Individual and Collective Mathematical Development: The Case of Statistical Data Analysis

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In the first part of this article, I clarify how we analyze students' mathematical reasoning as acts of participation in the mathematical practices established by the classroom community. In doing so, I present episodes from a recently completed classroom teaching experiment that focused on statistics. Against the background of this analysis, I then broaden my focus in the final part of the article by developing the themes of change, diversity, and equity.

In recent years, we have seen an increasing emphasis on the socially and culturally situated nature of mathematical activity. This trend encompasses a range of theoretical positions that include sociocultural theory, discourse theory, and symbolic interactionism. There are, of course, significant differences among these various perspectives that have, at times, been the subject of intense debate. However, rather than highlight differences, I focus on a central notion that I believe cuts across these positions and serves to differentiate them from purely psychological perspectives—that of participation in communal practices. In developing this notion, I ground the discussion in my own and my colleagues' work in classrooms. My immediate goal is to clarify how we analyze students' mathematical reasoning as acts of participation in the mathematical practices established by the classroom community. In doing so, I present episodes from a recently completed classroom teaching experiment that focused on statistics. Against the background of this analysis, I then broaden my focus in the final part of the article by developing the themes of change, diversity, and equity.

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ORIENTATION: DEVELOPMENTAL RESEARCH AND
THE CLASSROOM MICROCULTURE

The type of research that my colleagues and I conduct involves classroom teaching experiments of up to 1 year in duration (cf. P. Cobb, in press; Confrey & Lachance, in press; Simon, in press; Yackel, 1995). In the course of these experiments, we both develop sequences of instructional activities and analyze students' mathematical learning as it occurs in the social situation of the classroom. Research of this type falls under the general heading of *developmental research* (Gravemeijer, 1994) in that it involves both instructional *development* and classroom-based *research*. It should, therefore, not be confused with either child development research or with research into the development of particular mathematical concepts. The basic developmental research cycle is shown in Figure 1. The first aspect of the cycle involves developing instructional sequences as guided by a domain-specific instructional theory. In our case, we draw on the theory of realistic mathematics education developed at the Freudenthal Institute (Gravemeijer, 1994; Streefland, 1991; Treffers, 1987). Gravemeijer has written extensively about the process of instructional design in developmental research and clarifies that the designer initially conducts an anticipatory thought experiment. In doing so, the designer envisions how students' mathematical learning might proceed as the instructional sequence is enacted in the classroom, thereby developing conjectures about both (a) possible trajectories for students' learning and (b) the means that might be used to support and organize that learning. It is important to stress that the conjectures are tentative and provisional, and they are tested and modified on a daily basis during the teaching experiment. These adaptations and revisions are informed by an ongoing analysis of classroom events, and it is here that the second aspect of the developmental research cycle—classroom-based analyses—comes to the fore.
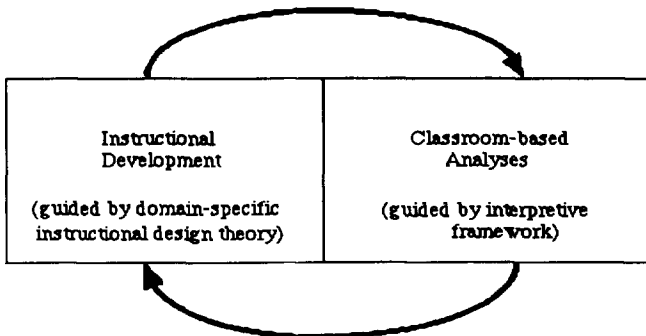

FIGURE 1   The developmental research cycle.

Interpretations of classroom events reflect suppositions and assumptions about learning, teaching, and mathematics as well as about the general relation between individual activity and communal processes. In my own case, for example, my colleagues and I initially intended to analyze students' mathematical reasoning in purely psychological terms when we began working intensively in classrooms 12 years ago. This is not to say that we ignored the role of social interaction in supporting mathematical learning. The classroom sessions in the first teaching experiment that we conducted, in fact, involved small-group work followed by whole-class discussions of students' mathematical interpretations and solutions. However, we treated social interaction and discourse as a catalyst for otherwise autonomous mathematical development and did not view them as influencing the products of learning—increasingly sophisticated mathematical ways of knowing.

Incidents that occurred at the beginning of the first teaching experiment, which was conducted with 7-year-old students in a second-grade classroom in the United States, led us to question our sole reliance on an individualistic, psychological orientation. Briefly, the teacher with whom we collaborated expected her students to engage in genuine discussions in which they explained and justified their mathematical reasoning. However, as a consequence of their prior experiences in school, the students assumed that their role was to infer the responses that the teacher had in mind all along rather than to articulate their own interpretations. The teacher coped with this conflict between her own and the students' expectations by initiating a process that we subsequently came to term the negotiation of classroom *social norms* (P. Cobb, Yackel, & Wood, 1989). Examples of social norms that became explicit topics of discussion included explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement or disagreement, and questioning alternatives in situations in which a conflict in interpretations had become apparent. In general, an analysis that focuses on social norms serves to delineate the classroom participation structure (Erickson, 1986; Lampert, 1990). These norms therefore constitute a crucial aspect of the classroom microculture that is continually regenerated by the teacher and students in the course of their ongoing interactions.

As this brief summary makes clear, our interest in classroom social norms and, more generally, in the classroom microculture did not arise as an end in itself. Instead, it emerged within the context of developmental research as we attempted to further our agenda of supporting students' mathematical learning in classrooms. It was within this context that we subsequently came to view one aspect of our analysis of classroom social norms as inadequate. In particular, we came to realize that these norms are not specific to mathematics; rather, they apply to any subject matter area. For example, one might hope that students would explain and justify their reasoning in science or history classes as well as in mathematics. We attempted to address this limitation by shifting our focus to normative aspects of students' activity that are specific to mathematics

(Lampert, 1990; Voigt, 1995; Yackel & Cobb, 1996). Examples of these so-called sociomathematical norms include what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation.

As we have noted elsewhere (Yackel & Cobb, 1996), the analysis of sociomathematical norms has proven useful in helping us understand the process by which teachers can foster the development of intellectual autonomy in their classrooms. This issue is particularly significant to us, given that the development of student autonomy was an explicitly stated goal of our work in classrooms from the outset. However, we originally characterized intellectual autonomy in individualistic terms and spoke of students' awareness of and willingness to draw on their own intellectual capabilities when making mathematical decisions and judgments (Kamii, 1985; Piaget, 1973). As part of the process of supporting the growth of autonomy, the teachers with whom we have worked initiated and guided the development of a community of validators in their classrooms, such that claims were established by means of mathematical argumentation rather than by appealing to the authority of the teacher or textbook. However, for this to occur, it was not sufficient for the students merely to learn that they should make a wide range of mathematical contributions. It was also essential that they become able to judge both when it was appropriate to make a mathematical contribution and what constituted an acceptable contribution. This required, among other things, that the students could judge what counted as a different mathematical solution, an insightful mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation. However, these are precisely the types of judgments that are negotiated when establishing sociomathematical norms. We therefore conjectured that students develop specifically mathematical beliefs and values that enable them to act as increasingly autonomous members of classroom mathematical communities as they participate in the negotiation of sociomathematical norms (Yackel & Cobb, 1996).

It is apparent from this account that we revised our conception of the most individualistic of notions—intellectual autonomy—as we worked in classrooms. At the outset, we defined autonomy in purely psychological terms as a characteristic of individual students' activity. However, as we developed the idea of sociomathematical norms, we came to view autonomy as a characteristic of an individual's way of participating in a community. In particular, the development of autonomy can be viewed as synonymous with the gradual movement from relatively peripheral participation in classroom activities to more substantial participation, in which students increasingly rely on their own judgments rather than on those of the teacher (cf. Forman, 1996; Lave & Wenger, 1991). The example of autonomy is paradigmatic in this regard in that it illustrates the general shift we have made in our theoretical orientation away from an initial psychological perspective toward what we call an *emergent perspective* (P. Cobb & Yackel, 1996).

Thus far, I have described two aspects of the classroom microculture that we have found useful to differentiate when conducting analyses that feed back to inform the ongoing instructional development effort. Our motivation for teasing out a third aspect of the classroom microculture, *classroom mathematical practices,* stems directly from our concerns as instructional designers. Recall that the approach we take to instructional design involves conducting an anticipatory thought experiment in the course of which the designer develops conjectures about the possible course of students' mathematical learning. However, these conjectures cannot encompass the anticipated mathematical learning of each and every student in a class, given that there are significant qualitative differences in their mathematical reasoning at any point in time. Descriptions of planned instructional approaches written so as to imply that all students will reorganize their mathematical activity in particular ways at particular points in an instructional sequence are, at best, highly idealized. It is, however, feasible to view a hypothetical learning trajectory as consisting of conjectures about the collective mathematical development of the classroom community. This proposal, in turn, indicates the need for a theoretical construct that allows us to talk explicitly about collective mathematical development. The construct that my colleagues and I have found useful is that of classroom mathematical practices that are established by the classroom community. Described in these terms, a learning trajectory then consists of an envisioned sequence of classroom mathematical practices together with conjectures about the means of supporting their evolution from prior practices.

As an initial illustration to clarify the notion of a classroom mathematical practice, consider the social norm of explaining and justifying interpretations. As I have noted, this and other social norms deal with facets of the classroom participation structure that are not specific to mathematical activity. In contrast, the related sociomathematical norms for argumentation deal with criteria that the teacher and students establish in interaction for what counts as an acceptable mathematical explanation and justification. For example, a criterion that became established during a teaching experiment that focused on place-value numeration was that explanations had to be clear, in the sense that the teacher and other students could interpret them in terms of actions on numerical quantities rather than, for instance, in terms of the mere manipulation of digits (Bowers, Cobb, & McClain, in press). Because sociomathematical norms are concerned with the evolving criteria for mathematical activity and discourse, they are not specific to any particular mathematical idea. Thus, the criterion that mathematical explanations should be clear could apply to elementary arithmetical word problems or to discussions about relatively sophisticated mathematical ideas that involve proportional reasoning. Classroom mathematical practices, in contrast, focus on the taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas. Consequently, if sociomathematical norms are specific to mathematical activity, then mathematical practices are specific to particular math-

ematical ideas. In the case of the teaching experiment that focused on place-value numeration, the analysis of mathematical practices focused on the specific arguments and ways of reasoning about quantities that the teacher and students treated as being clear and beyond further justification. In addition, the analysis described how each mathematical practice identified in this way emerged as a reorganization of prior practices, thereby providing an account of both the taken-as-shared understanding of place-value numeration that eventually was established in this particular classroom and the process by which it emerged.

It is apparent from this illustration that analyses of classroom mathematical practices account for the emergence of what traditionally is called *mathematical content* in terms of successive reorganizations of communal processes. This approach might seem controversial given that, in mathematics education, we typically view the development of mathematical ideas and concepts as a matter of individual learning. Thus, although it is now common to acknowledge seemingly nonmathematical aspects of the classroom microculture such as social norms, we typically think in terms of individual students' learning when we address issues that relate directly to mathematical content. In addition, it might seem counterintuitive to speak of the mathematical learning of the classroom community given the diversity of individual students' reasoning at any point in time. To address these concerns, I present episodes from a recently completed teaching experiment that focused on statistics to illustrate how an analysis of classroom mathematical practices characterizes changes in collective mathematical activity while taking into account the diversity in individual students' reasoning.

BACKGROUND TO THE TEACHING EXPERIMENT

The teaching experiment was carried out with 29 twelve-year-old students in a seventh-grade classroom in the United States and involved 34 lessons conducted over a 10-week period.[1] A member of the project staff served as the teacher for the first 21 classroom sessions, and two members of the research team shared the teaching responsibilities for the remaining 13 sessions. The overarching mathematical idea that served to orient our instructional design effort was that of distribution. We therefore wanted students to come to view data sets as entities that are distributed within a space of possible values (Hancock, in press; Konold, Pollatsek, Well, & Gagnon, 1996; Wilensky, 1997). Notions such as mean, mode, median, skewness, spread-outness, and relative frequency then would emerge as ways of describing how specific data sets are distributed within this space of values. Furthermore, in this approach, various statistical representations

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[1]Kay McClain, Koeno Gravemeijer, Jose Cortina, Lynn Hodge, Maggie McGatha, Beth Petty, Carla Richards, Michelle Stephan, and I conducted the teaching experiment.

or inscriptions would emerge as different ways of structuring distributions. For example, students who use box plots flexibly to compare data sets are reasoning about distributions that they have structured multiplicatively. Viewed in this way, the development of increasingly sophisticated ways of structuring and organizing data is inextricably bound up with the development of increasingly sophisticated ways of inscribing data (Biehler, 1993; de Lange, van Reeuwijk, Burrill, & Romberg, 1993; Lehrer & Romberg, 1996). In general, this focus on distribution allowed us to frame our instructional intent as that of supporting the gradual emergence of a single, multifaceted mathematical notion rather than a collection of, at best, loosely related concepts and inscriptions.

In preparation for the teaching experiment, we surveyed the relevant research literature and conducted a series of interviews and classroom performance assessments with seventh graders in the same school in which we planned to work. A broad distinction that emerged from these analyses was one between additive and multiplicative reasoning about data (cf. Harel & Confrey, 1994; Thompson, 1994). Briefly, the hallmark of additive reasoning about data is that students partition one or more data sets in ways appropriate to the question or issue at hand and then reason about the number of data points in the various parts of the data sets in part–whole terms. This can be contrasted with multiplicative reasoning about data, wherein students reason about the parts of a data set as proportions of the whole data set. Our goal for the learning of the classroom community was that reasoning about the distribution of data in multiplicative terms would become an established mathematical practice that was beyond justification.

Thus far, in discussing distribution as a key mathematical idea and distinguishing between additive and multiplicative reasoning about data, I have focused on what is traditionally termed *mathematical content.* It is therefore important to stress that our instructional focus also had a process aspect in that we attempted to ensure that the instructional activities as realized in the classroom had the spirit of genuine data analyses from the outset. To this end, we developed instructional activities that involved univariate data sets and that involved either describing a single data set for a particular purpose or comparing two or more data sets to make a decision or judgment. The importance of attending to process as well as to content in statistics becomes apparent once we acknowledge that anticipation is at the heart of data analysis. Proficient analysts anticipate that certain ways of structuring and inscribing data might reveal trends, patterns, and anomalies that bear on the questions at hand. These anticipations, in turn, reflect a deep understanding of central statistical ideas. For example, a student who decides that it might be productive to inscribe data sets as box plots anticipates the possibility of structuring the data sets multiplicatively. Similarly, a student who decides to create a scatter plot does so to investigate the covariation of two sets of univariate measures. The challenge as we formulated it, therefore, was to transcend what Dewey (1980) called the dichotomy between process and content by systematically supporting the emergence

of key statistical ideas while ensuring that the successive classroom mathematical practices that emerged in the course of the teaching experiment were commensurable with the activities of proficient data analysts. As Biehler and Steinbring (1991) noted, an exploratory or investigative orientation is not merely a means of supporting learning but is instead central to data analysis and constitutes an instructional goal in its own right.

The summary I have given of our instructional intent clarifies the potential end points of the learning trajectory that we envisioned for the classroom community. With regard to the starting points, the performance assessments that we conducted with seventh-grade students prior to the teaching experiment indicated that data analysis for them involved "doing something with the numbers," frequently by using methods derived from their prior instructional experiences with statistics in school (McGatha, Cobb, & McClain, 1998). In other words, these students did not view data as measures of particular aspects or features of situations that were judged to be relevant when addressing a particular question or issue. An immediate goal at the beginning of the teaching experiment, therefore, was to ensure that the first mathematical practices established in the classroom actually involved the analysis of data. In the approach that we took, the teacher talked through the data creation process with the students. This involved discussing the particular problem or question under investigation, clarifying its significance, delineating relevant aspects of the situation that might be measured, and considering various ways of measuring them. The data the students were to analyze were then introduced as resulting from this process. We conjectured that, as a consequence of participating in such discussions, the data would have a history for the students such that it was grounded in the situation and reflected particular purposes and interests (cf. Latour, 1987; Lehrer & Romberg, 1996; Roth & McGinn, 1998).

Beyond this general instructional strategy, we developed two computer-based minitools for the students to use as integral aspects of the instructional sequence.[2] Each minitool offered students several ways of structuring data. More important, these options do not correspond to a variety of conventional inscriptions as is typically the case with commercially available data analysis tools. Instead, we drew on the research literature to identify the various ways in which students structure data when given the opportunity to conduct genuine analyses. Therefore, the tools were designed to fit with taken-as-shared ways of reasoning at particular points in the envisioned learning trajectory while serving as a means of supporting the reorganization of that reasoning. The students used these minitools in 27 of the 34 classroom sessions. Typically, they worked at computers in pairs to conduct their analyses, and then the teacher organized a whole-class discussion using a computer projection system. I subsequently describe these two minitools when I pres-

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[2]Koeno Gravemeijer, Michiel Doorman, Janet Bowers, and I developed the two minitools.

ent an analysis of two of the classroom mathematical practices that emerged during the teaching experiment.

EMERGENCE OF THE FIRST MATHEMATICAL PRACTICE

The first computer minitool was not introduced until the fifth classroom session. The whole-class discussions in the preceding four sessions typically involved a sequence of separate reports in which different students described how they had completed the instructional activities. Furthermore, students did not appear to adjust their explanations by taking into account the interpretations of the listening students, and the listening students rarely asked clarifying questions on their own initiative. The classroom participation structure established at the beginning of the teaching experiment, therefore, delimited the possibility of developing a taken-as-shared basis for mathematical communication. It is also doubtful whether most of the students were actually analyzing data, in that the numbers they manipulated did not appear to signify measures of attributes of a situation about which a decision was to be made. The analyses that many students reported involved methods derived from their prior instructional experiences of doing statistics in sixth grade. For example, an appreciable number of the students initially calculated the mean of every data set irrespective of the question at hand. In general, the students' contributions to these initial whole-class discussions appeared to reflect their prior participation in the practices of traditional U.S. mathematics instruction.

A shift in the quality of classroom discourse occurred in subsequent discussions when the students explained how they had used the first minitool to conduct their analyses. This minitool was designed to provide students with a means of ordering, partitioning, and otherwise organizing sets of up to 40 data points in a relatively immediate way. When data are entered into the minitool, each individual data point is inscribed as a horizontal bar, the length of which signifies the numeral value of the data point. The students could select the color of each bar to be either pink or green, enabling them to enter and compare two data sets. For example, Figure 2 shows data generated to compare how long two different brands of batteries last. Each bar shows a single case, the life span of the tested batteries. The students could sort the data by size and by color. In addition, they could hide either data set and could also use what they called the *value tool* to find the value of any data point by dragging a vertical red bar along the horizontal axis. Furthermore, they could find the number of data points in any horizontal interval by using what they called the *range tool.*

The initial data sets the students analyzed were chosen so that the measurements had a sense of linearity and, thus, lent themselves to inscription as horizontal bars (e.g., the braking distance of cars, the length of time that batteries lasted, etc.). Nonetheless, the students spoke almost exclusively of "pinks" and "greens" during
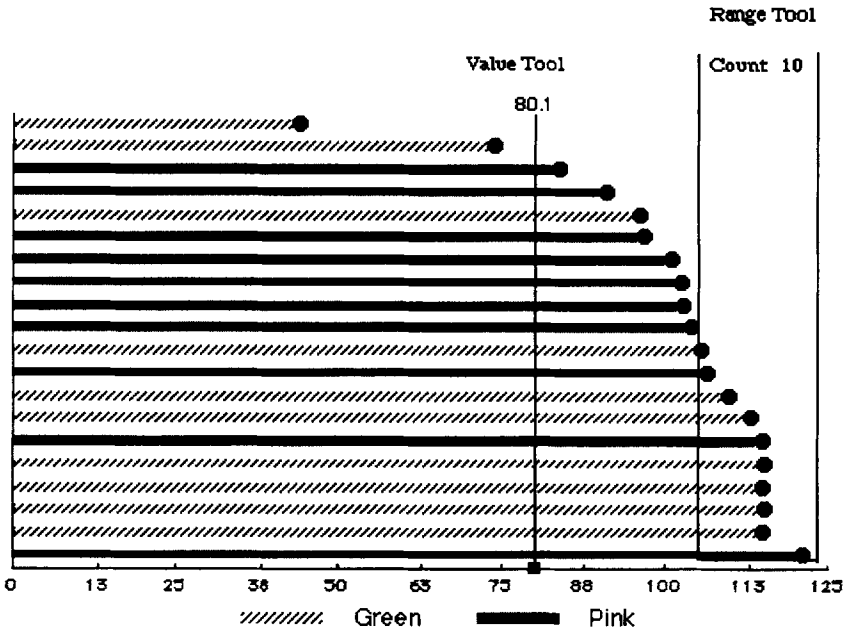
FIGURE 2   The first computer minitool.

the first whole-class discussion in which the minitool was used and did not offer conclusions with respect to the question at hand. It therefore seemed that they were describing differences in two sets of numbers inscribed as colored bars rather than analyzing data. However, the teacher was able to initiate a shift in the discourse during this session such that the students began to speak about the bars as attributes of individual cases that had been measured. This shift continued during the second discussion conducted with the minitool, when the students explained how they had analyzed the data shown in Figure 2. The green bars showed the data for a brand of battery called "Always Ready," and the pink bars showed the data for a brand called "Tough Cell." The first student who gave an explanation directed the teacher to use the range tool to bound the 10 highest values (see Figure 2).

Casey: And I was saying, see like there's 7 green that last longer.
Teacher: OK, the greens are the Always Ready, so let's make sure we keep up with which set is which. OK?
Casey: OK, the Always Ready are more consistent with the 7 right there, and then 7 of the Tough ones are like further back, I was just saying 'cause like 7 out of ten of the greens were the longest, and like ...
Ken: Good point.

Janice: I understand.

Teacher: You understand? OK, Janice, I'm not sure I do, so could you say it for me?

Janice: She's saying that out of 10 of the batteries that lasted the longest, 7 of them are green, and that's the most number, so the Always Ready batteries are better because more of those batteries lasted longer.

Although Casey spoke of "the greens," her comment that they lasted longer suggests that each bar signified how long one of the batteries lasted. Janice certainly understood Casey's explanation in these terms, and in revoicing it, both stated an explicit conclusion ("the Always Ready batteries are better") and justified it by summarizing the results of Casey's analysis ("because more of those batteries lasted longer"). In doing so, she contributed to the gradual emergence of an initial practice of data analysis.

As the episode continued, another student, James, challenged Casey's analysis by arguing that four of the pink bars (Tough Cell) were "almost in that area, and then if you put all those in you would have seven [rather than three pinks]." As James described features of the inscription, it is impossible to know whether the bars carried the significance of data for him. However, the teacher interpreted his challenge as calling into question the way in which Casey had organized the data.

Teacher: So maybe, Casey, you can explain to us why you chose 10, that would be really helpful.

Casey: All right, because there's 10 of the Always Ready and there's 10 of the Tough Cell, there's 20, and half of 20 is 10.

Teacher: And why would it be helpful for us to know about the top 10, why did you choose that, why did you choose 10 instead of 12?

Casey: Because I was trying to go with the half.

Significantly, Casey's justification for the way she organized the data, and thus her method for comparing the two types of batteries, did not make reference to the question at hand, that of comparing the two brands. It is also noteworthy that, with the possible exception of James, none of the students asked her for such a justification. This issue of justifying analyses became increasingly explicit as the discussion continued.

The next student to explain his reasoning, Brad, directed the teacher to place the value tool at 80 hr (see Figure 2).

Brad: See, there's still green ones [Always Ready] behind 80, but all of the Tough Cell is above 80. I would rather have a consistent battery

> that I know will get me over 80 hours than one that you just try to guess.
>
> Teacher: Why were you picking 80?
>
> Brad: Because most of the Tough Cell batteries are all over 80.

Possibly as a consequence of the questions that the teacher had asked Casey, Brad justified the method he had used without prompting. Furthermore, in doing so, he interpreted a feature of the inscription ("There's still green ones behind 80") as indicating a difference in the two brands of batteries that he considered significant, namely, whether the batteries of a particular brand would last consistently at least 80 hr. In this respect, his explanation involved a significant advance when compared with those that Casey and Janice had given.

Later in the discussion, Jennifer compared Casey's and Brad's analyses directly.

> Jennifer: Even though 7 of the 10 longest-lasting batteries are Always Ready ones, the two lowest are also Always Ready, and if you were using those batteries for something important, then you might end up with one of those bad batteries.

Significantly, Jennifer justified her preference for the statistic that Brad used by focusing on the pragmatic consequences of the two analyses. The obligation of justifying particular ways of organizing the data with respect to the practical issue at hand gradually became taken-as-shared during the remainder of the session. For example, toward the end of the discussion, one of the students observed the following:

> Barry: The other thing is that I think you also need to know something about that or whatever you're using them [the batteries] for.
>
> Teacher: You bet.
>
> Barry: Like, if you're using them for something real important and you're only going to have like one or two batteries, then I think you need to go with the most constant thing. But if you're going like, "Oh well, I just have a lot of batteries here to use," then you need to have most of the highest.

In making this comment, Barry explicitly clarified the situations in which the qualities of the two brands assessed by the two criteria (consistency vs. most of the highest) would be relevant.

It is important to note that, in the latter part of the discussion, 4 students volunteered that they had changed their judgments as a consequence of others' arguments. For example, Sally explained:

Sally: When you first look at the chart that you gave us like, oh, Tough Cell has more, 'cause look at all the high ones and it didn't hardly have any low ones. But when you compare them, they're a whole lot closer than what you think.

The chart that Sally referred to was a numerical table from which the students had entered the data into the minitool. Both Sally's comments and those of the other 3 students indicate that they experienced the discussion as an investigation in the course of which they had developed insights into the issue at hand, that of the relative merits of the two brands of batteries. In this respect, the discussion had the spirit of a genuine data analysis, even though the data sets were small and the methods the students proposed were relatively elementary.

The characteristics of data sets that emerged as significant in this discussion and in the subsequent classroom sessions in which the first minitool was used included the range and maximum and minimum values, the number of data points above or below a certain value or within a specified interval, and the median and its relation to the mean. The arguments that the teacher and students developed as they reasoned with the minitool, however, were generally additive rather than multiplicative in nature. In the first sample episode, for example, Casey, Janice, and the teacher jointly developed an argument that focused on how many of the 10 batteries that lasted the longest were of each brand. In doing so, they compared two data sets that they had structured in part–whole terms. This argument can be contrasted with one that focuses on the proportion of each data set that is among the 10 highest values. An argument of this type would involve comparing two data sets that have been structured multiplicatively. Crucially, such an argument is concerned with the relative amount of the data in each set that is above a certain value and, thus, with how each data set is distributed. Although additive reasoning is sufficient when comparing data sets with equal numbers of data points, the students failed to make arguments that involved reasoning about data proportionally when they experienced difficulties while comparing unequal data sets. This indicates that data sets were constituted in public classroom discourse as collections of data points rather than as distributions. The mathematical practice that emerged as the students used the first minitool might therefore be described as that of *exploring qualitative characteristics of collections of data points.* It is important to stress that these characteristics were treated as features of the situation from which the data were generated. For example, it was taken-as-shared in the sample episodes that the qualitative characteristic that the Tough Cell data were more "bunched up" indicated greater consistency. Participation in this first practice of data analysis therefore involved the fusion of inscriptions and the situations inscribed such that to use the minitool to structure data was to organize the inscribed situation (cf. Nemirovsky & Monk, in press).

EMERGENCE OF THE SECOND MATHEMATICAL PRACTICE

The students first used the second of the two computer minitools during the 22nd session of the teaching experiment. This tool was designed to allow students to analyze one or two data sets of up to 400 data points. Individual data points were inscribed as dots located on a horizontal axis of values (see Figure 3). The tool provides students with a variety of options for structuring data sets. The first, called "Create Your Own Groups," involved dragging vertical bars along the axis to partition the data set into groups of points. The number of points in each group was shown on the screen and adjusted automatically as a bar was dragged along the axis. The remaining four options were:

- Partitioning the data into groups of a specified size (e.g., 10 data points in each group).
- Partitioning the data into groups with a specified interval width.
- Partitioning the data into two equal groups.
- Partitioning the data into four equal groups.

The students also could hide the data, leaving only the axes and the vertical partition bars visible.
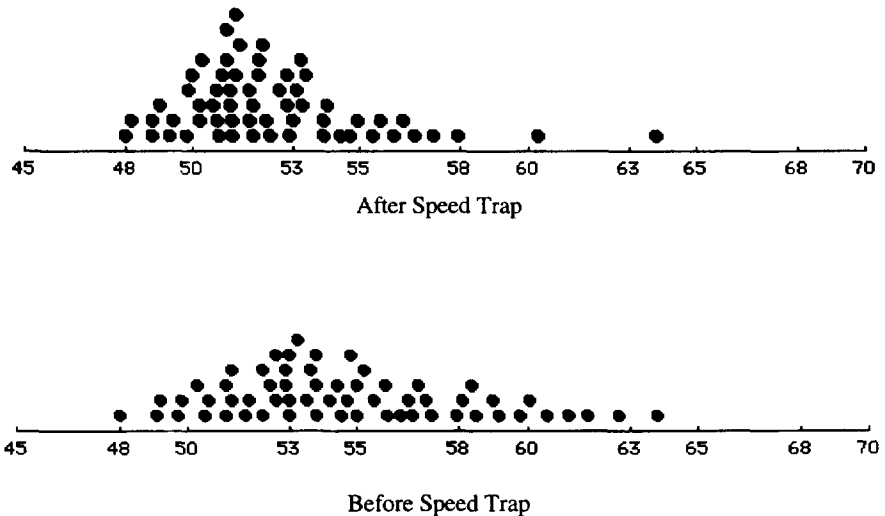

After Speed Trap

Before Speed Trap

FIGURE 3   The second computer minitool.

From the point of view of instructional design, the students' reasoning with this tool can be viewed as a progression from their activity with the first tool. For example, the dots at the end of the bars in the first tool, in effect, have been collapsed down onto the axis. The teacher, in fact, introduced the new line plot inscription by first showing a data set inscribed as horizontal bars, then removed the bars to leave only the dots, and finally transposed the dots onto the axis. In addition, the act of partitioning a set of data points into groups also had a history in students' prior use of the first minitool. Our general instructional intent when designing the second minitool was to build on the students' participation in the first mathematical practice by supporting the emergence of increasingly sophisticated ways of structuring data, particularly those that involve multiplicative reasoning. It is for this reason that the palate of options offered by the minitool does not correspond to a range of conventional inscriptions. We did, however, take into account the need for students eventually to use conventional inscriptions in powerful ways. In this regard, two of the five options—fixed interval width and four equal groups—are precursors to two important types of conventional inscriptions, histograms and box plots, respectively.

One of our initial concerns when the students began to use the second minitool was to ensure that the inscriptions did signify data sets that had been generated by measuring attributes of a situation rather than simply numbers on a line. As it transpired, the students' activity with this minitool did appear to involve reasoning with data from the outset. The practice of data analysis that emerged as the students used this tool can be illustrated by focusing on episodes from three whole-class discussions. In each of these discussions, two members of the project staff shared the teaching responsibilities. The first of these discussions occurred in the 26th classroom session and focused on the question of whether the introduction of a police speed trap in a zone with a 50 miles per hr speed limit had slowed down the traffic speed and thus reduced accidents. The data the students analyzed are shown in Figure 3. The bottom graph shows the speeds of 60 cars before the speed trap was introduced, and the top graph shows the speeds of 60 cars after the speed trap had been in use for some time.

To begin the discussion, one of the teachers asked Janice to read the report she had written of her analysis.

> Janice: If you look at the graphs and look at them like hills, then for the before group, the speeds are spread out and more than 55, and if you look at the after graph, then more people are bunched up close to the speed limit, which means that the majority of the people slowed down close to the speed limit.

This was the first occasion in public classroom discourse in which a student described a data set in global, qualitative terms by referring to its shape. One of the

teachers legitimized Janice's interpretation and indicated that it was particularly valued by drawing the "hills" on the projected data. Both teachers then capitalized on Janice's contribution in the remainder of the discussion, treating other students' analyses as attempts to describe qualitative differences in the data sets in quantitative terms. For example, Karen explained that she had organized the data sets by using a fixed interval width of 5.

> Karen: Like, on the first one [before the speed trap was introduced], most people are from 50 to 60, that's where most people were on the graph.

One of the teachers checked whether other students agreed with her interpretation. Karen then continued:

> Karen: And then on the top one [after the speed trap was introduced], most people were between 50 and 55, because, um, lots of people slowed down … so like more people were between 50 and 55.

The same teacher then recast Karen's analysis as a way of characterizing the global shift of which Janice had spoken. As a consequence of this revoicing, it gradually became taken-as-shared that the intent of an analysis was to identify global trends or patterns in data that were significant with respect to the issue under investigation. The history of this development can be traced to the first mathematical practice, in which, it will be recalled, the ways that collections of data points were organized had to be justified with respect to the question at hand.

A second illustrative episode that serves to clarify the nature of the emerging mathematical practice occurred in the next classroom session. The students' charge was to evaluate a special diet program that was designed to reduce the cholesterol levels of people who are susceptible to heart problems. In developing the situation, one of the teachers and the students talked through the data creation process that involved measuring the cholesterol levels of 60 people before and after they had followed the diet for 1 month. The data the students analyzed is shown in Figure 4. The bottom graph shows the cholesterol levels before the treatment, and the top graph shows the cholesterol levels after the treatment.

The first pair of students to describe their analysis, Sally and Madeline, explained that they had partitioned each data set into four groups of equal size. Throughout the discussion, the data were hidden so that only the axes and the partition lines were visible (see Figure 5). Their argument was that, even though the ranges of the two data sets were the same, the range of the middle two groups (or quartiles) was lower after the treatment, indicating that the diet was successful in lowering cholesterol. This argument indicates that, for Sally and Madeline, the two graphs revealed a global difference in the way that the two sets of data were distrib-
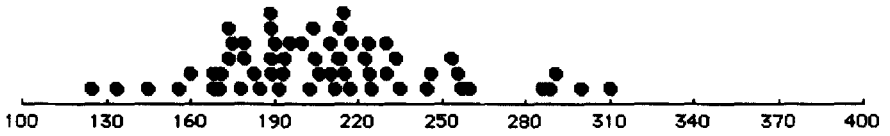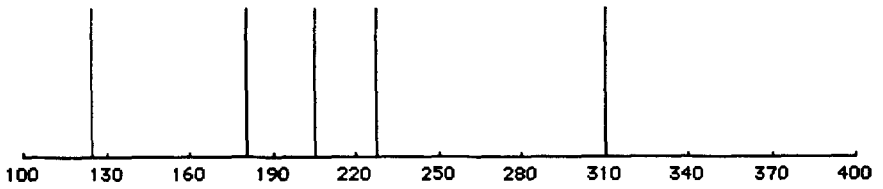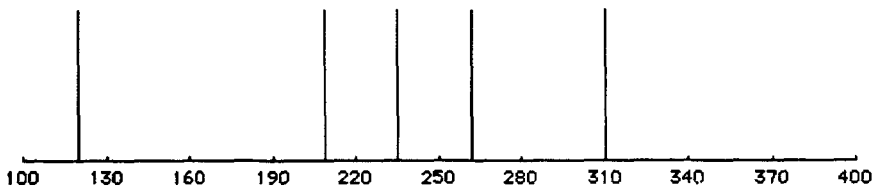
After Program

Before Program

FIGURE 4   The cholesterol data.

After

Before

FIGURE 5   The cholesterol data organized into four equal groups.

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uted. However, because the data were not visible, students had to infer how the data might be distributed from the graphs to understand their argument. Most had difficulty in doing so, and their analysis became the focus of a protracted exchange.

During their initial attempts to explain their reasoning, Madeline used the term "middle section" to refer to the middle two groups. One of the teachers asked the students if they knew what she meant by this term and subsequently established with them that, because one fourth of the data (or 15 data points) were in each group, half the data—or 30 data points—were in the middle section of each graph. Against this background, Valarie asked Sally and Madeline for clarification.

Valarie: What exactly were you talking about, the middle thing I didn't understand, I got everything else.
Sally: The middle half.
Madeline: There's four sections and we're talking about the two middle sections.
Sally: The middle half ... the second and third fourths.
Teacher 1:   Valarie, do you understand?
Valarie: No, now I understand the middle thing, but I don't understand how they used it in the problem.

In asking for further elaboration, Valarie indicated that, although she understood how Sally and Madeline had structured the data, she did not understand why they had done so. She therefore requested that they explain why the way that they had organized the data was relevant to the question at hand—that of assessing the effectiveness of the diet program. Significantly, although Sally and Madeline previously had explained that the middle section was lower on the after-treatment graph, they had not spoken explicitly about global differences in the way that the two sets of data were distributed when they described the results of their analysis.

As the discussion continued, James attempted to explain Sally and Madeline's reasoning, but he too spoke about groups or sections without interpreting them in terms of global, qualitative differences in the data sets. Later, Casey echoed Valarie in asking Sally and Melissa why they had focused on the middle parts of the graphs. The questions that both girls asked appeared to reflect the assumption that explanations should be interpretable in global, qualitative terms. In requesting clarification, they therefore contributed to the continual regeneration of data analysis as a practice that involved investigating trends or patterns in data that were considered relevant with regard to the issue at hand. In response to these questions, Sally and one of the teachers finally developed an explanation that made explicit reference to broad patterns in the way the data were distributed. They first established that the ranges of the two data sets were the same, and then Sally continued:

Sally: The range is the same, but like the median is what is different, like the median right here [points to the after-treatment graph], it means these [data points] move lower in the bottom half closer to the bottom than in here [points to the before-treatment graph].

In this explanation, Sally attempted to clarify what the difference in the medians meant in terms of how the data were distributed. The teacher then capitalized on her contribution:

Teacher 2: This is another way to think about it ... so we agree that before and after they are in about the same place [places his hands on the highest and lowest values of each distribution], they're in there somewhere, the range is the same. So what you're trying to focus on is where between the lowest and the highest they are before and after. Are they all up at one end, or have they all moved down to the other end?

In describing the intent of Sally and Madeline's analysis as that of investigating how the data were distributed, the teacher was attempting to orient other students' efforts to make sense of the graph. Although it is doubtful that all the students came to interpret these particular graphs in this way by the end of the episode, a third illustrative episode indicates that this interpretive stance did become taken-as-shared during subsequent classroom sessions.

The third illustrative episode occurred a week later during the 30th classroom session. The students had compared two treatment protocols for AIDS patients by analyzing the T-cell counts of people who had received one of the two protocols. Their task was to assess whether a new experimental protocol in which 46 people had enrolled was more successful in raising T-cell counts than a standard protocol in which 186 people had enrolled. The data the students analyzed are shown in Figure 6. The computer minitool was not used during the subsequent discussion. Instead, the discussion focused on the reports that the students had written of their analyses.

The inscription from the first report that was discussed showed global differences in the way the two sets of data were distributed (see Figure 7). The students judged this report to be adequate and made a number of comments.
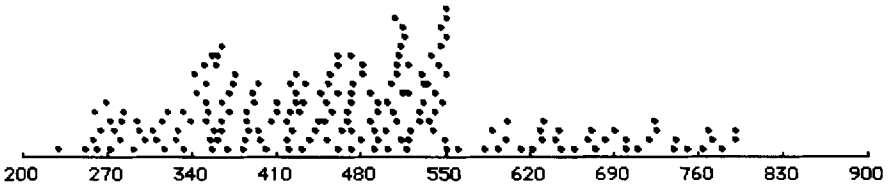
Janice: I think it's an adequate way of showing the information because you can see where the ranges were and where the majority of the numbers were.

David: What do you mean by majority of the numbers?

Teacher 1: David doesn't know what you mean by the majority of the numbers.
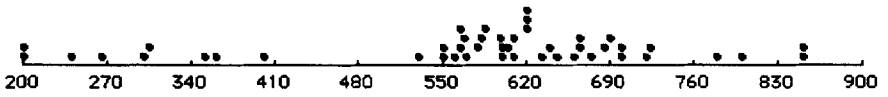
Experimental

Standard
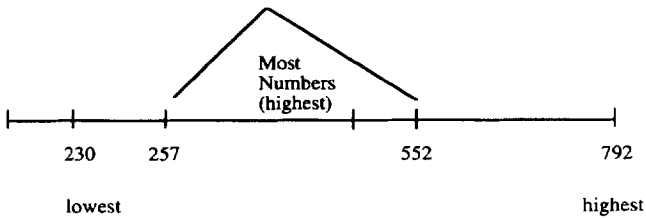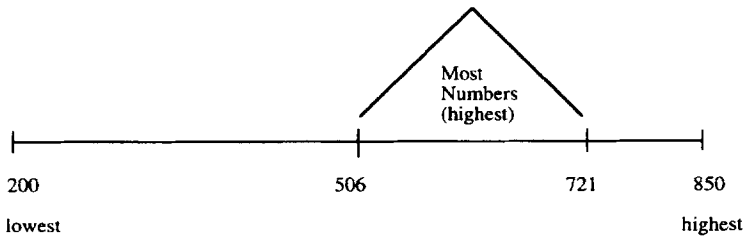
FIGURE 6   The AIDS protocol data.

FIGURE 7   First analysis of the AIDS protocol data.

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Janice:	Where most of the numbers were.
Teacher 1:	Sharon, can you help?
Sharon:	What she's talking about, I think what she's saying, like when you say where the majority of the numbers were, where the point is, like you see where it goes up.
Teacher 1:	I do see where it goes up [indicates the "hill" on the lower diagram].
Sharon:	Yeah, right in there, that's where the majority of it is.
Teacher 1:	OK, David?
David:	The highest range of the numbers?
Sharon:	Yes.
Teacher 1:	The highest range?
Several students:	No.
Teacher 1:	Valarie.
Valarie:	Out of however many people were tested, that's where most of those people fitted in, in between that range.
Teacher 1:	[Pointing to lower and upper bounds of one of the "hills"] You mean this range here?
Valarie:	Yes.

It is evident from this exchange that, when the students spoke about "the majority" or "most of the people," they were talking about data organized multiplicatively as qualitative proportions (P. Thompson, personal communication, September 1997). Janice first had introduced the term "the majority" during the discussion of the speed trap data when she had described hills in the data. A concern with global patterns in the way that data are distributed, in fact, assumes that the data are structured multiplicatively. In describing hills, Janice was reasoning about qualitative relative frequencies. However, this notion of the majority of the data did not become an explicit topic of conversation until the students analyzed data sets with unequal numbers of data points.

During the remainder of the discussion, the teachers attempted to guide the gradual refinement of the taken-as-shared notion of qualitative proportionality. For example, the third report discussed read as shown in Figure 8. One of the teachers clarified with the students during the subsequent exchange that the writers of the report had chosen the statistic of the number of patients with T-cell counts above 525 because the majority of the data points in the old treatment were below this value, and the majority in the new treatment were above it. Thus, they had developed a method for describing a global difference in two distributions. One of the students then suggested drawing graphs to show the results of the analysis (see Figure 9).

One of the teachers then made the following argument that reflected an additive interpretation of the graphs.

OLD PROGRAM   **NEW PROGRAM**

200 - 525   200 - 525

130   9

525 - 850   525 - 850

56   37

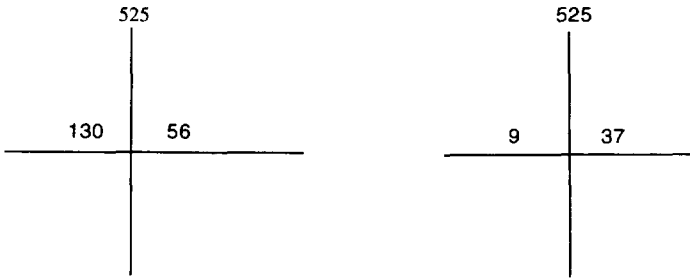FIGURE 8   Third analysis of the AIDS protocol data.

FIGURE 9   Graphs developed from the third analysis of the AIDS protocol data.

Teacher 2:   Could you just argue that this shows really convincingly that the old treatment was better, right, because there were 56 scores above 525, 56 people with T cell counts above 525, and here [points to the graph on the right] there's only 37 above, so the old one just had to be better, there's more people, I mean there's 19 more people in there, so that's the better one, surely.

The initial arguments the students made when rejecting this claim involved reasoning in terms of qualitative proportions. However, Ken made the following proposal:

Ken: I've got a suggestion. I don't know how to do it [inaudible]. Is there a way to make 130 and 56 compare to the 9 and 37, I don't know how.

Teacher 2:   I'll tell you. How many of you have studied percentages?

In the ensuing exchange, several students calculated the percentages of data points above the T-cell count of 525 in each distribution. As the discussion continued, it seemed to be taken-as-shared that the results of these calculations provided a way of describing global differences in the two distributions in quantitative terms.

In both the remainder of this session and in the final three sessions of the teaching experiment, discussions continued to focus on reasoning about data multiplicatively. Interviews conducted with the students shortly after the teaching experiment was completed indicate that most could readily interpret graphs of unequal data sets organized into equal interval widths, an analogue of histograms, and into four equal groups, an analogue of box plots, in terms of global characteristics of distributions. The classroom mathematical practice that had emerged as they developed these competencies can be described as that of *exploring qualitative characteristics of distributions.* Participation in this practice involved reasoning about data multiplicatively while using the computer minitool to identify global patterns and to describe them in quantitative terms. The transition from the first to the second mathematical practices involved a shift in the nature of discussions, such that the focus was on ways of organizing data that were relevant to the purpose at hand, rather on the practical decision or judgment per se. For example, during the discussion of the battery data, the students developed data-based arguments for why the batteries of one of the brands were superior. In contrast, the students agreed that the new treatment for AIDS patients was better than the standard treatment at the beginning of the discussion. The focus was instead on different ways of describing global differences in the two data sets. It might therefore be said that participation in the second mathematical practice involved analyzing data from a mathematical point of view. Taking this characterization of the second practice one step further, Konold et al. (1996) argued that a focus on the rate of occurrence of some set of data values within a range of values is at the heart of what they termed a *statistical perspective.* As participation in the second practice involved a concern for the proportion of data within various ranges of values, the students appeared to be well on the way toward developing this statistical perspective.

CHANGE

My overall purpose in presenting the sample episodes has been to illustrate a theoretical approach that involves analyzing the mathematical learning of the classroom community. The discussion of the two classroom mathematical practices documents how the taken-as-shared ways of reasoning and arguing about data changed in the course of the teaching experiments. It is important to stress that the account I have given does not focus on the mathematical development of any particular student. Instead, I have been concerned with changes in public mathematical activity and discourse. As Voigt (1995) observed, the taken-as-shared meanings and understandings inherent in classroom mathematical practices constitute a semantic domain in their own right and should not be equated with an overlap in individual meanings. The latter focus is essentially individualistic in that it is concerned with a relation between the mathematical interpretations of individual members of

the classroom community. In contrast, the approach I have illustrated takes the classroom community itself rather than the individuals that compose it as the unit of analysis and delineates changes in collective meanings and practices.

Turning now to clarify the general notion of a classroom mathematical practice, it should be apparent from the analysis of the sample episodes that the use of tools and symbols is integral to both the mathematical practices and the reasoning of the students who participate in them (cf. Dorfler, 1993; Kaput, 1991; Pea, 1993). For example, when the students explained their analyses of the battery data near the beginning of the teaching experiment, an act of moving the value tool or the range tool to a particular location was an act of structuring collections of data points. Similar comments can be made about the students' use of the second minitool. For example, when Janice spoke of hills in the speed trap data, she was describing the shape of data that had been inscribed as a line plot. The notion of data sets as distributions, rather than collections of data points, emerged and became taken-as-shared as she and the other students reasoned with line plots. In all likelihood, this central mathematical idea would not have arisen had data been inscribed differently. The analysis of the sample episodes is therefore consistent with the basic Vygotskian insight that the tools students use profoundly influence both the process of mathematical development and its products, increasingly sophisticated mathematical ways of reasoning (Meira, 1995; Saxe, 1991; van Oers, 1996; Wertsch, 1994).

A second aspect of classroom mathematical practices that complements the emphasis on tool use is that of argumentation. Norms or standards of mathematical argumentation were established relatively early in the teaching experiment. I can best substantiate this claim by following Krummheuer (1995) and Yackel (1997) in using Toulmin's (1969) scheme of conclusion, data, warrant, and backing. In this scheme, Toulmin referred to the support one might give for a conclusion as *data*. In the case of the analysis of the battery data, for example, a student might merely point to the two data sets and state the conclusion that one of the brands of batteries is superior. In doing so, the student treats the conclusion as a self-evident consequence of the data. If questioned, the student would be obliged to give a warrant that explains why the data support the conclusion. For example, Casey justified her conclusion that the Always Ready batteries were more consistent by explaining that she had focused on the 10 batteries that lasted the longest and noted that 7 of them were Always Ready batteries. In giving this warrant, Casey explained how she had structured and interpreted the data sets. In Toulmin's scheme, the warrant can be questioned, and it is then necessary to give a backing that indicates why the warrant should be accepted as having authority. Casey was, in fact, challenged by the teacher, who asked her why she had chosen to focus on the 10 batteries that lasted the longest. The backing that Casey gave, namely that 10 was half of the data set of 20 points, was delegitimized as the episode progressed. Instead, it gradually became taken-as-shared in the remainder of this episode and in

subsequent sessions that a particular way of structuring data had to be justified by explaining why it was relevant to the question or issue at hand. The backing that Casey gave proved to be unacceptable because she did not explain why focusing on the 10 longest-lasting batteries was an appropriate way of comparing the two brands. In contrast, Brad's explanation that he wanted a battery that he knew would last at least 80 hr was accepted as giving authority to his approach of partitioning the data sets into values above and below 80 hr.

The scheme of argumentation I have outlined is summarized in Figure 10. Because this scheme also captures the structure of argumentation established when the students used the second minitool and participated in the second mathematical practice, it constitutes a sociomathematical norm that cuts across specific practices. The distinction between argumentation as an aspect of the two practices concerns the nature of the data about which arguments were developed. Participation in the first mathematical practice involved developing arguments about collections of data points, whereas, in the second practice, the arguments were about distributions. The scheme of argumentation shown in Figure 10 is, in fact, quite general and applies to data analysis more broadly. This becomes apparent when we note that, in structuring and interpreting data, the students created methods that
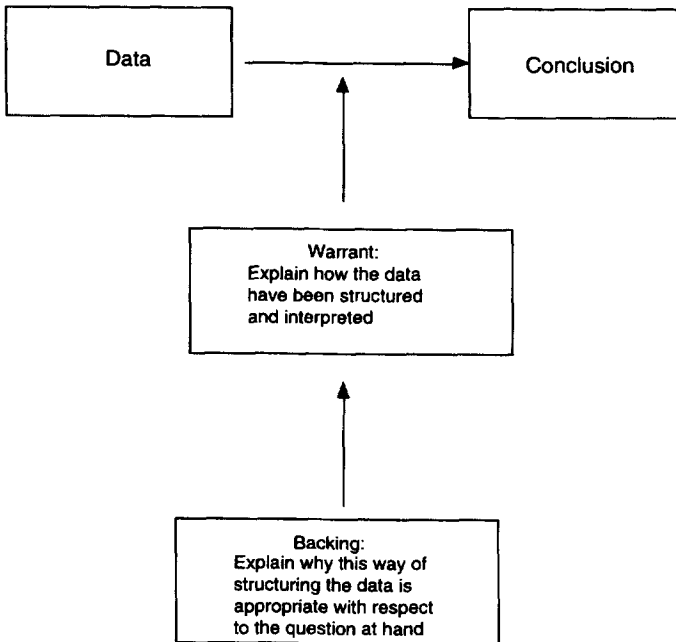

FIGURE 10   Scheme for argumentation that emerged during the teaching experiment.
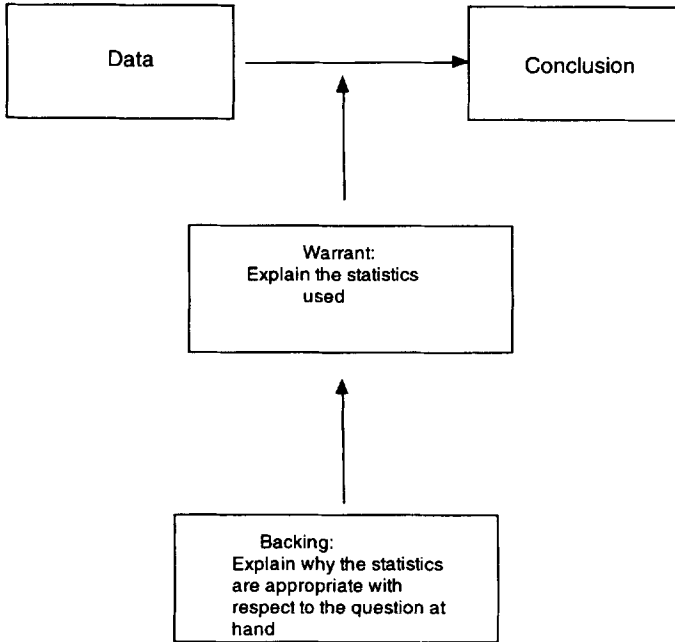
FIGURE 11   General scheme of argumentation for data analysis.

served the function of statistics. The compatibility between the general scheme of argumentation shown in Figure 11 and the norms for argumentation established during the teaching experiment indicates that the students were being inducted into what might be termed an *authentic data analysis point of view.*

As Yackel (1997) demonstrated, Toulmin's (1969) scheme has important methodological implications for the analysis of classroom mathematical practices. To illustrate this point, recall that, when Janice first introduced the notion of hills in the data, she had to explain her interpretation. In doing so, she gave a warrant for her interpretation of the data sets as distributions rather than collections of data points. In contrast, the legitimacy of a hills interpretation was not questioned when the students discussed analyses of the AIDS protocol data. It is precisely this lack of a need for a warrant that serves to indicate that the interpretation of data sets as distributions was taken-as-shared (Yackel, 1997). In general, an analysis of the evolution of mathematical practices focuses as much on what no longer needs to be said and done as it does on what the teacher and students actually say and do.

At the beginning of this article, I justified my focus on the learning of the class-room community by referring to the concerns and interests of developmental re-search. In doing so, I argued that the conjectures that instructional designers

develop when formulating hypothetical learning trajectories are about the learning of the group rather than of any particular student. Viewed in these terms, an analysis of the evolution of classroom mathematical practices documents the actual learning trajectory of a classroom community. This, in turn, implies that the theoretical notion of a classroom mathematical practice encompasses the two major aspects of developmental research—instructional design and classroom-based analyses. Analyses of the type I have illustrated are therefore cast in such a way that they can readily feed back to inform the ongoing instructional design effort. In the case of the statistics teaching experiment, for example, we are currently collaborating with a group of teachers to revise the instructional sequence.

It should be clear from the sample analysis that an approach of this type takes what are traditionally called *issues of mathematical content* seriously. For example, the contrast between the two mathematical practices is characterized, at least in part, by the distinction between additive and multiplicative reasoning about data. However, this approach also calls into question the metaphor of mathematics as *content*. The content metaphor entails the notion that mathematics is placed in the container of the curriculum, which then serves as the primary vehicle for making it accessible to students. In contrast, the approach I have illustrated characterizes what is traditionally called mathematical content in *emergent* terms. For example, the mathematical idea of distribution was seen to emerge as the collective practices of the classroom community evolved. This theoretical orientation clearly involves a significant paradigm shift in how we think about both mathematics and the means by which we might support students' induction into its practices. However, this approach does have the merit of being compatible with the view of mathematics as a socially and culturally situated activity (cf. Bauersfeld, 1992; John-Steiner, 1995; Lave, 1993; Sfard, in press).

This shift from the content metaphor to the emergence metaphor immediately brings issues of teachers' professional development to the fore. In this regard, it is important to observe that an analysis of the type that I have illustrated delineates a learning trajectory that culminates with overarching mathematical ideas that are the goal of an instructional sequence. The analysis therefore provides a justification for the instructional sequence that is cast in terms of (a) the collective development of particular mathematical ideas, and (b) the means of supporting that development. Such a justification, it should be noted, is not tied to specific instructional activities. Instead, the instructional activities used in, for example, a teaching experiment, illustrate one concrete enactment of the sequence. My colleagues and I conjecture that sequences justified in this manner might constitute an important means of supporting the development of professional teaching communities. When a sequence is justified solely in terms of traditional experimental data, teachers know that the sequence proved effective elsewhere, but they do not have the opportunity to develop an understanding that would enable them to adapt the sequence to their own situations. In contrast, the type of justification derived from

an analysis of classroom mathematical practices offers the possibility that teachers will be able to adapt, test, and modify the sequence in their own classrooms.

This conjecture about the potential role of instructional sequences is consistent with the view of implementation as idea-driven adaptation. In addition, the conjecture finds support in Ball and Cohen's (1996) argument that research-based instructional sequences can constitute important resources for teachers' as well as students' learning (see also Gearhart et al., 1994). At the time of writing, my colleagues and I were just beginning to investigate the viability of this conjecture in collaboration with a group of teachers. Our overall goal is to support the development of a professional teaching community that learns from its collective experience by analyzing, adapting, testing, and refining pedagogical ideas and processes that have led to the improvement of students' mathematical learning in other settings. These communal norms and practices can be thought of as the envisioned end points of a pedagogical learning trajectory, in which we and the teachers are joint participants. Part of the challenge we currently are attempting to address is that of developing the means of enabling the teachers to reconstruct justifications for particular instructional sequences by, to some extent, living through the design process.

In concluding this discussion of collective mathematical learning, it is worth noting that analyses that focus on communal mathematical practices in no way deny individual initiative and creativity. In the introductory section of this article, I discussed intellectual autonomy and argued that it can be viewed as a particular way of participating effectively in communal practice in which individuals rely on their own judgments. The main point of my argument was that, although we do not have to give up the notion of autonomy when we view individual reasoning as an act of participation in communal practices, we do need to reconceptualize it. A similar argument can be made about creativity. In the course of the seventh-grade teaching experiment, the students certainly made creative contributions. For example, we did not anticipate, when we designed the first computer minitool, that students would use the range tool to partition collections of data points. From our point of view, the first data analyses in which they used the range tool in this way were creative. Similarly, Janice's introduction of the hill metaphor to talk about global patterns in data was novel, as was Sally and Madeline's argument in which they focused on the middle section of data that had been structured into four equal groups. In the viewpoint I have outlined, these contributions are not seen as instances of the unbridled, purely individualistic creativity of the type that is sometimes glorified in mathematics education. Instead, each contribution is viewed as an act of participating in and contributing to the evolution of communal mathematical practices. More generally, creative acts carry with them the history of participation in previously established practices (cf. Hicks, 1996; Shotter, 1995). Viewed in this way, creativity is social through and through. Rather than being characteristic of a purely individual act, it is characteristic of a relation between individual ac-

tivity and the communal practices in which the individual participates. This conceptualization is nondeterministic in that it does not in any way trivialize or denigrate students' creativity. It does, however, challenge romantic views of creativity by locating it in a social context, one that teachers and students jointly constitute as they establish the practices in which they participate.

DIVERSITY

Thus far, in focusing on collective practices, I have emphasized the taken-as-shared ways of reasoning, arguing, and using tools that are established by a classroom community. It therefore is important to acknowledge that students participate in any particular mathematical practice in a variety of qualitatively different ways. Recall, for example, that a number of students had difficulty in understanding Sally and Madeline's reasoning when they explained that they had structured the cholesterol data into four equal groups. It appeared that, at that point in the teaching experiment, these students needed to know the actual location of individual data points to infer global patterns from graphs and, thus, reason multiplicatively about data sets as distributions. In contrast, Sally, Madeline, and a number of other students inferred global patterns in the distribution of the data directly from the graphs of four equal groups. In doing so, they seemed to structure data into quantitative rather than qualitative proportions. Consequently, whereas the first group of students interpreted the graphs by reasoning from the data points to distributions, the second group of students reasoned directly about the distributions. There were, therefore, significant differences in the ways in which the two groups of students participated in the second mathematical practice.

It might be thought that this diversity in students' thinking is specific to statistics, a domain that might appear to lend itself to alternative interpretations. Therefore, I should note that students also participated in classroom mathematical practices in a variety of different ways in previous teaching experiments that focused on elementary addition and subtraction (Gravemeijer, Cobb, Bowers, & Whitenack, in press), place-value numeration (Bowers et al., in press), and linear measurement (McClain, Cobb, Gravemeijer, & Estes, in press; Stephan, 1998). The need to clarify the relation between individual students' reasoning and the collective practices in which they participate is therefore a pressing one. As a first step, imagine as a thought experiment that we had interviewed not only the students in the teaching experiment classroom but also those from another seventh-grade classroom in the same school. I have no doubt that, if we shuffled the video recordings of these interviews, the reader could almost unerringly identify from which classroom each student had come. It is precisely this contrast in the mathematical reasoning of two groups of students that is accounted for by their participation in the differing mathematical practices established in the two classrooms.

To continue the thought experiment, imagine now that we focus only on the students in the teaching experiment classroom. The contrast is then not between one group of students as compared to another; instead, what is being contrasted is the reasoning among students who have participated in the same classroom mathematical practices. In focusing on differences in individual students' reasoning in this manner, we have adopted a psychological orientation of the type that is so prominent in mathematics education. This perspective is of value and complements a social perspective by bringing the diversity of students' reasoning to the fore. However, it is, by itself, inadequate for the purposes of developmental research in that it also blinds us to the taken-as-shared basis for mathematical communication established by the classroom community. Were we to adopt only this perspective, we would, like the proverbial fish, be oblivious to the water of communal practices. The challenge, therefore, is not that of choosing between social and psychological perspectives on mathematical activity but, instead, to develop ways of coordinating the two perspectives.

In the viewpoint that has emerged from my own and my colleagues' work in classrooms, the relation between the two perspectives is taken to be reflexive. This is an extremely strong relation and does not merely mean that individual students' reasoning and the practices in which they participate are interdependent. Instead, it implies that one literally does not exist without the other (Mehan & Wood, 1975). Thus, when adopting a psychological perspective, one analyzes individual students' reasoning as they participate in the practices of the classroom community. Conversely, when adopting a social perspective, one focuses on communal practices that are continually generated by and do not exist apart from the activities of the participating individuals. The coordination at issue is therefore not that between individual students and the classroom community viewed as separate, sharply defined entities. Instead, the coordination is between two alternative ways of looking at and making sense of what is going on in classrooms. In other words, we are coordinating different ways in which we can interpret classroom events. What, from one perspective, are seen as the norms and practices of a single classroom community is, from the other perspective, seen as the reasoning of a collection of individuals who mutually adapt to each others' actions. Whitson (1997) emphasized this point when he proposed that we think of ourselves as viewing human processes in the classroom, with the realization that these processes can be described in either social or psychological terms. In my view, both perspectives are relevant to the concerns and interests of classroom-based developmental research.

I already have hinted at the fact that this theoretical orientation has grown out of and remains deeply rooted in our attempts to support students' mathematical development while working in classrooms. I can best illustrate the way in which theory is grounded in the reality of the classroom by returning to the statistics teaching experiment. As we have seen, the students frequently worked in pairs at the computers and then explained their analyses in a subsequent whole-class discussion.

While the students were working at the computers, the teacher and a second member of the project staff circulated around the classroom to gain a sense of the diverse ways in which the students were organizing the data. Toward the end of the small-group work, the teacher and project staff member then conferred briefly to plan the whole-class discussion. In doing so, they routinely focused on the qualitative differences in students' analyses in order to develop conjectures about mathematically significant issues that might emerge as topics of conversation. The intent was to capitalize on the students' reasoning by identifying data analyses that, when compared and contrasted, might give rise to substantive mathematical discussions. The episodes I have presented from the seventh-grade teaching experiment exemplify such discussions. In the first episode, for example, the issue of justifying the statistic used with respect to the question at hand emerged from the contrast between Casey's and Brad's analyses. In the next episode that I presented, the two teachers decided to ask Janice to explain her hills metaphor at the beginning of a discussion because they conjectured that it might bring to the fore the issue of interpreting data in terms of qualitative proportions. In the final episode, the analyses of the AIDS protocol data that were discussed were sequenced so that an initial analysis that focused on hills in the data might provide the students with a point of reference when interpreting a subsequent analysis in which the two data sets were partitioned at a particular value.

In this opportunistic approach to instructional planning, students' diverse ways of participating in communal practices are a key resource upon which the teacher attempts to capitalize. The mathematically significant issues that become topics of conversation emerge from this diversity with the teacher's guidance. In reorganizing their thinking while participating in these discussions, students contribute to the evolution of the classroom mathematical practices. In the hands of a skillful teacher, the diversity in students' reasoning is, in many respects, the primary motor of the collective mathematical learning of the classroom community. In the last analysis, it is this realization that convinces my colleagues and me of the need to coordinate a social perspective on communal practices with a psychological perspective that takes into account the students' diverse ways of participating in them.

As a final point, it is important to note that the viewpoint I have outlined has two major ethical implications. The first is that all students must have a way to participate in the mathematical practices of the classroom community. In a very real sense, students who cannot participate in these practices are no longer members of the classroom community from a mathematical point of view. This situation is highly detrimental given that to learn is to participate in and contribute to the evolution of communal practices. Students who are excluded are deprived not merely of learning opportunities but of the very possibility of growing mathematically. One of our primary concerns when conducting a teaching experiment is therefore to ensure that all students are "in the game." To this end, we adjust the classroom participation structure, classroom discourse, and instructional activities on the ba-

sis of ongoing observations of individual students' activity. In doing so, we once again find ourselves coordinating psychological and social perspectives and contend that an approach of this type is necessary, if not sufficient, when addressing concerns of equity at the microlevel of classroom action and interaction.

The second ethical implication is closely related to the first and concerns the view one takes of students whose ways of participating in particular classroom practices are less sophisticated than those of other students. For example, by the end of the statistics teaching experiment, the majority of the students routinely developed arguments by inferring global patterns directly from complex graphs of data they had not themselves analyzed. However, a number of students developed arguments that indicated that they had interpreted the graphs in less-sophisticated ways. In the theoretical orientation I have presented, these differing interpretations are not viewed as cognitive characteristics of the individual students but as characteristics of their ways of participating in communal mathematical practices. In other words, the differences in the students' reasoning are seen to be socially situated and to reflect the history of their prior participation in particular practices. As a consequence, my colleagues and I do not take a cognitive deficit view of the students who made less sophisticated interpretations. Instead, our reflections on the teaching experiment have focused on the evolving mathematical practices that constituted the immediate social situation of their mathematical development as well as on the nature of their participation in those practices. In doing so, we have treated academic success and failure in the classroom as neither a property of individual students nor a property of the instruction they receive. Instead, we have cast it as a relation between individual students and the practices that they and the teacher coconstruct in the course of their ongoing interactions. In the last analysis, the ethical dimension of this perspective on success and failure in school is perhaps the most important reason for adopting a viewpoint that brings the diversity of students' reasoning to the fore while seeing that diversity as socially situated.

EQUITY

Throughout this article, I have focused on the issues of change and diversity as they relate to the concerns of instructional design at the classroom level. It therefore is important to acknowledge that the teaching experiment we conducted did not take place in a social vacuum. Instead, the classroom in which we worked was itself located within the sociopolitical setting of one particular school and community and ultimately within the activity system that constitutes schooling in the United States. At this broader level, the work of several scholars has made us aware that schooling involves a number of taken-for-granted policies and practices that foster inequity due to race, gender, class, and economic status (Apple, 1995; Zevenbergen, 1996).

Furthermore, as Lave (1996) observed, schooling as a social institution involves an inherent contradiction between the functions of universal socialization on the one hand and those of reproducing the unequal distribution of particular ways of knowing as cultural capital on the other hand. It was, in fact, with these global, structural analyses of schooling in mind that I said that the theoretical orientation I have presented is necessary but not sufficient when addressing issues of equity. My colleagues and I have argued elsewhere that this viewpoint on classroom activities and events must itself be complemented by a strong sociocultural perspective that places the classroom in a broader sociopolitical context (P. Cobb & Yackel, 1996). I therefore anticipate further insights from sociocultural analyses that, although not necessarily specific to mathematics education, cast into sharp relief social policies and practices that foster inequity.

At the more local level of the school and community in which we conducted the teaching experiment, the norms and practices established in the project classroom were potentially in conflict with those that the students experienced throughout the remainder of the school day. Furthermore, the students came from a number of different communities within the city in which the school was located and, therefore, had participated in a diverse range of out-of-school practices, some of which may have been inconsistent with the microculture established in the project classroom (cf. Ladson-Billings, 1995; Moll, 1997; Secada, 1992; Warren & Rosebery, 1995). In an initial attempt to address these concerns, two members of the project staff followed several of the students throughout the school day to develop an understanding both of general school norms and of the groupings that had been constituted within the student body. In addition, a member of the project staff has analyzed classroom interactions to identify possible inconsistencies between the classroom microculture and students' home cultures as well as to examine how the students perceived themselves and other members of groups within the classroom. The purpose of these explorations is to delineate issues whose investigations will contribute to our understanding of equity as it relates specifically to teaching and learning mathematics with understanding.

In addition to these concerns that take us beyond the classroom, issues of equity come to the fore when we restrict our focus to instructional design. In the case of the seventh-grade teaching experiment, a question that we had to address was why statistics should be taught in school. Two general types of justifications can be found in the literature. The first refers to developments in the discipline, many of which have been fueled by the use of computers as exploratory tools. The metaphor that emerges from these justifications is that of students as apprentice research statisticians. A second type of justification refers to the increasingly prominent role of statistical reasoning in both work-related activities and informed citizenship. The emphasis in this rationale is on social utility, and the image that emerges for students' roles is as that of consumers of analysis techniques developed by others.

In contrast to these two common rationales, we find a third justification to be far more compelling. Briefly, the increasing use of computers, not just within the discipline but in society in general, has placed an increasing premium on quantitative reasoning in general and on statistical reasoning in particular. There is much talk of preparing students for the "information age" but without fully appreciating that the information in this new era is largely quantitative in nature. This shift has dramatic implications for the discourse of public policy and, thus, for democratic participation and power (G. W. Cobb, 1997). It is already apparent that debates about public policy issues tend to involve reasoning with data. In this discourse, policy decisions are justified by presenting arguments based on the analysis of data. In many respects, this discourse is increasingly becoming the language of power in the public policy arena. Inability to participate in this discourse results in de facto disenfranchisement that spawns alienation from, and cynicism about, the political process. Cast in these terms, statistical literacy that involves reasoning with data in relatively sophisticated ways bears directly on both equity and participatory democracy. The image that emerges for the students' role is then not that of junior research statistician or utilitarian consumer of standard techniques. Instead, it is of students as increasingly substantial participants in the discourse of public policy. The important competencies for this participation are those of developing and critiquing data-based arguments.

I should stress that the rationale I have given for the importance of statistics in students' mathematics education is concerned with overall instructional goals. It does not in itself imply that a particular instructional approach such as one involving investigations should be taken. Nonetheless, our decision to focus on the competencies of developing and critiquing data-based arguments did lead us to make an important design decision when planning the teaching experiment. In particular, we ruled out an open-ended project approach in which students investigate issues of personal interest by generating data and instead developed instructional activities in which the students analyzed data sets created by others. However, we also were aware that data do not speak for themselves but instead are the product of a sequence of interpretive decisions and judgments (Latour, 1987; Roth, 1997). For example, data embody assumptions both about which aspects of the situation under investigation are relevant with regard to the issue at hand and about how they should be measured. We therefore anticipated that the students would not initially be able to "look through" data to the situation from which they were generated. It was for this reason that we developed the approach of talking through the data-creation process so that the data might have a history for the students. It is apparent from the sample episodes I have presented that this approach worked reasonably well. As the teaching experiment progressed, the students, in fact, assumed increasing responsibility for asking questions that related to the data creation process. Furthermore, although most of the classroom discussions focused on analyses that the students had conducted, in the last few classroom sessions

they developed arguments on the basis of graphs created by others. In the course of this transition, the students were developing the very competencies that make increasingly substantial participation in public policy discourse possible.

In terms of the broader literature on equity, the approach we have taken to statistics instruction is broadly compatible with Delpit's (1988) admonition that students should be taught explicitly what she calls the *culture of power*. Our approach also makes contact with the equity pedagogy of Banks and Banks (1995), which aims to help students from diverse cultural backgrounds develop the ways of knowing needed to participate effectively within and maintain a just, democratic society. Thus, although we look for further inspiration from scholars whose work focuses on global, structural characteristics of schooling and society, we also contend that a concern for equity is critical when considering issues traditionally addressed by mathematics educators. In particular, it is essential that we scrutinize the overall goals we have for students' mathematics education and examine whether they can be justified in terms of participation in a democratic society. I will be more satisfied if our work in the area of statistics can serve as a useful example in this respect.

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