bd} = \frac{ad \div bc}{1} = \frac{ad}{bc}$$

Tirosh shared these arguments with her subjects so they could see examples of generalized justifications for the algorithm. However, these formal arguments are based largely on manipulation of contextually empty symbols, and not based on quotitive and partitive meanings. Also, it is unclear whether some of these justifications would be accessible to elementary students.

In a study of 22 fifth grade students, Sharp and Adams (2002) had their subjects solve realistic problems that elicited quotitive conceptualizations for division, e.g., how many bows could be made from 11 feet of ribbon if each bow requires  $1\frac{1}{2}$  feet of ribbon? The subjects solved many such tasks, all involving quotitive division. The researchers anticipated that some of the subjects would be able to generalize an algorithm for numerical division with fractions in the following way.

1. Interpret  $\frac{a}{b} \div \frac{c}{d}$  as “how many times can  $\frac{c}{d}$  be subtracted from  $\frac{a}{b}$ ?”
2. Change to “how many times can  $\frac{bc}{bd}$  be subtracted from  $\frac{ad}{bd}$ ?”
3. Re-unitize to change to “how many times can  $bc$  be subtracted from  $ad$ ?”
4. Interpret as  $ad \div bc$
5. Interpret as  $\frac{ad}{bc}$

This justification is a blend of notions of repeated subtraction, unitizing, and fractions as quotients. Sharp and Adams observed that all the subjects demonstrated conceptual understanding of division but that none of them invented an algorithm that resembled the invert-and-multiply algorithm. They said the following:

Our students did not appear to be aware of an operational inverse relationship between division and multiplication. Hence, no knowledge existed on which they could construct the invert-and-multiply algorithm. It seems that trying to force a connection between the invert-and-multiply algorithm and whole-number operational knowledge would have been confusing to our students (Sharp & Adams, 2002, p.346).

This quote indicates that Sharp and Adams believe that the invert-and-multiply algorithm is conceptually dependent on an understanding that multiplication and division are inverses of each other. However, they did not elaborate on what they meant by that comment. In this quote, they also acknowledge their students' difficulty in connecting the invert-and-multiply algorithm to primitive meanings for division based on whole numbers. In their article, Sharp and Adams also cited research that suggested that the common denominator method for resolving a *quotitive* division task was most useful for establishing meaning behind the procedures (Sharp & Adams, 2002, p.336).

In a study of a pre-service teacher progressing through her first two years of student teaching, Borko, Eisenhart, Brown, Underhill, Jones, and Agard (1992) observed that despite having a basic quotitive meaning for division, their subject did not possess the schemes necessary to represent division of fractions using a visual model such as strip diagrams or pie charts. Also, their subject could use the algorithm and give a quotitive interpretation of the result but could not justify the algorithm itself. During the subject's

period of student teaching, the subject participated in a teaching methods course which included a unit on division with fractions. The instructor of this course unfortunately believed that his students (pre-service teachers) should learn that "(a) there is no direct relationship between stories or concrete and semi-concrete representations of the measurement interpretation of division of fractions and the standard algorithm; (b) representations can be used to verify a solution obtained through use of the algorithm, but not to derive the algorithm; and (c) the derivation of the algorithm demonstrated by the instructor was not within the knowledge constraints of many young learners" (Borko et al., 1992, p.214). I contend that none of these three statements is correct.

Other researchers have also provided evidence of a gap between procedural fluency and conceptual understanding in relation to division (Kribs-Zaleta, 2008; Perlwitz, 2005; Simon, 1993). During a class discussion on dividing fractions, Perlwitz (2005) gave her college students the task of finding how many pillow cases could be made from 10 yards of fabric if each pillow case required  $\frac{3}{4}$  of a yard. When using the invert-and-multiply algorithm, the class agreed on the numerical answer of 13 and  $\frac{1}{3}$ . However, when they used a  $\frac{3}{4}$ -yard measuring stick, they arrived at 13 pillow cases with  $\frac{1}{4}$  yard left over, thus claiming the answer should be 13 and  $\frac{1}{4}$ . Although eventually successful, it was not trivial for these students to reconcile the two answers. Kribs-Zaleta (2008) and Simon (1993) observed similar issues when their subjects also could not interpret the remainder in units of the divisor. In his study of preservice elementary teachers, Simon noticed that the subjects could accurately execute procedures for long division of whole numbers, but that these procedures were not well connected to the subjects' meanings for division. He stated that "their lack of conceptual understanding

given their algorithmic competence seems to challenge the idea that procedural practice eventually leads to understanding” (Simon, 1993, p.249). In other words, Simon observed that procedural fluency did not imply an understanding of the underlying quantities. Referring to the preservice teachers involved in his study, Simon said “the handling of remainders and fractional quotients in school was a procedural matter that did not help these subjects sort out the complex referential issues inherent in division” (Simon, 1993, p.249).

### **Partitive Division with Fractions in the Literature**

Several researchers have shared ways to think about partitive division with fractions, although without sharing relevant student data. Ott et al. (1991) presented situations which elicit partitive division with fractional values, such as determining the number of egg cartons per set and the number of dollars per ounce. They demonstrated proportional reasoning to determine the quotients, but without emphasizing partitioning and iterating. They further suggested a context-based learning trajectory that begins with whole number divisors, followed by fractional divisors less than one, culminating with fractional divisors greater than one that are presented as mixed numbers. Also, Gregg and Underwood Gregg (2007) provided justifications for the invert-and-multiply algorithm that are based on partitive conceptualizations. Although it is helpful to discuss ways to reason through contexts that elicit partitive division with fractions, none of the researchers mentioned above shared data of students thinking through such tasks.

Some studies did reveal students’ work. Kribs-Zaleta (2008) provided contexts that elicit partitive division with fractional values, such as cutting ribbon and pouring lemonade, and shared the work of 6<sup>th</sup> grade students who had not yet received formal

instruction on division with fractions. In one task, a student was trying to determine how many oranges make a serving, given that one and one-half oranges constitutes three-fifths of a serving. This student recognized that one-half of an orange corresponded to one-fifth of a serving and was able to reconstitute a whole serving of oranges. This study revealed that students can have productive meanings prior to formal instruction.

In another study, Jansen and Hohensee (2016) researched 17 prospective elementary teachers (PSTs) to identify challenges that PST's face specifically when solving partitive division tasks with fractional divisors. The researchers characterized a *productive conception* of partitive division as one that is both flexible and connected. I summarize their definitions for *flexible* and *connected*, as well as the opposite constructs *rigid* and *disconnected*, in Table 1.

Table 1

*Jansen's and Hohensee's Constructs for Conceptions of Partitive Division*

|           |                                                                                                                                                                                                                                |                                                                                                                                                                                                          |
|-----------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Flexible  | A person is "aware it is appropriate to partition the dividend for whole number divisors, iterate the dividend for unit fraction divisors, and both partition and iterate the dividend for non-unit proper fraction divisors." |                                                                                                                                                                                                          |
| Connected | Condition 1:<br>Translating<br>Between<br>Representations                                                                                                                                                                      | A person can "translate from a symbolic number sentence to a story problem situation that accurately represents that number sentence (or from a story problem situation to a symbolic number sentence)." |
|           | Condition 2:<br>Unit Rate<br>Awareness                                                                                                                                                                                         | A person has "awareness that partitive division generates a unit rate across tasks with a range of divisors, such as whole numbers and proper fractions."                                                |

Jansen's and Hohensee's definition of *flexibility* makes it seem that only awareness is required, but in their paper, they implied that they measured flexibility partly by whether a person could obtain right answers. The researchers used these constructs to analyze the PSTs' responses to three tasks prior to starting an instructional

unit on division. Since this study so closely relates to my own research interests, I share the three tasks they used, which I summarize in Table 2.

Table 2

*Tasks Used by Jansen and Hohensee in Their Study*

|    |                                                                                                                                                                                                                                                                   |
|----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1a | Kelly ran 4 miles in $\frac{2}{3}$ of an hour. How far did she run in 1 hour?                                                                                                                                                                                     |
| 1b | Choose the number sentence that you think represents the story in 1.a. the best, if any of them do. How did you decide?<br>$\frac{2}{3} \times 4 = ?$ $4 \div \frac{2}{3} = ?$ $4 + \left(\frac{1}{3} \times 4\right) = ?$ $\frac{2}{3} \div 4 = ?$ None of these |
| 2a | Write a realistic story problem for the following number sentence, using the given quantities: $24 \text{ DVDs} \div 4 \text{ hours} = ?$                                                                                                                         |
| 2b | Solve this problem using a drawing. Use a discrete, area, or linear model to show how you solved the problem.                                                                                                                                                     |
| 3a | Write a realistic story problem for the following number sentence, using the given quantities: $24 \text{ oz. of water} \div \frac{1}{4} \text{ hours} = ?$                                                                                                       |
| 3b | Solve this problem using a drawing. Use a discrete, area, or linear model to show how you solved the problem.                                                                                                                                                     |

Jansen and Hohensee used these tasks to measure the flexibility and connectedness of the PSTs' conceptions for partitive division. The researchers inferred flexibility if a PST was successful using partitioning and/or iterating to resolve a task. They inferred connectedness, in the sense of translating between representations, if a PST could establish a link between the situation and the symbolic statement of division. They inferred connectedness, in the sense of awareness that the quotient is a unit rate, if a PST revealed appropriate evidence of such awareness at any point during any part of the tasks.

Regarding connectedness, the data revealed that only three of the 17 PSTs identified the correct symbolic statement in Task 1. In Task 3, only four PSTs created a valid context, with the primary mistake being the description of a context for  $24 \times \frac{1}{4}$  instead of for  $24 \div \frac{1}{4}$ . In their paper, the researchers summarized two types of disconnected conceptions. First, a PST incorrectly translates between representations, and

inconsistently expresses awareness of the purpose of determining a unit rate across task components. Second, a PST still incorrectly translates between representations and does not demonstrate any awareness of the purpose of finding a unit rate. The researchers also described a new construct they called “emerging connected conception” by defining it as having a fully connected conception for some tasks, but not for all.

Regarding flexibility, the data revealed that 15 of the 17 PSTs accurately solved Task 1, but only six correctly solved Task 3. I am not surprised by the PSTs’ difficulty with Task 3 since many of them created an inappropriate context in the first place. In their paper, the researchers summarized two ways in which PSTs demonstrated rigid conceptions. First, a PST could partition and iterate to get a correct answer for Task 1a, but did not associate the iteration as a component of division, as evidenced by the results of Task 1b. Second, a PST correctly solved Task 3b by iterating, and connected this to the given division statement, but then could talk about division only in the context of reversing the process by partitioning the quotient to get the original dividend.

The dual-construct of connectedness, as the researchers defined it, presents some challenges. Can we say that a person has demonstrated the second type of connectedness if that person does not connect a context to a single symbolic statement of division, but succeeds at connecting the context to the goal of finding a unit rate? In addition, the data showed varying levels of success from task to task, suggesting that the two categories of connectedness are insufficient when talking about partitive division in general. For any one kind of value (e.g. whole divisors) I can imagine that a person may always, sometimes, or never demonstrate awareness of the purpose of finding a unit rate. Also, a person may always, sometimes, or never be able to translate between a context and a

single symbolic statement of division. As such, it would be possible to characterize a person as “always-sometimes” for whole divisors, meaning that the person can always connect a symbolic statement with a context, but only sometimes demonstrates awareness of the goal as finding a unit rate. This degree of categorization would allow for a more detailed analysis.

Concerning the tasks used in Jansen’s and Hohensee’s (2016) study, no improper fractions were used as divisors, thus missing opportunities to get additional data. Also, the researchers said they were looking for evidence that the PSTs were aware that the purpose of the division task was to find a unit rate, yet Task 1a asks for a distance, not a rate. This may have biased the PSTs to thinking about extensive quantities, and not intensive quantities.

### **Absence of Partitive Division in Curriculum**

As a critique, a survey of the Common Core State Standards in Mathematics (CCSSM) placements<sup>5</sup> for grades K-8 (National Governors Association, 2010) reveals almost no emphasis on partitive meanings. In fact, the only mention of partitive and quotitive meanings is found in grade 3, where division is limited to whole dividends, divisors, and quotients. Also, all examples listed in grades 5 and 6 that involve non-whole divisors are quotitive in nature. There are no examples listed in these placemats that involve partitive conceptualizations with non-whole divisors.

Not only does the Common Core gloss over partitive meanings, but a brief survey of textbooks and online resources reveals that many models for division with fractions

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<sup>5</sup> A summary of the CCSSM placements regarding meanings for division and division with fractional divisors can be found in the appendices.

are quotitive in nature. It is rare to come across curricular suggestions involving partitive division with fractions, although in recent years many researchers have done so (Beckmann, 2011; Gregg & Underwood Gregg, 2007; Kribs-Zaleta, 2008; Ott et al., 1991). Beckmann, in her textbook for prospective elementary teachers (2011), used amount of paint and number of walls that can be painted as the context for partitive division with fractional divisors. However, in her textbook, the partitive contexts are presented as optional. Gregg and Underwood Gregg (2007) used number of cakes and number of containers to model partitive division with fractional divisors. Ott et al. (1991) used unit prices, average speeds, and other contexts to discuss partitive division with fractional divisors. Lastly, Kribs-Zaleta (2008) shared four different contexts that elicit partitive division with non-whole divisors, including length of ribbon per bow and amount of lemonade per glass.

## Conclusion

In my review of the literature, most of the research regarding division with fractions involved quotitive conceptualizations, with just a few exceptions. I did not find any literature that specifically addressed the *advancement* of partitive meanings over the rational numbers, although Jansen and Hohensee (2016) said that they were in the process of conducting such a study. Also, a brief survey of the CCSSM (2010) and other curricula revealed little to no emphasis on partitive meanings with non-whole divisors. These are the primary reasons that motivated me to conduct this dissertation study. In the next chapter, I comment on my theoretical framework that guided the design and analysis of my study.

## CHAPTER 3

### THEORETICAL FRAMEWORK

My theoretical framework is rooted in constructivism. I believe that a person's mathematical understandings are constructed by adding to and refining existing understandings. This is done by harmonizing existing understandings with new experiences. A person can make sense of a concept if he or she has an adequate foundation of knowledge that supports the integration of the new concept. Sometimes a person will need to restructure existing understandings to make sense of the new experiences. Furthermore, I believe that a person's body of knowledge is constructed and structured in a way that is likely unique to that person. These constructivist beliefs are inspired by the work of Piaget, as described by Von Glasersfeld (1995). I also base my design and analysis of the data from this study on the quantitative theory as elaborated by Thompson (1994, 2011), with ideas for thinking with magnitudes as discussed by Thompson et al. (2014). In this chapter, I discuss important constructs for both constructivism and quantitative reasoning that guided my design of this study and the analysis of the results. I begin with a discussion of quantitative reasoning constructs in the following section.

#### **Quantitative Reasoning**

I believe that numbers are ultimately meaningless, unless they are thought about in some sort of context. For example, to say "I will give you four" is not meaningful until it is made clear to what the "four" is referring. When the context is more meaningful and connected to units-of-measure, the greater the opportunity a person has for making sense of operations and coordinating multiple levels of units. My dissertation study focused on

making meaning of operations by situating the operations in contexts that provided concrete things to think about, as well as the words to describe them. In other words, I provided opportunities for my subjects to engage in quantitative reasoning, as extensively described by Thompson over the years (1994b, 2011). The following section focuses on the following constructs: (1) quantity, (2) quantification, (3) value of a quantity, and (4) quantitative operation versus numerical operation.

## Quantity

Thompson (1994b, p.7) characterizes a quantity as the conception of an attribute of an object that admits a measuring process, or briefly, a measurable attribute of an object. For example, someone can conceive that a person has an attribute of tallness when standing, often referred to as that person's height. Thompson interprets the word *object* broadly. For example, the discrepancy between two people's heights is a quantity itself. In this case, the *object* is the collection of the two people, and the *measurable attribute* is the discrepancy in height. To be clearer, Thompson and Smith later added to the definition of quantity by stating that "quantities are attributes of objects *or phenomena* that are measurable" (Smith & Thompson, 2007, p.10). Quantities are not objects themselves – they exist in the mind of the person who is imagining them. Quantities can be imagined to be measured, combined, or compared. Also, combinations and comparisons can be additive or multiplicative. It is important to note that quantities can be imagined without being measured. Also, it is not necessary for two quantities to be measured before combining or comparing them. I can imagine a new quantity that is the difference between the heights of two people without needing to measure the height of either person.

## Quantification

Thompson defines quantification as “the process of conceptualizing an object and an attribute of it so that the attribute has a unit-of-measure, and the attribute’s measure entails a proportional relationship (linear, bi- linear, or multi-linear) with its unit” (2011, p.37). I interpret this as the act of establishing the relative size of a quantity with respect to some appropriate unit-of-measure. I use the word *appropriate* to emphasize that the unit-of-measure should<sup>6</sup> measure the same kind of attribute as the quantity in mind. For example, it is appropriate to use a unit of length to measure another length. Borrowing from Thompson et al. (2014), I consider the *magnitude* of a fixed quantity to represent an invariant amount, which is not dependent on the choice of unit-of-measure (e.g., 1 foot, 12 inches, and 1/3 yard all represent the same magnitude).

## Value of a Quantity

The value of a quantity is the numerical result of a measurement, which is a multiplicative comparison of the magnitude of the quantity in terms of the magnitude of some appropriate unit-of-measure. For example, 3.2 feet is the value of a length-quantity which represents a length that is 3.2 times as long as a 1-foot length. Given this definition, I consider the values of 1 yard and 3 feet to be two different values. However, they represent the same magnitude. I borrow from Thompson’s constructs by defining a *fractional value of a quantity* to be the value of a quantity whose numerical component is not a whole number. For example, a value of 3 gallons is not a fractional value but the

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<sup>6</sup> If a correspondence between two different kinds of quantities is established then it is possible to measure one kind of quantity in units of another kind of quantity, e.g. one light year measures a distance using time as the unit.

value of  $\frac{7}{8}$  of a gallon is a fractional value of a quantity. In summary, I adopt the concise descriptions that a *quantity* is something that can be measured, *quantification* is the act of measuring the quantity, and the *value of the quantity* is the result of the measurement. Several of these constructs regarding quantities are depicted in Figure 1.

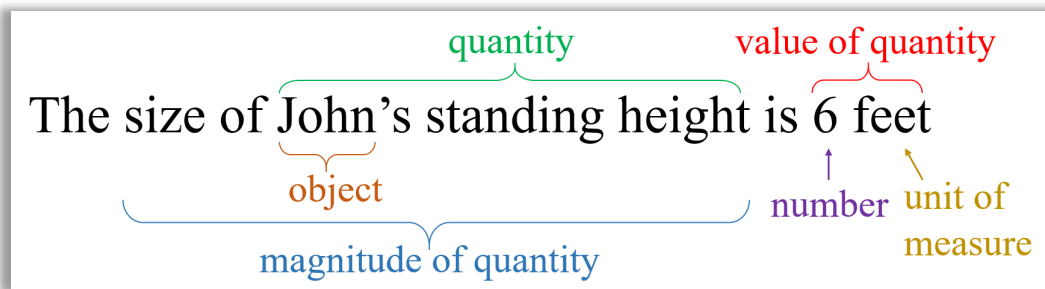


Figure 1. Constructs related to quantities.

### Quantitative Operation vs. Numerical Operation

Thompson (2011) describes a *quantitative operation* as the act of either combining two quantities (additively or multiplicatively) or comparing two quantities (additively or multiplicatively). The result of a quantitative operation is a new quantity which, when connected through the quantitative operation to the original quantities, forms a quantitative relationship. Again, I emphasize that measuring quantities is not a prerequisite to the creation of a quantitative relationship — I can imagine combining some number of buckets, each bucket containing some amount of liquid, and producing a total amount of liquid, without measuring any of these amounts. A network of connected quantitative relationships forms a quantitative structure. A quantitative operation should not be confused with a numerical operation, which is the act of calculating numbers by operating on numbers.

## Quantitative Reasoning Summary

In summary, Thompson defines quantitative reasoning as the act of reasoning with quantities and creating quantitative structures using quantitative operations. It is not to be confused with reasoning with numerals and the calculation of the results of numerical operations. Thompson and Thompson (1992) succinctly define quantitative reasoning as “reasoning about things and their attributes” (p.1). Quantitative reasoning is at the heart of productive modeling of realistic scenarios using mathematics. As such, it is at the heart of this study on meaningful operations. Valid reasoning with quantities should precede and ultimately dictate appropriate numerical operations on the values of those quantities. Thompson and Saldanha (2003) suggest that “rules and shortcuts for operating symbolically should be generalizations from conceptual operations instead of being taught in place of them” (p.36). The National Council of Teachers of Mathematics (2014) echoes this sentiment by stating that procedural fluency should emerge from conceptual understanding.

## Constructivism

Piaget (1954, 1972, 2001) is credited for advancing the meanings of scheme, assimilation, accommodation, and reflecting abstraction into robust constructs that are useful in describing behavior from a constructivist point of view. In this proposal, I provide my own interpretations of these constructs, as influenced by Piaget’s works and the works of many others who have followed (e.g., Chapman, 1988; Derry, 1996; Dubinsky, 1991; Montangero & Maurice-Naville, 1997; Steffe, 1994; Thompson, 1994b; Thompson et al., 2014; von Glasersfeld, 1995).

## Scheme

My interpretation of a *scheme* is the mental activity that underlies a repeatable, directional response to a stimulus (see Figure 2). I explain the components of this interpretation below, but first, I illustrate with an example. When I see the written expression “ $6 \div 2$ ” I can then draw (or imagine drawing) six apples, and then organize them into two groups of three apples per group. In this example, the stimulus is the written expression that entered my visual field. My scheme recognizes and interprets this sensory input and then triggers and guides my physical actions (or mental actions) to express my meaning of the written expression.

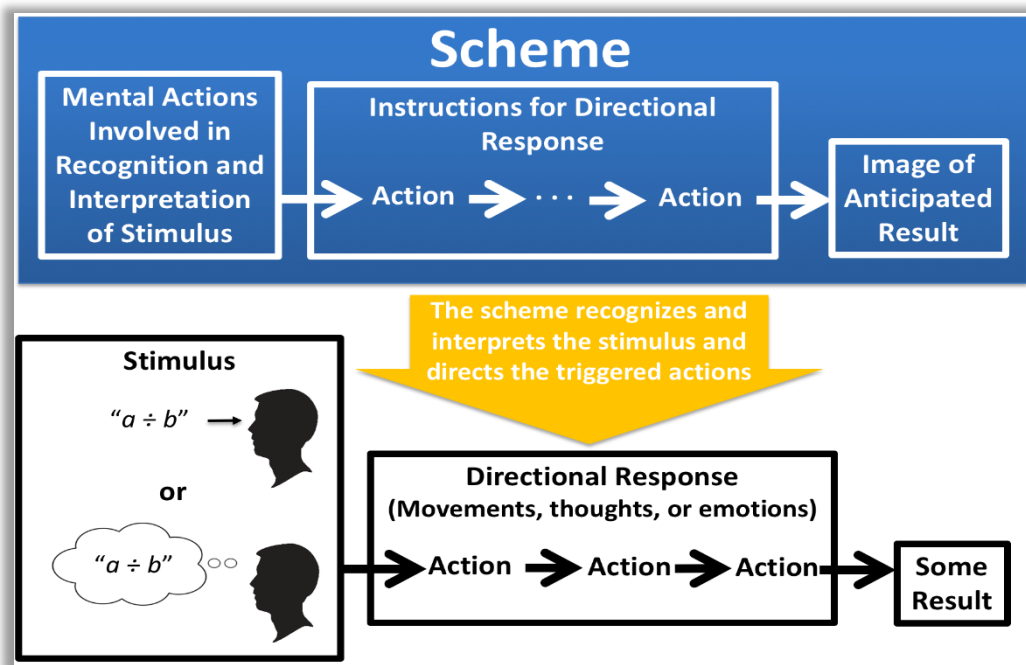


Figure 2. The nature of a scheme.

By *stimulus*, I refer to anything, such as an idea or something that is in a perceptual field (such as visual, auditory, etc.), that triggers some sort of response. If the *something* does not trigger a response, then I do not consider the *something* to be a stimulus. Furthermore, I phrase my description of stimulus so that it can be connected to

sensory input in the moment or to the memory (subconscious or conscious) of past sensory input. For example, in the example above, the expression  $6 \div 2$  can be currently visible, or it could be imagined from memory. Thus, my characterization of scheme is broad enough to include the notion of responses to evoked memories. This is an important specification because it allows for the possibility of chain reactions where a stimulus triggers a mental response that yields a familiar mental result, which triggers another response, etc. I consider the coordination of such chain reactions to be a composite scheme resulting from the combination of multiple schemes.

When I say *response* in my interpretation of a scheme, I refer to any action (thought, movement, or emotion (Piaget et al., 1977)), or combinations thereof, that is triggered by the stimulus. By *directional* response, I mean that the scheme directs the triggered action toward some “anticipated” result. This result can be a physical result or simply a mental result (a thought). By *repeatable* response, I mean that the mental activity pertaining to the response has a degree of permanence that enables repeated stimuli to trigger similar, if not identical responses. However, schemes can change over time, due to their modification or due to their inactivity.

Per my description, a scheme *is* an organization of mental activity, which happens to coordinate a directional response to stimuli. I emphasize this because I wish to make clear that I do not consider the stimulus, nor the directional response that is triggered by the stimulus, to be components of the scheme itself. The directional response (regardless of the types of actions involved) is simply a manifestation of the scheme. Indeed, I think of a scheme as a blueprint that includes instructions on recognizing and interpreting a stimulus, and on coordinating the directional response.

To an outside observer, there would be no evidence of a scheme if the response that is triggered by the stimulus involves only imperceptible actions (like thoughts or subtle emotions). However, it is an *observable* response to a stimulus that allows outside observers to make models of thinking. Steffe and Thompson (2000) use the constructs the *students' mathematics* to refer to students' actual schemes (which are inaccessible to us) and the *mathematics of the student* to refer to our models of their schemes based on our observations of student behavior (p.268). Since the actions involved in the directional response are sometimes observable, they can easily be confused with the underlying scheme itself. This is like confusing the movements of a construction worker, who is following a blueprint, with the blueprint itself. In my analogy, only the movements of the construction worker are observable; the blueprint is not. Also, using this blueprint metaphor, the coordination of schemes is like coordinating multiple blueprints into one blueprint – a scheme of schemes.

Schemes are malleable and subject to change or generalization. If a new stimulus is similar enough to a past stimulus, the recognitive and interpretive functions of the scheme may trigger identical (or nearly identical) directional responses. For example, if I see the written expression  $7 \div 3$ , the mental actions involved in the directional response differ slightly from those that are triggered when I see  $6 \div 2$ . I talk more about this in my discussions on assimilation, accommodation, and abstraction.

Schemes do *not* need to be organizations of deliberate mental activity. Many schemes that govern our responses to stimuli are instinctive, automatic, or subconscious, such as reflexes. Also, schemes can exist independent of our awareness of their existence. This is obvious in the case of infants; an infant's schemes guide his behavior well before

his intelligence has reached the capacity to obtain an awareness of these schemes. For example, Piaget (1954) ascribes the actions of an infant reaching for an object or sucking on the nipple of a bottle to the existence of schemes, yet an infant is not capable of metacognition about these schemes. In fact, it is reasonable to think that many of a person's schemes will elude his awareness for a long time, perhaps for the duration of his life.

In their article on schemes for thinking with magnitudes, Thompson et al. (2014) suggest that a person's *meanings* refers to that person's schemes. I adopt this convention by using the terms *scheme* and *meaning* interchangeably throughout my writing.

### **Assimilation**

I interpret *assimilation* as the mental activity involved in the recognition and interpretation of a stimulus, thereby accessing an existing scheme (see Figure 3). As mentioned earlier, I described a *stimulus* as an idea or something that is (or was) in a perceptual field, and that triggers a response. Given my characterizations of scheme, stimulus, and assimilation, assimilation is therefore responsible for the “correspondence” of a stimulus to a scheme.

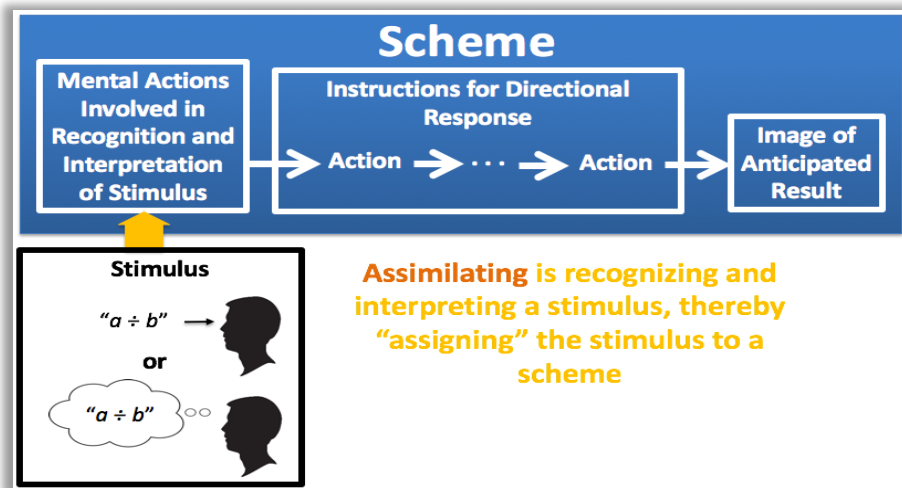


Figure 3. Assimilation of a stimulus to a scheme.

In the example of the expression  $6 \div 2$ , when I see, hear, or think about this expression, I recognize it as something familiar and interpret the symbols. In other words, I assimilate the expression. This assimilation provides access to multiple schemes, and, as such, can trigger multiple directional responses. Simply put, I assimilate the expression to multiple schemes. Assimilation of the entire expression is comprised of assimilation of each of the expression’s components. To illustrate, when I see the 6 in this context, I interpret it as a total amount of something. When I see the  $\div$  symbol in this context, I interpret this as a command to divide, which gives me access to numerous schemes involving division. Thus, assimilation of the symbol  $\div$  could trigger thoughts about fair-sharing (partitive division), thoughts about measuring (quotitive division), thoughts about doing the long division algorithm, thoughts about writing a fraction, or other thoughts that allow me to arrive as some result.

Piaget (as cited in Montangero & Maurice-Naville, 1997) provided a description of assimilation: “Assimilating an object to a scheme involves giving one or several meanings to this object, and it is this conferring of meanings that implies a more or less

complex system of inferences.” I take the liberty of rephrasing this quote to provide an interpretation of it that utilizes the terms and constructs that I have established in this chapter: “Assimilating a *stimulus* to a scheme involves giving one or several *interpretations* to this *stimulus*, and it is this conferring of *interpretations* that *grants access* to a more or less complex system of *schemes*.” This rephrasing describes assimilation in a way that is compatible with my characterization of assimilation. As an alternate characterization of assimilation, given that I consider meanings to be schemes, I could describe assimilation as the act of *attributing meaning to a stimulus*. To me, this new characterization of assimilation is compatible with my original characterization of assimilation.

### **Accommodation**

Whenever a stimulus cannot be attributed to a scheme, a modification of a current scheme, or the creation of a new scheme, is required for the stimulus to be assimilated. Similarly, when a directional response produces a result that contradicts the image of the anticipated result, a perturbation is experienced, which can be neutralized through a modification of a current scheme, or the creation of a new scheme. For both situations, I characterize an *accommodation* as the modification of a current scheme, or the creation of a new scheme<sup>7</sup>. Below, I expound on these two situations.

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<sup>7</sup> One could argue that I am being redundant because modifying a scheme results in a new scheme, so there is no need to characterize accommodation in two ways. I suspect that some schemes are innate, and that others are developed through experience. I also suspect that most newly created schemes are nothing more than combinations or modifications of existing schemes. But I do not rule out the possibility that a scheme can be created from “thin air”, hence my two-pronged characterization of accommodation.

*Type 1 accommodation:* An accommodation may occur in response to a perturbation when a stimulus cannot be attributed to a scheme (see Figure 4). Either there is no scheme to which the stimulus can be assimilated, or the stimulus does not trigger an existing scheme. I believe that when a person is bombarded with a wave of sensory input, only the stimuli that are assimilated are *understood*. All other pieces of sensory input are either discarded subconsciously or they cause a perturbation. To neutralize such a perturbation, an accommodation is required. To illustrate, suppose I have a fair-sharing scheme triggered by the stimulus  $6 \div 2$ . If I now see  $7 \div 2$  my fair-sharing scheme may not be able to cope with the odd numerator. One way to accommodate is to expand my existing scheme for fair-sharing six objects to include how to fairly share the 7<sup>th</sup> object. Another way for the mind to cope with a stimulus that cannot be assimilated is to interpret the stimulus as if it were a stimulus that *could* be assimilated, such as misinterpreting  $7 \div 2$  as  $6 \div 2$ . However, I do not consider such a coping mechanism as an act of accommodation.

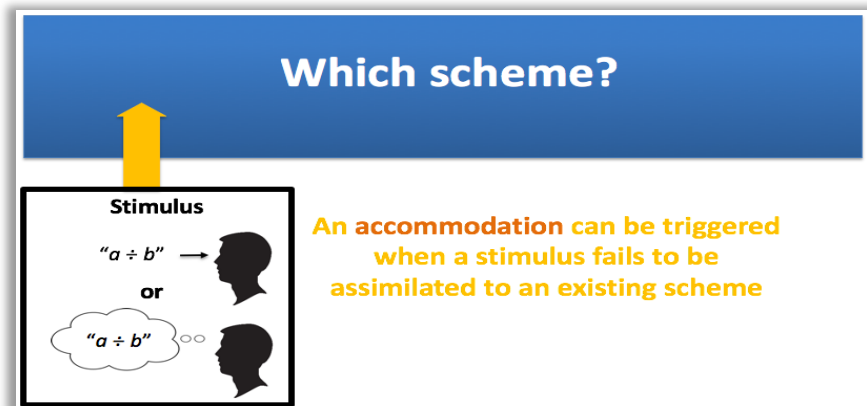


Figure 4. Type 1 accommodation.

*Type 2 Accommodation:* An accommodation may occur to neutralize a perturbation that is caused when the result of a directional response to a stimulus contradicts the image of the

“anticipated” result (see Figure 5). For example, some schemes for division include the expectation that *division makes smaller*. When a person with such a scheme divides nine by  $\frac{1}{2}$ , they get a result that contradicts the scheme’s image of the result. The resulting perturbation can be neutralized in many ways, such as modifying the scheme to expect division to sometimes increase a quantity, or by creating a new scheme specifically for divisors that have the value  $\frac{1}{2}$ .

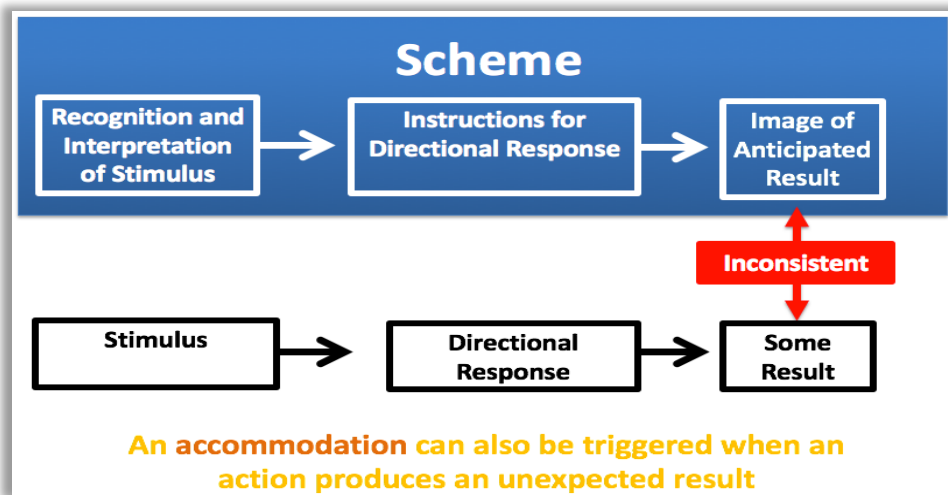


Figure 5. Type 2 accommodation.

Assimilation and accommodation are related to each other. Piaget (as cited in Montangero & Maurice-Naville, 1997) said that “assimilation and accommodation are the two poles of an interaction between organism and the environment.” I believe that Piaget is referring to the following relationship: assimilation is attributing meaning to a stimulus and accommodation is the modification (or creation) of a scheme brought about by a stimulus. As such, there exists a certain symmetry between assimilation and accommodation; in assimilation, the person is acting on the perception of the stimulus, and in accommodation the perception of the stimulus is acting on the person (on their schemes). Accommodation is a mechanism that *serves* assimilation. A person

accommodates a scheme or schemes *so that* they can assimilate what they could not assimilate before.

Why does accommodation occur? People have an innate desire for cognitive coherence. To this end, accommodation would occur to neutralize incoherence. Having briefly addressed *why* accommodation occurs, it is natural to discuss *how* it is brought about. I suspect that some schemes are created or refined through repetitive actions. It's also possible that some schemes are innate, embedded in our genetic code. In many cases, I would attribute the creation or modification of schemes to the act of reflection, which leads me to a discussion of the Piagetian construct of *reflecting abstraction*.

### **Reflecting Abstraction**

At the risk of oversimplification, I will treat *abstraction* synonymously with generalization. Piaget and others have described three different types of abstraction (e.g., Dubinsky, 1991; Piaget, 2001; von Glasersfeld, 1991) related to the scenario where a stimulus triggers an action which produces an expected result, which scenario is guided by an underlying scheme (see Figure 6).

*Type 1 Abstraction:* A person who focuses on the stimulus and generalizes from the attributes of the stimulus is engaging in *empirical abstraction*. For example, I could focus on a collection of even numbers (without acting on them) and generalize that all numbers are even. (Keep in mind that the generalizations do not have to be accurate.)

*Type 2 Abstraction:* A person who generalizes only from the results of the action is engaging in *pseudo-empirical abstraction*. For example, I could do several calculations of a certain type which all produce an even number and generalize that this type of

calculation always produces an even number, without focusing on the actions of the calculation.

*Type 3 Abstraction:* A person who generalizes from the actions is engaging in *reflecting abstraction*. For example, I could focus on the actions involved in the calculation that led to even-numbered results and sensibly generalize that such actions will always produce an even number.

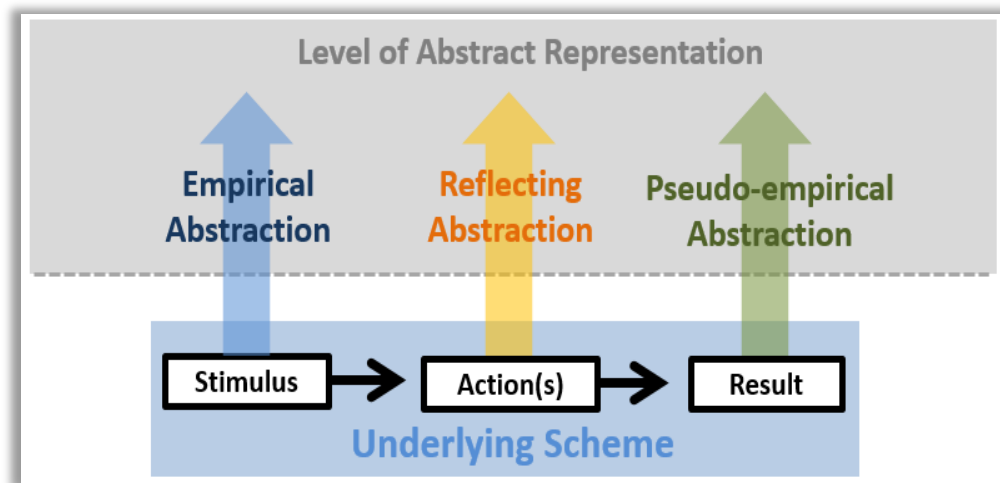


Figure 6. Three types of abstraction.

## Conclusion

I characterize the learning of mathematics as the advancement of schemes through accommodation and reflective abstraction. A primary objective of this dissertation study is to identify *cognitive obstacles* (or *cognitive barriers*) by which I mean attributes of mental activity that impair assimilation of stimuli to appropriate and productive schemes, or that impair the advancement of schemes. Since each person has a unique combination of schemes related to thinking about dividing the values of two quantities, I expect

teachers to demonstrate a variety of ways that they might assimilate the tasks I used in this study. In the next chapter, I discuss some of these ways to think about division involving fractional values, as well as meanings that teachers may hold related to fractions.

## CHAPTER 4

### CONCEPTUAL ANALYSIS

In Chapter 2, I shared several justifications for the invert-and-multiply algorithm identified by researchers (e.g., Sharp & Adams, 2002; Tirosh, 2000). However, these justifications were *not* primarily based on partitive and quotitive meanings. In this chapter, I will discuss the following: (1) meanings associated with fractions, (2) partitive and quotitive conceptualizations regarding division with non-whole values and how they relate to numerical division, including how they form a meaningful foundation for the invert-and-multiply algorithm, and (3) partitive and quotitive conceptualizations as a foundation for other mathematical meanings.

#### **Teachers' Meanings for Fractions**

To properly study how teachers might think about division with fractional values, it is important to consider how they might think first about the fractions. Researchers have identified and elaborated on several meanings triggered by the fraction symbol (Lamon, 2011; McCloskey & Norton, 2009; Norton & McCloskey, 2008; Siebert & Gaskin, 2006; Steffe & Olive, 2010; Thompson & Saldanha, 2003; Tzur, 1999, 2004; Tzur & Hunt, 2015). In this section, I share some of these meanings as well as other meanings that may be held by teachers. I will attempt to distinguish the more mature, coherent meanings from the more primitive, limiting meanings. It is reasonable to say that the extent to which someone can meaningfully conceptualize division with fractional values is largely determined by the extent to which that person has robust meanings for the fractions that are being divided. The meanings I will discuss are introduced in the list below. I listed the more primitive meanings first and the more mature meanings last, but I

do not suggest that this is an authoritative ranking of robustness, nor do I suggest that such a linear ranking is even possible.

1. Part-to-whole conception:  $2/3$  is interpreted as 2 out of 3 parts.
2. Fraction as an operation:  $2/3$  is interpreted as divide 2 by 3.
3. Fraction as a quotient:  $2/3$  is interpreted as the *result* of 2 divided by 3.
4. Fraction as an operator:  $2/3$  of a unit means partition the unit and iterate a part.
5. Fraction as a ratio:  $2/3$  is interpreted as the result of a multiplicative comparison between 2 of something and 3 of something else.
6. Fraction as a rate:  $2/3$  is interpreted as a constant intensive quantity.

### **Part-to-Whole Conception**

A teacher is engaged in *part-to-whole* thinking when they imagine  $2/3$  of a *quantity* as the combined size of two parts when that quantity is partitioned into three equally-sized parts (Steffe & Olive, 2010). This conception of fractions can be oversimplified to mean *2 parts out of 3 parts*. However, researchers such as Siebert and Gaskin (2006), Thompson and Saldanha (2003), and Tzur and Hunt (2015) have shown that this simplification impedes productive thinking related to fractions in ways that are included in the summary below.

1. When thinking *2 parts out of 3 parts*, it might not be significant to a person whether the three parts are equal in size.
2. When thinking *2 parts out of 3 parts*, the focus may be on counting. As such, the fraction  $2/3$  may have no more meaning than simply a counting exercise.
3. When thinking *2 parts out of 3 parts*, a teacher may construe a fraction as two separate values, instead of a single value.

4. When thinking *2 parts out of 3 parts*, it might not be clear to someone whether the three parts constitute a whole.
5. The word *part* implies that any one part (or collection of parts) must be smaller than the whole. As such, improper fractions seem nonsensical. For example, what does it mean to say *4 parts out of 3 parts*?
6. The word *part* implies a sense of inclusion. When thinking *2 parts out of 3 parts*, it is easily imagined that the two parts are a subset of the three parts. Prolonged exposure to such *part-to-whole* thinking can instill an intuitive sense for inclusion. However, such an intuition is limiting because fractions can also represent multiplicative comparisons between two disjoint or even dissimilar quantities.
7. Part-to-whole thinking does not support thinking about fractions with non-whole numerators or denominators. It is challenging to make sense of the fraction  $\frac{3/2}{2/7}$  by thinking *3/2 parts out of 2/7 parts*.
8. The part-to-whole conception can obscure comparisons of fractional values. It is reasonable to claim that *2 out of nine* identical candy bars are more than *1 out of 3* of these candy bars, but, referring to the same whole,  $2/8$  is less than  $1/3$ .
9. Finally, the part-to-whole conception can impede sensible operations with fractions. What sense is made of  $5/8 \times 3/4$  if you are thinking (*5 out of nine parts*)  $\times$  (*3 out of 4 parts*)? Also, *5 out of nine parts* combined with *3 out of 4 parts* can mean *nine out of 12 parts*, which conflicts with the calculation  $5/8 + 3/4 = 11/8$ .

Research has clearly identified several ways in which primitive part-to-whole conceptions of fractions can impede the development of robust reasoning with fractions.

## Fraction as an Operation

If  $m/n$  means nothing more to a teacher than divide  $m$  by  $n$  then the teacher will have trouble operating with fractions themselves. The fraction symbol gains maturity once it represents the *result* of division in the mind of the teacher.

## Fraction as a Quotient

This is interpreting a fraction as the *result* of division. I consider partitive and quotitive meanings for fractions to be mature. The partitive meaning is the conceptualization that  $c/a$  represents the amount per group such that  $a$  copies of that amount yield the amount  $c$ . The quotitive meaning is the conceptualization that  $c/b$  represents the number of copies of the amount  $b$  needed to constitute the amount  $c$ . The quotitive meaning can also be framed in terms of measurement;  $\frac{c}{b}$  represents the measurement of  $c$  in units of size  $b$ .

## Fraction as an Operator

This can be a powerful meaning for fractions, provided the numerous schemes related to operating with this meaning are adequately developed. Steffe and Olive (2010) have extensively discussed such schemes and their development. I will discuss fractions as operators for the case of  $m/n$  where  $m$  and  $n$  are positive integers. Thinking of a fraction as an operator means to think of  $m/n$  as  $m/n$  of *something*, even if that something is an unspecified whole. As for unit fractions, a mature conception of  $1/m$  of a unit is the single amount such that  $m$  iterations of that amount would collectively produce one whole unit. This is slightly different, but no less valid, than conceiving of  $1/m$  as the size of one piece following the partitioning of a whole into  $m$  equal pieces. Concerning non-

unit fractions, a person may imagine  $m/n$  of a unit as the single amount that results when a whole unit is partitioned into  $n$  equal parts, and one of those parts is iterated  $m$  times. As such,  $m/n$  of a unit is thought of as  $m$  copies of  $1/n$  of a unit. This last conceptualization was heavily emphasized during AMP workshops. For a teacher with more advanced schemes, the ordering of the partitioning and iterating is irrelevant, which yields the numerical equivalence of  $m \times 1/n$  and  $m \div n$ . Also, some schemes for partitioning and iterating are reversible. A teacher with reversible schemes realizes that  $n/m$  of ( $m/n$  of a unit) gives back one whole unit. Possessing reversible schemes is foundational to making sense of the invert-and-multiply algorithm using partitive and quotitive conceptualizations, which I will demonstrate in the next subsection. In the absence of a specified unit, a teacher with an operator conception of fraction still imagines  $m/n$  as  $m/n$  of 1 something. The maturity of the meaning of fraction as an operator is dependent on the extent to which the underlying schemes are developed.

### **Fraction as a Ratio or a Rate**

I adopt the meanings for the terms *ratio* and *rate* as articulated by Thompson (1994b) and Thompson and Thompson (1992). A *ratio* is the result of a multiplicative comparison between the magnitudes of two fixed quantities. Thompson (1994b) calls a *rate* a “reflectively abstracted constant ratio (p.18),” which I interpret as a ratio that is held constant in one’s mind as one imagines the corresponding quantities covarying within some broader scope. As such, a rate establishes a directly proportional relationship between the two covarying quantities. I emphasize that these meanings for the terms *ratio* and *rate*, which cast ratio and rate as *ways of thinking* about situations and not as *properties* of situations, are not ubiquitous in the field of mathematics education or

among teachers. To present alternative viewpoints, Thompson and Thompson (1992) provide a concise summary of the most frequent distinctions between ratio and rate found in the literature, as cited below (p.2).

1. A ratio is a comparison between quantities of like nature (e.g., pounds vs. pounds) and a rate is a comparison of quantities of unlike nature (e.g., distance vs. time (Vergnaud, 1983)).
2. A ratio is a numerical expression of how much there is of one quantity in relation to another quantity; a rate is a ratio between a quantity and a period of time (Ohlsson, 1988).
3. A ratio is a binary relation that involves ordered pairs of quantities. A rate is an intensive quantity—a relationship between one quantity and one unit of another quantity (J. Kaput, Luke, Poholsky, & Sayer, 1986; Lesh, Post, & Behr, 1988; Schwartz, 1988).

It is likely that the lack of consensus among researchers regarding the meanings for these two terms extends to teachers. Not only can definitions for these terms vary from teacher to teacher, but for any one definition, underlying meanings can also vary. Regarding fractions, the meanings a teacher holds for fraction as a ratio or a rate can vary from weak to mature. For some teachers, a fraction may simply represent a coordination of two values that are somehow related. This would be a weak meaning if this were the extent of their conceptualization. For others, a fraction may represent a coordination of two values which are multiplicatively compared, but they may not interpret the fraction as a symbol for the result of that comparison. Hence, a fraction remains a representation of two values instead of a single value. Borrowing from Thompson (1994b), when a person sees the

symbol “ $3/2$ ” (or “ $3:2$ ”), he or she could be thinking *3 of these always go with 2 of those*, which is a precursor to imagining a proportional relationship. Or, he or she could be thinking *this is  $3/2$  as large as that*, which is a relative size way of thinking. As discussed in my review of the literature, not all people have equally robust conceptions of rate. For some, a rate was an extensive quantity – for example, JJ considered speed to be a length (Thompson, 1994b). Thus, a teacher may also interpret a fraction symbol as a single value that is a rate, but a rate that is just an extensive quantity.

### **Summary of Teachers’ Meanings for Fractions**

Mature, coherent meanings for fraction are ultimately dependent on the sophistication of the underlying schemes. Above, I presented a categorization of meanings and suggested that within any one category, the level of maturity of that meaning can vary. I propose that a rich collection of meanings for fraction depends on the development of a multitude of the meanings I mentioned earlier, as well as establishing connections between these meanings. Thompson & Saldanha (2003) share in a similar belief by inextricably connecting fractions to other related concepts. They argued that a rich understanding of fractions is composed of developed meanings for measurement, multiplication, division, and relative size.

### **Partitive Conceptualizations when Dividing by Fractions**

The *partitive* (or *sharing*) model of division involves thinking about the size of a whole group when a known quantity is distributed among a known number of uniform groups (partial groups are possible). Partitive division has the general form:

$$\text{Total amount of stuff} \div \text{number of groups} = \text{amount of stuff per whole group}$$

Partitive division presumes a proportional correspondence between a varying number of groups (which changes from  $a$  to 1) and a varying amount of stuff (which changes from  $c$  to  $c \div a$ ). As discussed in Chapter 2, the primitive model of *fair-sharing* involves intuitive beliefs that the number of groups should be a whole number and that the resulting group size should be smaller than the original amount. However, these primitive intuitions are not amenable to thinking about partitive division over the rational numbers. For someone with more sophisticated meanings, the partitive division model is viable no matter the types of values that are involved, and it is possible to have a quotient that exceeds the dividend.

As discussed in my literature review chapter, partitive division with fractional values is largely absent from curriculum and the CCSSM. However, there are many instances where partitive division with a non-whole number of copies is required; for example, spreading soil over 2.3 acres, or pouring water into 2.25 containers. It is even sensible to split money between 2.5 people if I interpret this in such a way that two people get the same amount of money and that a third person gets only half as much. Furthermore, I can think more abstractly and distribute a certain amount of distance over 2.75 hours, which connects meanings for partitive division with schemes for rates and proportional correspondence.

To discuss the schemes relevant to a partitive conceptualization for division involving any positive rational number, I present two situations below, each involving water and containers, and share some ways that a person might think about them. I will also demonstrate how certain ways of thinking are not only productive, but conducive to the emergence of the invert-and-multiply algorithm.

**Situation 1:** Suppose six cups of water fill 2.25 containers. People could arrive at  $\frac{8}{3}$  cups in one container in a variety of ways. They could guess and check through multiplication or repeated addition trying to answer “2.25 copies of what make six cups?” Or, since 2.25 is nine quarters, they could imagine a container such that six cups fill nine of those containers, yielding  $\frac{6}{9}$  of a cup per container. If they then imagine a new container size such that the six cups fill  $\frac{1}{4}$  as many containers as before, then each of these new containers should contain 4 times as much as  $\frac{6}{9}$  of a cup. As a third way of thinking, perhaps a more elegant way of thinking, they begin by interpreting 2.25 containers as nine quarter-containers. Thus, they could split six cups into nine pieces and then copy 4 of those pieces to reconstitute a whole container. Thus,  $\frac{4}{9}$  of six cups corresponds to one whole container. This yields the numerical equivalence of  $6 \div \frac{9}{4}$  and  $\frac{4}{9} \times 6$  and is depicted in Figure 7.

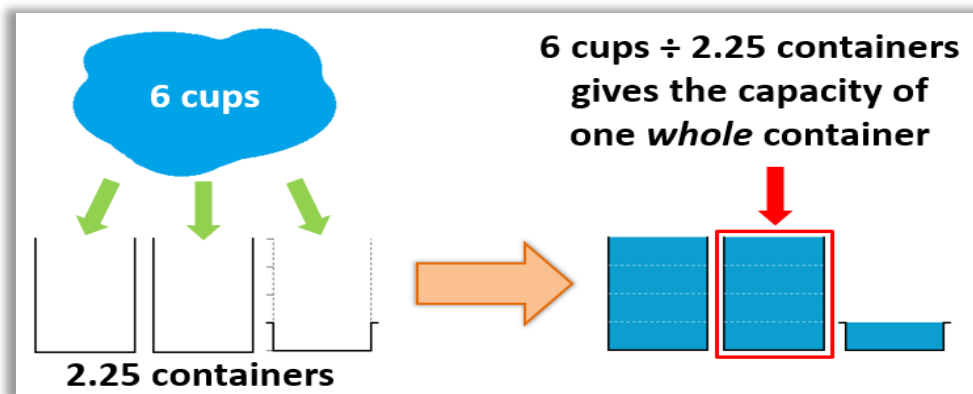


Figure 7. Six cups filling 2.25 containers.

**Situation 2:** Suppose six cups of water fill  $\frac{2}{3}$  of a container. People could arrive at nine cups in one container in a variety of ways, but I only present one solution. They could recognize that  $\frac{2}{3}$  of a container is the same as 2 one-third containers. Thus, they could split six cups into 2 pieces and then copy 3 of those pieces to reconstitute the

capacity of a whole container. This scheme establishes the numerical equivalence of  $6 \div \frac{2}{3}$  and  $\frac{3}{2} \times 6$  and is depicted in Figure 8.

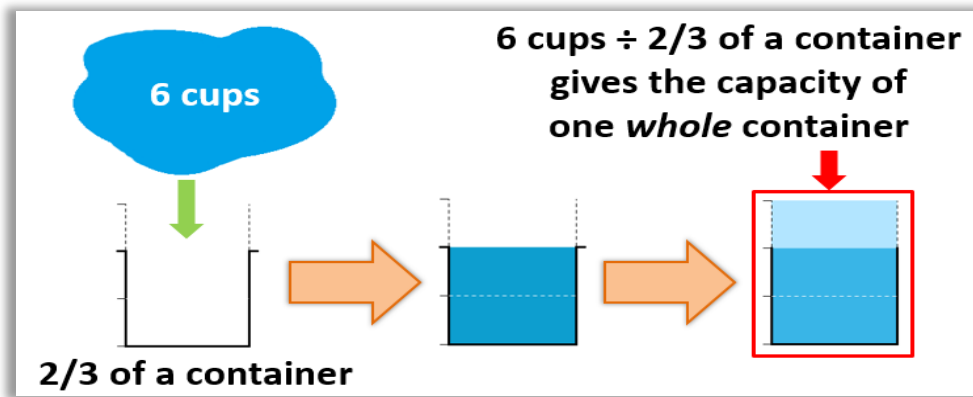


Figure 8. Six cups filling  $\frac{2}{3}$  of a container.

In both situations, six cups were considered, but for the schemes that were mentioned last in each situation, this amount of water could be replaced by an arbitrary amount of water, without affecting the scheme. As such, these examples illustrate partitive justifications for the invert-and-multiply algorithm, which I can generalize as follows. Imagine that  $a$  cups of water fill  $\frac{m}{n}$  containers, or  $m$  copies of  $\frac{1}{n}$  of a container. Partition  $a$  cups into  $m$  pieces and then copy  $n$  of one of those amounts to reconstitute a whole container. This shows that  $a \div \frac{m}{n}$  is equivalent in meaning to  $\frac{m}{n} \times a$ .

In situation 1, I illustrated a certain way of thinking that I comment on now. More sophisticated partitive reasoning includes images of the effect on the quotient as either or both the dividend and divisor vary. For instance, if the number of groups (divisor) is doubled, while the total amount of stuff (dividend) remains constant, then the amount of stuff per whole group (quotient) should be halved. This leads to the following numerical generalization.

$$c \div (ka) = \frac{1}{k}(c \div a) \text{ where } k \in \mathbf{R}^+$$

However, if the total amount of stuff (dividend) is doubled, while the number of groups (divisor) remains constant, then the amount of stuff per whole group (quotient) is also doubled. This leads to the following numerical generalization.

$$(kc) \div a = k(c \div a) \text{ where } k \in \mathbf{R}^+$$

### **Quotitive Conceptualizations when Dividing by Fractions**

The *quotitive* (or *measuring*) model of division involves thinking about the number of equally-sized groups needed to constitute some total amount, given that the group size and total amount are known. Quotitive division has the general form:

$$\text{Total amount of stuff} \div \text{amount of stuff per group} = \text{number of groups}$$

A person is describing quotitive division when he or she describes the meaning of  $c \div b$  as “*how many times does b go into c*”, or something similar. Note that unlike partitive division, with quotitive division, the dividend and divisor involve similar kinds of measurements, with the distinction that the divisor’s measure corresponds to one group. As such, quotitive division is akin to asking to measure some total amount of stuff in units of *groups*, which is why quotitive division is also referred to as *measuring division*. Thus,  $c \div b$  could be interpreted as the measurement of  $c$  in units of size  $b$ .

As discussed in Chapter 2, the primitive model of repeated subtraction involves intuitive beliefs that the total amount of stuff should be more than the amount of stuff per group. Some primitive intuitions also include an expectation that the resulting number of groups should be a whole number. However, these primitive intuitions are not amenable to thinking about quotitive division with rational numbers. For someone with more

sophisticated meanings, the quotitive division model is viable no matter the types of values that are involved. For example, it is equally sensible to consider how many groups of 2 make six as it is to consider how many groups of 2 make  $1/3$ .

Several researchers and educators have shared scenarios that elicit quotitive division with non-whole divisors. For example, Gregg and Underwood Gregg (2007) talked about finding the number of servings in a collection of cookies. Also, Kribs-Zaleta (2008) shared four different contexts that elicit quotitive division with non-whole divisors, such as the number of posters that can be made from a number of sheets of paper. Several researchers have also alluded to the common denominator method as a way to reason through a quotitive division task (Flores, Turner, & Bachman, 2005; Gregg & Underwood Gregg, 2007; Sharp & Adams, 2002; Tirosh, 2000). To elaborate on this method and discuss other schemes relevant to quotitive division, I present two situations below, each involving two measures of length in yards, and share some ways that a person might think about them. I will also demonstrate how certain ways of thinking, other than the common denominator method, are conducive to the emergence of the invert-and-multiply algorithm.

**Situation 1:** How many copies of  $3/4$  of a yard make six yards? People could answer this by engaging in repeated subtraction and be relieved to discover that a whole number of copies answers the question. Or, they could get a common denominator (hence, a common unit-of-measurement) so that the two resulting measurements could be more easily compared. Thus, they seek to determine how many 3-fourths make 24-fourths? This is equivalent to asking how many groups of 3 make 24, which yields  $24 \div 3$ , or eight groups. Often, the use of fraction bars or some other visual can aid in

determining how many 3-fourths make 24-fourths. Another way to think about this situation is to consider that six yards represent a measurement of a length using a 1-yard unit-of-measure. Finding how many copies of  $\frac{3}{4}$  of a yard make six yards, is equivalent to thinking about measuring the 6-yard length, but in units of measure that are  $\frac{3}{4}$  of a yard long. A person may realize that there are  $\frac{4}{3}$  of  $\frac{3}{4}$  yard in each yard, thus there are  $6 \times (\frac{4}{3} \text{ of } \frac{3}{4} \text{ yard})$  in six yards. This last scheme establishes the numerical equivalence of  $6 \div \frac{3}{4}$  and  $6 \times \frac{4}{3}$ .

**Situation 2:** How many copies of  $\frac{7}{4}$  of a yard make six yards? I only present two solutions. First, someone could engage in repeated subtraction and get to 3 copies with  $\frac{3}{4}$  of a yard unmeasured. This unmeasured portion is  $\frac{3}{7}$  of a copy of  $\frac{7}{4}$  of a yard. Hence, the conclusion is that 3 and  $\frac{3}{7}$  copies of  $\frac{7}{4}$  of a yard make six yards. Another way to approach this problem is to think about measuring the 6-yard length, but in units of measure that are  $\frac{7}{4}$  of a yard long. There is  $\frac{4}{7}$  of  $\frac{7}{4}$  yards in each yard. Thus, there are  $6 \times (\frac{4}{7} \text{ of } \frac{7}{4} \text{ yards})$  in six yards. This last scheme efficiently establishes the numerical equivalence of  $6 \div \frac{7}{4}$  and  $6 \times \frac{4}{7}$ .

In each situation, six yards were considered, but for the schemes that were mentioned last in each situation, this amount of distance could be replaced by an arbitrary length, without affecting the scheme. As such, these examples illustrate quotitive justifications for the invert-and-multiply algorithm, which could be generalized as follows. Suppose something measures  $a$  yards in length. The measurement of this same length in units of  $\frac{m}{n}$  yards is represented by  $a \div \frac{m}{n}$ . One yard is  $\frac{n}{m}$  copies of  $\frac{m}{n}$  yards, so  $a$  yards is  $a \times (\frac{n}{m} \text{ copies of } \frac{m}{n} \text{ yards})$ . This establishes that  $a \div \frac{m}{n}$  is equivalent in meaning to  $a \times \frac{n}{m}$ .

Furthermore, more sophisticated quotitive reasoning includes images of the effect on the quotient as either or both the dividend and divisor vary. For instance, if the amount of stuff per group (divisor) is doubled, while the total amount of stuff (dividend) remains constant, then the number of groups (quotient) should be halved. This is akin to saying that if some quantity is measured in some unit, and then the magnitude of the unit-of-measure is doubled and the quantity re-measured, then the new measurement of the quantity should be half of the original measurement. This leads to the following numerical generalization.

$$c \div (kb) = \frac{1}{k}(c \div b) \text{ where } k \in \mathbf{R}^+$$

However, if the total amount of stuff (dividend) is doubled, while the amount of stuff per group (divisor) remains constant, then the number of groups (quotient) should also be doubled. This is akin to saying that if some quantity is measured in some unit, and then the magnitude of that quantity is doubled and re-measured, without changing the unit-of-measure, then the new measurement of the quantity should be double the original measure. This yields the following numerical generalization.

$$(kc) \div b = k(c \div b) \text{ where } k \in \mathbf{R}^+$$

### **Relationship of Conceptualizations of Division to Ratios, Rates, and Proportions**

As stated earlier, I adopt the meanings for the terms *rate* and *ratio* as articulated by Thompson (1994b) and Thompson and Thompson (1992). A *ratio* is the result of a multiplicative comparison between the magnitudes of two fixed quantities. It does not matter whether the quantities that are being compared are similar or not (e.g., both could be lengths or one could be a length and the other could be a weight). Also, a ratio can be

evaluated by dividing the values of the fixed quantities that are being compared. As such, I consider a ratio to the result of an abstraction of either partitive (such as when comparing dissimilar quantities) or quotitive meanings (such as when comparing similar quantities). This conception of ratio aligns with the notion of *quotient as relative size* as discussed by Thompson and his colleagues, who use the term *relative size* even when comparing dissimilar kinds of quantities (Byerley et al., 2012; Thompson, 1994a; Thompson et al., 2014). I present below some examples of the meanings for division that underlie conceptualizations of ratio, first in cases when the compared quantities are perceived to be similar, then when they are considered to be dissimilar.

First, consider the case of comparing two *similar* kinds of quantities. Now consider that these quantities are measured with a common unit. For example, suppose you have \$12 and I have \$3, I can imagine each of my dollars as an abstract group, and the ratio of 4 could be interpreted as you have \$4 for every \$1 that I have (or \$4 per group). In this case, the ratio is the result of an abstraction of partitive meanings. Alternatively, the quotient of 4 can be interpreted as the number of copies of my amount of money that would constitute your amount of money, were my money to be copied. As such, the ratio of 4 is the result of quotitive meanings. If the units of measure are different (e.g., \$12 compared to 3€), then partitive meanings establish a \$4 to 1€ proportional relationship. In cases of comparing two similar kinds of quantities using quotitive meanings, ratios can be construed as measurements of one quantity's magnitude in units of the other quantity's magnitude. Additionally, a ratio, in such cases, can be determined without knowing the values of the compared magnitudes. For example, it is possible to determine that one length is three times as large as another length, without needing to

first measure the two lengths in terms of some other unit. Steffe and his colleagues (McCloskey & Norton, 2009; Norton & McCloskey, 2008; Steffe & Olive, 2010) have described one scheme that is triggered when trying to determine the ratio that represents one unmeasured magnitude in terms of another unmeasured magnitude. They called it the *partitive fractional scheme*, and, in the case of lengths, it is comprised of the following: (1) partitioning the smaller length, (2) iterating one piece of the partition to try to match the larger length, and (3) repeating until a successful partition and iteration is discovered. This scheme enables someone to describe the smaller magnitude in terms of the larger magnitude, as measured by a proper fraction. I also contend that the scheme could be adapted and used to describe the reverse relative size – large in terms of small – as measured by the reciprocal improper fraction.

Now, consider the case of comparing two *dissimilar* kinds of quantities. For example, if \$12 is multiplicatively compared to 3 pounds, then each pound can be thought of as an abstract group and the quotient reveals the dollar amount per group, which is \$4 per pound. This is an abstraction of partitive meanings. Thus, no matter whether the ratio is the result of comparing two similar (same or different units of measure) or two dissimilar quantities, the underlying meanings are abstractions of either partitive or quotitive conceptualizations of division.

Partitive meanings are intertwined with meanings for rate (as an abstracted constant ratio) and proportion. To illustrate, first consider a generic case of partitive division. Suppose 12 units fit evenly into 3 groups. Then the quotient of 4 is interpreted as 4 units per 1 group. This quotient can be thought of as a ratio that remains constant (i.e., a rate) as the number of groups varies, thus establishing a proportional

correspondence between the number of groups and the total number of units contained in the collection of groups. If the number of groups were to be tripled to become nine groups, then the total number of units would need to triple to become 36 units, so that the quotient of 4 units per group remains constant. Now consider a classic situation involving distance and time. Suppose that an object that is moving uniformly takes 3 seconds to travel 12 yards. The ratio that results from comparing 12 yards to 3 seconds is 4, which is interpreted as 4 yards per 1 second. Imagining each second as a group which contains yards is an abstraction of the partitive conceptualization. We now imagine that this ratio remains constant as the object moves (i.e., we imagine the ratio as a rate), which is to say that any amount of distance traveled is proportional to the amount of time that has elapsed up to that point. In other words, in  $x \cdot 1$  seconds, the object will have moved  $x \cdot 4$  yards, for any  $x \in \mathbf{R}^+$ .

### **Conclusion**

In this chapter, I provided a conceptual analysis for the two fundamental conceptualizations for division. I consider a person to have robust meanings for division if that person...

- 1) Can operate meaningfully with both partitive and quotitive conceptualizations over the positive rational numbers.
- 2) Possesses an awareness of the distinctions between partitive and quotitive conceptualizations regardless of number type.
- 3) Can recognize and/or invent situations that elicit both partitive and quotitive conceptualizations involving any kind of positive rational values.

A person who can operate meaningfully with both conceptualizations, no matter the types of values involved, has a strong conceptual foundation for other mathematical concepts such as rates and proportions, and for numerical algorithms such as the invert-and-multiply algorithm. Unfortunately, as discussed in my literature review chapter, the absence of partitive conceptualizations for division over the rational numbers in the mathematics curriculum is acutely felt, which I find to be detrimental to the development of robust mathematical meanings for students. These observations form a powerful motivation for my study. In the next chapter, I describe the methodology I used to investigate the advancement of partitive meanings for division.

## CHAPTER 5

### METHODOLOGY

I approached my dissertation study having already been involved with three pilot studies related to division with non-whole, positive rational numbers. These pilot studies provided me with insights into some of the cognitive obstacles which can impede the development of more robust schemes. For the pilot studies, I employed a variety of methodologies, some of which I leveraged in this study. In the next section, I briefly describe these types of methodologies.

#### **Descriptions of Methodologies**

In this section I provide a brief description of the types of methodologies used in the three pilot studies and in my dissertation study. Based on a constructivist perspective, I characterize the learning of mathematics as the advancement of schemes through accommodation and reflective abstraction. Each person's collection of schemes is, perhaps, as unique to that person as his or her fingerprint. As such, to investigate the advancement of schemes, it is essential for researchers to utilize methods of data collection that furnish qualitative data which allow the researcher to model the subject's meanings. One such method is through conducting task-based, semi-structured clinical interviews between one researcher and one subject. Borrowing from Goldin (2000), an interview is (1) *semi-structured* if a general protocol is developed beforehand, but the researcher is free to adapt the protocol in the moment, in response to the subject's actions (e.g., utterances, gestures, inscriptions) and probe beyond the reach of the established protocol and (2) *task-based* if mathematical tasks serve as the catalyst for the collection of the qualitative data. As described by other researchers (Clement, 2000; Hunting, 1997;

Steffe & Thompson, 2000), an interview is *clinical* if the researcher does not try to influence the subject's thinking but instead only interacts with the subject to probe for greater insights into the meanings that were underlying the subject's actions. In contrast to clinical interviews, during a *teaching experiment*, as described by Steffe and Thompson (2000), the researcher attempts to influence a subject's meanings. The clinical interview focuses on what a subject knows in the moment, but the teaching experiment aims at understanding the cognitive development of the research subject.

Teaching experiments tend to be guided by already existing theories (a *convergent* study as described by Clement (2000)), yet they may also generate new hypotheses (a *generative* study as described by Clement (2000)). If there are no existing theories regarding cognitive development, then Steffe and Thompson (2000) suggest conducting an *exploratory teaching* episode, whereby a researcher interacts with a group of subjects to experiment with lesson trajectories or pedagogies. From this exploration, hypotheses can be formulated about learning which can inform the design of subsequent teaching experiments. Alternatively, this kind of exploratory teaching need not occur with a multitude of students at once. If the exploratory teaching involves only one subject then I borrow the term *exploratory teaching interview*, a type of interview described by Steffe and Thompson, but named by Moore (2010). The distinction between a teaching experiment and an exploratory teaching interview is that during an exploratory teaching interview, the researcher's primary focus is on generating hypotheses, and during a teaching experiment, the researcher's primary focus is on testing existing hypotheses.

## Summaries of Pilot Study Methodologies

Prior to my dissertation study, I was a primary contributor to the design, implementation, and analysis of three pilot studies that directly informed the methodology of my dissertation study. In the section below, I summarize the methodologies of these pilot studies. Later in this chapter, I discuss the findings from the pilot studies that influenced the design of my dissertation study.

### Pilot Study A: Methods Summary

Eight middle school mathematics teachers from four different schools within one district volunteered to participate in a two-year study. They were all members of one out of four cohorts of teachers who were participating in the *Arizona Mathematics Partnership* (AMP), an NSF-funded, six-year professional development and research project that was based in the Phoenix metropolitan area. All eight teachers participated in six semi-structured, task-based clinical interviews conducted by researchers on the AMP project. Each interview was videotaped and included 6-10 questions and mathematical tasks, with each interview lasting approximately 50-70 min. The six interviews were spread out evenly over the course of the teachers' two-year involvement with the AMP project. The purpose of the study was to create models of teachers' schemes as they struggled through tasks that focused primarily on topics related to the multiplicative conceptual field, such as problems involving fractions, multiplication, division, proportional reasoning, and multiplicative comparisons.

## **Pilot Study B: Methods Summary**

I designed an exploratory teaching episode that involved 13 pre-service teachers in an elementary education course at a community college. I taught a three-session unit on division that I designed to help the students (1) have two conceptualizations for division (partitive and quotitive), (2) connect meanings for division with meanings for multiplication, (3) be able to use partitive and quotitive meanings (instead of algorithms) to divide two non-whole, positive rational numbers, (4) construct contexts which elicited both partitive and quotitive meanings for division with non-whole, positive rational numbers, and (5) make sense of the invert-and-multiply algorithm based on a partitive conceptualization for division. The three-session unit was preceded and interspersed with four task-based assignments to assess the cognitive progression of the students throughout the study. The three-session unit was also preceded by a session devoted to a review of fractions. Below is a summary of the timeline for the study.

- Students complete assignment 1 (a pre-assessment of students' meanings).
- Pre-session: Review of fractions; particularly fractions as multiplicative operators.
- Session 1: Discuss meanings for multiplication and division and discuss some problems from assignment 1.
- Students complete assignment 2.
- Session 2: Review meanings for multiplication and division and discuss some problems from assignments 1 and 2.
- Students complete assignment 3.
- Session 3: Review meanings for multiplication and division, draw attention to the connection between the invert-and-multiply algorithm and partitive division, and discuss some problems from assignments 1 through 3.
- Students complete assignment 4 (a post-assessment of students' meanings).

Of the 13 students in the course, I selected two of them, one low-performing and one high-performing, to do their four task-based assignments in a clinical and semi-structured interview setting. Concerning class time, I used some time to present material in a lecture format with whole class interaction and discussion. These lectures included dynamic visual models using Power Point of both partitive and quotitive conceptualizations of division with fractions. The basic trajectory of the unit progressed from specific to abstract, and from whole to non-whole values. Also, a significant portion of class time was devoted to small group discussion to allow students opportunities to solve problems at the white boards. Students showcased their solution strategies and critiqued the reasoning of their classmates.

### **Pilot Study C: Methods Summary**

For this study, I selected one middle school mathematics teacher who had completed the two-year AMP program. I met with this teacher on four separate occasions to gather data, each meeting lasting approximately 90-120 minutes. The first meeting consisted of a semi-structured, task-based clinical interview. The remaining three meetings were task-based exploratory teaching interviews, where we discussed the teacher's prior work and assessed her understanding with new tasks. The purpose of the study was to help me identify the subject's ways of operating, along with their affordances and limitations, when trying to resolve tasks that elicit partitive meanings for division. Additionally, I wanted to study how my deliberate interventions could help promote the teacher's development of more productive ways of operating. To better study the *advancement* of meanings, I selected a teacher with weaker mathematical meanings and who tended to operate numerically with decimal representations when presented with

non-whole values. I retained flexibility by planning only one interview in advance, so I could analyze the data after each interview in preparation for the next interview. I selected only tasks that involved trying to answer the question “so many copies of what amount make some other amount?” I framed these tasks in the context of attempting to identify the amount of water in one whole container given that a certain amount of water filled a certain number of containers. The tasks varied in complexity, which tended from whole to non-whole values, and from specified to unspecified values. They also included numbers of containers that were both less than and greater than one. For this study, I encouraged the teacher to use fraction circle manipulatives to coordinate the amount of water and the corresponding number of containers.

### **Methodology of Dissertation Study**

Having briefly discussed some general methods of qualitative data collection, as well as the methodologies specific to my three pilot studies, I move on to discuss the methodology of this dissertation study. I begin by restating my primary research questions and formulating secondary research questions that guided the methods and analysis of the data.

### **Primary and Secondary Research Questions**

The purpose of this dissertation study was to investigate the *advancement* of teachers’ partitive conceptualizations of division to accommodate fractional values. As established in my literature review chapter, very little insight is available about the *development* of partitive schemes. Additionally, partitive division over the positive rational numbers is underemphasized in curricula and in national standards for teaching

mathematics, even though there are many realistic scenarios where partitive division with a non-whole number of groups is required (e.g., spreading soil over 2.3 acres).

Additionally, I have established in my conceptual analysis chapter that partitive meanings are foundational to meanings for rates and proportional correspondence, and they form a conceptual foundation for the invert-and-multiply algorithm. For these many reasons, my dissertation study focused on the following primary research questions.

**Primary Research Question 1 (RQ1):**

What meanings, with their affordances and limitations, do in-service middle school mathematics teachers possess relative to partitive conceptualizations of division with non-whole divisors?

**Primary Research Question 2 (RQ2):**

How do these teachers' meanings change as a consequence of an instructional sequence that emphasized quantitative reasoning to aid in the advancement of these meanings?

Partitive schemes are but a cog in the grander machinery of multiplicative reasoning. The development of productive meanings for division with fractional values is dependent on a mature network of mathematical meanings, including, but not limited to, partitioning, iterating, measurement, fractions, multiplication, relative size, and proportionality. As part of my study, I needed to gain insight into teachers' meanings regarding several of these mathematical concepts.

Since my study involved an intervention, I needed to conduct both a pre- and post-assessment to furnish data that I could compare to look for advancements in the teachers' meanings. This necessitated three phases of my study. The pre-assessment

(Phase 1) allowed me to investigate and describe the affordances, limitations, and generalizability of the teachers' schemes related to partitive division. This data informed my intervention (Phase 2), after which a post-assessment (Phase 3) allowed me to discuss advancements and obstacles to advancement. I was also interested in observing whether the advancement of partitive schemes would naturally lead to a meaningful justification of the invert-and-multiply algorithm. For each phase of the study, I formulated secondary research questions that served to answer my primary research questions, which are found in Table 3 below.

Table 3  
*Secondary Research Questions*

|         |       |                                                                                                                                                                                                              |
|---------|-------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Phase 1 | RQ1.1 | What meanings do teachers reveal when they engage in tasks that I designed to elicit meanings for fractions as measures of relative size, with a focus on fractions as reciprocal measures of relative size? |
|         | RQ1.2 | What meanings do teachers reveal when they describe and model symbolic (decontextualized) statements of multiplication, both general and specific, with a focus on fractional multipliers?                   |
|         | RQ1.3 | What meanings do teachers reveal when they describe and model symbolic (decontextualized) statements of division, both general and specific, with a focus on fractional divisors?                            |
|         | RQ1.4 | What meanings do teachers reveal when they engage in tasks that I designed to elicit partitive conceptualizations of division, with varying degrees of abstraction, and with a focus on fractional divisors? |
|         | RQ1.5 | What justifications do teachers provide for the invert-and-multiply algorithm after working through the tasks mentioned in the previous research question?                                                   |
| Phase 2 | RQ1.6 | What cognitive obstacles do teachers further reveal as I actively attempt to promote the development of their meanings that are foundational to partitive division over the rational numbers?                |
| Phase 3 | RQ2.1 | How do the teachers' post-intervention meanings compare to their pre-intervention meanings?                                                                                                                  |
|         | RQ2.2 | What advancements to the teachers' schemes are evident and what challenges remain?                                                                                                                           |

For Phase 1, I conducted two task-based, semi-structured clinical interviews with six practicing middle school mathematics teachers. For Phase 2, I selected two of the six teachers and conducted four teaching experiment interviews. For Phase 3, I conducted one task-based, semi-structured clinical interview, with each of the two teachers from Phase 2, to assess the impact of my intervention. For all meetings in all phases, I planned on each interview to last about 90 minutes, and each meeting was with one teacher at a time. I summarize the structure of this study in Figure 9.

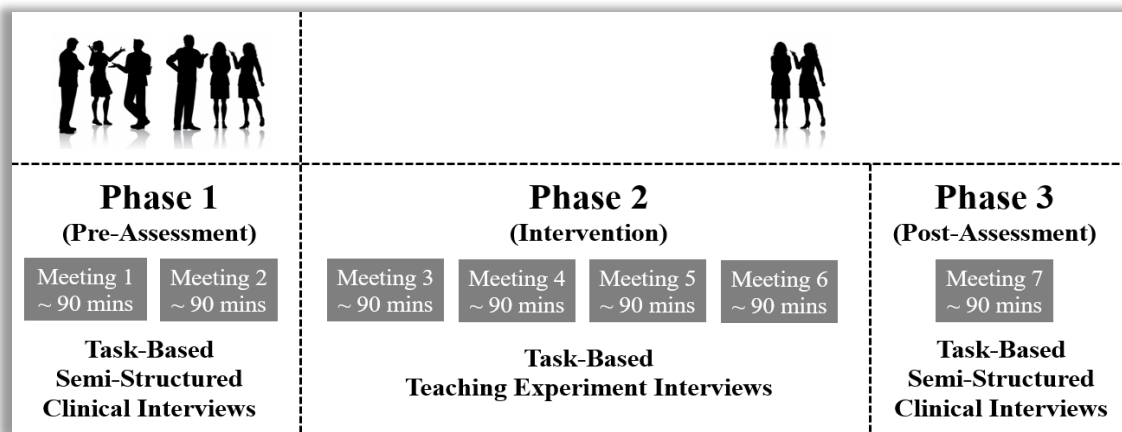


Figure 9. Summary of the three phases of this dissertation study.

## Research Subjects

I selected the subjects for this dissertation study from among a pool of middle school mathematics teachers who were former participants in the AMP project. These teachers had various amounts of experience teaching Kindergarten through 8<sup>th</sup> grade. During their involvement with the AMP project, each teacher attended 18 full-day workshops and participated in at least 16 after school professional learning community meetings, facilitated by a representative of the AMP project. One purpose of the AMP project was to help the teachers gain a deeper understanding of the *Common Core State Standards* for mathematical practice (National Governors Association, 2010), which

Arizona calls the *Arizona Mathematics Standards*. Another purpose was to help teachers develop their content knowledge related to numbers and operations, problem solving, statistics, and functions. Concerning schemes for division by fractional values, the teachers were exposed exclusively to quotitive conceptualizations during the project. However, it is not known the extent to which the teachers' participation in the AMP project impacted their schemes for division with fractions. I considered this lack of knowledge on my part to be insignificant because Phase 1 of this study would be a pre-assessment of their schemes going into the study. Following their participation in the AMP project, the pool of teachers was administered the *Learning Mathematics for Teaching* (LMT) post-test, which assessed their mathematical knowledge for teaching (MKT) with regards to Number Sense and Operations (NCOP), as well as other strands of mathematical content. The data for NCOP were analyzed using item response theory to classify the teachers as high, mid, or low. Scores less than negative one were categorized as *low*. Scores between negative one and positive one (inclusive) were categorized as *mid*. Scores above one were categorized as *high*.

For Phase 1, I selected six teachers; one who had performed in the low-level for numbers and operations on the LMT post-test, three teachers from the mid-level, and two teachers from the high-level. I did this to ensure that I had teachers with various capabilities regarding numbers and operations. The aliases I adopted begin with the same letter as the teacher's placement according to the LMT post-test – Linda was from the low-level; Mark, Mindi, and Mel were from the mid-level; and Uma and Ursa were from the upper(high)-level. For Phases 2 and 3, I narrowed my subjects down to Linda and Uma (the two highlighted rows) Phases 2 and 3, for reasons I describe at the end of

Chapter 6. The AMP project compensated the six teachers for any time they devoted to this study. The subjects, with their NCOP score level, teaching experience, and participation in this study are listed in Table 4.

Table 4  
*Research Subjects*

| Alias | NCOP Level   | Years of Teaching Experience |    |     | Current Grade | Phase 1 | Phase 2 | Phase 3 |
|-------|--------------|------------------------------|----|-----|---------------|---------|---------|---------|
|       |              | K-5                          | 6  | 7-8 |               |         |         |         |
| Linda | low          | 9                            | 5  | 4   | 6, 7, 8       | ✓       | ✓       | ✓       |
| Mark  | mid          |                              |    | 6   | 7             | ✓       |         |         |
| Mindi | mid          | 17                           | 3  |     | 6             | ✓       |         |         |
| Mel   | mid          | 6                            | 5  | 2   | 7             | ✓       |         |         |
| Uma   | upper (high) | 5                            | 10 |     | coach         | ✓       | ✓       | ✓       |
| Ursa  | upper (high) | 4                            |    | 8   | 7             | ✓       |         |         |

## Methods of Analysis

The nature of my research questions required me to conduct a qualitative study as described by Corbin and Strauss (2008). My primary data were video recordings of the teachers' hand gestures, utterances, and inscriptions as they worked through various tasks and activities. I scanned all written work and uploaded all video data and scanned artifacts to a secure server for storage and future analysis. Once the qualitative data was gathered, I analyzed it using open and axial coding in order to generate grounded theory, as described by Corbin and Strauss (2008). These authors described *coding* as “deriving and developing concepts from data” (p.66), and *grounded theory* as theory that “is derived from qualitative analysis of data” (p.1). They described *open coding* as maintaining an open mind so that analysts can “open up the data to all potentials and possibilities contained within them” (p.159), and *axial coding* as “relating concepts to

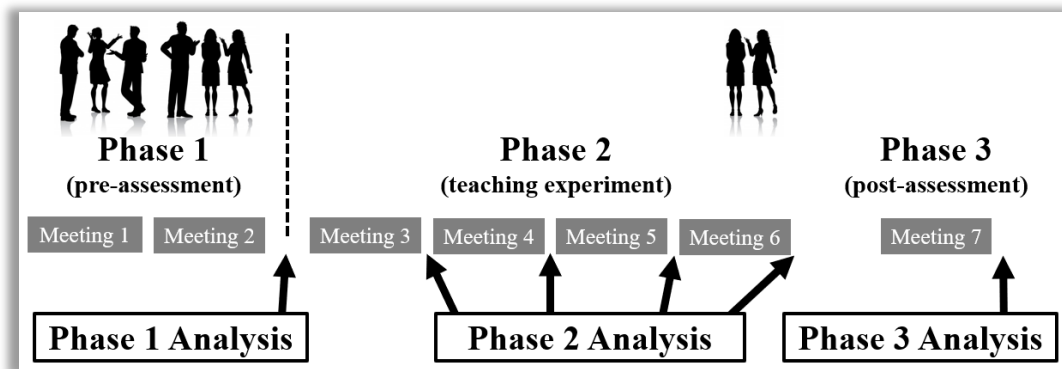
each other” (p.195). My analysis involved open coding as I tried to generate any constructs and theories that helped me answer my research questions, and it involved axial coding to the extent that (1) my analysis involving comparing data from Phases 1 and 3, and (2) my analysis was informed by findings and theories from my pilot studies. In particular, my research questions required me to perform a scheme-based analysis of the data, which is more challenging than a behavior-based analysis. Thompson and Saldanha (2003) commented on this complication.

“There is a practical drawback to scheme-based characterizations of learning objectives. Assessment of whether students have achieved a learning objective is more complicated when expressing it in terms of schemes of conceptual operations than when expressing it in terms of behavioral skills. When learning objectives are stated in terms of skills, determining whether a student has achieved them is straightforward. When learning objectives are stated as schemes of operations, students’ behavior must be interpreted to decide whether it reflects reasoning that is consistent with the objectives’ achievement. This complication is unavoidable” (p.37).

As part of my analysis of the teachers’ schemes, I was interested in whether their schemes were productive, generalizable, limiting, flexible, and connected. At the same time, I maintained a generative position, meaning that I was open to generating models, constructs, and frameworks that could inform future research, curriculum development, and educational policies. I intended to make models of the teachers’ schemes that explained the teachers’ observable actions regarding fractions, multiplication, division, relative size, partitioning, iterating, etc. These models would need to explain why the teachers could do what they could, why they were unable to do what they could not, and

the extent to which they could generalize their thinking. Furthermore, I needed to model any changes to their schemes that resulted from the teaching experiment.

Throughout the study, it was necessary to analyze the data at different junctions, as depicted in Figure 10. The first analysis came at the end of Phase 1, the pre-assessment. I needed to analyze the data from Phase 1 before beginning Phase 2 for two practical reasons. First, I used the data from Phase 1 to narrow the group of subjects from six teachers to two teachers based on which teachers needed development as supported by the data, but being careful to select teachers for whom it would be possible to benefit. I say more about who I selected, and why, following my discussion of the data from Phase 1. Second, the data from Phase 1 informed my pedagogical decisions regarding the impending teaching experiment in Phase 2 for these two teachers.



*Figure 10. Timing of my analyses of the data.*

Throughout the four-session teaching experiment, I needed to continually monitor the two teachers' progress, and wherever possible, draw them back to my learning goals. This required me to analyze data between each session of the four-session teaching experiment, so that I could make any necessary adjustments. At the end of Phase 2, I analyzed the data from the teaching experiment so I could properly adjust, if necessary,

the upcoming post-assessment in Phase 3. At the end of Phase 3, I performed a comparative analysis of the data from the pre- and post-assessments.

### **Phase 1 Methods: Pre-Assessment**

Before actively attempting to advance the teachers' schemes, I needed to conduct a pre-intervention investigation. I conducted two task-based, semi-structured clinical interviews, with each of the six teachers. For this phase of my study, I designed 25 tasks that adhere to the following guidelines.

1. The tasks primarily involve partitive division. I also included tasks to investigate the teacher's meanings regarding fractions, and reciprocal relative size, multiplication involving fractions, and division in general.
2. Most of the tasks are contextually grounded, typically using volume of liquid, number of equally-sized containers, and capacity of each whole container as the relevant quantities. I chose these quantities because they can comfortably be imagined with non-whole values. However, some tasks are not presented in a context, for reasons I discuss later in this chapter.
3. The tasks involve values of quantities that challenge primitive intuitive rules of partitive division. Specifically, I included situations with non-whole divisors which are, in some cases, greater than one, and in other cases less than one.
4. The tasks are generally devoid of words that trigger the numerical operation of division. Specifically, I generally avoid using the words *divide*, *partition*, and *split*. I am interested in teachers' meanings for partitive conceptualizations, and I'd like to minimize, if possible, any interference from existing meanings related to numerical division.

5. The tasks vary in level of generality. For some tasks, I am explicit about the values of the relevant quantities, and for other tasks I am not. This is done to gain insight into the limitations and generalizability of the teachers' schemes, and to test the teachers' ability to engage in quantitative reasoning.
6. Most of the tasks involve non-whole values in some way. At least one of the dividend, divisor, and quotient will have a non-whole value. For some tasks, the divisor will be whole and for others it will not. This will give me a basis for ascertaining the extent to which the type of numerical value for the divisor can affect a teacher's partitive schemes.

I administered the 25 tasks over the course of two separate meetings with each of the six teachers. For each task, I asked teachers to verbalize their meanings. Since I was interested in getting data on the teachers' quantitative reasoning, I also asked them to model their meanings with a picture. Thompson and Saldanha (2003) said "part of building conceptual operations is the attempt to express them in symbols and diagrams. Symbolic operations can become the focus of instruction once students have developed coherent and stable meanings that they may express symbolically" (p.37). During the first meeting, I presented each teacher with the first 13 tasks, and the rest of the tasks during the second meeting. The second meeting occurred no more than one week following the first meeting, so that the effects of elapsed time were minimal. A summary of the tasks is found in Table 5. In the section that follows the table, I give my reasons for including them and ways that I anticipated a teacher might reason through them and overcome certain cognitive perturbations. I also share how some findings from the pilot studies have influenced the design of some of these tasks.

Table 5  
Phase 1 Tasks

|           |                                           |    |                                                                                                                                                                                                                        |                                             |
|-----------|-------------------------------------------|----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------|
| Meeting 1 | Task Set 1<br>Fractions                   | 1  | Explain your meanings for the expression: $\frac{a}{b}$                                                                                                                                                                |                                             |
|           |                                           | 2  | The given line has a length that is $\frac{8}{5}$ of a unit. Draw a line that is one unit long.                                                                                                                        |                                             |
|           |                                           | 3  | Given these two strips of paper, how long is the red strip compared to the green strip? How long is the green strip compared to the red strip? (The red strip is 7 inches long, and the green strip is 4 inches long.) |                                             |
|           | Task Set 2<br>Multiplication              | 4  | Explain your meanings for the expression: $a \times b$                                                                                                                                                                 |                                             |
|           |                                           | 5  | Explain your meanings for the expression: $5 \times \frac{4}{3}$                                                                                                                                                       |                                             |
|           |                                           | 6  | Explain your meanings for the expression: $\frac{5}{3} \times 2$                                                                                                                                                       |                                             |
|           |                                           | 7  | Explain your meanings for the expression: $\frac{5}{3} \times 2$                                                                                                                                                       |                                             |
|           | Task Set 3<br>Division                    | 8  | Explain your meanings for the expression: $a \div b$                                                                                                                                                                   |                                             |
|           |                                           | 9  | Explain your meanings for the expression: $4 \div 3$                                                                                                                                                                   |                                             |
|           |                                           | 10 | Explain your meanings for the expression: $4 \div \frac{1}{3}$                                                                                                                                                         |                                             |
|           |                                           | 11 | Explain your meanings for the expression: $\frac{1}{2} \div \frac{3}{4}$                                                                                                                                               |                                             |
|           | Task Set 4<br>Missing<br>Multiplicand     | 12 | $\frac{10}{3}$ copies of what amount combine to make the amount 15?                                                                                                                                                    |                                             |
|           |                                           | 13 | 6 copies of what amount combine to make the amount 15?                                                                                                                                                                 |                                             |
| Meeting 2 | Task Set 5<br>Fair-Sharing                | 14 | How much is in one group if 15 of something is split into $\frac{10}{3}$ groups?                                                                                                                                       |                                             |
|           |                                           | 15 | How much is in one group if 15 of something is split into 6 groups?                                                                                                                                                    |                                             |
|           | Task Set 6<br>Whole<br>Divisor            | 16 | Suppose 13 gallons of water fill 5 equal containers. Describe the capacity of one whole container.                                                                                                                     |                                             |
|           |                                           | 17 | Suppose $\frac{3}{4}$ gallon of water fills 5 identical containers. Describe the capacity of one whole container.                                                                                                      |                                             |
|           |                                           | 18 | Suppose a certain amount of water fills 5 identical containers. Describe the capacity of one whole container.                                                                                                          |                                             |
|           | Task Set 7<br>Non-whole<br>Divisor<br>> 1 | 19 | Suppose 27 gallons of water fill $\frac{9}{4}$ identical containers. Describe the capacity of one whole container.                                                                                                     |                                             |
|           |                                           | 20 | Suppose 3 gallons of water fill $\frac{9}{4}$ identical containers. Describe the capacity of one whole container.                                                                                                      |                                             |
|           |                                           | 21 | Suppose a certain amount of water fills $\frac{9}{4}$ identical containers. Describe the capacity of one whole container.                                                                                              |                                             |
|           | Task Set 8<br>Non-whole<br>Divisor<br>< 1 | 22 | Suppose 5 gallons of water fill $\frac{2}{3}$ of a container. Describe the capacity of one whole container.                                                                                                            |                                             |
|           |                                           | 23 | Suppose $\frac{7}{4}$ gallons of water fill $\frac{2}{3}$ of a container. Describe the capacity of one whole container.                                                                                                |                                             |
|           |                                           | 24 | Suppose a certain amount of water fills $\frac{2}{3}$ of a container. Describe the capacity of one whole container.                                                                                                    |                                             |
|           | Final Task                                | 25 | Explain why it is that when you divide by a fraction, you can multiply by the reciprocal of the fraction instead. In other words, explain the following:                                                               | $a \div \frac{b}{c} = a \times \frac{c}{b}$ |

**Description of tasks.** In this section, I describe the Phase 1 tasks, give my reasons for including them, discuss ways that I anticipated a teacher might reason through them. I also share how some findings from the pilot studies have influenced the design of some of these tasks.

***Task Set 1: Fractions (Tasks 1-3).*** I designed the first set of tasks to help me model each teacher's meanings triggered by the fraction symbol, as well as meanings for fractions as operators and fractions as ratios. Much of what I expected teachers to do for the first two tasks is based on my findings from pilot studies A and B. However, the third task was not used in any of the three pilot studies. These tasks are as follows.

**Task 1:** Explain your meanings for the expression:  $\frac{a}{b}$

Task 1 helped me to gather data regarding each teacher's meanings associated with the fraction symbol. For this task, I told the teachers to limit the numerator and denominator to representing positive integers. I hoped that teachers would talk about their meanings in general terms, but I suspected that many teachers would want to use specific fractions so that they could discuss their meanings more concretely, such as with visual aids. For teachers that preferred to talk about specific fractions, I asked them to make drawings that represented their meanings. I made sure that teachers considered both proper and improper fractions in their descriptions. I asked teachers to imagine a context where they would need to think about a fraction. I also encouraged each teacher to give as many meanings as they could think of in the moment, while realizing that they likely would have a broader web of meanings than they could conjure in the moment, perhaps even some meanings they were not aware of. I suspected that all teachers would describe either part-to-whole meanings, fractions as commands to divide using partitive

conceptualizations, or fractions as the result of partitive division. I did not expect teachers to give quotitive interpretations, given that the denominator for this task represents a whole number which tends to elicit partitive meanings. I also anticipated that some teachers would use the term “ratio” and describe a fraction as a ratio to the extent that the fraction symbol represents the two values being compared, and not the single value that is the result of that comparison. Given the teachers’ involvement in the AMP project, I also expected teachers to interpret the fraction as “ $a$  copies of  $1/b$ .”

**Task 2:** The given line has a length that is  $8/5$  of a unit. Draw a line that is one unit long.

I designed Task 2 to specifically explore each teacher’s meanings for fractions as a reversible operator. The task involves an improper fraction to perturb part-to-whole meanings. Based on findings from Pilot Study A, I suspected that teachers would succeed in the task by partitioning the line into nine equal pieces and keeping only 5 of them. I planned to determine whether each teacher could construe a piece as both one fifth and one eighth, by having them explain the referents for these unit fractions.

**Task 3:** Given these two strips of paper, how long is the red strip compared to the green strip? How long is the green strip compared to the red strip?



7 inches long



4 inches long

I designed Task 3 to specifically explore each teacher’s meanings for fractions as reciprocal measures of relative size. Again, I emphasize that I did not use this task in any of my prior pilot studies. The green strip was 4 inches in length and the red strip was

seven inches in length. However, these measurements were not revealed to the teachers, nor did I immediately allow supplemental measuring tools to be used, because I was curious what strategies the teachers would use without them, such as folding the strips. I suspected that teachers would easily recognize that the green strip has more than half the length of the red strip, and that the length of the red strip is not quite twice the length of the green strip. I encouraged teachers to be as precise as possible with their comparisons. I expected that most teachers would use a *partitive fractional scheme*, which I described earlier in this proposal. For those teachers who struggled with determining the exact measures of relative size, I eventually allowed the use of a ruler. I suspected that some teachers would give additive comparisons by focusing on the difference in lengths. For these teachers, I prompted them to think multiplicatively by asking how many times as large is one strip as the other. Once a teacher gave one relationship (e.g., the green strip in terms of the red strip), I immediately asked about the reverse relationship, because I wanted to determine how naturally a teacher recognized that if the green strip is  $\frac{4}{7}$  of the length of the red strip, then the red strip is  $\frac{7}{4}$  of the length of the green strip. I also prompted the teachers to consider the unit-of-measure for each of these two values, to determine how aware each teacher is of the inconsistency of the referents of the inverse fractions.

***Task Set 2: Multiplication (Tasks 4-7).*** I designed the next set of tasks to explore each teacher's meanings for multiplication, specific to non-whole multipliers and multiplicands. These tasks are not placed in a quantitative context because I wanted to determine how the teachers contextualize the operations themselves, which can reveal the

limitations of their ways of thinking. Much of what I expected teachers to do for these tasks was based on my findings from Pilot Study A. The tasks are as follows.

**Task 4:** Explain your meanings for the expression:  $a \times b$

I used Task 4 to begin to explore each teacher's meanings for multiplication in general. I expected that some teachers would use the term *groups* in their language and that some would use the term *copies*, a term that workshop facilitators emphasized in the AMP project. I hoped that teachers would speak in general terms, but I believed that many teachers would use specific examples to explain their meanings, and that these specific examples would involve only positive integers for both the multiplier and multiplicand. I anticipated that many teachers would give general descriptions of multiplication that would *not* be conducive to non-whole multipliers. I suspected that all teachers would use some combination of array models, area models, equal groups models, and repeated addition models to explain multiplication. I prompted teachers to represent their meanings visually. I believed that some teachers would talk about commutativity, and that for some teachers the second number in the expression would be the multiplier.

**Task 5:** Explain your meanings for the expression:  $5 \times \frac{4}{3}$

**Task 6:** Explain your meanings for the expression:  $\frac{5}{3} \times 2$

I designed Task 5 and Task 6 to investigate issues caused by non-whole multipliers. I thought both tasks would be necessary since for some people the first number is the multiplier and for others, the second number is the multiplier. Thus, I anticipated that each teacher would be perturbed by at least one of the tasks. I suspected that some teachers would accommodate by invoking commutativity and using the whole value as

the multiplier regardless of its position, so that the equal groups model or the repeated addition model could be leveraged. However, for each of these tasks, I prompted each teacher to consider the first value as the multiplier. For Task 6, I thought some teachers would accommodate by abandoning terms such as *groups* and *repeated addition* in favor of using partitioning and iterating schemes that are part of their meanings for fraction as an operator. In Pilot Study A, I observed that for some teachers, it made no sense to talk about non-whole groups or to add something to itself repeatedly a non-whole number of times. For those teachers and in these cases, multiplication became nonsensical, something that they could do procedurally, but that they could not talk about meaningfully.

**Task 7:** Explain your meanings for the expression:  $\frac{2}{5} \times \frac{4}{3}$

I designed Task 7 to force teachers to work with a fractional multiplier and a fractional multiplicand, so that I could determine which meaningful models would emerge to make sense of the product. Based on findings from Pilot Study A, I anticipated that some teachers would have no meaning for this product and would be limited to procedural schemes only. I suspected that some teachers would use an area model or begin by finding and then combining two fifths of each of the four thirds.

**Task Set 3: Division (Tasks 8-11).** I designed the next set of tasks to explore each teacher's meanings for division, with a focus on non-whole dividends, divisors, and quotients. Since there are multiple conceptualizations of division, I planned to ask each teacher to discuss as many ways as possible to think about each task. Like the multiplication tasks in Task Set 2, these division tasks are not placed in a quantitative context because I wanted to determine how the teachers would contextualize the

operations themselves, which can reveal the limitations of their ways of thinking. Much of what I suspected teachers to do for this set of tasks is based on my findings from Pilot Study A, which showed that each of the eight teachers from the pilot study had both partitive and quotitive meanings, but that the meanings were not necessarily distinct in their awareness, let alone robust enough to be used interchangeably. For some of these eight teachers, the number type of the denominator dictated the triggered schemes. Some teachers were stumped for long periods of time, as they seemed to be trying to hold in mind both conceptions simultaneously. None of the teachers in Pilot Study A demonstrated that they could think flexibly between partitive and quotitive conceptualizations when non-whole divisors were present.

**Task 8:** Explain your meanings for the expression:  $a \div b$

I used Task 8 to begin to explore each teacher's meanings for division in general. I expected that most teachers would give partitive descriptions of division that are dependent on whole values for the divisor, and that some would give quotitive descriptions, but that few, if any, would give both. I also suspected that teachers would conjure specific examples of division to explain their thinking, but limited to whole dividends, divisors, and quotients. For such teachers, I planned to ask for a visual representation of their thinking so I could better ascertain whether they were relying on partitive or quotitive conceptualizations.

**Task 9:** Explain your meanings for the expression:  $4 \div 3$

For Task 9, I chose values for the dividend and divisor that are mostly conducive to primitive models for both partitive and quotitive division, with the added complication of a non-whole quotient, which violates one primitive intuition regarding quotitive division.

I suspected that all teachers would use fair-sharing models by either putting one whole in each of the three groups and then splitting the fourth whole into three to share among the groups, or by splitting each whole into thirds first and then putting a third from each whole into each group. I expected that some teachers would reveal quotitive meanings in addition to partitive meanings. In Pilot Study A, I observed that it was non-trivial for teachers to think about this task both partitively and quotitively. Of the eight teachers in that pilot study, only three attempted to give a quotitive interpretation in addition to the partitive interpretation. However, two of these three were unable to resolve the confusion that resulted when they drew a partitive representation but uttered a quotitive description.

**Task 10:** Explain your meanings for the expression:  $4 \div \frac{1}{3}$

With Task 10, I introduced a fractional divisor to perturb primitive partitive meanings. I did *not* expect any teacher to think about this partitively by wondering “one third of what amount makes the amount 4?” When appropriate, I challenged each teacher to rethink about both Tasks 9 and 10 using all the meanings they may have demonstrated during either of these two tasks the first time through. For example, if they gave a partitive explanation for Task 9, then I asked them to describe Task 10 using the same terms. This would help me determine any limitations in their partitive and/or quotitive thinking and language. During Pilot Study A, none of the eight subjects thought about this task partitively – they all used quotitive meanings. This was not surprising since quotitive meanings were highly emphasized during their involvement with the AMP project. During this pilot study, I observed one teacher become aware that they thought differently with this task than with the preceding task, but without being able to explain why the change happened. I challenged two other teachers in the pilot study to think

about this task in the same way as the preceding task by asking “what does it mean to you to split four into one-third groups?” One teacher said this made no sense, whereas the other reinterpreted what I said, but with quotitive language, by restating what I said as “split four into groups of one-third.” Another teacher seemed to be more aware of the distinction between the two conceptualizations of division by stating that for “ $4 \div 3$ ” the goal was to find “how much is in each part,” and for “ $4 \div 1/3$ ” the goal was to find “how many parts.” However, for this teacher, it seemed that the number type of the divisor dictated his way of thinking.

**Task 11:** Explain your meanings for the expression:  $\frac{1}{2} \div \frac{3}{4}$

I designed Task 11 to lend itself to quotitive interpretations by again choosing a non-whole divisor. I added to the complexity by choosing a non-whole dividend which will yield a non-whole quotient that is less than one. However, I selected the denominators of the fractions being divided to minimize the amount of partitioning required to quotitively arrive at the quotient – only the dividend needs to be reimaged in terms of fourths. I did not expect any teacher to approach this task with partitive meanings by asking the question “three-fourths of a copy of what amount makes the amount one-half?” Instead, I suspected that many teachers would try to model this quotitively, but that some would only be able to think about this task procedurally by invoking the invert-and-multiply algorithm. In Pilot Study A, I again observed conflation of the two conceptualizations of division when teachers attempted similar tasks, but with different fractions<sup>8</sup>. To my

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<sup>8</sup> In Pilot Study A, teachers attempted to describe the meaning of both  $\frac{4/3}{2/5}$  and  $\frac{1}{5} \div \frac{4}{3}$ . For the current study, I combined these into one task with more compatible denominators.

surprise, one teacher attempted a partitive approach by declaring that “division means splitting up” and attempting to distribute four-thirds of a cake to two-fifths of her class (she was thinking about  $\frac{4/3}{2/5}$ ). This was a valid partitive scenario, but she got lost in the goal when she said she was trying to find how much cake each student should get, as opposed to how much cake would be needed for the whole class. Several teachers from Pilot Study A attempted to quotitively model the task using strip diagrams but lost track of the quotitive requirement that the dividend and divisor must refer to the same whole. Many of the teachers from Pilot Study A simply did the invert-and-multiply algorithm and then gave a quotitive interpretation of the result.

***Task Set 4: Missing multiplicand meanings (Tasks 12-13).*** I used the next two sets of tasks to explore each teacher’s schemes that may emerge when I describe partitive division, but in two different ways – missing multiplicand language in Task Set 4 and fair-sharing language in Task Set 5. I designed the tasks based on my findings from Pilot Study B, in which each subject was asked “How much water is in one whole container when 10 cups of water are divided into five-thirds containers?” and “five-thirds copies of what amount of water combine to make 10 cups of water?” For the pilot study, these questions were placed one after the other, which may have caused some bias. Despite this, 11 of the 13 subjects did not show identical work in resolving the two tasks, and seven of the subjects answered only one of the two questions correctly. At least one of the subjects who answered the second question correctly, did so by recognizing it as a missing factor problem and by numerically calculating the answer of 6, with no unit. This student justified the numerical answer by showing that six copies of five-thirds combine to make ten, which is a justification of quotitive, not partitive, thinking. For my

dissertation study, I designed four tasks that are a refinement and extension of the tasks I used in Pilot Study B. To minimize bias, I separated these four tasks, two at the end of the first meeting, and two to begin the second meeting. I phrased the two tasks at the end of Meeting 1 in terms of finding a missing multiplicand. These two tasks are as follows.

**Task 12:**  $\frac{10}{3}$  copies of what amount combine to make the amount 15?

**Task 13:** 6 copies of what amount combine to make the amount 15?

In Task 12, I phrase the problem as a missing multiplicand task, but with a non-whole multiplier. In Task 13, I do the same, but with a whole multiplier. I presented these two tasks in this order because the non-whole divisor is less likely to trigger the numerical operation of division, which I wanted to avoid. Instead, I wanted each teacher to solve these tasks without relying on procedures connected to numerical division. If teachers began by doing numerical division, I intended to ask them to refrain and to reason through the task instead. I also asked the teachers to make a diagram representing their thought process, so that I could better assess their ways of operating. The completion of these tasks marked the end of the first meeting.

**Task Set 5: Fair-sharing meanings (Tasks 14-15).** The second meeting with each teacher began with this task set. These two tasks are a continuation of the two tasks that I placed at the end of the first meeting. I split the four tasks between the two interviews to minimize bias. The two tasks in this set are repeats of the preceding two tasks, but this time I phrase the tasks using language that elicits fair-sharing meanings.

**Task 14:** How much is in one group if 15 of something is split into  $\frac{10}{3}$  groups?

**Task 15:** How much is in one group if 15 of something is split into 6 groups?

Task 14 is a repeat of Task 12 but phrased this time in fair-sharing terms. Due to primitive intuitions regarding partitive division, I hypothesized that Task 14 would be nonsensical to some teachers but speculated that Task 12 would be meaningful to all of them. As such, I suspected that some teachers would *not* interpret Tasks 12 and 14 as asking the same thing. Task 15 is a repeat of Task 13 but phrased this time in fair-sharing terms. I suspected that each teacher would be able to solve Tasks 13 and 15, but I was curious if the phrasing of the tasks would trigger different schemes. As with Tasks 12 and 13, I wanted each teacher to solve Tasks 14 and 15 without relying on numerical procedures, and to make a diagram representing their thought process.

***Task Set 6: Partitive situations with a whole divisor (Tasks 16-18).*** I designed the next set of tasks to investigate each teacher's meanings for partitive division by asking the teacher to arrive at a quotient when supplied with various values for the dividend, but with the same whole divisor. For each task, I asked the teachers to draw a representation of their thinking. If a teacher immediately resorted to a numerical operation, I inquired about what triggered that operation. The tasks are as follows.

**Task 16:** Suppose 13 gallons of water fill 5 equal containers. Describe the capacity of one whole container.

**Task 17:** Suppose  $\frac{3}{4}$  gallon of water fills 5 identical containers. Describe the capacity of one whole container.

**Task 18:** Suppose a certain amount of water fills 5 identical containers. Describe the capacity of one whole container.

These tasks are situated in a specific context and they comply with primitive intuitions regarding partitive division. The tasks increase in complexity and I expected that teachers

would use partitioning schemes to arrive at answers. If needed, I planned to probe about how they got their answers so I could model the underlying partitioning schemes. Much of what I suspected teachers would do for this set of tasks was based on my findings from Pilot Study C. Task 18 is more abstract, and some schemes for partitioning are not helpful in thinking about this task. For example, a guess and check scheme for partitioning would not be productive here. To be successful, a teacher would need to unitize the entire amount of water into one whole, and then give a relative size to answer the question. I expected that some teachers would not be comfortable using an arbitrary, non-standard unit-of-measure (the total amount of water), as opposed to a standardized unit-of-measure (such as gallon or liter). Such teachers may think that the answer cannot be determined unless more information is provided. Also, a teacher may not possess developed schemes for fraction as a measure of relative size, thus rendering it difficult to think about “one-fifth” as a single amount that produces a whole when iterated five times.

***Task Set 7: Partitive situations with a non-whole divisor greater than 1 (Tasks 19-21).*** This set of tasks is like the previous set, except the divisor is now a non-whole value that is greater than one. Again, for each task, I asked the teachers to draw a representation of their thinking and to justify any triggered numerical operations. The tasks are as follows.

**Task 19:** Suppose 27 gallons of water fill  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 20:** Suppose 3 gallons of water fill  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 21:** Suppose a certain amount of water fills  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

I take a moment and comment on some linguistic nuances of these water-container tasks, by explaining the process I went through to settle on the phrasing. Consider the following phrasing of a division-by-a-fraction task.

Phrasing 1: *Find the size of a group if some amount of water is divided into  $\frac{9}{4}$  groups.*

I used the quantity *amount of water* because it is concrete and can easily be imagined and described. However, the notion of a *group* is more abstract. When I designed the tasks in this study, I wanted to use something concrete, like the water, to also represent a group, and so I chose to use a *container*. I also wanted to get away from words that trigger the operation of division, and so my phrasing became the following.

Phrasing 2: *Find the amount of water in one container if some amount of water fits into  $\frac{9}{4}$  containers.*

However, using the imagery of *fitting into containers* potentially introduces some complications. Like for groups, partial containers could be cognitively challenging. For some, partial containers do not exist, just containers of various sizes, so the notion of  $\frac{9}{4}$  containers is perturbing to the thinker. Such people may look at an image of  $\frac{9}{4}$  containers and see 3 containers, where one is simply smaller than the other two. Thus, I thought I should make it clear that the containers are identical, and that the water should be distributed, filling one container at a time, until there is no more water, which will potentially result in a partially filled final container. With this way of thinking, the potential for this perturbation is minimized, because there are no partial containers, just

*partially filled* containers. Thus, I decided to change from *fits into* to *fills*. This change would be propitious because it also solved another potential issue, which is to imagine containers that have a greater capacity than what is needed. For example, if I say that two cups of water *fit into*  $9/4$  containers, and a person is imagining gallon-sized containers, then it is likely not clear to that person what I mean. Thus, the phrasing now became the following.

Phrasing 3: *Find the amount of water in one container if some amount of water fills  $9/4$  containers.*

Even with this phrasing, it is still possible to be perturbed by *fills*, because of the partial container. If I am imagining all the containers to be identical, then technically, the last quarter container does not get filled, it only gets partially filled. If it became apparent that someone was bothered by this nuance, I could clarify it in one of two ways. I could either maintain the imagery of identical containers and explain that, yes, the last container is only partially filled, despite the language in the prompt. Or, I could encourage the person to imagine three containers, where two are identical, and the third is one quarter as big as either of the other two, and all three containers are filled. In either case, whether the person is imagining identical containers or one smaller container, I would need to make sure the person was attempting to describe the water in a *whole* container, as opposed to the smaller (or *partially filled*, if imagining identical containers) container, and so the phrasing became the following.

Phrasing 4: *Find the amount of water in one whole container if some amount of water fills  $9/4$  containers.*

I settled on this last phrasing, until I realized there could be another complication when the water filled less than one container, such as filling  $\frac{2}{3}$  of a container. If I say that 2 gallons fill two thirds of a container, and then ask how much water *is in* one whole container, the answer is trivially 2 gallons, unless more water is summoned. Thus, I wanted the reader to focus on the *capacity* of a container, and not its actual contents. And so, the phrasing took its final form as follows.

Phrasing 5: *Suppose some amount of water fills  $\frac{9}{4}$  containers. Describe the capacity of one whole container.*

This phrasing was as clear as I could muster, and it was versatile enough to apply to any number of containers, even less than one container. As my subjects engaged with these tasks, I was prepared to help them either imagine all equal containers, with one partially filled, or one smaller container and all of them filled. One downside to this phrasing, is that if I change the amount of water, but do not change the number of containers, then the person must adjust and reimagine new container-sizes. For example, if two gallons fill  $\frac{9}{4}$  containers, and then I change to two cups fill  $\frac{9}{4}$  containers, then the capacity of a whole container must also change. This could be a problem, because once a person imagines a container, which is a concrete unchanging physical object, then it might perturb the person when, suddenly, a different amount of water fills the containers. If this issue arose, then I figured I would simply suggest that the subject start afresh by imagining new containers. In the end, what really matters with these water-container tasks and a divisor of  $\frac{9}{4}$ , is that the subject is imagining three *groups* of water, two groups equal in size, and the third group is one fourth as large. If focused on this objective, then how the subject imagines the containers becomes insignificant – the

number of containers that a person imagines, the sizes of the containers, whether they are all equal in size, and whether the containers are filled, are ultimately unimportant, provided the subject imagines containers that are sufficient in both quantity and capacity. As I conducted the study and presented these water-container tasks to my subjects, I had to be mindful that just because I was using finely tuned language, it was still possible that some of these nuances would surface and cause perturbations for the subject, or would cause answers that seem unreasonable to me, but which are reasonable to the subject.

Now I return to discussing this task set. I suspected most, if not all, teachers would interpret “nine-fourths containers” as “two and one-fourth containers.” I expected more perturbations with this set of tasks than with the previous set, because intuitive rules about fair-sharing could interfere. In Pilot Study C, the teacher tended to convert non-whole numbers into mixed numbers or decimals. This impeded her dramatically. However, I encouraged her to avoid using mixed numbers and decimals. Ultimately, this helped her realize that “two and one-fourth containers” could be construed as nine equal pieces of a container. Once she had this epiphany, she then easily generalized her scheme of partitioning into nine pieces and iterating one of those pieces four times to resolve any task involving “nine-fourths containers.” I suspected that teachers in this study would have similar issues as they approached these tasks. However, since the interviews during Phase 1 were clinical, I did not intervene by prompting the teachers toward productive thinking.

Task 19 is the least challenging of the set because it yields a whole quotient following the initial partitioning. Task 20 introduces a complexity, in that the initial partitioning leads to a non-whole amount of water. Furthermore, some teachers may

become confused because one-quarter container can hold one-third of a gallon, and I anticipated that some teachers would confound the units of measure for these otherwise similar numerals. I expected that Task 21 would be problematic for any teacher who could not think abstractly about the total amount of water and who does not possess developed schemes for fractions as a measure of reciprocal relative size or fractions as an invertible operator, which includes robust schemes of partitioning and iterating to reconstitute a whole. In Pilot Study B, I observed one student solved Task 19 by using a guess and check scheme to partition the water. Not surprisingly, this same student was not able to think about Task 21 productively.

***Task Set 8: Partitive situations with a non-whole divisor less than one (Tasks 22-24).*** The next set of tasks is like the previous two sets, but this time the divisor is a non-whole value that is less than one. Again, for each task, I required the teachers to draw a representation of their thinking and to justify any triggered numerical operations. The tasks are as follows.

**Task 22:** Suppose 5 gallons of water fill  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

**Task 23:** Suppose  $\frac{7}{4}$  gallons of water fill  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

**Task 24:** Suppose a certain amount of water fills  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

These tasks violate the primitive intuition that division makes smaller. However, if someone does not connect these tasks to the numerical operation of division, then the expectation of a smaller quotient may not come into play. It is because of such tasks,

where the divisor is less than one, that I asked teachers to discuss the “capacity” of a whole container, as opposed to the “contents” of a whole container. When the given amount of water doesn’t fill a whole container, the focus must be on the whole container’s capacity, which would only equal the volume of its contents if more water were added. Since focusing on the capacity is valid regardless of the number of containers, and for the sake of consistency, I framed all the tasks involving water and containers in terms of describing container capacity. During Pilot Study C, I observed a source of confusion when the teacher was attempting to discuss such tasks with divisors less than one. These kinds of tasks, involving water and containers, require thinking about two different amounts of water, the given amount of water and the amount of water that would fill a whole container. The teacher in Pilot Study C always referred to the larger of these two amounts as the *total amount* of water. In cases where the number of containers filled was greater than one, the *total amount* of water represented to her the given amount of water. However, in cases where the number of containers filled was less than one, she used *total amount* to refer to the amount of water that would fill a whole container. Thus, in these latter cases, she was no longer able to focus on the size of the capacity relative to the given amount, and instead focused on the size of the given amount relative to the capacity – she could only focus on proper fractional amounts of the *total amount* of water. This was a barrier to her development in forming a meaningful foundation for the invert-and-multiply algorithm.

***Final Task: Invert-and-multiply algorithm.*** I concluded the second meeting (thus, concluding Phase 1) with one final task, as follows.

**Task 25:** Explain why it is that when you divide by a fraction, you can multiply by the reciprocal of the fraction instead. In other words, explain the following:

$$a \div \frac{b}{c} = a \times \frac{c}{b}$$

I included Task 25 at the end of the second interview, so I could have baseline data about each teacher's ability to meaningfully explain the invert-and-multiply algorithm. I speculated that the teachers would attempt to explain this in one of several different ways, as discussed in my review of the literature. Some of these ways include symbolic manipulations that rely on procedural or quotitive meanings. I did *not* suspect that any teacher would attempt to explain this algorithm using partitive meanings, despite having just completed several partitive tasks.

## **Phase 2 Methods: Teaching Experiment**

In Phase 2 of my study, I attempted to advance teachers' schemes regarding partitive division, but for only two of the six teachers from Phase 1. I narrowed my subjects down to two teachers based on which teachers needed development as supported by the data, but being careful to select teachers for whom it would be possible to benefit. I discuss the choices I made, and the reasons I made them, at the end of Chapter 6. I characterize Phase 2 as a *teaching experiment interview* because I worked one-on-one with each of the two teachers as I actively attempted to promote accommodations to their schemes, as guided by theories pertaining to their learning. From this point on, I refer to Phase 2 simply as a "teaching experiment," and I leave off the word "interview." As discussed by Simon and Tzur (2004), my hypothetical learning trajectory had three components: (1) my objectives for teachers' learning, (2) the mathematical tasks I will

use to promote that learning, and (3) hypotheses about the process of that learning. One focus of my teaching experiment will involve the development of schemes regarding relative size and fractions as multipliers. My general learning objective and seven specific learning objectives for the two teachers are in Table 6.

Table 6

*Learning Objectives for the Teaching Experiment in Phase 2*

|                                    |                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                                                                                                        |
|------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| General Learning Objective         | The teacher develops the schemes necessary for meaningful partitive division involving any positive rational divisor, which will ideally form a quantitative foundation for a generalized justification of the invert-and-multiply algorithm. |                                                                                                                                                                                                                                                                                                                                        |
| Seven Specific Learning Objectives | 1                                                                                                                                                                                                                                             | The teacher has schemes for multiplication over the positive rational numbers.                                                                                                                                                                                                                                                         |
|                                    | 2                                                                                                                                                                                                                                             | The teacher has schemes for fractions as reciprocal measures of relative size.                                                                                                                                                                                                                                                         |
|                                    | 3                                                                                                                                                                                                                                             | The teacher has two distinct meanings for division that are based on meanings for multiplication, by framing division as either a missing multiplier task or a missing multiplicand task. In particular, the teacher can use the <i>how much is in each copy</i> meaning to make sense of division with any positive rational divisor. |
|                                    | 4                                                                                                                                                                                                                                             | The teacher has generalizable schemes for finding the amount of water per container when some amount of water fills a <b>whole</b> number of identical containers.                                                                                                                                                                     |
|                                    | 5                                                                                                                                                                                                                                             | The teacher has generalizable schemes for finding the amount of water per container when some amount of water fills a <b>non-whole</b> , but <b>greater than one</b> , number of identical containers.                                                                                                                                 |
|                                    | 6                                                                                                                                                                                                                                             | The teacher has generalizable schemes for finding the amount of water per container when some amount of water fills a <b>non-whole</b> , but <b>less than one</b> , number of identical containers.                                                                                                                                    |
|                                    | 7                                                                                                                                                                                                                                             | The teacher arrives at the realization that regardless of the number of containers, the invert-and-multiply algorithm sensibly produces the amount of water in one whole container.                                                                                                                                                    |

There are several hypotheses that guided the design of my teaching experiment.

As I mentioned earlier, the partitive and quotitive conceptualizations of division are based on a quantitative, non-commutative model for multiplication whereby  $a \times b = c$  is taken to mean *a copies of the amount b produce the amount c*. The Common Core State

Standards for Mathematics (CCSSM) repeatedly refer to creating meaning for division by situating the values being divided in a multiplicative relationship (National Governors Association Center, 2010). As such, partitive division can be thought of as a missing multiplicand task characterized by the question “*a copies of what amount make the amount  $c$ ?*”, whereas quotitive division can be thought of as a missing multiplier task characterized by the question “*How many copies of the amount  $b$  make the amount  $c$ ?*” To be able to think about partitive division with non-whole divisors, it is imperative to lean away from primitive notions of fair-sharing. As such, my first hypothesis is as follows.

**Hypothesis 1:** *Connecting meanings for partitive division with meanings for multiplication by characterizing partitive division as a missing multiplicand task and steering away from the language and notions of fair-sharing is conducive to productive thinking about partitive division with non-whole divisors.*

*Furthermore, having developed meanings for fractions as multipliers and for fractions as measures of relative size is critical to forming one meaningful foundation for the invert-and-multiply algorithm.*

The second standard from the Common Core Standards for Mathematical Practice (National Governors Association Center, 2010) suggests that learners be able to “reason abstractly and quantitatively”. This means that learners should be able to reason about quantities, while simultaneously representing and manipulating quantitative relationships symbolically. In particular, this teaching experiment involves quantities that can comfortably take on non-whole values. I contend that mathematical procedures should always be connectable to contexts in such a way that these procedures are infused with

quantitative meaning. In my review of the literature, I shared research that demonstrated a general disconnect between procedures and conceptual understanding regarding division (Kribs-Zaleta, 2008; Perlwitz, 2005; Rizvi & Lawson, 2007; Sharp & Adams, 2002; Simon, 1993). To counter this issue, the National Council of Teachers of Mathematics has stated that procedural fluency should emerge from conceptual understanding (2014). For these reasons, my second hypothesis is as follows.

**Hypothesis 2:** *Reasoning about quantities (that can be comfortably imagined with non-whole values) and their relationships, not just with numbers, is propitious for the development of conceptually grounded schemes and procedures. Beginning with concrete representations of the quantities and ending with abstract representations fosters productive accommodation through reflective abstraction.*

To test this hypothesis, I have designed tasks that use physical manipulatives and which are conducive to the development of the schemes I am trying to promote. These manipulatives include working with actual containers of water, as well as working with foam fraction circles. The details are discussed later in this section

**Learning objectives.** A major principle that guided my interactions with the teachers relates to my constructivist views on the nature of learning. Borrowing from Piagetian theories, I characterize “learning” as accommodation, even if the learning is misguided and produces problematic schemes. I characterize “cognitive advancement” as learning that yields schemes that are more productive, general, and robust than before. I agree with the adage that there is value in struggling, and that learners should be given an appropriate amount of time to struggle. Perturbing the learners begins the struggle, but learners must be given enough time and guidance to accommodate on their own. Since

each learner has their own unique signature of schemes, I was required to adapt so that I could strategically and productively perturb each learner.

The teaching experiment began no sooner than two weeks following the end of Phase 1. This gave me adequate time to analyze the six teachers' pre-intervention schemes, and to narrow down to two teachers. For each teacher, the teaching experiment was spread over the course of four meetings, with about one meeting per week, each meeting lasting about 90 minutes. I spread the meetings out due to scheduling constraints and because I needed several days between each session to adjust to each teacher's personal development as I attempted to align their learning with my hypothetical learning trajectory. I focused on the seven learning objectives in order, but I was not particular about making sure a certain number of objectives was accomplished on any given day. I simply progressed through the objectives in order, and for each teacher, the four meetings were adequate to get through the trajectory. I now discuss each learning objective and the tasks I used to accomplish each learning goal.

***Learning Objective 1.*** The first learning objective is as follows. *The teacher has schemes for multiplication over the positive rational numbers. This includes having schemes for fractions as operators. Also, teachers can describe multiplication in a general way that is conducive to thinking with any positive rational multiplier.*

To develop schemes for fractions as an operator, I gave each teacher a strip of paper, no matter what length, and asked them to draw a strip that is  $a$  times as long for  $a = 3, 2\frac{1}{3}, 1.4, \frac{1}{5}, \frac{3}{5}, \frac{7}{5}$ . In Pilot Study C, I observed that a predisposition to think of non-whole numbers in terms of mixed numbers or decimals was not conducive to triggering the two-step scheme of partitioning and iterating. Thus, I encouraged each teacher to

consider the decimals and mixed numbers in terms of single fractions of whole numbers, and then to consider one instance of partitioning followed by one instance of iterating.

To promote the development of language to describe multiplication over the positive rational numbers, I asked the teachers to describe their meanings for  $4 \times 5$ ,  $4 \times 5/3$ , and  $4/3 \times 5$ , by using language and by modeling with pictures. I insisted that the teacher avoid invoking commutativity when trying to avoid a fractional multiplier. I guided the teachers toward describing  $a \times b = c$  as “ $a$  copies of the amount  $b$  gives the amount  $c$ ,” where fractional multipliers means to partition and then iterate.

***Learning Objective 2.*** The second learning objective is as follows. *The teacher has schemes for fractions as measures of relative reciprocal size, which enable him to describe the size of one given quantity relative to a second given quantity, as well as the reciprocal relative size.*

In Pilot Study B, I observed one student who could talk about the fraction “4/9” when I shaded four out of nine contiguous boxes, but she could not say that four-ninths of the total amount of water was in one container when that total amount of water filled nine-fourths containers, despite a drawing of this scenario that she made and that was identical to my drawing of the nine contiguous boxes. I theorized that this student was relying on a primitive part-to-whole scheme for fractions that, to her, was no more than an exercise of counting shaded boxes and total boxes. Thus, weak conceptions of fractions were a hindrance to her development. Objective 1 is for the teachers to develop schemes for fractions as a measure of relative size. To accomplish this, I repeated exercises like Task 3 in Phase 1, but this time I proactively guided each teacher’s

thinking towards partitioning followed by iterating. I also prompted the teachers to consider the reverse comparison as well.

***Learning Objective 3.*** The third learning objective is as follows. *The teacher has two distinct meanings for division that are based on meanings for multiplication, by framing division as either a missing multiplier task or a missing multiplicand task. Each teacher can characterize these meanings as trying to find either “how many copies” or “how much is in each copy.” Each teacher can use the “how much is in each copy” meaning to make sense of division with any positive rational divisor.*

In Pilot Study A, I observed that most of the teachers in the study did not have both meanings for division. For those teachers who did, they did not seem to be aware that the meanings were different and became confused when trying to toggle between these meanings. For a few teachers with both meanings, the value of the divisor dictated the meaning. As such, my goal in this part of the teaching experiment was to create an awareness of the two conceptualizations of division. I began by asking each teacher to describe the meaning of “ $a \div 4$ ” and then “ $a \div 1/4$ ” in the same way. I suspected that this would cause a perturbation, which would motivate the teachers to reframe their meaning for partitive division. To accomplish this, I needed to first help the teachers become aware of the two distinct conceptualizations of division. I did this by asking the questions “How much water is in one container when 20 gallons fill 4 identical containers?” and “How many containers are needed to hold 20 gallons of water if each container can hold 4 gallons?” I suspected that each teacher would resort to the numerical operation of division to arrive at a numerical answer of “5.” We discussed the different quantitative interpretations of this number, then reviewed the non-commutative structure of

multiplication. For each question, we discussed the proper positions in the multiplicative structure for the numbers “20,” “4,” and “5.” We then reframed each question by placing the numbers “20,” “4,” and the symbol “?” in their proper positions of the multiplicative structure, and rephrased the questions more generally in terms of either a missing multiplier task (e.g., “how many copies of 4 gallons are in 20 gallons?”) or a missing multiplicand task (e.g., “4 copies of what amount of water makes 20 gallons?”). I discussed the fact that the multiplicand and the product should be referring to the same unit, with the exception that the multiplicand is a unit rate. I then helped each teacher generalize further by describing division as trying to find either the *amount per copy* or the *number of copies*. From here, I return to the three previous examples,  $a \div 4$  and then  $a \div 1/4$ , and guided the teacher away from fair-sharing language toward missing multiplicand language. As such, each teacher should be able to think consistently, if not yet productively, about partitive division with any positive rational divisor.

***Learning Objective 4.*** The fourth learning objective is as follows. *The teacher has schemes for finding the amount of water per container when some amount of water fills a whole number of identical containers, connecting this to the “how much is in each copy” meaning for division. These schemes should allow the teacher to think productively through situations that involve both whole and non-whole dividends and/or quotients. These schemes should include ways to describe the amount of water in a whole container compared to the given amount of water, even when the given amount of water and the whole number of containers are not specified.*

Before I comment on this objective, I discuss a pedagogical decision regarding language. Earlier in this chapter, when I discussed Task Set 8 of Phase 1, I revealed a

cognitive obstacle that emerged during Pilot Study C when the divisor was less than one container. This issue related to the teacher who always referred to the larger amount of water as the *total* amount. For this reason, I encouraged the teachers to use language that helps distinguish the two relevant amounts of water in each task involving water and containers. During the teaching experiment, I tried to steer the teachers away from the expression *total amount of water* and toward using the expressions *the given amount of water* and *the amount of water in (or capacity of) one whole container* as they reasoned through each task.

I began the water/container tasks with cases where the whole number of containers is specified, but the given amount of water is unspecified, such as *some amount of water fills five containers*. I theorized that with robust schemes for fractions as measures of relative size, teachers would not have trouble reasoning at this level of abstraction, especially with a whole number of containers. If necessary, I planned to use fraction circles and actual bottles of water to aid in the tasks. However, it was not necessary to do this with either of the two teachers. My goal was for each teacher to conclude that one-fifth of the given water fills each container.

If necessary, I planned to repeat this activity, each time with an unspecified amount of water, but a different whole number of containers, until I had evidence that the teacher interpreted  $a \div n$  as the answer to the question “ $n$  copies of what amount give the amount of water  $a$ ?” and arrived at the answer  $1/n \times a$ . This would accomplish my objective of creating a generalized scheme, but if there was time, I had planned to help the teachers use partitioning schemes to apply this generalization to specified amounts of water, by working through the following cases.

1.  $a$  is whole and  $n$  divides  $a$
2.  $a$  is whole but  $n$  does not divide  $a$
3.  $a$  is a fraction of whole values and  $n$  divides the numerator of  $a$
4.  $a$  is a fraction of whole values but  $n$  does not divide the numerator of  $a$

**Learning Objective 5.** The fifth learning objective is as follows. *The teacher has schemes for finding the amount of water per container when some amount of water fills a non-whole, but greater than one, number of identical containers, connecting this to the “how much is in each copy” meaning for division. These schemes should allow the teacher to think productively through situations that involve both whole and non-whole dividends and/or quotients. These schemes should include ways to describe the amount of water in a whole container compared to the given amount of water, even when the given amount of water and the non-whole number of containers are not specified.*

Like for Objective 4, I began with cases where the non-whole number of containers is specified, but the given amount of water is unspecified. Again, I theorized that with robust schemes for fractions as measures of relative size, teachers would not have trouble reasoning at this level of abstraction. As an example of using containers, I might give the teacher two differently sized containers, one three-fourths as large as a second one that is marked as the *whole*. I would also have available a white-rimmed container, that is one-fourth the size of the whole container. This would provide something concrete to talk about, as we work toward the realization that we can measure one amount in multiple ways – the white-rimmed container is one-third of the smaller container, one-fourth of the whole container, and one-seventh of the collection. As such, the smaller container is three-sevenths of the collection.

I planned to repeat such activities, each time with an unspecified amount of water, but a different non-whole number of containers, each greater than one, until the teachers realize that they are giving meaning to the expression  $a \div n/m$  by reasoning through the question  *$n/m$  copies of what amount of water give the amount of water  $a$ ?* and arriving at the answer  $m/n \times a$ . Again, this would accomplish my objective of creating a generalized scheme, but if there was time, I had planned to help the teachers use partitioning schemes to apply this generalization to specified amounts of water, by working through the following cases.

1.  $a$  is whole and  $n$  divides  $a$
2.  $a$  is whole but  $n$  does not divide  $a$
3.  $a$  is a fraction of whole values and  $n$  divides the numerator of  $a$
4.  $a$  is a fraction of whole values but  $n$  does not divide the numerator of  $a$

**Learning Objective 6.** The sixth learning objective is as follows. *The teacher has schemes for finding the amount of water per container when some amount of water fills a non-whole, but less than one, number of identical containers, connecting this to the “how much is in each copy” meaning for division. These schemes should allow the teacher to think productively through situations that involve both whole and non-whole dividends and/or quotients. These schemes should include ways to describe the amount of water in a whole container compared to the given amount of water, even when the given amount of water and the non-whole number of containers are not specified.*

This objective is nearly identical to Objective 5, but with a proper fraction of a container. I placed this objective later in the learning trajectory because of a few cognitive obstacles that could hinder productive thinking, one of which is the issue

related to using the expression *the total amount of water*. I planned to do the same type of activities with this objective as I did with Objective 5.

***Learning Objective 7.*** The seventh learning objective is as follows. *The teacher arrives at the realization that regardless of the number of containers, the invert-and-multiply algorithm sensibly produces the amount of water in one whole container.*

I theorized that if a teacher conceived of  $a \div m/n$  as a command to find the capacity of one whole container when the amount  $a$  fills  $m/n$  containers, and if this teacher possesses the schemes to do so for any positive rational divisor, then this teacher will be able to quantitatively justify  $a \div \frac{m}{n} = \frac{n}{m} \times a$ . I am hopeful that the teachers will make this connection without me explicitly drawing attention to it. However, I intended to ask the teachers to reflect on the teaching experiment and to summarize anything they have learned. To facilitate this, I planned to provide them with copies of their work to stimulate recollection of what they have done. Apart from this, I planned to make no additional efforts to accomplish Objective 7. Instead, I assessed during Phase 3 whether the teachers could make this connection.

### **Phase 3 Methods: Post-Assessment**

Following the teaching experiment, I held one last meeting, a post-assessment, with each of the two teachers, which occurred about a week following the end of Phase 2. For the post-assessment, I repeated 12 of the tasks from Phase 1 (the pre-assessment) and compared the results from the two phases. These tasks are in Table 7. During Phase 3, I presented the tasks to the teachers in the order they are listed in the table, but numbered 1-12. In the table and in my writing, I refer to these tasks using Phase 1 numbering.

Table 7

*Phase 3 Tasks*

|    |                                                                                                                                                                                                         |
|----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 3  | Given these two strips of paper, how long is the red strip compared to the green strip? How long is the green strip compared to the red strip? (red = $\frac{8}{3}$ green)                              |
| 6  | Explain your meanings for the expression: $\frac{5}{3} \times 2$                                                                                                                                        |
| 8  | Explain your meanings for the expression: $a \div b$                                                                                                                                                    |
| 9  | Explain your meanings for the expression: $4 \div 3$                                                                                                                                                    |
| 10 | Explain your meanings for the expression: $4 \div \frac{1}{3}$                                                                                                                                          |
| 18 | Suppose a certain amount of water fills 5 identical containers. Describe the capacity of one whole container.                                                                                           |
| 16 | Suppose 13 gallons of water fill 5 equal containers. Describe the capacity of one whole container.                                                                                                      |
| 21 | Suppose a certain amount of water fills $\frac{9}{4}$ identical containers. Describe the capacity of one whole container.                                                                               |
| 20 | Suppose 3 gallons of water fill $\frac{9}{4}$ identical containers. Describe the capacity of one whole container.                                                                                       |
| 24 | Suppose a certain amount of water fills $\frac{2}{3}$ of a container. Describe the capacity of one whole container.                                                                                     |
| 23 | Suppose $\frac{7}{4}$ gallons of water fill $\frac{2}{3}$ of a container. Describe the capacity of one whole container.                                                                                 |
| 25 | Explain why it is that when you divide by a fraction, you can multiply by the reciprocal of the fraction instead. In other words, explain the following:<br>$a \div \frac{b}{c} = a \times \frac{c}{b}$ |

Below, I explain why I chose these tasks for the post-assessment in Phase 3. For each task, I asked the teachers to draw a representation of their thinking, and if a teacher immediately resorted to a numerical operation, I asked about what triggered that operation.

**Task 3:** Given these two strips of paper, how long is the red strip compared to the green strip? How long is the green strip compared to the red strip?



Since the notion of fractions as reciprocal measures of relative size is so fundamental to the meanings I was attempting to promote, I repeated Task 3 from Phase 1. But in Phase 3, I used different lengths because I had repeatedly referred to the original lengths throughout the teaching experiment, and so the original lengths could no longer serve as an adequate post-assessment task. I also pre-partitioned the strips with small marks, to mitigate any inaccuracies due to folding or some other tactic. This time around, I was hoping to see the teachers be able to fluidly switch from one comparison to the reverse comparison, accompanied by a sensible justification for the change in denominator. This would suggest an ability to switch from one perception of a whole to another, thus being able to relatively describe one magnitude using multiple unit fractional values, e.g. one third of the green strip has the same length as one eighth of the red strip.

**Task 6:** Explain your meanings for the expression:  $\frac{5}{3} \times 2$

I repeated Task 6 from Phase 1 because of the importance of robust meanings for fractions as operators in my learning trajectory. I wanted to determine if my subjects could sensibly think about fractions in the role of the multiplier. If they could not, then there would be virtually no chance of having developed the meanings that I intended.

**Task 8:** Explain your meanings for the expression:  $a \div b$

**Task 9:** Explain your meanings for the expression:  $4 \div 3$

**Task 10:** Explain your meanings for the expression:  $4 \div \frac{1}{3}$

I included Tasks 8 through 10 from Phase 1 because of my attempts to raise the teachers' awareness of the dual meanings for division, characterized as *how many copies* and *how much is in each copy*. When specific values were involved, I encouraged the teachers to

describe all ways they could think about these problems, and I was hoping that the teachers would be able to resolve them using both partitive and quotitive conceptualizations. I suspected that water and containers may be a context that the teachers would conjure (because of the teaching experiment) to facilitate any partitive conceptualizations.

**Task 18:** Suppose a certain amount of water fills 5 identical containers. Describe the capacity of one whole container.

**Task 16:** Suppose 13 gallons of water fill 5 equal containers. Describe the capacity of one whole container.

**Task 21:** Suppose a certain amount of water fills  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 20:** Suppose 3 gallons of water fill  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 24:** Suppose a certain amount of water fills  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

**Task 23:** Suppose  $\frac{7}{4}$  gallons of water fill  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

I presented the six tasks above in a different order from Phase 1, by asking the teachers to discuss general situations before discussing specific situations, which follows the order of the learning trajectory from Phase 2. I wanted to look for evidence of quantitative reasoning by requiring the teachers to think about general situations for which numerical

operations were not possible. I was hoping to see that the teachers had successfully developed a general meaning for partitive division with a rational divisor, before giving them a task for which numerical calculations would be performed, per the teachers' schemes. These tasks were foundational to providing me the evidence I needed to address my research questions regarding the development of meanings related to partitive conceptualizations for division.

**Task 25:** Explain why it is that when you divide by a fraction, you can multiply by the reciprocal of the fraction instead. In other words, explain the following:

$$a \div \frac{b}{c} = a \times \frac{c}{b}$$

Part of my learning trajectory was to promote the development of a meaningful foundation for the invert-and-multiply algorithm. As such, I included Task 25 from Phase 1 to determine if any of the teachers would naturally connect their experiences from the teaching experiment to this algorithm. I was hopeful that both teachers would. In the next chapters, I discuss the results of the study and present my analysis of the data.

## CHAPTER 6

### PHASE 1 RESULTS AND DISCUSSION

I based my analysis of the data from Phase 1 on my secondary research questions for this phase, which I repeat in Table 8.

Table 8

*Secondary Research Questions Corresponding to Phase 1*

|       |                                                                                                                                                                                                              |
|-------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| RQ1.1 | What meanings do teachers reveal when they engage in tasks that I designed to elicit meanings for fractions as measures of relative size, with a focus on fractions as reciprocal measures of relative size? |
| RQ1.2 | What meanings do teachers reveal when they describe and model symbolic (decontextualized) statements of multiplication, both general and specific, with a focus on fractional multipliers?                   |
| RQ1.3 | What meanings do teachers reveal when they describe and model symbolic (decontextualized) statements of division, both general and specific, with a focus on fractional divisors?                            |
| RQ1.4 | What meanings do teachers reveal when they engage in tasks that I designed to elicit partitive conceptualizations of division, with varying degrees of abstraction, and with a focus on fractional divisors? |
| RQ1.5 | What justifications do teachers provide for the invert-and-multiply algorithm after working through the tasks mentioned in the previous research question?                                                   |

These secondary research questions are sub-questions related to RQ1: *What meanings, with their affordances and limitations, do in-service middle school mathematics teachers possess relative to partitive conceptualizations of division with non-whole divisors?* In this chapter, I discuss the data relative to these secondary research questions.

#### **RQ1.1: Fractions as Reciprocal Measures of Relative Size**

I designed Tasks 2 and 3 to provide data to answer RQ1.1: *What meanings do teachers reveal when they engage in tasks that I designed to elicit meanings for fractions as measures of relative size, with a focus on fractions as reciprocal measures of relative size?* First, I discuss the data from Task 2, and then I discuss the data from Task 3.

**Task 2:** Do the following and justify your answers.

This line has a length that is  $\frac{8}{5}$  of a unit. → 

Draw a line here that is 1 unit long. →

While working on Task 2, all teachers imagined the given line as eight somethings and attempted to partition the given line into eight equal pieces. Mel and Mindi thought to do this by dividing the line in half, then each half into fourths, and then each fourth into eighths. The rest tried to do this by iterating a length, one iteration at a time, making minor adjustments so that the eighth iteration ended where the given line ended. I share the work of Mel, Uma, and Linda in Figure 11 below.

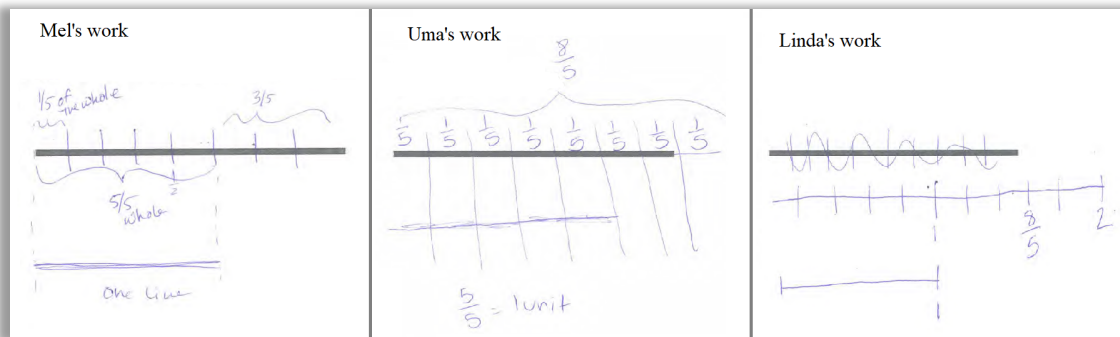


Figure 11. The work of Mel, Uma, and Linda on Task 2 in Phase 1.

Uma's first attempt at iterating led to seven pieces reconstituting the given line, so she just reimagined the given line to be a little longer to compensate. Once the teachers partitioned the given line, only Linda did not imagine the length of five of these segments. Instead, she increased the given line to be two units long, and then cut this new line in half. She guessed at how long to make the new line, saying that it should just be a little bit longer than the given line, because "I just need two-fifths of a unit more."

However, she did not reveal a rigorous way of thinking that guided her attempt to extend the given line an accurate amount. She divided the new line into 10 pieces that did not look equal in length, possibly due to an attempt to make it so that eight of the pieces would align with the given line. However, she did say that the 10 pieces *should* be equal in length. Linda's strategy is significant because it revealed that she did not think to divide the given line into eight pieces to begin with, which suggests that the two-step scheme of contracting by a factor of one-eighth, followed by expanding by a factor of five was not available to her. I now move on to the data from Task 3.

**Task 3:** Given these two strips of paper, how long is the red strip compared to the green strip? How long is the green strip compared to the red strip? (The red strip was seven inches long and the green strip was 4 inches long.)

I designed Task 3 to give me more insights about the teachers' meanings for fractions as reciprocal measures of relative size. I comment on each teacher below. Recall that the red strip was seven inches long and the green strip was four inches long, but I did not reveal this to the teachers.

For Task 3, Linda recognized that two greens would be longer than the red strip and decided to fold the green in fourths. She determined that she needed to add three of those fourths to the end of the green strip to match the length of the red strip. She said the "red strip is larger by seven-fourths of the green strip," which is a confusing blend of additive and multiplicative thinking – does this mean that the red strip is a green strip *plus* seven-fourths of a green strip, or is the red strip just seven-fourths of a green strip? She then said the "green strip is seven-fourths smaller than the red strip," which she was uncomfortable saying, but she could not think of another way to express it. Her last

utterance is a consequence of additive thinking, in that if something is a certain amount more than another thing, then the lesser thing is the same amount less. She did not have the language to describe the reverse multiplicative comparisons. She later wrote “ $\frac{7}{4}$  of the green strip is one red strip,” and when I pressed her to write down the reverse comparison, she wrote “1 red strip is  $\frac{7}{4}$  of the green strip,” which is a restatement of her first comparison, but in reverse order by mentioning the red strip before mentioning the green strip. She assimilated my suggestion to a scheme that produced a sensible answer for her, as if switching the order of the color references in her speech was adequate for doing the reverse comparison. Ultimately, she was unable to transition to the red strip being the unit-of-measure. I asked her to rethink the scenario supposing that the red strip was twice the length of the green strip, and even in this case, she did not say that the green strip would be half of the red strip. Instead, she kept repeating that two green strips would be one red strip. This task was revealing, in that Linda could not easily switch from one strip as the unit-of-measure to the other strip, which contributed to her inability to describe the reciprocal multiplicative relationships between the two strips.

For Task 3, Mark thought that the red strip was one-fifth of a green smaller than two greens so he concluded that the red strip was 1 and  $\frac{4}{5}$  green strips. His instinct for the reverse direction was to say that the green strip was  $\frac{5}{9}$  of the red strip but he said he was not confident. He said he got “ $\frac{5}{9}$ ” by thinking of 1 and  $\frac{4}{5}$  as  $\frac{9}{5}$  and finding the reciprocal, but he was not sure the reciprocal was what he needed. Even after he iterated one-fifth of the green strip nine times to get the red strip (done sloppily, hence the incorrect comparisons) he still said that he did not understand the “math of why the

reciprocal works.” He said he did the reciprocal because “something went off and made me do it.”

For Task 3, Mel took a convoluted approach. She constructed a unit-length, which was one green minus half of a red, and called it a “ted.” However, due to imprecision caused by the folding of the paper strips, she concluded that the green was seven teds and that a red was 12 teds. She then used her green strip as a ruler and measured the red as  $1\frac{5}{7}$  greens. Using her *ted* units, she was able to determine that one green was  $\frac{7}{12}$  of a red. At first, she did not recognize the reciprocal nature of the measurements due to the mixed number. Only when I asked whether she could find green in terms of red, but only from knowing that  $1\frac{5}{7}$  green made a red, that she noticed that expressing  $1\frac{5}{7}$  as  $\frac{12}{7}$  helped her recognize the reciprocal relationship.

For Task 3, Mindi folded the red in half repeatedly until it was partitioned into 16 pieces, then used the folded paper to conclude that green was  $\frac{9}{16}$  of red. For the other comparison, she said her “gut says to do the reciprocal of nine-sixteenths,” but she was unsure. She wrote that the answer was  $\frac{16}{9}$ , which she converted to  $1\frac{7}{9}$ , but she expressed concern about the nine in the denominator. I theorize that she was perturbed because she was unable to think of each segment as simultaneously being  $\frac{1}{16}$  and  $\frac{1}{9}$ . It is not until she imagined two extra segments added to the end of the red strip (making two whole greens) that she realized that the nine was an appropriate denominator because each whole green was nine pieces.

For Task 3, Uma cut off a piece of the green strip that appeared to be about  $\frac{1}{11}$  of the green, observed that 19 of them make the red, and concluded that the red was “eight-elevenths larger than the green.” For the reverse comparison, she said the green

was “eight-elevenths shorter than the red,” which indicates additive thinking, similar to what Linda said. I pressed Uma to provide a unit for the  $8/11$  in her reverse comparison and she acknowledged that the fraction was referring to the green strip and that it was weird to say, “The green strip is eight-elevenths of the green strip less than the red strip.” After some silence, I rephrased the question by asking, “What fraction of the red strip is the green strip?” She eventually said  $11/19$ , but admitted she was not very confident with this answer. When I asked her why, she stated that she had camera fright and was not thinking clearly. Ultimately, she accepted the reverse comparison of  $11/19$  but never seemed very confident with it.

For Task 3, Ursa sloppily folded the green into seven pieces and concluded by writing that the red was “1 and  $5/7$  larger than the green.” Her language suggests an additive comparison, but I am confident she was trying to say that the red is 1 and  $5/7$  greens. She used her partitioned green strip to conclude that the green is  $7/12$  of the red. I asked her if she could use her first answer to get her second answer, but she could not provide an explanation initially. However, she reasoned that if a strip is triple another, then the smaller strip is one-third as long, which led her to think that the two answers should be reciprocals of each other. She then confirmed that she could convert 1 and  $5/7$  to  $12/7$ , for which the reciprocal was  $7/12$ . It is noteworthy that she did not give a quantitative explanation for using the reciprocal of  $12/7$ , but instead she relied on a generalization of a simplified scenario.

When I presented Task 3 to the teachers, I asked for both comparisons at once. This was a design flaw because I was not able to determine which question each teacher thought they were answering first. All teachers, except Mindi, first expressed red in terms

of green. Due to imprecisions, such as those caused by folding the paper strips, only one teacher (Linda) determined the correct comparison of red in terms of green (red is  $\frac{7}{4}$  of green). But Linda could not reverse the comparison, which meant that no teacher determined the correct comparison of green in terms of red (green is  $\frac{4}{7}$  of red). No teacher immediately, and confidently, summoned the reciprocal of their first comparison as the reverse comparison. Mark, Mel, Uma, and Ursa each gave a mixed number to express red in terms of green, which prevented Mel and Ursa from seeing the reciprocal. Linda and Uma used additive language for the reverse comparisons, from which only Uma recovered by eventually giving a multiplicative comparison. Mindi and Mark both used the reciprocal for the reverse comparison on their own, but they did so without confidence, and only Mindi could eventually justify why.

Even though some teachers were eventually confident that the reciprocal was significant to the reverse comparison, initially it was not trivial for them to think so. Thompson and Saldanha (2003) used the construct *fractions as reciprocal measures of relative size* to refer to a reversible scheme of comparing the magnitudes of two quantities, one in terms of the other, as well as the reverse comparison. To illustrate, suppose that some amount  $A$  is  $\frac{3}{5}$  as large as some amount  $B$ . Like Thompson and Saldanha, I consider a person to have meanings for fractions as reciprocal measures of relative size when that person recognizes that  $A$  is  $\frac{3}{5}$  as large as  $B$ , while simultaneously recognizing that  $B$  is  $\frac{5}{3}$  as large as  $A$ . The development of such a reversible scheme is dependent on a person's ability to re-unitize by reconceiving a new whole and expressing the reverse comparison in terms of this new whole. It is essential to recognize that in the example above,  $\frac{1}{3}$  of  $A$  has the same magnitude as  $\frac{1}{5}$  of  $B$ . Once a reversible scheme is

sufficiently developed, a person would be able to take any fractional value of a quantity and automatically reconstitute one whole. In my conceptual analysis chapter, I explained how reciprocal comparisons are crucial to meaningfully operating with fractional divisors in such a way as to form a foundation for the invert-and-multiply algorithm for either conceptualization of division. As such, I consider the development of fractions as reciprocal measures of relative size as foundational to the advancement of meanings for division.

### **RQ1.2: Decontextualized Multiplication**

I designed Tasks 4-7 to address RQ1.2: *What meanings do teachers reveal when they describe and model symbolic (decontextualized) statements of multiplication, both general and specific, with a focus on fractional multipliers?* The tasks were as follows.

**Task 4:** Explain your meanings for the expression:  $a \times b$

**Task 5:** Explain your meanings for the expression:  $5 \times \frac{4}{3}$

**Task 6:** Explain your meanings for the expression:  $\frac{5}{3} \times 2$

**Task 7:** Explain your meanings for the expression:  $\frac{2}{5} \times \frac{4}{3}$

I summarize the observable behaviors from Tasks 5-7 in Table 9. The ✓ means the teacher successfully modeled the product with the indicated multiplier, the ✗ means the teacher tried but did not succeed, and the hash mark means the teacher did not attempt to model the product with the indicated multiplier.

Table 9

*The Multipliers in the Teachers' Thinking for Tasks 5-7*

|       | Task 5<br>$5 \times \frac{4}{3}$ |     | Task 6<br>$\frac{5}{3} \times 2$ |     |                                     | Task 7<br>$\frac{2}{5} \times \frac{4}{3}$ |     |
|-------|----------------------------------|-----|----------------------------------|-----|-------------------------------------|--------------------------------------------|-----|
|       | Multiplier                       |     | Multiplier                       |     | Contract by 1/3<br>then expand by 5 | Multiplier                                 |     |
|       | 5                                | 4/3 | 2                                | 5/3 |                                     | 2/5                                        | 4/3 |
| Linda | ✓                                | -   | ✓                                | ✗   | N/A                                 | ✗                                          | -   |
| Mark  | ✓                                | -   | ✓                                | ✓   | No                                  | ✓                                          | -   |
| Mindi | ✓                                | -   | ✓                                | ✓   | No                                  | ✓                                          | -   |
| Mel   | ✓                                | -   | ✓                                | ✓   | No                                  | ✓                                          | -   |
| Uma   | ✓                                | -   | -                                | ✓   | No                                  | ✓                                          | -   |
| Ursa  | ✓                                | -   | ✓                                | ✓   | No                                  | ✓                                          | -   |

In the paragraphs that follow, I discuss the data from Tasks 5 and 6. For Task 5 (Explain your meanings for  $5 \times \frac{4}{3}$ ), all six teachers imagined 5 as the multiplier, even Uma who consistently described the second factor as the multiplier, like she did when describing  $a \times b$  as “ $a$  is going to represent a number, times  $b$  representing how many groups of that number there are.” All teachers were successful at modeling the product by grouping five ones, three one-thirds, with two one-thirds remaining to get a result of six and  $\frac{2}{3}$ . Mel’s work on this task, as depicted in Figure 12, is exemplary of the schemes that all six teachers used. I did not ask the teachers for other ways of thinking on this task.

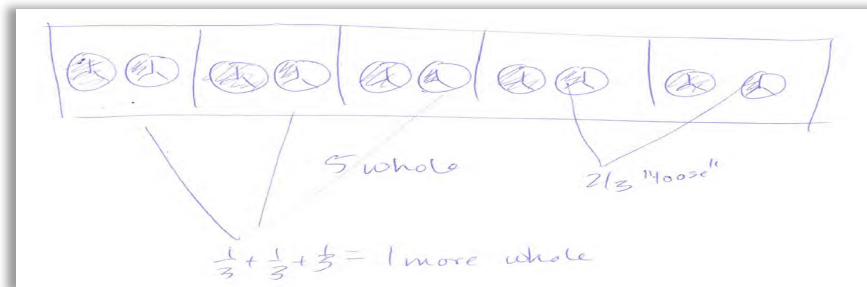


Figure 12. Mel's work on Task 5 in Phase 1.

For Task 6 (Explain your meanings for  $5/3 \times 2$ ), Linda could not productively think about  $5/3$  as the multiplier. She relied on commutativity and seemed satisfied to instead draw two copies of five-thirds, which gave an answer that she confirmed with the algorithm. She realized that this was not the same as representing  $5/3$  copies of 2, and she said she could not think about the meaning of “five-thirds copies.” Mark defaulted to 2 as the multiplier, but switched when I prompted him. He thought of  $5/3$  as a mixed number but paused when he tried to model  $2/3$  of 2. He then focused on  $1/3$  of 2 and procedurally obtained  $2/3$ . He then interpreted  $5/3$  of 2 as 5 copies of  $1/3$  of 2, or 5 copies of  $2/3$ . He then created a visual model by combining five copies of a representation of two-thirds to justify his answer of  $10/3$ . He then manipulated the expression until he had a whole number of copies that he finally modeled with a picture. It is significant to note that Mark used a blend of procedures and modeling to arrive at his answer. Mel, Mindi, Uma, and Ursa first modeled with 2 as the multiplier, switched when I prompted them, and interpreted  $5/3$  as 1 and  $2/3$  to arrive at a sensible answer. Mel had no trouble doing this, Mindi struggled only because she made a procedural error that was contradicting her models, and Uma and Ursa both struggled a great deal to arrive at an answer. I further discuss Uma’s and Ursa’s struggles in the next paragraphs. It is important to note that none of the teachers contracted 2 by a factor of  $1/3$ , followed by expanding this amount by a factor of 5. This was significant to me because the mixed number obscured the elegance of the dual-scheme of contraction followed by expansion. As such, the mixed number can be a cognitive obstacle to reversing the effects of a fractional multiplier by imagining the reciprocal fraction.

## Multiplier-Switch

I return to Ursa's difficulties while modeling  $5/3 \times 2$  in Task 6. In this task, Ursa used the language "five-thirds copies of two", paused to think about it, and then defaulted to 2 as the multiplier instead. I stopped her and urged her to consider the meaning of her own words "five-thirds copies of two." There was a long pause and after unsuccessfully trying to model this through drawings, she said that "five-thirds copies of one is five...thirds... hmm... so I have to do that twice." This led her to drawing two representations of five-thirds. Ursa was aware that she had switched the multipliers as evidenced when she said, "I feel like that's two copies of five-thirds and not five-thirds copies of two. I don't think that's fair. I think that's the easy way out." I represent her transition symbolically in the following way:

$$\frac{5}{3} \times 2 = \frac{5}{3} \times (1 + 1) = \frac{5}{3} + \frac{5}{3} = 2 \times \frac{5}{3}$$

Ursa started over and eventually she was successful at modeling  $5/3 \times 2$  by doing one copy of two, followed by clunkily showing that two-thirds of two resulted in one and one third. Despite her ultimate success, her method was so convoluted that it would have been cognitively taxing to reverse, obfuscating the simplicity of reversing the effects of a  $5/3$  scale factor by scaling by  $3/5$ . I call this phenomenon a ***multiplier-switch***, which I define to occur when a person imagines one factor as the multiplier, but then – during the process of modeling – the person instead views the model (with or without awareness) as a justification for the other factor as the multiplier. This phenomenon is likely to occur when the multiplicand is a whole number, due to a presence of deeply engrained schemes for multiplication as repeated addition. To illustrate, consider the symbolic representation of an example of a multiplier-switch below.

$$\begin{aligned}
 & a \times 5 \\
 &= a \times (1 + 1 + 1 + 1 + 1) \\
 &= a + a + a + a + a \\
 &= 5 \times a
 \end{aligned}$$

A person who cannot successfully model with a fractional multiplier will not be able to model partitive division. Although I consider a multiplier-switch to be a sensible way to provide a quantitative justification for the commutativity of multiplication in general, thinking this way would circumvent the two-step scheme of fractional multipliers as a contraction followed by an expansion. As such, I consider the multiplier-switch to be a cognitive barrier to the advancement of schemes relative to fractional multipliers, and hence to partitive division with fractional divisors.

### **Multiple-Unit Coordination**

Several teachers exhibited difficulties throughout Phase 1 related to re-unitization. To give an example of this, I return to Uma's difficulties while modeling  $5/3 \times 2$  in Task 6. For this task, Uma drew two contiguous rectangles and called them "two." She then split each rectangle into three columns and represented  $5/3$  of one of the rectangles by shading in five out of the six total pieces. I share her work in Figure 13. This led Uma to say that the result was  $5/6$ , which she instantly knew was wrong because the algorithm produced  $10/3$ .

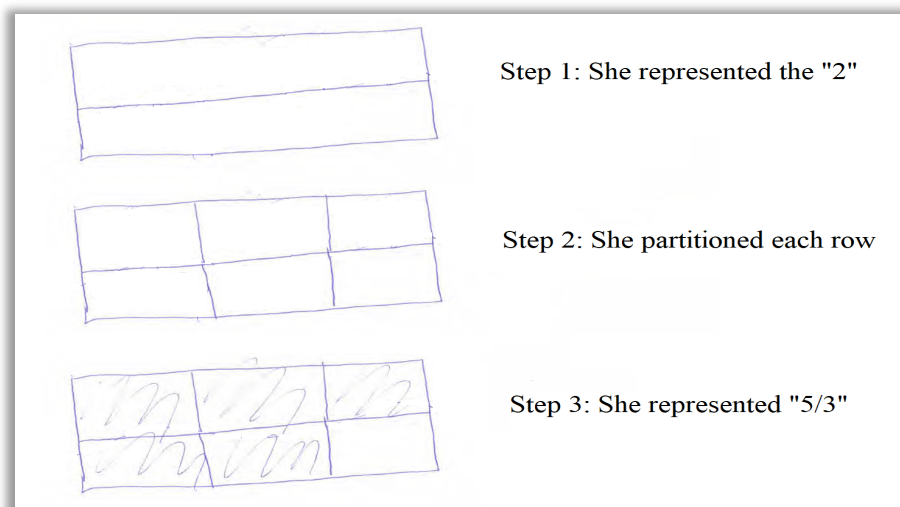


Figure 13. Uma's work on her first attempt with Task 6 in Phase 1.

There are at least two issues related to unitizing here. First, Uma represented five-thirds of one row, not five-thirds of the collection of two rows, due to maintaining the same unit for the  $5/3$  as for the 2. Each of these values was in terms of one row, and she was unable to view the two rows as a new whole. Second, she saw five out of six pieces shaded and gave an answer of  $5/6$ , which represents a change in unit, although she likely did not intend, nor was aware of this change. The confusion here lies with maintaining that each of the five shaded pieces represents one-third *of one row*, which is the same amount as one-sixth *of two rows*. She decided to start over by converting  $5/3$  to 1 and  $2/3$ . She drew one group of two rows, then a second group of two rows, and divided both groups into three columns. This resulted in six pieces per group. She labeled the first group as 1, became momentarily confused, and then also labeled it as 2, saying that the 1 was referring to the 1 in "1 and  $2/3$ ," and the 2 was referring to the fact that the group had a "value of 2." I share her work in Figure 14.

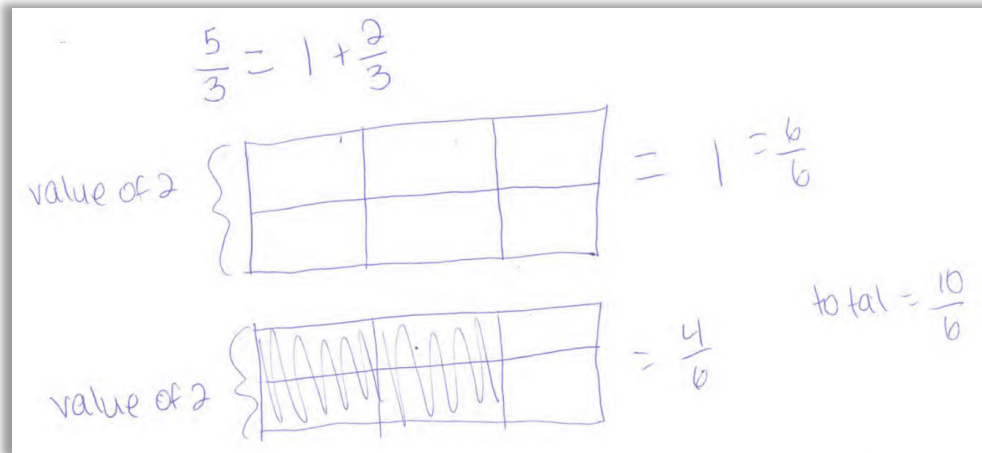


Figure 14. Uma's work on her second attempt with Task 6 in Phase 1.

Uma kept one whole group and shaded in two-thirds of the second group and said that her answer was  $\frac{6}{6}$  plus  $\frac{4}{6}$  which produced a total of  $\frac{10}{6}$ . This caused her to pause again because the algorithm produced an answer of  $\frac{10}{3}$ . Aided by the algorithm, she eventually realized that she should think of each piece as  $\frac{1}{3}$  and not  $\frac{1}{6}$ . Technically, her first answer of  $\frac{10}{6}$  was correct in terms of groups of two rows. Had it not been for the algorithm, she was very likely to have been confident with this answer and moved on. The data revealed that Uma lacked a strong awareness that, given the quantitative model for multiplication, the unit of the product should agree with the unit of the multiplicand. This is the reason why  $\frac{5}{3}$  of 2 is  $\frac{10}{3}$ , and not  $\frac{10}{6}$ . Contrarily, if the unit of the product agreed with the unit of the multiplier, then the numerical value of a product would always match the numerical value of the multiplier;  $\frac{5}{3}$  of 2 would just be  $\frac{5}{3}$ . Her confusion in this task was compounded by the fact that there was no particular context that provided clearer quantities about which she could easily reason. It is noteworthy that she had trouble articulating the meanings of the 1 and 2 when she was using them both to refer to the first group she drew. If I had presented a situation where cupcakes came in packages

of two, it would have been easier to discuss that one-third of a cupcake corresponded to one-sixth of a package, ultimately giving the statement:

$$5/3 \text{ packages} = 5/3 \times (2 \text{ cupcakes}) = 10/3 \text{ cupcakes.}$$

A task that is situated in a clear context not only provides something concrete to think about, but it also enables the use of vocabulary to keep the quantities distinct in the mind of the problem solver. As such, it is reasonable to conclude that a person would struggle less to coordinate multiple levels of units.

### **RQ1.3: Decontextualized Division**

I designed Tasks 8-11 to address RQ1.3: *What meanings do teachers reveal when they describe and model symbolic (decontextualized) statements of division, both general and specific, with a focus on fractional divisors?* These tasks provided data regarding the teachers' meanings for division in general, as well as their awareness of the dual conceptualizations. I repeat the tasks below.

**Task 8:** Explain your meanings for the expression:  $a \div b$

**Task 9:** Explain your meanings for the expression:  $4 \div 3$

**Task 10:** Explain your meanings for the expression:  $4 \div \frac{1}{3}$

**Task 11:** Explain your meanings for the expression:  $\frac{1}{2} \div \frac{3}{4}$

Recall that when each teacher finished Tasks 9 and 10, I challenged the teacher to rethink both Tasks 0 and 0 using all meanings they demonstrated during either of these two tasks the first time through. For example, if a teacher revealed a partitive meaning for Task 9 but a quotitive meaning for Task 10, then I asked that teacher to repeat each task using the other meaning. If a teacher only gave quotitive meanings for Tasks 9 and 10, then I

could not ask the teacher to reconsider the tasks using another meaning because the teacher revealed no other meaning. I summarize the observable behaviors from Tasks 8-10 in Table 10. For Task 8, the ✓ means the teacher described the indicated conceptualization. For Tasks 9 and 10, the ✓ means the teacher successfully modeled the quotient using the indicated conceptualization, the ✗ means the teacher tried but was unsuccessful, and the hash mark means the teacher did not attempt to model the quotient using the indicated conceptualization.

Table 10

*The Teachers' Conceptualizations for Division in Tasks 8-10*

|       | Task 8<br>$a \div b$ |     | Task 9<br>$4 \div 3$ |     | Task 10<br>$4 \div \frac{1}{3}$ |     | Task 9<br>revisited<br>$4 \div 3$ | Task 10<br>revisited<br>$4 \div \frac{1}{3}$ |
|-------|----------------------|-----|----------------------|-----|---------------------------------|-----|-----------------------------------|----------------------------------------------|
|       | Quo                  | Par | Quo                  | Par | Quo                             | Par | Quo                               | Par                                          |
| Linda | ✓                    | -   | ✗                    | -   | ✓                               | -   | No need                           | No need                                      |
| Mark  | -                    | ✓   | -                    | ✓   | ✓                               | -   | Did not ask                       | ✗                                            |
| Mindi | ✓                    | ✓   | ✓                    | ✓   | ✓                               | ✓   | No need                           | No need                                      |
| Mel   | ✓                    | -   | -                    | ✓   | ✓                               | -   | Did not ask                       | ✓                                            |
| Uma   | -                    | ✓   | -                    | ✓   | ✓                               | -   | ✓                                 | ✗                                            |
| Ursa  | ✓                    | ✓   | ✗                    | ✓   | ✓                               | -   | No need                           | ✗                                            |

I now discuss the data from each teacher, focusing on their awareness of the two meanings for division and their ability to model division using both conceptualizations.

### Partitive-Quotitive Awareness

Linda gave a quotitive description in Task 8 by saying, “How many copies of three can be taken from four?” Since she revealed only quotitive meanings for Tasks 9 and 10, I did not ask her to rethink either task. Of these two tasks, she was successful only at modeling  $4 \div 1/3$ , the task that yielded a whole number as the quotient. Later in the

study, she revealed partitive meanings when she was reasoning through contextualized tasks that elicited partitive thinking, which I discuss later in this chapter. This suggests that she had schemes (to some extent) for both meanings. Her one meaning that she was aware of – quotitive – was so strong that when given a decontextualized division statement, she could not summon a partitive meaning. This was still true even when she could not quotitively model  $4 \div 3$  – even then, she did not transition to a partitive model to cope with the perturbation. Also, at the start of Phase 2, even after finishing several tasks in Phase 1 that required partitive thinking, she still described division using only quotitive language. In fact, Linda’s quotitive meaning was so dominant that she was the only teacher who did not give a partitive interpretation for  $4 \div 3$ . Linda’s lack of partitive meanings in these tasks does not suggest that she did not have any. In fact, Linda demonstrated partitive meanings in subsequent tasks. Instead, the data implies that partitive meanings were simply not accessible to her when given decontextualized symbolic statements of division.

Mark described only a partitive meaning in Task 8 by saying “*a* broken up into *b* parts,” which allowed him to successfully model  $4 \div 3$ . When he finished this task, he said, “that’s all I got.” In the next task, he revealed a quotitive meaning when he successfully modeled  $4 \div 1/3$  by thinking about “four pies broken up into thirds.” These data suggest that he had schemes for both conceptualizations, but was not aware of them. I can think of two possible levels of awareness for Mark. First, it is possible he was aware of his changed thinking, but could not explain why or did not think it significant to mention. Second, he believed he was thinking in the same way each time. There is evidence to support the latter claim. It is important to note that his language for the two

meanings is very similar, “ $a$  broken up into  $b$  parts” versus “four pies broken up into thirds.” Despite the similarity in language, he demonstrated partitive reasoning when he modeled  $4 \div 3$  and quotitive reasoning when he modeled  $4 \div 1/3$ . The similarity in language is evidence that he thought his two models were conceptually equivalent. Soon after, I asked him to interpret  $4 \div 1/3$  as “four cut into one-third part,” which is how he phrased  $4 \div 3$ , and he interpreted this as “how many thirds are there in four wholes,” which is an assimilation of a partitive task to a quotitive scheme. This assimilation provided additional evidence that he thought his two models were conceptually equivalent. The data suggest that Mark did not have an awareness that there are two distinct quantitative meanings for division. Unfortunately, I neglected to ask him to reconsider  $4 \div 3$ , but instead using the quotitive meaning he displayed for  $4 \div 1/3$ .

From the onset, Mindi demonstrated a distinct awareness of the two conceptualizations for division. She gave two examples involving kids and groups, but which used only whole numbers, which is expected when partitioning children. She quantitatively distinguished between the quotients in each case, calling one the “group size,” and the other the “number of groups.” In Tasks 9 and 10, she had no trouble modeling  $4 \div 3$  and  $4 \div 1/3$  both quotitively and partitively. As such, there was no need to have her revisit these tasks. The only issue I observed was her language when she was modeling  $4 \div 1/3$  partitively. She said 4 was the group size, and she also said  $1/3$  was the group size. Despite her language, her partitive and quotitive schemes were correct for these tasks and she demonstrated an awareness of their distinction. She used “number of groups” and “group size” as her two quantitative options for thinking about the meaning of the divisor.

In Task 8, Mel gave a quotitive interpretation, using “group size” to think about the divisor. However, when modeling  $4 \div 3$ , she used a partitive conceptualization, then switched right back to a quotitive interpretation for  $4 \div 1/3$ . The data suggest that she had schemes for both conceptualizations, but lacked awareness that there were two meanings. When I asked her to compare her two solutions, she initially said her thinking was the same. When I rephrased  $4 \div 3$  as dividing four into three groups and asked if she was doing the same for  $4 \div 1/3$  she became perturbed. After a few moments, she came up with a strategy for how to divide four into one-third of a group, by saying that if four of something is one-third of a group then 12 would be the “new group size.” It seemed like she was inventing this way of thinking, in the moment, and she was fascinated with her discovery. She characterized this kind of thinking about division as a group-size transformation, going from a group that has four things in it, to a group that has 12 of those things in it. I wonder why she did not seem to think this about division when she gave a partitive model for  $4 \div 3$ . I agree that partitive meanings can be thought of as group-size transformations, but it could be problematic to characterize both the dividend and the quotient as “the group size.” I neglected to ask her to give a quotitive model for  $4 \div 3$ . The data suggested that Mel had a flexibility that allowed her to accommodate her “dividing into groups” division scheme when challenged with a fractional divisor so that she could assimilate “four divided into one-third of a group.” Prior to this realization, it would seem that, despite operating efficiently with both conceptualizations, she thought that the two meanings were the same.

For Task 8, Uma gave a partitive description of division saying, “ $a$  is the total amount, or beginning amount, and  $b$  is the number of parts.” She also said that division is

the “inverse of multiplication” and is used to find a missing factor, giving an example of \$4000 for 400 t-shirts and using division to find the amount per shirt, which is a partitive conceptualization. In the next task, she successfully drew a partitive model for  $4 \div 3$  and said she could not think of another way to do it. Then I asked her to model  $4 \div 1/3$ , to which she said she should be thinking, “How many copies of one-third make four?” Uma successfully drew a quotitive representation and she said that she could not think about it in another way. Again, these data suggest she had schemes for both conceptualizations, but she lacked awareness that there were two meanings. When I asked her to think about  $4 \div 1/3$  in the same way as  $4 \div 3$  she interpreted “break four into one-third groups” as partitioning four into groups of size one-third, and referred to her quotitive model as representing this meaning. This was a case of assimilation of a partitive task to a quotitive scheme. When I asked her to think about  $4 \div 3$  in the same way as she thought about  $4 \div 1/3$ , she said she needed to find “how many copies of three are in four.” She struggled at first but eventually drew four circles, circled three of the circles and wrote “1 copy of 3,” and then indicated that the fourth circle was  $1/3$  of a copy of 3. Uma demonstrated flexibility in adapting her quotitive scheme so she could sensibly justify the result of  $4 \div 3$ . However, she was not able to accommodate her partitive scheme adequately, and so she assimilated the partitive version of  $4 \div 1/3$  to her quotitive scheme.

For Task 8, Ursa immediately described both conceptualizations by saying “how many copies of  $b$  make  $a$ ” and “ $a$  divided up into  $b$  equal parts.” She gave a few examples that involved only whole numbers. In the next task, she had no trouble partitively modeling  $4 \div 3$  by drawing four circles, breaking the fourth circle into three parts, and then shading one whole circle and one-third of the fourth circle. She then tried

to use this same drawing to make sense of the quotitive question “how many copies of three make four?” While struggling with this she uttered “three copies of what make four,” which is a partitive question. Ursa identified “one copy of three” by identifying three whole circles, but then wondered why the fourth circle was broken into three pieces. After several minutes of struggling, she drew four new circles, enclosed three of them saying that is “one three,” which she described in writing as  $3/3$ , and then broke the fourth circle again into three pieces, shaded one piece and called it  $1/3$  saying the denominators have to be the same. The data seem to suggest that her issue had to do with a lack of unitizing, combined with interference from her partitive drawing. Eventually, she drew four circles side by side, cut them all into three rows, and satisfyingly said each row is four-thirds. She seemed relieved with this model, but which is partitive and not quotitive. It is not clear to me if she thought she had correctly produced a quotitive model, or if she was just content to draw any model that made sense to her. Her struggles at the end were an example of a task repeatedly failing to be assimilated to a quotitive scheme that could not accommodate it, only to finally be assimilated to a partitive scheme, bringing an end to her perturbation. Since she kept relying on her partitive drawing to make sense of the quotitive question, perhaps she thought the two different ways of describing division could be successfully modeled with one image, much like quantitative commutativity of multiplication can be modeled with a single image. In the end, she was unable to use her partitive drawing to justify her quotitive question. During the next task, she did not have trouble modeling  $4 \div 1/3$  with a quotitive conceptualization. She did not attempt a partitive conceptualization, so I prompted her to think of  $4 \div 1/3$  in the same partitive way she described  $4 \div 3$ . Like Mark and Uma, Ursa

also assimilated the task to a quotitive scheme. She phrased  $4 \div 1/3$  as “four split into one-third groups,” mimicking how she described  $4 \div 3$ , and then referred to her quotitive drawing and said, “I feel like that’s what I’m doing here. I have four, and I’m splitting it into one-third groups, and I have 12 groups.” Ursa is an interesting case because she demonstrated an awareness of two different ways to describe division, but then she lacked the necessary schemes to model them. In fact, her schemes seemed to interfere with each other, suggesting that she has not yet reflectively abstracted them to two distinct, generalized schemes that are flexible enough to cope with non-whole numbers.

The data presented above revealed that the teachers had varying abilities to model with both conceptualizations, and that only Mindi and Ursa had an awareness of the two meanings for division. I consider a person to have *partitive-quotitive awareness* if that person (1) is aware of the two quantitative conceptualizations for division, and (2) can operate under one conceptualization without interference from the other, while maintaining an awareness of the conceptualization with which they are operating. I use the construct *partitive-quotitive interference* to refer to any such interference described in condition 2. The data also revealed that awareness and capability were not connected. For example, Ursa acknowledged the two meanings in Task 8, but she was not able to quotitively model  $4 \div 3$ , and she could think of  $4 \div 1/3$  only quotitively. Whereas Mel did not initially demonstrate an awareness of the two meanings on her own, but she could successfully model both tasks with both conceptualizations when I prompted her, and even *invented*, in the moment, a partitive scheme for  $4 \div 1/3$ . These findings are similar to those of Simon (1993) who noted that the prospective elementary teachers in his study “were unable to think flexibly and consciously about division as partitive and quotitive...

Being unaware that they had selected a particular model of division, they were unable to reverse that decision when it proved unsuccessful (p.247).”

### **Partitive-to-Quotitive Assimilation**

The second condition of partitive-quotitive awareness is that a teacher can operate under one conceptualization without interference from the other. One type of partitive-quotitive interference revealed by the data occurred when teachers assimilated partitively-framed division tasks to quotitive schemata, a phenomenon I refer to as *partitive-to-quotitive assimilation*. This occurred for Mark, Uma, and Ursa, who all gave a quotitive description of  $4 \div 1/3$  even after I asked them to think about it in the same partitive way they thought about  $4 \div 3$ . In each of these three cases, the teacher experienced a moment of hesitation before giving their quotitive response. This could be because they perceived cues from either myself or from their own intuition that their second explanation of  $4 \div 1/3$  should somehow be different from their first quotitive explanation. They neutralized their momentary perturbation by interpreting the task as they had the first time, with a quotitive conceptualization. I do not consider this type of “mis-assimilation” as an act of accommodation. I simply consider such a mis-assimilation to be a coping mechanism to neutralize a perturbation. The teachers mentally changed the task so that they could operate with an existing scheme. For these teachers, partitive-to-quotitive assimilation prevented them from formulating a partitive meaning with a fractional divisor.

I was not surprised to witness partitive-to-quotitive assimilation when fractional divisors were present – I had witnessed this in several of my pilot studies. It was especially likely for Tasks 9 and 10, which involved decontextualized symbolic

statements of division. This lack of a context gave some measure of freedom so that the tasks could be assimilated to whatever schemes the teachers had.

### **Division Interpreted as another Operation**

Researchers have reported cases where subjects were asked to construct contexts that would elicit given symbolic division statements (e.g., Jansen & Hohensee, 2016; Simon, 1993). These researchers noticed that when fractional divisors were present, subjects had an increased likelihood to construct contexts that mistakenly elicited operations other than division. For this study, I also noticed this tendency during Task 11 of Phase 1 where the teachers were asked to model  $1/2 \div 3/4$ . Unlike other studies, I did not ask the teachers to invent a context that would elicit this instance of division, but instead I asked for their meanings of the expression as well as a drawing that represented their meanings. In doing so, the data revealed that teachers modeled multiplication rather than division. I describe this data next.

Linda gave a quotitive interpretation by writing, “how many copies of  $3/4$  (are) out of  $1/2$ ?” However, her model supported the operation “one-half of three-fourths” and she produced an answer of  $3/8$ . At first, she seemed confident with her conclusion but then became perturbed when she decided to check her answer procedurally by using the invert-and-multiply algorithm, which yielded  $2/3$ . Despite knowing the answer should be two-thirds, she was unable to justify this result using quotitive meanings.

Uma also initially interpreted the task as  $3/4$  of  $1/2$  and drew a picture. She performed the algorithm in her head (she admitted this later in the task) which exposed her misconception, and she changed to a quotitive meaning by asking, “How many copies of three fourths make one half?” She also admitted later that she was trying to find a way

of thinking that supported the answer produced by the algorithm. She was perturbed for some time, presumably because the divisor was larger than the numerator. When she finally realized she had to *reduce*  $3/4$  to  $1/2$  she changed her language to "what fraction of three-fourths makes one-half?" This suggests that Uma thought her first phrasing was not helpful, or perhaps not even sensible. Eventually, she succeeded – although not smoothly – at justifying the answer produced by the algorithm by using a quotitive conceptualization.

Ursa also drew a picture that supported *one-half of three-fourths*, she gave the answer  $3/8$ , and she ultimately caught her error through the algorithm. After several moments of reflection, I asked her what question she was trying to answer and she responded, "How many times does three-fourths go into one-half?" She kept trying to interpret her existing drawing in such a way that she could justify the answer of  $2/3$  as if her representation of *one-half of three-fourths* could also be used to represent  $3/4 \div 1/2$ . She said that she had been taught to get a common denominator and then divide only the numerators, which confirmed the answer of  $2/3$ . She stated that there "is a bunch of meaning behind this," but that she could not think of how to depict this meaning. Ultimately, she was unsuccessful with the task.

### **Procedural Contamination**

In the preceding section, I discussed the work of Linda, Uma, and Ursa who each errantly modeled multiplication for Task 11 (explain your meanings for  $1/2 \div 3/4$ ). These teachers arrived at an answer of  $3/8$ , but then caught their mistake through use of the invert-and-multiply algorithm. Of these three teachers, only Uma was successful at subsequently providing a quantitative justification of the result of the algorithm. I was

careful just now to say the *result* of the algorithm, as opposed to quantitatively justifying the algorithm itself, which is a much more challenging endeavor. Concerning the other teachers for this task, Mindi could not model the quotient, so she used the result of the algorithm to guide her to a quantitative justification. Mark, who normally defaulted to an algorithm when he could not reason productively, made a deliberate effort to avoid using the algorithm entirely on this task. Ultimately, he was unable to produce any result. Mel was the only teacher who did not appear to use the algorithm and was still successful at producing  $2/3$  (using a quotitive conceptualization). However, Mel did use the algorithm to confirm her correct result.

I wonder what data would have been revealed during Task 11, had none of these teachers known about the invert-and-multiply algorithm. One challenge for researchers who are working with pre- or in-service teachers is the fact that these teachers are already familiar with, and largely dependent on, the algorithms<sup>9</sup> for calculating the results of operations with fractions. It can be very difficult, maybe impossible, for a researcher to know the extent to which procedural results are guiding the thinking of their research subjects. Of course, this would not be a concern for research involving subjects who have not yet been exposed to the relevant procedures. However, in my study, it was an endemic concern. I define *procedural contamination* as the use of procedures to aid in conceptual understanding. I use this construct to describe the act of procedurally calculating a result (due to an unsuccessful attempt to reason quantitatively) followed by (or not followed by) trying to quantitatively justify the result (with or without success). I

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<sup>9</sup> I use the terms “algorithm” and “procedure” interchangeably.

acknowledge that the word *contamination* has a negative connotation, as if the use of procedures is detrimental to success, but this is not my intended meaning. In fact, procedural contamination tended to keep the teachers on track and helped them find and resolve errors in their thinking. Thus, the use of procedures *increased* the teachers' success at modeling calculations. I use the word "contamination" to suggest that the use of procedures tended to put an end to the teachers' attempts at quantitative reasoning. In this study, procedural contamination surfaced almost every time a teacher became confused by a task, either before arriving at an answer or when trying to verify an answer.

In a few cases, procedural contamination was the source of unnecessary perturbation when the procedure was done incorrectly, casting doubt on the teacher's answer. This occurred with Mindi while modeling  $5/3 \times 2$  on Task 6 in Phase 1. She procedurally calculated the result as  $10/3$  but then incorrectly converted this to the mixed number 1 and  $1/3$ . This caused several minutes of struggling, before I finally relieved her by directing her to convert  $10/3$  to a mixed number again. She smacked her forehead in disgust at the simple error, then she resolved the task to her satisfaction.

Most teachers were obvious when using a procedure to help them reason through a task. However, Uma was very discrete about it, as if ashamed to admit it. She knew that I was interested in researching her conceptual understanding of the operations, so she was fearful that using the algorithm to guide her thinking was somehow cheating. It took several tasks for me to begin to suspect that she was calculating in her head and then trying to quantitatively justify the result. At several moments during the study, I would ask her if she had first done the algorithm in her head, and at first, she would sheepishly admit that she had, only to boldly admit in later tasks that *of course* she had. This makes

me wonder how many times she was secretly trying to *reverse-engineer* a task that I did not catch, which makes me wonder to what extent this was happening with the other teachers as well. It is unfortunate that my subjects relied on the use of procedures to guide their quantitative reasoning, because the use of procedures offered the teachers an alternative approach, which seemed to quell the flow of data that would ordinarily accompany perseverance in problem solving.

#### **RQ1.4: Partitive Scenarios**

I designed Tasks 12-24 to address RQ1.4: *What meanings do teachers reveal when they engage in tasks that I designed to elicit partitive conceptualizations of division, with varying degrees of abstraction, and with a focus on fractional divisors?* I phrased Tasks 12-15 using language to elicit partitive meanings, but I used abstract wording – such as *amount* and *group*. Tasks 16-24 also evoked partitive conceptualizations but with less abstract quantities, such as *amount of water* and *number of containers*. I first discuss the data from Tasks 12-15, and then I discuss the data from Tasks 16-24.

**Task 12:**  $\frac{10}{3}$  copies of what amount combine to make the amount 15?

**Task 13:** 6 copies of what amount combine to make the amount 15?

**Task 14:** How much is in one group if 15 of something is split into  $\frac{10}{3}$  groups?

**Task 15:** How much is in one group if 15 of something is split into 6 groups?

Tasks 12 and 13 are examples of *missing multiplicand* tasks, which correspond to partitive division. Tasks 14 and 15 are *sharing* tasks, using language typically associated with partitive conceptualizations. All four tasks imply a quantitative basis for a partitive conceptualization of division, albeit the quantities are abstract. Despite the fact that Tasks 12-15 involved clearer partitive contexts than Tasks 8-11, there were still a few instances

of *partitive-to-quotitive assimilation*. I discuss such occurrences for Linda and Mel below.

### **Partitive-to-Quotitive Assimilation**

In Task 12, Linda rewrote the question as “how many copies of  $10/3$  make 15?” and she confidently stated that her rewritten question had the same meaning as the given question. She then unsuccessfully attempted to model the task quotitively. The partitive-to-quotitive assimilation was not surprising to me since, up to this point in the interview, Linda had demonstrated only an awareness of this one meaning for division. But in Task 13, Linda’s work revealed a partitive interpretation. I did not detect any perturbation due to the fact that she was now using a meaning that was different from her dominant quotitive meaning. Perhaps there was no perturbation because she did not connect Task 12 to the operation of division. Or perhaps, she did not sense that her partitive actions were a contradiction to her quotitive meanings. Whatever the reason, Linda did not interpret Task 12 with a partitive conceptualization.

In Task 14, Mel also demonstrated partitive-to-quotitive assimilation. She said “so this (pointing to  $10/3$ ) is telling me how many groups I have, and so... there’s 15 things and each group is this size (again pointing to  $10/3$ ), so if a group is three and one-third items then...” Her transition happened very quickly and, in the same sentence, she switched from one quantitative meaning for  $10/3$  to another. She then modeled the task using a quotitive interpretation by showing how many groups of three and one-third cookies make 15 cookies. However, in Task 15, she gave a partitive interpretation by splitting 15 kids into six elevators, which required some kids to be cut in half (she chuckled).

## Quotitive-Modeling Interference

I repeat that the second condition of *partitive-quotitive awareness* is that a teacher can operate under one conceptualization without interference from the other. As discussed earlier, one such type of interference is *partitive-to-quotitive assimilation*. Another type of partitive-quotitive interference is connected with the modeling itself. With quotitive meanings, the dividend and divisor each refer to the same unit, and so drawing representations of them is less challenging. When modeling quotitive division, where the quotient would be a whole number, it is often sufficient to draw a representation of the dividend with a representation of the divisor nestled within, and then count how many copies of the divisor constitute the dividend. A person can be successful doing this without an awareness of the re-unitization that occurs when the divisor amount is thought of as a single unit – in other words, without maintaining two levels of units. In such a case, successful division amounts to little more than a counting exercise. In my study, every teacher was successful at modeling quotitive division when the quotient was a whole number, even if the divisor was not. While the values of the dividend and divisor with a quotitive conceptualization each refer to the same unit, this is not true for a partitive conceptualization. This presents a challenge while modeling with a partitive conceptualization because two levels of units must be maintained and/or represented during the modeling process. Teachers can develop habits and/or expectations for modeling division that are productive for quotitive conceptualizations, but which are obstructions to successful partitive models. I use ***quotitive-modeling interference*** to refer to any difficulties in modeling partitive division that are caused by inappropriately applying meanings or strategies that are typically associated with quotitive division

models. In other words, quotitive-modeling interference occurs when aspects of quotitive modeling schemes inadvertently obstruct attempts at partitive modeling. I illustrate this construct by discussing some of the data for Linda, Uma, and Ursa.

While working on Task 13 in Phase 1, Linda was trying to answer “6 copies of what amount combine to make the amount 15?” She drew six rectangles, and her goal was to put something in each rectangle so that the total would be 15. She guessed and checked, and eventually discovered that if she partitioned each rectangle into five pieces and counted in pairs, that she would count to 15. I share Linda’s work in Figure 15.

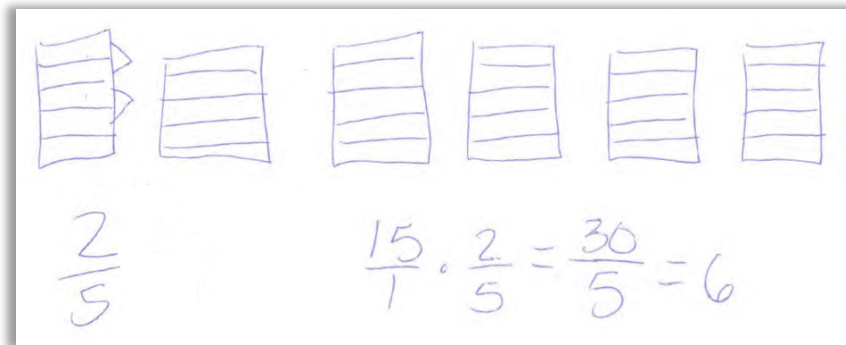


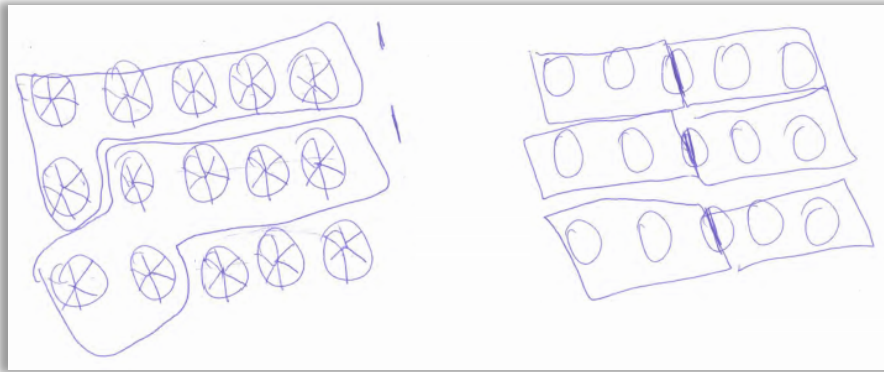
Figure 15. Linda’s work on Task 13 in Phase 1.

Linda should have consequently concluded that each pair represented an amount of 1, thus each rectangle had an amount of 2 and  $\frac{1}{2}$ . Instead, she concluded that the answer was “two-fifths copies of one.” By “copies of one,” the data suggest she meant *copies of one rectangle*. Her tone suggested she was confident in her answer of  $\frac{2}{5}$ , until she tried to check it by procedurally calculating  $6 \times \frac{2}{5}$ . The fact that she tried to verify her answer through this calculation demonstrated to me that she properly assimilated the task, in that she was indeed trying to find an amount, six copies of which would yield 15. Not surprisingly, she was perturbed when her procedural calculation did not yield 15. She eventually realized that she had established that 15 copies of  $\frac{2}{5}$  produced 6, which she

confirmed procedurally, while acknowledging that this was not the objective. To resolve this perturbation, she re-unitized by considering each pair of fifths of a rectangle as one, and adjusted her answer to 2 and  $\frac{1}{5}$ , but this whole number and this fraction do not have the same referent. Her answer should have been 2 and  $\frac{1}{2}$ , because one fifth of a rectangle is really one-half of the new unit (which new unit is two-fifths of a rectangle). She checked six times 2 and  $\frac{1}{5}$  and, again, did not get 15. Ultimately, Linda was unable to resolve this task to her satisfaction. Linda's difficulties with this task reveal her weakness with re-unitizing. It was difficult for her to look at one piece out of five contiguous pieces and think something other than *one-fifth*. This is why she initially gave an answer of  $\frac{2}{5}$  in each container, as if she was trying to answer the partitive question "15 copies of what amount make 6?" instead of "6 copies of what amount make 15?" In this case, Linda's difficulty with re-unitizing is an example of *quotitive-modeling interference*, in that she did not perceive two levels of units in her representation, which is also true of her quotitive models. One could argue that she did re-unitize when she counted in pairs to get to 15, but I contend that the act of counting does not imply an awareness of a second unit of reference – she did not view each of what she was counting as a new whole. Instead, Linda was limited to imagining that one of the six blocks served as the only unit-of-measure, but to be successful with this task, Linda needed to be able to coordinate two levels of units, the *number of copies* and the quantity referred to by the 15.

Now let us consider Uma's work on Task 13 in Phase 1. Uma began by drawing 15 circles, divided each circle into six pieces, then stopped. The data is not conclusive about why she stopped, but she adopted a quotitive meaning instead, by circling two

groups of six circles, with three circles remaining. After some fumbling, Uma eventually said that the three remaining circles were “three out of six” and so she gave an answer of  $2 \text{ and } 1/2$ . Despite her quotitive drawing, she then summarized her work using partitive language by saying, “six copies of two and one-half make 15,” which she demonstrated with a new visual of 15 circles. Figure 16 illustrates Uma’s work on Task 13.



*Figure 16. Uma’s work on Task 13 in Phase 1.*

I pointed out to Uma that her two pictures looked different, and she agreed and said that “it wasn’t until I came up with two and a half copies of six in 15 that I felt I could demonstrate that there are actually six copies of two and a half in 15.” To summarize her solution path, she abandoned a partitive model in favor of a quotitive model, arrived at an answer, invoked numerical commutativity, and then demonstrated a partitive conceptualization instead. This approach was only possible because of a dependence on numerical commutativity of multiplication, which was a quantitatively weak link in her strategy. I consider Uma’s work to reveal aspects of quotitive-modeling interference for the following reasons: (1) her strategy was dependent on quotitive meanings to generate an answer in the first place, (2) she used a quotitive modeling tactic by beginning with a representation of the dividend, and (3) she defaulted to quotitive modeling to cope with the confusion that resulted from being unable to think of the six in terms of the circles she

had already drawn. In other words, she was tempted to draw a representation of the *six* in terms of circles, but she instinctively knew that it was inappropriate to do this. She adjusted by deliberately reinterpreting the task with a quotitive conceptualization so that she could imagine that both the six and 15 were referring to circles. In reference to point number (2) above, it is perhaps more productive to model partitive division by drawing a representation of the divisor first, in order to devise a strategy for the partitioning and/or iterating of the dividend. Once a person devises such a strategy, that person could then focus on a representation of the dividend and on how to partition and/or iterate appropriately. I emphasized this approach during the teaching experiments in Phase 2.

Now let us consider Ursa's work on Task 14, where she was trying to determine "how much is in one group if 15 of something is split into  $10/3$  groups." Ursa began by drawing 15 circles and acknowledged that she was trying to make three and one-third groups. She suggested that if she put five circles in each group, then she would only have three groups, which is too few. If she put four circles in each of the three groups, then she would have three circles left over, and she said she did not know what to do with the three circles that remained. However, this task triggered numerical division and she procedurally calculated  $15 \div \frac{10}{3} = 4\frac{1}{2}$ . Ursa returned to her diagram and broke up the remaining three circles each in half, and she distributed one-half of a circle to each of the three groups so that each group would have four and one-half circles, with one and one-half circles remaining. She said she did not know what one-third of a group meant, nor did she recognize that the remaining circles constituted one-third of a group, likely because the size of the group seemed inconsistent as she kept toying with how many circles were in a whole group. Unsatisfied, she started again by drawing a picture of three

and one-third rectangles. She put “4” in each whole rectangle (not “4 and 1/2”) and wrote “3” next to the one-third rectangle, but she said that it did not make sense. Ultimately, Ursa was unsuccessful with this task and her work is illustrated in Figure 17.

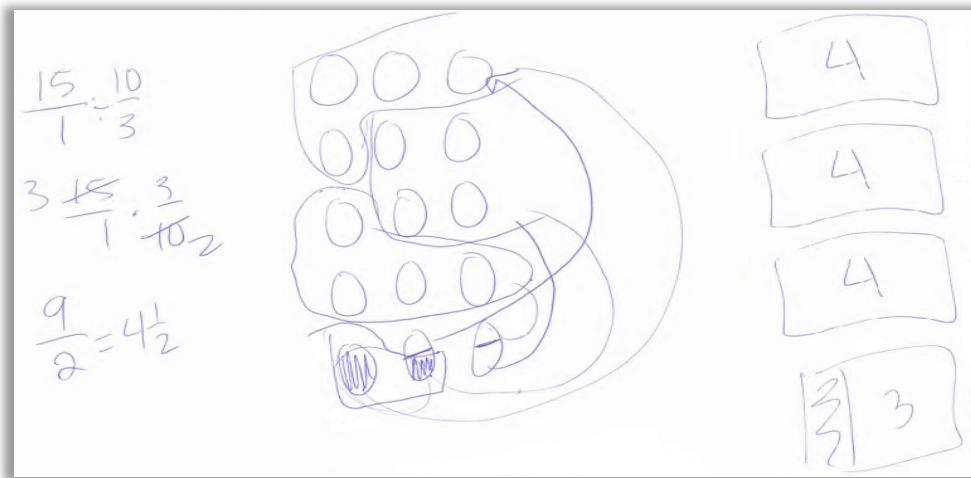


Figure 17. Ursa's work on Task 14 in Phase 1.

Ursa's difficulties had several sources, one of which was her insistence on drawing a representation of the dividend first, a form of quotitive-modeling interference. Other issues related to her understanding of what “one-third group” meant, and to her insistence on expressing  $10/3$  groups as a mixed number.

Quotitive-modeling interference hindered Linda, Uma, and Ursa from being able to productively construct partitive models. As such, these teachers did not possess *partitive-quotitive awareness*, which – as I discussed earlier – is a significant obstacle to advancements in partitive meanings for division.

### Procedural Contamination

In Task 12, Mark was trying to answer the question “ $10/3$  copies of what amount combine to make the amount 15?” He correctly identified this as a missing multiplicand task (i.e., a partitive conceptualization), but he could not model the division. Instead, he

resorted to using the algorithm to produce 4 and  $\frac{1}{2}$  and then demonstrated quantitatively that the product  $(3 \text{ and } \frac{1}{3}) \times (4 \text{ and } \frac{1}{2})$  yielded 15. When I asked if he would have been able to resolve this task without using the algorithm, he said that he would not. This example is interesting because he did not use the result of the algorithm to guide him toward a strategy that would have been helpful in resolving the task without the algorithm. Thus, the algorithm did not serve to advance his problem-solving abilities. Something similar happened in Task 13 when Uma was trying to answer the question, “6 copies of what amount combine to make the amount 15?” I discussed this in more detail in the preceding section of this chapter, but in summary, she deliberately converted to a quotitive conceptualization, arrived at an answer, invoked numerical commutativity, and then demonstrated a partitive conceptualization by showing that six copies of two and one-half make 15. I consider the procedural use of commutativity in her strategy to be an instance of procedural contamination. As I said elsewhere, it was the quantitatively weak link in her solution path, without which she may not have been able to resolve the task at all.

### **Specific Water-Container Task Results**

Tasks 16, 17, 19, 20, 22, and 23 involved distributing a known amount of water into a known number of containers. These tasks did not use typical language associated with division, so some teachers may not have realized that division was a relevant operation. I repeat these tasks below.

**Task 16:** Suppose 13 gallons of water fill 5 equal containers. Describe the capacity of one whole container.

**Task 17:** Suppose  $\frac{3}{4}$  gallon of water fills 5 identical containers. Describe the capacity of one whole container.

**Task 19:** Suppose 27 gallons of water fill  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 20:** Suppose 3 gallons of water fill  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 22:** Suppose 5 gallons of water fill  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

**Task 23:** Suppose  $\frac{7}{4}$  gallons of water fill  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

For these tasks, I was only interested in modeling the teachers' schemes, as revealed through their efforts to solve the problems, and whether these schemes were conducive to forming a foundation for the invert-and-multiply algorithm. As such, I did not generally ask the teachers to provide a symbolic operation that would answer the question, although in several cases, this data surfaced organically. For these tasks, no teacher demonstrated partitive-to-quotitive assimilation. This is certainly due to the explicit contexts of the tasks, where water and containers served to provide concrete things to think about, thus making it unlikely to conflate the two quantities and interpret the division tasks quotitively. Also, every teacher was eventually successful at creating a partitive model that supported a valid quotient. The only exception was Mark in Task 16 who did this task procedurally, without the aid of a visual model. This overall success by the teachers does not suggest that there were no cognitive barriers in the process, or that the schemes that were revealed were generalizable. I discuss these matters below.

For Task 16 (13 gallons fill 5 containers), all six teachers first distributed whole numbers of gallons to the five containers, and then figured out how to evenly distribute

any remaining gallons. They did this in slightly different ways, but none of the teachers gave a visual model that indicated a contraction of the 13 gallons by a factor of  $\frac{1}{5}$ . The work from Uma and Linda are provided in Figure 18.

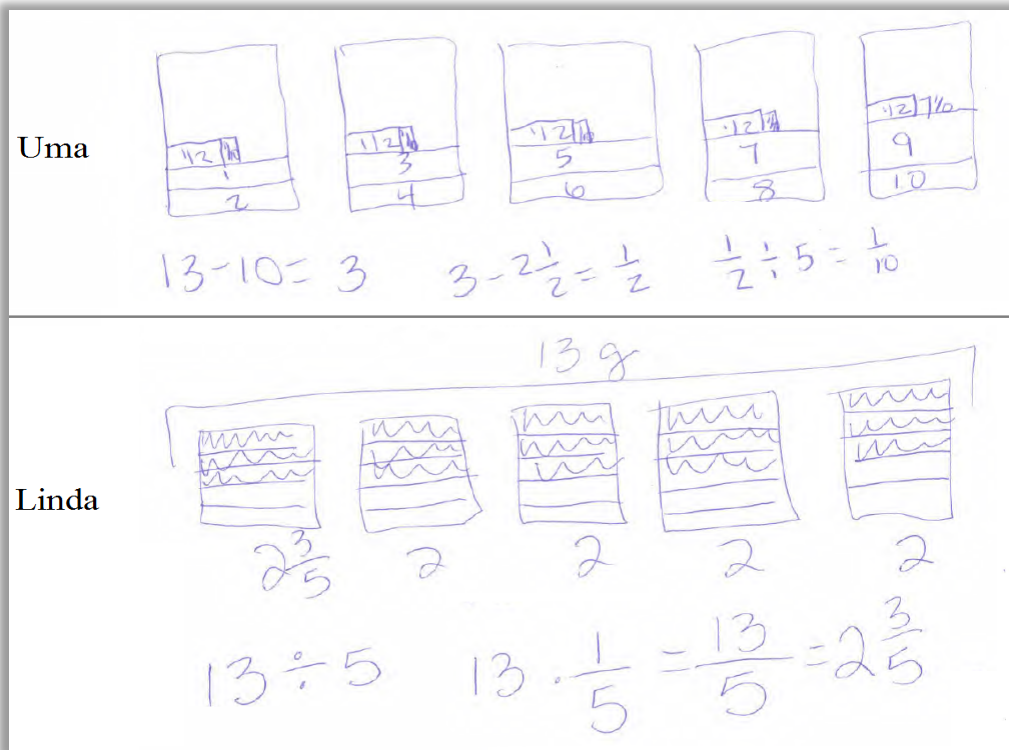


Figure 18. The work of Uma and Linda on Task 16 in Phase 1.

Unlike in Task 16, for Task 17 ( $\frac{3}{4}$  gallon fills 5 containers), some teachers (Mindi, Uma, and Ursa) gave a visual model that suggested a contraction by a factor of  $\frac{1}{5}$ . This is because in Task 17, the dividend was smaller than the divisor, so the teachers could not first share whole numbers of gallons in the same way they worked through Task 16. The work of Mindi, Uma, and Ursa is in Figure 19.

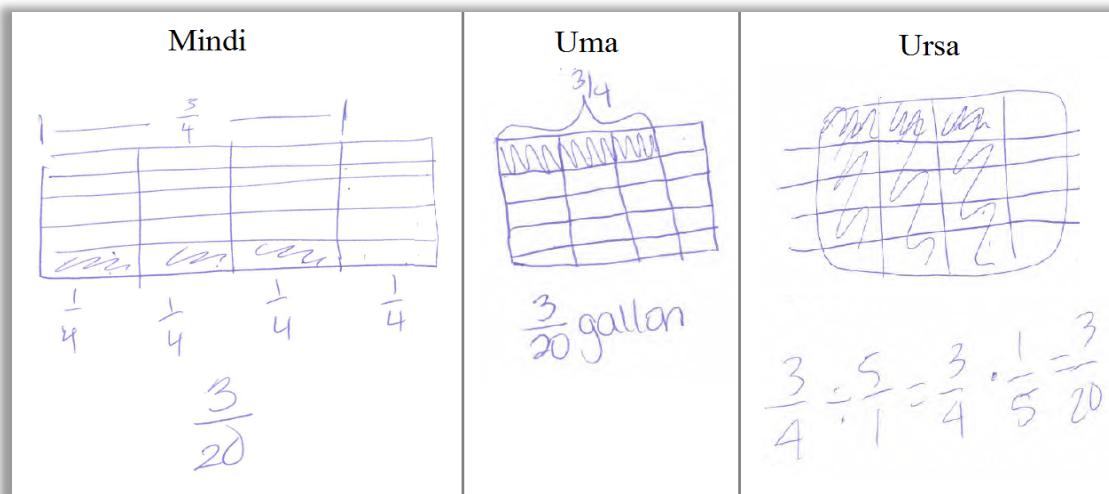


Figure 19. The work of Mindi, Uma, and Ursa on Task 17 in Phase 1.

For Task 19 (27 gallons fill  $9\frac{3}{4}$  containers), some confusion resulted from the non-whole number of containers. Linda began by interpreting nine-fourths of a container as nine one-fourths, which she drew as two and one-fourth containers. She then became perturbed and paused to reflect for several minutes. Speaking to herself, she confirmed that the water “should be shared equally.” I asked her what the goal was, and she said she was looking for the amount in a container, and she circled a whole container. She proceeded to deal out gallons, one at a time, until she arrived at nine gallons per container (including the partial container). I share Linda’s work in Figure 20.

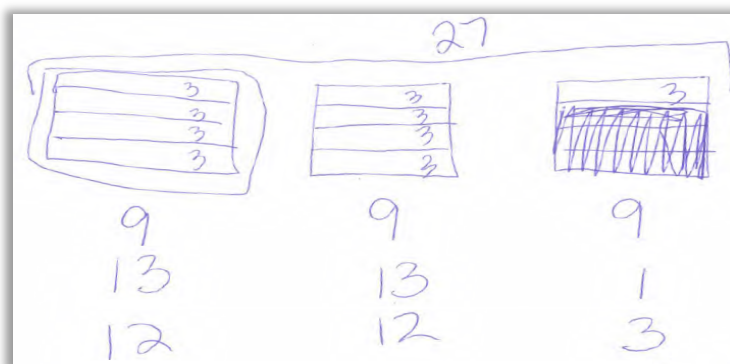


Figure 20. Linda’s work on Task 19 in Phase 1.

It appeared that she was not imagining the containers (and the partial container) to be full, because she was not bothered when she imagined the partial container to be big enough to hold nine gallons of water – or perhaps she was reimagining the containers as three equal containers so that she could do fair-sharing. At this point, I emphasized that the 27 gallons should fill up the partial container and the two whole containers. After thinking, she said that 15 gallons was too much, presumably thinking that 15 gallons in each whole container would exceed the allotted 27 gallons. Linda adjusted her answer and concluded that 13 gallons would be in each whole container, and that one gallon would be in the “quarter container.” She was guessing and checking until she had some same amount of water in each of the whole containers, and some smaller amount of water in the quarter container. Her answer suggested that she did not realize that if the nine quarter containers were filled, then the quarter container should contain one-fourth as much water as a whole container. Perhaps she was content with this answer because she lost sight of the fact that the containers were filled and she reasoned that if 13 gallons could fit in a whole container, then 1 gallon would be able to fit in a quarter container. I guided her by commenting, “the quarter container has to be filled and so if there are 13 gallons here (gesturing at a whole container) and only one gallon here (gesturing to the quarter container) then...”, and I trailed off leaving her to think about the reasonability of what I was saying. Linda then changed her answer to 12 gallons in each whole container, and three gallons in the quarter container, at which point she filled in each of the nine quarter containers with a 3. She was using a guess-and-check strategy, tweaking her answer until the conditions were met (with a little help from me), and she was fortunate that the task involved numbers that were conducive to success with this strategy. It is important to

note that Linda did not divide 27 by nine, perhaps because she was no longer thinking *nine* of anything, but rather *two and a quarter* of something.

Ursa also was initially confused by the non-whole number of containers in Task 19. She first calculated  $27 \div 9/4$  to get an answer of 12, but then I asked for a visual model. She drew two whole containers and one-fourth of a container more. She was momentarily confused, and she was not sure how the 12 related to the picture. This revealed that she did not initially know how to interpret the question. After several moments of silence, she divided the whole containers into quarters as well and counted nine quarter-containers in total, put “3” in each quarter container, and confirmed that 12 is in one whole container.

For Task 19, Mindi also used a guess-and-check strategy. She drew three containers and knew that only one-fourth of the third container would be full. She did not initially partition the two whole containers into quarter pieces. She began to guess and check, first with “10” in each whole container, then with 12 in each whole container. She stopped here because “ $12 + 12 + 3 = 27$ .” At this point, she partitioned the wholes into quarters and labeled each quarter as “3.” She mentioned that she only had a guess-and-check strategy. The data suggests that Mindi was so focused on 2 and  $1/4$ , instead of  $9/4$ , that it did not occur to her to divide 27 by 9.

In Task 19, only Uma and Mel divided 27 by nine to find the amount in one quarter-container. For the other teachers, it was a hindrance to think of  $9/4$  as *2 and 1/4*. This prevented them from dividing 27 gallons by nine to get the amount of water in one quarter-container, and forced Linda and Mindi to use a guess-and-check strategy, which

is not generalizable to the point of forming a meaningful foundation for the invert-and-multiply algorithm.

For Tasks 22 (5 gallons fill  $\frac{2}{3}$  container) and 23 ( $\frac{7}{4}$  gallons fill  $\frac{2}{3}$  container), no teacher had trouble partitioning the given amounts of water in half and then combining the three halves. As an example, I share Linda's work for Task 22 in Figure 21. For Linda, and others as well, it was not clear to me whether she thought to combine the three amounts of water through multiplication.

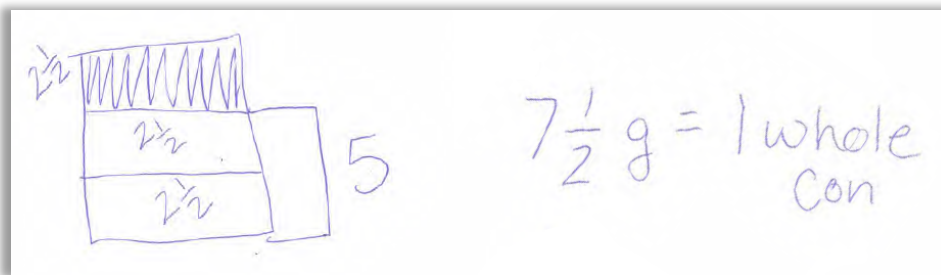


Figure 21. Linda's work on Task 22 in Phase 1.

### General Water-Container Task Results

Tasks 18, 21, and 24 involved distributing an unknown amount of water into a known number of containers, which I repeat below.

**Task 18:** Suppose a certain amount of water fills 5 identical containers. Describe the capacity of one whole container.

**Task 21:** Suppose a certain amount of water fills  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 24:** Suppose a certain amount of water fills  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

For these tasks, I was looking for generalizations based on the teachers' strategies they used for the same situations but with known amounts of water. I summarize the results in

Table 11. The ✓ means the teacher suggested the indicated operations, and the hash mark means they did not.

Table 11

*The Teachers' Results on the General Water-Container Tasks*

|       | Task 18<br>? water<br>5 containers |                | Task 21<br>? water<br>9/4 containers |                |               | Task 24<br>? water<br>2/3 container |                |               |
|-------|------------------------------------|----------------|--------------------------------------|----------------|---------------|-------------------------------------|----------------|---------------|
|       | $? \div 5$                         | $1/5 \times ?$ | $? \div 9/4$                         | $4/9 \times ?$ | $4(? \div 9)$ | $? \div 2/3$                        | $3/2 \times ?$ | $3(? \div 2)$ |
| Linda | ✓                                  | ✓              | ✓                                    | -              | -             | -                                   | -              | ✓             |
| Mark  | -                                  | ✓              | -                                    | -              | -             | -                                   | -              | -             |
| Mindi | ✓                                  | ✓              | -                                    | -              | ✓             | -                                   | -              | ✓             |
| Mel   | ✓                                  | ✓              | ✓                                    | ✓              | ✓             | -                                   | ✓              | ✓             |
| Uma   | ✓                                  | -              | ✓                                    | -              | ✓             | -                                   | -              | ✓             |
| Ursa  | ✓                                  | -              | -                                    | -              | ✓             | -                                   | -              | ✓             |

For Task 18, everyone identified at least one operation that would resolve the task. For Task 21, Linda and Mark could not generalize to something other than division because their schemes relied on knowing the given amount of water. For Task 24, only Mark could not generalize to something other than division. In several cases for Tasks 21 and 24, the teachers could describe the two-step process, but they did so without explicitly connecting this to one-step multiplication with a fractional multiplier. It is noteworthy that for Task 24, no teacher indicated division as an operation. In fact, while working on Task 22 (5 gallons fill 2/3 container), I asked Mel if there was an “operation that this (task) triggers for you,” and she answered “no.” She said that she could solve the problem by dividing by two and then tripling, but she did not recognize that she could solve the problem by a single act of division. She said, “Until I thought about it, I did not know what I was gonna do with the numbers...I wouldn't think of a problem like this as ‘I'll just go to an algorithm.’” This finding lends support to the findings of Jansen and

Hohensee (2016) who observed that, with the same rational divisor of  $\frac{2}{3}$ , only three out of 17 subjects identified the correct symbolic statement of division given a context.

Another issue that surfaced during the water-container tasks was the improper use of symbols. In Task 21, Ursa was trying to generalize a process for describing the capacity of one whole container when some amount of water fills  $\frac{9}{4}$  containers. She correctly explained that she could divide the amount of water by nine, and then copy this amount four times. However, when expressing her thinking symbolically, instead of writing  $(x \div 9) \times 4$ , she wrote  $(x \div \frac{9}{4}) \times 4$ , which would numerically produce an incorrect amount. I share her work in Figure 22.



Figure 22. Ursa's work on Task 21 in Phase 1.

As another example, in Task 24, Linda committed a similar error when she was trying to describe the capacity of one whole container when some amount of water fills  $\frac{2}{3}$  of the container. The data suggested that she realized she could divide the amount of water in two, but when she tried to express this symbolically, she errantly wrote  $x \div \frac{2}{3}$  as representing the amount of water in one-third of the container. I share Linda's work in Figure 23.

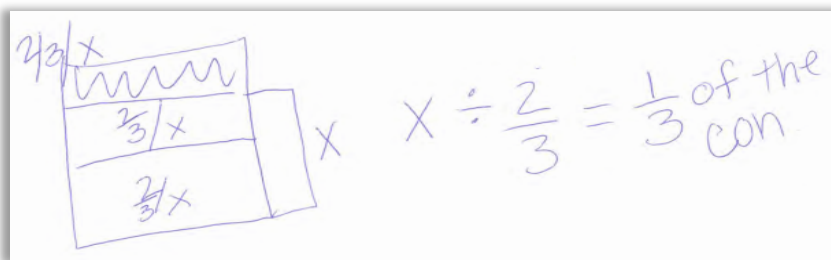


Figure 23. Linda's work on Task 24 in Phase 1.

This kind of error is not surprising given that a fraction such as  $9/4$  can be interpreted as nine one-fourths. As such, some people may interpret “divide the water by nine quarter-containers” into the partly symbolic expression “ $x \div 9$  quarter containers,” which is then inappropriately expressed as  $x \div 9/4$ . I consider this kind of error to be less significant because it is not the consequence of an error in quantitative reasoning. However, such an error poses a problem when trying to build a unit rate meaning for partitive division by connecting the symbol  $x \div 9/4$  to the capacity of one whole container, as opposed to the capacity of one quarter-container.

### RQ1.5: Invert-and-Multiply Algorithm

I intended Task 25 to help answer RQ1.5: *What justifications do the teachers provide for the invert-and-multiply algorithm after working through the (water-container) tasks?* By placing this task at the end of Phase 1, I expected that the preceding tasks would influence the teachers' attempts to justify the algorithm. Specifically, I expected that the teachers would leverage their generalizations of the general water-container tasks (Tasks 18, 21, and 24) to justify the algorithm. In Table 12, I summarize whether each teacher was successful with a generalization for Tasks 21 and 24, and whether each teacher was subsequently successful at justifying the invert-and-multiply algorithm. The ✓ indicates that the teacher successfully justified the algorithm with the

indicated conceptualization, the ✖ indicates the teacher tried unsuccessfully, and the hash mark indicates that the teacher did not attempt with the indicated conceptualization. I discuss each teacher separately in the paragraphs that follow.

Table 12

*The Teachers' Results when Trying to Generalize Division by a Fraction*

|       | Task 21<br>? water<br>9/4 containers | Task 24<br>? water<br>2/3 container | Task 25<br>$a \div \frac{b}{c} = a \times \frac{c}{b}$ |           |
|-------|--------------------------------------|-------------------------------------|--------------------------------------------------------|-----------|
|       | Generalized to<br>4/9×? or 4(?÷9)    | Generalized to<br>3/2×? or 3(?÷2)   | Partitive                                              | Quotitive |
| Linda | no                                   | yes                                 | ✖                                                      | ✖         |
| Mark  | no                                   | no                                  | ✖                                                      | -         |
| Mindi | yes                                  | yes                                 | ✓                                                      | -         |
| Mel   | yes                                  | yes                                 | ✓                                                      | -         |
| Uma   | yes                                  | yes                                 | ✖                                                      | -         |
| Ursa  | yes                                  | yes                                 | -                                                      | ✖         |

Ursa successfully generalized for Tasks 21 and 24, however, for Task 25, she tried to justify the algorithm using a quotitive conceptualization. She did not seem aware that this was not the same conceptualization as the preceding tasks, which is indicative of a lack of *partitive-quotitive awareness*. I do not discuss Ursa's data on Task 25, because quotitive justifications for the algorithm were not relevant to my research questions. Mel and Mindi were both able to provide a quantitative justification of the algorithm. I also do not discuss their work here because I am reporting on cognitive obstacles, and it suffices to say that I was satisfied with their explanations, which were generalizations from the water-container tasks. For the remainder of this section, I discuss the data for Task 25 from Linda, Mark, and Uma.

Linda did not possess a general scheme for partitive division with a divisor of  $9/4$ , but she did possess such a scheme for a divisor of  $2/3$ . Furthermore, she revealed in Task 22 (5 gallons fill  $2/3$  container) that the operation  $5 \div 2/3$  would resolve the task, which correctly connected this type of task with symbolic division. She began Task 25 by reflecting on the water-container tasks she had just finished, but without looking back at her work on the previous tasks. Consider Linda's utterance below.

So, we were talking about water and containers and...when I divide by the fraction, that's (she did not point at anything) giving me what part of it is, and then I add all the parts. So, that's multiplication because I add the amount of parts.

Although Linda was thinking aloud, this quote provides an example of unclear language, something that plagued Linda repeatedly throughout the study. She constantly used vague pronouns, mentioned numbers without referents, and phrased quantities ambiguously. For example, when she said, "part of it is," she was not clear about the meaning of *it* in her sentence. Also, her language "add the amount of parts" sounds like the quantity, *number of parts*, instead of the quantity, *amounts of stuff contained in those parts*. These examples of vague language are especially prominent in all of her work involving decontextualized symbolic operations. Returning to Task 25, at one point, she said, "*a* is dividing by *b*... times *c*...(pause)...no." I could not tell if she was on a productive path of quantitative reasoning, or if she was simply stating the meaning of the multiplication statement in the prompt. Regardless, she trailed off and began to fumble with some other symbolic statements, as depicted in her writing in Figure 24.

$$a \div \frac{b}{c} = a \times \frac{c}{b} \quad a \div b \cdot c \quad \frac{a/b}{c} = \frac{ac}{b}$$

Figure 24: Linda's work on Task 25 in Phase 1.

After a few minutes of struggling to say anything she had confidence in, we had the following dialogue.

*Excerpt 1*

---

|   |        |                                                                   |
|---|--------|-------------------------------------------------------------------|
| 1 | Linda: | It's so hard because I just know that the reciprocal works. I     |
| 2 |        | wouldn't know how to explain why.                                 |
| 3 | MW:    | Explain this part (pointing only to the division operation in the |
| 4 |        | prompt). What is division trying to accomplish?                   |
| 5 | Linda: | How many parts of $b$ $c$ can I pull out of $a$ .                 |
| 6 | MW:    | Okay, I have no questions.                                        |

---

When I asked Linda what the division statement was trying to accomplish, she gave a quotitive explanation, despite practicing partitive meanings during the water-container tasks. At this point, I knew that she would not be able to conceptually generalize the invert-and-multiply algorithm by relying on the partitive division tasks she had reasoned through moments earlier.

In Tasks 21 and 24, Mark did not give a partitive generalization for a divisor of  $9/4$ , nor for a divisor of  $2/3$ . Not surprisingly, he decided to use a specific example to help him reason through Task 25. He imagined six candies and wrote  $6 \div 3/2$ , which he changed to  $6 \div 1.5$ . He drew two rows of three, said the first row is one group, and “1.5 candies” from the second row is a half of a group, so he said there were “4.5 candies per group.” Mark's work is illustrated in Figure 25.

Handwritten work on Task 25 in Phase 1. The left side shows two divisions:  $6 \div \frac{3}{2}$  and  $6 \div 1.5$ . Below these are two circles, each containing the number 4. The right side shows a multiplication:  $6 \cdot \frac{2}{3} = \frac{12}{3}$ , with a circled 4 below it.

Figure 255. Mark's work on Task 25 in Phase 1.

Mark did not seem aware that his notion of a group changed from a group-size of three to a group-size of four and one-half. Regardless, it was clear he was using a partitive conceptualization. However, when I asked him what the unit should be on the 1.5, he said, “candies per group,” which was troublesome for two reasons: (1) this is the type of unit for a divisor with a quotitive conceptualization, and (2) this implied a third group-size. It is likely he answered this way because his image highlighted both 1.5 groups and 1.5 candies (in the second row). He then procedurally calculated 4, which contradicted his 4.5. Mark was unable to resolve this perturbation, and ultimately, he gave up.

Uma, on the other hand, was able to generalize for Tasks 21 and 24, so I expected her to be able to describe the algorithm. Uma used Task 20 (3 gallons fill  $\frac{9}{4}$  containers) to guide her thinking on Task 25. Her work on Task 20 is shown in Figure 26.

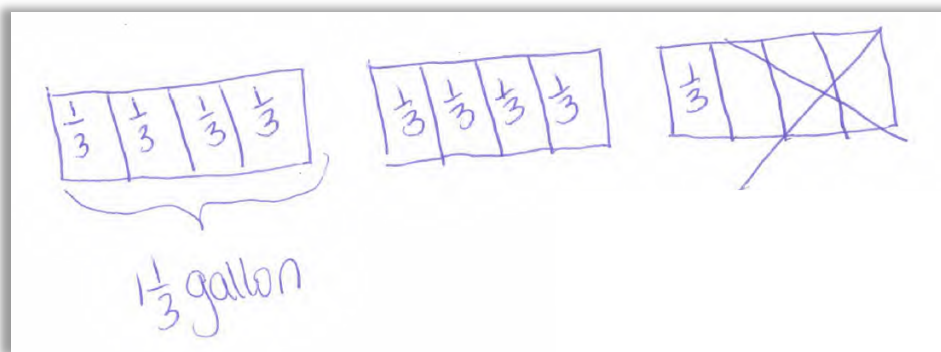


Figure 266. Uma's work on Task 20 in Phase 1.

Reflecting on her work from Task 20, Uma realized that she would divide 3 by 9, then multiply by 4. Unfortunately, she did not recognize this as a single step of multiplying by  $\frac{4}{9}$ , because she said that “3 times  $\frac{4}{9}$  means to multiply by 4, then divide by 9,” which is the reverse of the order she was expecting. Regrettably, this compelled her to try to find another way to make sense of the algorithm. After 20 minutes of grappling with this task, she gave up. It was unfortunate that Uma did not realize that multiplication by  $\frac{4}{9}$  can be construed in two ways: (1) multiply by four and then divide by nine, or (2) divide by nine and then multiply by 4.

### Summary of Phase 1 Findings

The data from Phase 1 provided evidence of a variety of meanings held by the teachers. Since I centered my overall study on researching cognitive advancement, I focused my analysis on the aspects of the teachers’ schemes that were not conducive to acquiring advanced meanings for partitive division. I summarize these findings in the paragraphs that follow.

Concerning *fractions as reciprocal measures of relative size*, five of the teachers (all except Linda) were ultimately able to compare the red and green strips to each other in Task 3 by imagining reciprocal fractions, but only Mel and Mindi were able to provide

a quantitative justification. However, for none of the teachers was imagining the reciprocal self-evident, suggesting that for none of the teachers was the scheme reversible. The data revealed that the teachers' use of mixed numbers was a severe hindrance to imagining the reciprocal.

Concerning fractional multipliers, of the five teachers who had a meaning for  $5/3$  as a multiplier, none of them thought to contract 2 by a factor of  $1/3$ , followed by expanding this amount by a factor of 5. Four of the teachers thought of  $5/3$  as a mixed number in order to model the product. This was significant to me because mixed numbers obscured the elegance of the dual-scheme for fractional multipliers of contraction followed by expansion. While trying to reason with a fractional multiplier, Ursa demonstrated a ***multiplier-switch***, which occurs when someone begins to model a product, imagining one factor as the multiplier, but then, during the process of modeling, the person instead views the model as a justification for the other factor as the multiplier. This can be an impediment to formulating a productive meaning for a fractional multiplier because it provides a sensible way to circumvent the formulation of such a meaning.

Concerning the creation and coordination of multiple levels of units, the data in my study revealed that the lack of a specific context contributed significantly to the teachers' inability to maintain multiple levels of units. Robust meanings for multiplication, and hence division, are dependent on being able to maintain awareness of multiple levels of units. All teachers encountered perturbations at some point due to issues related to re-unitization.

Concerning the dual meanings for division, no teacher demonstrated *partitive-quotitive awareness*, which is when a person possesses the following two characteristics: (1) awareness of the two quantitative conceptualizations for division, and (2) ability to operate under one conceptualization without interference from the other, while maintaining an awareness of which conceptualization with which they are operating. The data also revealed that awareness and capability did not correlate. That is, a teacher could be aware of the two meanings but be unable to operate meaningfully with them both. Or, a teacher could operate with both conceptualizations when prompted by a context, but without an awareness of a change in meaning. Concerning condition 2, I identified two kinds of *partitive-quotitive interference*. The first is *partitive-to-quotitive assimilation*, which occurs when a person assimilates partitively-framed division tasks to quotitive schemata. Four of the teachers exhibited these “mis-assimilations” at some point in Phase 1, and it occurred when the division tasks were abstract, and when the divisors were not whole. The second is *quotitive-modeling interference*, which refers to any difficulties in modeling partitive division that are caused by inappropriately applying meanings or strategies that are typically associated with quotitive division models. Due to quotitive-modeling interference, three teachers were impaired while trying to produce partitive models.

### Selecting the Two Subjects for Phases 2 and 3

At the completion of Phase 1, I aimed to narrow my study from six to two teachers. Since I was interested in studying the *advancement* of schemes, I had planned to choose two teachers for whom the data revealed to have weak meanings, yet could benefit from the teaching experiment in Phase 2. I now summarize my assessments of

each teacher based on the data from Phase 1, and I discuss how I selected both Linda and Uma as the Phase 2 and 3 subjects.

Linda had very weak meanings for fractions, multiplication, and division. She could not meaningfully model fractional multipliers. She demonstrated an awareness of only one meaning for division, quotitive division, which she could not model consistently. She demonstrated issues with re-unitizing, in that she could not maintain a consistent unit, nor multiple units simultaneously. Of all the teachers, she had the most potential to gain from participating in the teaching experiment. In addition, she had extensive experience teaching at the 5<sup>th</sup> and 6<sup>th</sup> grade levels, and she was currently teaching at the 6<sup>th</sup> grade level, which is when students are building meanings for division with fractional divisors. Given all these reasons, Linda was a good candidate for Phase 2.

Mark also demonstrated weak meanings, but he was prone to procedural contamination every time he encountered difficulties in completing the tasks. Mark could benefit from the teaching experiment, but he did not exhibit perseverance in reasoning through tasks. Additionally, he had experience only at the 7<sup>th</sup> grade level, which meant he did not work with students in building initial meanings for division with fractions.

Mindi possessed schemes that were quite advanced. During Phase 1, she demonstrated an acute awareness of the two models for division, and managed to give partitive and quotitive models to represent decontextualized statements of division, even with fractional divisors. Consequently, the teaching experiment with Mindi would not have contributed to my research goal of investigating cognitive advancement.

Mel also possessed schemes that were very advanced. Even though she did not initially demonstrate an awareness of the two meanings for division, she possessed robust

schemes related to each conceptualization. She was able to invent in the moment a partitive meaning for dividing by  $\frac{1}{3}$  in Task 10. Also, in Task 21, she developed a perfectly generalized scheme, providing her with a conceptual foundation for the fact that  $m \div \frac{9}{4} = \frac{4}{9} \times m$ . She struggled with her explanation of Task 25, but she was on the verge of an abstracted conceptual justification for the invert-and-multiply algorithm. Consequently, Mel would also not have helped me answer my research questions.

Uma demonstrated weaker schemes, and she was also prone to procedural contamination, although she was very discrete about it. However, she exhibited a sincere desire to have a conceptual understanding of the procedures she was using, and she was very persistent in trying to infuse these procedures with meaning. She had some trouble coordinating multiple levels of units in both multiplication and division tasks, and she demonstrated partitive-to-quotitive assimilation. Additionally, she had extensive experience at the 5<sup>th</sup> and 6<sup>th</sup> grade levels, and was currently acting in the role of a mathematics coach to other teachers. For these reasons, Uma was a good candidate for Phase 2.

Ursa also demonstrated weaker schemes. Even though she expressed an awareness of the two meanings for division, she could not model them consistently when fractions were involved. She also demonstrated partitive-to-quotitive assimilation when fractional divisors were present. Additionally, she relied heavily on proportional reasoning schemes, e.g. if  $x$  containers corresponded to  $y$  gallons, then one container corresponded to  $y \div x$  gallons. This method was pseudo-procedural because she resorted to it when other attempts at reasoning were not successful. Ursa was capable of benefitting from the teaching experiment, but she had no experience teaching at the 6<sup>th</sup> grade level.

In conclusion, the data suggested that Mindi and Mel were too advanced in their mathematical abilities to contribute to addressing my research questions. The other four teachers were good candidates, but Mark and Ursa did not have extensive experience with students first learning division with fractions, which occurs in the 5<sup>th</sup> and 6<sup>th</sup> grade levels. Linda and Uma did have such experience, and they both demonstrated a willingness to persevere in making sense of their work. Thus, I decided to use Linda and Uma for Phases 2 and 3. They both completed the next two phases, but for the purposes of this dissertation, I only reveal and discuss the data from Linda.

## CHAPTER 7

### PHASE 2 RESULTS AND DISCUSSION

This chapter describes the results and provides a discussion of findings from a four-session teaching experiment with Linda. I based my analysis of the data from the teaching experiment on my secondary research question for this phase (RQ1.6), which is, *“What cognitive obstacles do teachers further reveal as I actively attempt to promote the development of their meanings that are foundational to partitive division over the rational numbers?”*

The hypothetical learning trajectory I described in Chapter 5 on methodology provided opportunities for me to adapt in order to meet Linda’s needs. Based on the weaker meanings that Linda revealed in Phase 1, I revised the teaching experiment trajectory specifically because she: (1) could not meaningfully model multiplication with fractional multipliers, (2) held a predominant quotitive conceptualization of division, and (3) demonstrated issues with re-unitizing in that she could not maintain a consistent unit, nor multiple units simultaneously. For Linda’s teaching experiment in Phase 2, I followed the ordering of the seven learning objectives (found in Chapter 5), but I devoted more time to activities that focused on Linda’s meanings for unitizing, fractional multipliers, and division. By the end of the teaching experiment, Linda and I had engaged in discussions that centered on twelve learning activities. I summarize the timeline of our discussions over the four-session teaching experiment in Table 13. In this table, I also identify the length of the session and the learning objective to which the activities aligned. I repeated some activities over multiple sessions so that I could give Linda several opportunities to reflect on the activity.

Table 13

*Timeline of the Teaching Experiment with Linda*

| Session | Obj. | Activity |                                                                                                                                                                                                          |
|---------|------|----------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1       | 1    | 1        | Given the pink strip of paper, draw a strip of paper that is $a$ times as long. (For $a = 3, 1/5, 3/5, 7/5, 2\frac{1}{3}, 1.7$ )                                                                         |
|         |      | 2        | Describe your meanings for the expressions: $4 \times 5, 4 \times 5/3, 4/3 \times 5$                                                                                                                     |
|         | 2    | 3        | The green strip is how many times as long as the red strip? The red strip is how many times as long as the green strip? (red = $7/4$ green)                                                              |
| 2       | 1    | 4        | Describe your meanings for the expressions: $a \times b, m/n \times b$                                                                                                                                   |
|         | 2    | 3        | (Repeat task - see row above for task description)                                                                                                                                                       |
|         |      | 5        | The yellow strip is how many times as long as the blue strip? The blue strip is how many times as long as the yellow strip? (yellow = $9/2$ blue)                                                        |
|         | 3    | 6        | Describe your meanings for the expressions: $a \div b, a \div 4, a \div 1/4$                                                                                                                             |
|         |      | 7        | How much water is in one container when 20 gallons fill 4 identical containers? How many containers are needed to hold 20 gallons of water if each container can hold 4 gallons?                         |
| 3       | 1    | 4        | (Repeat task - see row above for task description)                                                                                                                                                       |
|         | 2    | 3        | (Repeat task - see row above for task description)                                                                                                                                                       |
|         | 3    | 7        | (Repeat task - see row above for task description)                                                                                                                                                       |
|         |      | 6        | (Repeat task - see row above for task description)                                                                                                                                                       |
|         | 4    | 8        | Suppose some amount of water fills 5 containers. Describe how much water would fill a whole container.                                                                                                   |
| 4       | 5    | 9        | Suppose some given amount of water fills the blue-rimmed containers. Describe how much of the given amount of water would fill a whole container. (There are 1 and $\frac{3}{4}$ blue-rimmed containers) |
|         | 2    | 3        | (Repeat task - see row above for task description)                                                                                                                                                       |
|         | 3    | 7        | (Repeat task - see row above for task description)                                                                                                                                                       |
|         | 4    | 8        | (Repeat task - see row above for task description)                                                                                                                                                       |
|         | 5    | 10       | Suppose some given amount of water fills 2 and $\frac{2}{3}$ containers. Describe how much of the given amount of water would fill a whole container.                                                    |
|         | 6    | 11       | Suppose some given amount of water fills $\frac{3}{5}$ containers. Describe how much of the given amount of water would fill a whole container.                                                          |
|         | 3    | 12       | Describe your meanings for the expressions: $a \div 9/5, a \div 3/7, a \div 1/3$                                                                                                                         |

### RQ1.6: Additional Cognitive Obstacles

To address this research question, I discuss the relevant data from the four sessions of the teaching experiment below.

#### Session 1 of the Teaching Experiment

The first session of the teaching experiment focused on three primary activities.

**Activity 1: Finding multiples of the pink strip.** This activity required Linda to use a given pink strip of paper to draw more strips whose lengths were multiples of the length of the pink strip. When Linda was attempting to draw a strip that was 2 and  $\frac{1}{3}$  times as long, she drew 2 whole strips and  $\frac{1}{3}$  of a strip, but with gaps in between, as if the goal was to produce 2 and  $\frac{1}{3}$  strips. I prompted her to consider drawing one strip that was 2 and  $\frac{1}{3}$  times as long. She redrew her image, but as one strip, shown in Figure 27.

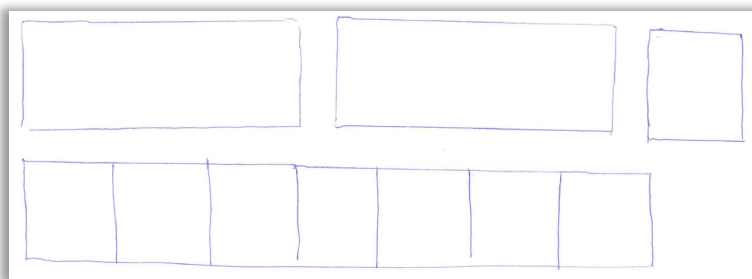


Figure 27. Linda's work on Activity 1 in Phase 2.

I asked her to express 2 and  $\frac{1}{3}$  in a different way and she said  $\frac{7}{3}$  and showed that her new strip was seven-thirds of the pink strip. At the end of this activity, Linda and I discussed expressing non-whole multipliers as single fractions, which means to partition and then iterate. The data from this activity suggest that the mixed number contributed to Linda losing sight of the task's objective – she misinterpreted the task as *draw 2 and  $\frac{1}{3}$  pink strips*, instead of as *draw one strip that is 2 and  $\frac{1}{3}$  times as long*. This provided

additional data to suggest that mixed numbers can be cognitive barriers to productive reasoning.

**Activity 2: Meanings for multiplication.** At the start of this activity, I explained to Linda that we would adopt the convention that the first number of a multiplicative statement is the number of copies, or *multiplier*, and the second number is the amount that is being copied, or *multiplicand*. When I gave Linda the task of modeling  $\frac{4}{3} \times 5$ , she said she would use the commutative property and model  $5 \times \frac{4}{3}$ . I stopped her and insisted that she not use the commutative property. She wrote “ $\frac{4}{3}$  copies of 5,” and I asked her to explain what that meant to her. After a long pause, she said the following.

It's hard for me to visually see it when it's in this format. In each one would be five, but I would only take four-thirds of each one, which I don't see how that's possible, visually. I would need, in each one is five, but I would need four-thirds of each of the groups. But then I'm switching it again to the commutative property, because I'm trying to say five groups of four-thirds.

Initially I did not know how to interpret what she was saying, but upon reviewing the data, I determined that she was describing a multiplier-switch. I will return to this a little later. After her comments, I gave her the pink strip and asked her to consider the task of thinking about  $\frac{4}{3}$  of the pink length. She drew four contiguous, equally-sized rectangles, where three of the rectangles combined matched the length of the pink strip. I asked her to consider the length of the new strip if the pink strip was five centimeters long. She said 5.33 (no unit) which she rounded from 5 and  $\frac{1}{3}$ , revealing her difficulty with seeing one block as anything but one-third. I asked her what the length of one rectangle was (one-third of the pink strip), and she calculated 5 divided by 3 to get 1 and  $\frac{2}{3}$  centimeters. I asked her if she could find the length of one rectangle using multiplication instead of

division, and she said 5 times  $\frac{1}{3}$ . She then wrote  $\frac{5}{3}$  in each rectangle, confirming that  $\frac{5}{3}$  was the same as 1 and  $\frac{2}{3}$ . I encouraged her to work with improper fractions instead of mixed numbers. Then she filled in each of the four rectangles with  $\frac{5}{3}$  and said her overall picture represented the meaning of the expression “ $\frac{4}{3}$  copies of 5.” Her work is provided in Figure 28.

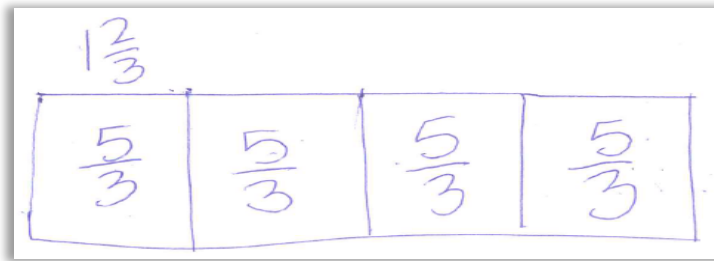
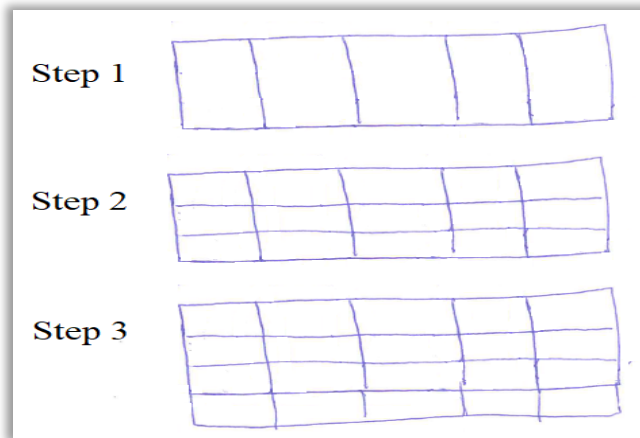
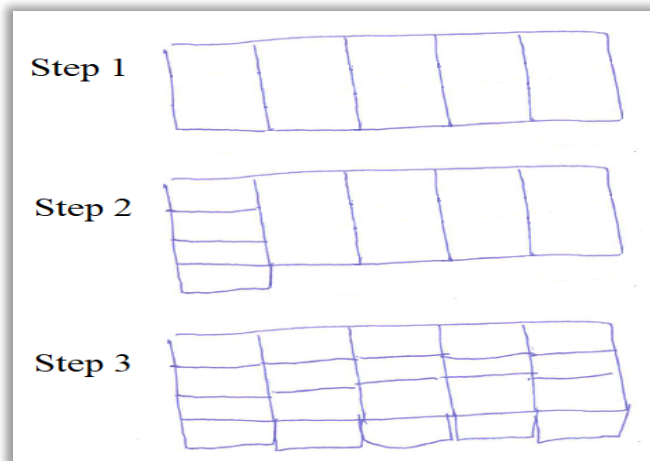


Figure 28. Linda's first attempt at modeling  $\frac{4}{3}$  of 5 during Activity 2 in Phase 2.

I hid her drawing from view and asked her to describe the meaning of  $\frac{4}{3}$  copies of 5. She said it means “I have 4 one-third copies, 5 times” but then she caught her own mistake, realizing that this meant *five copies of four-thirds*, not *four-thirds copies of five*. She became perturbed saying that she kept reverting to the five as the multiplier. I thought she was too focused on representing the  $\frac{4}{3}$  and so I asked her to focus on a representation of the five first, and then to operate on that based on her meanings for  $\frac{4}{3}$  as a multiplier. She drew a long box with five columns, split the five columns into three rows, and added one more row at the bottom, as illustrated in Figure 29.



*Figure 29.* Linda's second attempt at modeling  $\frac{4}{3}$  of 5 during Activity 2 in Phase 2. I optimistically thought she viewed each row in Step 2 of Figure 29 as one-third of five, but realized I was wrong when she said, "I have five of something and I'm gonna make each one of the five, four-thirds." She then redrew her model to explain to me that she was thinking of  $\frac{4}{3}$  of each of the five columns, one column at a time. I share her second explanation in Figure 30.



*Figure 30.* Linda's representation of  $\frac{4}{3}$  of 5 during Activity 2 in Phase 2. The data revealed that Linda did not view each row as one-third of the whole collection of five, but rather she imaged one-third of one column repeated five times. This data helped me understand her initial statements at the start of this task. She was describing a

multiplier-switch, which circumvented the two-step scheme of contracting five by a factor of  $1/3$ , followed by expanding the contracted amount by a factor of 4. I represent this multiplier-switch symbolically below.

$$\begin{aligned}
 & \frac{4}{3} \times 5 \\
 &= \frac{4}{3} \times (1 + 1 + 1 + 1 + 1) \\
 &= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} + \frac{4}{3} + \frac{4}{3} \\
 &= 5 \times \frac{4}{3}
 \end{aligned}$$

When Linda modeled  $4/3$  of the pink, she partitioned and iterated, but when she modeled  $4/3$  of 5, she partitioned and iterated five times, instead of just once. I pointed this out to her and explained that I aimed for her to always think of  $4/3$  of *whatever* as the same two-step process. I then asked her to describe  $4/3$  of  $x$ , and she emphasized first multiplying by four. I suggested that she should think about and describe the contraction first, followed by the expansion. When I asked her to model  $4/3$  of 18, she confidently calculated  $1/3$  of 18 to get 6, and then combined four copies of that to get 24. She generalized this process by saying she would first divide by the denominator and then multiply by the numerator. I asked her to describe this two-step process without using the word *divide*, and she said she would multiply by the reciprocal of the denominator instead. I then asked her to write down a symbolic representation of this two-step process, and I share her response in Figure 31.

$$(5 * \frac{1}{3}) * 4$$

Figure 31. Linda's first attempt to represent  $\frac{4}{3}$  of 5 during Activity 2 in Phase 2.

In her symbolic expression, the ordering of her writing was left to right, to match the ordering of her thinking. However, this is problematic given the convention that the multiplier should be left of the multiplicand. Therefore, Linda and I discussed that we should always place the multiplier in the first position, thinking and saying *partition* then *iterate*, but writing it right to left so that the first numbers are always multipliers and using parentheses to dictate the ordering of the two operations. She then produced the symbolic statement that is in Figure 32.

$$4 (\frac{1}{3} * 5)$$

Figure 32. Linda's second attempt to represent  $\frac{4}{3}$  of 5 during Activity 2 in Phase 2.

I then drew her attention back to her drawing of  $\frac{4}{3}$  times 5 (seen in Step 3 of Figure 29) and used this image to support  $4 \times (\frac{1}{3} \times 5)$  instead of  $5 \times (\frac{4}{3})$ , by explaining that each whole row represented  $\frac{1}{3} \times 5$ . She admitted that she did not interpret her picture this way and that this was a new way of thinking for her. To summarize our time together on this activity, Linda and I discussed how to think about, speak about, and symbolically represent the two-step process of contraction followed by expansion, when modeling fractional multipliers.

The data from Activity 2 revealed recurring cognitive obstacles for Linda, as well as some new ones. In Phase 1, Linda had recurring issues related to re-unitizing (in Tasks 3, 9, 13, and 19), and she exhibited similar issues during this activity. She could represent

$\frac{4}{3}$  of the pink strip, but when I told her to also consider that the pink strip was 5 centimeters long, she had difficulty coordinating the two levels of units. This was evidenced when she said the length of the new strip was 5 and  $\frac{1}{3}$  – she easily coordinated one whole pink strip and 5 centimeters, but she did not coordinate  $\frac{1}{3}$  of the pink strip and  $\frac{1}{3}$  of 5 centimeters. The teaching experiment also revealed some new cognitive obstacles regarding fractional multipliers. In Phase 1, during Task 6 (Explain your meanings for  $\frac{5}{3} \times 2$ ), Linda said she did not know how to think about  $\frac{5}{3}$  *copies of* 2, and she could only represent 2 *copies of*  $\frac{5}{3}$ . Thus, the data from Phase 1 did not reveal much about the challenges that fractional multipliers posed for her. However, when I encouraged her to try to model  $\frac{4}{3}$  of 5 during the teaching experiment, she exhibited a multiplier-switch. She also revealed difficulty with fractional multipliers when she was trying to model  $\frac{1}{3}$  of 5, as shown in Figure 30. She could only imagine  $\frac{1}{3}$  of one column at a time, and did not imagine the entire row as  $\frac{1}{3}$  of all five columns. These data suggest that she favored thinking of fractions as fractions of *one*, and her difficulties surfaced when she had to imagine fractions of values other than one. The data from Activity 2 also suggested that Linda's reliance on the commutativity of numerical multiplication was a hindrance to developing meanings for fractional multipliers. When the multiplicand was a whole number, Linda deliberately tried to switch the roles of the multiplier and multiplicand, with little regard for the change in quantitative meaning. Linda was also comfortable placing the multipliers in either position, such as shown in Figure 31. She was not careful about the positioning of the multiplier in her symbolic statements, which is likely the effect of her reliance on numerical commutativity. Similarly, it was difficult for her to be consistent in her thinking, speech, and writing

regarding the contraction-expansion dual-scheme for fractional multipliers. These data suggest that Linda did not possess a consistent, quantitative (hence non-commutative) conceptualization of multiplication.

**Activity 3: Comparing the red and green strips.** I presented Linda with the activity of comparing a red strip (7 inches long) and a green strip (4 inches long). This was an exact repeat of Task 3 in Phase 1, but this time I asked for each comparison separately. The first comparison was, “The green strip is how many times as long as the red strip?” She interpreted this incorrectly and produced an answer of the red strip in terms of the green strip. Like in Phase 1, she said that 1 and  $\frac{3}{4}$  green strips make the red strip, which she expressed as “ $\frac{7}{4}$  times green gives red.” She interpreted this last expression as “7 copies of  $\frac{1}{4}$  of green gives the red,” and she used the strips as a manipulative to explain the contraction of the green strip, followed by expansion to produce the red strip. Also, while reading the task aloud, she said, “The green strip is how many times longer than the red strip,” which is not what was written. At this point, I did not yet intervene, and we proceeded to the second comparison, which was, “The red strip is how many times as long as the green strip?” While reading the task aloud, she again said “how many times longer than” instead of “how many times as long as.” Like the first question, she also interpreted this question backwards, as trying to scale the red strip to produce the green strip. She first wrote “(blank space)  $\cdot$  red = green,” and then stopped to ponder. We then had the following dialogue.

*Excerpt 2*

---

|    |        |                                                                               |
|----|--------|-------------------------------------------------------------------------------|
| 1  | Linda: | I can't do the reciprocal $\frac{4}{7}$ because that doesn't make sense (long |
| 2  |        | pause). If red equaled seven-fourths of green (she writes $red = \frac{7}{4}$ |
| 3  |        | <i>green</i> )...it <i>has</i> to be the inverse (said without conviction).   |
| 4  | MW:    | Why do you say it has to be?                                                  |
| 5  | Linda: | Because...(long silence, some dialogue omitted). So, I know the               |
| 6  |        | red is seven-fourths of green, and green is four-sevenths of red              |
| 7  |        | (said without confidence). Right? I can't take the inverse (changes           |
| 8  |        | her mind again).                                                              |
| 9  | MW:    | Can I ask you some questions?                                                 |
| 10 | Linda: | Yes, I'm stuck.                                                               |

---

I am not sure why she had an instinct to say that the green strip should be  $\frac{4}{7}$  of the red strip – perhaps she solved the equation she wrote for green by multiplying both sides by  $\frac{4}{7}$ . I intervened and confirmed that seven copies of  $\frac{1}{4}$  of the green strip make the red strip. We then had the following dialogue.

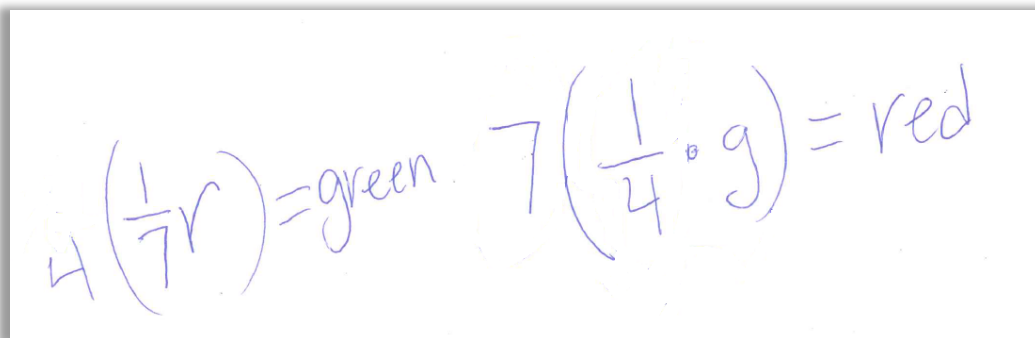
*Excerpt 3*

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|    |        |                                                                      |
|----|--------|----------------------------------------------------------------------|
| 1  | MW:    | What do you have to do to the red, to get something this long        |
| 2  |        | (pointing at one-fourth of the green strip)?                         |
| 3  | Linda: | I have to divide it.                                                 |
| 4  | MW:    | By what?                                                             |
| 5  | Linda: | By one-fourth. Or by four.                                           |
| 6  | MW:    | So, if you divide the red in four, you'll get this long (pointing at |
| 7  |        | one-fourth of the green)?                                            |
| 8  | Linda: | No.                                                                  |
| 9  | MW:    | I think you were thinking of the green again, right? What you have   |
| 10 |        | to do to the green to get this long, is divide it by four.           |
| 11 | Linda: | Yes.                                                                 |

- 12 MW: What do you have to do to the red to get this long (pointing at one-  
13 fourth of the green)?  
14 Linda: I have seven copies of that, right?  
15 MW: Yes, there were seven of them that went across (the red).  
16 Linda: Seven of them go across. What do I have to do to the red to get  
17 that? I divide it.  
18 MW: By what?  
19 Linda: By seven.
- 

Next, I asked Linda to put her thinking on paper and she wrote “ $1/7$  copies,” but couldn’t progress further. I then had the idea that we needed to have a word for the length that is  $1/4$  of green, so that Linda and I could talk about it more easily. I took a blue piece of paper and cropped it to the same size as one-fourth of the green. This gave us a third color – blue – that could serve as an intermediary for going from green to red and back again. We spent the rest of the time discussing going from green to blue to red, as a two-step process which we represented symbolically, then going from red to blue to green, as a two-step process which we represented symbolically. I did this by having her write down the letter  $g$  and then operate on it using only multiplication to produce red. We then did the same, but starting with the letter  $r$ . I share her work in Figure 33.



The image shows a piece of paper with two handwritten equations in blue ink. The first equation is  $4\left(\frac{1}{7}r\right) = \text{green}$  and the second equation is  $7\left(\frac{1}{4}g\right) = \text{red}$ . The equations are written in a cursive, handwritten style.

Figure 33. Linda's work during Activity 3 in Phase 2.

In both cases, Linda expressed the correct order of thinking – contraction then expansion – and the correct symbolic notation keeping the multipliers to the left. This marked the beginning of building a scheme for fractions as reciprocal measures of relative size.

Several challenges were revealed in the data above. Linda's difficulties with re-unitization were again apparent when she had trouble imagining one length as both  $\frac{1}{4}$  and  $\frac{1}{7}$ . When I introduced the blue strip, it became easier for us to describe this new length, and to focus on it independently of the colors red and green. This color-disassociation freed Linda from only imagining this new length in terms of green. This is evidenced by the fact that prior to introducing the blue strip, when I asked Linda to describe how to transform the red into  $\frac{1}{4}$  of the green (as I was holding the green strip folded in fourths), she answered she needed to divide the red strip by four. She could only imagine the folded green paper in terms of the green paper. The data also revealed issues with Linda's interpretation of mathematical language, in that she did not answer the questions with the appropriate comparisons. Also, she used vague speech when she mistakenly said "times longer than" while reading "times as long as." The phrase she said is vague because an expression such as "2 times longer than 3 feet" could be interpreted as six feet, or it could be interpreted as 3 feet *plus* six feet.

## **Session 2 of the Teaching Experiment**

I started this session by asking Linda to reflect on our last session. During her reflection, she did not specifically mention fractional multipliers or the strip activity. Thus, I chose to begin the second session with Activities 4, 3 (repeated), and 5, which focused on fractional multipliers and reverse comparisons. Linda also participated in an activity that introduced her to the two meanings for division.

**Activity 4: Meanings for fractional multipliers.** During Activity 4, I asked Linda to model  $\frac{3}{4}$  of  $b$ . She chose  $b = 2$ , said “three copies of one-fourth... two times,” and then modeled  $2 \times \frac{3}{4}$ . This was a case of a multiplier-switch. I pointed out that she used the commutative property, and I asked her to make sense of  $\frac{3}{4}$  *copies of 2*. Despite our discussions during Session 1, she did not know what to do, so I introduced a context involving a bar of chocolate that weighed two pounds. I drew a rectangle and then we had the following dialogue.

*Excerpt 4*

---

|    |        |                                                                              |
|----|--------|------------------------------------------------------------------------------|
| 1  | MW:    | Can you pretend that this rectangle represents two pounds of                 |
| 2  |        | chocolate? How would you find three-fourths of that?                         |
| 3  | Linda: | So, it would be three copies of one-fourth. I'd break it up into             |
| 4  |        | fourths (she divides the rectangle into four columns).                       |
| 5  | MW:    | Okay, so one-fourth of what?                                                 |
| 6  | Linda: | Of two.                                                                      |
| 7  | MW:    | Okay, so how much is one-fourth of two pounds?                               |
| 8  | Linda: | One-fourth of two pounds would be...half (no unit). So, the                  |
| 9  |        | inverse. So, it'd be three copies of one-fourth times half? Half             |
| 10 |        | copies? I don't know.                                                        |
| 11 | MW:    | Could you tell me...(she cuts me off)                                        |
| 12 | Linda: | Because this (pointing to two columns) would be one pound and                |
| 13 |        | then one-fourth (pointing to one column) would be a half a pound             |
| 14 |        | (writes $\frac{1}{2}$ above each column). Or you can just cut it in two (she |
| 15 |        | draws a horizontal line making two rows, as shown in Figure 34).             |
| 16 | MW:    | Okay, so three-fourths of two pounds is...                                   |
| 17 | Linda: | Six-eighths.                                                                 |
| 18 | MW:    | Of what?                                                                     |
| 19 | Linda: | No, it'd be...yeah, it'd be six-eighths.                                     |

- 20 MW: Can you shade in three-fourths of two pounds? (Linda shades the  
21 first three columns). Could you tell me how many pounds that  
22 represents?  
23 Linda: Six-eighths pounds, or three-fourths pounds (said without  
24 confidence) ...uh, no....a pound and a half (said with confidence)!  
25 MW: How did you get that?  
26 Linda: Half plus half plus half (tapping each  $\frac{1}{2}$  above the first three  
27 columns)
- 

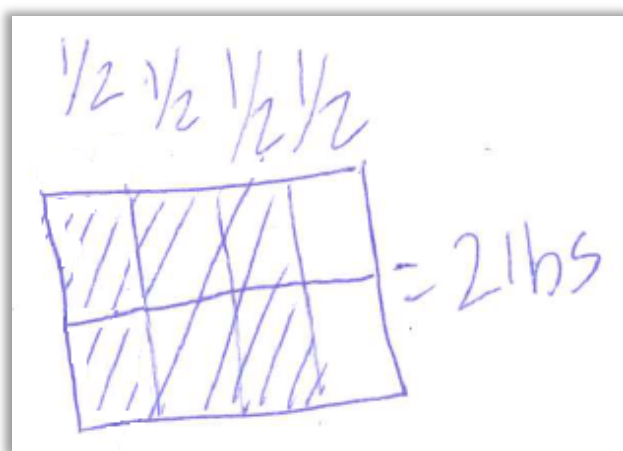


Figure 34. Linda's first attempt at modeling  $\frac{3}{4}$  of 2 during Activity 4 in Phase 2.

Linda was successful at re-unitizing in this instance. The data suggest that the context helped her to focus on pounds, which allowed her to recognize and correct her mistake. I then asked her to summarize the two-step process for scaling 2 pounds to 1 and  $\frac{1}{2}$  pounds. She said she found *one-fourth copies of 2*, and she then described finding three copies of this amount. She then tried to summarize more concisely, but she errantly wrote, “3 copies of  $\frac{1}{4}$  of  $\frac{1}{2}$  copies of 2.” I was not sure why she wrote the extra  $\frac{1}{2}$  – perhaps due to her picture where she unnecessarily split the rectangle into two rows. I thought that I should walk her through her statement to help her realize that it would produce an unreasonable answer. I focused on the portion of her written statement  $\frac{1}{2}$

*copies of 2* and I asked her for the result of  $1/2$  copy of 2, and she said  $1/4$ . This startled me, so I decided to use units, anticipating she would realize the unreasonableness of her statement. To my bewilderment, she confirmed several times that you obtain  $1/4$  pound when you “do  $1/2$  copy of 2 pounds.” She seemed very convinced which led me to speculate that she was interpreting the prompt “what is  $1/2$  copies of 2 pounds?” as “what is the size of  $1/2$  copy of a pound relative to 2 pounds?” I phrased the question differently by asking, “What is  $1/2$  of 2 pounds?” This time, she responded with “one.” Returning to her incorrect statement (3 copies of  $1/4$  of  $1/2$  copies of 2), I began with the 2 pounds and then I asked her to follow the steps as indicated by her notation. She calculated  $1/2$  of 2 pounds to get 1 pound. She then began to calculate  $1/4$  of one pound, at which point she said, “I would not get my answer.” I asked, “what answer,” and she said, “three copies of one-fourth.” I wondered to myself whether her fractions were in terms of *chocolate bars* or *pounds*. I pointed out that earlier she said the answer was 1 and  $1/2$  pounds. This perturbed her because she was now trying to show that the answer should be  $3/4$ . The data suggest that her confusion was because she was conflating the halves and fourths, despite having a context to help her keep chocolate bars and pounds distinct. The data also reveal that she may never have been thinking about chocolate bars because she never explicitly referred to a *bar*. At this point, I emphasized to her that each time she said a fraction, the referent should be clear in her thinking and in her speech. We turned the page over and started again. The next segment of Linda’s work is illustrated in Figure 35.

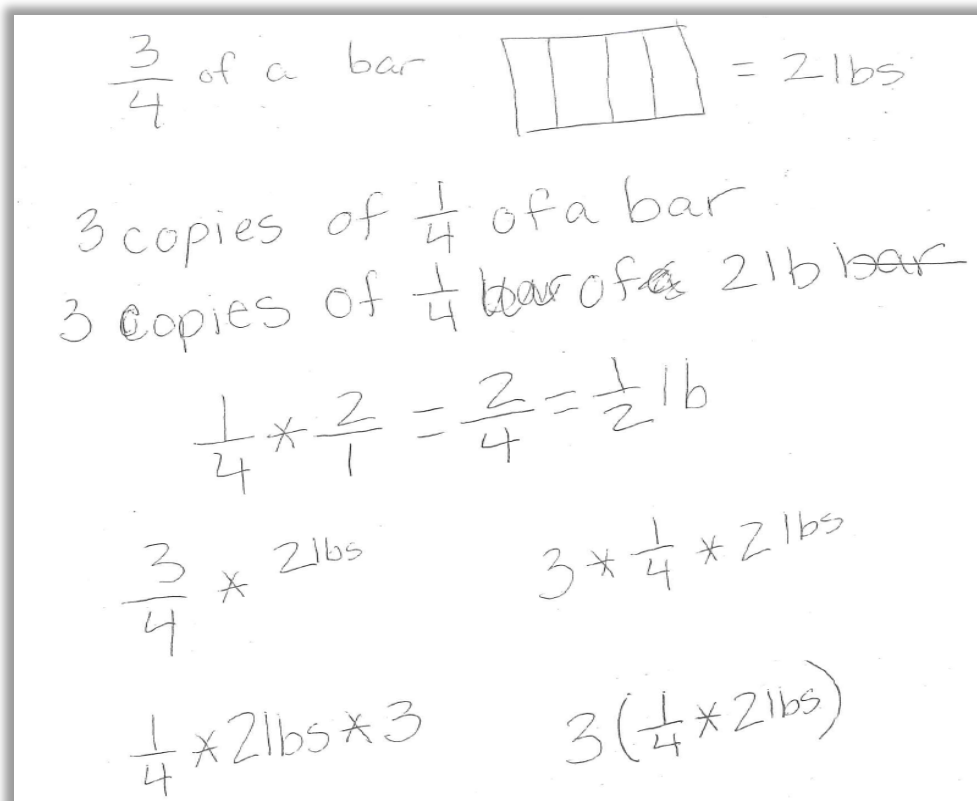


Figure 35. Linda's second attempt at modeling  $\frac{3}{4}$  of 2 during Activity 4 in Phase 2.

I drew a rectangle to represent a bar. Linda modeled three-fourths of it and wrote “3 copies of  $\frac{1}{4}$  of a bar.” I told her the bar weighed 2 pounds. She became momentarily perturbed, not knowing how to rewrite the statement using the 2-pounds information. She eventually decided on writing, “3 copies of  $\frac{1}{4}$  of a 2-lb bar.” I suggested she should just focus on pounds and she edited her writing to say, “3 copies of  $\frac{1}{4}$  of 2 pounds.” She then calculated  $\frac{1}{4}$  of 2 pounds to get  $\frac{1}{2}$  pound.

This activity revealed some of the challenges in trying to advance Linda’s thinking, speaking, and symbolic writing regarding fractional multipliers. Linda rarely mentioned units for the numbers she was saying and writing, which suggests that she was quantitatively disengaged from the scenario. She also did not mention *bar* as a unit, suggesting that she was only able to think about pounds, which is additional evidence of

her inability to maintain two levels of units simultaneously. This explains why she had trouble interpreting one column in her picture as both  $\frac{1}{2}$  (of a pound) and  $\frac{1}{4}$  (of a bar). The data also raised some concerns for me regarding Linda's perception of the word *copies*, because when I kept asking her to find  *$\frac{1}{2}$  copy of 2 pounds*, she kept answering  $\frac{1}{4}$  of a pound. However, when I omitted the word *copy* and asked her to find  *$\frac{1}{2}$  of 2 pounds*, she answered 1 pound. This suggests that the word *copy* may not have a strong quantitative significance for her.

**Activity 3: Comparing the red and green strips (second occurrence).** Linda repeated Activity 3, first with the question, "The green strip is how many times as long as the red strip?" She misread this as "how many times longer," as she did during Session 1. I pointed this out right away and tried to explain why her phrasing was vague, but Linda moved on without really responding to what I was saying. She concluded, "7 copies of  $\frac{1}{4}$  of the green strip make the red strip," which is correct, but not the measurement implied by the question – she misread the task as she did during Session 1. Consequently, I shifted my focus to help her recognize that she was incorrectly answering the question. I asked her to translate the wording of the prompt into a mathematical statement. At first, she interpreted the word *is* as division, such as interpreting "*a is 2*" as  $a \div 2$ . To move her in the right direction, I recommended that she consider the statement "2 plus 3 is 5," and she decided *is* suggested an equality. This led her to correctly translate the prompt into the mathematical statement,  $g = ? \times r$ . She then recognized that in her first attempt, she misread the task and had actually determined that  $\frac{7}{4} \times g = r$ . She then focused on determining the correct comparison, but at first, she did not know what to do. To help her, I drew a box that was the same length as one-fourth of the green strip and asked her

to describe its length. She answered in two ways by saying one-fourth of the green strip and one-seventh of the red strip. This led her to conclude that  $\frac{4}{7} \times r = g$ . We then discussed how it is possible to measure one fixed amount in different ways depending on the unit-of-measure, much like the fact that 2 feet is the same length as 24 inches. This second exposure to Activity 3 revealed that Linda struggled with translating a written expression into an appropriate symbolic expression, which suggests a deficiency in quantitative reasoning, either at the level of the written expression, or at the level of the symbolic expression, or both.

**Activity 5: Comparing the yellow and blue strips.** To ascertain Linda's progress regarding fractions as reciprocal measures of relative size, I gave her two new strips – a yellow strip that was nine inches long and a blue strip that was 2 inches long. She still said “longer” instead of “as long as” when she read the prompts, but this time she correctly aligned the questions and the comparisons. She explained and wrote both comparisons in sensible ways. At this point, the data suggested that she had developed a productive scheme for reversing a multiplicative comparison. Additionally, she was able to explain why the reciprocal fraction was sensible to her. However, I wondered if her newly developed dual-scheme of forward and reverse comparisons was permanent, adaptable to situations that did not involve strips, and reciprocal in nature.

**Activity 6: Meanings for division.** I designed the tasks in this activity to form a baseline of Linda's thinking that we could revisit later in the teaching experiment. As such, I did not intervene during this first attempt at this activity. I asked her to describe her meanings for  $a \div 4$  and she said, “How many copies of four can I pull out of  $a$ ?” I then asked for another meaning, but she could not think of one, which did not surprise me

because of what she could and could not do in Phase 1. I then asked her to describe her meanings for  $a \div 1/4$  and she said, “How many one-fourth copies can I pull out of  $a$ ?” I drew her attention to the location of the word *copies*, and she rephrased her question as, “How many copies of one-fourth can I pull out of  $a$ ?” I asked for another meaning, and not surprisingly, she could not give one. The results here were identical to the results of the analogous tasks from Phase 1. At this point, I moved on to the next activity, planning to return to this activity after Linda had formulated a partitive meaning for division.

**Activity 7: Quotitive and partitive water-container tasks.** I designed two tasks (Task A and Task B) in Activity 7 to introduce Linda to the two conceptualizations for division. I first showed her Task A, “How much water is in one container when 20 gallons fill 4 identical containers?” She said, “I want to find how many 20 gallons would fill just one.” Her language was odd, but her modeling revealed that she interpreted the task as I intended. She distributed the 20 gallons, one gallon at a time to one container at a time, and concluded that 5 gallons would fill one container. She also mentioned that she could calculate  $20 \div 4$ . Her work is in Figure 36.

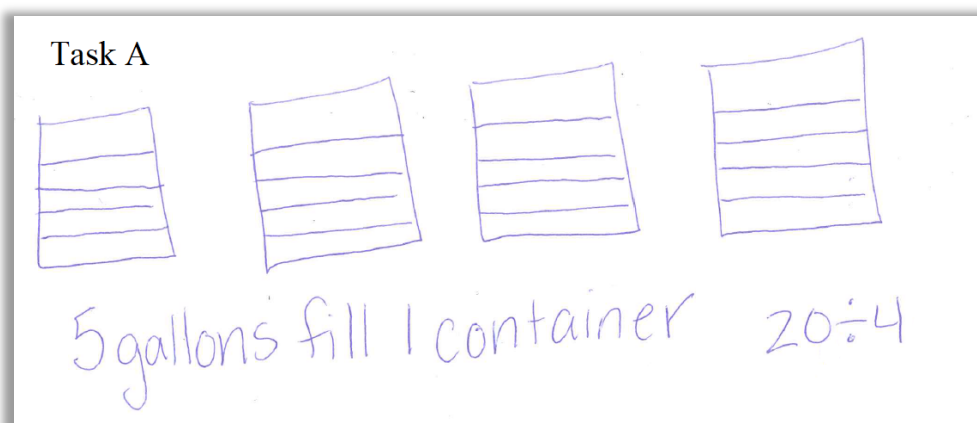


Figure 36. Linda's work on a partitive model for  $20 \div 4$  during Activity 7 in Phase 2.

I then showed Linda Task B, “How many containers are needed to hold 20 gallons of water if each container can hold 4 gallons?” She immediately calculated  $20 \div 4$  and gave an answer of 5 *containers*. She modeled this by drawing five rectangles, each split in four rows, as depicted in Figure 37.

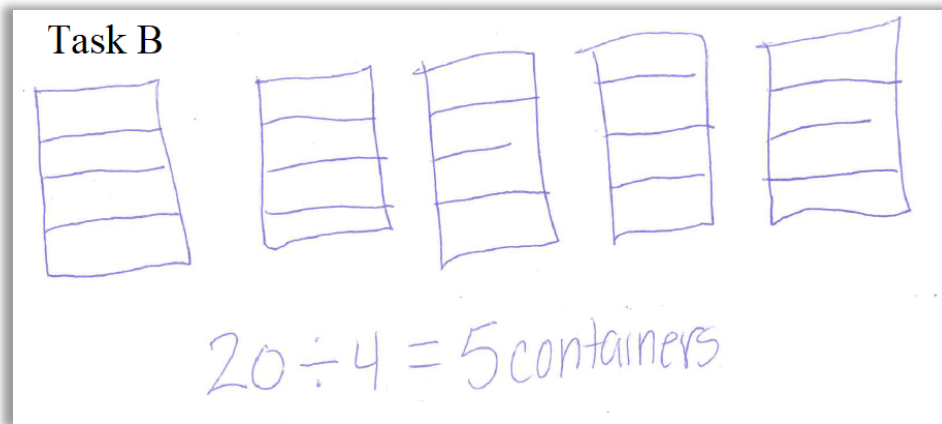


Figure 37. Linda's work on a quotitive model for  $20 \div 4$  during Activity 7 in Phase 2.

When I placed her work side-by-side, she acknowledged the following three things: (1)  $20 \div 4$  answered each question, (2) the *fives* had different meanings, and (3) her visual representations were different. I then hid her work from view and asked her to describe the meaning for  $20 \div 4$ , and she said, “How many groups of four can I pull out of 20?” I then placed her work back in view and asked her to identify the task that corresponded to this question, and she correctly identified Task B. We then had the following dialogue.

*Excerpt 5*

|   |     |                                                                   |
|---|-----|-------------------------------------------------------------------|
| 1 | MW: | So, what question does this division answer (pointing to Task A)? |
| 2 |     | It's not the same right? This picture (Task A) doesn't seem to be |
| 3 |     | showing how many fours are in 20. What is it showing instead?     |

4 Linda: It *does* show four into 20 because I have four containers...four  
5 containers of...would fill 20 gallons. But, this (Task B) shows that I  
6 have four gallons in each container.

7 MW: When you did it this way (Task B) and you focused on gallons only  
8 and you pulled them off in groups of four...When you thought *how*  
9 *many fours can I pull away from 20*, the *four* and the *20* had to  
10 have the same unit in that case. Because you can't pull a different  
11 thing away from something. I can't pull bricks away from feathers.  
12 I have to pull feathers away from feathers.

13 Linda: Right.

---

The oddness of Linda's phrasing aside, her language suggests that she thought her quotitive question was general enough to be interpreted partitively. I asked her to produce units for the 20 and 4 in each task. For Task B she correctly indicated gallons and gallons, and for Task A she correctly indicated gallons and containers. I then suggested that her typical division question – which is quotitive – felt inappropriate for Task A, because “it feels weird to pull containers away from gallons.” She then tried to phrase an appropriate question for Task A.

*Excerpt 6*

---

1 Linda: Out of 20 gallons, how much would four containers fill in the  
2 gallons? But, then I'm just restating that.

3 MW: Back before we had gallons and containers, you said, “How many  
4 copies of four can I pull out of 20?” Could you talk about the  
5 meaning of 20 divided by four using the word *copies* that seems  
6 more suitable for this situation (Task A).

7 Linda: How many copies of...How many...(trails off)

8 MW: Okay, that's fine. Right now I'm exploring where the boundaries  
9 are. This gives me the focus of our next activity.

---

Next, I showed Linda the expression,  $a \times b = c$ , and asked her to give a general description of the factors and the product. She said, “ $a$  is the number of copies,  $b$  is what you are copying, and  $c$  is the number of copies you are copying.” I suggested that her meaning for  $c$  sounded like her meaning for  $a$ . She adjusted and said, “ $c$  is the total number of copies you are copying...the total number of groups you are copying.” Her difficulty in giving a general description of the product indicated a weak quantitative foundation for multiplication. It is significant to notice that she did not give a general quantitative description for  $b$  either, hence her difficulty to do so for  $c$ . I asked her to consider her values in Task A, and to place the 20, 4, and 5 in the correct positions of the multiplicative structure  $a \times b = c$ . Her work is in Figure 38.

The image shows handwritten work on a white background. At the top, the equation  $a \times b = c$  is written in purple ink. An arrow points from the text "what you are copying" to the variable  $b$ . Another arrow points from the text "the number of copies" to the variable  $a$ . Below this, the equation  $4 \text{ container} \times 5 \text{ gallons per container} = 20 \text{ gallons}$  is written. The phrase "5 gallons per container" is enclosed in a hand-drawn rectangular box.

Figure 38. Linda's attempt to create a partitive structure during Activity 7 in Phase 2. Linda placed the values in the correct positions, although she expressed 5 in terms of gallons, and so I guided her to add *per container*. She also put a box around the 5 to

suggest that this was the unknown factor for the task. We then had the following dialogue.

*Excerpt 7*

---

|    |        |                                                                               |
|----|--------|-------------------------------------------------------------------------------|
| 1  | MW:    | So, with this example in place, can you think of a general way to             |
| 2  |        | describe $c$ ? (long pause)                                                   |
| 3  | Linda: | How many copies of what you are copying.                                      |
| 4  | MW:    | So, $c$ is the number of copies of what you are copying? I thought            |
| 5  |        | that's what $a$ was. Isn't $a$ the number of copies of what you are           |
| 6  |        | copying?                                                                      |
| 7  | Linda: | No, that's (pointing to $a$ ) the <i>number</i> of copies...multiply what you |
| 8  |        | are copying gives you the total...amount...of copies (laughs). No.            |
| 9  | MW:    | So, the 20 is the total amount of containers?                                 |
| 10 | Linda: | No.                                                                           |
| 11 | MW:    | What is the 20?                                                               |
| 12 | Linda: | Gallons.                                                                      |
| 13 | MW:    | It's the total amount of what?                                                |
| 14 | Linda: | Of gallons per container.                                                     |
| 15 | MW:    | 20 doesn't tell me anything about how many containers there are.              |
| 16 |        | So, we need a way to describe $c$ without saying it's the number of           |
| 17 |        | copies.                                                                       |
| 18 | Linda: | Okay. (pauses)                                                                |
| 19 | MW:    | Forget the general one (I cover the expression $a \times b = c$ ). How        |
| 20 |        | would you describe the 20 gallons?                                            |
| 21 | Linda: | 20 gallons ...uh... in four containers.                                       |
| 22 | MW:    | I like that, so 20 gallons is the amount <i>in</i> the four containers total. |
| 23 | Linda: | Okay, so $c$ would be the total amount of $a$ ...in $a$ . The total amount    |
| 24 |        | in $a$ .                                                                      |
| 25 | MW:    | I think we're definitely closer. I see where you are going with it.           |

- 26 Linda: So, because 20 gallons is the total amount in four containers, so  $c$   
27 would be the total amount in  $a$ .  
28 MW: Right. Okay, I'm with you on that. You agree though, it's hard to  
29 talk about when you don't have a context in which to discuss it.
- 

This dialogue revealed that Linda had a weak general meaning for multiplication. Or at the very least, she lacked the language to describe a general meaning. When she said initially that “ $c$  is the number of copies you are copying,” I realized that, for her, the word *copies* was not necessarily a generalization of *containers*, else I could restate her description as “ $c$  is the number of *containers* you are copying,” which would suggest that  $c$  was measured in containers, contradicting that  $c$  was measured in gallons. At the onset of this activity, I hypothesized that Linda would realize that *containers* was a concrete representation for the word *copies*. However, the data suggest that this was not the case. I then asked Linda to place the 20, 4, and 5 in the correct positions of the general multiplicative structure for Task B. She placed the numbers in the correct positions, indicated the correct units, and placed a box around the factor that was previously unknown. I share her work in Figure 39.

The image shows a handwritten mathematical equation at the top:  $a \times b = c$ . Below this, there is a more specific equation. The first factor, "5 container", is enclosed in a hand-drawn rectangular box. This is followed by a multiplication sign, then "4 gallons per con", an equals sign, and finally "20 gallons". The word "con" is written on a new line under "per".

Figure 39. Linda's attempt to create a quotitive structure during Activity 7 in Phase 2.

I finished this session by pointing out to Linda that in Task B, we were trying to find the multiplier, which is the number of copies, and in Task A, we were trying to find the

multiplicand, which is the amount per copy. I reminded Linda that her dominant meaning for division was the meaning corresponding to Task B, and that I was planning on spending the next two sessions of the teaching experiment discussing the meaning for division corresponding to Task A.

The data from this activity suggested that Linda had a weak quantitative meaning for multiplication. This was evidenced by her difficulty in describing the quantitative significance of the product, both generally and in the case where the product was 20 gallons. Furthermore, Linda did not reveal evidence that she was bothered by the contradiction caused when she said, “ $c$  is the number of copies you are copying,” while she had written on paper that  $c$  represented 20 gallons of water. This suggests that Linda was either not connecting *copies* with *containers*, or that she was not keeping *containers* distinct from *amount of water*. This activity also revealed instances of odd language such as, “I want to find how many 20 gallons would fill just one (container)” and “I have four containers...four containers of...would fill 20 gallons.” These utterances suggest that Linda was not engaged in quantitative reasoning, or at the very least that she was not attentive to using meaningful phrases to convey her thoughts.

### **Session 3 of the Teaching Experiment**

During this session of the teaching experiment, Linda and I reviewed fractional multipliers, reciprocal comparisons, and meanings for division. We then had time to discuss some water-container tasks.

**Activity 4: Meanings for fractional multipliers (second occurrence).** I asked Linda to describe the meaning of  $a \times b$  and she said, “ $b$  copies of  $a$ ,” which is backwards of what he had been practicing. I told her to reverse it so that  $a$  is the number of copies. I

reminded her that even though multiplication is numerically commutative, I did not consider it conceptually commutative, and we reviewed the meanings of the terms, *multiplier* and *multiplicand*. I asked Linda to describe her meanings for  $m/n \times b$  and she said, “this (pointing at the fraction) number of copies of  $b$ .” I asked, “what does a fraction number of copies mean,” and she paused. Eventually she said that it is  $m$  copies of  $b$  and then divided by  $n$ . I told her that what she said is valid but that I wanted her to focus on the reverse order, so she said to divide by  $n$  then multiply by  $m$ . This time, she wrote it correctly by writing  $m \cdot \left(\frac{1}{n} \times b\right)$ , with the multipliers on the left and the parentheses to indicate the order of operations. I reemphasized the notion of contraction followed by expansion, which we practiced by calculating  $5/3 \times 12$  and  $5/3 \times 10$ . The data from this activity revealed that Linda demonstrated meanings that contradicted our earlier discussions regarding the position and meaning of a multiplier, and the ordering of contraction then expansion. This raises the question as to what is required to effectively and permanently alter a person’s web of schemes.

**Activity 3: Comparing the red and green strips (third occurrence).** I repeated Activity 3, but this time I phrased the tasks differently. I first asked, “What do you multiply the red strip by to get the green strip?” In response, she wrote  $r \cdot x = g$ , presumably due to the wording of the task. I told her we should reverse the multiplier and multiplicand because we were trying to find “how many copies of red make green” and she said, “yes, you are right,” and wrote  $\frac{4}{7} \cdot r = g$ . Later in the activity, I used one-fourth of the green strip to trace a marking on the paper and asked Linda to describe its length. She said only, “one-fourth” but without a saying a unit. She did not give me a

measurement in terms of the red strip. I wondered if this was because she watched me use the green strip to make the marking. I further wondered if I had instead taken the red strip, folded it down to one-seventh, and used that to trace the marking, whether Linda would have said “one-seventh.” Regardless, I asked her to describe the length of the marking, but in a second way. She held the red strip next to the marking, hesitated, pulled it away, opened the green paper, and said, “it has to be based on just this one (the green paper).” This suggests that Linda felt that the length of the marking could only be represented in terms of one strip, which unsettled me because she did not reveal this concern during this activity in the previous sessions of the teaching experiment. But I insisted that she measure the length of the marking using the red strip and she said “one-seventh.” I reemphasized that the length could be either value depending on the unit-of-measure. The data from this activity is interesting because it raises again the question as to what is required to effectively and permanently alter a person’s web of schemes. Despite extensive discussions between us, Linda still placed the multiplier in the wrong position. Also, even though this was the third time through this activity, Linda still hesitated to imagine the contracted length as both  $1/4$  and  $1/7$ .

**Activity 7: Quotitive and partitive water-container tasks (second occurrence).** Returning to Activity 7, I showed Linda her own work for Tasks A and B at the same time (Figure 36 through Figure 39), and asked her reflect on what she saw. She acknowledged that she could use numerical division to find either a missing multiplier or a missing multiplicand. Furthermore, she acknowledged that in each case she calculated  $20 \div 4$ , but the two *fives* had different meanings. She expressed that this made sense to

her, so I returned to the tasks from Activity 6 to determine the extent to which her schemes for division had undergone accommodation.

**Activity 6: Meanings for division (second occurrence).** Now that Linda had formulated two meanings for division during Activity 7, I returned to Activity 6 by first asking her to describe her meanings for  $x \div y$ . She responded, “how many copies of  $y$  make  $x$ ,” which was slightly different from the *pulling out of* language she had been using regularly. When I asked her for a second meaning, she said, “how many copies of  $x$  make  $y$ .” I retorted by saying that  $20 \div 4$  “does not mean the number of copies of 20 that make four.” She acknowledged this and then said, “ $y$  copies of  $x$ ?” Her tone suggested that this was a complete question, so I pointed out that this was not a coherent question. She tried again by saying, “how many copies of  $x$ ...,” and then she stalled. We returned to her work from Tasks A and B in Activity 7 and I asked her to identify which task was better aligned with the quotitive question she was asking. After she correctly identified Task B, I asked her to construct a general question that aligned with Task A. She thought for a moment and then said, “ $y$  copies of what make  $x$ ?”

Now that Linda had formulated two questions for the two meanings of division, I asked her to describe her meanings for  $a \div 4$  and she said, “how many copies of 4 make  $a$ ” and “4 copies of what make  $a$ .” She indicated the questions were reasonable. I then asked her to describe her meanings for  $a \div 1/4$  and she said, “how many copies of  $1/4$  make  $a$ ” and “ $1/4$  copies of what make  $a$ ,” and said both questions were also reasonable. To determine what these last two questions meant to her, I asked her to model  $3 \div 1/4$  both ways. She had no trouble with the quotitive representation. She drew three rectangles, partitioned each into four parts, then counted 12 total parts. For the partitive question, she

said right away, “I don't know how to draw a picture of that though,” but she tried anyway. I share her work for both models in Figure 40.

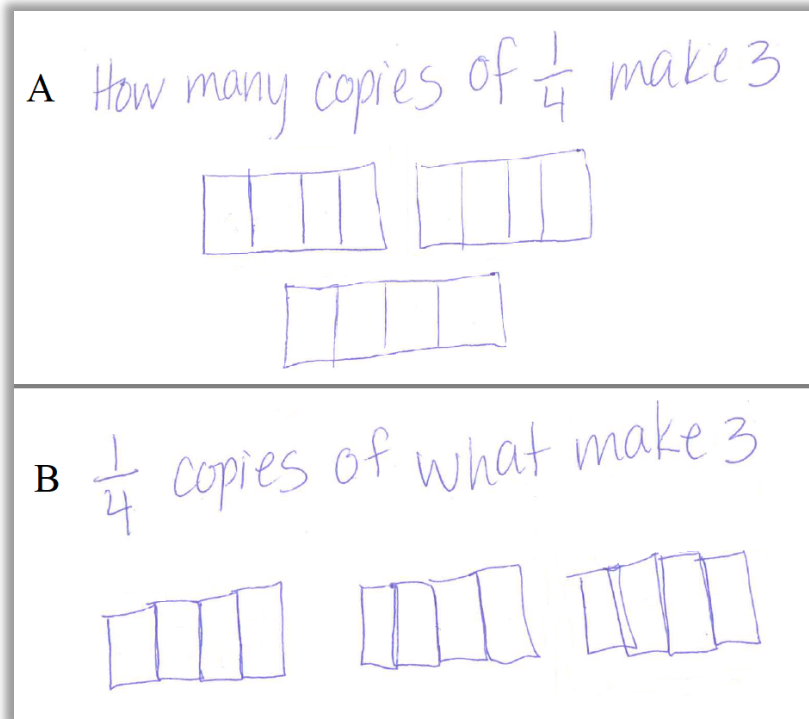


Figure 40. Linda's quotitive (A) and partitive (B) models during Activity 6 in Phase 2.

While looking at the images and questions depicted above, Linda and I had the following dialogue.

| Excerpt 8 |        |                                                                           |
|-----------|--------|---------------------------------------------------------------------------|
| 1         | Linda: | So, I have one-fourth...I made 12 copies of one-fourth to make            |
| 2         |        | three. So, (referring to question B) one-fourth (points at $1/4$ ) copies |
| 3         |        | of 12 (points at <i>what</i> ) make three (points at 3).                  |
| 4         | MW:    | So, this picture (picture B) helps me with which question?                |
| 5         | Linda: | This one (points to question B).                                          |
| 6         | MW:    | Why?                                                                      |
| 7         | Linda: | Because I made one-fourth (points to one piece of a rectangle in          |
| 8         |        | picture B) copies of something, and I had to make three. (pauses)         |
| 9         |        | Am I still saying it backwards?                                           |

10 MW: Where is the *one whole* in this picture (picture B)? (Linda circles  
 11 the four pieces of the left rectangle). Where is the *three*? (she  
 12 circles the other two rectangles, one at a time) So far, we're going  
 13 down the same route as this one (pointing at picture A).

14 Linda: Yeah, so I made, I drew them separate (light chuckle).

15 MW: Where is the 12 (referring to picture B)?

16 Linda: Each one of these (she taps each of the 12 pieces in picture B).

17 MW: Where is the one-fourth? (Linda circles one piece in picture B). So,  
 18 the 12 is answering which question, with this picture (B)?

19 Linda: It's not answering this one (points to question B)? It's answering  
 20 that one again (points to question A).

21 MW: Why?

22 Linda: Well, I think it's answering this one (switches back to question B)  
 23 because I drew one-fourth copies of something (I cut her off)

24 MW: Where's the *something* in this picture (B)? Where's the *x* in this  
 25 picture (B)?

26 Linda: Each one of these (puts an *x* in each of the four pieces of the left  
 27 rectangle in picture B).

28 MW: So, you're saying (pointing to the start of question B) that if I do  
 29 one-fourth of that (pointing to one piece of a rectangle), then I'll  
 30 get three? (pause) It sounds more like if I do 12 *x*'s then I'll get  
 31 three.

32 Linda: Right. Okay, then I don't know how to draw a picture of that.

---

The data is not conclusive about what was causing Linda's trouble. One possibility is that the plurality of the word *copies* may have been the source of the confusion. Since the word is plural, perhaps Linda was trying to imagine many of them. I wonder what she might have done if I had phrased the partitive question, "1/4 copy of what is 3?" The data from Activity 4 suggest that Linda did not have a strong quantitative

meaning for the word *copy*, making me wonder what she might have done if I had removed the word *copy* and simply asked, “ $1/4$  of what is 3?” Her trouble during this activity could also be rooted in a lack of re-unitizing due to quotitive-modeling interference. With her quotitive model, the  $1/4$  and the 3 each refer to the same unit. But to produce a partitive model, she would have needed to re-unitize and imagine a new unit for the  $1/4$ . It is difficult to say anything conclusively, but the overall data from the study revealed Linda’s endemic weakness with re-unitizing and with fractional multipliers, which suggest that re-unitizing was the primary source of her difficulty with this activity.

Linda tried a few more times to produce a partitive model but each attempt led to a quotitive model. I decided to take a different approach by asking her to model  $1/4 \times 20$ . She exhibited a multiplier-switch and modeled 20 copies of  $1/4$ . To help her make progress, I suggested that she describe doing  $1/4$  of anything, like a pizza, and she said she would break the pizza up into four groups. I asked her how many pepperonis would be in each group if the pizza had 20 pepperonis. In response, she drew 20 circles in four rows of 5 to illustrate that  $20 \div 4$  is 5. We discussed how this new drawing was not the same as her drawing of *20 copies of  $1/4$* . Once we had arrived at  $1/4 \times 20 = 5$ , I asked her how she would “reverse-engineer” this process if she was given  $1/4 \times ? = 5$ . She drew a box, split it in four pieces, put five circles in one piece, then copied this for each piece, making 20 circles in total, which is depicted in Figure 41.

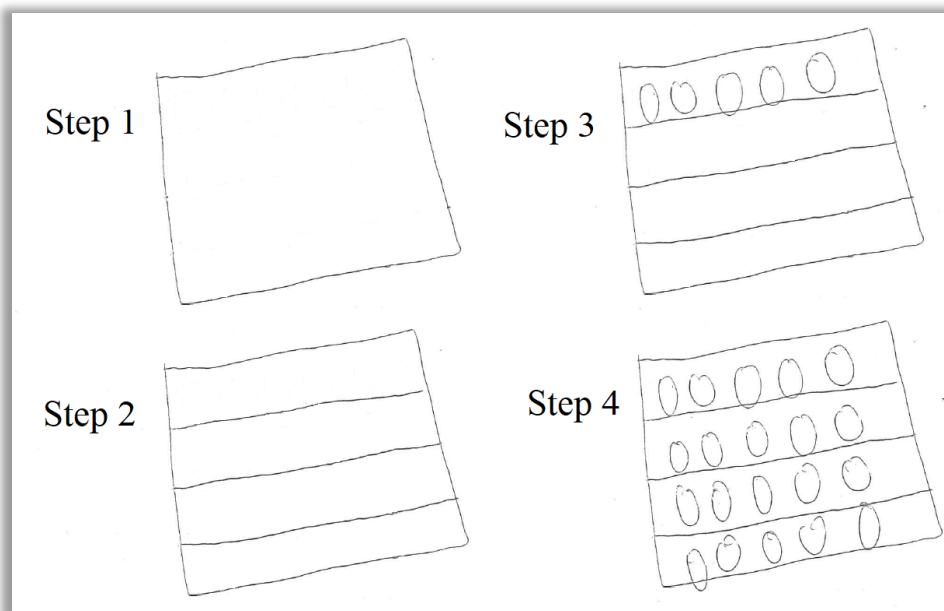


Figure 41. Linda's work modeling " $\frac{1}{4}$  of what is 5" during Activity 6 in Phase 2.

I suggested to Linda that this process is the way I was aiming to have her model a partitive meaning for division. We discussed the following four steps: (1) draw a box to represent the unknown value of the multiplicand, (2) operate on it according to the value of the multiplier, (3) populate the result of the operation with the value of the dividend, and (4) determine the original unknown amount. We returned to the task of modeling  $\frac{1}{4}$  *copies of what make 3*, which she did correctly, in the same way she solved the pepperoni task.

This activity resurfaced several cognitive difficulties for Linda. She again exhibited a multiplier-switch when modeling  $\frac{1}{4} \times 20$ . Also, there was further evidence that Linda had a weak quantitative meaning for the word *copies*. Furthermore, Linda again demonstrated an inability to re-unitize in that she could not imagine a unit for the  $\frac{1}{4}$  other than the same unit she was imagining for the 3, which may have been caused by quotitive-modeling interference. This caused her to be unsuccessful at producing a

partitive model for  $3 \div 1/4$ . The kind of re-unitization needed to produce a valid partitive model for this task required Linda to construct in her imagination an abstract unit. This is certainly more cognitively demanding than coordinating multiple levels of units in contexts where the two quantities are more concrete. For example, consider a context where the relevant units are one *gallon of water* and one *container*, or a context where the units are a *green strip* and a *red strip*. In such contexts, the units are easier to imagine and easier to keep distinct in one's thinking. In such contexts, the primary issue is *maintaining* two levels of units, not necessarily *creating* them. However, producing a partitive model for the decontextualized statement  $3 \div 1/4$  required Linda to *create* two levels of units in her thinking. Linda had already demonstrated weaknesses in maintaining two levels of units in contexts where the units were more concrete. Thus, it is not surprising that she struggled in cases that required her to conjure and *then* coordinate two levels of abstract units.

**Activity 8: Unknown amount of water fills 5 containers.** I asked Linda to describe how much water would fill a whole container, if some amount of water filled 5 containers. She drew five rectangles and wrote  $x \div 5 = ?$  We then had the following dialogue.

*Excerpt 9*

---

|   |        |                                                                        |
|---|--------|------------------------------------------------------------------------|
| 1 | MW:    | Which flavor of division is this?                                      |
| 2 | Linda: | (Long pause) It would be the <i>how many copies of x make 5</i> .      |
| 3 | MW:    | That would be if I had five divided by $x$ . Remember, division, you   |
| 4 |        | want to read it this way (pointing right-to-left) so you gotta use the |
| 5 |        | <i>five</i> before you use the $x$ .                                   |

6 Linda: Right, so...it would be *5 copies of what make  $x$*  (she also writes this  
7 expression).

8 MW: What was the other option for describing this division (pointing at  
9  $x \div 5$ )?

10 Linda: *How many copies of 5 make  $x$ .*

11 MW: Why does that feel like an inappropriate question for this situation?

12 Linda: Because that's not how it's written.

13 MW: How what's written?

14 Linda: *How many copies of  $x$  make 5* (she also writes this expression).  
15 Right?

16 MW: Okay, what's wrong with this question (referring to her last  
17 expression).

18 Linda: Because it doesn't match. It should be *how many copies of 5 make*  
19  *$x$ .*

20 MW: Okay, right, so that question (*how many copies of 5 make  $x$* ) is  
21 different than this question (*5 copies of what make  $x$* ).

22 Linda: Yes.

23 MW: Why is this question (*how many copies of 5 make  $x$* ), it's still a  
24 good question, but why is it inappropriate for this scenario?

25 Linda: Because I'm not dividing up the *five*. I'm dividing up the  $x$ .  
26 Because that's the given amount.

27 MW: I think you're answering, "why is the  $x$  first and the five second?"  
28 (pointing at  $x \div 5$ ). And I agree with that. The  $x$  should be first and  
29 the five should be second. But even when the  $x$  is first and the five  
30 is second, there are still two ways to think about division. It's either  
31 *5 copies of what make  $x$* , or *how many copies of 5 make  $x$* . Those  
32 are both valid ways to think about this division (pointing at  $x \div 5$ ).  
33 But given the context now, only one of those questions seems  
34 relevant.

35 Linda: It's this one (*5 copies of what make x*)! Because it's five copies of  
 36 what makes  $x$ . So, I want to know how many copies of this  
 37 (pointing at the *what*) will make  $x$ .  
 38 MW: You *know* how many copies. You want to know how much is being  
 39 copied.  
 40 Linda: Right, how much is being copied. Yes.

---

I then asked Linda to produce units for the 5 and  $x$  in the quotitive question and she wrote, "How many copies of 5 containers make  $x$  water?" I suggested that this question "feels weird, because how do you copy containers and get water out of it? It's like turning straw into gold. But it does make sense to copy water and get water." I took this opportunity to discuss that for a multiplicative statement, the multiplicand and product should be referring to the same thing. I said, "You copy apples to get apples, children to get children, and water to get water." I then asked her for another operation that would answer the question and she wrote  $\frac{1}{5} \cdot x = ?$  We discussed that since  $x \div 5$  and  $\frac{1}{5} \cdot x$  both answer the same question, then they must be the same amount, and she suggested this was the invert-and-multiply algorithm.

This activity revealed some cognitive obstacles that I did not anticipate. Linda was conflating my questions regarding two meanings for division, with questions regarding commutativity of division. When I was trying to get her to think about the two meanings for division (*5 copies of what make x* and *how many copies of 5 make x*), she thought that I was referring to *how many copies of 5 make x* and *how many copies of x make 5*. However, Linda did recognize and explain why  $x \div 5$  was more appropriate than  $5 \div x$  when she said, "I'm not dividing up the *five*. I'm dividing up the  $x$ ." The data revealed another cognitive obstacle when Linda tried to give a general description of the

missing multiplicand by saying, “I want to know how many copies of this (referring to the missing multiplicand) will make  $x$ .” This was a flawed way to describe the task, because she *did* know how many copies were involved – there were *five* copies. Her poor language suggests a weak quantitative conceptualization for division, which is not surprising given how she struggled so much during Activity 7 to give a general quantitative description of multiplication.

**Activity 9: Unknown amount of water fills  $7/4$  containers.** For this activity, I used actual containers as manipulatives. I gave Linda two cylindrical containers with blue rims, one was labeled *whole*, and the other was  $3/4$  as large as the whole container. I gave her a third container with a white rim that was  $1/4$  as large as the whole container. I depict an image of the three containers in Figure 42.

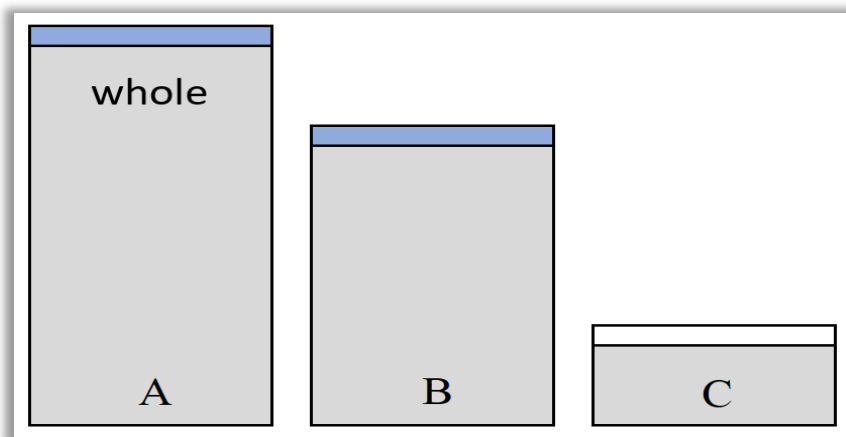


Figure 42. The three containers that we used during Activity 9 in Phase 2.

The containers depicted above are labeled A, B, and C in the image so that I may refer to them in the discussion below. The actual containers that Linda used were not labeled A, B, and C. Linda determined that the blue-rimmed containers (A and B) represented 1 and  $3/4$  containers, or seven quarter-containers. She wrote *7 copies of  $1/4$* , and then we had the following dialogue.

*Excerpt 10*

---

- 1 MW: One-fourth of what?
- 2 Linda: One-fourth of the small container (C).
- 3 MW: So, this (C) is one-fourth of itself?
- 4 Linda: No, so this together (A and B) is seven copies of one-
- 5 fourth...(trails off)
- 6 MW: And the one-fourth is referring to what?
- 7 Linda: The whole, both containers (A and B).
- 8 MW: So, this (C) is one-fourth of all of this (A and B)?
- 9 Linda: Yes.
- 10 MW: If I put them together (I stack A and B), this (C) is one-fourth of all
- 11 of that (A and B)?
- 12 Linda: Yeah.
- 13 MW: (I place C next to the vertical stack of A and B) Copy that (C) four
- 14 times.
- 15 Linda: One, two, oh, seven times. (chuckles)
- 16 MW: What fraction is this (C) of all of this (A and B)?
- 17 Linda: Um, one-seventh.
- 18 MW: Okay, so, one-fourth of what?
- 19 Linda: Seven copies of one-fourth of...of this (grabs C)!
- 20 MW: So, this (C) is one-fourth of itself?
- 21 Linda: No, it should be one-fourth copies of seven (writes *1/4 copies of 7*).
- 22 MW: So, you take seven and you divide it by four?
- 23 Linda: Because I have one-fourth of these (C), it should be *make* seven,
- 24 not *of* seven. (pause) One-fourth of these (C) make seven (said with
- 25 confidence).
- 26 MW: One-fourth of this (C)? If I do one-fourth of this (C), it's going to
- 27 get smaller. Right? If I break this (C) into four pieces...
- 28 Linda: I need seven copies of this (C).

29 MW: Okay, I like that first sentence (*7 copies of 1/4*). I'm not sure I like  
 30 that second sentence (*1/4 copies of 7*). (Linda crosses out *1/4 copies*  
 31 *of 7*). We're still hung up on one-fourth of what? This (C) is one-  
 32 fourth relative to what?

33 Linda: A whole, of this (stacks A and B).

34 MW: And when you say a *whole*, do you mean the whole collective of  
 35 the two containers put together?

36 Linda: Yes.

37 MW: So, in other words, if I copy this (C) four times (moving C up the  
 38 stack, one iteration at a time), I'll get all of it (A and B)?

39 Linda: Yes, well no. Seven of these (C) equal that (A and B).

40 MW: Okay, so why then are you calling this (C) one-fourth? This (C) is  
 41 one-fourth of what?

42 Linda: Of this one (A).

43 MW: Not the combined?

44 Linda: No, not the combined.

45 MW: So, what word can you put after that (pointing to *7 copies of 1/4*)?

46 Linda: (after a long pause, she writes *7 copies of 1/4(1 container)*)

---

This dialogue reveals Linda's difficulty with not just re-unitizing, but with *unitizing*. She was fixated on the value  $1/4$  but she could not establish its unit-of-measure. It was very intriguing that she suggested many times that container C was  $1/4$  of itself. And when I did not seem satisfied, she almost appeared to be guessing until she could read my body language for approval. Her next erroneous suggestion was that container C was  $1/4$  of containers A and B combined. After she had finally settled on " $1/4$  of one container," I suggested that she more clearly write " $1/4$  of one *whole* container." We then discussed that the quarter-container could be measured in three different ways –  $1/4$  of the whole container,  $1/3$  of the partial container, or  $1/7$  of both containers – and I related

this idea to the red and green strips. This again reveals Linda's weaknesses with re-unitizing, but it also reveals a more fundamental void in her quantitative reasoning. It was not until I suggested to her that she iterate to verify her claims, that she recognized her fallacious units. Why did this manner of verification not occur to her? In other words, why did Linda not simply iterate container C four times to determine which container was the unit-of-measure? The data does not support an answer to this question, which could be the focus of future research. We then returned to the task at hand, and we had the following conversation.

*Excerpt 11*

---

|    |        |                                                                             |
|----|--------|-----------------------------------------------------------------------------|
| 1  | MW:    | If you know how much water is in both of these combined (A and              |
| 2  |        | B vertically stacked), how could you figure out how much water is           |
| 3  |        | just in this one (A)?                                                       |
| 4  | Linda: | Well combined is seven-fourths,...times...well to find the water in         |
| 5  |        | just this one (A), it would be times one-fourth (writes $7/4 \times 1/4$ ). |
| 6  | MW:    | That's seven-fourths what?                                                  |
| 7  | Linda: | Seven-fourths of the whole container (A), uh, of the containers             |
| 8  |        | combined (A and B, writes <i>containers combined</i> after $7/4$ ). Because |
| 9  |        | it's seven-fourths of the water, of the given amount of water.              |
| 10 | MW:    | What is seven-fourths of the given amount of water?                         |
| 11 | Linda: | That's how much, if you break it into fourths, how many copies of           |
| 12 |        | one-fourth there are.                                                       |
| 13 | MW:    | So, (I stack A and B) the given amount of water fills all of this           |
| 14 |        | right? What do you have to do to all of this (A and B) to make it           |
| 15 |        | just this (removes B, leaving just A)?                                      |
| 16 | Linda: | I have to (puts B back on top of A) take this part off (removes B again).   |

- 17 MW: That's one way to do it, but let's think of contract and expand. How  
18 do I contract all of this (A and B vertically stacked) down to this  
19 amount (C)?  
20 Linda: I divide by seven.  
21 MW: How do I expand it back up to that amount (A only)?  
22 Linda: Times it by...four.
- 

One of my objectives with this activity was to help Linda determine the statement of division that would answer the task. She looked at her work for Activity 8 (some amount of water fills 5 containers), and said, "I don't know how much water (writing  $x$ ) and I'm gonna divide it by seven-fourths of the containers...to equal..." She did not finish this statement, although she wrote  $x \div \frac{7}{4} = ?$  It is noteworthy how she described the number of containers – she said, "seven-fourths *of* the containers (plural)." Her words make it sound like she is describing seven-fourths of seven-fourths of a container. This is yet another example of unclear language that beset her communication.

#### **Session 4 of the Teaching Experiment**

For the last session of the teaching experiment, I aimed to review reciprocal comparisons and meanings for division as discussed in the preceding sessions. Then I concluded with water-container tasks with fractional numbers of containers.

**Activity 3: Comparing the red and green strips (fourth occurrence).** This was the fourth time that Linda compared the red and green strips. She was successful at making both comparisons, but not with cognitive ease. Thus, at this point I characterized her comparison scheme as *reversible* but not at an abstracted level.

**Activity 7: Quotitive and partitive water-container tasks (third occurrence).**

This was the third time I gave Linda Activity 7 to reflect on. I asked her to describe two meanings for the expression  $20 \div 4$  and she quickly said, “How many groups of four can I pull from 20?” She then said, “The other way...would be...I can pull...blank number of copies...from 20?” To guide her, I put in her view her work from her first exposure to Activity 7 (Figure 36 through Figure 39). While looking at her previous work she said, “Out of 20 copies, how many groups of 4 can I make?” but she recognized that she just reworded her first question. She tried again saying, “4 copies of how many containers equal 20 gallons?”, then she settled on the question, “4 containers of how many gallons can make 20 gallons?” These utterances revealed Linda’s lingering issues related to units and what is sensible in a quantitative multiplication statement. To determine her level of attention to quantities, I asked her to identify – pointing at  $a \times b = c$  – which two components should be referencing the same unit. She said, “the number of copies (pointing at  $a$ )” and “what the answer is (pointing at  $c$ ).” This again revealed a misconception regarding the quantitative structure of multiplication that we had been discussing extensively. She sensed that I disapproved, or perhaps she sensed her own error, and expressed frustration by saying, “I started breaking through last time. Why can’t it come back?” We worked together and she eventually settled again on the two questions, *how many copies of 4 make 20?* and *4 copies of what amount make 20?* The data from this activity again reveal Linda’s weakness with quantitative multiplicative structures, which necessarily impairs her meanings for division. The data also reaffirm the difficulty in trying to build permanent mathematical meanings.

**Activity 10: Unknown amount of water fills 2 and  $\frac{2}{3}$  containers.** For this activity, Linda used  $a$  to represent the water in all the containers, and  $b$  to represent the water in “each one.” Her phrasing made me suspect that she was not imagining different amounts in the three containers, which she confirmed. Thus, she was not imagining the containers being full, despite the language in the prompt. I suggested that there was enough water to fill the whole containers, but only enough water to partially fill the third container. This was a revelation to her and she said things “make more sense now.” She was then able to resolve the task arriving at  $\frac{3}{8} \times a = b$ . The data from this activity confirmed that despite my careful phrasing of the water-container tasks, there was no guarantee that readers of the task would imagine what I intended them to imagine. For Linda, this presented a minor, and easily correctable, cognitive barrier. The data for Activities 11 and 12 did not reveal any additional cognitive barriers so I conclude this section of the chapter and summarize my findings for Phase 2 in the next section.

### **Summary of Phase 2 Findings**

To answer RQ1.6, I focused on identifying additional cognitive obstacles that Linda exhibited during the teaching experiment. In this section I briefly summarize some of the challenges from the teaching experiment that she also exhibited during Phase 1, followed by summaries of the new cognitive barriers.

### **Cognitive Obstacles that Linda Revealed in Phases 1 and 2**

During the teaching experiment, Linda exhibited several cognitive obstacles that she had also revealed during Phase 1. In Activity 1 of the teaching experiment, her focus on a mixed number led her to lose sight of the task’s objective, and she misinterpreted the

task as *draw 2 and 1/3 pink strips*, instead of as *draw one strip that is 2 and 1/3 times as long*. Additionally, she again revealed challenges related to re-unitization, as evidenced in Activity 2 where she was trying to find  $\frac{4}{3}$  of 5 centimeters and she claimed that the new length was 5 and  $\frac{1}{3}$ . She successfully coordinated one whole pink strip with 5 centimeters, but she did not coordinate  $\frac{1}{3}$  of the pink strip with  $\frac{1}{3}$  of 5 centimeters. Also, during the first three exposures to Activity 3 (comparing the red and green strips), Linda had trouble imagining the contracted length as both  $\frac{1}{4}$  and  $\frac{1}{7}$  because she could initially only imagine the contracted length in terms of the green strip. And lastly, during Activity 4 Linda could not coordinate the quantities, *bars of chocolate* and *weight of chocolate*, which explained why she had trouble simultaneously imagining both  $\frac{1}{2}$  of a pound and  $\frac{1}{4}$  of a bar.

### **Additional Cognitive Obstacles Revealed in Phase 2**

The data from the teaching experiment allowed me to expound on some of Linda's difficulties from Phase 1 and to identify new cognitive barriers. I summarize these findings in the paragraphs below.

**Fractional multipliers.** In Phase 1, Linda did not know how to think about  $\frac{5}{3}$  *copies of 2*, which did not provide much data regarding her meanings for fractional multipliers. However, in Phase 2, when she tried to model  $\frac{4}{3}$  of 5 during Activity 2, she exhibited a multiplier-switch. Linda again exhibited a multiplier-switch in Activity 6 when she was trying to model  $\frac{1}{4} \times 20$ . Linda revealed additional difficulty with fractional multipliers during Activity 2 when she was trying to model  $\frac{1}{3}$  of 5 wholes, and she could only imagine  $\frac{1}{3}$  of each whole, one at a time.

**Unitization vs. re-unitization.** During Activity 9 (unknown amount of water fills  $7/4$  containers), Linda had difficulty with not just re-unitizing, but with *unitizing*. She was trying to operate with the value  $1/4$ , but she could not establish its unit-of-measure. She made several errant suggestions regarding an appropriate unit, but she did not catch her errors because she did not attempt to verify her claims through iteration. She was trying to operate with a value, for which she could not establish an accurate unit-of-measure, which suggested a quantitative detachment from the task. Furthermore – during Activity 6 (meanings for division) – Linda was not able to *unitize*, in that she could not imagine a unit for the  $1/4$  other than the same unit she was imagining for the 3, which caused her to be unsuccessful at producing a partitive model for  $3 \div 1/4$ . In Phases 1 and 2, Linda had revealed a weakness with *maintaining* two levels of units, but in Phase 2, she also revealed an inability to *create* levels of units.

**Weak meanings for multiplication.** During Activity 7 (quotative and partitive water-container tasks), Linda revealed a weak general meaning for multiplication, when she said initially that the product was “the number of copies you are copying,” which better aligned with her description of the multiplier. She had trouble describing the product in both the general case and in the specific case of *4 containers with 5 gallons per container make a total of 20 gallons*. Linda did appear to be perturbed by the contradiction caused when she said, “*c* is the number of copies you are copying,” while she had written on paper that *c* represented 20 gallons of water. The data also seem to suggest that Linda’s reliance on numerical commutativity contributed to her difficulties with maintaining a consistent, quantitative (hence non-commutative) conceptualization of multiplication. During Activity 2 (meanings for multiplication), Linda’s reliance on the

commutativity of numerical multiplication made it difficult for her to be consistent in her thinking, speech, and writing regarding the contraction-expansion dual-scheme for fractional multipliers.

**Weak meanings for division.** Since Linda struggled to give a general quantitative description of multiplication, I was not surprised to observe that she also struggled with general meanings for division. During Activity 8 (unknown amount of water fills 5 containers), when discussing the two meanings for division (*5 copies of what make  $x$*  and *how many copies of 5 make  $x$* ), Linda referred to the two questions, *how many copies of 5 make  $x$*  and *how many copies of  $x$  make 5*. As such, she conflated questions relative to the two meanings for division, with questions regarding commutativity of division. Also, when Linda tried to give a general description of the missing multiplicand task ( $5 \cdot ? = x$ ), she said, “I want to know how many copies of this (referring to the missing multiplicand) will make  $x$ .” This was a flawed way to describe the task, because she *did* know how many copies were involved – there were *five* copies. These data suggested that Linda possessed a weak quantitative conceptualization for division, which was not surprising, since her dual meanings for division were nascent at the time that this data was gathered.

**Using vague and confusing language.** During Activity 3 (comparing the red and green strips), Linda consistently and mistakenly said “times longer than” while reading “times as long as.” Her phrasing was vague because an expression such as “2 times longer than 3 feet” could be interpreted as six feet, or it could be interpreted as 3 feet *plus* six feet. During Activity 4 (meanings for fractional multipliers), Linda rarely mentioned units for the numbers she was saying and writing, which suggested that she was

quantitatively disengaged from the scenario. During Activity 7 (quotitive and partitive water-container tasks), Linda uttered more confusing statements such as, “I want to find how many 20 gallons would fill just one (container),” and “I have four containers...four containers of...would fill 20 gallons.” These utterances suggest that Linda was not engaged in quantitative reasoning, or at the very least that she was not attentive to using meaningful phrases to convey her thoughts. As a final example, during Activity 9 (unknown amount of water fills  $\frac{7}{4}$  containers), Linda described  $\frac{7}{4}$  containers as, “seven-fourths *of* the containers (plural).” Her words make it seem as though she was describing seven-fourths of seven-fourths of a container.

**Understanding language.** During Activity 4 (meanings for fractional multipliers), when I repeatedly asked Linda to find  $\frac{1}{2}$  *copy of 2 pounds*, Linda kept answering  $\frac{1}{4}$  of a pound. However, when I omitted the word *copy* and asked her to find  $\frac{1}{2}$  *of 2 pounds*, she answered *1 pound*. This suggested that she did not have a strong quantitative significance for the word *copy*. During Activity 3 (comparing the red and green strips), Linda repeatedly struggled to give the appropriate comparison, as required by the task. She also struggled with translating a written expression into an appropriate symbolic expression, which suggested a deficiency in quantitative reasoning, either at the level of the written expression, or at the level of the symbolic expression, or both.

**The development of schemes.** During the teaching experiment, Linda had multiple opportunities to reason through certain tasks. She repeated Activity 3 four times, once per session, and her trouble with re-unitizing persisted through many of the attempts. By the fourth attempt, she had exhibited a newly developed dual-scheme of forward and reverse comparisons, but I wondered whether the scheme was permanently

situated in her network of meanings. I also wondered if the scheme had been adequately reflectively abstracted to be applied to analogous situations that did not involve paper strips. The data from Phase 3 suggested that she had not abstracted adequately. There were other instances where Linda was not consistent in her actions. For example, despite extensive discussions between us, she still placed the multiplier in the wrong position, as evidenced in Activities 3 and 4. Linda captured her frustration best during Activity 7 when she said “I started breaking through last time. Why can’t it come back?” Linda’s difficulty to repeat operations of thought suggested that she had not formed permanent schemes. Thompson (2013) commented on this when he said, “to construct a scheme requires applying the same operations of thought repeatedly to understand situations being made meaningful by that scheme...Put another way, we construct stable understandings by repeatedly constructing them anew” (p.61). Linda’s inability to retain information, and to repeat certain ways of thinking, was a recurring cognitive obstacle throughout the teaching experiment.

## CHAPTER 8

### PHASE 3 RESULTS AND DISCUSSION

I based my analysis of Linda's thinking from Phase 3 on my secondary research questions for this phase, which I repeat in Table 14.

Table 14

*Secondary Research Questions Corresponding to Phase 3*

|       |                                                                                             |
|-------|---------------------------------------------------------------------------------------------|
| RQ2.1 | How do the teachers' post-intervention meanings compare to their pre-intervention meanings? |
| RQ2.2 | What advancements to the teachers' schemes are evident and what challenges remain?          |

These secondary research questions were necessary to address RQ2: *How do these teachers' meanings change as a consequence of an instructional sequence that emphasized quantitative reasoning to aid in the advancement of these meanings?*

#### **RQ2.1: Comparative Analysis of Phases 1 and 3**

In this section, I share Linda's results from Phase 3 and compare them to her results from Phase 1 to address RQ2.1, which is, "How do the teachers' post-intervention meanings compare to their pre-intervention meanings?" I began my analysis by looking at changes in Linda's behavior, which I summarize in Table 15. This table also contains summaries of the 12 tasks that were used in Phase 3. The exact tasks are listed in Appendix 3. In this table, the ✓ indicates that I observed the behavior, and the ✗ indicates that I did not observe the behavior. I highlighted in green six rows in the table to indicate that I observed the corresponding behavior in Phase 3 but not in Phase 1. I highlighted in red two rows to indicate that I did not observe the behavior in Phase 3, even though I did observe it in Phase 1. The rest of the rows correspond to no change in the indicated behavior. This table does not reflect all changes in Linda's behavior, but it

does highlight the primary behaviors to indicate that the teaching experiment had an effect on Linda's actions.

Table 15

*Comparison of Linda's Behaviors from Phases 1 and 3*

| Task |                                       | Potential Behaviors                               | Phase |   |
|------|---------------------------------------|---------------------------------------------------|-------|---|
|      |                                       |                                                   | 1     | 3 |
| 3    | Comparing red and green strips        | Successfully gave long in terms of short          | ✓     | ✓ |
|      |                                       | Successfully gave short in terms of long          | ✗     | ✓ |
| 6    | $5/3 \times 2$                        | Successful model of $5/3$ as multiplier           | ✗     | ✗ |
|      |                                       | Contracted the value 2 and expanded               | ✗     | ✗ |
| 8    | $a \div b$                            | Gave a quotitive description                      | ✓     | ✓ |
|      |                                       | Gave a partitive description                      | ✗     | ✓ |
| 9    | $4 \div 3$                            | Successful quotitive model                        | ✗     | ✗ |
|      |                                       | Successful partitive model                        | ✗     | ✓ |
| 10   | $4 \div 1/3$                          | Successful quotitive model                        | ✓     | ✓ |
|      |                                       | Successful partitive model                        | ✗     | ✗ |
| 18   | Some amount fills 5 containers        | Suggested the operation $\text{water} \div 5$     | ✓     | ✓ |
|      |                                       | Suggested the operation $1/5 \times \text{water}$ | ✓     | ✓ |
| 16   | 13 gallons fill 5 containers          | Successful partitive model                        | ✓     | ✓ |
|      |                                       | Suggested the operation $13 \div 5$               | ✓     | ✓ |
|      |                                       | Suggested the operation $1/5 \times 13$           | ✓     | ✓ |
| 21   | Some amount fills $9/4$ containers    | Suggested the operation $\text{water} \div 9/4$   | ✓     | ✗ |
|      |                                       | Suggested the operation $4/9 \times \text{water}$ | ✗     | ✗ |
| 20   | 3 gallons fill $9/4$ containers       | Successful partitive model                        | ✓     | ✓ |
|      |                                       | Suggested the operation $3 \div 9/4$              | ✓     | ✗ |
|      |                                       | Suggested the operation $4/9 \times 3$            | ✗     | ✓ |
| 24   | Some amount fills $2/3$ container     | Suggested the operation $\text{water} \div 2/3$   | ✗     | ✗ |
|      |                                       | Suggested the operation $3/2 \times \text{water}$ | ✗     | ✓ |
| 23   | $7/4$ gallons fill $2/3$ container    | Successful partitive model                        | ✓     | ✓ |
|      |                                       | Suggested the operation $7/4 \div 2/3$            | ✗     | ✓ |
|      |                                       | Suggested the operation $3/2 \times 7/4$          | ✓     | ✓ |
| 25   | Explain invert-and-multiply algorithm | Successful quotitive explanation                  | ✗     | ✗ |
|      |                                       | Successful partitive explanation                  | ✗     | ✗ |
|      |                                       | Successful other explanation                      | ✗     | ✗ |

Concerning Linda’s meanings, I conducted a closer examination of the data to make conjectures about whether the teaching experiment had an effect on her underlying schemes. I share my analysis for each task in the sections that follow.

### **Task 3: Compare an 8-inch Red Strip and a 3-inch Green Strip**

In Phase 1, Linda said the “red strip is larger by seven-fourths of the green strip,” and the “green strip is seven-fourths smaller than the red strip.” Linda could not switch from the green strip as the unit-of-measure to the red strip, and as such, she could not describe the reciprocal multiplicative relationships between the two strips. However, in Phase 3, Linda could express both comparisons verbally and symbolically. Her language is below.

#### *Excerpt 12*

---

|   |        |                                                                      |
|---|--------|----------------------------------------------------------------------|
| 1 | Linda: | The red strip is eight copies of one-third...equal red. And three    |
| 2 |        | copies of one-eighth is green.                                       |
| 3 | MW:    | Okay, and this (pointing to her written “1/3”) is one-third of what? |
| 4 | Linda: | One-third of green (writes in “green”)...and red (writes “red” next  |
| 5 |        | to her written “1/8”)                                                |

---

For each comparison she wrote down a double-product symbolic representation (contract and expand) and a single-product symbolic representation, as depicted in Figure 43. Additionally, she expressed one-third of the green strip as one-eighth of the red strip. This was a significant advancement in her thinking as compared to Phase 1, and I became optimistic that she would be able to coordinate multiple levels of units and consequently reason productively through the upcoming tasks.

$$\begin{array}{ll} 8\left(\frac{1}{3}\right) = \text{red} & \frac{8}{3} \text{ green} = \text{red} \\ 3\left(\frac{1}{8}\right) = \text{green} & \frac{3}{8} \text{ red} = \text{green} \end{array}$$

Figure 43. Linda's work on Task 3 in Phase 3.

### Task 6: Explain your Meanings for $5/3 \times 2$

In Phase 1, Linda could not productively think about  $5/3$  as the multiplier saying that she could not think about the meaning of  $5/3$  copies. In Phase 3, she again did not produce a model with  $5/3$  as the multiplier. Linda defaulted to interpreting the task as 2 copies of  $5/3$ , despite multiple discussions between us during the teaching experiment about imagining the first number as the multiplier, even if it is a fraction. I did not remind her that the first number should be the multiplier. Instead, I asked her to reverse the order and reconsider her model. She wrote “ $5/3$  copies of 2,” but then modeled two copies of  $5/3$  again, but in a slightly different way than before. She then said, “two copies of five-thirds,” while pointing at the written expression “ $5/3$  copies of 2,” as depicted in Figure 44. The data suggest that Linda interpreted  $5/3$  copies of 2 as  $5/3$  copied twice, which led her to produce the second drawing in which the two copies of  $5/3$  were more obvious. This is similar to a multiplier-switch as I described it in Chapter 6. However, in this case, the switch from  $5/3$  as the multiplier to  $5/3$  as the multiplicand happened immediately, as opposed to partway through the modeling process. Since Linda assimilated a task with a fractional multiplier to a scheme for a whole number multiplier, she circumvented the dual-scheme of contraction and expansion that we had extensively worked on during the

teaching experiment. If the teaching experiment was successful in developing the contraction-expansion scheme for Linda, then the data suggest that a decontextualized statement of multiplication involving a whole number as one factor did not trigger the scheme, even when Linda phrased the statement using language that would suggest to think of  $5/3$  as the multiplier.

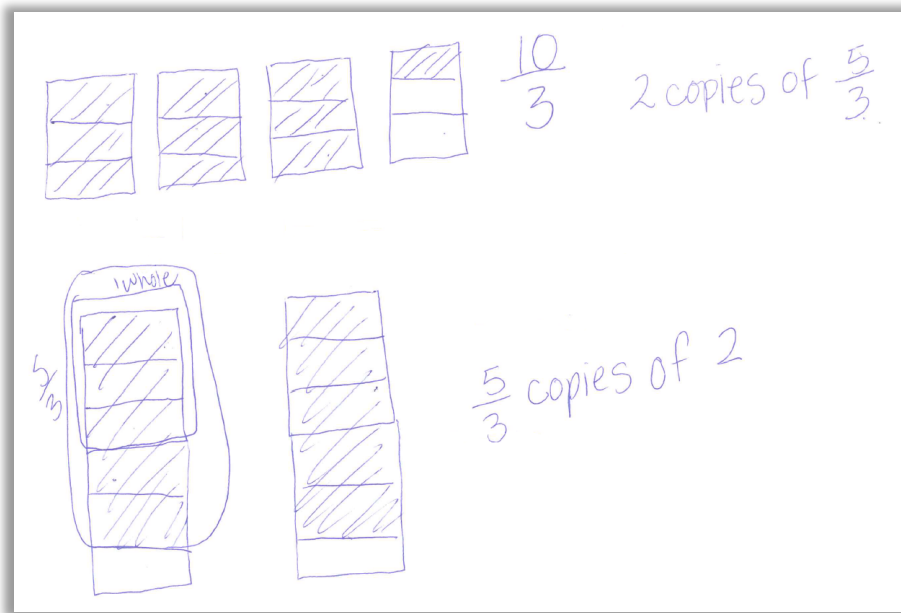


Figure 44. Linda's work on Task 6 in Phase 3.

### Task 8: Explain your Meanings for $a \div b$

In Phase 1, Linda provided the quotitive interpretation, “How many copies of  $b$  can I take from  $a$ ?” In the moment, she could not think of another meaning for the division expression. However, in Phase 3, Linda demonstrated an awareness of two different meanings for division by writing, “how many  $b$  copies are in  $a$ ?” and “ $b$  copies of what =  $a$ ?” She then attempted to demonstrate each meaning by modeling  $10 \div 5$  both ways. Her work for a quotitive conceptualization is illustrated in Figure 45.

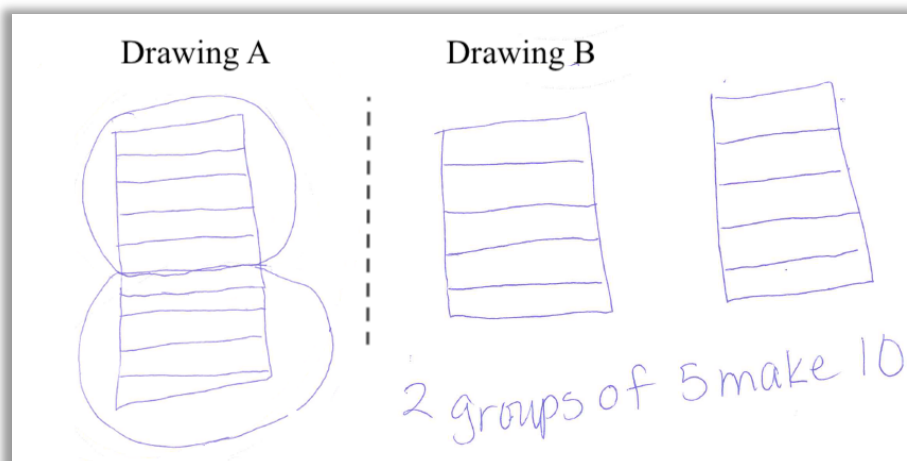


Figure 45. Linda's first attempt to model both conceptualizations on Task 8 in Phase 3. Linda easily modeled the quotitive conceptualization (Drawing A), but when trying to model the partitive conceptualization, she redrew two copies of five (Drawing B) and wrote "2 groups of 5 make 10." This reminded me of what she did in the prior task, confusing a different representation of one way of thinking as a second way of thinking. However, this time, she caught herself. She realized that Drawing B answered her quotitive question, so she tried to convince herself that Drawing A must therefore answer the partitive question. She wrote "5 copies of what make 10," paused, replaced the "what" with "2," then declared that Drawing A answered this question. I asked her how Drawing A was different from Drawing B. She got perturbed and eventually concluded that both of her drawings were answering the question, "How many copies of five make ten?" I asked her about how she might depict the question, "Five copies of what make 10?" She paused and after much silence, she drew five boxes, split each in half, and said "five copies of two equal ten." Her drawing is shown in Figure 46.



Figure 46. Linda's second attempt at a partitive model on Task 8 in Phase 3.

I asked Linda where in her picture was the *two* she uttered and she pointed at half of one of the boxes, then she labeled each half of a box as  $\frac{1}{2}$ . She said there were two halves in each box, so one whole in each box. She re-unitized but the data suggest she was unaware of the re-unitization. I asked her if “five copies of one make ten?” This perturbed her, and so she redrew five new boxes, put a 2 in each box, and then appeared to be satisfied. Her work is in Figure 47.

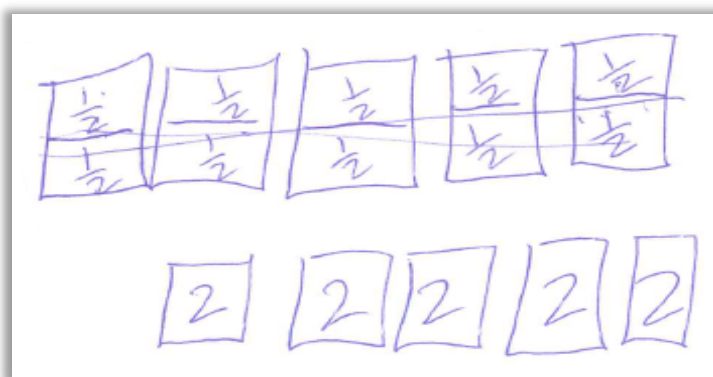


Figure 47. Linda's third attempt at a partitive model on Task 8 in Phase 3.

Her confusion with this task was related to her weakness with maintaining multiple units simultaneously. She was unable to think of each half of a box as a whole unit. This lack of attention to multiple levels of units is due to quotitive-modeling interference. The data in this study revealed that when Linda modeled quotitively, her drawings all referred to the same unit, and she simply counted groups of units, without any evidence that she was thinking of each group itself as a new unit-of-measure. The fact that her quotitive models

did not require multiple levels of units interfered with her ability to produce a partitive model that does require a coordination of multiple levels of units.

### Task 9: Explain your Meanings for $4 \div 3$

In Phase 1, Linda wrote “how many copies of 3 can be taken from 4,” but she did not know how to proceed. I then asked her to think about six divided by 3, which then prompted her to draw a quotitive model that supported the answer of 2. We returned to the task of 4 divided by 3. She drew four squares, circled three of them, and then said she had “one copy and one-third left over.” I wondered why she said “one-third.” Later data suggest that she procedurally divided four by three in her head, obtained an answer of one and one-third, and then tried to produce a drawing that showed this result. Due to this procedural contamination, I was not convinced that she recognized that one square can be thought of as one-third of a copy of three squares, so I asked her to provide a unit for one-third, and in response she labeled each square as  $\frac{1}{3}$ ,” with no unit. I asked her how this drawing helped her answer her own question of “how many copies of 3 can be taken from 4?” She responded by explaining that she had “one copy of 3 and one-third left.” I share her work up to this point in Figure 48.

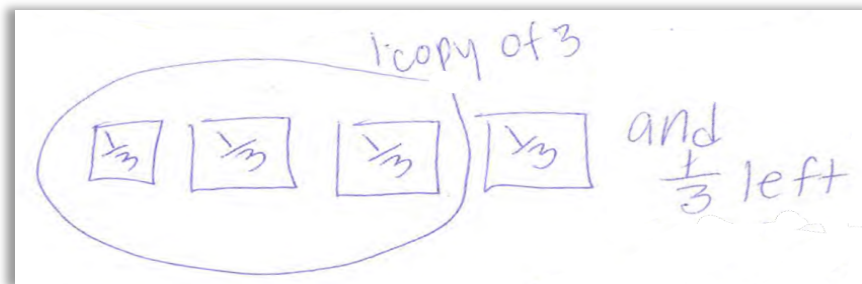


Figure 48. Linda’s attempt at a quotitive model on Task 9 in Phase 1.

If she was maintaining only one level of unit, then her drawing and utterances would be contradictions of each other. However, no contradiction is present if two levels of units are maintained (e.g., three squares is the same amount as one group of three squares). I was not convinced she was aware of the two levels of units so I asked her again, “one-third copy of what?” She responded in an unexpected way by changing her answer by writing “3 copies of  $\frac{1}{3}$  and  $\frac{1}{3}$  left.” Her response suggested she was not aware of the two levels of units, nor that she imagined the fourth square as one-third of a copy of three squares. Instead, Linda thought of the circled copy of three squares as *one*, not *one group of three wholes*, so the squares each became one-third, and only one-third. These data suggest that inadvertent re-unitization is something that is more likely to happen when doing mathematics with no context. For example, each square can be thought of as one whole, as one-fourth of the whole collection, or as one-third of a group of three squares, and without a context it can be challenging to describe these various perspectives, assuming you even manage to keep them distinct in your thinking. The data suggest that she settled on each square being one-third only so that the last square would be one-third, which is what she obtained by doing the calculation procedurally. I asked her whether “1 copy of 3” was interchangeable with “3 copies of  $\frac{1}{3}$ ” and she said yes. Even though this is numerically false (although it could be true depending on the units), I suspect she said yes because both are represented by her single picture. I do not suspect, based on the data, that she said yes because she was aware that the two statements can be quantitatively identical when appropriate units are used.

In Phase 3, Linda had no trouble phrasing both meanings for division, whereas she gave only a quotitive description in Phase 1. For the partitive model, she drew three

boxes and labeled each as  $\frac{4}{3}$ , showing that one whole and another third was in each box.

She confirmed her work by procedurally adding three copies of four-thirds to get four.

Figure 49 illustrates Linda's partitive model for Task 9 during Phase 3.

The figure shows a handwritten partitive model. At the top, three boxes, each containing  $\frac{4}{3}$ , are grouped by a bracket underneath with the number 4. To the right of the boxes is an equals sign. Below this, the equation  $\frac{4}{3} + \frac{4}{3} + \frac{4}{3} = \frac{12}{3} = 4$  is written.

Figure 49. Linda's attempt at a partitive model on Task 9 in Phase 3.

Linda then attempted a quotitive model. She drew four rows, shaded the top three rows, labeled each row as  $\frac{1}{3}$ , then said confidently “four copies of one-third make four.” It would have been more sensible to say “four copies of one-third make four-thirds,” but she said it the way she did because she was trying to answer the question, “How many copies of three make *four*?” Here work is shown in Figure 50.

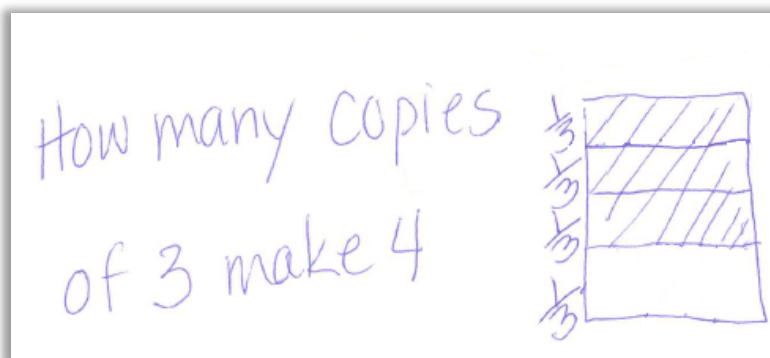
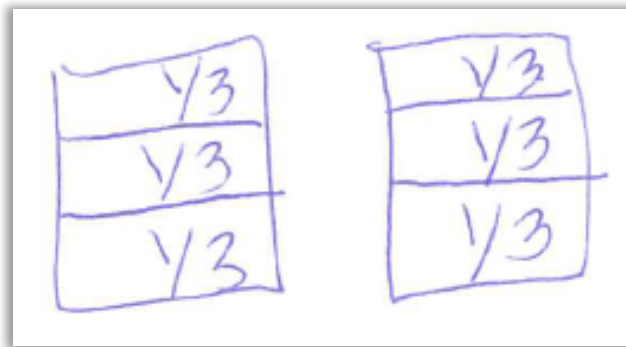


Figure 50. Linda's first attempt at a quotitive model on Task 9 in Phase 3.

I was not convinced she was maintaining multiple levels of units so I asked her “one-third of what?” She responded by circling all four pieces. I asked where in her picture is one copy of three, and she circled one row only. These actions seemed incoherent to me,

as if she was just guessing and looking for approval from me. She admitted that she knew the answer should be four-thirds, a case of procedural contamination. Perhaps she sensed that I was unsettled, or perhaps she was unsettled herself, but she tried again. She drew two blocks of three rows, labeled all of them  $\frac{1}{3}$  and repeatedly said that each block is one whole and that there are three one-thirds in each whole. This work is in Figure 51.



*Figure 51.* Linda's second attempt at a quotitive model on Task 9 in Phase 3.

At this point, the data suggest that Linda was simply drawing a representation of the answer of  $\frac{4}{3}$  that she got procedurally. These issues of re-unitizing are the same issues that she exhibited during this task in Phase 1. In both phases, Linda could not a piece of a block as both one and as one-third.

At this point, I interjected by supplying a context. I suggested that three peanut butter cups come in a package. I asked her to imagine that the 3 and 4 in  $4 \div 3$  each refer to peanut butter cups. I asked her how many packages make four peanut butter cups and she instantly answered "one and a third." I pointed at one peanut butter cup and asked her "how many cups" and she said "one." I asked "how many packages" and she said "one-third of the package." When there was a context, that supplied distinctive imagery and vocabulary to describe both types of unit, it became easier for her to re-unitize.

### Task 10: Explain your Meanings for $4 \div \frac{1}{3}$

In Phase 1, Linda easily produced a quotitive model to determine that 12 copies of one-third make four. However, she did not attempt a partitive conceptualization because she was not aware that this was a possibility for this task. However, in Phase 3, Linda wrote “ $\frac{1}{3}$  copies of what = 4” and “how many  $\frac{1}{3}$  copies make 4.” She focused on the first statement, but then partitive-to-quotitive assimilation occurred when she immediately drew a quotitive representation, as depicted in Figure 52.

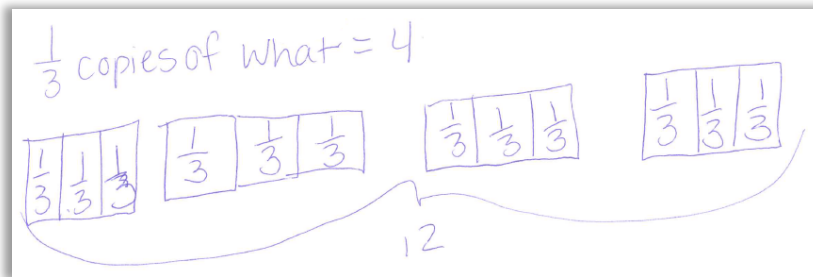


Figure 52. Linda’s first attempt at a partitive model on Task 10 in Phase 3.

When I asked her which question she answered, she acknowledged that she actually answered the second question. She spent about 15 minutes trying to break free of her quotitive thinking, but each attempt ultimately led to another version of a quotitive model. During her confusion, she wrote, said, and drew many incoherent statements. At one point, she drew an image that I thought would be helpful for her, which is highlighted in Figure 53. She drew four boxes first and labeled each  $\frac{1}{3}$ , then extended each box to account for the other two-thirds, putting a “?” in each new box. If she had been able to view the first four boxes as one-third of a larger picture, and if she had maintained that each box was one whole, then she might have been successful. However, once she labeled each box as  $\frac{1}{3}$ , she could only view each row as one, instead of three.

Ultimately, she was unsuccessful producing a partitive model that supported the answer of 12.

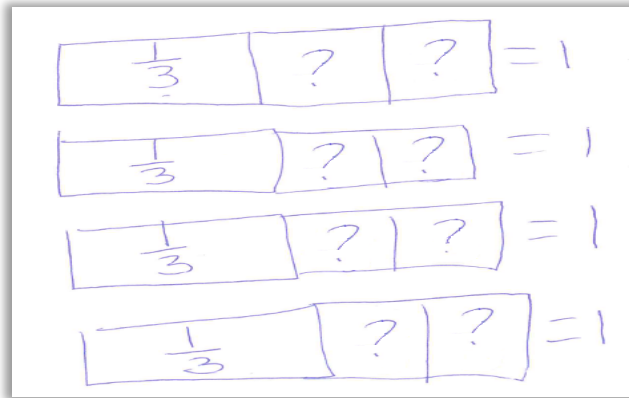


Figure 53. Linda's second attempt at a partitive model on Task 10 in Phase 3.

### Task 18: An Unknown Amount of Water fills 5 Containers

In Phase 1, Linda said she would “divide it (the given amount of water) by five.” I asked if there was anything else that she could do to resolve the task and she said no. I pushed her by asking what she would do if she could only use the operation of multiplication, and she responded that she could multiply by the reciprocal of five, which is one-fifth. I was seeking evidence that she thought that division by five and multiplication by its reciprocal are quantitatively equivalent, so I asked her to describe why she could just multiply by one-fifth, to which she responded, “I would have to multiply by the inverse of five because I’m only looking for one-fifth of the amount of water.” This suggested she was thinking quantitatively, not just procedurally.

In Phase 3, Linda drew five blocks and used “ $a$ ” to represent the given amount of water. She said each block would have one-fifth of all the water, which she wrote as “ $1/5$  of  $a$ .” When I asked her for a division expression, she deliberated between  $5 \div a$  and  $a \div 5$ , eventually settling on the latter, writing “ $a \div 5 = \text{whole container}$ .” She explained that the

first expression ( $5 \div a$ ) was wrong because it was “asking how many copies of  $a$  can I pull out of five, when it should be if I have five containers, how much  $a$  is in each of those containers.” When I asked her for the meaning of the expression that she preferred ( $a \div 5$ ) she said, “How many copies of  $a$  make five?” This is troubling for the following three reasons: (1) it is quotitive language in a partitive situation, (2) the divisor and dividend are interchanged in her speech, and (3) it contradicts what she said a few seconds prior. She seemed perturbed, and eventually asked if  $a \div 5$  should be interpreted as, “Five copies of what equal  $a$ ?” She repeated this question to herself several times and said, “I don’t know which flavor this is.” I then asked her to repeat the two “flavors” of  $a \div 5$ , to which she said, “how many copies of five make  $a$ ” and “five copies of what equal  $a$ .” I asked her which of the two questions seemed relevant to the task and she said something new, “Five copies of what will give you one container?” Ultimately, she never did resolve her confusion and she said, “I don’t know why this is hard.”

### **Task 16: 13 Gallons Fill 5 Containers**

In Phase 1, Linda distributed the gallons, two gallons per container, leaving three gallons undistributed. Then she split each of the remaining three gallons into 5 equal parts (each part  $\frac{1}{5}$  of a gallon), distributed the parts among the containers one gallon at a time, and concluded that  $2\frac{3}{5}$  gallons were in each container. I asked her to procedurally calculate an answer and she wrote  $13 \div 5$ , indicating that she recognized this as a division task. She then wrote  $13 \cdot \frac{1}{5}$  and confirmed her answer through a procedural calculation. Her work is shown in Figure 54. In this image, the five rectangles were a tool to help her distribute the extra three gallons, one gallon (one row) at a time.

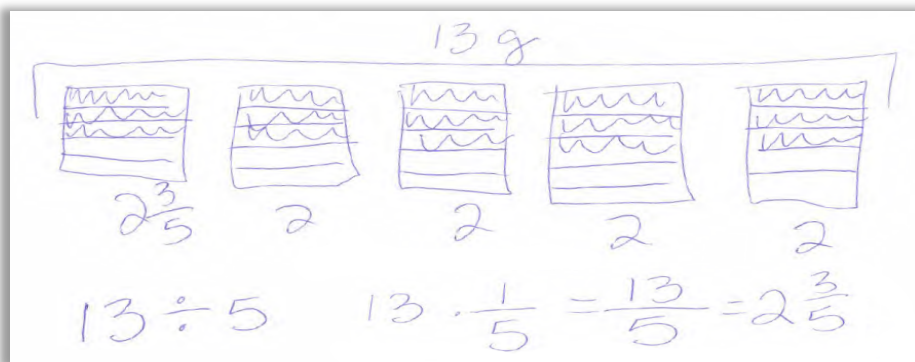


Figure 54. Linda's work on Task 16 in Phase 1.

It was encouraging that she considered  $13 \div 5$  and  $13 \cdot \frac{1}{5}$  as equivalent, but I could not determine from the data whether she realized that these expressions are quantitatively equivalent or whether she was vacuously employing the invert-and-multiply algorithm. Also, I could not determine from the data whether she imagined the containers full and was describing each container's capacity, or whether she was not imagining the containers full was describing each container's contents.

In Phase 3, Linda said "one-fifth" is the capacity of each container, to which I had to ask "one-fifth of what?" She had no problem responding, "one-fifth of 13 gallons." I asked for another operation that would yield the answer, and she wrote  $13 \div 5$ . I prompted her to identify the appropriate "flavor of division," and she laughed and said "I don't know, I confuse the two." But then she tried and said the following sequences of statements.

*Excerpt 13*

---

|   |        |                                             |
|---|--------|---------------------------------------------|
| 1 | Linda: | How many gallons fill five containers?      |
| 2 |        | How many copies of five make...(trails off) |
| 3 |        | How many...(trails off again)               |
| 4 |        | It would be the other one.                  |
| 5 |        | Five copies of what make 13?                |

---

She settled on this last expression and explained her choice in a sensible way by saying, “five copies of each container make a total of 13 gallons.” I interpreted her language as “five copies of *one* container make a total of 13 gallons” and not “five copies of *all the containers* make a total of 13 gallons.” This data reveals the non-trivial nature of phrasing a partitive division task as a missing factor problem, even when the divisor is a whole number. Also, this is an example of an advancement in her meanings for division. In Phase 1, she was able to phrase division questions that only elicited quotitive conceptualizations.

### **Task 21: An Unknown Amount of Water Fills $\frac{9}{4}$ Containers**

In Phase 1, I suspected that Linda would not be able to generalize a process to accomplish this task because she had used guess-and-check strategies on the previous two tasks that involved a known amount of water filling  $\frac{9}{4}$  containers. For this task, she drew a picture of the nine quarter containers and indicated that  $x$  (which represented the unknown total amount of water) was contained in these containers. I asked what she would do with  $x$  to answer the question and she said she would calculate  $x \div \frac{9}{4}$ , as depicted in Figure 55.

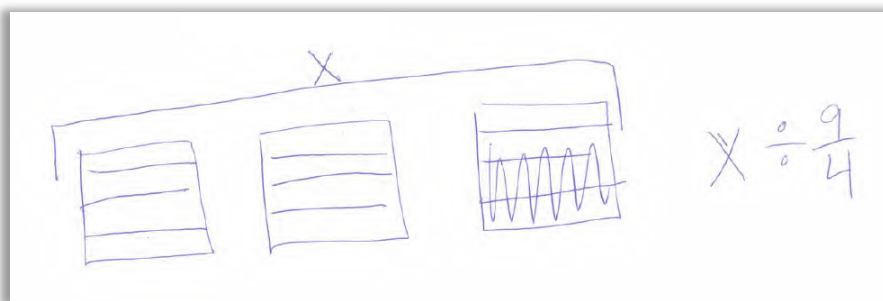


Figure 55. Linda's work on Task 21 in Phase 1.

I asked what she would do to  $x$  to find the amount of water in one quarter container, and she said she would multiply it by one-fourth. She said this because she could not coordinate multiple levels of units and so she was only able to imagine each row of a box as one-fourth.

In Phase 3, Linda drew a representation of the containers, wrote “9 copies of  $\frac{1}{4}$  container,” she wrote “ $a$ ” to represent the given amount of water, but then she wrote “ $\frac{9}{4}$  copies of  $a$ ” to answer the task. On one hand, she said “ $a$ ” is the given amount of water, but her expression “ $\frac{9}{4}$  copies of  $a$ ” implies that “ $a$ ” was one container, or the water in one container. She did not seem perturbed by this contradiction. I asked her to explain her expression “ $\frac{9}{4}$  copies of  $a$ ”, and she redrew the containers side-by-side, as depicted in Figure 56. Drawing A is her first drawing, and Drawing B is her second drawing.

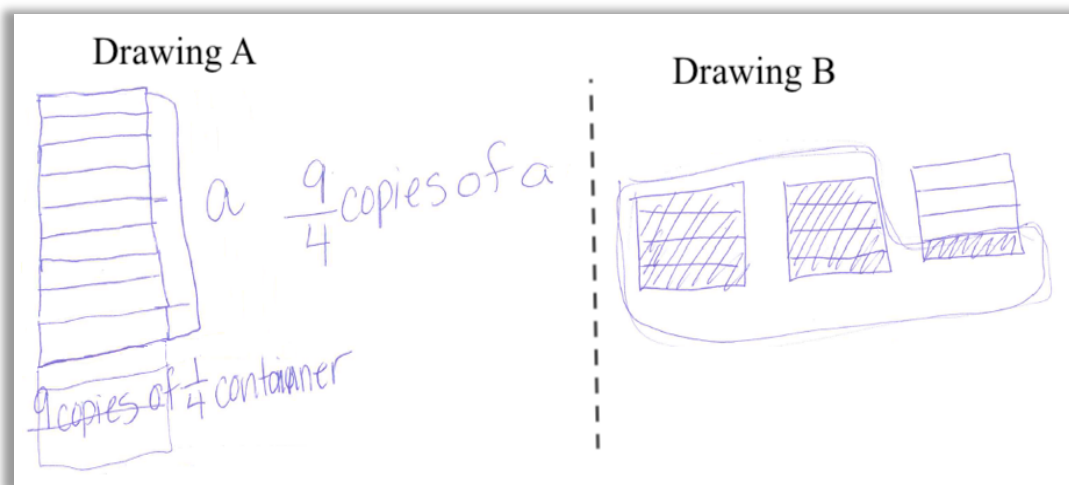


Figure 56. Linda's work on Task 21 in Phase 3.

Linda said “two and one-quarter containers is the answer... nine-fourths copies of the water I have.” She temporarily lost sight of the goal and was trying to describe how much water she started with. We then engaged in the following dialogue.

Excerpt 14

---

|    |        |                                                                           |
|----|--------|---------------------------------------------------------------------------|
| 1  | MW:    | What does “ <i>a</i> ” represent for you again?                           |
| 2  | Linda: | The amount of water I used.                                               |
| 3  | MW:    | And where in this picture is “ <i>a</i> ” (pointing to Drawing B)?        |
| 4  | Linda: | Um...there (circling the nine shaded portions in Drawing B).              |
| 5  | MW:    | And what’s our objective, to find, in this?                               |
| 6  | Linda: | The capacity of one container. So, one container would fill nine-         |
| 7  |        | fourths of that...of <i>a</i> (finger circles the nine shaded portions in |
| 8  |        | Drawing B).                                                               |
| 9  | MW:    | Why nine-fourths?                                                         |
| 10 | Linda: | Because that’s how much I filled each container with? (she reads          |
| 11 |        | the prompt aloud) “Describe the capacity of one whole container.”         |
| 12 |        | Four copies of one-fourth.                                                |
| 13 | MW:    | One-fourth of what?                                                       |
| 14 | Linda: | Of <i>a</i> . So four copies of one-fourth of <i>a</i> .                  |

---

In this conversation, the meaning for “*a*” toggled from the *given amount of water* to the *water in one container*. She did not reveal any indication that her work was perturbing to her. I think she was not bothered by her words and inscriptions because she could only imagine “*a*” as nine-fourths containers, so describing one container as “ $9/4$  of *a*” was the only fractional comparison that was available to her. But when she re-read the prompt, she changed her focus to one container and said “four copies of one-fourth of *a*,” but this is just “*a*” again. She seemed unsatisfied because either she knew “*a*” could not be the amount of water in one container *and* in all the containers, or because she sensed that I was bothered by her statements. She decided to use a specific value by choosing “*a*” to be 16 gallons. She calculated  $9/4 \times 16$  and obtained 36 gallons, which caused her to pause. I asked her to identify the 16 gallons in Drawing B, and she said that the 16

gallons constituted all the shaded portions in Drawing B. One explanation for the incorrect multiplier is that she did not have a quantitative image of what it means to multiply by nine-fourths. This is likely, since she was not able to model a fractional multiplier in Task 6. She then had an epiphany realizing that she was trying to find the amount in one container, which led her to realize that 36 gallons was unreasonable since she only had 16 gallons to begin with. She then thought each quarter container should be one-fourth of 16 gallons, giving 4 gallons per quarter container, which added again to a total of 36 gallons. She remained perturbed and asked, “But how can I end with more gallons than I started with?” The data suggests that she held steady in her thinking that 16 gallons was the amount in all the shaded portions in Drawing B, despite her statements that implied 16 gallons was the amount in one container. She could only think to find one-fourth of the 16 gallons to find the amount in one quarter-container – the fraction  $\frac{1}{4}$  was too engrained in her thinking and she could not summon the necessary  $\frac{1}{9}$ .

At this point, I considered the post-assessment on this task to be completed, and so I decided to intervene. I prompted her to shift her focus back to Drawing A and I asked her to use the symbol “?” to identify the capacity of one container. I thought it would be helpful for Linda to look at the vertically stacked containers because she could more easily compare a vertical stack to the strip exercises. I asked her to think of “ $a$ ” as the red strip and “?” as the green strip. This began a chain reaction of thinking for Linda that ultimately led her to confidently conclude that “ $\frac{4}{9}$  of  $a$ ” would yield the correct answer. We briefly discussed that *one-fourth of a container* was the same amount as *one-ninth of the given water*, similar to the ideas we had discussed in the strips tasks. She said, “It wasn't until you said the green and red strips. That made sense to me.” I believe the strip

activity was helpful because it allowed Linda to realize that the water in one container and the given amount of water could be perceived as two distinct quantities that could be the unit-of-measure.

### Task 20: 3 Gallons Fill $\frac{9}{4}$ Containers

In Phase 1, Linda drew the nine quarter containers used a guess-and-check process to find the amount in each quarter. She tried one half gallon, but this resulted in too much water. She then tried one quarter gallon, but this resulted in not enough water. She then tried one third of a gallon in each quarter container, which was just the right amount, and she concluded that  $\frac{4}{3}$  gallons were in each whole container, as depicted in Figure 57.

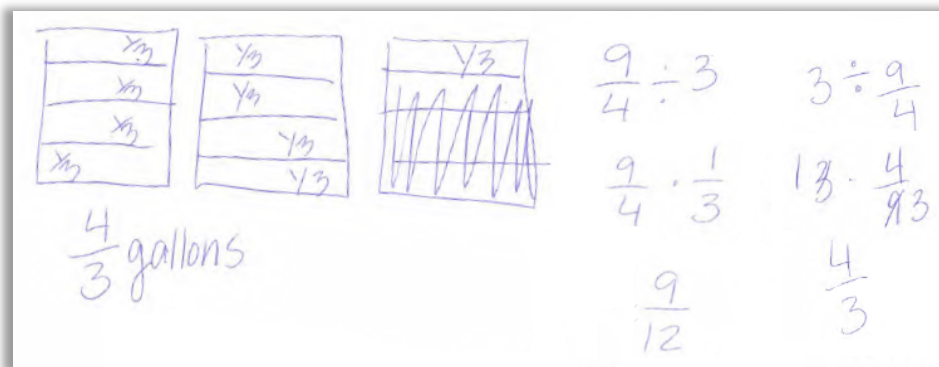


Figure 57. Linda's work on Task 20 in Phase 1.

Linda did not divide the amount of water by nine to get the amount in a quarter container. I prompted her to consider an operation that would answer the question and she suggested  $3 \div \frac{9}{4}$ . When I asked her to explain her choice she hesitated and then switched to  $\frac{9}{4} \div 3$ . She calculated both and then settled on her first operation, which she indicated was reasonable because she was “dividing the three gallons into nine-fourths containers.”

In Phase 3, Linda modeled this task well, due to my intervention during the previous task. She concluded that  $\frac{4}{3}$  gallons of water was the capacity of one whole container. Her work is illustrated in Figure 58.

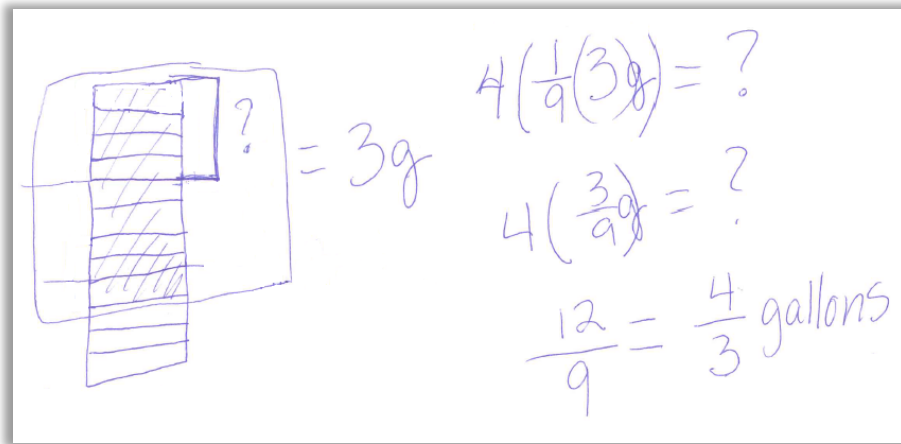


Figure 58. Linda's work on Task 20 in Phase 3.

I asked Linda to describe the meaning of the “ $\frac{3}{9}$ ” in the second line of her work and she said “ $\frac{3}{9}$  of a gallon.” But she then errantly suggested that this value corresponded to three out of the nine shaded portions in her drawing and said “I have three copies of one-ninth.” It is important to note that she did not mention a unit for the one-ninth, a common type of omission by her. As a result, I asked her to confirm whether four copies of what she just described would give a whole container as expected. She was perturbed when she iterated this amount only three times which reconstituted all the shaded portions in her drawing. These data seemed to suggest that she thought she should be reconstituting the given amount of water, and not the capacity of one container. This is an interesting task because the same amount of water can be measured with three relevant unit fractions; one-third of a gallon, one-fourth of a container, or one-ninth of the given amount of water. This is a coordination of three units, which may have been the source of her confusion. Also, her picture did not explicitly reveal what one gallon is, so she did not

have a frame of reference to think about one-third of something. From her drawing alone, she was stuck with either one-fourth or one-ninth. She gained confidence in her answer of “ $\frac{4}{3}$  of a gallon” by calculating two and one-fourth copies of four-thirds gallons to confirm that the total amount was three gallons. Thus, I returned to focusing on the meaning of the  $\frac{3}{9}$  in her middle step. After several moments of silent thinking, she said the  $\frac{3}{9}$  would be “just one of these (pointing at a single piece),” and wrote that one piece (a quarter container) is “ $\frac{1}{9}$  of 3 gallons.” I suggested to her that there were three ways to measure this amount, as if there were three different strips. She then correctly identified one piece (a quarter container) as one-third of a gallon, one-fourth of a container, and one-ninth of the given amount of water. She then laughed and said, “Why do they gotta make it so complicated?”

#### **Task 24: An Unknown Amount of Water Fills $\frac{2}{3}$ container**

In Phase 1, Linda demonstrated a conceptual understanding of what to do, to divide the amount of water by two and then combine three copies of this amount. However, her symbolic representations were not correct – she wrote the amount of water in each third of the container was  $x \div \frac{2}{3}$ . In Phase 3, Linda again had no trouble modeling this. She had a quantitative scheme, and her notation was accurate. I compare her work from both phases for this task in Figure 59.

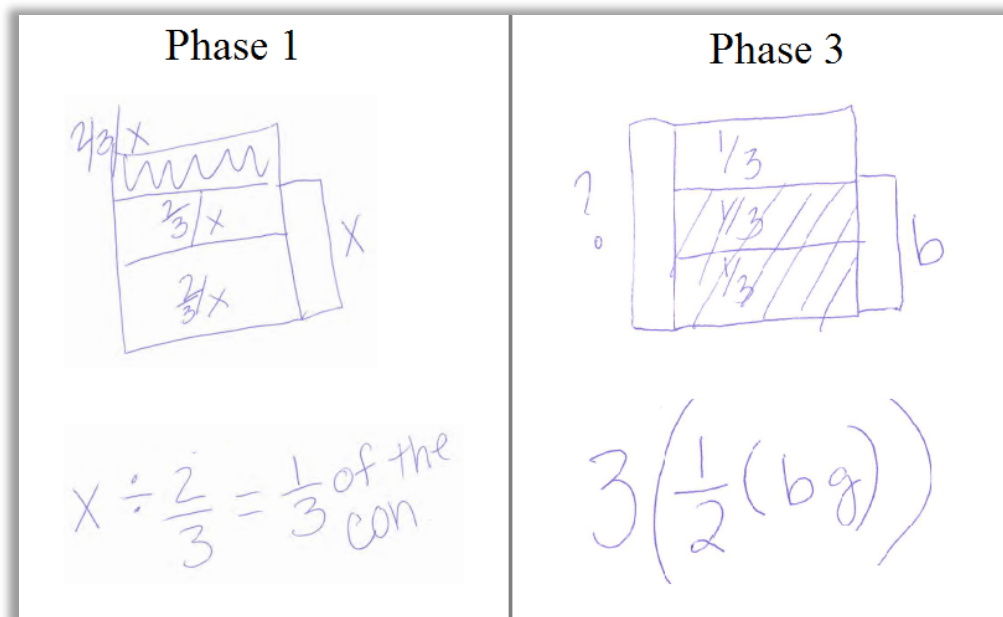


Figure 59. Linda's work on Task 24 in Phases 1 and 3.

### Task 23: 7/4 Gallons Fill 2/3 Container

In Phase 1, Linda had no trouble with this task. In Phase 3, she had no trouble with language, symbols, or conceptual operations on this task. She recognized that she should find one-half of the given amount of water, and then triple this amount. There was a brief moment of confusion for her when she labeled each third of a container as “1/2” and I asked her “one-half of what?” She said “container,” then changed to “gallon,” and then settled on “1/2 of 7/4 gallons.” This demonstrated a flexibility to re-unitize, even with a non-standard unit. I suggested that she could use the general language of “one-half of the given water.” I asked her what one-step calculation would give the answer instead, and she wrote  $3/2(7/4)$ . I asked for another operation that would give the result, to which she said “I guess division” and wrote  $7/4 \div 2/3$ . I asked her to identify which meaning for division was appropriate and she wrote “2/3 copies of what is 7/4 g.” She gave a good

quantitative explanation for this question, expressing it as a missing multiplicand statement “ $\frac{2}{3}(x) = \frac{7}{4}$ .” Her work on this task is highlighted in Figure 60.

Handwritten work by Linda for Task 23 in Phase 3. The work includes a diagram of three stacked containers, each labeled  $\frac{1}{2}$ , with a total of  $\frac{7}{4}g$ . To the right, there are several equations:  $3\left(\frac{1}{2}\left(\frac{7}{4}g\right)\right)$ ,  $3\left(\frac{7}{8}g\right) = \frac{21}{8}g$  could fill 1 container,  $\frac{3}{2}\left(\frac{7}{4}\right)$ ,  $\frac{7}{4} \div \frac{2}{3}$ , and  $\frac{2}{3}(x) = \frac{7}{4}$ .

Figure 60. Linda's work on Task 23 in Phase 3.

### Task 25: Explain the Invert-and-Multiply Algorithm

In Phase 1, Linda was unable to provide a meaningful justification for the algorithm, as discussed in Chapter 6 in response to RQ1.5. In summary, she began by unsuccessfully attempting to connect the variables in the general statement of the algorithm to the water-container quantities. However, since she could not generalize for known numbers of containers in the prior tasks, I did not expect her to generalize for an unknown number of containers in this task. When I asked her what division meant, she gave a quotitive explanation, at which point her efforts concluded.

In Phase 3, Linda spent much more time grappling with this task. She chose specific numbers ( $a = 10$ ,  $b = 20$ , and  $c = 4$ ), but said she was not thinking of a context, just numbers. She wrote, “20/4 copies of what is 10?” which is a partitive conceptualization. She rewrote this as  $5x = 10$ , and then solved by dividing by 5 to get 2. She, then showed that the multiplication also gave 2, while acknowledging that she did not demonstrate why this works. She said, “I proved it. Now I gotta justify it.” She then tried to rephrase “ $b/c$  copies of what is  $a$ ?” by writing “ $a$  copies of what is  $b/c$ ?” We then had the following dialogue.

*Excerpt 15*

---

|   |        |                                                                               |
|---|--------|-------------------------------------------------------------------------------|
| 1 | Linda: | Yeah, that works.                                                             |
| 2 | MW:    | What do you mean?                                                             |
| 3 | Linda: | So $a$ copies of what is $b/c$ , but it's not explaining the question. I just |
| 4 |        | rewrote it a different way, but it's not the way you have it (pointing        |
| 5 |        | at the algorithm in the prompt). (Pause) It's everything we've been           |
| 6 |        | doing but it's in the abstract (said with frustration)!                       |

---

Linda's comments suggest she thought the question, “ $a$  copies of what is  $b/c$ ?” was not appropriate for the symbolic division statement in the prompt. Linda then turned the page over, rewrote the algorithm from the prompt, and started over with a context. She wrote “ $a = 4$  containers” and “ $b/c = 3/4$  gallons,” and then wrote, “How many  $3/4$  gallons fill 4 containers?” She drew four containers, divided each in four pieces, and paused. She redrew four containers, but divided each in three pieces, and stopped again. She concluded, “I'm not asking the right question, that's why.” However, she did not attempt to ask a different question and she could not progress further.

At this point, I concluded that she would not be able to provide a justification of the algorithm. However, I decided to determine what she could do, if I guided her slightly. We had the following dialogue.

*Excerpt 16*

---

|   |        |                                                                   |
|---|--------|-------------------------------------------------------------------|
| 1 | MW:    | What is the result of doing containers divided by gallons?        |
| 2 | Linda: | You can't, because you're doing two different units. Don't I need |
| 3 |        | gallons to gallons?                                               |
| 4 | MW:    | I guess it sort of depends on which question for division you are |
| 5 |        | asking.                                                           |

---

I encouraged her to model  $a \div b$  with whole numbers and she chose  $a = 4$  containers and  $b = 5$  gallons. But then she then modeled  $5 \div 4$  instead of  $4 \div 5$ . I suggested to her that something was wrong and she recognized that  $5/4$  is not the result of  $4 \div 5$ . We discussed how *water* divided by *containers* gives *water-per-container*, and I told her to return to the algorithm, but to imagine the dividend as the amount of water and the divisor as the number of containers. She tried again and wrote  $a = 4$  gallons and  $b/c = 3/4$  containers, and made a model of the situation, as depicted in Figure 61.

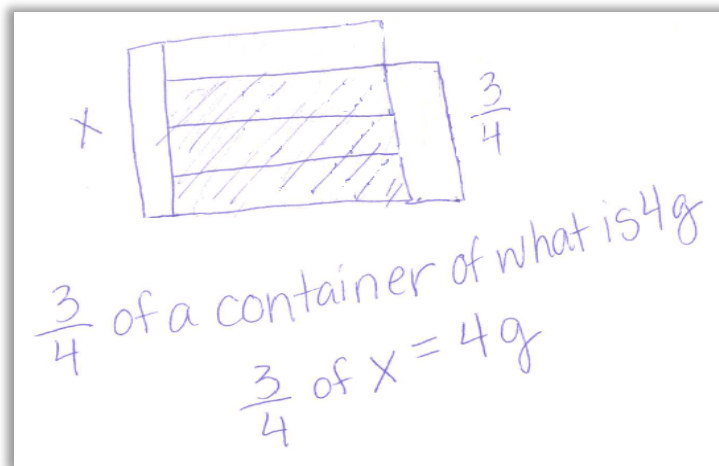


Figure 61. Linda's work on Task 25 in Phase 3.

She labeled each shaded piece as  $\frac{1}{3}$  and acknowledged that the whole container is four of those. I asked “ $\frac{1}{3}$  of what?” and she said “one-third of three-fourths of a container” and “one-third of four,” but she settled on the former. In the latter statement Linda did not put a unit on the four, so I could not determine if she was referring to gallons or pieces of the container. Since Linda chose an example where 4 is the number of pieces that make of the container *and* the number of gallons in three of those pieces, she set herself up for conflation of the two quantities, which became apparent as she continued through the task. She then said, “One-third of three-fourths of the container equals  $x$ ,” which was not consistent with her drawing. Adding to the confusion, she then wrote the statement  $\frac{1}{3} \left( \frac{3}{4} x \right) = 4$ . Her utterances and inscriptions continually contradicted each other. In one moment, she implied the  $x$  was the water in the whole container, and in another moment, she implied the  $x$  was the water in one-fourth of a container. Also, in one moment, she implied the *four* was the amount of water in three-fourths of the container, and in another moment, she implied the *four* was the amount of water in one-fourth of the container. This type of inconsistency happened for several more minutes as we had a dialogue about what she was trying to do, what is one-third of what, and what the *fours* are. Eventually, I asked her to switch her label of  $\frac{3}{4}$  (in Figure 61) to 4 gallons and we had the following dialogue.

*Excerpt 17*

---

|   |        |                                                                      |
|---|--------|----------------------------------------------------------------------|
| 1 | MW:    | What do you do to four gallons to reconstitute the entire container? |
| 2 | Linda: | I times it by one-third.                                             |
| 3 | MW:    | And that gives you the whole container?                              |
| 4 | Linda: | Yes, because I have four one-third copies (pointing at the four      |
| 5 |        | pieces of the container one at a time).                              |

- 6 MW: So, is this four (pointing at the label 4g next to three-fourths of the  
7 container) the same thing as the number of copies that are there?  
8 Linda: No, this is four gallons (pointing at the label 4g).  
9 MW: So what do you do to four gallons to reconstitute the whole  
10 container?  
11 Linda: I times it by three-fourths, no one-fourth because there's four  
12 copies of one-third.  
13 MW: One-third of what?  
14 Linda: Of three-fourths of the container.  
15 MW: Think in terms of gallons and not containers.
- 

Linda then wrote  $4\left(\frac{1}{3}\left(\frac{3}{4}\right)\right)$ , calculated the result 1, and said, "This doesn't work."

After some more moments of unproductive thinking, I asked her to think of the shaded portion as the green strip, the whole container as the red strip, what she would do the green strip to make the red strip. We then had the following dialogue.

*Excerpt 18*

- 
- 1 Linda: I break the green up into thirds and I have four of those one-thirds.  
2 MW: And how many gallons is the green strip?  
3 Linda: (repeats my question to herself twice) Would be four gallons.  
4 MW: Okay, so what do you do to four gallons to make the amount of  
5 gallons in the red strip?  
6 Linda: Times it by four-thirds!  
7 MW: Why does that make sense?  
8 Linda: Because I have four copies of one-third.  
9 MW: One-third of what?  
10 Linda: Of four gallons.  
11 MW: And how would you write that symbolically?
-

Linda then produced the image in Figure 62. She then said that had shown that the algorithm works because she noticed that Expressions A and B were the same. However, the *fours* in Expression A have the reverse quantitative meaning as the *fours* in Expression B. Linda did not catch this, and we discussed this issue together, and concluded the task.

Handwritten work showing two expressions:

Expression A:

$$4 \text{ copies of } \frac{1}{3}(4)$$

$$4\left(\frac{1}{3}(4)\right)$$

$$4\left(\frac{4}{3}\right)$$

Expression B:

$$4 \div \frac{3}{4} = 4 * \frac{4}{3}$$

Figure 62. Linda's concluding work on Task 25 in Phase 3.

## RQ2.2: Characterizing Linda's Advancements

Linda's data from Phase 3 revealed some changes in behavior as compared to Phase 1. In this section, I address RQ2.2, which is, "What advancements to the teachers' schemes are evident and what challenges remain?" I begin by focusing on Linda's three areas of weakness that I identified from the data in Phase 1: fractional multipliers, re-unitization, and meanings for division. I then discuss other obstacles that the data from Phase 3 revealed.

### Fractional Multipliers

Linda demonstrated some advancements in meanings for fractional multipliers, but not at a robust level. For example, consider Task 6 (explain your meanings for

$5/3 \times 2$ ). In Phase 1, Linda could not provide a meaning for  $5/3$  copies, and she could only make sense of the product imagining two as the multiplier. In Phase 3, Linda circumvented demonstrating a meaning for  $5/3$  as a multiplier through a multiplier-switch, by interpreting  $5/3$  copies of 2, as  $5/3$  copied twice. In other words, she assimilated a task with a fractional multiplier to a scheme for a whole-number multiplier. While working on Task 21 in Phase 3, Linda indicated that contracting an amount by a factor of  $1/9$ , and then expanding this new amount by a factor of 4, was the same thing as scaling the original amount by a factor of  $4/9$ . These data suggest that Linda had made an advancement since Phase 1, while the data from Task 6 in Phase 3 suggest that her improved schemes were not robust enough to facilitate assimilation of all fractional multiplier tasks. In both phases, Linda struggled with multiplicative tasks of the form:

$$(\text{known non-unit fractional amount}) \times (\text{known amount}) = (\text{unknown amount})$$

This structure (involving a non-unit fractional multiplier) is the same structure that caused her problems, but in the context of division. In other words, she struggled with quotitive models of the form:

$$(\text{unknown non-unit fractional amount}) \times (\text{known amount}) = (\text{known amount})$$

She also struggled with partitive models of the form:

$$(\text{known non-unit fractional amount}) \times (\text{unknown amount}) = (\text{known amount})$$

Otherwise stated, Linda struggled with every quotitive division task that yielded fractional quotients (e.g.,  $4 \div 3$  in Task 9), and she struggled with every partitive division task with fractional divisors (e.g.,  $3 \div 9/4$  in Task 20). However, she was successful at every quotitive task that yielded whole-valued quotients, and she was successful at every partitive task with whole-valued divisors. For Linda, the presence of fractional

multipliers was a perfect predictor of perturbations regarding division. Her weak meanings for fractional multipliers were suffocating her meanings for division. This suggests that robust schemes for fractional multipliers is crucial to the development of robust meanings for both partitive and quotitive division.

### **Multiple-Unit Coordination**

Linda demonstrated some advancements in re-unitizing, but there were several instances during Phase 3 where I observed persistent problems regarding multiple levels of units. In the next three paragraphs, I summarize the following: (1) the data from Phase 1 that supported my claim that Linda's challenges were caused by a lack of re-unitization, (2) the data from Phase 3 that provided evidence of advancement in her schemes, and (3) the data from Phase 3 that suggested that Linda continued to struggle with unitizing.

In Phase 1, several instances provided evidence that Linda struggled with re-unitizing. In Task 3 (comparing the red and green strips), Linda could not switch from the green strip as the unit-of-measure to the red strip. Once she imagined one-fourth of the green as *one-fourth of the green*, she could not re-imagine this length in terms of the red. This may be because she only folded the green and held a mental image that the contracted amount was green, and she did not recognize that relative length did not depend on the color of the contracted amount. In Task 9 (explain your meanings for  $4 \div 3$ ), Linda was trying to show that 1 and  $\frac{1}{3}$  is the result of answering, "How many copies of 3 are in 4?" She drew four squares and circled three of them. The data indicated that Linda thought of the circled copy of three squares as *one*, not *one group of three wholes*, so each square became one-third, and one-third only. In Task 13 (6 copies of what amount make 15?), it was difficult for Linda to look at one piece out of five

contiguous pieces and think something other than one-fifth, even though she was counting these *one-fifths* in pairs. In Task 19 (27 gallons fill  $9/4$  containers), Linda did not think to divide 27 by nine, perhaps because she could imagine one-fourth of a container as *one-fourth*, and not as *one-ninth*.

In Phase 3, the data suggested that Linda had made some advancements. In Task 3 (comparing the red and green strips), she described that one-third of the green strip could also be expressed as one-eighth of the red strip. However, during Task 21 (some amount of water fills  $9/4$  containers), Linda was not successful initially at seeing one-fourth of a container as one-ninth of the given amount of water. Yet, when I prompted her to think about the red and green strips, she was able to recognize both values for the one quantity, saying that thinking about the strips helped her to make sense of the solution. Similarly, in Task 25 (justify the invert-and-multiply algorithm), Linda credited the strips for helping her realize how to appropriately scale the water in three-fourths of a container to reconstitute the whole container. These data suggest that Linda did not reflectively abstract her nascent strip-comparison schemes to the point of facilitating assimilation for other kinds of analogous tasks. During Tasks 21 and 25, she still required minimal guidance from me to help trigger her newly formed schemes.

The data in Phase 3 also revealed that Linda continued to struggle with re-unitizing. In Task 8 (explain your meanings for  $a \div b$ ), Linda was trying to partitively model  $10 \div 5$  when she drew five separate boxes, each cut in half. She was perturbed because she labeled each semi-box as one-half, yet she knew that each whole box should represent two. In Task 9 (explain your meanings for  $4 \div 3$ ), Linda had the same re-unitizing issue she demonstrated in Phase 1, in that she could not recognize a piece of a

block as both one and as one-third. In Task 20, one amount of water could be measured with three relevant unit fractions; one-third of a gallon, one-fourth of a container, or one-ninth of the given amount of water. Linda could imagine the one-fourth and the one-ninth, but was perturbed by the one-third. This is because her picture did not explicitly reveal a representation of one gallon, so her picture supported only imagining one-fourth or one-ninth.

As the data suggest, the teaching experiment did not adequately help Linda develop robust schemes related to re-unitizing. Linda had difficulty in cases where she had to coordinate two values of one quantity, where neither value was one, such as  $\frac{1}{4}$  of the green strip is  $\frac{1}{7}$  of the red strip. She did not have trouble when one of the values was 1, such as 1 group is the same as  $\frac{10}{3}$  of something. In these latter cases, re-unitizing amounts to a counting exercise, and Linda had no difficulty when the count was a whole number, such as discovering that 12 is the number of copies of  $\frac{1}{3}$  that make 4. These data suggest that counting a whole number of things is not at the same level of cognition as consciously coordinating multiple units.

Linda's difficulties with re-unitizing were mitigated by using more specific contexts. For example, recall that while working on Task 9 in Phase 3, Linda struggled with quotitively modeling  $4 \div 3$ . I suggested that three peanut butter cups came in a package. When she had concrete things to imagine, *and* the words to describe them, she did not struggle to give accurate measurements when I toggled between packages and peanut butter cups as the units-of-measure. Additionally, it was beneficial to Linda to compare novel tasks to the red and green strip activities. The strips also provided concrete things to imagine, and the words to describe them.

## Dual Meanings for Division

Due to the teaching experiment, Linda gained an awareness that there are two meanings for division, and she acquired reasonable language to describe them. Consider Task 8 (explain your meanings for  $a \div b$ ). In Phase 1, Linda described division only with a quotitive conceptualization. However, in Phase 3, Linda expressed both meanings for division by writing “how many  $b$  copies are in  $a$ ?” and “ $b$  copies of what =  $a$ ?” This was also true for Task 9 (explain your meanings for  $4 \div 3$ ) and Task 10 (explain your meanings for  $4 \div 1/3$ ). However, Linda’s awareness, and her abilities with language, did not ensure success at producing meaningful models for each question, as evidenced in Tasks 9 and 10 from Phase 3, where Linda was not successful at quotitively modeling  $4 \div 3$ , nor at partitively modeling  $4 \div 1/3$ . Additionally, she experienced quotitive-modeling interference during Task 8 in Phase 3, and partitive-to-quotitive assimilation in Task 10 of Phase 3. However, she recognized the partitive-to-quotitive assimilation and attempted, although unsuccessfully, to return to a partitive conceptualization. The fact that she caught herself using the wrong conceptualization is an advancement because it would not have been possible for her to do this in Phase 1 with an awareness of only one meaning for division. Recall that I consider a person to have *partitive-quotitive awareness* if that person (1) is aware of the two quantitative conceptualizations for division, and (2) can operate under one conceptualization without interference from the other, while maintaining an awareness of the conceptualization with which they are operating. Linda’s *quotitive-modeling interference* and the momentary *partitive-to-quotitive assimilation* demonstrated that she was not able to operate under one conceptualization without interference from the other. In summary, the data from Phase 3

revealed that Linda had advanced significantly regarding condition 1, but that she still did not meet condition 2. Consequently, I do not characterize her as having acquired partitive-quotitive awareness, despite her advancements.

### **Other Cognitive Obstacles**

My analysis of the data revealed another cognitive obstacle related to meanings for division that my study did not focus on. This obstacle concerns the realization that the amount in one group when some amount  $a$  is split (divided, partitioned, distributed) into  $b$  groups, is the same amount that is the answer to the question, “ $b$  copies of what amount make  $a$ ?” In other words, does a person realize that typical partitive language related to “fair-sharing” is connected to the characterization of partitive division as a missing multiplicand task? The data in my study suggest that such a realization is non-trivial. Linda demonstrated a weakness in being able to identify a division task as a missing multiplicand task on several occasions, even when reasoning about a whole number of groups. In Task 18 (some amount of water fills 5 containers) from Phase 3, recall that Linda identified that  $a \div 5$  represented the amount of water in one container, but she could not summon the correct language to characterize this as a missing multiplicand task.

Another cognitive obstacle that Linda demonstrated is related to the quantitative structure of a division statement. For Task 20 (3 gallons fill  $\frac{9}{4}$  containers) in Phase 1, after Linda used a guess-and-check strategy to determine that there were  $\frac{4}{3}$  gallons per container, she did not know whether  $3 \div \frac{9}{4}$  or  $\frac{9}{4} \div 3$  was the correct operation. She procedurally calculated both expressions as a basis for her decision. For Task 18 (some amount of water fills 5 containers) in Phase 3, Linda could not initially decide between  $a \div 5$  and  $5 \div a$ . For Task 25 (justify the invert-and-multiply algorithm) in Phase 3, Linda

was trying to connect the statement  $a \div b/c$  to the water-container tasks. She wrote “ $a = 4$  containers” and “ $b/c = 3/4$  gallons,” and then wrote, “How many  $3/4$  gallons fill 4 containers?” Later in the task she attempted to model  $4 \div 5$  by distributing 5 gallons among 4 containers. These data seem to suggest that she was unaware of the significance of the quantities for the dividend and divisor.

Another cognitive obstacle for Linda regarded with her language. Often, when she said (wrote) a fraction, she did not say (write) a referent. I constantly had to ask, “of what?” This issue was endemic throughout all phases. For example, in Phase 3, it occurred – as reported in the data in this chapter – during Tasks 3, 9, 16, 21, 23, and 25. Linda also used vague language when describing quantities, which made it difficult at times for me to model her thinking. For example, for Task 10 (explain your meanings for  $4 \div 1/3$ ) in Phase 1, Linda said, “How many  $1/3$  copies make 4?” which is not as clear as, “How many copies of  $1/3$  make 4?” For Task 16 (13 gallons fill 5 containers) in Phase 3, Linda said, “Five copies of each container make a total of 13 gallons.” This language could mean, “Five copies of *one* container make a total of 13 gallons,” or “Five copies of *all the containers* make a total of 13 gallons.” For Task 25 (justify the invert-and-multiply algorithm) in Phase 1, Linda said “part of it is,” but she was not clear about the “it” in her sentence. She also said, “add the amount of parts,” which can be referring to the quantity, *number of parts*, or to the quantity, *amounts of stuff contained in those parts*. As an example of another language issue, Linda sometimes used the phrase “divide into,” which is a *chameleon* expression that can be used to describe two different quantitative operations. For example, does “two divided into six” mean  $2 \div 6$  partitively or  $6 \div 2$  quotitively? Alternatively, Linda may have been using the phrase to mean a

numerical operation, devoid of quantitative meaning. These are just a few examples of the many instances where her communication was vague. Clark, Moore, and Carlson (2008) discussed the importance of clear mathematical communication, and they said it perfectly when they said, *speak with meaning*.

## CHAPTER 9

### CONCLUSION

The findings from this study testify of the difficulties that professional development providers face when helping practitioners acquire productive mathematical meanings. This study shed light on some of the cognitive obstacles that hinder mathematical development, but there is still much to learn. In this chapter, I share the following:

- (1) Summary of the background of this study
- (2) Summary of my findings
- (3) Emergence of a framework for characterizing robust meanings for division
- (4) Limitations of the study
- (5) Implications for curriculum and instruction
- (6) Directions for future research

#### **Summary of the Background of this Study**

Researchers have described two fundamental conceptualizations for division, known as *partitive* and *quotitive* division. Several researchers have also identified many cognitive obstacles that have inhibited the development of robust meanings for division involving non-whole values, while other researchers have commented on the challenges related to such development. Regarding division with fractions, much research has been devoted to quotitive conceptualizations of division, or on symbolic manipulation of variables. Research and curricular activities have largely avoided the study and development of partitive conceptualizations involving fractions, as well as their connection to the invert-and-multiply algorithm.

I designed this dissertation study to focus on the *advancement* of teachers' partitive meanings to enable them to productively assimilate fractional divisors. This study was motivated for the following reasons: (1) partitive division over the positive rational numbers is detrimentally underemphasized in contemporary curriculum and standards for teaching mathematics, (2) partitive meanings form a conceptual foundation for other mathematical meanings, such as rates and proportional correspondence, and for the invert-and-multiply algorithm, and (3) limited research is available on this topic. I investigated six middle school mathematics teachers' meanings related to partitive conceptualizations of division over the positive rational numbers. I also investigated the impact of a teaching experiment that I designed with the intent of advancing one of these teachers' meanings.

### **Summary of Findings**

I designed the methodology and analysis of the data in an effort to answer two primary research questions, which encompassed eight secondary research questions. In this section, I summarize my findings with regards to these research questions. I analyzed the data from Phases 1 and 2 to answer my first primary research question: *What meanings, with their affordances and limitations, do in-service middle school mathematics teachers possess relative to partitive conceptualizations of division with non-whole divisors?* The data from Phase 1 corresponded to secondary research questions RQ1.1-RQ1.5 and the data from Phase 2 corresponded to secondary research question RQ1.6.

### **RQ1.1: Fractions as Reciprocal Measures of Relative Size**

I analyzed Phase 1 data to answer RQ1.1: *What meanings do teachers reveal when they engage in tasks that I designed to elicit meanings for fractions as measures of relative size, with a focus on fractions as reciprocal measures of relative size?* I used *reversible scheme* to refer to a dual-scheme that constitutes schemata for enacting and reversing a process, such that the reverse scheme is instantly accessible as a companion to the forward scheme. I borrowed from Thompson and Saldanha (2003) by characterizing the scheme for *fractions as reciprocal measures of relative size* as a reversible scheme specific to comparing the magnitudes of two quantities – in both directions – through means of imagining reciprocal fractions. The data from this study revealed that imagining the reciprocal of one comparison to determine the reverse comparison was not self-evident for any of the teachers, which suggested that none of the teachers possessed the scheme for fractions as reciprocal measures of relative size. The data revealed that for some teachers, the use of mixed numbers was a severe hindrance to imagining the reciprocal fraction. Ultimately, five of the six teachers were ultimately able to compare the red and green strips to each other in Task 3 of Phase 1 imagining reciprocal fractions, but only two teachers were able to provide a quantitative justification.

### **RQ1.2: Decontextualized Multiplication**

I analyzed Phase 1 data to answer RQ1.2: *What meanings do teachers reveal when they describe and model symbolic (decontextualized) statements of multiplication, both general and specific, with a focus on fractional multipliers?* No teacher experienced difficulty modeling multiplication with a whole-valued multiplier. However, for Task 6

$(5/3 \times 2)$  in Phase 1, one teacher had no meaning for  $5/3$  as the multiplier, and the remaining five teachers thought of  $5/3$  as a mixed number in order to model the product. None of the teachers contracted 2 by a factor of  $1/3$ , followed by expanding this amount by a factor of 5. This was significant to me because the mixed number obscured the elegance of the dual-scheme of contraction followed by expansion. As such, the mixed number can be a cognitive obstacle to reversing the effects of a fractional multiplier by imagining the reciprocal fraction. During this same task, one teacher demonstrated a ***multiplier-switch***, which I defined to be the phenomenon when a person imagines one factor as the multiplier, but then – during the process of modeling – the person instead views the model (with or without awareness) as a justification for the other factor as the multiplier. In cases where the multiplicand is a whole number, a *multiplier-switch* can be an impediment to formulating a productive meaning for a fractional multiplier because it provides a sensible way to circumvent the formulation of such a meaning.

The data also revealed the teachers' difficulties with fractional multipliers caused by inattention to re-unitization. During Task 6, one teacher experienced repeated perturbations due to conflating  $1/3$  of 2 with  $2/6$ . Such issues were widespread throughout the study – all teachers encountered perturbations at some point because of issues related to re-unitization. The data also suggested that the lack of a specific context contributed significantly to the teachers' inability to maintain multiple levels of units.

### **RQ1.3: Decontextualized Division**

I analyzed Phase 1 data to answer RQ1.3: *What meanings do teachers reveal when they describe and model symbolic (decontextualized) statements of division, both general and specific, with a focus on fractional divisors?* In this study, I observed that

some teachers could operate with both conceptualizations, yet they did not seem to realize that their ways of operating were inconsistent. These teachers became perturbed when I asked questions that alluded to their inconsistency. Another teacher was aware of the dual meanings, but conflated the meanings while trying to model division tasks, which led to cognitive dissonance. Other teachers did not seem to have any awareness of multiple conceptualizations for division. Concerning the dual meanings for division, no teacher demonstrated *partitive-quotitive awareness*, which is when a person possesses the following two characteristics: (1) awareness of the two quantitative conceptualizations for division, and (2) ability to operate under one conceptualization without interference from the other, while maintaining an awareness of the conceptualization with which they are operating. Concerning the second condition, I identified two types of *partitive-quotitive interference*. The first type was *partitive-to-quotitive assimilation*, which I defined to occur when a person assimilates partitively-framed division tasks to quotitive schemata. Four of the teachers exhibited these *mis-assimilations* at some point during Phase 1, and it occurred when the division tasks were abstract and when the divisors were not whole. For Task 11 ( $1/2 \div 3/4$ ) in Phase 1, three of the teachers created models for multiplication instead of division, a phenomenon reported by other researchers (e.g., Jansen & Hohensee, 2016; Simon, 1993).

#### **RQ1.4: Partitive Scenarios**

I analyzed Phase 1 data to answer RQ1.4: *What meanings do teachers reveal when they engage in tasks that I designed to elicit partitive conceptualizations of division, with varying degrees of abstraction, and with a focus on fractional divisors?* The data from the tasks corresponding to RQ1.4 revealed more instances of *partitive-to-quotitive*

*assimilation*, which was the first type of *partitive-quotitive interference*. Additionally, I identified a second type, which I called ***quotitive-modeling interference***, to refer to any difficulties in modeling partitive division that are caused by inappropriately applying meanings or strategies that are typically associated with quotitive division models. The data suggested that three teachers were impaired because of quotitive-modeling interference. This cognitive obstacle manifest itself in three different ways: (1) deliberately creating a quotitive model to cope with confusion during an attempt to model partitively, (2) confusion due to beginning a partitive model with a representation of the dividend (commonly done for quotitive modeling), and (3) modeling the dividend and divisor using a common unit (required for quotitive models, but errant for partitive models). During the water-container tasks, no teacher demonstrated partitive-quotitive interference, due to the specificity of the contexts. The primary cognitive obstacles for these tasks were related to mixed-number interference, re-unitization, and fractions as reciprocal measures of relative size.

### **RQ1.5: Invert-and-Multiply Algorithm**

I analyzed Phase 1 data to answer RQ1.5: *What justifications do the teachers provide for the invert-and-multiply algorithm after working through the (water-container) tasks?* Leading up to Task 25 (explain the algorithm), all six teachers had the opportunity to generalize with two different fractional divisors in the water-container tasks; Task 21 (some amount of water fills  $\frac{9}{4}$  containers) and Task 24 (some amount of water fills  $\frac{2}{3}$  container). The data revealed that successful generalization in Tasks 21 and 24 did not guarantee a meaningful explanation for Task 25. Mindi, Mel, Uma, and Ursa were able to generalize both Task 21 and 24, but only Mindi and Mel were able to

use the same water-container context to generalize even further to explain Task 25. Uma was also technically successful although she believed she was not because she unfortunately said that  $a \times c / b$  implied that multiplication by  $c$  should precede division by  $b$ , which was not the order she wanted. Ursa did not connect Task 25 to Tasks 21 and 24 and she tried unsuccessfully to resolve Task 25 using a quotitive conceptualization. Mark was not successful at generalizing in either water-container task, and consequently he could not generalize for Task 25. Linda generalized successfully only for Task 24, but she unsuccessfully attempted a quotitive explanation for Task 25. In summary, two teachers used the water-container tasks (partitive conceptualizations) to justify the algorithm, two teachers unsuccessfully attempted a quotitive explanation, and two teachers unsuccessfully attempted a partitive explanation. Additionally, three of the unsuccessful teachers exhibited *partitive-quotitive interference* during their attempts at Task 25.

#### **RQ1.6: Additional Cognitive Obstacles**

I analyzed Phase 2 data to answer RQ1.6: *What cognitive obstacles do teachers further reveal as I actively attempt to promote the development of their meanings that are foundational to partitive division over the rational numbers?* My findings reported in this dissertation only refer to Linda's schemes. Phase 1 revealed that she had weak meanings regarding re-unitization, that she could not operate with fractional multipliers, and that did possess partitive-quotitive awareness. As I engaged her in activities that I designed to strengthen these schemes, I was able to discern additional cognitive obstacles for Linda. When she tried to model products with fractional multipliers and whole-valued multiplicands, she repeatedly experienced *multiplier-switches*. This happened even in

cases where she was deliberately trying to avoid it. Linda also revealed issues with not only re-unitizing, but with *unitizing*. By this I mean that she could not conjure appropriate units for the fractions she was trying to operate with, which suggested a disengagement from quantitative reasoning. Also, given one value and its unit, she could not conjure a new unit for a second value, which was detrimental to her success at a partitive model for  $3 \div 1/4$ . Furthermore, she struggled to give a quantitative description of a general multiplicative statement, using language to describe the product that was effectively the same as the language she used to describe the multiplier. Consequently, she struggled with providing consistent quantitative meanings for division, including confusion about the correct ordering of the dividend and divisor. Additionally, Linda revealed weakness with language. She was vague when she described quantities and operations, and she nearly always omitted units when saying and writing values. She also struggled with interpreting the correct direction for a multiplicative comparison from written text. Lastly, she revealed an inability to retain information and ways of thinking, which raised questions about what is required to establish permanence for schemes.

I now move on to summarize my findings for RQ2.1 and RQ2.2, which corresponded to Phase 3 of my study. These findings addressed my second primary research question: *How do these teachers' meanings change as a consequence of an instructional sequence that emphasized quantitative reasoning to aid in the advancement of these meanings?*

### **RQ2.1: Comparative Analysis of Phases 1 and 3**

I analyzed Phase 3 data to answer RQ2.1: *How do the teachers' post-intervention meanings compare to their pre-intervention meanings?* I compared Linda's work in

Phase 3 to her work on the same tasks in Phase 1. Behaviorally, there were several differences as show in Table 15, which indicated changes to the underlying schemes. The most obvious changes related to fractions as reciprocal measures of relative size and partitive-quotitive awareness. I elaborate on these changes in response to my final secondary research question, summarized in the following subsection.

### **RQ2.2: Characterizing Linda's Advancements**

I analyzed Phase 3 data to answer RQ2.2: *What advancements to the teachers' schemes are evident and what challenges remain?* I focused my analysis on Linda's schemes for fractional multipliers, re-unitization, and meanings for division. In Phase 1, Linda said she did not know how to think about a fractional multiplier, but in Phase 3 she described partitioning then iterating, which was an advancement. However, she exhibited a multiplier-switch while modeling  $5/3 \times 2$ , while believing that she had accurately modeled the fractional multiplier. My analysis of Linda's data from all phases revealed that the number type of the multiplier was a perfect predictor for whether she could meaningfully operate using multiplication and both meanings of division. She was always successful at producing a model when the multiplier was a whole number, and she always struggled at producing a model when the multiplier was not a whole number.

Regarding re-unitization, Linda demonstrated difficulties during Phase 1. The data from Phase 3 revealed some advancements, but some persistent struggles as well. In Phase 1, she struggled recognizing one magnitude as two different values depending on the unit-of-measure. Once she had a unit in mind, she was virtually unable to transition to another unit. In Phase 3, she was able to re-unitize for the strip-comparison task by recognizing that  $1/3$  of the green strip was the same length as  $1/8$  of the red strip.

However, her nascent reciprocal-comparison scheme was not reflectively abstracted to the point of applying to analogous situations in the water-container tasks. For example, she could not recognize on her own that  $\frac{1}{4}$  of a container was  $\frac{1}{9}$  of the given amount of water, which prevented her from being able to reason productively through the task.

Concerning division, the data revealed that Linda had advanced significantly regarding the first condition of *partitive-quotitive awareness*. In Phase 1, she could only operate under a quotitive conceptualization, but in Phase 3, she expressed both meanings for division by writing “how many  $b$  copies are in  $a$ ?” and “ $b$  copies of what =  $a$ ?” However, Linda’s awareness, and her abilities with language, did not ensure success at producing meaningful models for each question, as evidenced in Tasks 9 and 10 from Phase 3, where Linda was not successful at quotitively modeling  $4 \div 3$ , nor at partitively modeling  $4 \div \frac{1}{3}$ . Also, during Phase 3, there were instances of *quotitive-modeling interference*, and *partitive-to-quotitive assimilation*. Thus, Linda’s schemes were not advanced enough to satisfy the second condition of partitive-quotitive awareness.

### **Emergence of a Framework for Robust Meanings for Division**

Prior to this study, I considered a person to have robust meanings for division if that person

- 1) Operated meaningfully with both partitive and quotitive conceptualizations over the positive rational numbers.
- 2) Possessed an awareness of the distinctions between partitive and quotitive conceptualizations regardless of number type.
- 3) Recognized and/or invented situations that elicited both partitive and quotitive conceptualizations involving any kind of positive rational values.

As a consequence of my findings in this study, I have created a more extensive framework for characterizing robust meanings for division, which is shown in Figure 63.

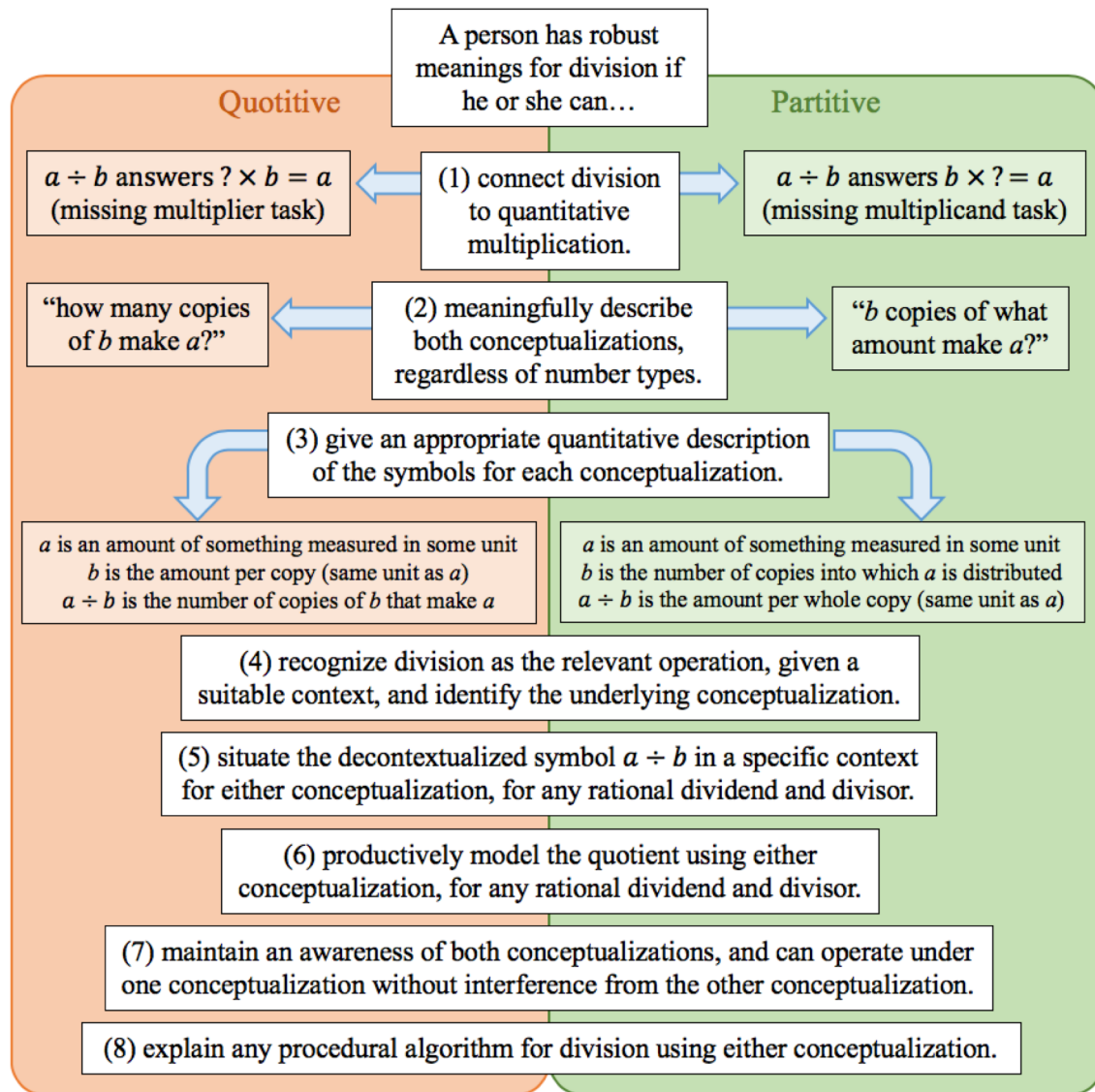


Figure 63. Framework for robust meanings for division.

Even though this framework focuses primarily on division, the development of other mathematical meanings (e.g., multiplication, relative size, re-unitization) is essential for a person to have all the abilities I described above. For example, in this study, several teachers could not productively think about fractions as reciprocal measures of relative size. This issue is related to issues with re-unitizing, which plagued

several teachers throughout the entire study. Also, teachers with weak or no meanings for fractional multipliers have little chance at developing partitive meanings with fractional divisors. This framework is founded on four underlying principles, which I discuss in the paragraphs that follow.

*Principle 1:* The operation of division must be a quantitative operation, not just a numerical operation. If the operation is not already connected to some sort of context, and is presented symbolically, then a person should be able to conjure a quantity-based scenario that would elicit the relevant operation. The quantities in this scenario can be abstract (e.g., the number of groups) or specific (e.g., the number of cookie packages). The more specific the quantities, the less likely that a person will conflate units. For example, instead of thinking about a group of cookies, a person should think about a package of cookies. In this example, a *package* provides the thinker with something more concrete to imagine, as well as the language to describe the re-unitized collection of cookies. As another example, to add specificity to a scenario, I used colored strips in my tasks and activities to allow easy communication about which strip was the object of my focus. The data from this study revealed that unitizing issues tended to occur when teachers imagined quantities that were vague or abstract.

*Principle 2:* Division must be appropriately connected to a non-commutative, quantitative model for multiplication. Numerical commutativity for multiplication was a hindrance to my efforts in this study to help teachers build meaningful operations with fractional values. For example, it allowed some teachers to circumvent the meaning of a fractional multiplier, which is crippling to the development of division schemes. It is not adequate for a person to think that division only solves a missing factor problem, without

regard to which factor. That person must construe division as either solving a missing multiplicand problem, or as solving a missing multiplier problem, and recognize that these objectives are not the same. Without distinct meanings for the factors in multiplication, a dual-meaning for division is not possible.

*Principle 3:* Robust meanings for division require an elevated awareness of the quantitative distinctions between the two conceptualizations, which is what I call *partitive-quotitive awareness*. The data revealed that a lack of partitive-quotitive awareness was a leading source of the teachers' dissonance and invalid reasoning. As such, robust meanings for division are dependent on reflectively abstracted schemes to the point of possessing a categorization of the two conceptualizations. A person should be able to operate with one conceptualization without interference from schemes related to the other conceptualization, while maintaining a realization of the conceptualization with which they are operating.

*Principle 4:* Meaningful division must be accompanied by meaningful language. A person with weak meanings is more likely to use unclear, inconsistent language. And conversely, unclear language may cause mis-assimilations. For example, partitive language such as "split four into three groups" is very clear because the number of groups is a whole number. However, during this study, I witnessed partitive-to-quotitive assimilation by some teachers when I repeated this partitive language, but with a fractional divisor. For example, some teachers interpreted "split four into one-third groups" as "split four into groups of size one-third." This demonstrated that language, that was meaningful with one type of number for a divisor, caused perturbations and mis-assimilations when that same language was used with a different type of number for the

divisor. A person should also have meaningful language to describe the two conceptualizations themselves. They do not necessarily need to use the words *quotitive* and *partitive*, but they should be able to say something that is sensible for any rational dividend and divisor. As such, I recommend avoiding common labels, such as *fair-sharing* and *repeated-subtraction*, which are only sensible in certain situations. Since I consider it important to maintain a connection to non-commutative multiplication, I recommend characterizing the two conceptualizations as “how-many-copies-division” and “how-much-in-each-copy-division.”

### **Limitations of the Study**

There were several aspects of this study that I would improve were I to repeat the study. There were several instances where I strayed from my protocol and neglected to ask certain questions. Some examples of this include me forgetting to ask teachers to provide quotitive models for tasks they had only attempted with a partitive conceptualization, and forgetting to ask some teachers to connect contexts with symbolic operations. In particular, I would have liked to know what division operation the teachers connected with the water-container tasks (especially for  $\frac{2}{3}$  container) so that I could better ascertain whether the teachers connected these tasks to the invert-and-multiply algorithm. For Task 3 (comparing the red and green strips) in Phase 1, I asked for both comparisons at the same time, but I should have asked for each comparison separately to determine how the teachers interpreted the prompts. Concerning the teaching experiment, I wonder what would have resulted if I had given the teachers take-home activities, so that they had extra opportunities to reflect on their newly formed ways of thinking. Or, I could have planned for more time with the teachers during the teaching experiment to let

them engage in repeated reasoning in hopes of establishing permanence for their recently developed schemes. Also, the data was replete with examples of *procedural contamination*, which was unavoidable working with in-service practitioners. Procedural contamination casts a faint shadow on the quality of the data in this study, and in any similar studies with practitioners. I say this because the use of procedures to guide (or to replace) quantitative reasoning offered the teachers an alternative approach to resolve tasks, which impacted (or circumvented) their efforts at quantitative reasoning.

### **Implications for Curriculum and Instruction**

Robust quantitative reasoning is essential to meaningful mathematics. The findings of this study affirm the importance of grounding mathematics in quantity-based contexts. I support contemporary efforts, such as the CCSSM (National Governors Association Center, 2010) and NCTM (2014), to focus mathematics curriculum on meaning-making. If possible, all mathematical ideas should be introduced, discussed, and fleshed out by reasoning about specific quantities that are relatively easy to describe and imagine.

Concerning re-unitization, as soon as possible in the cognitive development of the students, educators should frequently implement activities that require multiple units-of-measure. For example, educators could engage their students in activities such as the following.

Activity 1: Describe the size of different collections of squares of a 3-by-4 array in units of *row*, *column*, and *array*.

Activity 2: Describe the length of a strip of paper in units of several differently colored strips of paper with different lengths. These colored unit-strips could also be combined to form new units of measure.

Activity 3: Describe different amounts of packages and cupcakes, in units of *cupcake*, *package*, and *crate of packages*.

With regards to multiplication, educators should introduce children to multiplication only in contexts with specific quantities. When a context suggests multiplication, reversing the order of multiplication is not a sensible action. Thus, the notion of numerical commutativity of multiplication should be delayed until after robust quantitative meanings for the multiplier and multiplicand are well established. Furthermore, a convention should be adopted to establish a consistent position for the multiplier.

To build robust meanings for fractional multipliers, curriculum should focus on contraction (partitioning) followed by expansion (iterating) of *unit* quantities. For example, an educator could require students to find fractional multiples of one strip of paper with an emphasize on length, of one piece of paper with an emphasis on area, or of one block of clay with an emphasis on weight or volume. Subsequent activities could include the values of these quantities as measured in some other unit. For example, an educator would first require students to find  $\frac{5}{3}$  of a strip of paper, then reveal the length of the paper as nine inches, and then ask for the length of the scaled paper in inches. Additionally, curriculum should emphasize immediately reversing the effect of scaling an amount by a fractional multiplier. Repetition with appropriate activities, such as the red- and green-strip activities, can be conducive to the development of fractions as reciprocal

measures of relative size. When appropriate and often, these reciprocal-comparison activities should be revisited in other contexts, such as comparisons of volumes, areas, or counts of people.

With regards to division, I believe there is much room for improvement in K-12 education, post-secondary pre-service programs, and professional development efforts. I acknowledge that division may be most accessible to a young child in terms of *fair-sharing* or *repeated-subtraction*. However, at the appropriate moment in the cognitive development of the students, educators should only connect division to multiplication by characterizing division as either a *missing-multiplier* or a *missing-multiplicand* task. It would be advisable to connect these new characterizations of division to the extant notions of *fair-sharing* and *repeated-subtraction*, but I recommend that educators then permanently abandon these early notions. Furthermore, I contend that partitive and quotitive division should be equally emphasized by educators, even for fractional divisors. Educators could use the water-container activities discussed in this paper, using actual containers as manipulatives. In such contexts, partitive and quotitive meanings for division could be developed for whole- and non-whole-valued divisors. I feel strongly that implementing such changes over the course of a student's K-12 education would yield powerful results.

### **Directions for Future Research**

The framework for robust meanings of division that I presented in the previous section provides several opportunities for future research. Some such research has already been conducted by Jansen and Hohensee (2016), who used the construct *connected conception* to discuss whether a person could connect symbolic statements with partitive

contexts, and connect partitive division with finding a unit rate. This is related to conditions 3 through 5 (specific to partitive conceptualizations) in the framework I presented above. Jansen and Hohensee also used the construct *flexible conception* to discuss whether a person could appropriately partition and/or iterate a dividend to produce accurate quotients, which is related to condition 6 in the framework. Other research opportunities include investigating a person's ability to connect notions of fair-sharing, or notions of finding a unit rate, with missing-multiplicand tasks. The data in this dissertation study revealed that it was not trivial for some teachers to connect the thought of *splitting  $a$  into  $b$  groups* with the thought  *$b$  copies of what amount make  $a$* . Additional research is required to investigate these types of connections (which relate to conditions 1 and 2 in the framework), and especially for cases of division involving fractional divisors. This dissertation study revealed several findings regarding partitive-quotitive awareness (condition 7), but more research is required to investigate overcoming partitive-quotitive interference. For example, Ursa was aware of the two meanings, but she exhibited many instances of partitive-quotitive interference. However, Ursa did not participate in the teaching experiment and so I was unable to investigate attempts to elevate her partitive-quotitive awareness. As another research opportunity, during this dissertation study I lightly investigated how water-container tasks contributed to the teachers forming a meaningful, partitive foundation for the invert-and-multiply algorithm (condition 8). However, connecting the water-container tasks to this algorithm was not a primary objective of the teaching experiment. Thus, additional research is required to more fully investigate partitive conceptions and the invert-and-multiply algorithm. Furthermore, I did not research, nor have I found any research that has specifically

investigated how partitive conceptualizations can form a meaningful foundation for the long-division algorithm.

Other questions surfaced during this study for which I did not have adequate data to address. During Phase 1, no teacher in this study revealed an existing scheme for fractions as reciprocal measures of relative size. Concerning the development of such a scheme, Thompson and Saldanha (2003) said, “The fact that this understanding happens so rarely among U.S. students makes it quite hard to research its development. But the fact that these understandings of fractions exist so rarely is a significant problem for U.S. mathematics education (p.33).” During the teaching experiment, I used the red- and green-strip activities to help Linda develop such a reversible scheme. During Phase 3, she was successful with the strip-comparison task, but her newly formed schemes were not abstracted enough to allow assimilation for analogous tasks that did not involve strips of paper. As such, more research is required to investigate how to facilitate the abstraction of schemes for fractions as reciprocal measures of relative size.

The data in this study also revealed that newly formed schemes were often ephemeral for Linda. Frequently, she would practice some way of thinking, but then forget or conflate it later during the teaching experiment. She noticed this about herself when she said, “I started breaking through last time. Why can’t it come back?” This raises questions about what it takes to establish permanence of schemes, for which more research is required, especially concerning schemes related to partitive meanings for division with fractional divisors.

Linda was successful with every multiplication and division task that involved whole-valued multipliers, but she struggled with every multiplication and division task

that involved non-whole-valued multipliers. For her, the type of number for the multiplier was a perfect predictor of success for both multiplication and division. More research is required to determine if these findings extend to the general population, and to further investigate cognitive advancement in this regard.

During this study I attributed some of the teachers' difficulties creating partitive models of division to *quotitive-modeling interference*. However, this was conjecture, as supported by the data. More research is required to establish whether quotitive modeling schemes truly interfere with attempts at partitive modeling.

Finally, I would be interested in conducting components of the teaching experiment that I described in this paper, with young students, who have not yet been exposed to the procedures for operating with fractions. This would eradicate the concerns I have expressed concerning procedural contamination. Such a study would certainly yield interesting and illuminating findings.

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## APPENDIX A

### COMMON CORE STANDARDS REGARDING DIVISION

Below I list the standards regarding meanings for division as found on the CCSSM placements for grades K-8 (2010).

**3.OA.A.2**

Interpret whole number quotients of whole numbers (e.g., interpret  $56 \div 8$  as the number of objects in each group when 56 objects are partitioned equally into 8 groups, or as a number of groups when 56 objects are partitioned into equal groups of 8 objects each).

**3.OA.A.3**

Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities.

**3.OA.B.6**

Understand division as an unknown-factor problem (e.g., find  $32 \div 8$  by finding the number that makes 32 when multiplied by 8).

**5.NF.B. 7**

Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. (Note: Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division.)

a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. For example, create a story context for  $\frac{1}{3} \div 4$ , and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that  $\frac{1}{3} \div 4 = \frac{1}{12}$  because  $\frac{1}{12} \times 4 = \frac{1}{3}$ .

b. Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for  $4 \div \frac{1}{5}$ , and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that  $4 \div \frac{1}{5} = 20$  because  $20 \times \frac{1}{5} = 4$ .

c. Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. For example, how much chocolate will each person get if 3 people share  $\frac{1}{2}$  pound of chocolate equally? How many  $\frac{1}{3}$ -cup servings are in 2 cups of raisins?

#### **6.NS.A.1**

Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for  $\frac{2}{3} \div \frac{3}{4}$  and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that  $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$  because  $\frac{3}{4}$  of  $\frac{8}{9}$  is  $\frac{2}{3}$ . In general,  $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ . How much chocolate will each person get if 3 people share  $\frac{1}{2}$  pound of chocolate equally? How many  $\frac{3}{4}$  cup servings are in  $\frac{2}{3}$  of a cup of yogurt? How wide is a rectangular strip of land with length  $\frac{3}{4}$  mile and area  $\frac{1}{2}$  square mile?

APPENDIX B

PHASE 1 TASKS

**Task 1:** Explain your meanings for the expression:  $\frac{a}{b}$

**Task 2:** Do the following and justify your answers.

This line has a length that is  $\frac{8}{5}$  of a unit.  $\rightarrow$  

Draw a line here that is 1 unit long.  $\rightarrow$

**Task 3:** Given these two strips of paper, how long is the red strip compared to the green strip? How long is the green strip compared to the red strip?

7 inches long

4 inches long

**Task 4:** Explain your meanings for the expression:  $a \times b$

**Task 5:** Explain your meanings for the expression:  $5 \times \frac{4}{3}$

**Task 6:** Explain your meanings for the expression:  $\frac{5}{3} \times 2$

**Task 7:** Explain your meanings for the expression:  $\frac{2}{5} \times \frac{4}{3}$

**Task 8:** Explain your meanings for the expression:  $a \div b$

**Task 9:** Explain your meanings for the expression:  $4 \div 3$

**Task 10:** Explain your meanings for the expression:  $4 \div \frac{1}{3}$

**Task 11:** Explain your meanings for the expression:  $\frac{1}{2} \div \frac{3}{4}$

**Task 12:**  $10/3$  copies of what amount combine to make the amount 15?

**Task 13:** 6 copies of what amount combine to make the amount 15?

**Task 14:** How much is in one group if 15 of something is split into  $10/3$  groups?

**Task 15:** How much is in one group if 15 of something is split into 6 groups?

**Task 16:** Suppose 13 gallons of water fill 5 equal containers. Describe the capacity of one whole container.

**Task 17:** Suppose  $3/4$  gallon of water fills 5 identical containers. Describe the capacity of one whole container.

**Task 18:** Suppose a certain amount of water fills 5 identical containers. Describe the capacity of one whole container.

**Task 19:** Suppose 27 gallons of water fill  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 20:** Suppose 3 gallons of water fill  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 21:** Suppose a certain amount of water fills  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 22:** Suppose 5 gallons of water fill  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

**Task 23:** Suppose  $\frac{7}{4}$  gallons of water fill  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

**Task 24:** Suppose a certain amount of water fills  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

**Task 25:** Explain why it is that when you divide by a fraction, you can multiply by the reciprocal of the fraction instead. In other words, explain the following:

$$a \div \frac{b}{c} = a \times \frac{c}{b}$$

## APPENDIX C

### PHASE 2 ACTIVITIES FOR LINDA

**Activity 1:** Given the pink strip of paper, draw a strip of paper that is  $a$  times as long.

(For  $a = 3, \frac{1}{5}, \frac{3}{5}, \frac{7}{5}, 2\frac{1}{3}, 1.7$ )

**Activity 2:** Describe your meanings for the expressions:  $4 \times 5, 4 \times \frac{5}{3}, \frac{4}{3} \times 5$

**Activity 3:** The green strip is how many times as long as the red strip? The red strip is how many times as long as the green strip?

7 inches long

4 inches long

**Activity 4:** Describe your meanings for the expressions:  $a \times b, \frac{m}{n} \times b$

**Activity 5:** The yellow strip is how many times as long as the blue strip? The blue strip is how many times as long as the yellow strip?

9 inches long

2 inches long

**Activity 6:** Describe your meanings for the expressions:  $a \div 4, a \div \frac{1}{4}$

**Activity 7:** How much water is in one container when 20 gallons fill 4 identical containers? How many containers are needed to hold 20 gallons of water if each container can hold 4 gallons?

**Activity 8:** Suppose some amount of water fills 5 containers. Describe how much water would fill a whole container.

**Activity 9:** Suppose some amount of water fills the blue-rimmed (1.75) containers. Describe how much of the given amount of water would fill a whole container?

**Activity 10:** Suppose some given amount of water fills  $2\frac{2}{3}$  containers. Describe how much of the given amount of water would fill a whole container.

**Activity 11:** Suppose some given amount of water fills  $\frac{3}{5}$  containers. Describe how much of the given amount of water would fill a whole container.

**Activity 12:** Describe your meanings for the expressions:  $a \div \frac{9}{5}, a \div \frac{3}{7}, a \div \frac{1}{3}$

APPENDIX D

PHASE 3 TASKS

The numbering below is the numbering from Phase 1. In Phase 3, I presented the tasks in the same order as they are presented below.

**Task 3:** Given these two strips of paper, how long is the red strip compared to the green strip? How long is the green strip compared to the red strip?



**Task 6:** Explain your meanings for the expression:  $\frac{5}{3} \times 2$

**Task 8:** Explain your meanings for the expression:  $a \div b$

**Task 9:** Explain your meanings for the expression:  $4 \div 3$

**Task 10:** Explain your meanings for the expression:  $4 \div \frac{1}{3}$

**Task 18:** Suppose a certain amount of water fills 5 identical containers. Describe the capacity of one whole container.

**Task 16:** Suppose 13 gallons of water fill 5 equal containers. Describe the capacity of one whole container.

**Task 21:** Suppose a certain amount of water fills  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 20:** Suppose 3 gallons of water fill  $\frac{9}{4}$  identical containers. Describe the capacity of one whole container.

**Task 24:** Suppose a certain amount of water fills  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

**Task 23:** Suppose  $\frac{7}{4}$  gallons of water fill  $\frac{2}{3}$  of a container. Describe the capacity of one whole container.

**Task 25:** Explain why it is that when you divide by a fraction, you can multiply by the reciprocal of the fraction instead. In other words, explain the following:

$$a \div \frac{b}{c} = a \times \frac{c}{b}$$

APPENDIX E

LETTER OF CONSENT

The following is one example of the letter of consent I used for each meeting with each teacher.

## **Meeting 1**

### **AMP Research Team**

*Thank you for participating in this interview. I am collecting data as part of a research project for the Arizona Mathematics Partnership (AMP) to better understand how individuals reason mathematically. I will use the data for research purposes only.*

*As we progress through the interview, I ask that you verbalize all your thoughts so that I can gather information about your thinking on various tasks. Additionally, I ask that you express your thoughts on paper by showing all your work and by drawing pictures which convey what you may be visualizing.*

*The interview is expected to last no more than 90 minutes and you will be compensated at the rate of \$25 per hour. If at any time you wish to stop the interview, please let me know. I want you to be as comfortable as possible.*

|                                                  |               |
|--------------------------------------------------|---------------|
| <b>Teacher's Name:</b>                           |               |
| <b>Teacher's Signature:<br/>(giving consent)</b> |               |
| <b>Interviewer:</b>                              | Matthew Weber |
| <b>Interview Date:</b>                           |               |