2 Multiplicative Conceptual Field: What and Why?

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INTRODUCTION

It is a trivial idea to consider that children (and students) develop more and more complex competencies and conceptions by using their former knowledge to make sense of new situations and try to grasp them. This process can be viewed as a general process of adaptive behavior: assimilation and accommodation, as Piaget first stressed it.

But most psychologists have tried to theorize about the progressive complexity to children's competencies within content-free frameworks: logic, information processing, linguistics, or factor analysis. Piaget himself has sometimes paid attention to the conceptual components of children's knowledge (space, time and speed, probability . . .) and sometimes tried to reduce the conceptual complexity progressively mastered by children to some kind of general logical complexity, represented in his theory by such structures as those of grouping, the INRC group, and combinatorics.

The conceptual field theory asserts that a more fruitful approach to children's cognitive development is provided by using a framework referring to the contents of knowledge themselves and to the conceptual analysis of the domain. This approach has already provided enlightening results for the acquisition of elementary arithmetic (additive structures, multiplicative structures); for elementary physics, biology, or economics; for elementary algebra and geometry; and for different technological domains. As far as MCF (multiplicative conceptuals field) is concerned, it is now clear that one cannot reduce proportional reasoning, or the concepts of fraction and ratio, or the algorithms of multiplication and division, to any logical, information processing, or linguistic reasoning. Logic, computer science and linguistics do not provide us with concepts sufficient to conceptualize the world and help us meet the situations and problems that we experience. This is true for the acquisition of rational knowledge as well as for science itself. This epistemological point of view is in line with, but goes far beyond the consequences one can draw from, Godel's theorem.

The conceptual field theory also stresses that the acquisition of knowledge is shaped by the situations and problems first mastered and that knowledge has therefore many local features. All concepts have a restricted domain of validity, which varies with experience and cognitive development. The conceptual field theory is a pragmatist theory, though that does not mean it is empiricist. A problem is not a problem for an individual unless the individual has concepts enabling him or her to consider it as a problem for him or herself; the process of simulating a problem goes far beyond the abstraction of regularities from the observable world. Problems are practical and theoretical, and not merely empirical, even for young children. When a class of problems is solved by an individual (this means that he or she has developed an efficient scheme to deal with all or nearly all the problems of the class), the *problematic* character of that specific class passes away. This new power enables one to tackle new situations and objects and try to understand new properties and relationships, and therefore to pose and recognize or consider new problems for oneself. Thus, this process is a continuing cycle.

From my point of view, the most general features of the multiplicative conceptual field are the following:

- Its framework is mathematical, in a wide sense of mathematics. But the mathematics used to analyze MCF takes account of the contrasts and interconnections among the conceptual operations needed to progressively master this field. I will illustrate this point later.
- The situations and problems that offer a sound experiential reference for MCF are not purely mathematical, especially at the elementary and early secondary levels. The child's early experience of buying goods and sharing sweets, and his or her first understanding of speed, concentration, density, similarity, or probability are essential. The didactical consequences of this are important.

- The identification of the bulk of concepts needed to analyze the bulk of MCF situations is an essential theoretical problem for researchers. I use the word *bulk* instead of *set*, because the frontiers of MCF are not strictly defined, neither in terms of the concepts nor the situations and problems. There is also a dialectical tie between situations and concepts in the sense that each bulk depends heavily on the other. The concept of a conceptual field represents some kind of equilibrium between the need to classify the objects we have to study in psychology and mathematics education and the need to understand something about learning, development, and teaching. Obviously, students move in the whole repertoire of their mathematical competencies, yet I consider MCF as a reasonable-sized object for research and theory.
- Finally, one must not minimize the role of language and symbols in the development and the functioning of thinking. This is, of course, true for MCF as well as for additive structures, algebra, or mechanics. It is therefore essential to classify and analyze the variety of symbolic and linguistic signifiers that we may use when communicating and thinking about MCF, even though signifiers are not concepts or conceptual operations but only stand for them. An essential theoretical and empirical task for researchers is to understand why a particular symbolic representation can be helpful, under which conditions and when and why it can be profitably replaced by a more abstract and general one. The necessity to educate all students to a reasonably proficient level in algebra makes these considerations essential.

The conceptual field theory is therefore a complex theory. This complexity is inevitable because we need to embrace, in one single theoretical glance, the whole development of the situations progressively mastered, of the concepts and theorems required to operate efficiently in those situations, and of the words and symbols that can effectively represent these concepts and operations to students, depending on their cognitive levels.

Teaching is essential. Situations, verbal and symbolic mediations, and scaffoldings of all kinds are the usual ways by which teachers help students learn. Therefore one can hardly hold a radical constructionist approach. I prefer to speak of the appropriation process by which students make social knowledge their own, personal knowledge, with the help of teachers, parents, and peers (Vygotsky, 1962). This point of view is a constructionist point of view, but in a restricted meaning of the word: nobody, in place of the student, can grasp the meaning of a problem (and eventually its solution), make sense of a mathematical sentence, or develop a new mathematical scheme to be part of the student's repertoire. The role of teachers is nevertheless essential, but I will not develop this point in this chapter.

Intuitive Knowledge and Formal Knowledge

Because it contains an explicit reference to the idea of *concept*, some researchers consider that the conceptual field theory concerns the learning and teaching of explicit and formalized concepts. This is not true. Its first aim is rather to account for the knowledge contained in most ordinary actions, those performed at home, at work, at school, or at play by children and adults. It also refers to the knowledge involved in problem solving. Specifying the complete meaning of the theory requires several kinds of clarification.

The conceptual field theory asserts that one needs mathematics to characterize with minimum ambiguity the knowledge contained in ordinary mathematical competences. The fact that this knowledge is intuitive and widely implicit must not hide the fact that we need mathematical concepts and theorems to analyze it. I have introduced the ideas of *concepts in action* and *theorems in action* for that very purpose. The expression *intuitive knowledge* clarifies nothing, except that the subject uses his or her knowledge spontaneously, without reflecting much on its contents and groundings. A cognitive approach requires a more precise analysis, which has to be mathematical for mathematical competencies.

However, and this is a second point to be clarified, one cannot actually achieve this analysis using an a priori framework. The thesis that the analysis must be mathematical does not mean that it can be found ready-made in mathematics. The classification of relationships and situations, and the distinctions concerning the formation of the concepts of fraction and rational number that the reader will find in this chapter, have not been a primary focus of mathematicians. However, these areas of study interest psychologists and researchers in mathematics education. Great attention must be devoted to the comparative difficulty of different classes of problems and procedures and the different verbal expressions and writings produced by students. Psychology has led me, for example, to stress the fact that multiplication is not usually conceived by children as a binary operation, that the isomorphic properties of the linear function are more easily grasped than the constant coefficient properties, and that many ways of reasoning concern relationships between magnitudes or quantities, rather than pure numbers.

Another point deserves clarification. Today, the literature is full of papers concerning real-life mathematical competencies and real-life learning, as opposed to school mathematics and school learning. This opposition is misleading in the sense that no mathematical procedure observable in real-life situations cannot potentially be found in the classroom, provided students are offered a wide variety of situations to deal with, rather than stereotyped algorithms. It is a real problem that school (especially in some countries) does not offer students a variety of meaningful situations and problems. But this problem must not be confused with my claim that we need formal mathematics to characterize real-life competencies.

The last point that may require clarification concerns the need to establish more clearly the kinds of relationships that connect the formation of intuitive knowledge with consciousness, and make it more explicit. Vygotsky made a useful distinction between consciousness before and consciousness after, showing examples in which widely automated and unconscious competencies could be developed first, followed by some reflection and analysis, whereas in other examples, consciousness and explication, were conditions for the emergence of new competencies. The explication, symbolization, and even the formalization of mathematics may be more crucial for certain competencies than for others, even at the elementary and early secondary levels. It is a crucial point for a theory of teaching.

What Is MCF?

MCF is simultaneously a bulk of situations and a bulk of concepts. A concept is made meaningful through a variety of situations, and different aspects of the same concepts and operations are involved in different situations. At the same time, a situation cannot be analyzed with the help of just one concept; at least several concepts are necessary. This is the main reason that researchers should study conceptual fields and not isolated situations or isolated concepts.

Another reason comes from the fact that students master certain classes of situations before they master others; it may take up to ten years for a student to go from the simplest to the most complex ones. During that process, she or he will have to deal with a variety of things: situations, words, algorithms and schemes, symbols, diagrams and graphs . . . and will learn sometimes by discovering, sometimes by repeating, sometimes by representing and symbolizing, sometimes by differentiating, sometimes by reducing different things to one another. Because the landscape of knowledge acquisition is so complex, the theoretical framework of researchers must also be complex.

From a *conceptual* point of view, MCF has the following essential ingredients:

- multiplication and division;
- linear and bilinear (and *n*-linear) functions;
- ratio, rate, fraction, and rational numbers;
- dimensional analysis;
- linear mapping and linear combinations of magnitudes.

From a *situational* point of view, MCF comprises a rather large number of situations that need to be classified and analyzed carefully, so that one may describe a hierarchy of possible competencies developed by students, inside and outside school. It is the problem of analyzing the cognitive tasks underlying these situations, and the procedures used by students to deal with them, including erroneous procedures, that has pushed me to use such sophisticated concepts as those just listed.

By considering the situations used in the classroom to introduce multiplication and division, one is first compelled to consider that multiplication and division are only the most visible part of an enormous conceptual iceberg. School overestimates explicit knowledge and underestimates or even devalues implicit knowledge; and one cannot readily analyze simple multiplication or division problems within the panoply of MCF. And yet the simple multiplication involved in the calculation of the price of five miniature cars at the cost of $4 each, raises crucial questions.

1. The result is given in dollars, not miniature cars. Why?

2. One can understand multiplication of 4×5 as the interaction of paying $4, 5 times; but it would be impossible to explain to 7- or 8-year-olds that multiplication of 5×4 is 4 iterations of 5: one cannot add miniature cars and find dollars, and there is no reason to iterate 5, as only 5 miniature cars have been bought.

3. "5 times more" is meaningful, as it is a scalar relationship and has no dimension. "4 times more" is meaningless. Of course the multiplication of 5×4 is meaningful, but it represents a functional relationship between different possible quantities of cars and their costs.

4. These two multiplications rely upon different theorems:

$$\text{(a)} \qquad \text{scalar } f(5) = 5f(1)$$

It is usually introduced through iterated addition and therefore relies upon the additive isomorphism property

$$f(1 + 1 + 1 + 1 + 1 + 1) = f(1) + f(1) + f(1) + f(1) + f(1)$$

from which the multiplicative isomorphism property $f(n \cdot 1) = nf(1)$ is conceptually derived.

$$\text{(b)} \qquad \text{functional } f(5) = 4 \cdot 5$$

It uses the constant coefficient property $f(x) = ax$, instead of the previous isomorphism property.

5. The constant coefficient represents neither cars nor dollars but dollars per car. Dimensional analysis is implicitly present.

I have already analyzed this example elsewhere (Vergnaud, 1988), and I have also explained about the classification of problems in earlier papers (Vergnaud 1983; 1988). Therefore, here I will just review the main categories of multiplicative structures and stress the epistemological points that appear to me to be the most essential (for more details, see Vergnaud, 1983).

- Simple proportion:

M_1	M_2
a	b
c	d

Calculate one of these four magnitudes knowing the other three.

- Concatenation of simple proportions:

M_1	M_2	M_3
a	b	
	c	d
f		g

Calculate one of these six magnitudes knowing the other five.

- Double proportion: calculate one of the six magnitudes in the following table, knowing the other five.

M_2 \ M_1	a	b
	M_3	
c	f	
d		g

- Comparison of rates and ratios:

M_1	M_2	M_1	M_2
a	b	c	d

Which rate (b/a, d/c) is bigger? Or which ratio (c/a, d/b)?

It is clearly impossible to analyze the procedures used by students in these situations without the framework of linear and bilinear functions and the clear identification of the magnitudes involved: elementary, quotient, and product magnitudes.

The following example of a procedure used by some 10- to 13-year-olds exemplifies the need for a sophisticated mathematical framework to theorize about the intuitive knowledge of students.

Given: The consumption of flour is on average 3.5 kg per week for ten persons. *Question:* What quantity of flour is needed for fifty persons over twenty-eight days? *Answer:* 5 times more persons, 4 times more days, 20 times more flour; therefore $3.5 \times 20 = 70$ (kg)

It is impossible to give account of that reasoning without making the hypothesis of the following implicit theorem in the subject's head:

$$f(n_1 x_1, n_2 x_2) = n_1 n_2 f(x_1, x_2)$$

$$\text{Consumption } (5 \times 10, 4 \times 7) = 5 \times 4 \text{ Consumption } (10, 7)$$

Of course, this theorem is available because the ratio of 50 persons to 10 persons, and the ratio of 28 days to 7 days are simple and visible. It would not be so easily applied to other numerical values. Therefore, its scope of availability is limited. Yet it is a mathematical theorem, and can be expressed in different ways:

1. In words: The consumption is proportional to the number of persons when the number of days is held constant; it is proportional to the number of days when the number of persons is held constant.

2. By a double-proportion table, as shown in Figure 2.1.

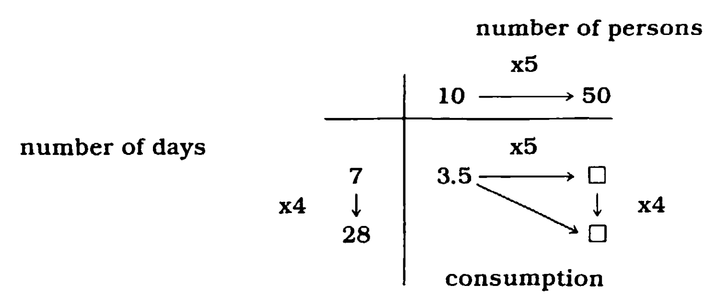


Fig. 2.1
Double-proportion table.

3. By a formula:

$$C = KP \cdot D$$

where
C = consumption,
P = number of persons,
D = numbers of days,
$K = C(1, 1)$, consumption per person and per day.

C is proportional to P when D is constant and to D when P is constant. Therefore, it is proportional to the product.

It is clear that these different modes of expressing the same reasoning are not cognitively equivalent: The last one is more difficult. These modes rather are complementary and illustrate different ways of making explicit the same hidden mathematical structure at different levels of abstraction.

Situations, Schemes, Concepts, and Symbols

The conceptual field theory is a psychological theory of cognitive complexity. There are several ways to gain cognitive complexity and several ways to fail at an attempt to gain cognitive complexity.

The first, and very essential, way to make progress is to learn to manage a new class of situations. The hierarchical classification of multiplication and division problems, which takes into account the conceptual structure, the domain of experience used, and the numerical values, therefore, is important to the study of the growth of cognitive complexity. For instance, the distinction between multiplication, division 1 (partition), and division 2 (quotation) is commonly accepted as the first basis of MCF.

Multiplication		*Division type 1*		*Division type 2*	
M_1	M_2	M_1	M_2	M_1	M_2
1	a	1	?	1	a
b	?	b	c	?	c

But this is true only when the domain of experience referred to is conceptually easy (sharing discrete objects, buying

goods) and when the numerical values are whole numbers (small whole numbers for b). These situations provide the first meaning for multiplication and division:

b times more, b times less

Division 2 already demands two more cognitive steps: Find how many times a goes into c, or apply to c the inverse functional coefficient $/a$. These two operations are equivalent mathematically, but not conceptually: The first one consists of finding a scalar ratio, and the second one an inverse quotient of dimensions.

Along similar developmental considerations, the primitive conception of fractions comes from the partition structure and is usually available for very simple values: 1/2 first, 1/4 one or two years later, and $1/n$ (for $n < 10$) by the end of elementary school. Archimedian fractions are therefore viewed as both operators and quantities: $1/n$ is first viewed as dividing by n some discrete or continuous quantity—it is therefore an operator—but the result is a fractionary quantity $1/n$.

At the same time, young students use scalar ratios and functional rates, as shown in fig. 2.2 They can therefore combine Archimedean fractions and scalar ratios into non-Archimedean fractions, p/q, provided $p < q$. Some examples would include the following:

- Sharing a pastry cut into eight parts. What fraction is eaten by 5 children who each eat one part?
- Sharing a bag of twenty-four sweets, divided into eight parts. What fraction is eaten by five children? How many sweets does each child eat? How many sweets are eaten by all five children?

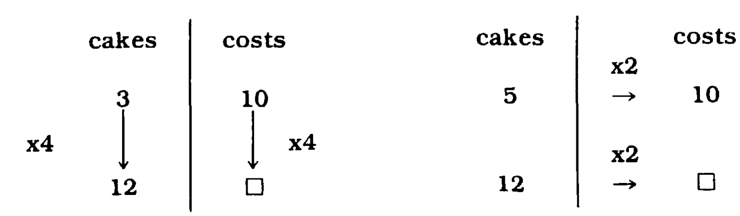
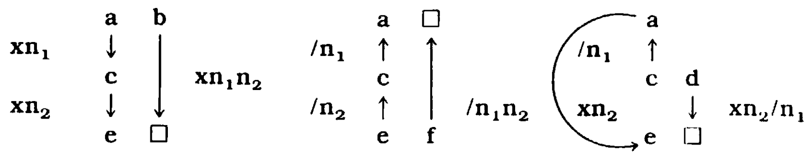


Fig. 2.2
Scalar ratios and functional rates.



Note: n_1 *and* n_2 *are any whole numbers.*

Fig. 2.3
Combining scalar ratios.

They also combine scalar ratios in several different ways, as shown in figure 2.3. The main conceptual difficulties met by students in these combinations concern the commutativity of division and multiplication

$$/n_1 \text{ and } xn_2 = xn_2 \text{ and } /n_1$$

and the need to multiply when one combines two divisions

$$/n_1 \text{ and } /n_2 = /(n_1 \times n_2)$$

It is worth noticing that during the elementary school years, students are also introduced to double proportion in two different domains:
as combinatorics

> How many possible different colored houses can be painted with three colors for the roof and four colors for the walls?
> How many possible couples of dancers can be formed with five boys and 7 girls?

and as area and volume

> What is the area of a room 5 meters long and 4 meters wide?
> In the latter case, formulas are usually taught

$$A = L \times W \qquad V = L \times W \times H$$

Therefore, by the end of elementary school, students have already been faced with some essential aspects of multiplicative structures. They have had to deal with different problems of proportion, with different kinds of operations involving ratios and rates, and with different types of symbolisms. However, they can master only a small part of the conceptual field; they still have a long way to go to understand it fully. For instance, they have yet to build such high-level concepts as those of rational number, function, and variable, dependence and independence.

They also have to achieve more modest steps, such as extending the scope of validity of their intuitive knowledge to complex ratios and rates and to nonwhole numbers. It is now well known that there are strong epistemological obstacles to such extension: The beliefs that one cannot divide a number by a larger one, multiplication makes bigger and division smaller, and so on. Students also have to extend their knowledge of multiplicative structures to such difficult domains as geometry (similarity and homothety), physics (density, mechanics), probability, and so forth. This extension raises a sharp theoretical issue. From a cognitive point of view, applying multiplicative structures to new domains of experience is both necessary to conceptualize them properly, and made possible only if some specific conceptualization of the domain has taken place. It seems to be a vicious circle.

The vicious circle can be disrupted only if one develops a reasonably complex theory of cognitive development and learning, especially of the relationship between schemes, concepts, and symbols.

What is a scheme? A scheme is defined as the invariant organization of action for a certain class of situations. This dynamic totality, introduced by Piaget (after Kant), to account for both "sensory-motor skills" and "intellectual skills" requires a strict and deep analysis if one wishes to understand the relationship between competences and conceptions.

A scheme is finalized; goals imply expectations. A scheme generates actions; it must contain rules. A scheme is not a stereotype, as the sequence of actions depends on the parameters of the situation: Its application involves hic et nunc computations. A scheme also involves operational invariants: categories to pick up relevant information (concepts in action) and propositions from which inferences are made (theorems in action). All these aspects of a scheme are illustrated in Figure 2.4.

This analysis makes it clear that no action is possible without operational invariants that enable the subject to pick up information and compute what to do and expect.

The theory of conceptual fields offers a way to under-operational invariants (concepts in action and theorems in action) —inferences→ { rules of action; goals and expectations }

Fig. 2.4
What does a scheme consist of?

stand how the intuitive knowledge contained in behavior works. Intuitive knowledge is made essentially of operational invariants, that is, concepts in action and theorems in action. They are the "conceptual" part of schemes, however implicit or explicit, conscious or unconscious they may be. If a scheme addresses a class of situations, it must contain invariants that will be relevant over the whole class. This is especially visible when a scheme is extended to a larger class of situations. Transfer presupposes invariants as well as differentiation and restriction.

Among the most important theorems in action developed by students, one finds the isomorphic properties of the linear function

$$f(x + x') = f(x) + f(x')$$
$$f(x - x') = f(x) - f(x')$$
$$f(c_1x_1 + c_2x_2) = c_1f(x_1) + c_2f(x_2)$$

and the constant coefficient properties of the linear function

$$f(x) = ax$$
$$x = 1/af(x)$$

and some specific properties of bilinear functions

$$f(c_1x_1, c_2x_2) = c_1c_2f(x_1, x_2)$$

Among the most important concepts in action developed by students, one find those of quantity and magnitude, unit value, ratio and fraction, function and variable, constant rate, dependence and independence, quotient and product of dimensions.

Concepts in action are necessary ingredients of theorems in action, in the same way that propositional functions and arguments are necessary ingredients of propositions. But concepts are not theorems. They allow no derivation (or inference, or computation); a derivation requires propositions. Propositions can be true or false; concepts can be only relevant or irrelevant. Yet, there are no propositions without concepts.

Reciprocally, there are no concepts without propositions, as it is the need to derive action from the representation of the world and have true (or at least truer) conceptions of the world that makes concepts necessary. A computable model of intuitive knowledge must comprise concepts in action and theorems in action as essential ingredients of schemes.

Schemes play the most essential part as they generate actions. (Intellectual operations also are actions.) They can generate actions because they contain operational invariants, which constitute the core of representation.

But a concept is not fully a concept unless it is explicit. Moreover, the process of making concepts and theorems explicit helps one identify the relevant or irrelevant invariants. Therefore, linguistic expressions, symbols, and symbolic representations that may accompany, at the signifier level, the formation of concepts and theorems, must also be studied. Explication and symbolization are an important path through which cognitive complexity is gained.

Not only is it important that students be faced with a variety of occasions to extend or restrict the scope of validity and availability of their schemes and to develop new schemes, but also they be helped by external means, like linguistic and extralinguistic signifiers, in recognizing the invariant structure of different problems and therefore the possibility of using the same schemes or similar ones. Not only is it important that situations be clearly and exhaustively classified from the point of view of their conceptual structure, but also that the invariants (concepts and theorems) be worded, symbolized, diagramed, or graphed so that they become elements of explicit rational conceptions and do not remain elements of only implicit schemes. This is probably a necessary condition for the transference of concepts and theorems to any numerical values and to any domain of experience.

As a matter of fact, transfer and generalization necessarily require the recognition of the "same" structure in different situations. Does it help to associate specific words and sentences or specific symbols to similar problems and relationships? In other words, is it helpful for students that hidden invariants and structures be made explicit? There is probably no universal answer to this question, but it is likely that making some relationships explicit can be helpful. This is the reason why the conceptual field theory considers language and symbols important. For instance, it is important to express and eventually symbolize the structure of data and questions, and to use words and symbols that can be used by students. In the previous example of the consumption of flour by fifty students over twenty-eight days, the double-proportion table can be handled easily by a 10- to 15-year-old, whereas the algebraic notation cannot.

The "table-and-arrow diagram" offers many advantages. It uses the properties of the two-dimensional space to represent some relevant properties of simple proportion and double proportion. A list of its strengths for each type of proportion follows.

- Simple proportion:
  Parallelism is used to represent different kinds of quantities or magnitudes.
  Vertical arrows indicate ratios.
  Horizontal arrows indicate functions and quotients of magnitudes.
  Combinations of arrows represent products of operators.
- Double proportion:
  Orthogonality is used to represent independence.
  Parallelism line to line or column to column represents proportion.
  Margins are used to represent the values of the elementary magnitudes, and the inside of the table is used to represent the values of the product magnitude.
  Arrows are used to represent ratios, functions, and combinations of ratios.

Words are important, but mathematical sentences are usually complex when expressed in natural language. Algebraic symbols are economical and powerful, but they make theorems very abstract for children. The table-and-arrow diagram is a prealgebraic representation that is less abstract than algebra while still representing the essential relationships. This is why I find it important to communicate about MCF with the help of such a symbolic system. Students find it fairly easy to use at the primary and early secondary level, and many adults find it easier to use than algebra. This does not mean that algebra is not necessary. Most students seem to find both algebra and proportion tables useful for a long while.

The status of an explicit and symbolized theorem is different from the status of an implicit local operational invariant. But the former has no meaning if it is not grounded in the latter, and the latter is available only in a limited range of situations. Moreover, scientific concepts and theorems are debatable and public. Implicit concepts and theorems by their nature cannot be explicitly debated. Therefore an enormous amount of the discussion that is expected from students in the learning of mathematics could not take place if mathematics consisted only of schemes. There is a need for symbolizing and formalizing, which makes mathematics different from a bulk of schemes addressing a bulk of situations. The concept of linear function cannot emerge from dealing with proportion problems only. It has to be worded and analyzed as a general and comprehensive concept.

And yet situations and schemes are essential from a psychological point of view. They are also essential for didactics, as the capacity to invent complex and meaningful didactic situations is probably the most genuine activity of professional teachers. As it is presented in this chapter, MCF is reduced to its main elementary components. It is another enterprise to combine these components to provide students with more challenging and motivating situations. I have not addressed that problem here.

CONCLUSION

The multiplicative conceptual field can be viewed as

- a set (bulk) of situations that require multiplication, division, or combination of such operations;

- a set (bulk) of schemes that are needed to deal with these situations. Schemes are invariant organizations of behavior for well-defined classes of problems; but they can also be evoked to solve new problems;
- a set (bulk) of concepts and theorems that make it possible to analyze the operations of thinking needed: linear and nonlinear functions, fraction, ratio, rate and rational number, dimensional analysis, vector space theory. (These three concepts maybe explicit, but they are very often implicit only in schemes.)
- a set (bulk) of formulations and symbolizations.

All four sets are necessary to understand how students master more and more complex situations, more and more profoundly and reliably, and to understand how teachers can help them by presenting appropriate situations to them and giving them appropriate explanations.

Especially important is the choice of situations that can make new concepts or new aspects of a concept more meaningful. We need to develop a powerful theory of teaching situations tied to both the epistemology of mathematics and the psychology of learning mathematics.

Operational knowledge is an answer to genuine practical and theoretical problems. This is apparent when one considers the history of science and the history of techniques and technology. How much of this idea can we transpose into the classroom?

Even if it is each individual student's cognitive decision to recognize or discover a new property of a concept, there is a large set of possible ways for teachers to help students: organize interesting and mathematically fruitful situations and activities, focus attention, explain and symbolize the relevant relationships and operations of thinking, or reduce the gap between the problem and its solution.

The practical competence of teachers must be analyzed in terms of a strong cognitive theory of learning and teaching. This is what the conceptual fields theory tries to provide. This theory asserts that the core of cognitive development is conceptualization. Therefore, we must devote all our attention to the conceptual aspects of schemes and to the conceptual analysis of the situations for which students develop their schemes, in school or in real life. Words and symbols are

nevertheless essential. Therefore we must also devote our attention to the adequacy of linguistic and extralinguistic means by which we help students identify invariants and recognize them as mathematical objects.

REFERENCES

Behr, M., and J. Hiebert (eds.). 1988. *Number concepts and operations in the middle grades*. Reston, VA: National Council of Teachers of Mathematics. Hillsdale, NJ: Lawrence Erlbaum.

Lesh, R., and M. Landau (eds.). 1983. *Acquisition of mathematics concepts and processes*. New York: Academic Press.

Piaget, J. 1967. *Biologie et connaissance*. Paris: Gallimard.

Vergnaud, G. 1983. Multiplicative structures. In *Acquisition of mathematics concepts and processes*, ed. R. Lesh and M. Landau, 127–174. New York: Academic Press.

———. 1988. Multiplicative structures. In *Research agenda in mathematics education: Number concepts and operations in the middle grades*, ed. J. Hiebert and M. Behr, 141–161. Hillsdale, NJ: Lawrence Erlbaum.

——— et al. 1990. Epistemology and psychology of mathematics education. In *Mathematics and cognition*, ed. J. Kilpatrick and P. Nesher, 2, 17. Cambridge: Cambridge University Press.

Vygotsky, L. S. 1962. *Thought and language*. Cambridge, MA: MIT Press.