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Secondary mathematics teachers' meanings for measure, slope, and rate of change☆

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ABSTRACT

This article reports an investigation of 251 high school mathematics teachers' meanings for slope, measurement, and rate of change. The data was collected with a validated written instrument designed to diagnose teachers' mathematical meanings. Most teachers conveyed primarily additive and formulaic meanings for slope and rate of change on written items. Few teachers conveyed that a rate of change compares the relative sizes of changes in two quantities. Teachers' weak measurement schemes were associated with limited meanings for rate of change. Overall, the data suggests that rate of change should be a topic of targeted professional development.

1. Introduction

We agree with Copur-Gencturk (2015), Zaslavsky (1994), and Thompson (2013) that how teachers understand a mathematical idea is an important factor in the mathematical understandings that students actually form. To the extent that teachers listen to and adapt to what they understand their students to mean, teachers who understand an idea they teach coherently provide greater opportunities for students' to learn that idea coherently. Inversely, the less coherently teachers understand an idea they teach, the fewer are students' opportunities to learn that idea coherently.

Rate of change is a central idea in the secondary mathematics curriculum. It is therefore important to understand the extent to which teachers' meanings for rate of change are sufficient to support them in helping students make sense of rate of change and related ideas.

Prior studies of secondary teachers' meanings for slope, rate of change, and quotient typically focused on small numbers of teachers in an effort to model their meanings or to characterize their proficiency with these ideas (Ball, 1990; Coe, 2007; Fisher, 1988; McDiarmid & Wilson, 1991; Stump, 1999, 2001; Thompson, 1994b; Thompson & Thompson, 1994). Large scale investigations of mathematical knowledge for secondary teaching, such as the TEDS-M study of mathematical knowledge and pedagogical content knowledge did not release any items related to quotient, rate of change, fraction or measurement (Tatto et al., 2012).

Since little is known about secondary teachers' meanings for the content they teach, we developed a diagnostic instrument, *Mathematical Meanings for Teaching secondary mathematics* (MMTsm), to help researchers and professional development leaders diagnose groups of secondary teachers' mathematical meanings (Thompson, 2016). Our aim was to help professional development leaders design interventions that would address weaknesses in teachers' meanings so that teachers can better help students. Studies of elementary teachers have demonstrated that teachers' scores on assessments of Mathematical Knowledge for Teaching are related to improvement in their students' performance (Hill, Ball,

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Blunk, Goffney, & Rowan, 2007). At the same time, while the aim of the MMTsm is diagnostic, using it on a large scale allows us to examine the prevalence of particular meanings and ways of thinking in larger populations of teachers.

Our development of the rate of change items on the MMTsm was guided by qualitative work that characterized students' and teachers' thinking about rate of change, quotient, and slope (Coe, 2007; Lobato & Thanheiser, 2002; Martínez-Planell, Gaisman, & McGee, 2015; Nagle, Moore-Russo, Viglietti, & Martin, 2013; Planinic, Milin-Sipus, Katic, Susac, & Ivanjek, 2012; Stump, 1999, 2001; Thompson, 1994a, 1994b; Thompson, Carlson, Byerley, & Hatfield, 2014; Thompson & Saldanha, 2003; Thompson & Thompson, 1994; Walter & Gerson, 2007; Zaslavsky, Sela, & Leron, 2002). Construction of quality items and rubrics required articulating productive meanings for rate of change that are useful in many contexts such as calculus, science and economics. We also needed specific descriptions of common unproductive meanings for rate of change. In addition, other studies provided evidence that many secondary teachers' meanings for quotient are only productive in limited situations and suggested that investigation of teachers' meanings for "elementary" ideas is important (Ball, 1990; McDiarmid & Wilson, 1991).

This article reports 251 teachers' responses to MMTsm items that focused on high school teachers' meanings for slope and rate of change. We were interested in whether teachers' meanings for rate of change were additive, multiplicative, or both and how they coordinated additive and multiplicative meanings for slope. We examined whether teachers were able to differentiate between situations best modeled with subtraction versus division. We also were interested in the extent to which teachers' meanings for rate of change appeared to be connected to meanings for quotient as a measure of relative size.

1.1. *Summary of article*

1.1.1. *Literature review*

Identifies common meanings for rate of change, slope, measurement, and quotient that have been identified in qualitative studies and were used to write items and categorize teachers' responses.

1.1.2. *Methods*

Discusses the participants in study, rubric creation, and explains why open ended items with multiple acceptable answers should not be used to evaluate individual teachers.

1.1.3. *Results*

Includes six rate of change, slope and measurement items, associated rubrics, and teachers' responses. The results indicate that a majority of teachers' meanings for these concepts are only productive in limited circumstances.

1.1.4. *Looking across items*

Shows the correlation between teachers' measurement responses and multiplicative responses for rate of change and slope. Provides qualitative evidence of the limitations of chunky meanings for slope and rate of change.

1.1.5. *Conclusion*

Teachers need more opportunities to develop productive mathematical meanings, including meanings for slope and rate of change, in undergraduate programs and professional development.

2. Literature review

2.1. *Mathematical meanings versus mathematical knowledge*

Thompson and Thompson (1996) used the phrase Mathematical Knowledge for Teaching (MKT) to describe teachers' schemes for ideas they teach and which they hold at a reflected level. They described teachers' reflected schemes as guides for their interactions with students whom they hope will develop the meanings and ways of thinking that the teacher intends. Silverman and Thompson (2008) expanded this scheme-based meaning of MKT by examining how teachers might create what they called Key Pedagogical Understandings from a basis of their personal, well-formed schemes—schemes which Simon (2006) called Key Developmental Understandings. A Key Pedagogical Understanding is a mini-theory that a teacher holds regarding how to help students create the schemes that the teacher intends. In other words, A. Thompson, P. Thompson, and Silverman used *knowledge* in the sense of Piaget and von Glasersfeld—as schemes and ways of coordinating them that enable people to function adaptively in light of their goals and experienced situations. We see teachers' schemes as more than a set of declarative facts that the teachers learned about students and mathematics. An example of a declarative fact is "when students add two fractions many add the numerators and denominators." We want to model teachers' more general schemes for fraction, measure, quotient and rate of change that would allow us to predict how teachers' might respond in a large variety of situations and not just in a specific context such as teaching the procedure to add fractions.

Ball, Hill and colleagues (Ball & Bass, 2002; Hill et al., 2007; Hill, Schilling, & Ball, 2004) used the phrase MKT differently than did A. Thompson, P. Thompson, and Silverman. Ball et al. observed elementary school teachers and documented the "mathematical knowledge and skill used in the work of teaching" (Ball, Hill, & Bass, 2005, pp. 16–17). Schilling, Blunk, and Hill (2007) discussed the evolution of their understanding of what their instrument measured from "declarative knowledge" to "a kind of close reasoning":

When we began developing items in this domain, we hypothesized that teachers' knowledge of students existed separately from their mathematical knowledge and reasoning ability. We thought of such knowledge as "declarative," or factual knowledge teachers have of student learning. Results from these validation studies, however, suggest that this "knowledge" may actually contain both elements of mathematical reasoning and knowledge of students and their mathematical trajectories. From this standpoint, Knowledge of Content and Students is less declarative knowledge than a kind of close reasoning in which teachers engage, flexibly, about students and their work. (Schilling et al., 2007, p. 121).

We believe teachers' mathematical schemes and their ability to reflect on their schemes, are one of the most important factors in determining their success in reasoning about students and their work. The productivity and generalizability of their mathematical schemes will either constrain or afford teachers' decision making in the mathematics classroom. We see the specific declarative facts Ball, Hill, and Shilling identified as "doing work for" teachers in the classroom as one part of teachers' mathematical schemes. Instead of designing items to measure teachers' declarative knowledge about specific teaching situations, we designed items that would allow teachers to reveal their schemes for major concepts in the hope that understanding their schemes would help us understand how they might convey mathematics in a wide variety of teaching situations. The teachers' more general schemes help us understand the teachers' "close reasoning" they engage in when presented with teaching situations. The difference in Thompson et al.'s and Ball et al.'s goals as well as different theoretical perspectives on knowledge led to different methods in studying teachers' thinking (Byerley et al., 2015).

In 2013, P. Thompson began to use the phrase mathematical *meanings* for teaching instead of the phrase mathematical *knowledge* for teaching for three reasons: (1) readers often failed to understand that he was using *knowledge* in the sense of Piaget and Glasersfeld and not in the sense of Ball and colleagues (Thompson, 2013, p. 85); (2) to Piaget, knowledge and meaning were largely synonymous and both were imbued with the idea of scheme (Montangero & Maurice-Naville, 1997), and (3) readers understood easily that *meaning* connotes something personal and that a person's meanings are intertwined, whereas *knowledge* seemed less personal and more declarative, standing apart from the knower. We used the term *meanings* instead of *knowledge* because we did not want readers to think our diagnostic instrument was designed to measure whether or not teachers had mastered a set of declarative facts related to teaching mathematics.

In light of the above, we named our instrument *Mathematical Meanings for Teaching secondary mathematics* (MMTsm) because we wanted to assess the meanings a teacher holds for the mathematics they teach. We hasten to say that we intend mathematical meanings for teaching to be understood as a teacher-centric construct, not a normative construct. Put another way, every teacher has meanings for the mathematics they teach. An individual teacher's meanings might be incoherent, superficial, coherent, or productive, but they are her meanings for the mathematics she teaches. We categorize teachers' meanings according to how productive they are. We agree that "productive or unproductive is a more appropriate criterion than right or wrong, and final assessments of particular conceptions will depend on the contexts in which we evaluate their usefulness" (Smith, diSessa, & Roschelle, 1993, p. 147). We define "productive mathematical meanings" to be meanings that a teacher holds which would be productive for students' long-term mathematical learning were they to hold them also. We acknowledge immediately two concerns: (1) whether a meaning is actually productive for students' learning depends on schemes available to the students at the moment of instruction, and (2) determining whether or not a meaning is actually productive for students requires collecting empirical evidence from students. A particular meaning might seem productive from the perspective of a more advanced knower, but there could be unforeseen consequences when attempting to teach this meaning to someone first experiencing the idea.

2.2. Schemes and meaning

We use the definition of scheme offered by Thompson et al. (2014):

We define a scheme as an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity. (p. 11)

Unlike *knowledge* which people often understand to mean knowing something that is agreed upon to be true, a teachers' scheme can be productive in many situations or only in a few. For example, we cannot say thinking of a fraction as a part of a whole is untrue, but we can say this scheme is productive only in limited circumstances (Norton & Hackenberg, 2010).

One hallmark of a productive meaning for an idea such as rate of change is having a meaning that can be used to understand a wide variety of contexts and problems. The MMTsm includes several items that address contexts that we see as involving slope or rate of change. Our theory of meanings predicts that, for a teacher who has disconnected meanings for slope and rate of change, different contexts that involve slope or rate from our perspective could trigger different schemes in a teacher's thinking. Our theory also predicts that, for a teacher who has a coherent system of meanings for slope and rate of change, these same contexts would trigger different aspects of one scheme.

Our theory of meanings has strong implications for issues of reliability and validity of assessments and their items. If teachers conveyed similar meanings on multiple items intended to measure meanings for rate of change, the set of items would be considered to have internal consistency reliability. An assessment that validly assesses subjects' meanings with regard to ideas for which the population of subjects has a wide variety of disconnected meanings will likely have low internal consistency reliability. Subjects will respond differently to items that the assessment's writers see as tapping the same idea. This is exactly the case reported by Carlson, Oehrtman, and Engelke (2010). They conducted hundreds of interviews to establish the validity of their instrument's items with regard to their interpretations of students' answers, but most students chose the best response on some items that targeted a key idea but not on other items targeting the same idea—they had disconnected ways of thinking about different contexts that (to Carlson et al. (2010)) involved the same idea. To understand teachers' meanings it is critical to look at their responses to a variety of rate of change items because we expect that their responses to one item will not be a strong predictor of their responses to other similar items unless they have strongly connected meanings for slope and rate of change.

The construct *meaning* is similar, at the surface, to the construct *concept image*. Tall and Vinner (1981) explained "the concept image consists of all the cognitive structure in the individual's mind that is associated with a given concept. This may not be globally coherent and may have aspects which are quite different from the formal concept definition" (p. 151). Even (1993) used the constructs *concept image* and *concept definition* to describe how secondary mathematics teachers used their experiences with functions to determine whether or not a given formula or graph was a function. The teachers often did not use the modern mathematical definition of function to determine if a graph was a function. Meaning, as explained in Thompson et al. (2014) has a different theoretical foundation and therefore different entailments than concept image. Tall and Vinner spoke of cognitive structures associated with a *concept.* Meaning, to us, is more restricted. It is, to use Tall and Vinner's language, the cognitive structures associated with (implied by) a person's current *understanding.* Teachers sometimes had multiple meanings for slope that were consistent with the formula $\Delta y/\Delta x$, but some meanings were more helpful than others in particular contexts as teachers understood them.

2.3. Quantitative meanings for quotient, measure, covariation, and rate of change

In line with our stance that all meanings are personal, we attempt to convey the meanings of quotient, measure, covariation, and rate of change that we used in this study. We also explain why these particular meanings are productive and coherent for student learning. The meanings that we summarize here are described in greater detail in (Thompson, 1994a, 1994b; Thompson & Carlson, 2017; Thompson et al., 2014; Thompson & Saldanha, 2003).

2.3.1. Quotient

A quantitative meaning for quotient entails a multiplicative comparison of two quantities with the intention of determining their relative size. Determining the relative size of two quantities means thinking of and expressing the magnitude of one quantity in terms of a multiple of the magnitude of another. A person who understands a rate's value as a quotient understands that a rate gives the relative size of changes in two quantities.

Many students and teachers understand quotient (without knowing the word) only as the numerical result of division, without having an affiliated sense that they have determined a relative size. Other students and teachers understand the word quotient only as the name of a figural configuration that involves a vinculum—a horizontal division bar. Some students and teachers think that the mathematical meaning of a quotient is bound to the context in which it is used, such as "3/4" being the slope of a line means "up 3 and over 4" (Stump, 2001). If they see "3/4" as referring to a part of a whole they may understand that 3 is a subset of 4. Several studies show that many school students', future teachers', and teachers' meanings of division are non-quantitative and have little to do with ideas of relative size (Ball, 1990; Byerley & Hatfield, 2013; Byerley, Hatfield, & Thompson, 2012; McDiarmid & Wilson, 1991; Simon, 1993). A person with an image of multiplication as making multiple or partial copies of a quantity is positioned to understand the connection between multiplication and quotient as described by Thompson and Saldanha (2003).

2.3.2. Measure

Thompson et al. (2014) characterized various schemes for measure by discussing levels of reasoning about magnitudes of quantities. They distinguished among six meanings of magnitude: gross perception of size, size as measure being a count of a specific unit, size as measure relative to a specific unit (*Steffe Magnitude*), size as independent of specific units (*Wildi Magnitude*), relative size of measures in specific units, and relative size independent of units (*Relative Magnitude*). The last four meanings of magnitude (*Steffe Magnitude*, *Wildi Magnitude*, and two forms of *Relative Magnitude*) are all based in multiplicative comparisons of quantities' measures. The last three meanings involve the understanding that the amount of a measured quantity is invariant across changes in unit. For example, people with a *Wildi Magnitude Scheme* understand that a container's volume measured in gallons and liters is the same size even though the respective measures of the container's volume are different. Further, they understand the reciprocal relationship of relative size—that because a gallon is 189/50 times as large as a liter, the measure of a container in gallons is 50/189 times the measure of the container in liters. *Wildi Magnitude* and *Relative Magnitude* schemes are foundational for mature understandings of rate of change and understanding intensive quantities in science.

2.3.3. Variational and covariational reasoning

Both Confrey and colleagues and Thompson and colleagues have written extensively about the covariation construct (Confrey & Smith, 1995; Saldanha & Thompson, 1998; Thompson, 1990, 1994a, 1994b, 1994c; Thompson & Thompson, 1992, 1994). In this study, we use Thompson's meaning of covariation, expanded as in Thompson and Carlson's (2017) framework for describing different levels of variational and covariational reasoning. This expanded framework includes a distinction introduced by Castillo-Garsow, Johnson, and Moore (2013) and Castillo-Garsow (2012) between what he called "chunky continuous reasoning" and "smooth continuous reasoning". A person reasons about a quantity or variable varying in "continuous chunks" by thinking that it attains a next value, that intermediate values exist, but without thinking that the quantity or variable actually attained any of those values. Thinking with smooth continuous variation is defined as,

> The person thinks of variation of a quantity's or variable's value as increasing or decreasing by intervals while anticipating that within each interval the variable's value varies smoothly and continuously (Thompson & Carlson, 2017, p. 440).

Thompson and Carlson (2017) then defined smooth continuous covariation as a person conceptualizing the values of two quantities varying simultaneously, while also having conceived of the quantities values varying smoothly and continuously.

2.3.4. Rate of change

Thompson and colleagues described students' productive rate of change schemes as emerging through the progressive coordination and integration of schemes for quantity, variation, covariation, change, accumulation, and proportionality (Silverman & Thompson, 2008; Thompson & Thompson, 1996; Thompson, 1994b; Thompson, Byerley, & Hatfield, 2013; Thompson & Thompson, 1994). A mature meaning for rate of change involves imagining covariation of quantities as well as a relative size or relative magnitude scheme. This is consistent with Thompson and Carlson's (2017) argument that "for students to conceptualize rates of change requires that they reason covariationally, but it also requires conceptualizations that go beyond covariational reasoning, such as conceptualizations of ratio, quotient, accumulation, and proportionality" (p. 441). Understanding constant rate of change entails imagining two quantities covarying such that an accumulation of changes in one quantity is proportional to the associated accumulation of changes in the other quantity (Thompson, 1994b). Students must understand the difference between the amount of a quantity and the change in the amount of the quantity. They must make additive comparisons to determine the change in quantities and then multiplicatively compare those changes. Norton and Hackenberg (2010) provided evidence collected in teaching experiments with students that the development of productive meanings for rate of change and proportion are dependent on having meanings for fractions that are more advanced than part-whole meanings.

2.4. Teachers' meanings for slope, rate of change, measure and quotient

Prior studies give examples of a variety of meanings for slope, rate of change, and quotient that we used to create rubrics for categorizing teachers' written responses to MMTsm items. The meanings we will highlight from this literature are chunky meanings for slope and rate, thinking about slope/rate as an index of slantiness/fastness, the disconnect between meanings for rate and division, and difficulty in creating a quantitative image of division by a fraction. We believe most teachers' meanings for these ideas are multi-faceted and vary based on the situation they encounter.

2.4.1. Chunky meanings for slope

Coe (2007) and Stump (2001) interviewed in-service and preservice secondary teachers who conveyed a chunky meaning for slope. While teaching a lesson on slope, Joe, a preservice teacher, defined slope as, "'vertical change/horizontal change,' and presented a graph of the line passing through the points (0,0) and (3,2). He emphasized that the slope as a fraction, 2/3, up 2, over 3" (Stump, 2001, p. 216). Joe conveyed a chunky, non-multiplicative meaning for slope and never said that for any sized change in x the change in y is 2/3 as large. We believe his language in interviews and teaching would convey to a student that the vinculum (division bar) serves to separate numbers that tell us how to move in horizontal and vertical directions. This meaning is limited to Cartesian coordinate systems and cannot be applied to polar coordinate systems. One consequence of the meaning for slope Joe conveyed was that a student in his class did not understand that "the two fractions $5/-6$ and $-5/6$ could both represent the same slope" (Stump, 2001, p. 216). Joe noted in a post-teaching interview, "They think you are describing a movement as opposed to you describing a number, a measurement" (Stump, 2001, p. 216). Another consequence of this chunky meaning for slope is that individuals experience difficulty in reasoning about the values of the points in between the two points at either end of the "chunk." Thompson (2013) gave an example of a teacher who could not find the values of points on a line in between the two points she produced by moving up and over in chunks on a graph. For this teacher, "division did not produce a quotient that has the meaning that the dividend is so many times as large as the divisor—3/4 as a slope was not a number that gave a rate of change." (Thompson, 2013, p. 81).

The three experienced secondary mathematics teachers whom Coe (2007) studied also conveyed chunky meanings for slope. Peggy was asked "why do we use division to calculate slope?" and she replied that she didn't know because "she never really thought of it as the division operation" (Coe, 2007, p. 207). Peggy understood slope as directions on how to move up and over on a graph and did not imagine comparing the relative size of numerator and denominator. A chunky meaning for slope is adequate to solve many textbook problems, such as graphing a line given the equation of a line.

2.4.2. Slope as an index of steepness

Coe (2007) and Stump (2001) described teachers who employed a meaning for slope as an *index of slantiness* to various degrees of success. A teachers' sense of slantiness does not have to involve comparing changes in x and changes in y, but simply associating particular numerical values with particular graphs based on repeated exposure to graphs of linear equations. One limitation of remembering what a particular slope "looks like" is the dependence on the graphs being displayed in a rectangular coordinate system whose axes are in the same scale. Natalie, a preservice teacher was unable to extend her *index of slantiness* meaning for slope to rate of change situations. At the beginning of the methods course, Natalie said, "Slope is a term used to associate the incline of a line with a numerical value" (Stump, 2001, p. 217). Despite the instructors' emphasis on the connection between slope and real world comparisons of changing quantities such as distance and time Natalie choose to focus on steepness, inclined plane examples and developing "rise over run" as a measure of steepness in her lessons (Stump, 2001, p. 221). Natalie was "resistant to including the notion of slope as a measure of rate of change in her work" (Stump, 2001, p. 221). The physical situations Natalie used in instruction included real-world examples such as inclined planes and ski slopes where steepness was visually apparent in the situation. She did not help students understand that slope could be thought of as the rate of change of any two quantities.

Mary, an in-service teacher who described slope as "how steep a line is" (Coe, 2007, p. 115) was unable to use her meaning for slope to make sense of multiple questions involving basic applications of slope. Instead of saying that a decreasing, concave up graph was decreasing at an increasing rate, she said that the graph was decreasing at a decreasing rate. Her reason was that the tangent lines appeared less steep as x increased. She did not consider that the changes in y in the graph were negative, and simply looked at how steep the graph appeared, as if it were a hill. Though a slope of -2 is smaller than a slope of -1, because -2 is less than -1, Mary saw a slope of -1 as less steep than a slope of -2 from the perspective of thinking about the slantiness of a hill.

2.4.3. Teachers' meanings for quotient

All three studies we found that investigated secondary teachers' meanings for quotient showed that teachers had significant difficulties with the idea (Ball, 1990; Byerley & Hatfield, 2013; McDiarmid & Wilson, 1991). We hypothesize that teachers with weak meanings for quotient are less likely to think of rate of change and slope multiplicatively, and more likely to resort to chunky or "index of steepness" meanings.

McDiarmid and Wilson (1991) presented 55 alternatively certified secondary teachers with four story problems that prompted them to choose which story problem could be solved by dividing by 1/2. Only 33% were able to identify a quantitative situation that involved division by a fraction. In interviews, some teachers in their study could see no real world application for division by fractions. Similarly, Ball and McDiarmid (1989) asked prospective teachers "to develop a representation—a story, a model, a picture, a real-world situation—of the division statement $1\frac{3}{4} \div \frac{1}{2}$" (p. 21). Five of nine prospective secondary teachers and zero of nine elementary teachers responded appropriately (p. 22). If a quotient is considered to be the relative size of two quantities' measures, it is irrelevant how one represents their measures.

Byerley and Hatfield (2013) asked 17 preservice secondary teachers who were taking an upper division teaching methods course to draw a picture representing a given division problem (7.86 divided by 0.39 equals 20.15). The results suggested that the preservice teachers did not have strong quantitative meanings of quotient as a measure of relative size. Two said "I don't know." Nine of 17 gave an explanation such as, "20.15 times 0.39 is 7.86", which suggests they understood division as reversing multiplication calculationally. This, by itself, does not imply that they understood 20.15 as a quotient—that 7.86 is 20.15 times as large as 0.39. Six of 17 students represented the relative size of 7.86 and 0.39 in an image to explain the meaning of a quotient. See Fig. 1 for an example.

Only one of the seventeen preservice teachers in the study was able to explain why division was used to calculate slope. We hypothesized that without an image of slope as a measure of relative size of changes, students' meaning for quotient would not help them explain the use of division in the slope formula.

2.4.4. Teachers' meanings for measure

Two previously reported MMTsm items provide information about teachers' meanings for measurement (Byerley & Thompson, 2014; Thompson et al., 2014). One item asked teachers to convert between liters and gallons given a conversion factor (Fig.2).

The second item asked teachers to convert between measures in the imaginary units "Nerds" and "Raps" given a conversion factor (Fig. 3).

As of the publication of those articles, we had collected 100 secondary teachers' responses during the MMTsm's development phase. Only 24% of secondary mathematics teachers demonstrated the understanding that a fixed container measured in liters will have a larger measure than the same container measured in gallons (Byerley & Thompson, 2014). Fifty of 100 teachers appropriately converted between the imaginary units *Nerds* and *Raps*. Only 17% of the 100 teachers gave higher-level responses to both measurement problems. Most teachers did not imagine that since a gallon is larger than a liter, the number of gallons in a container must be smaller than the number of liters in the same container. Quantitative measurement meanings are critical for developing the ability to conceive of the change in one quantity measured in terms of the change in the other. We hypothesize that a person who does not reason quantitatively about a measurement situation, is also less likely to develop a meaning for rate of change that entails measuring one change in terms of another change. Other researchers have also found that students' and teachers' difficulties with multiplicative situations are due to a lack of orientation to reason about the measures of the quantities in the problem (Lobato & Siebert, 2002; Simon & Blume, 1994a,1994b).

We believe that responses on the measurement items reveal difficulty reasoning about measurement and are not simply due to

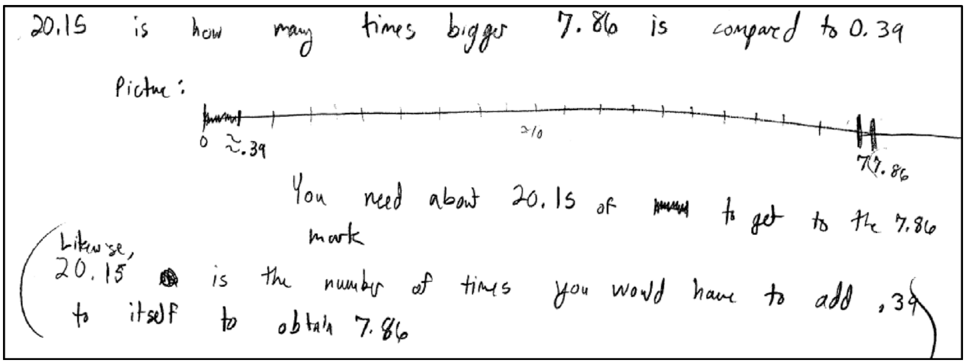


Fig. 1. Diagram depicting relationship between numbers in division problem.

A container has a volume of m liters. One gallon is $\frac{189}{50}$ times as large as one liter. What is the container's volume in gallons? Explain.

Fig. 2. MMTsm item Gallons to Liters. © 2014 Arizona Board of Regents. Used with permission.

In Nerdland they measure lengths in Nerds. The highlighted arc measured in Nerds is 12 Nerds. In Rapland they measure lengths in Raps. One Rap is 3/4 the length of one Nerd. What is the measure of the highlighted arc in Raps?

Fig. 3. Item Nerds and Raps. © 2014 Arizona Board of Regents. Used with permission.

"careless mistakes." We received many responses similar to the one in Fig. 4 that indicated that teachers did not rush through the problem. Rather, their responses suggested that they considered the situation.

We interviewed teachers on this item and discussed it in classes for calculus students and preservice secondary teachers. Many students and preservice teachers retained their original response that conflicted with a gallon being larger than a liter even after we asked the question, "Which is bigger? A gallon or liter?" The discussions included drawing an image of the situation, and asking them to reconsider their answers in light of the lack of consensus on the correct answer. In one class of preservice secondary teachers, it took approximately 40 minutes of discussion until everyone agreed that the container's volume is (50/189)m.

2.4.5. Teachers' meanings for rate of change

At least four studies have investigated secondary teachers' meanings for rate of change (Bowers & Doerr, 2001; Coe, 2007; Person, Berenson, & Greenspon, 2004; Thompson, 1994b). Data from each study supported the claim that many secondary teachers have meanings for rate that are chunky or indexical. This mirrors the findings on teachers' chunky and indexical meanings for slope.

Brian, a secondary preservice teacher, described rate of change as "the amount something changes in a given time" (Person et al., 2004, p. 21). Using Castillo-Garsow's language we notice Brian's description conveys an image of completed chunks of change. Brian conveys that a rate of change is an *amount* of change. He did not convey comparing a measure of an amount of change with a measure of amount of time. Time does pass in the background, but Brian's meaning for rate of change was not about multiplicative comparisons of two changes. Later, Brian described a constant speed with the idea of cruise control. Building on observations made by Stroud (2010), we notice that Brian conveyed the idea of speed as the number to which a speedometer points instead of as a relationship of relative size between number of miles traveled and number of hours elapsed. Thinking of speed as the number a speedometer points to is similar to thinking of slope as an index of slantiness-both are indexes that relate to an aspect of the situation.

Thompson (1994a) reported that senior and graduate mathematics education students had difficulties understanding the rate of change of a cone's volume with respect to its height partially because they confused "changing" with "rate of change" and "amount and change with amount" (Thompson, 1994b, p. 257). One student, Adam, struggled to explain the relationship between two quantities because he identified the idea of "rate" with the idea of "change" (Thompson, 1994b, p. 261). Confusing rate with amount or rate with a change is consistent with schemes that do not entail considering the multiplicative comparison of two quantities.

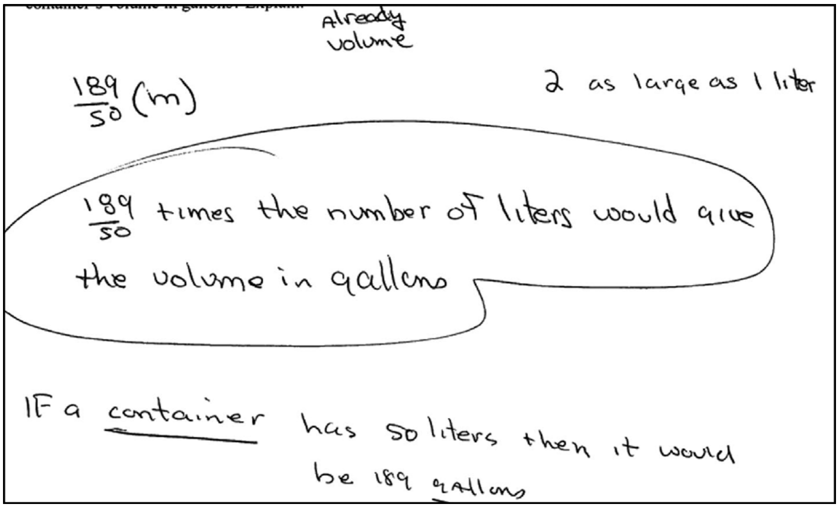


Fig. 4. A teacher who noted that gallons are larger than liters and yet said that the container would contain more gallons than liters.

Bowers and Doerr (2001) reported over half of the fifteen secondary mathematics teachers in a university course inappropriately applied the formula $d = rt$ in situations with non-constant rates of change (Bowers & Doerr, 2001, p. 124). Despite the mathematical relationship between $\Delta d/\Delta t = r$ and $\Delta d = r\Delta t$, we suspect that many people apply these formulas calculationally in different circumstances without connecting their quantitative meanings.

2.5. Connections between literature review and our methodology

The literature review reported conceptual analyses of productive meanings for rate of change, fraction, quotient, and slope. We used these conceptual analyses to design items and create rubrics for each item. In the results section we describe the rubrics we used to categorize responses to items. For the vast majority of responses, we found evidence that the response was consistent with one of the ways of thinking reported in prior qualitative research and categorized it accordingly. The majority of these qualitative studies relied on interviews or teaching experiments in which the researchers used multiple sources of evidence to infer a teachers' meaning for slope or rate of change. We drew upon these descriptions of teachers' thinking, in addition to our own teacher interviews, to help us infer a teachers' potential meanings based on their sometimes terse written responses.

A summary of the most important constructs is in Table 1.

Table 1
Various Types of Reasoning about Slope, Quotient, and Rate of Change.

Construct	Evidence of this reasoning
Smooth continuous covariational reasoning	A slope of three means that as x and y covary, for any sized change in x the associated change in y is three times as large.
Chunky covariational reasoning	A slope of three means that every time x changes by 1, y changes by 3.
Slope is an index of steepness/rate is an index of fastness	Slope is a number we assign to a slant to describe how steep a line looks.
Formulaic meaning for slope/rate of change	$\Delta y/\Delta x$, rise/run, $\Delta d = r\Delta t$, etc.
Quotient is the result of a division computation.	When I use long division and follow the steps I get a quotient.
Quotient is a measure of relative size	I can estimate the quotient (A/B) by comparing the measure of quantity A to the measure of quantity B.

3. Methods

This section describes the methods used during the four-year NSF-funded Project Aspire to design and validate the MMTsm diagnostic instrument. The motivation for Project Aspire was to develop a diagnostic instrument that would identify teachers' meanings in a way that would be useful for designing and evaluating professional development. The MMTsm was *not* designed for evaluating individual teachers' competencies. Because the scoring of our items focuses on meanings and not competency, it would be unfair to judge a teacher on his or her responses. Professional development leaders receive a report of group outcomes on each item. No information about individual teachers is provided to them. The MMTsm has been used by several Math/Science Partnership projects as a pre-post assessment to determine whether professional development had a positive impact on the group's mathematical meanings.

3.1. Item development

One of the primary goals for the items was to give teachers the opportunity to convey the sense they made of an item, which then gave us grounds to discern meanings they employed in making that sense. The relationship between our theory of meanings and the methods are discussed in greater detail in Thompson (2016). It was important that teachers could interpret a question in their own way, and that the question would prompt them to display their meanings explicitly enough that we could interpret and categorize them confidently. We also had to create items that prompted teachers to use higher-level meanings if they were able to do so. For example, if we wanted to determine whether a teacher thought about slope as the relative size of the change in y and the change in x, we could not ask a question that could be solved by routinely using the slope formula.

Thompson (2016) summarized the process of creating items and rubrics for the MMTsm. We followed common instrument construction guidelines such as making many revisions to items based on interviews with teachers and pilot administration of the instrument. Feedback from our advisory board and other mathematicians and mathematics educators was also essential to item development.

3.2. Rubric development

In Summer 2012 we administered draft versions of the slope and rate of change items discussed in the results sections to 144 secondary mathematics teachers. We created rubric levels for items using a modified grounded-theory approach (Corbin & Strauss, 2007). The modification was that we began our data analysis with the conceptual analysis of magnitudes and rates of change described in the literature review, as well as multiple descriptions of teachers' meanings from prior qualitative studies.

We developed rubrics by grouping grounded codes used to describe responses into levels based on our interpretation of the mathematical meanings teachers expressed. By reading a teacher's written response to an item we do not believe it is possible to model their meanings with the same assurance as if we interviewed the teacher. However, we did interview a subset of teachers to check whether the meanings we attributed to their written comments were consistent with the meanings they expressed later to us. Further, we hypothesize that teachers' use the same meanings to respond to the items as they use while teaching. This is not to say that their written responses will exactly reflect what they say in classroom, only that their written and spoken descriptions of mathematics will be based on their meanings in either context.

When scoring responses we did not attempt to determine the depth of the teachers' understanding of mathematics that they left unarticulated. Instead, we read the teacher's response literally and asked, "If this is what they said to a class, what meanings for the mathematical idea might students construct?" We relied heavily on prior research on student thinking to make determinations about the meanings for particular ideas that would be productive in the largest number of situations. For example, we pointed to Coe (2007) to demonstrate that a primary meaning for slope as an index of steepness is useful in more limited contexts than a multiplicative meaning for slope. We scored responses that conveyed that slope is an index of steepness at a lower level than responses that convey a multiplicative meaning for slope, even though we acknowledge that a meaning for slope as steepness is useful in some contexts. A lower level response should not necessarily be considered incorrect or without merit.

Advisory board members and consultants scored randomly selected subsets of teacher responses at multiple stages in rubric construction and refinement. Scorers included one or more statisticians, mathematicians, high school math teachers, and math education researchers. Scorers came from various institutions and countries. The diversity ensured the items and rubrics could be used internationally. Translated versions of the instrument and scoring rubrics have been used successfully with over 400 teachers in Korea (Thompson, Hatfield, Yoon, Joshua, & Byerley, 2017; Thompson & Milner, in press; Yoon, Byerley, & Thompson, 2015).

3.3. Sample and scoring

In Summer 2013 and 2014 we administered a revised version of the MMTsm to 251 high school teachers in two different states. The teachers were participating in Math Science Partnership professional development programs (NSF or state funded). The sample had 63 teachers with at least a mathematics BA, 81 teachers with at least a mathematics education BA, and 107 with a BA in another subject. Many of these teachers also had masters degrees in a variety of fields. The number of years they had taught high school math varied from one year to over fifteen years.

The Project Aspire team scored all teacher responses. To estimate interrater reliability (IRR) an outside collaborator scored 50 overlapping responses for each item. Non-perfect agreement was scored as disagreement. Items with complex responses had lower IRR than items with simple or numerical responses. Table 2 shows all items reported in this article had high levels of interrater reliability.

The interrater reliability is strong enough to support inferences about what types of meanings we could expect to find in a group of teachers. We do not use the instrument to give individual teachers' an overall score.

4. Results

We present each item, its abbreviated scoring rubric, and the distribution of teachers' responses on that item. The actual rubrics had numerous examples of responses at each level and detailed instructions on how to resolve tricky issues. After presenting all of the items and results we examine what the responses to the set of items convey about teachers' meanings for slope and rate of change.

4.1. Meaning of slope

We designed the item in Fig. 5 to reveal teachers' meanings for slope. We wrote Part A anticipating that many teachers would say that 3.04 means that every time that x changes by 1, y will change by 3.04. We designed Part B to reveal teachers' ability to use constant rate of change to determine the change in dependent variable for any change in the independent quantity. Because of the qualitative research on chunky thinking we wanted to see how teachers applied their meaning of slope when the change in x was not one.

Table 2
Interrater Reliability Scores for MMTsm Items.

Item Name	Number of responses scored by two scorers	Percent Agreement	Cohen's Kappa
Gallons to Liters	50	0.94	0.917
Nerds to Raps	50	0.94	0.905
Meaning of Slope Part A	50	0.84	0.773
Meaning of "Over" Part A	50	0.86	0.814
Meaning of "Over" Part B	50	0.9	0.849
Slope from Blank Graph Part A	31	0.968	0.957
Increasing or Decreasing from Rate Part B	49	0.959	0.926

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.

Convey to Mrs. Samber's students what 3.04 means.

(next page)

Part B.

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.

A student explained the meaning of 3.04 by saying, "It means that every time x changes by 1, y changes by 3.04." Mrs. Samber asked, "What would 3.04 mean if x changes by something other than 1?"

What would be a good answer to Mrs. Samber's question?

Fig. 5. The item Meaning of Slope was designed to reveal meanings for slope. ©2014 Arizona Board of Regents. Used with permission.

The summary rubric for *Meaning of Slope* Part A is given in Table 3.

Level A2a responses are considered slightly more productive for students than A2b responses because the meaning of slope in Level A2a responses is not constrained to horizontal and vertical motion on a Cartesian graph. Level A1 responses gave a formula for slope or a one-word description of slope such as "slantiness" but did not describe changes in x and y. Because we are focused more on teachers' meanings than their knowledge some responses scored at A1 have incorrect formulas such as $\Delta x/\Delta y$ or y/x. Even though the formulas are incorrect, the teachers are still considered to have conveyed a formulaic meaning for slope (Level A1). We scored responses that did not fit any other category at level A0. In cases where one teacher responded with multiple meanings for slope we categorized the response according to the highest level meaning.

The most common meaning conveyed in our sample was a chunky, additive meaning for slope (See Table 4). Only ten teachers of 250 conveyed a multiplicative, relative size meaning for slope (Level A3). Approximately 78% of teachers' responses conveyed a chunky or additive meaning for slope (Levels 2a and 2b). About 81% of teachers who majored in mathematics and 80% of teachers who majored in mathematics education conveyed a chunky, additive meaning. There is no evidence of a statistically significant relationship between response and degree type ($\chi^2(6, n = 242) = 10.71, p = 0.097$). The eight teacher responses scored at level A0 or "no response" were excluded from the chi squared analysis because of low cell counts.

Even though we view a multiplicative meaning for slope as the most productive for students, we do not view responses at level A1 and A2 as mathematically incorrect or unproductive in every respect. Many of the teachers during their summer professional development workshops used such meanings for slope to solve the majority of problems in common high school curriculums that they teach. Although we have not explored this issue, it is possible that students and teachers can develop multiplicative meanings for slope by building on chunky and formulaic meanings for slope. However, results to Part B of *Meaning of Slope* and on other items suggest chunky and formulaic meanings for slope are not adequate in some important contexts and teachers would benefit from receiving instruction about multiplicative meanings for slope.

4.1.1. Results for part B meaning of slope

Part B gave teachers an additional opportunity to convey a multiplicative meaning for slope. After many attempts at rubric and item revision, we only had 72% agreement for scores on Part B. The teachers' responses gave interesting insight into their meaning for slope, but were hard to categorize reliably. We report limited results to show how many teachers with mathematically acceptable meanings in Part A struggled to explain slope when x did not change by one.

Table 3
Abbreviated version of rubric for Part A of Meaning of Slope. © 2014 Arizona Board of Regents. Used with permission.

Level	Example of what the response conveyed:
A3-Relative Size	x can change by any amount and y changes by 3.04 times the change in x.
A2a-Chunky	For every change of 1 in x, there is a change of 3.04 in y.
A2b-Chunky graphical	Slope gives information about how to move horizontally and vertically. For example: If x moves to the right 1 space, y moves up by 3.04.
A1-Formula or One-Word description	– Gave slope formula – Used a phrase such as "slantiness." – Used "rise over run" without describing the changes.

Table 4
Responses to Part A Meaning of Slope.

Response	Math Majors	Math Ed Majors	Other Majors	Total
A3-Relative size	3	3	4	10
A2a-Chunky	27	18	37	82
A2b-Chunky graphical	24	47	41	112
A1-Memorized	7	12	19	38
A0-Other/IDK	1	1	2	4
No response	1	0	3	4
Total	63	81	107	250

- We included 250 teachers instead of 251 because one teacher did not state his major.
- IDK means "I don't know" in all tables.

Approximately ten (8%) of 120 teachers who gave chunky (level 2a/2b) responses to part A conveyed multiplicative meanings for slope in Part B. Approximately 41 (34%) of the 120 teachers who conveyed a chunky meaning on Part A gave a level zero response to Part B. Although level zero responses varied widely, they all failed to deal with the Part B prompt coherently. Level zero responses did not explain a meaning for slope, nor explain how to find a change in y given a non-unit change in x. This suggests that having a chunky meaning for slope is insufficient to deal meaningfully with situations where the input variable changes by something other than one.

4.2. Relative rates

Although additive and formulaic meanings for slope and rate seem to be sufficient to solve many common problems in the secondary curriculum, these meanings can lead to invalid models of physical situations. The response to the item *Relative Rates* (Fig. 6) shows one context where thinking of a rate of change additively is less productive than thinking of a rate of change multiplicatively. It is plausible that teachers who responded with a multiplicative comparison may have also reasoned additively about one second chunks and coordinated additive and multiplicative reasoning to chose j/s. We first discussed this item in Byerley and Thompson (2014). This article includes responses from 150 additional teachers.

We suspect that thinking of rate of change additively makes it more difficult to solve *Relative Rates* because the common additive choice "*j–s*" corresponds to the change in distance for a chunk of size one second. If the item had stated how many seconds had elapsed multiplying *j-s* by the number of seconds would have been an appropriate solution and additive reasoning might have been adequate.

There are multiple ways of thinking about the problem that result in choice (a), so there is no way to determine precisely what type of thinking a teacher actually engaged in to pick (a). We do know based on interviews that some teachers who noticed and highlighted the word "any" still gave the response "j-s". Despite the variety of potential solution paths, the interview data suggests that teachers who picked (a) were thinking about the situation additively. For example, some teachers thought of "*j*" as a changing quantity that represents Julie's distance for any given amount of time. With *j* representing a changing quantity instead of the value of a fixed unknown rate, the additive response *j-s* made sense to these teachers. Another teacher drew a velocity versus time graph and thought of the total distance traveled as the area under the curve. They named the areas "*j*" and "*s*" so that the distance between them was equal to "*j-s*". Some teachers appear to solve the problem for one-second intervals of time (see Fig. 7).

Fig. 7 shows one consequence of having an additive meaning for rate of change ("1 unit of distance for each 1 unit of time"). For those with an additive meaning, speed is the distance travelled in a 1-unit interval (i.e. chunk) of time as opposed to the relative size of the measure of distance travelled and the measure of elapsed time to travel that distance. In interviews and prior research some teachers used the formula $d = rt$ without considering quantitative relationships that this formula entails (Bowers & Doerr, 2001). We included choices (b) and (d) for teachers who might expect to see a product as part of the answer (Bowers & Doerr, 2001). Table 5 Presents the Results for *Relative Rates* by Major.

The majority of the 250 teachers (54%) used an additive model (*j-s*) of a situation that requires a multiplicative comparison. Only 27% of teachers used multiplicative language as well as a multiplicative comparison. There is no evidence of a relationship between

Every second, Julie travels j meters on her bike and Stewart travels s meters by walking, where $j > s$. In *any* given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a. Julie will travel $j - s$ meters more than Stewart.
b. Julie will travel $j \cdot s$ meters more than Stewart.
c. Julie will travel j / s meters more than Stewart.
d. Julie will travel $j \cdot s$ times as many meters as Stewart.
e. Julie will travel j / s times as many meters as Stewart.

Fig. 6. Item called Relative Rates. © 2014 Arizona Board of Regents. Used with permission.

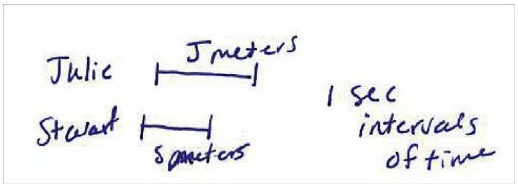


Fig. 7. A teacher's response to Relative Rates. © 2014 Arizona Board of Regents. Used with permission.

Table 5
Responses to item Relative Rates.

Response	Math Majors	Math Ed Majors	Other Majors	Total
j/s times (e)	18	21	30	69
j/s more (c)	4	8	8	20
j-s (a)	36	44	56	136
*j*s* (b or d)	2	6	9	17
Other	2	2	3	7
No response	1	0	0	1
Total	63	81	107	250

response and degree type ($\chi^2(8, n = 249) = 2.63, p = 0.955$). The teacher without a response to *Relative Rates* was excluded from the Chi-square analysis.

4.3. Meaning of "Over"

The phrase "Δy is a change in y over an interval of length Δx" appears commonly in precalculus and calculus textbooks. Here, "over" does not mean division. It means that a change in the value of y happened while the value of x varied through an interval. In early development of the MMTsm, we saw many teachers using the phrase "change in … over …", but it was ambiguous whether they meant "change … while …" or they were describing a spatial arrangement, as in "a over b".

The word "over" often cues the use of slope or rate of change formulas wherein one number or expression is "over" another. This usage pattern allows teachers to choose the operation of division and solve many problems without having to conceptualize rate of change as a multiplicative relationship between changes in two quantities. The responses to the item *Meaning of "Over"* reveal teachers' tendencies to be cued by the word "over" to model a situation with division (See Fig. 8).

Thompson (2016) first presented results to this item to illustrate our methodology for writing items that reveal teachers' meanings. Here, we use it to investigate teachers' meanings for quotient and rate. This article reports additional responses, interview data, and examples of teacher work.

4.3.1. Rubric for part a Meaning of "Over"

The rubric is summarized in Table 6. The highest level response to Part A was "during" or similar. Level A2 responses conveyed the meaning of "over" as "elapsed time" or "amount of time". Teachers who gave level A2 responses noted that "over" was better interpreted as having to do with the passage of time than as division. However substituting the word "elapsed time" into the sentence for over does not make as much as much sense as substituting the word "during." Level A1 responses conveyed that "over" meant division. Responses conveyed that over meant division in a variety of ways including words such as ratio or using mathematical symbols for division. Level A0 responses were about something other than the teacher's meaning for the word "over."

A college science textbook contains this statement about a function *f* that gives a bacterial culture's mass at moments in time.

> The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this statement?

Part B. Express the textbook's statement symbolically.

Fig. 8. The item Meaning of "Over". © 2014 Arizona Board of Regents. Used with Permission.

Table 6
Abbreviated rubric for Part A of Meaning of "Over". © 2014 Arizona Board of Regents. Used with Permission.

A3-During	The response conveys that "over" means "during," or otherwise refers to the passage of time while the culture's mass is changing.
A2-Elapsed Time	The response conveys the meaning of "over" as the equivalent of "elapsed time" or "amount of time" but does not relate it to the culture's mass.
A1-Division	The response conveys that "over" means division.

4.3.2. Rubric for part B Meaning of "Over"

Part B asked teachers to rewrite the sentence in mathematical notation. The rubric focused on whether the teacher wrote a difference or a quotient (See Table 7). Level B2a responses represented a difference and took into account the passage of time in some way (See Fig. 9). Level B2b responses such as $\Delta m = 4$ were considered to be mathematically acceptable, but distinct from B2a responses because the expressions did not incorporate the passage of time.

There were many variations of responses that involved quotients (Level B1). Some B1 responses additionally confounded mass with change in mass or used function notation inappropriately. The response "$f(x)$ = mass/Δtime" was considered B1 because it used division. We emphasized in scorer training that they were not grading a test and that mathematically incorrect statements should be ignored or noted separately if the mistakes were not relevant to categorizing the teacher's meaning for the word over. We scored responses such as "$f(x) = 4$" at Level B0. Typically, we could not categorize level zero responses based on literature or our experiences and we took great pains to ensure any response we could make sense of was not categorized at level zero.

Table 7
Abbreviated rubric for Part B Meaning of "Over". © 2014 Arizona Board of Regents. Used with Permission.

B2a-Difference	The teacher represented the difference of 4 g in the culture's mass at beginning and end of a time period.
B2b- $\Delta m = 4$	The teacher represented a change in the culture's mass, but does not refer to the passage of time. (e.g. $\Delta m = 4$)
B1-Quotient	The response contains a quotient or an algebraically equivalent statement (e.g., $m/\Delta x = 4$, $m = 4\Delta x$).

4.3.3. Results for Meaning of "Over"

One hundred thirteen (113) of 251 teachers (45%) gave the higher-level response of "during" or equivalent (See Table 8).

Table 8 also shows that 71 of 251 teachers (28.3%) said that "over" means division in response to Part A. This is not surprising, because "over" frequently means division in textbooks. However, it is surprising that only 18 of 113 teachers (15.9%) who said that "over" means during also represented the statement as describing a change in mass, and that 46 of 113 teachers (40.7%) who said that "over" means during used division to represent the statement symbolically. The latter teachers' meanings for quotient did not contradict their notion of duration even though from our point of view one concept is multiplicative and the other is not. They were willing to say that a quotient of a change in mass and a change in time produces a change in mass.

We suspect that these teachers' meanings for quotient are not multiplicative, that they see the vinculum as a symbol used to separate two numbers that are related to each other. They evidently experience no conflict in describing "over" as duration in words and as division symbolically. Furthermore, 40 of 113 teachers (35.3%) who appropriately described over as meaning "during" gave a level zero response when they attempted to represent the sentence symbolically.

The response from the teacher Naneh in Fig. 10 conveyed that the word "over" was part of the definition of slope, "change in y over change in x," and thus meant division. Naneh's work suggests that she translated parts of the sentence to mathematical symbols using key words. For example she seems to have written "$\Delta x = 4$" as a direct translation of the last four words of the sentence. This translation does not take into account that, in the context of the complete sentence, Δx refers to an unspecified interval of time and four is a number of grams.

Naneh's marks suggest strongly that she parsed the statement as (The change in) (the culture's mass over the time period Δx) (is 4 g). Other teachers parsed it as (The change in the culture's mass) divided by (the time period Δx) is 4 g. The only way for teachers to avoid either reading is to constrain themselves by the fact that the result is four grams, and not four grams per time unit.

Notice that in Part B of Fig. 10, Naneh's symbolic expression equates a quotient of two extensive quantities with four grams. We interviewed three teachers who said that a number of grams equals a quotient of mass and time. One expressed concern and two did not. When a teacher's meaning for slope is primarily focused on the change in y, it allows room for him or her to understand the statement "Δmass/Δtime = 4 g" unproblematically. To them, the vinculum does not signal a measure of relative size of changes. Rather, the vinculum signals that mass changed and time changed. There is no evidence that having a math or math education degree was a statistically significant predictor of representing the statement with subtraction or with division ($\chi^2(8, n = 250) = 9.756, p = 0.282$).

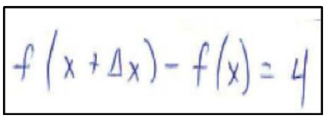


Fig. 9. One type of higher-level representation of the situation in Meaning of "Over".

Table 8
Responses to Part A and Part B for Meaning of "Over".

Response	B2a-Difference	*B2b $\Delta m = 4$*	B1 Divide	B0 Other	NR/IDK	Total
A3 "during"	12	6	46	40	9	113
A2	1	1	14	15	2	33
A1 "divide"	0	4	67	0	0	71
A0	1	1	6	5		13
NR/IDK	1	0	0	0	20	21
Total	15	12	133	60	31	251

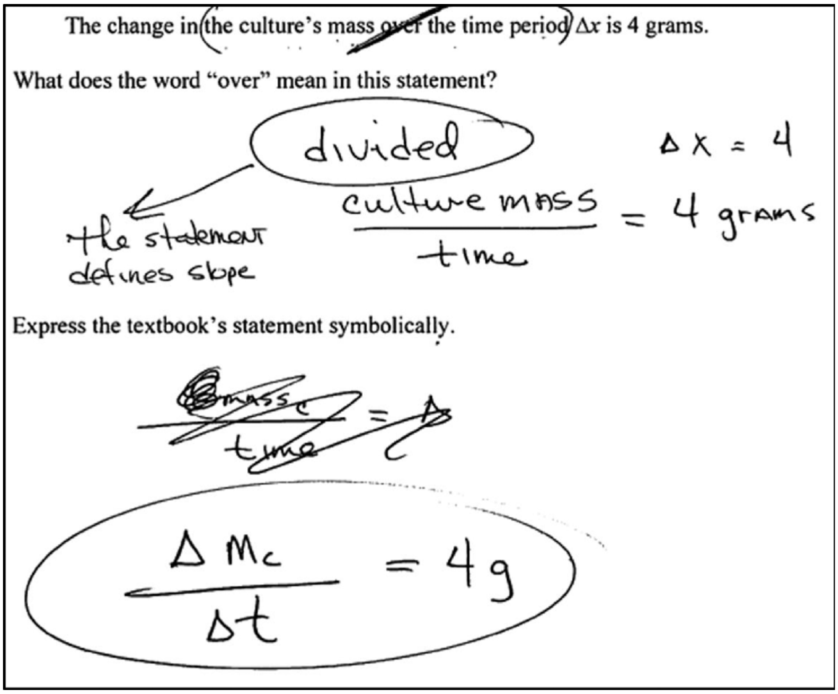


Fig. 10. Naneh's response to Meaning of "Over" which conveys "over" is a key word indicating division.

4.4. Slope from blank graph

We designed the item, *Slope from Blank Graph* in Fig. 11 to see the extent to which teachers would use a meaning of slope as a relative size of changes in y and changes in x to estimate a numerical value of slope given a graph without labeled axes. We believe that if a teacher "understands the quotient $\Delta y/\Delta x$ as the measure of Δy in units of Δx, then he or she would be more likely to estimate the numerical value of m simply by physically measuring Δy using Δx as a unit" (Thompson, 2016, p. 443).

Thompson (2016) reported 96 teacher responses to this item from a pilot sample. Fifty-two percent of the teachers in his sample

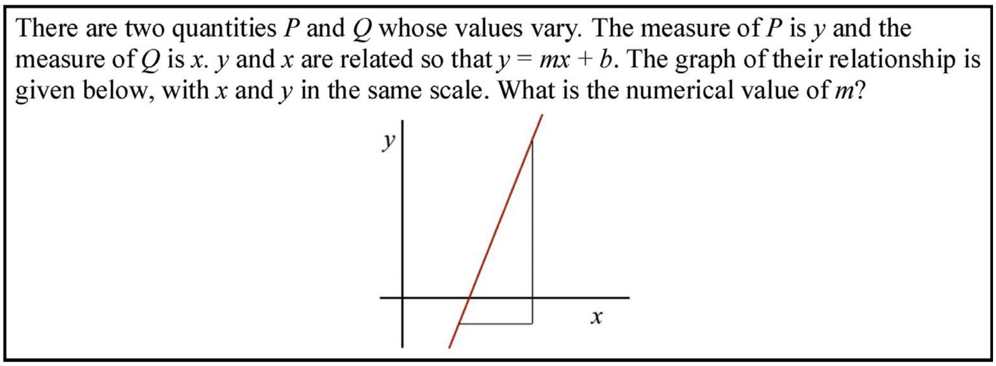


Fig. 11. Item *Slope from Blank Graph*. Diagram is larger in actual Item. © 2014 Arizona Board of Regents. Used with permission.

Table 9
Rubric for Slope from Blank Graph. © 2014 Arizona Board of Regents. Used with permission.

3-Accurate approximation	*Any* of the following: – The teacher estimated a value between 2 and 3.
2-Formula	*Any* of the following: – The teacher solved for m in $y = mx + b$, getting $m = (y - b)/x$. – The teacher wrote a mathematically valid formula that could be used to determine the value of the graph's slope, such as Δy/Δx.
1-Inaccurate approximation	The teacher gave an estimate for the slope smaller than 2 or larger than 3.

gave an approximation of slope between two and three (the actual slope is 2.5). Being able to estimate slope from a blank graph is associated with higher scores on calculus tests about rate of change functions (Byerley, in preparation).

The Aspire team scored the responses to *Slope from Blank Graph* using the rubric in Table 9.

Responses scored at level three often conveyed that the numerical value was approximate and included explanations such as "it appears the length of the vertical changes is 2.5 times the horizontal change." If a teacher gave both a formula and a numerical approximation they were scored at level three. No teachers in our sample computed a slope by assigning points to the graph using an arbitrary scale and then using the slope formula with those two points. The teachers with accurate approximations measured the change in y in the same measurement units as the change in x without subtracting x and y coordinates of two points.

Teachers who gave estimates outside of the range two to three provided a variety of reasons; their reasons were generally not appropriate in the context of the item. For example one teacher said "looks like $m = 1$ since nothing in front of m." Another said "$m = ½$ because that is what is missing from the equation if you solve for area of a triangle".

Thirty-three of 158 high school teachers (21%) estimated a value of the slope from two to three. Consistent with findings by Coe (2007) and Stump (2001), almost half (71 of 158) of the teachers provided a symbolic formula for slope (Table 10).

In addition to categorizing responses with the rubric we also noted whether the teachers' formula was of the form $\Delta y/\Delta x$ or y/x. We considered the distinction between y and Δy important because in Thompson's (1994b) study of advanced college mathematics students' understanding of the Fundamental Theorem of Calculus, students who failed to distinguish between amount and change in amount struggled to understand the relationship between accumulation and rate of change. We found that 39 of 158 high school mathematics teachers (25%) confounded the change in a quantity's value with the quantity's value. The slope of the line can only be calculated with the formula y/x when the line intersects the origin. On many items teachers compared two amounts when a comparison of two changes would have been appropriate for the situation.

We did not interpret lack of response as a teacher not understanding the question. This item was near the end of the instrument; 28 of the 32 teachers who left the item blank also left questions after it blank. In an earlier pilot of the MMTsm, this item was not at the end of the instrument and only 4 of 112 teachers left the question blank.

4.4.1. Item validation for slope from blank graph

In six item interviews we asked teachers and calculus students what "numerical value" meant and everyone responded that we meant a number, not a formula. For example, one calculus student said the question asked her "to actually find a number, but I don't know how I'm going to do that right now." This student knew an estimate was acceptable but could not provide one. One teacher wrote on his test, "no numbers are given so no numerical answer can be found" and another wrote, "I don't see any numerical values." A third teacher wrote "how can I find a numerical value when there are no numbers?" We did not have any interviews where someone said they were confused about the meaning of numerical value.

Some of the 71 teachers who gave a formulaic response (Level 2) might have been able to estimate the slope had they been pressed to do so by an interviewer. We can not infer from this item that teachers who did not provide an estimate could not provide an estimate if pressed to do so. However, their tendency to give a formula when asked for a numerical value has implications for teaching. We considered rewriting the item to make it clearer we were asking for an estimated numerical value, but we wanted teachers to be free to express the meanings they might convey in teaching. We were more interested in how teachers would respond to a somewhat vague item than if they could estimate slope from a blank graph after being told that a formula is unacceptable. Giving a formula is not incorrect, but it finesses the issue of slope as relative size of changes and could convey to students that symbolic answers are preferable even in contexts where a question asks for a meaning.

Many calculus students at a large public university have internalized the idea that slope cannot be determined without numbers.

Table 10
Results of Slope from Blank Graph (n = 158).

Response	Total
Level 3: "2 to 3"	33
Level 2: formula	71
Level 1: Inaccurate	5
Level 0 or IDK	17
No Response	32
Total	158

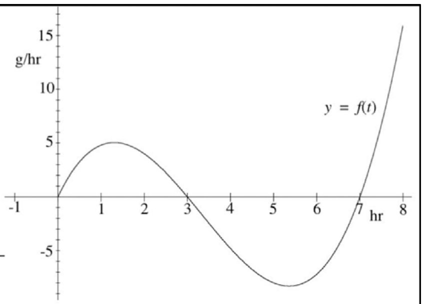
Calculus students were asked to "*estimate* a numerical value of slope" on a course test, after their instructors discussed in class how to estimate slope given a graph with blank axes scaled identically. The calculus test item had a similar image as the MMTsm item but was multiple choice and only one estimate of the slope of the line was remotely plausible. Only 96 of 170 students answered the item correctly even after explicit instruction on how to answer the question.

4.5. Increasing or decreasing from rate

The Aspire team designed the item *Increasing or Decreasing from Rate* to require teachers to differentiate between the idea of an increasing rate of change and an increasing mass (See Fig. 12). This item is distinct from other items because it involves a non-constant rate of change and understanding instantaneous rate of change is useful to interpret the graph.

The values of function f give the rate of change (in grams/hr) of a bacterial culture's mass t hours after measurements began.
Over what intervals within the first 8 hours is the culture's mass increasing? Explain.

a) $0 < t \leq 1.4$ and $5.5 < t \leq 8$
b) $0 < t < 8$
c) $0 < t < 3$ and $7 < t \leq 8$
d) None of the above. My answer is ____________________
e) I don't know



(Part's B and C appeared on teachers' next page)

Part B. The graph from the prior page is repeated below. Highlight the point (2.5, 2.25) on the graph of f. What does this point represent?

Part C.

Would you like to change your answer to the question on the prior page? Make the appropriate selection.

a) $0 < t \leq 1.4$ and $5.5 < t \leq 8$
b) $0 < t < 8$
c) $0 < t < 3$ and $7 < t \leq 8$
d) None of the above. My answer is ____________________________
e) I don't know
f) I do not want to change my answer.

Fig. 12. Item Increasing or Decreasing from Rate. © 2014 Arizona Board of Regents. Used with permission. (Spacing reduced and graph for Part B omitted to reduce size).

The purpose of Parts B and C, displayed on the next page after Part A, were to probe the possibility that teachers interpreted the graph in Part A as an amount function and picked (a) just because they did not attend to the unit labels on the axes—they did not realize the graph was for a rate of change function.

In item-interviews during development of the MMTsm, some teachers showed a persistent tendency to confound mass with rate of change of mass due to their meanings for rate of change. If someone considers a rate of change to be a change in mass, it is more difficult to differentiate between an increasing bacterial mass and an increasing rate of change of bacterial mass with respect to time. In some teachers' minds the rate of change of bacteria and the amount of bacteria are both extensive quantities directly tied to the number of bacteria. Secondary mathematics teacher Annie said that the function is a rate of change function but associated a positive slope on the rate of change graph with a positive rate of change. Annie also associated slope of zero on the rate of change graph with a value of zero for rate of change (See Excerpt 1).

Excerpt 1. Annie's Explanation of Increasing or Decreasing from Rate.

[Annie reads problem aloud, emphasizes *grams/hour.*]
We interpret increasing. umm…let's see the function gives the rate of change in grams per hour… and so umm…what we are going to look at I would look at the rate of change being positive or negative, if we have a positive rate of change the grams per hour the mass is increasing per hour, is getting larger, so I look at where I have a positive rate of change, and I try to identify where I have no rate of change [*highlights maximum at (1.25, 5) where the rate of change is approximately positive 5, but the acceleration is zero*], this is telling me where the mass is staying the same, and then I have a negative slope so mass is getting small down to a zero rate of change so I'm not getting any smaller or larger…
[Teacher chooses (a)]
Interviewer: So just a little bit ago you said on the interval from 0 to 1.25 you said the change in the rate of change was positive.

Annie: Positive, right.
Interviewer: So the change in the rate of change of the bacterial culture's mass was increasing, so that meant that the mass was increasing as well?
Annie: Right, right.

Annie's realized that she was looking at a rate of change graph. She tended to think of the slope of the rate of change graph at a point as more relevant to the problem than the value of the rate of change function. Her agreement that an increasing rate of change corresponded to an increasing mass, and that the mass decreases where the graph's slope is negative, is consistent with a meaning for rate of change of mass that is non-multiplicative and easily confounded with the idea of change in mass.

Table 11 shows the results on *Increasing or Decreasing from Rate*.

Eighty-seven of 239 teachers (36.4%) chose (c) on Part A. Only one of these 87 teachers changed his or her mind on Part C. On the other hand, 105 teachers kept their same non-(c) response after being asked to focus on the meaning of a point in Part B. This suggests that teachers' choices of non-(c) answers were not due to inattention to quantity labels on the axes.

Table 12 focuses on 153 teachers who did not choose (c) on Part A. Fifty (50) of these 153 teachers said that the point represented a rate of change at a moment in time. Of those 50, only 26 changed their answer to (c). The other 24 of these 50 teachers chose other than (c) even after stating that the point represents a rate of change at a moment in time. This suggests that although Part B drew their attention to the axes' labels, their meaning for graphs and rate of change did not help them select (c).

Ninety-eight (98) of these 153 teachers (62%) identified the point in Part B as representing something other than a rate of change at a moment in time. Of these 98 teachers, 12 changed their answer to (c). We suspect that these 12 teachers' choices were motivated by something other than realizing the implications of the graph being of a rate of change function.

Descriptions of teachers' thinking from qualitative studies helps us hypothesize reasons for the teachers' responses (Coe, 2007; Thompson, 1994b). Teachers who consider slope as an index of slantiness and not as a comparison of two changes could relate the word "increasing" with a graph that is slanted up. It is possible many teachers consider the shape of the graph instead of the quantities that covaried to make it (Moore & Thompson, 2015). Having an additive meaning for rate of change makes it more difficult to distinguish between a rate of change and an amount of change and could have caused Annie to say that a negative slope in the graph of rate of change of mass with respect to time implied a decreasing mass (Excerpt 1).

Table 11
Results for Part A and Part C on Increasing or Decreasing from Rate.

	Response to Part A						
		Chose (c) on Part A	Chose (a) on Part A	Chose (b) on Part A	Other/IDK	Blank	Total
Response to Part C	Chose (c) on Part C	86	35	0	3	0	124
	Chose (a) on Part C	1	77	0	1	0	79
	Chose (b) on Part C	0	3	11	1	0	15
	Other/IDK	0	2	0	17	1	18
	Blank	0	0	0	0	1	1
	Total	87	117	11	22	2	239[a]

[a] There are only 239 teachers in this table because one group took a shorter version of the MMTsm that did not include *Increasing or Decreasing from Rate*.

Table 12
Responses to Part B and Part C by teachers who chose other than (c) on Part A of Increasing or Decreasing from Rate.

	Point means Rate of Change at Moment in Time	Other Meaning of Point (e.g. Mass at Moment in Time)	Blank/IDK	Total
Chose (c) on Part C	26	12	0	38
Chose (a) on Part C	1	0	0	1
Chose (b) on Part C	3	1	0	4
Other/IDK	16	85	4	105
Blank	0	0	1	1
Total	50	98	5	153

5. Looking across items

In this section we look across items for consistencies and inconsistencies in teachers' meanings and ways of thinking about slope and rate of change. The higher-level meanings on each rubric are related to an image of rate of change as multiplicative comparison of changes in two quantities. The lower levels of the rubric arose from grounded coding of teachers' responses and are not the same across rubrics.

5.1. Relationship between measurement responses and rate of change responses

Coe's (2007) interviews suggested that teachers' meanings for rate of change are loosely linked to their measurement schemes. This section investigates the hypothesis that teachers with weak measurement schemes do not convey multiplicative meanings for

Table 13
Responses to slope from blank Graph part A compared to number of higher-level responses to measurement items.

	Number of Higher-level Responses to Two Measure Items			
	Higher-level Responses to 2 items	Higher-level Response to 1 item	Higher-level Responses to 0 items	Total
Estimate “2 to 3”	14	11	8	33
A2 “formula”	16	22	33	71
A1/A0/IDK	2	4	16	22
No Response	7	12	13	32
Total	39	49	70	158

slope and rate that are based on a measure of the relative size of two changes.

We use teachers' responses to *Gallons to Liters* and *Nerds to Raps* (Figs. 2 and 3) to quantify the strength of their measurement schemes. These items required teachers to convert between two units given a conversion factor. In this analysis we will group all low-level responses to *Gallons to Liters* and *Nerds and Raps* into one category. Ten of the 251 teachers left *Gallons to Liters* blank but answered *Nerds to Raps.* We replaced their score of “No response” with a zero (low-level) because *Gallons to Liters* was not near the end of the test and the majority of teachers had enough time to finish the entire test.

Table 13 compares teachers' tendency to estimate a numerical value of slope by measuring Δy with Δx to their success on two measurement items.

Scanning the columns of Table 13, we see that 35% (14/39) of teachers who answered two measurement items estimated an appropriate numerical value for slope. Only 5% (2/39) of teachers who gave two higher-level responses to measurement items said something incoherent on *Slope from Blank Graph* (Levels A1/A0/IDK). In contrast, only 11.4% (8/70) of teachers who gave two low-level responses to the measurement questions estimated the slope of the line appropriately. Although some teachers with two low-level responses on the measure items provided acceptable formulas for slope without a numerical estimate, 22.9% (16/70) said something mathematically incoherent about the slope of the line. The association between responses to *Slope from Blank Graph* and responses to the two measurement items is statistically significant (JT = 4739.50, z = 2.41, p = 0.0079). This result is consistent with the hypothesis that being able to imagine a measurement process was important for estimating slope from a graph with blank axes.

We briefly summarize the other results of comparing teachers' measurement responses to their rate of change item responses as the tables were similar. The teachers with higher levels on the measurement tasks were more likely to convey multiplicative meanings on other items. There is a statistically significant association between the number of higher-level answers to measurement problems and responses to *Relative Rates* (JT = 11538.00, z = 2.60, p = 0.005), *Increasing or Decreasing from Rate* (JT = 11417.50, z = 4.3497, p < 0.0001), and *Meaning of “Over”* (JT = 11835.00, z = 3.15, p = 0.0017).

The correlations are consistent with the hypothesis that meanings for measurement ideas are important for productive rate of change meanings. In all cases, teachers' responses to two middle school items were more predictive of their rate of change meanings than holding a degree in mathematics or mathematics education. Our study cannot show that teachers developed more productive rate of change meanings because they had stronger measurement schemes. However, had we seen no relationship between responses to the items we could have inferred that something was wrong with our hypothesis or our items. As with the rest of the item responses, it was never the case that a high score on one item was highly correlated with a high score on another item. There were statistically significant correlations, but the correlations were not high enough to be meaningful. On the whole, the tables typically demonstrated a lack of connections between teachers responses to items.

5.2. Limitations of chunky, additive meanings for slope and rate of change

A number of teachers conveyed chunky, additive meanings for slope and rate on *Meaning of Slope* and *Relative Rates.* Few teachers used language that conveyed measuring one change in terms of another in slope or rate contexts. Further, the results of the two measurement items suggested that many teachers have measurement schemes that seem unlikely to support a meaning for rate of change that involves measuring one change in terms of another change. This section uses qualitative data gathered as part of rubric and item validation to discuss the consequences of the prevalence of chunky, additive meanings for slope and rate in our sample.

We hasten to say that our comments in this section are epistemological. We do not mean them as criticisms of teachers' mathematical meanings. Instead, we address the implications for teaching and for coherent understanding that particular meanings and ways of thinking hold.

5.2.1. Slope is the change in y

On *Meaning of Slope* some teachers confounded Δy with slope. Kristen, a HS math teacher who has taught algebra nine times conveyed a chunky meaning for slope on Part A (Fig. 13) because the changes occur in chunks of one and 3.04. She has a MA in educational leadership and a BA in elementary education.

The Part B response in Fig. 13 provides confirmation that, for Kristen, 3.04 is more strongly associated with the change in y than with a comparison of the relative size of corresponding changes in y and x. The response in Fig. 13 foregrounded the change in y and kept the change in x in the background. Kristen's additive meaning for slope that confounded Δy and Δy/Δx prevented her from

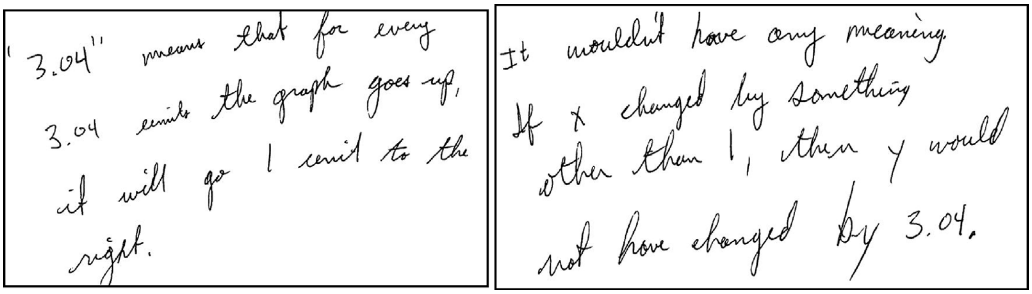


Fig. 13. Kristen's chunky response to Part A (Left) and Part B (Right) of Meaning of Slope.

explaining the meaning of 3.04 for non-unit changes in *x*.

Rani's response (Fig. 14) conveys that the meaning of slope is dependent on a one unit change in *x*. Rani had a BA in mathematics education and a MA in secondary education. She taught Algebra I and Algebra II 15 times each, Precalculus nine times, and Differential Equations nine times.

Labeling the change in *y* as *m* conveys that slope is not a comparison of two changes, but rather the change in *y* that is associated with a one unit change in *x*. Many teachers, like Kristin (Fig. 13), did not give a meaning of a slope of 3.04 when *x* did not change by one.

5.2.2. Slope is the distance between two points

Chunky thinking leads to a variety of problems beyond confounding Δy with Δy/Δx. Nari's chunky responses conveyed that the only points on the line that "mattered" were the points obtained by the process of moving over and up in fixed chunks (see Fig. 15). Nari is a HS teacher with a BA in mathematics education. She taught algebra once and Geometry once. Nari's response is inconsistent with imagining that between any two points on the line there are infinitely many points.

There are many consequences of thinking that points on the line only occur at fixed intervals. If points only occur at fixed intervals it is possible to conceptualize slope as the distance between two points on a line. Mike explicitly said that the slope is a distance between two points (see Fig. 16).

Mike's response in Fig. 16 conveys that slope gives directions on how to get from one point to the next and that 3.04 is a distance. Unfortunately, Mike's meaning for slope as distance overpowered the visually obvious fact that the hypotenuse of the triangle is longer than either leg. We suspect that Mike's meaning for slope as distance arose from taking the hypotenuse of a "rise over run" triangle as itself being the slope of a line. Mike taught Algebra I five times, Algebra II twice, and Geometry three times. He also has a PhD in education.

The tendency to isolate the meaning of slope from a quantitative meaning for division (Coe, 2007), is a potential explanation for viewing slope as a distance. Daniel, a university calculus student interviewed on *Meaning of Slope* as part of its rubric's validation, repeatedly explained that slope was a distance. In the same interview Daniel explained that "A divided by B" means "the amount of B's that would fit into A" and he knew the slope formula contained division.

Excerpt 2 shows that Daniel did not connect his meanings for division with his meanings for slope and that this disconnect allowed him to think of slope as a distance.

Excerpt 2. Daniel Explains his Meaning for Slope of 3.04.

Interviewer: So the slope is the length between the two points.
Daniel: Right.
I: Okay. So why do you divide the change in *y* and the change in *x* to get a length?
D: Because, it's… you've got the one *x* here and the other one here and so you are trying to find the way which they both get to each other basically. That's…
I: Okay. [Daniel laughs] Is that at all related to seeing how many B's fit into A or is that like a separate thing in your

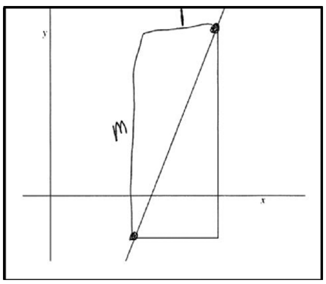


Fig. 14. Rani's response that conveys that slope is the change in y.

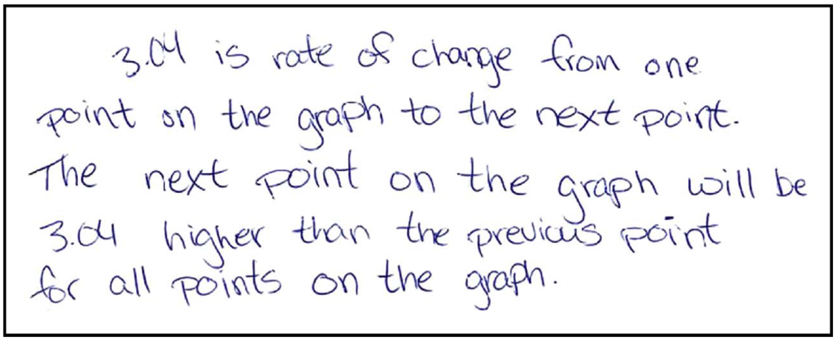


Fig. 15. Nari's chunky response to Meaning of Slope conveys that the points on the line only occur at fixed intervals.

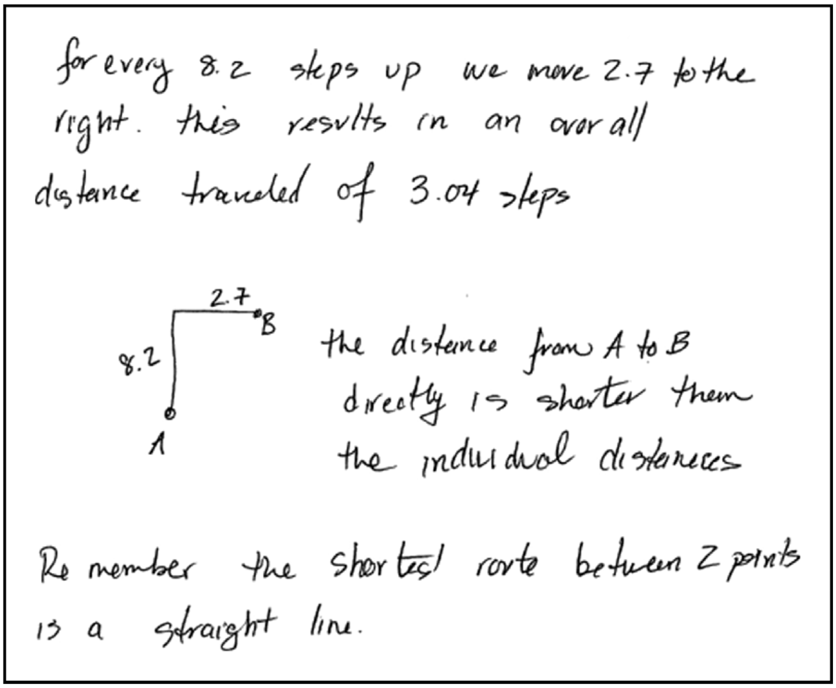


Fig. 16. Mike's additive response to Meaning of Slope conveys that slope is a distance between two points.

brain?
D: If you are doing the slope it's different, I guess, I'm seeing it different in my brain, I guess it is because of the word slope gave this a different meaning.
I: What does the bar in between them mean to you?
D: I just… divide [laughs]
I: Alright. It's just that you were not using the how many times B fits into A language at all when describing the slope so that is why I was asking.
D: Yeah. No, not with slope.

Daniel connected slope and division after subsequent instruction, but the meaning he carried from high school into calculus was that the meaning of division in the slope formula differed from the meaning of division he learned in school. A teacher who conveys an additive, chunky meaning for slope allows students to assimilate the teacher's instruction without connecting the idea of slope of a line to the idea of a quotient of changes. Instead, they think that the numerator gives the number of steps to take in one direction and the denominator gives the number of steps to take in another directions, and the vinculum merely separates them.

5.2.3. Teachers' slope and rate meanings are tied weakly to quotient

Other interviews confirm that some teachers did not connect their meaning of the vinculum in formulas for slope with their meaning of division. Ross has a BS in construction and a MSE in industrial engineering. He taught Algebra II for ten years, Precalculus for seven years, AB Calculus for 2 years and AP Statistics for 7 years. We asked Ross, "Why do you divide to calculate the slope of the line?" Ross first repeated the definition of slope to himself and then explained:

Excerpt 3. Ross's Explanation of the use of Division in the Slope Formula.

> The division can be used because we talk about the slope being the average rate of change and "average" is the total [taps fingers] uhh… the total of the observations divided by the number of observations. And then of course we have to make a distinction of what is… what we interpret the total observations and the number of observations, so if I want to talk about… i.e., for example, total distance traveled by some total time during the travel so umm…total distance traveled would be delta distance over delta time but we can also see this also as a ratio of the two differences.

Ross associated slope with average rate of change and then average rate of change with arithmetic mean. He appeared to connect slope and division using the arithmetic mean formula. It is fairly common for secondary teachers to think average rate of change is computed using an arithmetic mean (Yoon et al., 2015).

The interviewer pressed Ross to explain his meaning for ratio in Excerpt 4.

Excerpt 4. Ross's Explanation of his Meaning for the Word "Ratio."

> The problem I'm (pause) unfortunately from the different stuff I've been looking at now, we can unfortunately use the word ratio unfortunately to mean both a comparison of two different units to each other but also the terminology as a fraction a part to whole, there is this way we use ratio and fraction together, but the idea is you know, I'm just trying to get at the idea of what slope is, it is this ratio, for every change in this I do the change in this. The reason I don't want to use the word fraction for this is because again we think of a fraction having the same units. You know, one fourth of a pie, two thirds of a gallon. But when I talk of a ratio of distance to time those are two different units. And so it's for every change of this element I have a change in the other element. For every one hour, change of one hour I drive down the road I go an additional 65 miles down the road, so it is not a fractional concept like we normally think of it, it's part to whole, because unfortunately when we see division you can also interpret that as fraction. But the fact is because we don't have the same two units, it doesn't fit as neatly with some of those notions we've been taught about the differences between fractions and ratios.

Ross knew common meanings for ratio such as 'comparison of two quantities with different units' and fraction such as "part of whole." Ross' statements about "the different stuff I'm looking at now" and "notions we've been taught about differences between fractions and ratios" suggest that he had recently been led to consider distinctions in meanings for fractions and ratios. He saw that in the case of slope, the comparison was often between changes in quantities measured with different units and this made him uneasy about his idea that ratios, slope, division and fractions all involve comparisons. Focusing on the units being the same or not appeared to keep Ross from elaborating on the structural similarities of various situations that use division. When two quantities are measured in the same unit their sizes can be compared directly. When two quantities are measured in different units, such as miles and hours, the measures of the quantities can be compared.

Some responses to *Meaning of "Over"* illustrated how Daniel and Ross might have used slope hundreds of times to get a correct answer without needing to connect it to their meaning for quotient. The response in Fig. 17 conveys that the vinculum in a rate of change formula merely separates the numbers that tell how to move on a Cartesian graph.

One problem with a disconnect between the idea of slope and quotient is that it allowed teachers to comfortably model one situation in *Meaning of "Over"* in two inconsistent ways. Many teachers understood that the word "over" could mean duration in some contexts but still believed the word over in symbolic contexts should be converted to a spatial arrangement of symbols.

James thought the word "over" could mean both division and duration in the same context. James had a BS in mathematics education and taught Algebra II, Geometry, and Precalculus two times each. We interviewed James to better understand why someone would say over meant both during and divide in the same statement. The interviewer first asked James to respond anew to a blank version of the item. We interviewed James six months after he took the MMTsm.

Excerpt 5. James Discusses Part A of Meaning of "Over".

> James: [Reads question carefully aloud.] [Over means] during or duration. You could also think of it as a ratio, so change in mass over, yeah so during or duration, so in your math class when they say "something over something," they always mean a divide sign so a ratio.
> I: Do you think they are both saying the same thing?
> J: Well, yeah, I think that. Well yeah, they are saying. I think the during or duration is more saying conceptually what is going on, and the divided by or over I see the reason behind that, I think I'm more pointing out mathematically what we mean when we say over with no explanations as to why, it is just the way it is.
> I: So is the mass, the change in mass divided by the change in time, is that how you write the idea of duration?

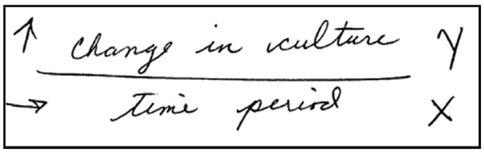


Fig. 17. Response to Meaning of "Over" where the vinculum merely separates two numbers that tell how far to move horizontally and vertically.

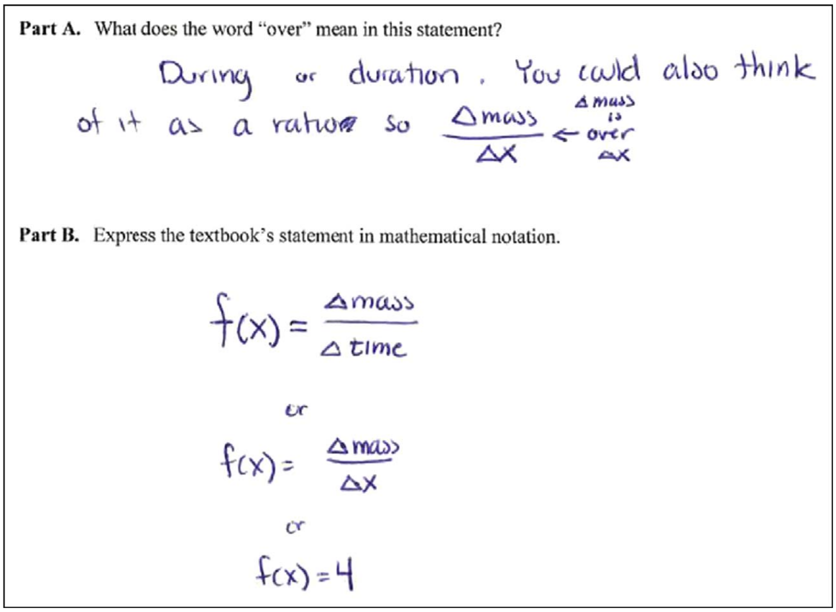


Fig. 18. James' original response to Meaning of "Over".

J: Can you repeat the question?
I: Is the "delta mass divided by delta *x*" a mathematical way of saying duration?
J: I want to say the change in *x* is the way of saying duration. I want to say the change in *x* is representing duration. But maybe we could include the division sign. So no, I would not say that "delta mass over delta *x*" is a way of saying duration. So this is funny.

On Part A James gave a remarkably similar response on the written test and the interview conducted six months later. After James responded to a blank item the interviewer showed him his original response in Fig. 18.

James recognized his problematic use of function notation in his original written work. However, he retained his use of division in his algebraic representation of the statement (Excerpt 6).

Excerpt 6. James' Interview on Part B.

J: [James read Part B. James looked at his old answers from when he took the MMTsm six months earlier, found them all problematic and crossed them out and explained he used function notation incorrectly.]
I: What would you say today?
J: I like the idea of the function. I would keep the function.
I: You can work it out on a paper.
J: Yeah, just give me a second. The change in the culture's mass…[pause]
J: Change in mass over [divided by] change in *x* equals 4. That would be my new thing.
I: Four what?
J: Four grams. [James showed no discomfort with a quotient being equal to 4 g]

James accepted that "over" could mean both divide and during in the same situation by saying that "during" is the conceptual meaning and "divide" is the mathematical meaning. Even though James realized that divide and duration are not expressed in the same way mathematically, he did not say that a quotient being equal to four grams was problematic—even after the interviewer called attention to the units. James's understanding that "over" always means divide in mathematical situations (because "that is just the way it is") was so prominent that he kept the division sign despite the inconsistencies he noticed and discussed. A combination of James' strong association of "over" and division, and his additive meaning for quotient, allowed him to inappropriately model an additive situation with division.

Taken as a whole, the responses to this study's rate of change questions show a common lack of connection between a meaning for quotient as a measure of relative size and the meanings for slope and rate. The interviews show how this lack of connection might contribute to teachers' difficulties in helping students develop coherent meanings for slope and rate of change.

6. Conclusion

6.1. Why attend to shortcomings in teachers' meanings?

Most responses to the rate of change items in this article suggest that teachers held meanings for slope that are only useful in limited circumstances. This strongly suggests a need for attention to these ideas in preservice secondary mathematics education programs and additional professional development for inservice high school teachers. Only four percent of teachers described slope as involving multiplicative comparisons between two changes on *Meaning of Slope*. Many teachers gave chunky (77.6%) and formulaic (15.2%) descriptions of slope that are suitable in restricted contexts often encountered in textbooks. We found substantial qualitative and quantitative evidence in our study that these additive and computational meanings were unproductive in a variety of situations. We also found that some students and teachers with chunky or formulaic meanings for slope had other problematic meanings, such as slope is the distance between two points on line.

The data in this article demonstrates that many teachers have meanings for slope and rate of change that work poorly in many situations that are highly related to the secondary mathematics standards they are teaching. Approximately half of teachers appropriately determined where the mass of a bacterial culture was increasing given a rate of change of bacteria with respect to time graph. Only ten percent of teachers appropriately used subtraction instead of division to model a change in mass. Twenty-one percent of teachers applied the concept that a fixed quantity will have a larger measure when measured with a smaller unit on both measurement problems. We also presented evidence that computational approaches to the measurement problems were correlated with problematic responses on all rate of change items. Much of the qualitative work done by us and other researchers suggests that secondary teachers have meanings for quotient that are productive in limited circumstances and that their meanings for rate of change and slope are weakly tied to their meanings for quotient.

We emphasize that this study is not about teachers' misconceptions. We agree with Smith et al. (1993) that "now that misconceptions are recognized as a pervasive phenomenon in mathematics and science learning, research that simply documents them in yet another conceptual domain does not advance our understanding of learning" (p. 125). Rather, our focus is on meanings that teachers have for mathematics they teach and their potential productivity for students' learning.

Although Smith et al. explained why the field should move beyond researching students' misconceptions they also explained the ways in which this research was a useful starting point. Research documenting unproductive and incoherent student thinking demonstrated the scope of the problem that needed to be addressed and prompted the development of curriculum, instruction, and research methods to improve student learning. We hope that faculty who teach mathematics and mathematics education majors will be prompted by our data to investigate and attend to their undergraduates' mathematical meanings for rate of change, slope, and measurement. Mathematics faculty have been uniformly surprised by their mathematics and mathematics education majors responses to MMTsm items. They are surprised that their students' answers strongly resemble responses from teachers in our study.

Although rate of change is a foundational concept in calculus and important higher mathematics courses such as differential equations, earning a degree in mathematics and mathematics education was not correlated with higher-level responses to any of the items in this article. This suggests that a teacher can pass university-level mathematics classes, teach thousands of secondary textbook problems, and still struggle with meanings that are foundational for ideas they teach.

It is important to remind the reader again that our sample of 251 high school mathematics teachers was a convenience sample. It was not selected randomly. We therefore cannot claim that the sample is representative of any specific population. However, the sample is large for an educational study and the teachers in it were from many regions in their states. We therefore feel that there is a distinct possibility that the meanings we found are common within the United States, and therefore that it is possible that a high percentage of high school mathematics teachers are likely to convey meanings for slope and rate of change to students that are useful for solving common high school textbook problems, but not to understand situations quantitatively.

6.2. Cultural regeneration of mathematical meanings

The results of our study, if they apply broadly, demonstrate a serious problem with U.S. high school mathematics teachers' mathematical meanings for teaching secondary mathematics. Results regarding other areas assessed by the MMTsm show that the problem is much broader than meanings for slope and rate of change (Musgrave & Thompson, 2014; Thompson, Hatfield, Byerley, & Carlson, 2013; Thompson et al., 2017; Thompson & Milner, in press; Yoon et al., 2015).

We urge readers not to view these results as a condemnation of teachers' capabilities, but rather as pointing to a systemic, cultural problem within U. S. mathematics education. Our cautionary note is in line with Stigler and Hiebert (1999), who saw the results of their video study as providing insights into cultures of teaching rather than as a critique of individual teachers.

We interacted with many of the teachers who responded to our instrument. They all were highly motivated to improve their mathematics teaching, which was their reason for participating voluntarily in NSF Math/Science Partnership professional development programs. They wanted to talk about mathematics, and many teachers were disturbed that the questions we asked made them aware that their meanings were not sufficient to provide satisfactory (to them) answers. They also agreed that the meanings emphasized in the MMTsm are important for their students to develop. We therefore do not see our results as pointing to teachers' individual failings. In fact, the first author was a secondary teacher who, despite a commitment to teaching conceptually, primarily conveyed chunky and formulaic meanings for slope and rate of change to her students. She would have appreciated the chance to learn about the limitations of the mathematical meanings she conveyed but this issue was not discussed in her mathematics education classes nor in her professional development experiences.

The reason we believe that our results point to a systemic, cultural problem in U.S. mathematics is that the higher-level meanings we identified are rarely an explicit part of school mathematics or undergraduate mathematics curricula. Most mathematics and mathematics education majors take classes such as calculus and real analysis; textbooks for these classes commonly do not revisit quantitative meanings of measurement, slope, or rate of change.

Based on personal interactions with teachers in our sample, we believe most of them want to teach well and worked hard in college to learn the mathematics that was expected of them. We believe the problematic meanings teachers expressed on our instrument are meanings that they developed as school students and became reinforced by their experiences in teaching from mathematics textbooks that support, directly or inadvertently, the same meanings that they developed as students.

Our results are related to what Lortie (1975) described as the cultural regeneration of schools. Lortie claimed that school students who identified positively with their schooling and with their teachers were likely to enter teaching, thus regenerating for future students the schooling experiences they internalized. While our data says nothing about why teachers enter teaching, it gives another perspective on the issue of cultural regeneration. It seems quite plausible that a process like the following regenerates the problem of mathematical meaning in U.S. school mathematics:

- Many students leave high school with poorly formed meanings for ideas of the middle- and secondary-school mathematics curriculum.
- Students take mathematics courses in college that are designed with the presumption that students have basic mathematical meanings they in fact do not have.
- Instructors of these college mathematics courses presume students have basic mathematical meanings they in fact do not have, or do not know how to address students' basic mathematical meanings given the constraints of the course.
- Students apply coping mechanisms (e.g., memorization) in college mathematics that allowed them to succeed in high school.
- Students return to high schools to teach ideas they understood poorly as school students, rarely revisited in college, and for which they still have poorly-formed meanings.

It is beyond the scope of this article to address the systemic problem we've described. Thompson (2013) outlines one research and political agenda that addresses this issue partially. We believe a cultural regeneration cycle can only be broken with

- sustained, intensive professional development that is aimed helping teachers develop and teach for productive mathematical meanings,
- parallel efforts to redesign preservice high school teacher preparation programs, and
- a refocus of undergraduate mathematics programs on having students learn mathematics meaningfully.

The professional development effort, in our opinion, must also focus on helping teachers select curriculum materials that cohere with their effort to re-conceptualize their mathematics in terms of supporting students' construction of coherent mathematical meanings. This effort also should be informed by future research. We end with a list of potential research questions:

1. Is this sample representative of teachers in other states?
2. What is the relationship between what teachers convey on the written instrument and what they convey in teaching?
3. Do teachers with productive mathematical meanings emphasize productive mathematical meanings in their instruction?
4. Do teachers who orient their instruction to producing coherent mathematical meanings have students who learn mathematics coherently?
5. To what extent can an individual teacher have a positive impact on high school students' weak meanings? Must an emphasis on coherent meanings exist across grades to have a satisfactory impact?
6. How might we design professional development to help teachers improve their meanings?
7. Does taking science classes help teachers develop stronger meanings for rate of change and measurement?
8. What modifications of undergraduate mathematics curricula and instruction might help teachers develop stronger mathematical meanings for teaching secondary mathematics?

Conflicts of interest

None.

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