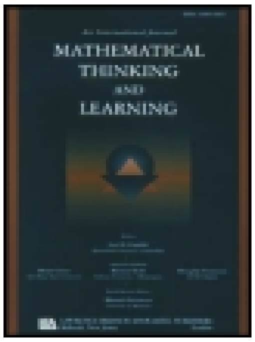

Mathematical Thinking and Learning

ISSN: 1098-6065 (Print) 1532-7833 (Online) Journal homepage: http://www.tandfonline.com/loi/hmtl20

Perspectives on Advanced Mathematical Thinking

Annie Selden & John Selden

To cite this article: Annie Selden & John Selden (2005) Perspectives on Advanced Mathematical Thinking, Mathematical Thinking and Learning, 7:1, 1-13, DOI: 10.1207/s15327833mtl0701_1

To link to this article: https://doi.org/10.1207/s15327833mtl0701_1

Published online: 18 Nov 2009.

MATHEMATICAL THINKING AND LEARNING, *7*(1), 1–13

Perspectives on Advanced Mathematical Thinking

Annie Selden and John Selden
Department of Mathematical Sciences
New Mexico State University

This article sets the stage for the following 3 articles. It opens with a brief history of attempts to characterize *advanced mathematical thinking,* beginning with the deliberations of the Advanced Mathematical Thinking Working Group of the International Group for the Psychology of Mathematics Education. It then locates the articles within 4 recurring themes: (a) the distinction between identifying kinds of thinking that might be regarded as advanced at any grade level, and taking as advanced any thinking about mathematical topics considered advanced; (b) the utility of characterizing such thinking for integrating the entire curriculum; (c) general tests, or criteria, for identifying advanced mathematical thinking; and (d) an emphasis on advancing mathematical practices. Finally, it points out some commonalities and differences among the 3 following articles.

This introduction and the following three articles discuss several diverse views of advanced mathematical thinking (AMT). We first provide a brief overview of the landscape, without developing any particular aspect in detail, and then locate these three articles within four recurring themes found in the growing literature. All three articles discuss ways of thinking about or doing mathematics that the authors consider beneficial for students. Sometimes referred to as "mathematical habits of mind" or "mathematical practices," these ways of thinking about and doing mathematics may be fairly widely regarded as productive, but are often left to the implicit curriculum. That is, they are usually not taught explicitly, and in current school curricula, may not be considered by teachers as part of their responsibility. Indeed, some teachers may not see such habits of mind as capable of being taught. Perhaps that is one reason the recent RAND Mathematics Study Panel advocated the teach-

Requests for reprints should be sent to Annie Selden, Dept. of Mathematical Sciences, New Mexico State University, Box 30001 Dept 3MB, Las Cruces, NM 88003–0001. E-mail: aselden@emmy.nmsu.edu

ing and learning of *mathematical practices* as one of three large focal areas for future research in mathematics education (Ball, 2003). We conclude with a brief comparison of the three articles.

We hope that by laying out some recurring themes in the discussions of AMT that further progress in designing and implementing curricula will be encouraged, and that by introducing aspects of advanced mathematical thinking or its precursors earlier, the K–16, or even the K–graduate school curricula can become better integrated. In this, we are encouraged by the most recent *NCTM Principles and Standards for School Mathematics* (2000) that recommend Problem Solving, Reasoning and Proof, Communication, Connections, and Representation Standards for all instructional programs from prekindergarten through Grade 12. These process standards include the recommendation that all instructional programs should enable students to: (a) apply and adapt a variety of appropriate strategies to solve problems; (b) select and use various types of reasoning and methods of proof; (c) analyze and evaluate the mathematical thinking and strategies of others; (d) understand how mathematical ideas interconnect; and (e) select, apply, and translate among mathematical representations to solve problems. Parts of the discussions of AMT in this issue can be interpreted as if they were an attempt to come to a better understanding of what these five NCTM recommendations mean, and perhaps, also to supplement them.

A BRIEF HISTORY

From initial considerations thereof, the term *advanced mathematical thinking* has been fraught with ambiguity—does the term *advanced* refer to the mathematics, or to the thinking, or to both? Clearly, more advanced topics in the curriculum, such as calculus or differential equations, cannot be grasped without a solid understanding of more elementary topics, such as function and rate of change. These, in turn, depend on an understanding of proportion and number. Also, the thinking lies on a continuum—processes such as analyzing, conjecturing, defining, formalizing, proving, generalizing, and synthesizing, although more frequent in advanced mathematics, can and should develop from elementary grades onward (See Dreyfus, 1990).

Research into cognitive, and other aspects, of mathematical thinking and learning began with elementary topics, such as the acquisition of early number concepts. Indeed, research on more elementary concepts was predominant in the work of the International Group for the Psychology of Mathematics Education (PME) from its beginning in 1976 until the mid-1980s (Dreyfus, 1990). Then, in 1985, a PME Working Group on Advanced Mathematical Thinking was formed, and it continued meeting until the late-1990s. For practical purposes, when initial discussions failed to coalesce around a single satisfactory definition, this Working Group focused its efforts on the teaching and learning of mathematics at the tertiary level.[1]

Three products resulted from the deliberations of this PME Working Group: (a) a chapter on advanced mathematical thinking authored by Tommy Dreyfus in an International Commission on Mathematical Instruction Study Series volume featuring the work of PME (Nesher & Kilpatrick, 1990); (b) a volume that considered the nature of advanced mathematical thinking, cognitive theory, and overviews of research into the teaching and learning of such advanced topics as limits, differential equations, infinity, and proof (Tall, 1991); and (c) a special issue of *Educational Studies in Mathematics* devoted to advanced mathematical thinking (Dreyfus, 1995).

In the second of these, Tall (1991, p. 20) asserted that "The move from elementary to AMT involves a significant transition: that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on definitions." Expanding on this in a subsequent PME plenary address, Tall (1995) stated that cognitive growth from elementary to AMT can be hypothesized as starting "from 'perception of' and 'action on' objects in the external world, building through two parallel developments—one visuo-spatial to verbal-deductive, the other successive process-to-object encapsulations using manipulable symbols" (p. 63) leading eventually "from the equilibrium of visual conviction and proceptual manipulation to defined objects and formal deduction…. The full range of creative advanced mathematical thinking is mainly the province of professional mathematicians and their students" (p. 71).

Somewhat later, beginning in 1998, related issues were taken up by a Working Group of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME–NA) titled, The Role of Advanced Mathematical Thinking in Mathematics Education Reform. This PME–NA Working Group began by discussing such questions as what kinds of earlier experiences might help students make the transition to the kinds of AMT that postsecondary students are often asked to engage in (Heid, Ferrini-Mundy, Graham, & Harel, 1998). This rather naturally metamorphosed into efforts at characterizing AMT and looking for seeds thereof that are, or could be, planted early in students' mathematical careers. For example, the tendency to interpret a concept in multiple ways can be useful for problem solving at various levels (Heid et al., 1999). Three perspectives emerged:

1. AMT deals with the kind of thinking that occurs mainly at the collegiate or graduate levels and requires precise reasoning about ideas that are not entirely accessible to the five senses. This view is expanded upon in the Edwards, Dubinsky, and McDonald article (this issue).

[1]This is our own observation, based on participating in all but the first few meetings of the Working Group.

2. A consideration of AMT as involving the overcoming of epistemological obstacles, together with ways of thinking that are helpful in this. This view has been extended to the current Harel and Sowder article (this issue).

3. A preference for focusing on "advancing mathematical activity" centered around expanded definitions of horizontal and vertical mathematizing (Treffers, 1987) as exemplified by the mathematical practices of symbolizing, algorithmatizing, and defining. This view has been expanded into the current Rasmussen, Zandieh, King, and Teppo article (this issue; see Heid, Harel, Ferrini-Mundy, & Graham, 2000).

Over the past 10 or so years, "advanced mathematical thinking" has come to be a descriptor, or key word, that authors can, and often do, use to describe their research for journals such as the *Journal for Research in Mathematics Education (JRME)*. As with all such descriptors, it is left to authors to determine which key words describe their research, and thus, the phrase has come to mean "whatever the author chooses it to mean" (E. Silver, past editor of *JRME*, personal communication, April 8, 2003). The main effect of this, perhaps resulting from authors taking their cue from the bulk of the research reported in Tall's (1991) seminal volume, seems to have been that the term is often used to signal mathematics education research at the tertiary level. Although Thompson (1993) titled his review of the Tall (1991) volume, *Yes, Virginia, Some Children Do Grow Up to Be Mathematicians*, he also noted that Tall had emphasized that "advanced mathematical thinking does not begin after high school" and that "this thinking must begin in the first grade." Pimm (1995), in his rather critical review of the Tall (1991) volume, observed that the adjective "advanced" had been applied, by the book's chapter authors, variously to describe both the mathematics and the thinking. Pimm, noting that one can always advance beyond the state one is currently in, questioned whether there is even such an entity as AMT.

That AMT should have something to do with the nature of advanced mathematics, toward which one can view elementary mathematics as aiming, as well as something to do with the practices of mathematicians, does not seem in doubt. However, exactly what features might characterize that mathematics and that thinking continues to be deliberated. Furthermore, exactly how one might foster such thinking in a seamless way so that the seeds, or precursors, of that thinking are planted and nurtured from early on is still an open question—a question addressed in this issue.

Although the *NCTM Principles and Standards for School Mathematics* (2000) advocate that the process standards (problem solving, reasoning and proof, communication, connections, and representation) be integrated across instructional programs from prekindergarten through Grade 12—in effect, that students' mathematical thinking should become progressively more advanced—accomplishing this is no easy task. However, there are some indications of how early exposure to challenging mathematical ideas can provide experiences upon which to draw for subsequent, sometimes much later, mathematical generalizations and abstractions (cf. Maher & Martino, 1996a, 1996b, 1997; Maher & Speiser, 1997).

What special kinds of thinking, by undergraduates, graduate students, and mathematicians are especially associated with advanced mathematics? Indeed, what kinds of mathematics might reasonably be regarded as advanced within the kindergarten through graduate school curricula?

FOUR RECURRING THEMES

Our review of the literature, together with the three articles herein, suggests four general themes regarding the nature of, and possible characterizations of, advanced mathematical thinking.

Advanced Thinking Versus Advanced Mathematics

The first theme concerns the distinction between, on the one hand, directly identifying the kinds of mathematical thinking that could be regarded as advanced at any age or grade level, and on the other hand, of taking as advanced the kinds of thinking characteristic of mathematical topics that could themselves be identified as advanced. For example, Edwards et al. (this issue) discuss this distinction in the introduction to their contribution. In practice, however, these two distinct points of view often support each other. That is, to argue that some kind of thinking, say the habits of mind and abilities associated with handling abstraction, is advanced, one can note that such thinking often occurs in thinking about advanced topics, such as abstract algebra. Conversely, if one needed to argue that abstract algebra was advanced, one might note that much of the thinking involved is indeed abstract.

The Utility of Characterizing Advanced Mathematical Thinking

A second theme concerns the utility of identifying or characterizing various kinds of AMT. Once a kind of thinking, such as generalizing, has been identified as advanced, it seems more likely that some form of it, or at least a precursor to it, could be analyzed and taught earlier in the kindergarten through graduate school curriculum. This would allow the curriculum to be better integrated by providing long-range goals for instruction that go beyond satisfying the mathematical needs of everyday life in a technological society. This theme appears to have played a considerable role in the development of the current *NCTM Principles and Standards for School Mathematics* (2000). There is also an “existence proof” of how a series of relatively small, but coherent, long-term interventions with one group of students over a number of years can lead to remarkable instances of AMT, including the development by students, on their own, of the idea of proof (Maher & Martino, 1996a, 1996b).

Consider the case of Stephanie, one of a number of children with whom Maher and Martino (1996a, 1996b, 1997) began their long-range, but occasional, interventions commencing in Grade 1. By Grade 3, the children had begun building physical models and justifying their solutions to the following problem: How many different towers of heights 3, 4, or 5 can be made using red and yellow blocks? Stephanie, not only justified her solutions, she validated or rejected

> her own ideas and the ideas of others on the basis of whether or not they made sense to her. … She recorded her tower arrangements first by drawing pictures of towers and placing a single letter on each cube to represent its color, and then by inventing a notation of letters to represent the color cubes. (Maher & Speiser, 1997, p. 174)

She used spontaneous heuristics like guess and check, looking for patterns, and thinking of a simpler problem, and developed arguments to support proposed parts of solutions, and extensions thereof, to build more complete solutions. Occasional interventions continued for Stephanie through Grade 7. Then in Grade 8 she moved to another community and another school and her mathematics was a conventional algebra course. The researchers interviewed her that year about the coefficients of $(a + b)^2$ and $(a + b)^3$. About the latter, she said "So there's *a* cubed … And there's three *a* squared *b* and there's three *ab* squared and there's *b* cubed. … Isn't that the same thing?" Asked what she meant, she replied, "As the towers." It turned out, upon further questioning, that Stephanie had been visualizing red and yellow towers of height 3 to organize the products $a^i b^j$. (For a more complete discussion, see Maher & Speiser, 1997.) Stephanie then used the towers of blocks metaphor to develop the coefficients of expressions such as $(a + b + c)^n$, a remarkable achievement that prompted Speiser, upon presenting his paper at PME–21 in Lahti, Finland, to remark, "I wish some of my [university] students were able to reason that well."

Criteria for Advancedness

A third theme focuses on finding what could be called very general tests, or criteria, for identifying AMT that might be considered at many grade levels throughout the curriculum. Harel and Sowder (this issue) suggest such a very general test. Their central idea is that AMT involves, to some degree, at least one of the three characteristics of epistemological obstacles. An epistemological obstacle is a bit of knowledge, rather than a lack of knowledge, that somehow inherently stands in the way of acquiring subsequent, more general, knowledge. It is sufficiently robust to withstand occasional contradictions. Finally, such an obstacle should turn up in the historical development of mathematics. Harel and Sowder consider the influence of an epistemological obstacle to be a matter of degree and state that its influence is often blended with that of one or more didactic obstacles, that is, obstacles somehow arising from the teaching itself rather than from the nature of what is to be learned.

Edwards et al. (this issue) also take this tack, proposing a different, although not necessarily conflicting test. They develop the idea that thinking might be regarded as advanced if it depends on deductive reasoning and does not depend directly upon sensory perception. While they acknowledge that "exemplary mathematical thinking may occur at any age of student and level of mathematics," they wish to reserve the term "advanced mathematical thinking" for thinking that involves "rigorous and deductive reasoning about mathematical objects that are unavailable to our five senses." Although this kind of advancedness can occur at a number of levels and we have previously mentioned the remarkable degree to which young students can engage in deductive reasoning, it seems unlikely that such students would often reason about mathematical objects unavailable to the five senses. Indeed, we see students as slowly progressing from first tacitly viewing the objects of mathematics as part of, or closely associated with, the physical world, and consequently as possessing descriptive definitions. Eventually some students come to view mathematical objects as abstract and brought into existence by analytic definitions,[2] such as that of group in abstract algebra. To illustrate how mathematical objects might be seen at the beginning of such a progression, consider a mathematically naive individual observing 3 red apples. That there are 3 and that they are *red* would have, more or less, the same status, namely, that of properties of physical objects. Somewhat later, the number 3 might come to be seen as an object in its own right, but one that is part of, or associated with, the physical world. Such objects have descriptive definitions that should correctly mirror the corresponding physical objects or situations. These initial perspectives of mathematical objects, and their corresponding descriptive definitions, are quite different from perspectives taken much later when using

[2]Although analytic definitions can be inspired by physical situations, they are often considered as ultimately reducible to undefined terms; hence, one cannot regard them as right or wrong. By contrast, although descriptive, or synthetic, definitions describe existing objects or situations often associated with the physical world, such as democracy or whole number addition, they *can* be regarded as right or wrong. Moreover, when using an analytic definition, it is essential to attend to all parts of the definition. However, by contrast, although synthetic definitions often provide incomplete and redundant descriptions, it not always essential to attend to *all* parts of such definitions. Difficulties arising from undergraduate students treating mathematical definitions as descriptive, rather than analytic, have been described by Edwards and Ward (2004).

analytic definitions that, although they may be inspired by physical observations, are more or less reducible to undefined terms.[3]

The utility of the Edwards et al., idea of advanced mathematical thinking as "unavailable to the five senses," and the idea of a progression of students' views regarding the status of mathematical objects, can be illustrated by examining a suggestion offered at a recent seminar. We were discussing various systematic errors that some fourth- and fifth-grade pupils' make when comparing the size of decimal numbers (Resnick et al., 1989). The offered suggestion was: Why not simply avoid all of the cited problems by defining, and directly using, $x < y$ if and only if $x - y$ is negative? However, it seems unlikely that one could clear up pupils' difficulties stemming from their previous whole number and fraction knowledge by simply defining away the problem. As noted by Resnick et al. (1989), fourth- and fifth-grade pupils making such errors are likely to see decimals, and their comparisons, as extensions of whole numbers or fractions studied earlier. Thus, they are likely to see decimals as numbers derived from the physical world. For such pupils, whole numbers, fractions, and even decimals may be "out there" in the physical world, and as such, only capable of being described. Thus, for these pupils, it seems inappropriate to attempt to redefine the concept of "less than" by simply introducing this abstract, analytic definition. Trying to do so might result in making little change in how the pupils actually make decimal comparisons.

Advancing Mathematical Practices

Finally, a fourth theme consists of elucidating specific kinds of mathematical practices, together with describing the development of the associated thinking. One very general practice, that occurs across the K–16 curriculum and in the work of mathematicians, might be called *structuring* real world and mathematical problems. Mathematicians such as Hadamard (1945) and Thurston (1994) have tried, often through introspection, to understand their own, and other mathematicians' creative structuring processes. By structuring, we mean the introduction of notation, diagrams, definitions, analyses, etc., to a class of problems to facilitate their solution—in some cases by converting them to other kinds of problems. This may involve the creation of structures, for example, symbols new to the practitioners.

---

[3]One could ask: What is the utility of having students move from viewing the objects of mathematics as descriptively defined aspects of the physical world, to viewing them as analytically defined abstract objects? One answer is: In constructing and understanding proofs, it is essential to attend to *all* parts of a definition—something unnecessary for other kinds of arguments based on descriptive definitions of physical objects or situations. Another answer is: Mathematics today is exceptionally reliable—barring the later discovery of errors, when a theorem is proved, "it stays proved." This reliability, and indeed independence from place and time, depends not only on careful logic, but also on the use of analytic definitions. The kind of unreliability that can result from treating mathematical definitions as descriptive can be seen in *Proofs and Refutations* (Lakatos, 1976).

An example of structuring is provided by the mathematizing, in particular, the symbolizing, elucidated in the article by Rasmussen et al. (this issue). Starting with either a real world or a mathematical problem, mathematizing is a process of consideration and reflection that alternates between two forms—*horizontal*, during which the mathematics at hand becomes more familiar and is broadened, and *vertical*, during which new mathematics (notation, algorithms, definitions, etc.) is created. Such mathematizing can be seen especially well in the kind of inquiry-oriented classrooms, in which students routinely come to explain and justify their thinking, from which Rasmussen et al. (this issue) draw their examples. Although their article describes the mathematical activity of undergraduates, the authors propose that the notion of mathematizing that they develop is "not limited to grade or content levels."

Another more commonly taught kind of structuring is modeling that may not involve the creation of new mathematical objects. In the traditional view of modeling, a student typically starts with a problem in a familiar setting (real world or mathematical) and adds (or focuses on some aspect of) structure, such as variables, diagrams, equations, functions, etc., with a view to converting the given problem into a more tractable one in a (or another) familiar mathematical domain. For example, tertiary students might be asked to show that, in Euclidean geometry, angles cannot be trisected (using only straightedge and compass) first by converting the problem to one about fields, and then to one about Galois groups in the hope that the new problem might be easier to solve. Students are usually not asked to "see" for themselves the relationship between two such disparate domains as Euclidean geometry and abstract algebra. If students *are* asked to "see" for themselves such relationships, then even problems in applying first calculus (without explanations of how to solve them, or even whether to use calculus) can become nonroutine modeling problems.

That such "seeing" is difficult at any level can be observed in the work of Lobato and Siebert (2002) in the case of Terry, a student who had recently completed Algebra 1 in Grade 8. In a summer teaching experiment, Terry and other students were asked what measurements they would take to determine the steepness of a wheelchair ramp (without explicitly asking them to consider slope). Terry initially focused on the height of the ramp and considered its length to be a dependent variable. As it turned out, Terry's reasoning evolved through quite a number of stages (described in detail by the authors), and it took instructor-facilitation to get him to see steepness as a function of the two independently varying quantities, height and length. Perhaps in Terry's defense and certainly as a caution to others who might attempt a similar teaching experiment, the authors note that "there are five other ratios that also provide mathematically consistent, albeit unconventional, measures of steepness (namely slant height to length, slant height to height, length to height, height to slant height, and length to slant height)" making it less likely that "a student would naturally focus on the particular ratio of height to length" (p. 111).

In elucidating specific kinds of mathematical practices and the creative ways students might arrive at them, we suggest that examinations of structuring, and in particular, the examples of symbolizing, algorithmatizing, and defining in Rasmussen et al. (this issue), do not nearly exhaust the possibilities. Much of the mathematics education research literature at the undergraduate level, although not directly about the nature of advanced thinking, or couched in terms of mathematical practices, nevertheless suggests promising areas for investigation. For example, one of our articles (Selden & Selden, 2003) is about mid-level undergraduates' ability to check the correctness of proofs and the ways they go about it, but does not give a detailed delineation of the kind of practices or thinking involved. However, it does note that this complex kind of thinking, that we called *validation*, is widespread among mathematicians and suggests it may eventually be fairly well described. The empirical part of the study suggests that current mid-level U.S. university students are not good at validating proofs, but our experience suggests many graduate students and mathematicians can do so quite reliably. Thus, the article implicitly suggests that students who continue in mathematics need to improve the practice of validating proofs. While there is currently little direct instruction in this practice, students who learn to validate proofs on their own might be considered as engaged in a form of mathematizing.

Much of the literature at the undergraduate level provides similar indications of features of mathematical practices or thinking. This includes examinations of problem solving (Arcavi, Kessel, Meira, & Smith, 1998; Schoenfeld, 1985), studies of students dealing with definitions (Dahlberg & Housman, 1997; Edwards, 1997; Rasmussen & Zandieh, 2000), investigations of students dealing with such abstract algebra concepts as isomorphism and quotient group (Dubinsky, 1997; Leron, Hazzan, & Zazkis, 1995), considerations of reasoning in linear algebra (Sierpinska, Defence, Khatcherian, & Saldanha, 1997) and reflections on unifying and generalizing concepts (Dorier, 1995), among many others.

COMMONALITIES AND DIFFERENCES

In addition to locating the following articles in this issue within the four recurring themes—advanced thinking versus advanced mathematics, the utility of characterizing such thinking, criteria for advancedness, and advancing practices, it might be useful to compare them directly.

Harel and Sowder (this issue) have a very general test, or criterion, for AMT, namely that one should be able to see in a student's "way of thinking" at least one of the three characteristics of an epistemological obstacle. While such thinking refers to individual students, it is likely to occur in association with a wide variety of mathematical topics and at many levels throughout a student's entire mathematical education. For example, AMT is likely to occur not only in reasoning about higher dimensional vector spaces but also, much earlier, in proportional reasoning.

Edwards et al. (this issue) also have a very general test, but a different one. In their view, AMT depends on rigorous deductive reasoning about mathematical notions inaccessible to the five senses. Just as with Harel and Sowder (this issue), this kind of thinking refers to individuals, but is likely to occur in studying certain topics and at many levels. However, unlike Harel and Sowder, it is unlikely to occur very often during preuniversity studies. As with Harel and Sowder's view, such AMT is likely to occur in reasoning about higher dimensional vector spaces; however, in contrast to that view, it is unlikely to be involved in proportional reasoning.

Rasmussen et al. (this issue) take a very different perspective. The other two articles propose differing views of AMT, but such thinking is always about mathematical objects—limits, uncountable sets, groups, proportions, even proofs. In contrast, Rasmussen et al. are more concerned with how students can invent, or reinvent, for themselves at least some portions of such mathematical objects. In other words, their view of "advancing" is not so much about a student's thinking at a given time, as about the way the student develops that thinking over time and constructs some of the accompanying mathematics. Such development and construction of mathematical ideas is often encouraged when a teacher and fellow students cooperate in maintaining a classroom culture suitable for mathematizing, especially vertical mathematizing. As with Harel and Sowder's perspective, this can occur at many levels throughout a student's entire education. However, unlike the other two views on advanced mathematical thinking, it is unlikely that the topic alone, whether it be higher dimensional vector spaces or proportionality, would result in students engaging in vertical mathematizing, that is, in their advancing.

To draw a metaphor from linear algebra, the three perspectives in the following three articles move the discussion of advanced mathematical thinking forward in independent ways—ways that we hope will inspire much more work.

REFERENCES

Arcavi, A., Kessel, C., Meira, L., & Smith, J. P., III. (1998). Teaching mathematical problem solving: An analysis of an emergent classroom community. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education. III* (CBMS Issues in Mathematical Education, Vol. 7, pp. 1–70). Providence, RI: American Mathematical Society.

Ball, D. L. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education* (RAND Mathematics Study Panel Report No. MR–1643.0–OERI). Santa Monica, CA: RAND Corporation.

Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics*, *33*, 283–299.

Dorier, J.-L. (1995). Meta level in the teaching of unifying and generalizing concepts in mathematics. *Educational Studies in Mathematics, 29,* 175–197.

Dreyfus, T. (1990). Advanced mathematical thinking. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education* (pp. 113–134). Cambridge, UK: Cambridge University Press.

Dreyfus, T. (Ed.). (1995). Advanced mathematical thinking [Special issue]. *Educational Studies in Mathematics, 29*(2).

Dubinsky, E. (Ed.). (1997). An investigation of students' understanding of abstract algebra (binary operations, groups and subgroups) and the use of abstract structures to build other structures (through cosets, normality and quotient groups) [Special issue]. *The Journal of Mathematical Behavior, 16*(3).

Edwards, B. (1997). An undergraduate student's understanding and use of mathematical definitions in real analysis. In J. A. Dossey, J. O. Swafford, M. Parmantie, & A. E. Dossey (Eds.), *Proceedings of the 19th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17–22). Columbus, OH: The ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Edwards, B. S., Dubinsky, E., & McDonald, M. A. (2005). Advanced mathematical thinking. *Mathematical Thinking and Learning, 7,* 15–25.

Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definitions. *The American Mathematical Monthly*, *111*(5), 411–424.

Hadamard, J. (1945). *The psychology of invention in the mathematical field.* Princeton, NJ: Princeton University Press.

Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical Thinking and Learning, 7,* 27–50.

Heid, M. K., Ferrini-Mundy, J., Graham, K., & Harel, G. (1998). The role of advanced mathematical thinking in mathematics education reform. In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 53–58). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Heid, M. K., Ferrini-Mundy, J., Graham, K., Harel, G., Edwards, B., Ivey, K., et al. (1999). The role of advanced mathematical thinking in mathematics education reform. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 164–169). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Heid, M. K., Harel, G., Ferrini-Mundy, J., & Graham, K. (2000). Advanced mathematical thinking: Implications of various perspective on advanced mathematical thinking for mathematics education reform. In M. L. Fernández (Ed.), *Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 33–37). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* Cambridge, UK: Cambridge University Press.

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29*, 153–174.

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *The Journal of Mathematical Behavior, 21*, 87–116.

Maher, C. A., & Martino, A. M. (1996a). The development of the idea of proof. A five-year case study. *Journal for Research in Mathematics Education*, *27*, 194–219.

Maher, C. A., & Martino, A. M. (1996b). Young children invent methods of proof: The "Gang of Four." In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 431–447). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Maher, C. A., & Martino, A. M. (1997). Conditions for conceptual change: From pattern recognition to theory posting. In H. Mansfield & N. A. Pateman (Eds.), *Young children and mathematics: Concepts and their representation.* Sydney, Australia: Australian Association of Mathematics Teachers.

Maher, C. A., & Speiser, R. (1997). How far can you go with a tower of blocks? In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 174–183). Jyväskylä, Finland: Gummerus.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* Reston, VA: The National Council of Teachers of Mathematics.

Nesher, P., & Kilpatrick, J. (Eds.). (1990). *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education*. Cambridge, UK: Cambridge University Press.

Pimm, D. (1995). The advance party. [Review of the book *Advanced mathematical thinking*]. *Educational Studies in Mathematics*, *29*, 97–122.

Rasmussen, C., Zandieh, M. (2000). Defining as a mathematical activity: A realistic mathematics analysis. In M. L. Fernández (Ed.), *Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 301–305). Columbus, OH: The ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Rasmussen, C., & Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning, 7,* 51–73.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education*, *20*, 8–27.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, *34*, 4–36.

Sierpinska, A., Defence, A., Khatcherian, T., & Saldanha, L. (1997). A propos de trois modes de raisonnement en algèbre linéaire [Three modes of reasoning in linear algebra]. In J.-L. Dorier (Ed.), *L'enseignement de l'algèbre linéaire en question* (pp. 259–268). Grenoble, France: La Pensée Sauvage.

Tall, D. (Ed.). (1991). *Advanced mathematical thinking*. Dordrecht, The Netherlands: Kluwer Academic.

Tall, D. (1995). Cognitive growth in elementary and advanced mathematical thinking. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 61–75). Recife, Brazil: Universidade Federal de Pernambuco.

Thompson, P. W. (1993). Yes, Virginia, some children do grow up to be mathematicians. [Review of the book *Advanced mathematical thinking*]. *Journal for Research in Mathematics Education, 24*, 279–284.

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the AMS*, *30*, 161–177.

Treffers, A. (1987). *Three dimensions. A model of goal and theory description in mathematics education: The Wiskobas project.* Dordrecht, The Netherlands: Kluwer Academic.