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Teaching Experiment Methodology: Underlying Principles and Essential Elements[1]

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> The constructivist is fully aware of the fact that an organism's conceptual constructions are not fancy-free. On the contrary, the process of constructing is constantly curbed and held in check by the constraints it runs into Ernst von Glasersfeld, 1990, p. 33.

A primary purpose for using teaching experiment methodology is for researchers to experience, firsthand, students'[2] mathematical learning and reasoning. Without the experiences afforded by teaching, there would be no basis for coming to understand the powerful mathematical concepts and operations students construct or even for suspecting that these concepts and operations may be distinctly different from those of researchers.

The constraints that researchers experience in teaching constitute a basis for understanding students' mathematical constructions. As we, the authors, use it, "constraint" has a dual meaning. Researchers' imputations to students of mathematical understandings and operations are constrained by the language and actions they are able to bring forth in students. They also are constrained by students' mistakes, especially those mistakes that are essential; that is, mistakes that persist despite researchers' best efforts to eliminate them. Sources of essential mistakes reside in students' current mathematical knowledge. To experience constraints in these two senses is our primary reason for doing teaching experiments. The first type of constraint serves in building up a "mathematics of students" and the second type serves in circumscribing such a mathematics within conceptual boundaries.

In our teaching experiments, we have found it necessary to attribute mathematical realities to students that are independent of our own mathematical realities. By "independent of our own" we mean that we attribute mathematical concepts and operations to students that they have constructed as a result of their interactions in their physical and sociocultural mileau[3]. In this attribution, we remain aware that we may not, and probably cannot, account for students' mathematics using our own mathematical concepts and operations. Although our attribution of mathematical realities to studens is a conceptual construct, it is grounded in the mathematical realities of students as we experience them. We use the phrase "students' mathematics" to refer to whatever might constitute students' mathematical realities; we use the phrase "mathematics of students" to refer to our interpretations of students' mathematics.[4] The distinction we make between students' mathematics and mathematics of students has been captured by Ackermann (1995) in speaking of human relations:

> In human relations, it is vital to attribute autonomy to others and to things--to celebrate their existence independently from our current interaction with them. This is true even if an attribution (of existence) is a mental construct. We can literally rob others of their identity if we deny them an existence beyond our current interests (p. 343).

Students' mathematics is something we attribute to students independently of our interactions with them. In doing so, we reason as follows: If we acknowledge that students are autonomous organisms not unlike ourselves, and if we claim that our consensual mathematical reality is the only one, then certain difficulties arise. If our, the researchers', consensual mathematical reality is the only one, then the mathematical reality of any student would be solely in our imagination. But a student, being not unlike ourselves, can insist that his or her mathematical reality is the sole mathematical reality and that the mathematical reality of everyone else is in his or her imagination, including our, the researchers', consensual reality. Being autonomous also, we would not want to accept that our consensual mathematical reality is merely a concoction of the student's imagination. So, we have to accept the student's mathematical reality as being distinct from ours. We call those mathematical realities "students' mathematics," whatever they might be. Students' mathematics is indicated by what they say and do as they engage in mathematical activity, and a basic goal of the researchers in a teaching experiment is to construct models of students' mathematics. "Mathematics of students" refers to these models, and it includes the modifications students make in their ways of operating (Steffe, 1988).

We regard the mathematics of students as a legitimate mathematics to the extent we can find rational grounds for what students say and do. Looking behind what students say and do in an attempt to understand their mathematical realities is an essential part of a teaching experiment. The process involved in looking behind what students say and do has been called conceptual analysis by von Glasersfeld (1995), and it is here one becomes explicitly aware of one's own engagement in a kind of mathematical research. For us, this awareness is essential because teaching experiment methodology is based on the necessity of providing an ontogenetic justification of mathematics; that is, a justification based on the history of its generation by individuals. This kind of justification is different from the impersonal, universal, and ahistorical justification often provided for mathematics, but we believe it is an appropriate way to regard mathematics, especially in the case of specifying a school mathematics. In fact, we contend an ontogenetic justification of mathematics is generally no less rational than its alternatives. But it is different to regard mathematics as a product of the functioning of human intelligence (Piaget, 1980) than to regard it as a product of impersonal, universal, and ahistorical reason. Our view of mathematics as a product of functioning human intelligence defines mathematics as a living subject rather than as a subject of being; this is the core belief from which our research methodology flows.

By regarding mathematics as a living subject, we are faced with a different mathematics than appears in contemporary school mathematics. Our practical stance is that the better we understand it, the better positioned we are to affect students productively. Indeed, it is our intention that the mathematics of students replace contemporary school mathematics (Steffe & Wiegel, 1992). This is the main thrust of our work, and we want it to be understood in this way. We strive to specify the mathematical concepts and operations of students and to make them the conceptual foundations of school mathematics.

Historical Perspectives

Reasons for the Emergence of Teaching Experiments

Teaching experiments were not always an accepted way of doing research in mathematics education. It was not until approximately 1970 that teaching experiments in mathematics education emerged in the United States. They emerged for two reasons. First, models out of which one might make a mathematics of students were developed outside of mathematics education and for purposes other than educating students (e.g., Piaget & Szeminska, 1952; McLellan & Dewey, 1895; Brownell, 1928). After intensive efforts to use these models to study the mathematical development of students, it became clear that new models were needed that had their roots in mathematics education. Some researchers came to understand that mathematics educators could not simply borrow models from the fields of genetic epistemology, philosophy, or psychology and use them with the expectation that they could be used to explain students' mathematical learning and development in the context of teaching. Models were needed that included an account of the progress students make as a result of interactive mathematical communication. Although it was not the intention of researchers to explain students' mathematics using known mathematical systems, it was felt researchers needed to learn how to use their own mathematical knowledge in actual interactions with students. The models of students' mathematics available at the time were not sensitive either to the issues involved in sustaining students' mathematical activity over extended periods of time or to the issues involved in how teachers might participate in students' modifications of that activity.

Second, a large chasm existed between the practice of research and the practice of teaching. Essentially, the experimental methodology used prior to the emergence of teaching experiments had its roots in the "agriculture paradigm." In this paradigm, the researcher selects one or more samples from a target population and subjects it or them to various treatments. The effect of one treatment is compared to the effects of others, with the intention of specifying differences between or among them. This seemed to be a reasonable way to proceed in research, and it was the one advocated by Campbell and Stanley (1966):

> By experiment we refer to that portion of research in which variables are manipulated and their effects upon other variables are observed (p. 1).

Researchers' attempts to formulate possible factors that could be varied systematically in such a way that a corresponding variation in other variables might be observed seemed logical enough. But the research essentially failed. In hindsight, classical experimental design in the spirit of Campbell and Stanley (1966) suppressed conceptual analysis in the conduct of research. One reason for this suppression was the assumption that an experimental manipulation would causally affect other variables—such as measures of students' mathematical achievement—quite apart from the individuals involved in the treatment. The subjects in experiments were recipients of treatments and usually were not the focus of conceptual analysis. The subjects were subjected to treatments; they did not participate in the coconstruction of the treatments in the context of teaching episodes. How students made meanings or the meanings they made was not of primary interest.

Classical experimental design inhibited efforts to investigate students' sense-making constructs. It also inhibited efforts to engage in the kind of research we believe is essential in establishing mathematics education as an academic field. Researchers found it all too easy to appear "scientific" by using advanced statistical methods and by writing in the passive voice—the latter giving an objective, "hard science" appearance to claims. Too often, conceptual analyses of mathematical understanding and mathematical performance were absent.

A strong reliance on psychometrics was another significant difficulty researchers came to have with classical methods. Psychometrics was founded upon the idea that a student's actual score on an item is composed of a "true" score and some amount due to error. For this to make sense, one needed to assume a cause-effect relationship between the item presented and a student's resulting performance. This led to all sorts of pseudo-scientific constructs, such as test items having "dimensions" that would account for different students behaviors (Thompson, 1982, p. 156).

Psychometrics lent support to classical experimental design—that to experiment means to manipulate variables. In classical experimental research, tasks, items, dimensions, and so forth were objects to be manipulated. It was through such manipulation that researchers controlled students' environments and, hence, uncovered the reality of their knowledge. From a constructivist perspective, however, there is no such thing as a "constant" stimulus; students construct for themselves the tasks in which they actually engage, and it is the constructive process and the constructed task that are interesting scientifically (Powers, 1973, 1978).

Reasons for Acceptance of Teaching Experiments

There seem to be several reasons teaching experiments were accepted almost at face value by mathematics educators without being developed and analyzed more extensively. First, the methodology seemed intuitively correct. The word "teaching" in the title appealed to the common sense of mathematics educators and resonated with their professional identification as mathematics teachers.

Second, versions of the methodology were being used already by researchers in the Academy of Pedagogical Sciences in the then Union of Soviet Socialist Republics. Reports of this research became available in the United States through the efforts of Izaack Wirszup at the University of Chicago (Wirszup & Kilpatrick, 1975-1978) and provided academic respectability for what was then a major departure in the practice of research in mathematics education. The Soviet versions of the teaching experiment were examined by a small group of researchers in the United States in their formulation of a new methodology for mathematics education research.[5]

Finally, the field of mathematics education was entering a postmodern period (von Glasersfeld, 1987). Major shifts in the way in which mathematical knowing was understood were under way, and they became expressed in the paradigm wars of the next decade (e.g., Kilpatrick, 1987; Vergnaud, 1987; Sinclair, 1987; Wheeler, 1987; Brophy, 1986; Confrey, 1986). Mathematics education researchers began to change their concept of normal science, and the teaching experiment filled a void in the methodologies available for investigating mathematical learning and development[6]. In fact, over the past twenty years, growing numbers of researchers have sought to understand students' mathematical experience. They have tried to account for students' mathematical activity in the context of teaching as manifestations of their current imagery, reasoning, and understandings. They also have tried to build accounts of how students learn specific mathematical concepts. Rather than become interested in these issues in a pure form, researchers explicitly acknowledged that mathematical activity in school occurs as a result of students' participation in teaching. Experimental methodologies used in the 1970s were inadequate for addressing these issues. In particular, the foundational assumptions of psychometrics conflicted profoundly with assumptions that gave meaning to the idea of understanding another's mathematical experiences without assuming a God's-eye-view of those experiences (Thompson, 1982).

So, new research methodologies emerged to support avenues of inquiry that current methodologies could not sustain, and the teaching experiment marked a revolution in the practice of mathematics education research (Cobb & Steffe, 1983; Hunting, 1983; Thompson, 1979, 1982).

The methodology of the teaching experiment will unavoidably continue to evolve among the researchers who use it. It certainly did not emerge as a standardized methodology nor has it been standardized since. Rather, the teaching experiment is a conceptual tool that researchers use in the organization of their activities. It is primarily an exploratory tool, derived from Piaget's clinical interview and aimed at exploring students' mathematics. Because it involves experimentation with the ways and means of influencing students' mathematical knowledge, the teaching experiment is more than a clinical interview. Whereas the clinical interview is aimed at understanding students' current knowledge, the teaching experiment is directed toward understanding the progress students make over extended periods. It is a dynamic way of operating, serving a functional role in the lives of researchers as they strive to organize their activity to achieve their purposes and goals. In this, it is a living methodology designed initially for the exploration and explanation of students' mathematical activity. Our comments concerning the teaching experiment should be interpreted in this context because they are not meant to be prescriptive.

The Elements of Teaching Experiment Methodology

A teaching experiment involves a sequence of teaching episodes (Steffe, 1983). A teaching episode includes a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episode. These records, if available, can be used in preparing subsequent episodes as well as in conducting a retrospective conceptual analysis of the teaching experiment. These elements are germane to all teaching experiments.

Exploratory Teaching

There is a case where it may be appropriate to emphasize the experiential aspects involved in teaching and not to concentrate on hypothesis testing or retrospective analyses. For example, one of the first teaching experiments conducted in the United States was done for a reason that does not appear in the report of the experiment (Steffe, Hirstein, & Spikes, 1976). After working for a period of approximately eight years trying to apply Piaget's cognitive-development models to mathematics education, a point had been reached where children's construction of number, as explained by Piaget & Szeminska (1952), was not useful in furthering the researchers' goals. New models for children's construction of number were needed that were more sensitive to their mathematical experience (cf. Steffe, Cobb, & von Glasersfeld, 1988). So, the researchers in the Steffe et al.

(1976) study returned to their professional commitment as mathematics teachers in order to learn students' mathematics firsthand. Based on their past experience as mathematics teachers, they understood that teaching students for short periods of time could not serve as a basis for a solid understanding of their thinking and how it might be influenced. They taught two classes of first-grade students for one school year, an experience that contributed some of the basic ideas about teaching experiment methodology.

Any researcher who hasn't conducted a teaching experiment independently, but who wishes to do so, should engage in exploratory teaching first. It is important that one become thoroughly acquainted, at an experiential level, with students' ways and means of operating in whatever domain of mathematical concepts and operations are of interest. In understanding this, one must adopt a certain attitude if substantial progress is to be made toward learning a mathematics of students. The teacher-researcher must attempt to put aside his or her own concepts and operations and not insist that the students learn what he or she knows. Otherwise, the researcher might become caught in what Stolzenberg (1984) called a "trap" -- focusing on the mathematics the researcher takes as given instead of focusing on exploring students' ways and means of operating. The researcher's mathematical concepts and operations can be orienting, but they should not be regarded, initially at least, as constituting what the students should learn.

Meanings of Experiment in a Teaching Experiment

Testing Research Hypotheses. Other than the goal of becoming thoroughly familiar with children's mathematics in an experiential sense, our purpose for engaging in exploratory teaching was to begin making essential distinctions in students' ways and means of operating. For example, when teaching the two classes of first-graders, we observed children who could count but who needed objects in their visual field in order to carry out the activity. We also observed other children who could count as these children did, but who also willfully created substitute countable items such as tapping their finger on a table when the items to be counted were hidden from view. Children of the second type were obliged to always count from "one" in order to give meaning to a number word. There was another group of children who could count as children of the first two types did, but they also could start from any number word and count further when the items they intended to count were not in their visual field.

These distinctions proved to be essential in two senses. First, we did not teach these children to count in the ways they did. Rather, their ways of counting arose from within them and we as their teachers had nothing to do with it.[7] Second, the ways in which the children counted were resistant to our efforts to teach them to count in ways that, to us, were more advanced. We tried to teach the children in the first group to count as the children in the second group did, and the children in the second group to count as the children in the third group did, but we were essentially unsuccessful. We also tried, again with little success, to teach the children in the third group to count forward or backward interchangeably in what, to us, were missing addend situations.

So, we had found a problem that seemed to have educational significance. Consequently, a two-year teaching experiment was mounted to explore if progress could be made in solving this problem. By using individual interviews (Ginsburg, 1981), six children were selected; three who gave every indication of counting as the children in the first group did and three who gave every indication of counting as the children in the second group did (Steffe et al., 1988). The experiment was to explore the consequences of being in one of the two groups. Would the three children in each group remain fundamentally alike over the duration of the teaching experiment and distinctly different from the three children in the other group? Or, would all of the children tend toward homogeneity in their mathematical understandings and interactions?

We use “experiment” in “teaching experiment” in a scientific sense. The hypotheses in the teaching experiment mentioned above were that the differences between children of different groups would become quite large over the two-year period and that the children within a group would remain essentially alike. That the hypotheses were confirmed is important, but only incidental to our purposes here. What is important is that teaching experiments are done to test hypotheses as well as to generate them. One does not embark on the intensive work of a teaching experiment without having major research hypotheses to test.

The research hypotheses one formulates prior to a teaching experiment guide the initial selection of the students and the researchers’ overall general intentions. However, the researchers do their best to “forget” these hypotheses during the course of the teaching episodes, in favor of adapting to the constraints they experience in interacting with the students. The researchers’ intention is to remain aware of the students’ contributions to the trajectory of teaching interactions and for the students to test the research hypotheses seriously. Researchers have students "test the research hypotheses seriously" by teaching them with the goal of promoting the greatest progress possible in all participating students. The researchers return to the research hypotheses retrospectively after completing the teaching episodes. This method -- setting research hypotheses aside and focusing on what actually happens -- is basic in the ontogenetic justification of mathematics.

Generating and Testing Hypotheses. In addition to formulating and testing major research hypotheses, another modus operandi in a teaching experiment is for the researchers to generate and test hypotheses during the teaching episodes. Often, these hypotheses are conceived "on the fly," a phrase Ackermann (1995) used to describe how hypotheses are formulated in clinical interviews. Frequently, they are formulated between teaching episodes as well. The teacher-researcher, through reviewing the records of one or more earlier teaching episodes, may formulate one or more hypotheses to be tested in the next episode.

In a teaching episode, the students' language and actions are a source of perturbation for the teacher-researcher. It is the job of the teacher-researcher to continually postulate possible meanings that lie behind students' language and actions. It is in this way that students guide the teacher-researcher. The teacher-researcher may have a set of hypotheses to test before a teaching episode and a sequence of situations planned to test the hypotheses. But, because of students' unanticipated ways and means of operating as well as their unexpected mistakes, the teacher-researcher may be forced to abandon these hypotheses while interacting with the students and to create new hypotheses and situations on-the-spot. The teacher-researcher also might interpret the anticipated language and actions of the students in ways that were unexpected prior to teaching. Impromptu interpretations occur to the teacher-researcher as an insight that would be unlikely to happen in the absence of experiencing the students in the context of teaching interactions. Here, again, the teacher-researcher is obliged to formulate new hypotheses and to formulate situations of learning to test them.

Through generating and testing hypotheses, boundaries of the students' ways and means of operating can be formulated--where the students make what to us are essential mistakes. These essential mistakes are of the same nature as those Piaget found in his studies of children, and we use them for essentially the same purpose he did. They arise from students' failures to make adaptations when interacting in a medium. Operations and meanings we impute to students within the boundaries of their essential mistakes constitute what we call living models of the students' mathematics. The boundaries of a living model are usually fuzzy, and what might be placed just inside or just outside them is always a source of tension and often leads to creative efforts on the part of the researchers.

> Essential mistakes provide stability in a dynamic living model of students' mathematics. We understand better what students can do if we understand what they cannot do. We understand what students can understand better if we understand what they cannot understand. It also helps to understand what a child can do if we understand what other students, whose knowledge is judged to be at a higher or lower level, can do. In this, we are in accordance with Ackermann (1995)that: The focus of the clinician [teacher] is to understand the originality of [the child's] reasoning, to describe its coherence, and to probe its robustness or fragility in a variety of contexts (p. 346).

When a student makes what appears to be an essential mistake, our purpose is not "to judge or evaluate the child's performance in relation to performances of other children who might come up with the right answer" (Ackermann, 1995, p. 346). In this, we interpret Ackermann as speaking about the attitude of the teacher-researcher toward the child. Rather than believing that a student is absolutely wrong or that the student's knowledge is immature or irrational, the teacher-researcher must attempt to understand what the student can do; that is, the teacher-researcher must construct a frame of reference in which what the student can do seems rational. This is a basic challenge facing the researcher.[8]

Meanings of Teaching in a Teaching Experiment

Teaching actions occur in a teaching experiment in the context of interacting with students. However, interaction is not taken as a given--learning how to interact with students is a central issue in any teaching experiment. The nuances of how to act and how to ask questions after being surprised are among, in our experience, the most central issues in conducting a teaching experiment.

The researchers may have research hypotheses to test at the beginning of a teaching experiment, but even researchers experienced in teaching may not know well enough what progress students will make or know well enough their mathematical thinking and power of abstraction to formulate learning environments prior to teaching. Wholly unexpected possibilities may open up to the teacher-researcher in the course of the teaching experiment .

If the researchers knew ahead of time how to interact with the teaching experiments' students and what the outcomes of those interactions might be, there would be little reason for conducting a teaching experiment. So, frequently, the researchers are obliged to engage in responsive and intuitive interactions with the students when they are, in fact, puzzled about where the interactions are headed. As the teaching experiment progresses, the researchers become more experienced with the students and often change from interacting in a responsive and intuitive way to interacting analytically.

Responsive and Intuitive Interaction. In responsive and intuitive interactions, the teacher-researcher is usually not explicitly aware of how or why he or she acts as he or she does and the action appears without forethought. He or she acts without planning the action in advance of the action. In this role, we see ourselves as agents of action (or interaction). As agents of action, we strive to harmonize ourselves with the students with whom we are working to the extent that we "lose" ourselves in our interactions. We make no intentional distinctions between our knowledge and the students' knowledge, and, for us, experientially, everything is the students' knowledge as we strive to feel at one with them. In essence, we become the students and attempt to think as they do (Thompson, 1982, 1991; van Manen, 1991). Researchers do not adopt this stance at the beginning of a teaching experiment only. Rather, they maintain it throughout the experiment.

By interacting with students in a responsive and an intuitive way, the goal of the researchers is to explore the students' reasoning. For example, when working with two third-grade children named Jason and Patricia,[9] Jason drew a stick spanning the computer screen, as shown in Figure 1. The teacher asked the two children if they could cut the stick into two equal pieces. This question was asked with no apparent expectation of what the children might do.

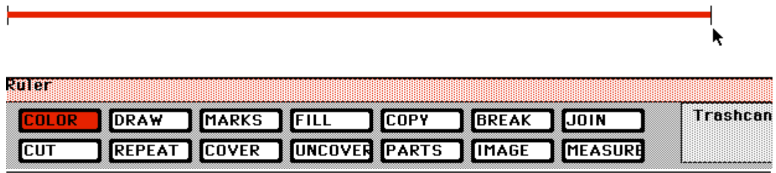

Figure 1. A Stick in TIMA: Sticks[10]

Protocol I

> Jason: (Using the CUT command, he makes a cut in the stick, then leans back to inspect his choice visually.)
>
> Teacher: How do you know these two sticks are equal? How do you know they are the same size?

The two pieces were obviously of different lengths, and the two children made a visual comparison between the two pieces. But, Jason went further and suggested: "Copy the biggest one and then copy them again," and then said "No," shaking his head. After that, the two children sat in silent concentration.

The teacher-researcher tried to imagine finding a way of cutting the stick into two equal pieces as if he were one of the children. Based upon his interpretation that the children could find no action in their computer microworld that would yield two equal pieces of the stick,[11] he suggested to Patricia that she draw a shorter stick that would be "easier to divide" In this suggestion, the tone of the researcher's voice indicated to the students that even he could not find where to cut this rather long stick. The teacher-researcher had no foreknowledge of where his suggestion might lead the students. That they led to productive interactions and spontaneous contributions by Patricia and Jason is exhibited in Protocol II.

Protocol II

> Teacher: (After Patricia had drawn a stick about one decimeter in length.) Now, I want you to break that stick up into two equal pieces of the same size.
>
> Patricia: (She places her right index finger on the right endpoint of the stick, then places her middle finger to the immediate left of her index finger. She continues in this way, walking her fingers along the stick in synchrony with uttering) One, two, three, four, five. (She stops when she is about one-half of the way across the stick.)
>
> Jason: (He places his right index finger where Patricia left off; he uses his right thumb rather than his middle finger to begin walking along the stick. He changes to his left index finger

rather than his right thumb after placing his thumb down once. He continues in this way until he reaches the left endpoint of the stick, uttering) Six, seven, eight, nine, ten. (Then) they're five and five. (He smiles with satisfaction).

P: (She smiles also).

The actions that the children took after the researcher's directive in Protocol II were contributed by the students and were not suggested by the teacher-researcher. Patricia introduced independently the action of walking her fingers along the stick until she arrived at a place that she regarded as one-half of the way across the stick. Jason picked up counting where Patricia left off, which indicates that he shared her goal to find a way to establish equal pieces of the stick. Patricia's counting activity was meaningful to him, and he could be said to engage in cooperative mathematical activity with Patricia.

In a teaching episode, as in a clinical interview, the students' reasoning is the focus of attention (Ackermann, 1995). When the students' reasoning proves to be rich and full of implications for further interaction by the teacher-researcher, he or she often turns to analytical rather than to responsive and intuitive action. Researchers' abilities to engage in analytical action frequently follows an insight into the mental operations that make students' language and actions possible. The teacher-researcher formulates an image of the students' mental operations and an itinerary of what they might learn and how they might learn it.[12] Initially, this itinerary is articulated loosely or not at all. Nevertheless, the teacher-researcher has a sense of direction and a sense of possibilities for where the he or she might try to take the students. The teacher-researcher now has initial goals along with a sense of possibilities for how the goals might be achieved in future teaching episodes. As the teacher-researcher engages the children in further teaching episodes, this goal structure becomes extended and articulated. The most important feature of the extension and articulation is that the teacher-researcher modifies the goal structure constantly while developing it to fit the students' mathematical activity. Extending and modifying the goal structure lasts until the students' schemes seem to be well established and the students seem to have reached a plateau. At this point, where the teacher-researchers might try to take the students and how they might take them again become major issues.

<u>Analytical Interaction</u>. The teacher-researcher sometimes will have a precise hyptothesis about students' schemes or operations. These are testable through an analytic interaction -- an interaction

with students inititated for the purpose of comparing their actions in specficic contexts with actions consonant with the hyptothesis. For example, because of the independence of Patricia's and Jason's contributions in Protocol II, a witness of the teaching episode inferred that the numerical operations of both children were activated and that they used these numerical operations in their attempts to partition the stick into equal pieces of indefinite size. This inference proved to be crucial in this particular teaching experiment because it served as a foundation for generating and testing a hypothesis of how these two children constructed fraction schemes.

Based on our interpretation that Jason and Patricia partitioned the stick mentally in Protocol II into an indefinite numerosity of sticks of equal but indefinite size before counting, the hypothesis was formulated that these children could establish an equipartitioning scheme. We generated the situation of learning of Protocol III in a test of our hypothesis in a teaching episode just three days after the one of Protocol II. The two children drew a stick of approximately one decimeter in length and the teacher-researcher asked the children to find the share for one of four people.

Protocol III

Teacher: Let's say that the three of us are together and then there is Dr. Olive over there. Dr. Olive wants a piece of this candy (the stick), but we want to have fair shares. We want him to have a share just like our shares, and we want all of our shares to be fair. I wonder if you could cut a piece of candy off from here (the stick) for Dr. Olive?

Jason: (Using the MARKS command, he makes three marks on the stick, estimating the place for the marks visually.)

Patricia: How do you know they are even? There is a big piece right there.

Jason: I don't know. (He clears all the marks and then makes a mark indicating one share. Before he can continue making marks, the teacher-researcher intervenes.)

Teacher: Can you break that somehow? (The teacher-researcher asks this question to explore the nature of Jason's partitioning operations.)

Jason: (Using the BREAK command, he breaks the stick at the mark; then he makes three copies of the piece and aligns them end-to-end under the remaining piece of the stick starting from the left endpoint of the remaining piece as in Figure 2.)

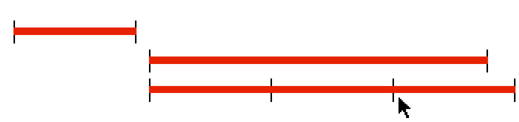

Figure 2: Jason Testing If One Piece Cut From A Stick Is One Of Four Equal Pieces

Teacher: Why don't you make another copy? (This suggestion was made to explore if Jason regarded the piece as belonging to the three copies as well as to the original stick.)

Jason: (He makes another copy and aligns it with the three others. He now has the four copies aligned directly beneath the original stick, which itself is cut once. The four pieces together are slightly longer than the original stick, as illustrated in Figure 3.)

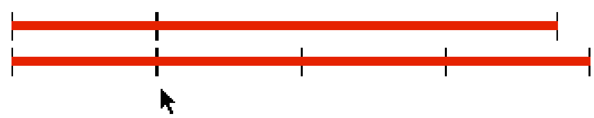

Figure 3: Jason's Completed Test.

We regard the scheme that Jason established in Protocol III as an equipartitioning scheme because he copied the part he broke off from the stick three times in a test to find if three copies would reconstitute the remaining part. After the suggestion to make another copy, Jason joined the four copied parts to ascertain if they were the same length as the original stick. This indicated that he was trying to find if the piece that he broke off from the stick was a fair share. This way of working was a modification of his numerical operations displayed in Protocol II.[13] It was presaged by his comment: "Copy the biggest one and copy them again," following Protocol I.

To claim that a modification constitutes an accommodation, it is necessary to observe the modification in subsequent interactions. Toward this end, after Jason had compared the copies with the original in Protocol III, we observed Patricia draw a stick a bit longer than the piece that Jason had cut off, and she, too, made copies and compared them with the original stick. This was not merely an imitation. After finding that her copies made a stick that was too long, Jason drew a shorter stick, and, after copying and joining, he found that the four pieces together were too short. The two children proceeded in this way until they finally drew an appropriate stick. These sustained attempts were executed independently of the teacher's directives, and they confirm the internal necessity that the children must have felt that the four pieces together must be exactly the same length as the original stick. The two children had constructed an equipartitioning scheme out of the raw material of the numerical operations manifest in Protocol II. In the teaching experiment, this was a critical event and we could see possibilities for the children in their construction of fraction schemes that were unavailable to us before. We now had a definite direction to pursue, even though we fully expected to generate and test many more hypotheses.

Any knowledge claim that we make concerning students' mathematical knowledge is based on what students contribute independently to the situations that they establish. This is indispensable, but unless we had presented Jason and Patricia with the situation of Protocol III, we would not have experienced their equipartitioning scheme. This observation is crucial because adults' concepts and operations modify their interaction in the same way that students' concepts and operations modify their interaction. Not only are situations of learning a function of what we impute to students, but also their zones of potential construction are essentially a function of our interpretations. We are in accordance with Maturana (1978) that "We literally create the world in which we live by living it. If a distinction is not performed, the entity that this distinction would specify would not exist" (p. 61).

The Role Of A Witness Of The Teaching Episodes. Communication with students can be estabilished more easily if the teacher-researcher has a history of interactions with students similar to the students involved in the teaching experiment. This is, in part, why we recommended strongly that the teacher-researcher engage in exploratory teaching before attempting to conduct a teaching experiment. Recognizing mathematical language and current students's actions in an interaction that one has experienced before is a source of confidence for the teacher-researcher that communication is indeed being established. However, the teacher-researcher should expect to encounter students operating in unanticipated and apparently novel ways as well as their making unexpected mistakes and becoming unable to operate. In these cases, it is often helpful to be able to appeal to an observer of the teaching episode for an alternative interpretation of events.

Being immersed in interaction, the teacher-researcher may not be able to step out of it, reflect on it, and take action on that basis. This is very difficult because the teacher-researcher would have to "be" in two places in a very short time frame--in the interaction and outside it. It is quite impossible to achieve this if there are no conceptual elements available to the teacher-researcher that can be used in interpretating the current situation. In this case, the teacher-researcher would be caught up in trying to respond to what the student just said or did and would not reflect on the student's contribution. In this case, the observer may help the teacher-researcher both to understand the student and to posit further action.

There are also occasions when the observer might make an interpretation of a student's actions that is different from that of the teacher-researcher for any one of several reasons. The observer also might catch important elements of a student’s actions that apparently are missed by the teacher-researcher. In these cases, the observer may ask the teacher-researcher if he or she would like another opinion. The teacher should have the right of refusal if he or she is confident of having a clear understanding of what the student is doing or has plans for action.

The experiences of being a teacher in a teaching episode and being an observer of a teaching episode are quite distinct. We have mentioned already the difficulties of a teacher becoming an observer of his or her own circumstances of observation, and this is why there should be a witness of every teaching episode. It is helpful for the witness and the teacher to interchange roles in an attempt to improve their communication. It also is helpful for both of them to be involved in planning the next teaching episode.

Teaching As A Method Of Scientific Investigation. As indicated earlier, a primary goal of the teacher in a teaching experiment is to establish living models of students’ mathematics, as exemplified in the protocols. The goal of establishing living models is sensible only when the idea of teaching is predicated on an understanding of human beings as self-organizing and self-regulating. If students were not self-regulating and self-organizing, a researcher would find that they would make no independent contributions in the way Jason contributed to what traspired in Protocol II. That is, without being told to make three copies of the stick and compare the copies with the part remaining and without having observed such activity, Jason engaged in these operations purposefully. We did not observe Jason’s operations in any social interaction in which he engaged. Rather, they were initially observed in his interactions in the microworld.

Moreover, if students were not self-regulating or self-organizing, they would not modify their numerical schemes independently, as Jason and Patricia did in the establishment of an equipartitioning scheme. Clearly, in the absence of independent contributions from the students, there would be no scientific reason for conducting a teaching experiment. The researchers would not be constrained to the students’ ways and means of operating, and there would be no basis for attributing to students a mathematical reality independent of the researchers'. On the other hand, the experience that students are self-organizing systems in the sense explained by Maturana (1978) suggests that they do have their own mathematical reality.

It is easy to undervalue teaching as a method of scientific investigation where the emphasis is on the researchers' learning the students' mathematical thinking and modifications of that thinking. In their attempts to learn students' mathematics, the researchers create situations and ways of interacting with students that encourage the students to modify their current thinking. It is with respect to these situations and interactions that the students emerge as self-regulating and self-organizing.

Stressing self-regulation and self-organization does not mean that the students are not teachable or that we do not attempt to induce learning in them. Quite to the contrary, we consider students to be teachable if, first, we can engage them in situations in which they use their mathematical schemes independently and, second, if we can observe independent modifications in their use of these schemes. We contrast this idea of students being teachable with Maturana's (1978) idea of an instructable system:

> The general idea of an instructable system is this: If the state a system adopts as a result of an interaction were specified by the properties of the entity with which it interacts, then the interaction would be an instructive interaction. Systems that undergo instructive interactions cannot be analyzed by a scientific procedure. In fact, all instructable systems would adopt the same state under the same perturbations and would necessarily be indistinguishable to a standard observer (p. 34).

If students were instructable in Maturana's sense, they would be uninteresting scientifically because it would be possible to explain their mathematical knowledge by appealing to the teacher-researcher's mathematical knowledge. The teaching experiment's coherence resides in what the teacher-researcher can say about bringing forth, sustaining, and modifying students' mathematical schemes.

Toward this end, we establish schemes of action and operation that fit within our constraints and attempt to induce changes in these schemes. This production of schemes is compatible with El'konin's (1967) assessment of Vygotsky's research that the essential function of a teaching experiment was the production of models:

> Unfortunately, it is still rare to meet with the interpretation of Vygotsky's research as modeling, rather than empirically studying, developmental processes (p. 36).

So, we can begin to appreciate how important the independent contributions of our students are to us as we work with them in teaching experiments. Just as important are those boundaries that we establish on students' ways and means of operating by using their essential mistakes. These essential mistakes serve as constraints for the researchers and are a basic source of their problems. In fact, it is our goal to eliminate the constraints that we experience when teaching students, and the only way that can happen is if we succeed in helping students to eliminate the constraints that they might experience by modifying their schemes. The only way that we can learn to solve our problems is for the students to learn to solve the problems that we present to them. This is a most sobering realization because we have found that it can take students long periods of time to construct the knowledge necessary to overcome their essential mistakes.[14]

Learning and Development in a Teaching Experiment

A virtue of a teaching experiment is that it allows the study of constructive processes, which is in part understood as the accommodations that students make in their functioning schemes. Because of the researcher's continued interaction with students, he or she is likely to observe at least the results of those critical moments when major restructuring is indicated by changes in a student's language and actions. Documenting such major restructuring of mathematical schemes is compatible with a vital part of Vygotsky's (1978) emphasis on studying the influence of learning on development. Further, our emphasis on modeling such mathematical development is compatible an essential part of Vygotsky's experimental-genetic method, which El'konin (1967) described as allowing "a dissection in abstract form of the very essence of the genetic process of concept formation" (p. 36).

A way to think about development in the context of teaching is that the essential mistakes of students most often disappear through the processes involved in development. These processes are set in motion in particular interactions but continue beyond any specific interaction. Learning, on the other hand, occurs in particular interactions in which students modify their current schemes using available operations in new way. A study of both development and learning is involved in teaching experiments. Neither is caused by teaching actions and, in this sense, both are spontaneous.[15]

Our focus on mathematical development in the context of teaching does not exclude us from taking advantage of the products of spontaneous development. In Piaget's (1964) view, spontaneous development was regarded as the results of children's interactions in their physical and sociocultural milieu. To make these results accessible to us in teaching experiments, we focus on what students contribute independently to our interactions with them. In this way, we are able to bring the history of students' spontaneous development into their mathematical education and to formulate and test hypotheses concerning the contributions of this history to their mathematical education. In particular, our goal is to bring forth the schemes that students have constructed through spontaneous development and to use them in the formulation of the major research hypotheses of the teaching experiment.

Because we focus on the schemes that students construct through spontaneous development, we are able to take advantage of the products of spontaneous development in our study of learning. In particular, we are able to regard learning as spontaneous in the frame of reference of the students. However, Piaget (1964) regarded learning as provoked as opposed to spontaneous:

> In general, learning is provoked by situations--provoked by a psychological experimenter; or by a teacher with respect to some didactic point; or by an external situation. It is provoked in general, as opposed to spontaneous. In addition, it is a limited process--limited to a single problem, or to a single structure (p. 8).

Learning is not spontaneous in the sense that the provocations that occasion it might be intentional on the part of the teacher-researcher. In the child's frame of reference, though, the processes involved in learning are essentially outside of his or her awareness. This is indicated by the observation that what students learn often is not what was intended by the teacher-researcher. It also is indicated when a child learns when the teacher-researcher has no such intention. Even in those cases where students learn what a teacher-researcher intends, the event that constitutes learning arises not because of the teacher's actions. Teaching actions only occasion students' learning (Kieren, 1994). Learning arises as an independent contribution of the interacting students. In the case of Jason learning the equipartitioning scheme in Protocol III, asking him to cut one of four equal pieces from a stick would have been totally ineffective if he were yet to construct iterable units. Moreover, although Jason's actions followed from the teacher's query, we could not say that the query caused his actions. There was a break between Jason's actions and the teacher's query in that his (Jason's) actions were contributed by him, not by the teacher-researcher. To an observer, the conjunction of Jason's actions and the teacher's query may appear related, but the actions are distinct from the query. Other queries might occur that would seem fully as causal.

We do not use “spontaneous” in the context of learning to indicate the absence of elements with which the students interact. Rather, we use the term to refer to the noncausality of teaching actions, to the self-regulation of the students when interacting, to a lack of awareness of the learning process, and to its unpredictability. Because of these factors, we regard learning as a spontaneous process in the students’ frame of reference.

If learning is placed in the context of accommodation of the products of spontaneous development, it need not be regarded as limited to a single problem or as a limited process. In fact, in a teaching experiment, it is never the intention of the teacher-researcher that the students learn to solve a single problem, even though situations are presented to them that might be a problem for them. Rather, the interest is in understanding the students’ assimilating schemes and how these schemes might change as a result of their mathematical activity. Neither Jason nor Patricia was limited to the situation in which they established their equipartitioning schemes. Rather, their schemes applied to any situation in which a whole was shared into a specific number of parts.

We can now make our reasons for engaging in exploratory teaching more clear. Through exploratory teaching, our goal is to bring forth the schemes that students have constructed through spontaneous development. It is essential for researchers to have constructed a topography of students’ spontaneous schemes prior to engaging in the individual interviews at the beginning of a teaching experiment for the purpose of selecting student-participants. Our recommendation that the spirit of exploratory teaching be continued throughout a teaching experiment is based in part on the reality that schemes different from those that were identified before the experiment began may emerge during it.

Learning as Accommodation.

In Piaget's (1964) position that learning is subordinate to spontaneous development, "learning" referred to the results of specialized interactions. In teaching experiment methodology, "learning" also refers to the results of specialized interactions. But, here, learning is regarded as being based on spontaneous development rather than being subordinate to it. What is of interest to us is how students might modify their spontaneous schemes in the context of specialized mathematical interactions.

Because it is a goal of the teacher-researcher to bring forth students' spontaneous schemes, an important function of the teacher-researcher in a teaching experiment is to foster students' successful assimilation. In this, we emphasize students' mathematical play as a particular form of cognitive play (Steffe & Wiegel, 1994) . In mathematical play, the teacher-researcher tries to engender generalizing assimilation.

An assimilation is generalizing if the scheme involved is used in situations that contain sensory material that is novel for the scheme (from the point of view of an observer), but the scheme does not recognize it (until possibly later, as a consequence of the unrecognized difference), and if there is an adjustment in the scheme without the activity of the scheme being implemented (cf. Steffe & Wiegel, 1994). Another case would be where giving meaning to utterances entails a coordination of the schemes involved in interpretation again without the activity of the schemes being implemented. Stressing generalizing assimilation is compatible with Ackermann's (1995) rule that, in a clinical interview, constraints of the situations used in a test of the limits of the child's thinking should be varied. But, it is only compatible because, rather than testing the limits of the child's thinking initially, we encourage students to use their schemes in situations that not only include novel elements from the researchers' point of view, but also that are so similar to situations constituted by students' current schemes that they may not identify the novelty (Thompson, 1994). In doing this, limitations in students' thinking may occur unexpectedly.

The situations used to encourage generalizing assimilation should be interesting and challenging for students and excite their mathematical imagination, but they should not be so far beyond their current schemes that they require students to make major accommodations in the scheme. The teacher-researcher intentionally varies the context, the material, and the scope of the situations, but not the conceptual operations involved .[16]

Mathematical play, as a form of cognitive play, is a necessary prelude for students' engagement in independent mathematical activity. Independent mathematical activity is goal-directed,[17] and it can evolve out of mathematical play with the subtle guidance of the teacher-researcher. Independent mathematical activity can be either an individual or a social activity. As social activity, from the researchers' perspective, students' independent mathematical activity comprises a self-regulating and possibly self-sustaining social system in the sense that Maturana (1978) spoke of a consensual domain of interactions.[18]

Learning how to bring forth and sustain students' independent mathematical activity is a part of learning how to interact with students in a teaching experiment. A goal in this is for students to make their mathematical knowledge explicit and to find limits in their ways and means of operating. Another goal is for students to come to understand mathematics as something that belongs to them. In other words, two of the goals of the teaching experiment are to establish the zones of actual construction of the participating students and to specify the independent mathematical activity of the students in these zones.

<u>Functional Accommodations</u>**.** By an accommodation of a scheme, we mean any modification of the scheme that is permanent.[19] An accommodation is functional if it occurs in the context of the scheme being used (Steffe, 1991). To encourage accommodation, the teacher-researcher chooses learning situations within the assimilatory power of the students that contain elements that might engender perturbation in the use of the schemes. The elements might block use of the schemes, they might lead to inadequacies in the schemes' activity, or they might lead to ambiguities in the results of the schemes. The accommodations that we have in mind differ from generalizing assimilation (which also can be regarded in the context of accommodation) in that they consist of a novel composition of the operations available or changes in the activity of the scheme. They go beyond use of the scheme in a situation in which it has not been used previously, which is an essential characteristic of generalizing assimilation.

When fostering accommodation, the teacher-researcher's intention is for the students to use their schemes in novel ways. In fact, the teacher-researcher must decide when he or she can pose situations of learning[20] that take them appropriately beyond their current independent mathematical activity. The teacher-researcher is engaged in hypothesis generation and testing now, where hypotheses are formulated about what the students might learn mathematically beyond their current ways and means of operating.

A major part of the teaching experiment, then, is for the teacher-researcher to generate situations of learning systematically and to test conjectures and local hypotheses about the mathematical learning of the students. In these activities, conjectures and local hypotheses of the teacher-researcher are documented along with how they were tested. Protocol III is an example of a test of the hypothesis that Jason and Patricia had constructed partitioning operations. It is important to note that a test of this hypothesis did not require a simple reproduction of the observation that the children used their counting schemes in partitioning in Protocol II. Rather, it entailed a test of the generative power of the children in learning situations quite distinct from the original situation of observation. This is a major modus operandi in a teaching experiment. We are not as concerned with replicating an observation to establish the viability of the interpretation involved in the observation as we are with establishing what else the student can do that follows on from the interpretation.

Developmental accommodations. Sometimes, the teacher-researcher chooses situations within the students' assimilatory power that, from the point of view of the teacher-researcher, contain elements that are outside of the students' schemes. The students' would notice these elements only if their schemes did not work or if their attention were drawn to them. In either case, from the researcher's perspective, these elements would block a resolution of the situation unless a major reorganization of the students' schemes occurred. For example, a child may compare the boys and the girls when asked if there are more children or more boys in the classroom because of a lack of hierarchical classification operations. If this were the case, class inclusion would be irrelevant to the child's scheme of classification, and the child would not establish relations between the boys and the children, or the girls and the children. From the child's perspective, comparing the boys and the girls directly solves the situation, and it may not be possible for the teacher-researcher to engender an awareness of the inclusion relation in the child.

These kinds of situations constitute currently unsolvable problems for students from the researchers' perspective, and we use them to check students' current developmental levels. We also use them in an attempt to induce an awareness in the student of the possibility of a result different from the one that they may have established. In doing so,, the teacher-researcher might ask a question or make a comment that is intended to induce an element of doubt in the students: for example, the teacher-researcher might make counter suggestions such as “Another child we saw yesterday thought that. . . . do you think this makes sense” (Ackermann, 1995, p. 347). The teacher-researcher also might ask the students to explain what situation they solved. Then, the teacher-researcher can repose the situation in an attempt to make a contrast between it and the students’ situations. In making the contrast, it is the teacher’s goal that the students reorganize their thinking in a way that will lead to a solution of the situation.

Still another technique that we use is to ask students to anticipate the outcome of their operations. This technique is similar to Ackermann’s (1995) idea of inviting the child to make guesses (anticipations) and expressing these guesses in various ways. Using this approach, we often encourage the students to take action as Ackermann (1995) does: ”Let’s try and see what happens!” (p. 347). In any event, the teacher-researcher must be prepared to abandon a situation when it becomes apparent that the students cannot find an action that would lead to the reorganization envisioned

Retrospective Analysis and Model Building

Retrospective Analysis. Retrospective analysis of the public records made of the interactive mathematical communication in a teaching experiment is a critical part of the methodology. It is even more labor-intensive than the activity of teaching. In fact, most researchers who propose teaching experiments fail to plan adequately for retrospective analyses of the teaching episodes. Through teaching and witnessing, researchers have mental records of the interactions with students, and it becomes clear to the researchers who engage in videotape analyses that much of what was learned when working with the students was learned spontaneously and outside their awareness. Careful analysis of the videotapes offers the researchers the opportunity to activate the records of their past experiences with the students and to bring them into conscious awareness.

When the researchers recognize an interaction as having been experienced before, past interpretations of the students’ activity that were made on the fly may reoccur to the teacher-researcher. However, through watching the video-tapes, the teacher-researcher has the advantage of making an historical analysis of the students’ mathematics retrospectively and prospectively, and both of these perspectives provide insight into the students' actions and interactions that were not available to the teacher-researcher when the interactions took place. It is especially important that the teacher-researcher be able to take a prospective view of the interacting child and interpret the significance of what the students may or may not have been doing from that perspective. In this way, the researcher can set the child in a historical context and modify or stabilize the original interpretations, as the case may be.

There are also those inevitable cases where the researchers do not recognize an interaction as having been experienced before. In these cases, the teacher-researchercan make novel interpretations in terms of his or her evolving concept of the students' mathematics. In any event, what the researchers are trying to do is to construct elements of the models of the constructing students over the course of the teaching experiment. It is a distinct advantage to understand these constructing minds as being occasioned by the teacher-researcher's own ways and means of operating because that understanding holds the potential of bridging the gap between research and practice that plagued us earlier on.

Model-building. Through retrospective analyses, we attempt to bring to the fore the activity of model building that was present throughout the teaching episodes. In the modeling process, we use concepts in the core of our research program like assimilation, accommodation, cognitive and mathematical play, communication, development, interaction, mental operation, self-regulation, scheme, zone of potential construction, and others. These concepts emerge in the form of specific and concrete explanations of students' mathematical activity. In this regard, the modeling process in which we engage is compatible with how Maturana (1978) regards scientific explanation:

> As scientists, we want to provide explanations for the phenomena we observe. That is, we want to propose conceptual or concrete systems that can be deemed intentionally isomorphic to the systems that generate the observed phenomena. (p. 29)

Our modeling process is only compatible because we have no access to students' mathematical realities outside of our own ways and means of operating when bringing the students' mathematics forth. So, we cannot get outside our observations to check if our conceptual constructs are isomorphic to the students mathematics. But we can and do establish viable ways and means of thinking that fit within the experiential constraints that we established when interacting with the students in teaching episodes.

These ways of thinking allow us to "get inside the heads" of students and specify explanatory concepts and operations. So, through the use of the core concepts of our research program, we make concrete claims about the mental functioning of students, and these claims draw their operational character from the framework itself. The unavoidable circularity in model-building between the framework and the models is what drives and sustains our research programs in that the core concepts are subject to modification in their use, and new concepts arise out of model-building that modify our core concepts substantially.

Model-building involves the creativity of the researchers, and the processes involved are themselves open for investigation. One important aspect is trying to think as students think (Thompson, 1982):

> [A researcher] constructs a model just as [he or she would construct] any other conceptual system—by reflectively abstracting and relating operations which serve to connect experientially derived states. Here I am applying Piaget's notion of reflective abstraction to the researcher. As he or she watches a student ease through some problems and stumble over others, or successively ease and blunder through parts of a problem, the researcher asks himself, "What can this person be thinking so that his actions make sense from his perspective?" This is the ground floor of modeling a student's understanding. The researcher puts himself into the position of the student and attempts to examine the operations that he (the researcher) would need and the constraints he would have to operate under in order to (logically) behave as the student did (p. 161).

One does this for each student in an investigation, and, as soon as one begins to see a pattern in one's mode of explanation, the job must be expanded to abstracting reflectively the operations that one applies in constructing explanations. When the researcher becomes reflectively aware of these operations, and he can relate one with another, he or she has an explanatory framework. This explanatory framework usually opens new possibilities for the researcher who turns to using it for new purposes.

Because the models that we formulate are grounded in our interactions with students, we fully expect that the models will be useful to us as we engage in further interactive mathematical communication with other students. The models are also useful instruments of communication with others interested in the mathematical education of students. However, unless the models reemerge in interactions with students, they would not be useful to anyone, including the model-builders.

Are Teaching Experiments Scientific?

As we have indicated, teaching experiments are concerned with conceptual structures and models of the kinds of change that are considered learning or development. This focus entails two specific conditions. First, it is important to emphasize that no single observation can be taken as an indication of learning or development. Change is the transition from one point to another and therefore requires at least two observations made at different times.[21] Second, when we speak of learning or development, we have in mind not any change, but change in the students current mathematical schemes that occur in interactive mathematical communication. Such change might be anticipated by the researchers, but, often, it can be known only in retrospect because the researchers might experience a different change than the one anticipated, or even no observable change. In any case, it is the researchers who learn as well as, perhaps, the students. This point is fundamental in a teaching experiment; it is the researchers who are striving to learn what change they can bring forth in their students and how to explain such change.

Regardless of whether students change as anticipated or change in ways that are known only in retrospect, the researchers do become aware of a directionality of change. This awareness has a consequence that would be considered unusual in other kinds of research. It is quite often the case that an observation can be reinterpreted from the vantage point of a later one as a preliminary step of a change that was not discernible at the time. Such reinterpretation of past findings might be deemed improper in other branches of science, but in an investigation of mental operations that relies more on the microanalysis of videotapes than on the brief live observation of the students' activity, it is no less legitimate than, for example, the reevaluation of microbiological evidence on the basis of enlargements of a microscopic image.

Teaching experiments serve the construction of models of conceptual change. In this regard, they follow what two founders of the discipline of cybernetics have described as accepted scientific procedure:

> An intuitive flair for what will turn out to be the important general question gives the basis for selecting some of the significant among the indefinite number of trivial experiments which could be carried out at that stage. Quite vague and tacit generalizations thus influence the selection of data at the start. The data then lead to more precise generalizations, which in turn suggest further experiments and progress is made by successive excursions from data to abstractions and vice versa (Rosenblueth & Wiener, 1945, p. 317).

In the teaching experiment, hypothesis formulation, experimental testing, and reconstruction of the hypothesis form a recursive cycle. The researcher may begin with a hypothesis, a preliminary model, constructed on the basis of his or her theoretical assumptions and prior experience. Whatever the students say or do in the context of interacting with the researchers in a medium is potential data for inferences about the students' conceptual operations and serves as confirming or disconfirming the hypothesis. Initially, these inferences are necessarily vague and often spring from as yet tacit assumptions. As a rule, they suggest further experimentation that may help to make the assumptions more explicit and the inferences more precise. As the cycle continues, a firmer model of the students' mental activity begins to take shape and the cycle proceeds.

There are two basic goals that seemingly stand in opposition as the cycle of model building proceeds. First, we want to find if the model remains viable in the face of the experiments conducted by the researchers. That is, the researchers strive to generate observations that would force them to modify their models of the students' mathematics. If no such observations are forthcoming, then the researchers can regard their model as being at least temporarily viable. On the other hand, given that the researchers strive to work at the boundaries of the students' knowledge for the purpose of testing hypotheses, they always stand ready to modify their model to account for perhaps unexpected observations. Here, it is important to note that we have found that students' mathematical schemes change slowly over time and that students work at the same learning level for extended periods (cf. Steffe et al., 1988).

Even if other researchers find the inferences made about students' mathematics compatible with their observations, this would not mean that the researchers have found an objective insight into the workings of the students' minds. This other mind remains as fundamentally inaccessible to observation as the clockwork of the watch that Einstein chose as a metaphor for the universe:

> Physical concepts are free creations of the human mind, and are not, however it may seem, uniquely determined by the external world. In our endeavor to understand reality we are somewhat like a man trying to understand the mechanism of a closed watch. He sees the face and the moving hands, even hears its ticking, but he has no way of opening the case. If he is ingenious he may form some picture of a mechanism which could be responsible for all the things he observes, but he may never be quite sure his picture is the only one which could explain his observations. He will never be able to compare his picture with the real mechanism and he cannot even imagine the possibility or the meaning of such a comparison (Einstein & Infeld, 1967/1938, p. 31).

If it was scientific in that case to invent a mechanism and to test it against the observable features of a watch, it is no less so for an educational researcher to posit conceptual structures and mental operations and to investigate their fit with whatever seems relevant in the students' observable behavior. In doing this, the researcher engages in the process of experiential abstraction in a retrospective analysis of videotaped teaching episodes. The researcher may have engaged in a conceptual analysis already in establishing the conceptual constructs used in the experiential abstraction. But this does not mean that the products of such a conceptual analysis would constitute the mathematics of students without the constructs emerging in their mathematical language and actions. It is essential to note that the construction of the mathematics of students is based on a conceptual analysis of the mathematical language and actions of students as well as on theoretical constructs perhaps established in an earlier conceptual analysis not involving students. There is a dialectical relationship between the two kinds of analyses. Theoretical constructs are used in analyzing students' language and actions and thus modify the teacher-researcher's ways of interacting with students. Conversely, the theoretical constructs are modified in their use. In fact, a need may be established for constructing novel theoretical constructs. The idea of establishing a fit, then, is not merely one of confirming a previously conceived conceptual construct. Rather, establishing a fit may involve major accommodations by the teacher-researcher.

What Einstein and Infeld (1967/1938) called a "picture of a mechanism" (that could be held responsible for what one observes) is precisely what cyberneticians now call a "model"; the construction of models that can be seen in some way as analogous to a student's thought processes is the main scientific purpose of teaching experiments. From the researcher's standpoint, the conceptual structures and their change in the heads of students are very much like the workings inside a closed watch. We believe that the understanding and solving of problems that involve abstract entities such as numbers and their relations is dependent on mental operations, and there is no way to observe mental operations directly. At best, they can be inferred from a variety of observable manisfestations. From the side of the researchers, a teaching experiment includes the generation and testing of hypotheses to see whether or not the experiential world that the students' language and actions comprise allows the current interpretation that the developing model proposes. It is a question of fit rather than of match. Thus, a model is viable as long as it remains adequate to explain students' independent contributions. But no amount of fit can turn a model into a description of what may be going on. It remains an interpretation that seems viable from a particular perspective.

There is a difference between this point of view and Einstein's. In spite of the fact that he saw clearly that the scientist could no more compare his theoretical models with the real universe than the man in his metaphor could compare his invented mechanism with the closed watch, Einstein had the metaphysical belief that "God was not playing dice." In other words, he believed that, in principle, an intrinsic order could be found in reality independent of the investigator.[22] As constructivists, we are more modest. We believe that a model's viability pertains to a domain of our experience and we make no ontological claims about the nature of students' mathematics.

Final Comments

A claim that one has constructed a model of students' mathematics is a strong claim. One could question the replicability of a teaching experiment on which a model was based and the generalizability of the model itself. These are important issues. However, we do not recommend that teaching experiments be replicated in the sense that Piaget's work was replicated (Lovell, 1972). The Piagetian replication studies were useful in the sense that they encouraged the researchers doing the replications to learn the theory. Moreover, Piaget's research became widely known through the work of his replicators, but it became known in a rather distorted way. Few replication studies extended his theory because the replicators did not seem to understand that Piaget was concerned with conceptual development and not with performance. The concept of replication seemed predicated on the concept of theory-testing rather than on theory-building--using aspects of a theory to build models that supersede current ones.

On Replication

At the very minimum, researchers in a teaching experiment who make a claim about what students know are obliged to make records of the living models of students' mathematics that illustrate aspects of the claim available to an interested public. Second, researchers can build an important aspect of replication into a teaching experiment. One example is to select at least three students whose language and actions indicate similar spontaneous schemes. In this way, one can attempt to replicate case studies of individual students in the same teaching experiment. Replicate case studies contribute significantly to building models of students' mathematics (Steffe, 1991).

Third, when the researcher's goals change sufficiently to warrant conducting another teaching experiment, the current model can be used as input--as conceptual material to be reorganized. However, the primary emphasis should be on constructing superseding models.

One model is said to supersede another if it is a reorganization of the preceding model, if it solves all of the problems that the preceding model solved but solves them better, and if it solves problems that the superseding model did not solve. It is in the sense of constructing superseding models that we advocate that researchers replicate their own teaching experiment or a teaching experiment of other researchers. In our opinion, this way of thinking about replication will serve in the scientific progress of mathematics education and will provide stability in what we regard as the mathematics of students.

On Generalizability

The intent to build models that supersede current ones encourages communication among researchers. It is a way of making public the community's more or less private results as well as of checking those results. The intent to build superseding models also serves generalizability--the aim of building models that apply in principle to settings beyond the ones that gave rise to the models originally.

Asking a question about the generalizability of any model of students' mathematics is similar to asking about the generalizability of number and quantity. The issue is not that number and quantity are ideas that are generalizable. Rather, the pertinent issue is that they are concepts that prove useful in settings other than those in which they are built. It does not make sense to demand of teaching experiments that they "generalize" in the way in which one might hope that claims thought to be true about a random sample would be true as well about the population from which the sample was drawn.

The issues involved in generalizability are quite different for us. By being explanatory, the researcher's conceptual schemes are dynamic concepts in his or her life that can be used and even modified in further interactions with students. Any conceptual scheme that is constructed through experiential abstraction has this quality. If it does not, it has no place as a part of the mathematics of students. So, at the bottom line, if we find our way of thinking about students' mathematics useful in interpreting the mathematical activity of students other than those in the original teaching experiment, this provides critical confirmation of our way of thinking. It is not a matter of generalizing the results in a hypothetical way, but of the results being useful in organizing and guiding our experience of students doing mathematics.

Further, if we can reorganize our previous ways of thinking in a new teaching experiment, that is, if we can learn, aspects of the old model become involved in new relations in the new model and, thus, become generalized conceptually. When it is possible to communicate with other researchers doing teaching experiments independently of us, this also serves as a vital confirmation of our way of thinking and perhaps as a site for each of us to construct a superseding model. The element of generalization that is involved is strengthened if that other researcher launched his or her teaching experiment for the purpose of constructing a superseding model of our current model of students' mathematics.

There are cases when sampling procedures can be useful. For example, after the hard work that has gone into constructing a model of students' mathematics, the researchers may become interested in interviewing a group of students of the same age as those in the teaching experiment that gave rise to the model. The goal of the researchers would be to find disconfirmations of aspects of the model with an eye toward building a superseding model in a future teaching experiment. Questions such as finding the ratios of students entering their first grade in school with certain spontaneous schemes are of educational interest. These questions can be answered by an expert clinician using clinical interviews and appropriate sampling procedures, but we do not regard such queries as being related to questions of the generalizability of a model of students' mathematics.

On Self-Reflexivity

At every point when interacting with students in a teaching experiment, the students' and teacher's actions are codependent. The realization that the researchers are participants in the students' constructions and the students are active participants in the researcher's constructions is precisely what recommends the teaching experiment methodology. Rather than being regarded as a weakness of the methodology, it is one of its greatest strengths because it provides researchers the possibility of influencing the education community's images of mathematics teaching, learning, and curricula. In fact, the teaching experiment was designed for the purpose of eliminating the separation between the practice of research and the practice of teaching.

But the principle of self-reflexivity (Steier, 1995)--applying the principles of the methodology first and foremost to oneself-- is even more deeply embedded in the methodology than portrayed above. For example, the researchers must look within themselves to make explicit what is analogous to the practice of establishing students' spontaneous schemes. Neither the researchers nor the students start as blank slates in a teaching experiment, and the knowledge of the researchers become the most critical issue. Although we have appealed to researchers to set aside their own mathematical knowledge that they would not attribute to students for the purpose of learning from them, this does not mean that the researchers who are more or less successful in doing this will be devoid of knowledge. On the contrary, it is the researchers who attribute mathematical meaning to students' language and actions using their own instruments of assimilation.

This idea may seem to contradict almost everything that we have said. But although students' language and actions constrain the researchers' constructions, it is the researchers who bring this language and action forth in the students. Of course, what is brought forth is always a function of the researcher's current knowledge and this is why we find von Foerster's (1984) aesthetical and ethical imperatives so important in doing a teaching experiment: If you desire to see, learn how to act, and act always so as to increase the number of choices.

In previous sections, we spoke of researchers being involved in teaching experiments. One can "be involved" in teaching experiments in many ways, some direct and some indirect. But the essential involvements in a teaching experiment are as a witness and as a teacher-researcher in teaching episodes. We want to make our position completely clear in this regard. It is essential that at least some of the people doing the theorizing are active integrally both as witnesses and as teacher-researchers in any experiment's teaching. Those remaining are, at a minimum, active integrally in the retrospective analysis conducted at the end of the teaching episodes. A teaching experiment run by an "executive researcher"—someone who supervises but is essentially uninvolved in teaching or witnessing—is fundamentally flawed on methodological grounds. Such teaching experiments are unacceptable.

Notes

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[2] We use the term "student" generically to emphasize that our remarks apply to, for example, investigating children's understanding of number or college mathematics majors' understandings of the fundamental theorem of calculus.

[3] Although we stress construction as a result of interaction, we do not assume that information is transferred directly from an environment to the individual in an interaction. Whatever constitutes information for the individual is constructed by the individual using his or her own conceptual constructs.

[4] We formulated this sentence to highlight the fact that there is no one "mathematics of students." In principle, there could be variations in what researchers emphasize in teaching experiments as they study students' mathematics. We believe that these variations, rather than being a weakness of teaching experiments, are one of their greatest strengths because different possibilities for students' mathematics education emerge.

[5] For example, a symposium entitled, "The Soviet Teaching Experiment: Its Role and Usage in Americal Research" was presented at the 1978 annual meeing of the National Council of Teachers of Mathematics in San Diego, California.[6]See Zweng, Green, Kilpatrick, Pollack, & Suydam (1983) for an interesting discussion of alternative methodologies.[7] This assertion may seem to contradict the idea of a mind-dependent reality. However, the comment is made to acknowledge our awareness that these students had a history of interactions of which we were not a part. It also acknowledges that in our interactions, the students contributed their ways of counting to the situations without being told by us how to count or even to count.

[8] Establishing a mathematics of students is essential in support of the current reform efforts in mathematics education. It is insufficient to focus on reforming mathematics teaching and learning without also reforming the mathematical aspects of those activities (Steffe & Wiegel, 1992).

[9] Jason and Patricia were two children in the teaching experiment entitled "Construction of the Rational Numbers of Arithmetic", National Science Foundation Project No. RED-8954678.

[10] The child was using TIMA: Sticks, a computer microworld that we designed especially for the teaching experiment. In this microworld, the child could use the button labeled "CUT" to cut the stick.

[11] The children were not familiar yet with "PARTS", which could be used to partition a stick into equal sized pieces.

[12] We focus on learning as accommodation in the context of scheme theory. In this, what students learn is defined in terms of changes in their schemes rather than in terms of the mathematical knowledge of the researchers.

[13] Although it is possible to regard Jason's behavior in Protocol III as unrelated to his behavior in Protocol II, this would miss Jason's use of his number concepts in partitioning.

[14] The "mistakes" of Patricia and Jason in Protocol II were not essential mistakes because the children modified their ways and means of operating only three days later.

[15] Learning may be provoked by instruction, providing an impetus for students' attention and reflection, but it is not caused by instruction. See Piaget's quote, below, and ensuing discussion.

[16] Varying the context and material of the situations is similar to Dienes' (1960) perceptual variability principle. But it is similar only because the teacher also can vary certain mathematical parameters of a situation without varying the mathematical operation. For example, the teacher may use number words in the teens for students who can count only to ten with the goal of encouraging

them to learn the number words past "ten" without changing anything else about counting.

[17] We speak of shared goals in the case of cooperative mathematical activity.

[18] A consensual domain is established when the individuals of a group adjust and adapt their actions and reactions to achieve the degree of compatibility necessary for cooperation. This involves the use of language and the adjustments and mutual adaptations of individual meanings to allow effective interaction and cooperation.

[19] By "permanent," we mean that the modification reemerges independently in situations where the scheme is used.

[20] The way in which we generate these situations is similar to Dienes' (1960) mathematical variability principle. It differs in that we base our decisions on the students' mathematical schemes as well as on our own mathematical knowledge.

[21] We do not specify any minimal duration between the two times. The two times may be distinct moments within a single experiential episode or distinct moments across two different experiential episodes. What happens in between the two moments of time is critical and constitutes the study of the interactions involved. Rather than merely document the observations at the two times, what is of interest is to specify a trajectory of changes from one time to another.

[22] This is not to say that we do not seek order and regularity in students' mathematical activity. Quite to the contrary. By making the comment, we intend to highlight that what we experience as students' mathematics is unavoidably dependent upon our actions as researchers. This is the main reason we believe that the researchers in a teaching experiment must be experientially involved themselves in the students' mathematical activity.

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