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THE CYCLIC NATURE OF PROBLEM SOLVING: AN EMERGENT MULTIDIMENSIONAL PROBLEM-SOLVING FRAMEWORK

ABSTRACT. This paper describes the problem-solving behaviors of 12 mathematicians as they completed four mathematical tasks. The emergent problem-solving framework draws on the large body of research, as grounded by and modified in response to our close observations of these mathematicians. The resulting *Multidimensional Problem-Solving Framework* has four phases: orientation, planning, executing, and checking. Embedded in the framework are two cycles, each of which includes at least three of the four phases. The framework also characterizes various problem-solving attributes (resources, affect, heuristics, and monitoring) and describes their roles and significance during each of the problem-solving phases. The framework's sub-cycle of conjecture, test, and evaluate (accept/reject) became evident to us as we observed the mathematicians and listened to their running verbal descriptions of how they were imagining a solution, playing out that solution in their minds, and evaluating the validity of the imagined approach. The effectiveness of the mathematicians in making intelligent decisions that led down productive paths appeared to stem from their ability to draw on a large reservoir of well-connected knowledge, heuristics, and facts, as well as their ability to manage their emotional responses. The mathematicians' well-connected conceptual knowledge, in particular, appeared to be an essential attribute for effective decision making and execution throughout the problem-solving process.

KEY WORDS: affect, conceptual knowledge, control, mathematical behavior, mathematics practices, metacognition, problem solving

1. INTRODUCTION

The mathematics education community has shown interest in understanding the nature of problem solving for over 45 years. While early work in problem solving focused on describing the problem-solving process (Pólya, 1957), more recent investigations have focused on identifying attributes of the problem solver that contribute to problem-solving success. In a review of the problem-solving literature from 1970 to 1994, Lester (1994) noted a consistent finding that problem-solving performance appears to be a function of several interdependent factors (e.g., knowledge, control, beliefs, and sociocultural contexts) that overlap and interact in a variety of ways. More recent studies have cited planning and monitoring as key discriminators in problem-solving success (Schoenfeld, 1992; DeFranco, 1996; Geiger and Galbraith, 1998; Carlson, 1999a). Other studies have revealed the influence of various affective dimensions (e.g., beliefs, attitudes, and emotions) on the problem-solving process. Although many studies have investigated

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and compared the characteristics of novice and expert problem solvers (e.g., Krutetskii, 1969; Lesh and Akerstrom, 1982; Lesh, 1985; Schoenfeld, 1985a, 1989; Geiger and Galbraith, 1998; Stillman and Galbraith, 1998), many aspects of the problem-solving process still do not appear to be understood. While the literature supports that control and metacognition are important for problem-solving success, more information is needed to understand how these behaviors are manifested during problem solving, and how they interact with other problem-solving attributes reported to influence the problem-solving process (e.g., resources, heuristics, affect).

Building on the existing body of literature, we focused our study on gaining new information about the interaction of major aspects of problem solving that have been identified as important for problem-solving success. In doing so, we wanted to gain a better understanding of the cognitive and metacognitive processes involved in problem solving, and to acquire more specific information about the major problem-solving attributes that have been reported to influence the problem-solving process. We also wanted to bring improved clarity and coherence to the body of problem-solving literature. We thus began by developing a broad taxonomy to characterize major problem-solving attributes. As our principal method of data collection, we elected to investigate the behaviors of 12 experienced problem solvers, all mathematicians, while they worked through four mathematics problems. Our initial analysis revealed that our taxonomy was limited in its ability to characterize some of the critical behaviors being exhibited by the mathematicians in our study. We then reanalyzed the data using a grounded approach. This paper describes the emergence of our *Multidimensional Problem-Solving Framework* and reports the major findings from our study.

2. Background

2.1. *What is a mathematics problem?*

Early studies (1970–1982) in problem solving were concerned with determining the aspects of a task or problem that contributed to its difficulty level (Lester, 1994), while more recent research has had a different focus. "Today there is general agreement that problem difficulty is not so much a function of various task variables as it is the characteristics of the problem solver" (Lester, 1994). This view was echoed by Geiger and Galbraith (1998) who claimed, "it is the relationship between the learner and a problem that is of significance, not the perceived level of the problem as viewed within some hierarchy of abstraction."

Our view of a mathematics problem and the notion of problem solving includes problems at all levels in any mathematical context. We do not restrict mathematical problems or the notion of problem solving to work on a specific class of problems that are encountered in a problem-solving course. Rather, based on Schoenfeld's definition of a problem, we regard problem solving as including situations in which an individual is responding to a problem that he or she does not know how to solve "comfortably" with routine or familiar procedures:

> A problem is only a problem (as mathematicians use the word) if you don't know how to go about solving it. A problem that has no 'surprises' in store, and can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an exercise. (Schoenfeld, 1983, p. 41)

2.2. *Phases of the problem-solving process*

In his renowned publication *How to Solve It*, Pólya (1957) suggested that solving a problem involved: (i) understanding the problem; (ii) developing a plan; (iii) carrying out the plan; and (iv) looking back. He described the problem-solving process as a linear progression from one phase to the next and advocated that when solving a problem,

> [First,] we have to see clearly what is required. Second, we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a plan. Third, we carry out our plan. Fourth, we look back at the completed solution, we review it and discuss it. (Pólya, 1957, pp. 5–6)

Twenty-five years later, Garofalo and Lester described problem-solving behavior as consisting of four phases of distinctly different metacognitive activities: orientation, organization, execution, and verification. In describing their framework, Garofalo and Lester indicated that shifts from one phase to the next commonly occurred when metacognitive decisions resulted in some form of cognitive action.

2.3. *How do experts behave?*

Summarizing Schoenfeld's work (Schoenfeld, 1985, 1987b), Lester (1994) characterized "good" mathematical problem solvers as possessing more knowledge, well-connected knowledge, and rich schemata. "They regularly monitor and regulate their problem solving efforts," Lester observes, and "they tend to care about producing elegant solutions." In addition, they appear to have a high level of self-awareness of their strengths and weaknesses and tend to focus on the underlying structure and relationships in the problem (Stillman and Galbraith, 1998). Good mathematical problem solvers also exhibit flexibility during problem solving and tend to use powerful content-related processes rather than general heuristics alone (Geiger and Galbraith, 1998).

2.4. *The role of resources, heuristics, control, and affect*

The literature on problem solving routinely describes *resources* as formal and informal knowledge about the content domain, including facts, definitions, algorithmic procedures, routine procedures, and relevant competencies about rules of discourse (Pólya, 1957; Schoenfeld, 1989; Geiger and Galbraith, 1998). The utility of a problem solver's resources depend on the factor of *control*: Many studies on problem solving have reported that even when individuals appear to possess the resources to solve a particular problem, they often do not access those resources in the context of producing a problem solution. In Schoenfeld's words (1992), "It's not just what you know; it's how, when, and whether you use it."

Many researchers have joined Schoenfeld in illuminating the crucial role that control plays in achieving problem-solving success (Schoenfeld, 1985a, 1992; DeFranco, 1996; Vinner, 1997; Carlson, 1999a). In a 1992 study, Schoenfeld noticed that undergraduate students who demonstrated poor control were unlikely to notice when their efforts were unproductive. In a study involving mathematicians, DeFranco (1996) reported that not only was effective control a positive force for successful problem solvers, control was either lacking in less successful subjects, or it acted as a negative force in their problem-solving efforts. In Vinner's (1997) theoretical framework for addressing pseudo-conceptual and pseudo-analytical thought processes, he noted that students who showed evidence of pseudo thought processes often lacked the control mechanisms that good problem solvers possess. Evidence of pseudo thought processes includes behaviors such as random associations, lack of validation efforts, and absence of inquiry about meaning.

Metacognition has also been a focus of problem-solving studies (e.g., Lesh and Akerstrom, 1982; Schoenfeld, 1982; Silver, 1982; Lester et al., 1989a) and has been defined to include knowledge about and monitoring of one's thought processes and control during problem solving. We note that the terms control, metacognition, and monitoring have held varying and overlapping meanings in the literature. Our initial taxonomy uses the term *control* to encompass metacognition and monitoring and all associated behaviors. Following Schoenfeld (1992, p. 355), we will use the term *monitoring* to mean the mental actions involved in reflecting on the effectiveness of the problem-solving process and products. We also define *self-regulation* to refer to the actions that are taken in response to assessments of "on-line" progress.

Research has also substantiated that affective variables such as beliefs, attitudes, and emotions have a powerful influence on the behavior of the problem solver (Schoenfeld, 1989; Lester et al., 1989b; McLeod, 1992;

DeBellis and Goldin, 1997). Although emotions are more evident than beliefs during problem solving, beliefs (deep-seated convictions such as "learning mathematics is mostly memorization") also play an important role (Schoenfeld, 1989, 1992; Carlson, 1999a, b). Schoenfeld claims that purely cognitive behavior is rare, and that thinkers perform most intellectual tasks within the context established by their perspectives on the nature of those tasks. Belief systems, says Schoenfeld (1992), shape cognition and determine the perspective with which one approaches mathematics and mathematical tasks, and should therefore be included in any investigation of why individuals succeed or fail in their attempts to solve mathematics problems. In a 1999 study that investigated the background, beliefs, and problem-solving behaviors of graduate students, Carlson found that effective problem solvers expressed beliefs that: doing mathematics requires persistent pursuit of a solution; the solution process may require many incorrect attempts; problems that involve mathematical reasoning are enjoyable; mathematical ideas should be understood instead of just memorized; learning mathematics requires sorting out information on one's own; and verification is a natural part of the problem-solving process (pp. 254–255).

DeBellis and Goldin (1997) have focused their investigations on local affective responses, described as the responses that occur during the process of solving a problem. Local affect has been found to have an impact on both cognitive processing (Hannula, 1999) and the construction of mathematical knowledge (DeBellis and Goldin, 1999). Both positive feelings, such as satisfaction and pride, and negative emotions, such as anxiety and frustration, are common, as local affect changes frequently during the process of solving a problem. Local affective pathways of struggle, success, and elation typically lead to motivation and interest, whereas local pathways of struggle, failure, and sadness typically lead to anxiety (Hannula, 1999).

Affective responses, however, are seen to be extremely complex, consisting of much more than the expression of positive and negative feelings or the exhibition of confidence. They entail structures of intimacy, integrity, and meta-affect that promote deep mathematical inquiry and understanding. Complex networks of affective pathways both contribute to and detract from powerful mathematical problem-solving ability. Intimate mathematical experiences generate bonding between the learner and her mathematics. Such bonding has been characterized by behaviors such as cradling one's work in one's hands or arms, or speaking passionately about one's mathematical products. High levels of intimacy have also been associated with negative emotional responses, as exhibited by expressions of anger or frustration. *Mathematical integrity*, another dimension of affect, refers to an individual's standards for validating that a solution is correct, a problem is solved satisfactorily, or the learner's understanding is sufficient.

Mathematical integrity has also been described as an individual's expression of honesty relative to his understanding of a solution. *Mathematical intimacy* and *integrity* are seen to be reflexively related in that absence of integrity raises an obstacle to intimacy, and absence of intimacy reduces the individual's need for integrity (DeBellis and Goldin, 1999).

3. THEORETICAL FRAMEWORK

Drawing from the large amount of literature related to problem solving, we devised an initial taxonomy (Table I) that would allow us to characterize the various problem-solving attributes that have been identified as relevant for problem-solving success. The dimensions of the taxonomy are resources, control, methods, heuristics, and affect. We define *resources* to be the conceptual understandings, knowledge, facts, and procedures used during problem solving. *Control* refers to the metacognitive behaviors and global decisions that influence the solution path. This includes the selection and implementation of resources and strategies, as well as behaviors that determine the efficiency with which facts, techniques, and strategies are exploited (e.g., planning, monitoring, decision making, conscious metacognitive acts). Motivated by our interest in gaining a better understanding of control, and by calls for more work to investigate aspects of control in problem solving, we categorized control into three sub-dimensions: initial cognitive engagement, cognitive engagement during problem solving, and metacognitive behaviors. *Initial cognitive engagement* includes activities such as putting forth effort to read and understand the problem and establishing givens and goals. *Cognitive engagement* during problem solving includes both attempts to fit new information with existing knowledge and the construction of logically connected statements. *Metacognitive behaviors* include reflection on the efficiency and effectiveness of the cognitive activities and subsequent self-regulatory behaviors.

The methods dimension of the taxonomy describes the general strategies used when working a problem, while the heuristics dimension describes more specific procedures and approaches. The affective dimension includes attitudes, beliefs, emotions, and values/ethics (including mathematical integrity and mathematical intimacy as described above). Since our original intention was to use the taxonomy as a coding scheme, we assigned each attribute a two- or three-letter coding label. The taxonomy guided the major design decisions for our study, including the selection of the interview tasks, interview protocol, and plans for our data analysis. It also served as our initial lens for coding our data and identifying the behaviors that were exhibited. However, our initial analysis using our problem-solving taxonomy revealed that it was limited in its ability to characterize some of

TABLE I
Initial Problem-Solving Taxonomy

Resources	
RK	Knowledge, facts, and procedures
RC	Conceptual understandings
RT	Technology
RW	Written materials
Control	
CP Initial Cognitive Engagement	
CPE	Effort is put forth to read and understand the problem
CPO	Information is organized
CPG	Goals and givens are established and represented
CPS	Strategies and tools are devised, considered, and selected
CE Cognitive Engagement During Problem Solving	
CES	Evidence of sense making
CEM	Effort is put forth to stay mentally engaged
CEL	Effort is put forth to construct logically connected statements
CM Metacognitive Behaviors During Problem Solving	
CMQ	Reflects on the efficiency and effectiveness of cognitive activities
CMM	Reflects on the efficiency and effectiveness of the selected methods
CMC	Exerts conscious effort to access resources/mathematical knowledge
CMG	Generates conjectures
CMV	Verifies processes and results
CMR	Relates problem to parallel problem
CMP	Refines, revises, or abandons plans as a result of solution process
CME	Manages emotional responses to the problem-solving situation
CMI	Engages in internal dialogue
Methods	
MT	Constructs new statements and ideas
MC	Carries out computations
MR	Accesses resources
Heuristics	Uses heuristics during the problem-solving process (e.g., HW: Works backwards; HO: Observes symmetries; HS: Substitutes numbers; HR: Represents situation with a picture, graph, or table; HC: Relaxes constraints; HD: Subdivides the problem; HA: Assimilates parts into whole; HL: Alters the given problem so that it is easier; HE: Looks for a counter example; HI: investigates boundary values)

Affect

AA	*Attitudes*	*AB*	*Beliefs*
AAE	Enjoyment	ABC	Self-confidence
AAM	Motivation	ABE	Pride
AAI	Interest	ABP	Persistence
		ABM	Multiple attempts are needed in problem solving
AE	*Emotions*	*AV*	*Values/Ethics*
AEF	Frustration	AVI	Mathematical intimacy
AEA	Anxiety	AVG	Mathematical integrity
AEJ	Joy, pleasure		
AEI	Impatience, anger		

the problem-solving behaviors we were observing—e.g., the interactions between the various elements in the taxonomy. (We describe this finding in more detail in Section 4.2.)

4. The study

4.1. *Methods*

The subjects for this study were eight research mathematicians (all male) and four Ph.D. candidates (three male and one female) from two large public universities in the southwestern and western United States. The four Ph.D. candidates completed their degrees soon after participating in this study, so we refer to the subjects collectively as mathematicians or experienced problem solvers. We chose to work with mathematicians because we hypothesized that by observing individuals with a broad, deep knowledge base and extensive problem-solving experience, we would learn more about the problem-solving process and interactions of various problem-solving attributes (e.g., cognitive processes, metacognitive behaviors, and affective responses).

The mathematicians were asked to complete four (of five) problems that required knowledge of foundational content and concepts such as basic geometry, algebra, and proportions. The problems were selected because: (i) they were challenging enough to engage a research mathematician, yet required fundamental mathematical concepts and knowledge that is accessible to any mathematician, regardless of the area of specialization; (ii) the nature of the problem would produce a variety of solution paths, thus eliciting various metacognitive and cognitive behaviors and prolonged engagement during the solution process; and (iii) the problems were sufficiently complex to lead to dead ends and elicit strong affective responses. These problems (Appendix I) did, in fact, afford us the opportunity to witness a variety of cognitive and metacognitive behaviors and affective responses. Problems 2 and 3 were especially illuminating.

Each subject was interviewed separately by one of the authors. At the start of the session, the interviewer asked the mathematician to verbalize his thought processes as he completed the problems (Appendix I). The interviewer observed the subject's behaviors and also periodically reminded the solver to verbalize his thinking and articulate a rationale for specific behaviors. The interviewer gave no indication regarding the correctness of the solution. The interviews were audio-taped and later transcribed. The amount of time required to complete each problem varied across mathematicians and across problems, with completion of each problem varying from about 10 to 45 minutes. (We also note that completion of a problem did not always result in a correct solution.)

4.2. *Data analysis*

In the first pass through the transcribed interviews, we coded the data using the initial problem-solving taxonomy (Table I). This coding, although effective for identifying, labeling, and classifying various problem-solving attributes in our data, did not fully explain the reasoning patterns and interactions that we were observing. In particular, we noticed that our framework was limited in its ability to characterize specific *interactions* between the problem-solving process and aspects of the subjects' cognitive processes, metacognitive behaviors, and affective responses. As a result, our next attempt to analyze the data involved a grounded approach, employing open coding techniques (Strauss and Corbin, 1990).

Both researchers analyzed the transcripts independently to ascertain the mathematical thinking and behaviors exhibited, while attempting to identify emerging trends in the problem-solving process (see Tables III–VI; Column 2). It was in this phase of the analysis that we first became aware of a recurring pattern of strategies and conjecture, followed by computations, checking, and a decision (Figure 1). We also noticed that a second type of cyclic reasoning was being exhibited when the mathematician was considering the viability of various solution approaches. The patterns that we observed guided a more fine-grained analysis of the cycles. This analysis resulted in our classifying the four major phases that these mathematicians moved through when completing a problem as: orienting, planning, executing, and checking (Tables III–VI; Column 3). We observed that once the mathematicians oriented themselves to the problem space, the *plan–execute–check* cycle was then repeated throughout the remainder of the solution process. We also noticed that, when contemplating various solution approaches during the planning phase of the problem-solving process, the mathematicians were at times engaged in a *conjecture – imagine – verify* cycle (Figure 1).

In the next stage of our data analysis, we negotiated the nuances of characterizing the phases of the cycles. We also attempted to better understand the nature of the cognitive and metacognitive processes and their interaction with other aspects of problem solving defined in our initial taxonomy. Interview transcripts of the solvers' problem-solving sessions were scrutinized to identify uses and effects of heuristics, resources, actions of control, and various affective responses (using the coding from our initial framework as appropriate). Original audio-recordings were revisited for expressions of anger, anxiety, frustration, pleasure, and other

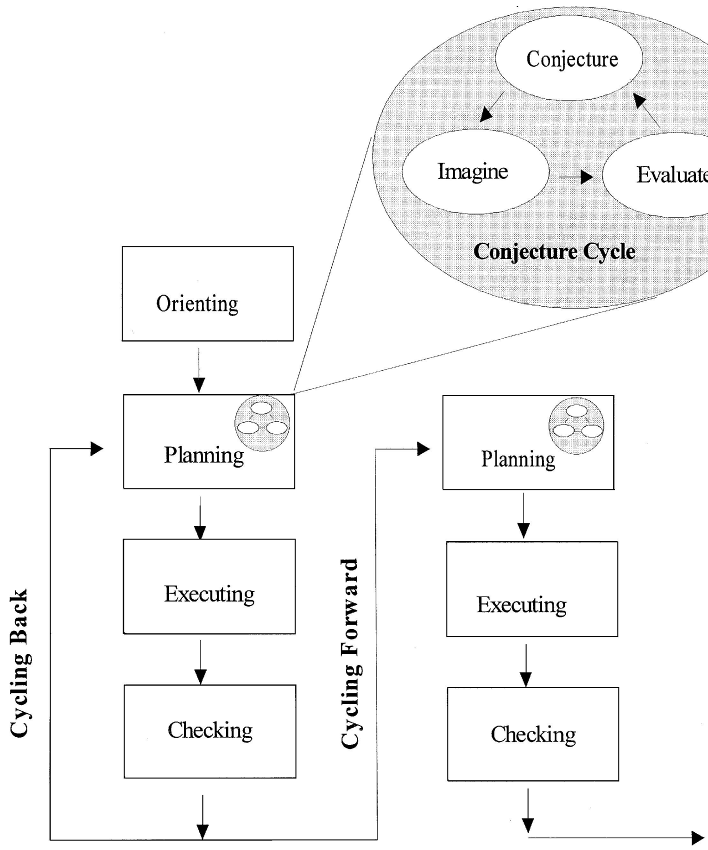

Figure 1. The Problem-Solving Cycle.

emotional responses. We also reviewed all coded transcripts to identify and characterize the *most common ways* that resources, heuristics, affect, and control were exhibited during each of the problem-solving phases. It was at this point that we also noticed patterns in how different control mechanisms were used during problem solving. In particular, we noticed that the mathematicians monitored their thought processes and products regularly during all four problem-solving phases, although the specific nature of what was monitored varied from phase to phase. We also observed that strategic control behaviors such as accessing knowledge, managing emotional responses, and verifying processes and results (CMC, CME, CMV in our original taxonomy) were behaviors that were exhibited during specific phases. We further observed that these behaviors, reasoning patterns, and knowledge influenced the mathematicians' problem-solving success. To make this distinction in our reporting we will refer to *monitoring* as *reflection on* and *regulation of* one's thought processes and products at any point in the solution process. All other control behaviors defined in our original taxonomy will be referred to as *strategic* control. What emerged was a more structured, coherent, and descriptive characterization of the interplay between the problem-solving phases, cycling, and problem-solving attributes. We illustrate this characterization in the form of a multidimensional problem-solving framework (Table VII).

5. RESULTS

In this section, we describe the problem-solving process and behaviors exhibited by four mathematicians when completing the Paper-Folding Problem (Table II). We chose this problem because, of the problems used in our study, it provided the richest exhibition of problem-solving behaviors.

TABLE II
The Paper-Folding Problem

A square piece of paper ABCD is white on the frontside and black on the backside and has an area of 3 in^2. Corner A is folded over to point A' which lies on the diagonal AC such that the total visible area is $1/2$ white and $1/2$ black. How far is A' from the fold line?

References to the interview transcript (noted with a numbered reference to the specific excerpt) that support the observation are indicated. In addition, we have included four coded transcripts (Tables III–VI) that illustrate the labeling strategies we used to identify and characterize the problem-solving behaviors. The first column of each table displays the transcription with each expression referenced with a number. The second column contains the labels (using the language of our original framework as appropriate) that were most descriptive of the exhibited behaviors. The third column of each table illustrates the four problem-solving phases that emerged subsequent to our labeling the specific behaviors.

5.1. *A description of Aaron's solution*

Aaron was able to orient himself to the problem space by reading through the problem statement once (Table III, Excerpt 1). He initially conjectured that he needed to find an area function (2, 3). As he continued to imagine this conjecture playing out, he recognized that finding A' would involve finding the altitude of the triangle he formed by folding the paper over (4). After some frustration and contemplation (5) he expressed awareness that the triangle is a right triangle (6). He then reflected on this conjecture by visualizing the triangle formed by folding the paper over. After monitoring his thinking, he concluded that the right triangle also contained a

TABLE III
Aaron's transcription and coding

Excerpt		Behavior	Phase
(1)	Reads problem...	• *Initial engagement*	Orienting
(2)	So I'm probably going to try to do a similar thing here.	• *Strategy* • *Heuristic*	Planning
(3)	So I'm probably going to try to find an area function, or a function for how far A' is from the fold line.	• *Conjecture*	
(4)	It's just the altitude of this triangle. Let me draw you a picture. . . .try letting the altitude of the black triangle be x.	• *Mathematical knowledge* • *Organizing and labeling information*	
(5)	See if I end up... (struggle, long pause)	• *Affect*	
(6)	Then I guess we're dealing with a right angle triangle.	• *Conjecture* • *Mathematical knowledge*	
(7)	So this is all black. So this angle will always be... I just wanted to make sure that I'm not making any implicit assumption about the current position and assume that we had a right triangle here, but that's always going to be a 45-degree angle.	• *Conjectures/imagines transformation of triangle* • *Accesses mathematical knowledge* • *Monitoring*	
(8)	So this is... I'm not even sure that I need these sides yet. I'm just trying to fill in what I know...	• *Organizing information*	
(9)	et's start with the area of the dark triangle. The area of the little square is $2x$ squared.	• *Conjectures/imagines* • *Accesses mathematical knowledge*	
(10)	Wait. So here... $\frac{1}{2}$ base is x squared.	• *Executes/verifies*	
(11)	Now the area of the white triangle... well this should be simple.	• *Strategy* • *Affect – Confidence*	
(12)	It starts off with 3 and we're subtracting the area of the dark triangle. Actually the square which would be 3 minus $2x$ squared. (mumbling about squares)... it can't be negative 1. Positive 1.	• *Accesses mathematical knowledge* • *Executing*	Executing
(13)	And now I check to make sure that I believe my answer, or that the answer seems unreasonable in any way.	• *Self monitoring*	
(14)	Well it's 1 inch... so it's all right. Well, I don't see any reason why that's not a reasonable answer.	• *Reflecting on process and solution*	Checking *Cycles forward*
(15)	In fact, I didn't use that as much as I could have. I was just noticing that to find the area of the white. The whole area is equal to 3, minus the dark and minus this and this equal to that. If I would have realized it that that meant given what we are looking for just meant that the area of the triangle had to be 1... that would have been yet simpler.	• *Reflecting on the problem and his solution*	

45-degree angle (7). He continued to contemplate the viability of his approach by reflecting on what he needed to do to move toward a solution (8). After labeling the altitude of the triangle with an x, he verified that the area of the black triangle is x^2 (9, 10). This is correct based on his labeling; however, he did not verbalize all of the thinking that allowed him to arrive at this conclusion. His next expression (11), along with his following statement (12), suggested that he quickly recognized that the area of the white triangle could also be represented by x^2. He proceeded to piece together his knowledge and various deductions to conclude that 3 (the entire area) minus $2x^2$ is equal to 1. He appeared to perform mental calculations prior to concluding that x must be equal to 1 (the correct answer) (12). He then reflected on the reasonableness of his answer and concluded that there were no conflicts. (He appeared to test his deduction against his mathematical knowledge and knowledge of the problem) (13, 14). His reflections on his approach resulted in his concluding that, had he initially recognized that the area of the triangle had to be equal to 1, he could have arrived at his answer with a simpler approach (15).

5.2. *A description of James' solution*

James' initial engagement with the problem involved his reading the problem statement while attempting to make sense of it (Table IV, Excerpt 1). Once he was able to picture the problem, he proceeded to construct a drawing of the situation (he drew a square and labeled each side as having a length of $\sqrt{3}$ [2]). After drawing an illustration of the folded paper, his verbalizations suggested that he was imagining the folding of corner A onto the diagonal of the square (3). While appearing to transform the figure in his mind, he conjectured $\frac{3}{2}$ as the total area of the triangle formed by folding the corner (7, 8). He then reflected on the correctness of this statement and concluded that his conjecture was incorrect. His reengagement with the problem was illustrated by his restating of known facts (11) and drawing the fold on his illustration. He then labeled each of the three pieces of his figure with an A (i.e., each triangle formed by the fold and the strip representing the difference of the original square and the square formed by the fold). His calculations resulted in his conclusion that $A = 1$ (14). After reestablishing known facts he again focused on his ultimate goal of finding the distance of A' from the fold (15). Using his knowledge of a 45–90–45 triangle relationship, he concluded that the distance from the fold was $\frac{1}{\sqrt{2}}$. He then reasoned that if $x = \frac{1}{2}$, the area of the square would be $\frac{1}{4}$, which he immediately rejected as incorrect (17, 18). He then made another conjecture, which led to his conclusion that the area of the triangle is $\frac{1}{2}$ (19, 20). After testing this conclusion against facts that he knew to be true,

TABLE IV
James' transcription and coding

Excerpt		Behavior	Phase
(1)	Ok, a square piece of paper is white on the front side and black on the backside and has an area of 3 square inches. Corner A is folded... so the total visible area is half white and half black. How far is A from the fold line?	• *Initial engagement* • *Sense making*	Orienting
(2)	Ok, so each side is square root 3.	• *Sense making—organizing information* • *Mathematical knowledge*	
(3)	And then fold it over, so that each of these guys are the same,	• *Heuristic—modeling*	
(4)	So the total area here is $\frac{3}{2}$.	• *Conjecture*	Planning
(5)	So, this is x and this is x, then this is x squared over 2.	• *Imagine, verify* • *Strategizing* • *Mathematical knowledge*	
(6)	What's that supposed to mean?	• *Monitoring progress—does this make sense?*	
(7)	Total area is 3. I fold it over so this is half. And....the whole square is 3.	• *Sense making*	
(8)	I fold it in, so *that* is $\frac{3}{2}$.	• *Conjecture*	
(9)	I don't... oh (he then draws the fold on the paper and labels the sides of the smaller square x and labels the point A' on the diagonal).	• *Testing conjecture* • *Monitoring quality of thinking*	Executing
(10)	I'm sorry, that's not correct. It's not $\frac{3}{2}$.	• *Rejecting conjecture*	Checking *Cycles back*
(11)	This area is A, this area is A, and that area out there is supposed to be A (gesturing towards figures on his diagram).	• *Reengagement* • *Conjecturing*	Planning
(12)	So, we're supposed to have half white and half black.	• *Sense making* • *Conjecture/imagine/verify*	
(13)	So, this area x squared is $2A$ (pointing to sketch).	• *Strategy*	
(14)	$2A$ plus A is $3A$. So, A is going to be 1. So, x squared is $\frac{1}{2}$...	• *Executing strategy*	Executing
(15)	How far is A' from the fold line, so I want from A' to the fold line...	• *Monitoring progress*	
(16)	So, x is 1 over the square root of 2. And then, that is that divided by that, so you have.....1 over the square root of 2 divided by the square root of 2 is $\frac{1}{2}$...	• *Executing strategy* • *Mathematical knowledge*	
(17)	let me check this and make sure... so if x is 1 over the square root of 2, this area is $\frac{1}{4}$ (pointing to the area of the triangle formed from folding corner A over to A').	• *Verifying work* • *Mathematical knowledge*	Checking *Cycles back*
(18)	No,	• *Rejecting solution* • *Mathematical knowledge*	

(*Continued on next page*)

TABLE IV
(*Continued*)

Excerpt	Behavior	Phase
(19) that is, is $\frac{1}{2}$...	• *New conjecture*	Planning
(20) $\frac{1}{2}$ ...That's $\frac{1}{2}$, that's $\frac{1}{2}$, that's $\frac{1}{2}$, so that's $\frac{1}{4}$ so this area would be $\frac{1}{2}$.	• *Imagine conjecture unfolding;* • *Tests conjecture*	Executing
(21) No!	• *Rejects conjecture* • *Affect— Frustration, impatience*	Checking *Cycles back*
(22) What am I doing wrong?	• *Reflects on thinking* • *Affect—Pride, ego, frustration*	
(23) Ok, a square piece of paper is white on the front side and black on the backside has an area of 3 square inches. Corner A is folded... so the total visible area is half white and half black. How far is A from the fold line?	• *Reengages with problem text*	Planning
(24) What am I,	• *Conjecture*	
(25) You can't...	• *Tests conjecture*	Executing
(26) (pushes paper aside).	• *Rejects conjecture* • *Affect — Frustration*	Checking *Cycles back*
(27) There's nothing wrong with my brain, it's my calculations.	• *Affect—aha!*	Planning
(28) The total area is 3. That's the total...Yes	• *Sense making*	
(29) Now I fold it and then this area, which is black, is the same as this area. So, this is some area A.	• *Heuristic — modeling the problem*	
(30) And that's A. That area is the same as this area. This is the lost area.	• *Sense making*	
(31) So, $3A$ equals 3. The area is 1.	• *Conjecture*	
(32) So, I want the area of this animal here to be 1.	• *New strategy*	
(33) So, if that's x and that's x, the area of the whole square is 2...x is the square root...	• *Executing strategy* • *Mathematical knowledge*	Executing
(34) That makes much more sense...	• *Self-monitoring*	
(35) So this one I can do in my head...	• *Affect—Ego/Pride*	
(36) Let's see ... if that's 1 and that's 1, that's $\frac{1}{2}$, that's $\frac{1}{2}$ and that's 1 that's 1 that's 1.	• *Executing*	
(37) So, it's, and the answer was how far is A' from the fold line, so I take it by that you mean this line and that distance is 1.	• *Verifying solution*	Checking Completion
(38) Sloppy calculations...	• *Affect—embarrassment*	

he rejected this conjecture (21). After reflecting on his process (22), he again reengaged by rereading the problem (23) and making another conjecture that he quickly rejected (24, 25). His reengagement again resulted in his attempt to make sense of the problem and reestablish known facts

(26–30). This was followed by his stating confidently that the area of the black triangle (backside of what is folded over) is *1* (31). After reestablishing that the side of the smaller square is x, and observing that the area of this square is 2, he concluded (by drawing on his knowledge of the relationship between the area and side of a square) that the length of x must be equal to $\sqrt{2}$ (32, 33). He then reflected on the reasonability of his conjecture and expressed that he believed it was correct (34). After expressing that the last few calculations would be trivial (35), his final mental computations resulted in his correctly concluding that the distance from A' to the fold is 1 (36). His final step involved verification that his answer satisfied the original question that was posed (37).

5.3. *A description of Marco's solution*

Marco initially read the problem and made multiple attempts to make sense of the given information and the statement of the question (Table V, Excerpts 1–5). In addition to drawing a picture, he constructed a square from a piece of paper and folded the upper left-hand corner over and slid the corner of the paper along the diagonal (Excerpt 6). (His behavior and comments suggested that he was trying to estimate what fold would generate equal areas.) Following his initial efforts to orient himself to the problem, he stated various conjectures regarding possible solution approaches. These conjectures marked the first of several *plan–execute–check* cycles the interviewer observed in Marco's solution process. After each conjecture,

TABLE V
Marco's transcription and coding

Excerpt	Behavior	Phase
(1) Let's see, *A* square piece of paper is white on the front and black on the back and has an area of 3 square inches.	• *Reads problem*	Orienting
(2) (He draws the square and labels on each side.)	• *Sketches a diagram (heuristic)* • *Mathematical knowledge*	
(3) OK, now corner *A* is folded over to point A' which lies on the diagonal *AC* (he pauses and labels the corner A, B, C, D)...	• *Organizes information*	
(4) Such that the total visible area is $\frac{1}{2}$ white and $\frac{1}{2}$ black. How far is A' from the fold line?	• *Continues to read problem*	
(5) (He then reads the problem a second time—this time even more slowly and carefully, emphasizing relevant information.)	• *Continues to read problem*	

(*Continued on next page*)

TABLE V
(*Continued*)

Excerpt		Behavior	Phase
(6)	(He went back to the picture and redrew it; then he pulled out a piece of paper and constructed a square and shaded the back with his pencil; he then proceeded to fold it along the diagonal, so that the corner labeled A was folded over; he then slid the corner up and down the diagonal until he finally settled on a point that appeared to meet the requirements of the problem (the area was $\frac{1}{2}$ black and $\frac{1}{2}$ white.) (long pause) Is this where they are equal?	• *Models the problem (heuristic)* • *Conjecture-imagine-evaluate*	Planning
(7)	So, let me begin by playing real dumb...	• *Affect—Pride/Ego*	
(8)	It looks like one of those problems if you label everything and crunch numbers, probably you'll get an answer (pause)...	• *Conjecture/classifies problem*	
(9)	Also, it looks like one of those problems where there is maybe some slick way of figuring out...	• *Aesthetic concern*	
(10)	using what we know about the sides of a 45-45-90 triangle and the relationship of the triangle and the strip (long pause)...	• *Mathematical knowledge* • *Conjecture/imagine*	
(11)	But, I'm not going to do that.	• *Rejects conjecture*	
(12)	I'm going to look for the ugly solution (pause)...	• *Aesthetic concern*	
(13)	I can't think of any slick way to do it, so I'm going to just figure out where the areas are equal.	• *Affect—Pride/Ego* • *Strategy*	
(14)	The whole thing is 3 square inches, so it has to be $\sqrt{3} \pm x$ on a side and because this is a square, this also has to be $\sqrt{3-x}$ (cradles work)...	• *Executing strategy* • *Expression of intimacy*	Executing
(15)	So that triangle is going to be half of the square, so that's $\frac{x^2}{2}$ and this strip is going to be 3 minus the square that got cut off which is x^2,	• *Accesses mathematical knowledge* • *Executing strategy*	
(16)	So $3 - x^2$ has to be equal to So... computing, you get...	• *Executing strategy*	
(17)	Now, does that make any sense?	• *Monitoring*	
(18)	Now, if we plug numbers back in just to check. The triangle would be...	• *Checking reasonableness of the solution*	Checking *Cycles forward*
(19)	So, how does this help me?	• *Monitoring*	
(20)	Now, I need to go back and find the length of the piece of the triangle so I know how far it is from the fold line.	• *Strategy*	Planning
(21)	I guess I can...	• *Monitoring*	
(22)	et me see, will the properties of right triangles help?	• *Conjecture*	

Marco was typically silent for a short period during which he appeared (based on the comments and actions that followed) to be considering how the conjectured solution would play out (8–12). Once he decided on a strategy, he appeared to draw on his knowledge of basic geometric properties to determine how to algebraically express the areas of the two equal figures (13–15). He then performed computations to generate a solution that he subsequently checked (16–18). After completing the verification, he carried the newly generated information forward as he began the plan–execute–check cycle again (20–22). Marco worked on this problem for about 15 more minutes, but was never able to resolve the computational conflicts that he encountered. His decision to move on was based upon his expressed desire to attempt the other problems in the restricted time frame of the interview.

The forgoing transcripts from the Paper-Folding Problem sessions reflect the general patterns we found throughout our interviews with the mathematicians. After coding the collection of interview transcripts from all 12 mathematicians for all problems, we recognized that, although the solvers' specific paths in arriving at an answer were unique—as exemplified above—their general problem-solving approaches were surprisingly consistent. Encouraged by this result, we continued our search for patterns in the coding. At this point we also observed that as the mathematicians moved toward a solution, four distinct phases (characterized by distinct shifts in the mathematicians' cognitive behaviors) uniformly surfaced in their problem-solving processes (Tables III–V; Column 3). We have labeled these four phases as *orienting*, *planning*, *executing*, and *checking*.

6. An emerging framework

During the *orienting* phase, the predominant behaviors of sense making, organizing, and constructing were identified. Although the length of time needed to complete this phase varied from mathematician to mathematician, they were all remarkably efficient in constructing either a picture or a mental image of the problem situation as they attempted to make sense of the question. Some specific behaviors that were commonly exhibited during the orienting phase included defining unknowns, sketching a graph, constructing a table, etc. These behaviors were typically accompanied by intense cognitive engagement, as exhibited in the subjects' focused movement toward the construction of a personal representation of the problem situation.

During the *planning* phase, the mathematicians initially devised conjectures about a viable solution approach. They appeared to contemplate various solution approaches by imagining the playing-out of each approach, while considering the use of various strategies and tools. The sequence of behaviors that were exhibited included (a) the construction of a *conjecture*; (b) either the verbalization of a solution approach or silence, with the subject appearing to *imagine* how the solution approach would play out; and (c) *evaluation* of the viability of the conjectured approach. After completing each sub-cycle the mathematician returned to the planning phase until a solution approach was selected. This analysis also revealed that reflections on and decisions about the general problem solving approach typically occurred just before the mathematician entered the executing phase of the problem-solving process (e.g., Table V, Line 13).

During the *executing* phase, the mathematicians predominantly engaged in behaviors that involved making *constructions* and carrying out *computations*. Some specific behaviors included writing logically connected mathematical statements, accessing resources (including conceptual and factual knowledge), executing strategies and procedures, and carrying out computations.

During the *checking* phase, the subjects shifted to *verification* behaviors as they spontaneously assessed the correctness of their computations and results. Their exhibited behaviors included spoken reflections about the reasonableness of the solution and written computations. We also witnessed the mathematicians contemplating whether to accept the result and move to the next phase of the solution, or reject the result and cycle back. These *decisions* were exhibited prior to their moving to a new problem-solving cycle.

It is important to note that the mathematicians rarely solved a problem by working through it in linear fashion. These experienced problem solvers typically cycled through the plan–execute–check cycle multiple times when attempting one problem (Figure 1). Sometimes this cycle was slow and tedious; at other times the solver appeared to move through the cycle with little effort. When the checking phase resulted in a rejection of the solution, the solver returned to the planning phase and repeated the cycle (Figure 1). When the checking resulted in an acceptance of the solution, the subject continued to another plan–execute–check cycle until the problem was completed.

The plan–execute–check cycle involved a deliberate and complete execution, usually produced by writing, and a more formal checking that included written computations and calculations. In contrast, the *conjecture – imagine–evaluate* sub-cycle of the planning phase often involved no writing and either an acceptance or rejection of an approach without any visible verification. The act of conjecturing typically involved the subject imagining or stating a hypothetical solution approach. This was followed by the subject either verbally or silently playing the solution out in her mind, as evidenced by her utterances or behaviors following brief periods of silence (for example, refer to Table V, Excerpt 6). At some point while imagining the playing-out of the solution, the individual evaluated if the solution approach was viable. If the conjectured solution approach was seen as potentially productive, the mathematician moved to the next phase; otherwise the conjecture–imagine–evaluate sub-cycle was repeated until a viable solution path was identified (Figure 1).

In addition to making decisions about their solution approaches, the mathematicians regularly engaged in metacognitive behaviors that involved reflecting on the effectiveness and efficiency of their decisions and actions. These reflections were exhibited frequently during each of the four problem-solving phases, and they appeared to move the mathematicians' thinking and products in generally productive directions. These reflective behaviors included pauses in the executing phase to determine the reasonableness of the constructions (i.e., whether the emerging results fit with the mathematicians' current knowledge and understandings) and reflections about whether the approach was productive (e.g., Is this approach getting me anywhere? What does this tell me?). The mathematicians tended to act on their monitoring in ways that moved them forward in the solution process. In analyzing the transcripts further, we recognized that the metacognitive acts within each problem-solving phase were best characterized as acts of monitoring (e.g., reflection on one's thought processes and products). We also saw that the specific monitoring behaviors sometimes influenced the mathematicians' strategic control decisions, which we can capture by examining their transcripts as they move through the problem-solving cycles.

6.1. *Uses of resources, heuristics, affect, and monitoring during the problem-solving phases*

After identifying the problem-solving phases and confirming the consistency of the problem-solving cycle, we systematically examined the uses of other attributes of problem solving (e.g., resources, heuristics, affect, and monitoring) during each problem-solving phase. During this phase of our data analysis, we labeled (highlighted in bold print in Table VI) these attributes and looked for consistent uses of these attributes. This involved our reanalyzing each coded transcript to identify common uses of resources (knowledge, facts, and procedures) and heuristics (constructing a diagram, attempting a parallel problem, etc.) and consistent displays of affect (enjoyment, pride, frustration, and mathematical integrity) and monitoring (reflections on the effectiveness and efficiency of the solution process) for each problem-solving phase. In this examination, we noticed

TABLE VI
Select excerpts from Fred's transcription

Excerpt		**Behavior**	Phase
(7)	... has an area of 3 square inches... so each side of the square is root 3, cornering this right here is gonna be the diagonal, is root 6. Ok.... lies on the diagonal *AC* such that the total visible area is half that...	• *Organizing information* • ***Mathematical knowledge***	Orienting
(8)	So, is it saying, may I ask you a question? Is it saying that this area right here and this area right here is all white...	• ***Sense making***	
(9)	Ohhhh, I see... there ya go	• ***Self-monitoring***	Planning
(10)	Alright. Now we're rocking.	• ***Affect—aha, excitement***	
(11)	Ok, so I'm gonna call this x. That means that this distance right here is gonna be x minus... it's gonna be root 3 minus...... root 3 minus x... (long pause as he performs calculations on paper)...	• *Organizing information* • ***Sense making*** • ***Mathematical knowledge***	Executing
(12)	So, what does that mean?	• ***Monitoring—does this make sense?***	
(13)	That means that, this....	• *Verification*	Checking *Cycles forward*
(14)	What do I know about that?	• ***Monitoring—where am I?***	Planning
(15)	This is root 3, this is root 3. How am I gonna find this? (pause)	• ***Reflecting on information*** • ***Mathematical knowledge***	
(16)	Ok, start with something simpler...	• *Strategy* • ***Heuristic—look for a simpler problem***	
(17)	This distance right here is x... that area right here is x times root 3 and this area right here is gonna be... this distance is... that's x... this right here is gonna be root 3 minus x... that's the area of the white stuff.	• ***Reflecting on work so far*** • *Organizing information* • ***Mathematical knowledge***	Executing
(18)	Such that the total visible area...	• *Revisits problem text*	*Cycles forward* (no overt checking)
(19)	And that's gotta equal a half of what the original was (long pause)...	• *Conjecture* • ***Affect—hesitation***	Planning
(20)	Alright. Now, we're rocking.	• ***Affect—excitement*** • ***Monitoring—where am I?***	
(21)	So this is 2 root $3x$ ... squared (so 2, 3, 2 squared minus 2 root $3x$ plus root 3...solve this using quadratic formula (gets out calculator)... Ok, so all I do is I graph this quadratic and I'm gonna find what the roots are... so it looks like this one is probably the right one.	• *Executing* • ***Mathematical Knowledge*** • ***Heuristic, sketch a graph***	Executing

(*Continued on next page*)

TABLE VI
(*Continued*)

Excerpt		Behavior	Phase
(22)	There's another one over here, but it's gonna be too big	• *Verification* • ***Mathematical knowledge***	Checking *Cycles forward*
(23)	So, let's think. . .	• ***Monitoring***	Planning
(24)	How far is A from the folded line?	• *Revisits problem text*	
(25)	Ha, so much more.	• ***Affect—impatience***	
(26)	So, let's think. Is there a quick way to figure this out? Or do I actually have to. . .	• ***Monitoring—where am I?*** • ***Heuristic—refers to sketch*** • *Considers strategies*	

that the mathematicians consistently manifested these attributes within a phase; we also saw that the use and interactions of the attributes between phases were often distinct. We illustrate the product of this analysis in the following excerpt from the transcript of Fred's Paper-Folding Problem interview. This coding makes explicit Fred's use of specific mathematical resources (7, 11, 15, 17, 21, 22) and heuristics (16, 21). It also illuminates his expressions of affect (10, 19, 20, 25) and monitoring (9, 12, 14, 20, 23, 26).

When we applied this analysis to all the coded interviews, we found consistent patterns in the mathematicians' use of content knowledge, heuristics, monitoring, and affect in each problem-solving phase. The interactions we observed among these attributes led us to develop more detailed descriptions of the role each major problem-solving attribute seems to play in each problem-solving phase (Table VII, columns 2–5). These observations grounded our development of the Multidimensional Problem-Solving Framework, which we describe below.

6.2. *The Multidimensional Problem-Solving Framework*

The *Multidimensional Problem-Solving Framework* provides a detailed characterization of phases and cycles that occurred consistently in the problem-solving process, as well as the specific behaviors that our 12 mathematicians exhibited as they attempted to solve four mathematics problems. The framework describes how resources and heuristics interacted with the general behaviors exhibited during the four problem-solving phases (orienting, planning, executing, and checking). It also characterizes how monitoring and affect were expressed during each of the phases. The framework further illustrates the cyclic nature of the problem-solving process

Table VII
Multidimensional Problem-Solving Framework

Phase • Behavior	Resources	Heuristics	Affect	Monitoring
Orienting • Sense making • Organizing • Constructing	Mathematical concepts, facts and algorithms were accessed when attempting to make sense of the problem. The solver also scanned her/his knowledge base to categorize the problem.	The solver often drew pictures, labeled unknowns and classified the problem. (Solvers were sometimes observed saying, "this is an X kind of problem.")	Motivation to make sense of the problem was influenced by their strong curiosity and high interest. High confidence was consistently exhibited, as was strong mathematical integrity.	Self-talk and reflective behaviors helped to keep their minds engaged. The solvers were observed asking, "What does this mean?"; "How should I represent this?"; "What does that look like?"
Planning Conjecturing Imagining Evaluating	Conceptual knowledge and facts were accessed to construct conjectures and make informed decisions about strategies and approaches.	Specific computational heuristics and geometric relationships were accessed and considered when determining a solution approach.	Beliefs about the methods of mathematics and one's abilities influenced the conjectures and decisions. Signs of intimacy, anxiety, and frustration were also displayed.	Solvers reflected on the effectiveness of their strategies and plans. They frequently asked themselves questions such as, "Will this take me where I want to go?", "How efficient will Approach X be?"
Executing • Computing • Constructing	Conceptual knowledge, facts and algorithms were accessed when executing, computing and constructing. Without conceptual knowledge, monitoring of constructions was misguided.	Fluency with a wide repertoire of heuristics, algorithms, and computational approaches were needed for the efficient execution of a solution.	Intimacy with the problem, integrity in constructions, frustration, joy, defense mechanisms and concern for aesthetic solutions emerged in the context of constructing and computing.	Conceptual understandings and numerical intuitions were employed to reflect on the sensibility of the solution progress and products when constructing solution statements.
Checking • Verifying • Decision making	Resources, including well-connected conceptual knowledge informed the solver as to the reasonableness or correctness of the solution attained.	Computational and algorithmic shortcuts were used to verify the correctness of the answers and to ascertain the reasonableness of the computations.	As with the other phases, many affective behaviors were displayed. It is at this phase that frustration sometimes overwhelmed the solver.	Reflections on the efficiency, correctness and aesthetic quality of the solution provided useful feedback to the solver

and the points at which strategic control influenced the mathematicians' problem-solving decisions and actions.

The general behaviors (e.g., sense making, organizing) appear directly under the phase name (e.g., orienting) (Table VII). The cells to the right contain a description of the primary role of four problem-solving attributes (i.e., resources, heuristics, affect, and monitoring) during that problem-solving phase.

During the orientation phase the mathematicians initially engaged in intense efforts to make sense of the information in the problem. They all displayed confidence, general curiosity, reflective behaviors (e.g., How should I represent this? What does this mean?), and high mathematical integrity; these qualities were all evident as they went about constructing logical representations of the problem situation using diagrams, charts, tables, etc. As they did this, they spontaneously accessed their concepts, facts, and algorithms as needed to represent the problem situation. Their constructions were also aided by heuristics such as categorizing the problem as an X kind of problem and working backwards.

During the planning phase the mathematicians were frequently observed accessing conceptual knowledge and heuristics as a means of constructing, imagining, and evaluating their conjectures. We also observed that although they displayed negative emotional responses of frustration and anxiety, their high confidence and effective coping mechanisms kept them engaged and focused. Their ongoing monitoring of their strategies and plans was exhibited when they verbalized questions about their expenditure of mental resources (e.g., "What will happen if I try X?"). Our analysis revealed that their frequent displays of self-talk that included verbalizations of their conjectures, questions, and comments contributed to their efficient movement toward a solution plan.

During the execution phase these mathematicians were also observed accessing their conceptual knowledge, facts, and algorithms when constructing statements and carrying out computations. The efficiency and effectiveness of their actions appeared to be strongly influenced by their fluency in accessing a wide repertoire of heuristics, algorithms, and computational approaches. The ongoing monitoring of their solution approach kept their work moving in generally productive directions, and their strong conceptual and procedural knowledge assured the effectiveness of their monitoring. Their strong bond with the problem coincided with their strong affective responses (e.g., frustration, anxiety, elation, and joy). Their effective management of these affective responses, using a variety of defense and coping mechanisms, was instrumental in their persisting toward a solution to the problem. In retrospect, our data also support that these mathematicians held beliefs that doing mathematics requires sorting out information on one's own, being persistent, and being willing to tolerate many false attempts to finally attain a correct solution.

The checking phase again involved the mathematicians' drawing on their conceptual and procedural knowledge to verify the reasonableness of their results and the correctness of their computations. During this stage intense negative affective responses sometimes resulted in their setting the problem aside. Their monitoring took the form of their reflecting on the quality of their processes (e.g., verification strategies) and products (e.g., correctness of their solution).

7. Concluding Remarks

This study has contributed new insights into the problem-solving process by offering a multidimensional framework to be used for investigating, analyzing, and explaining mathematical behavior. Our framework includes four phases with two embedded cycles and provides detailed characterizations of how resources, affect, heuristics, and monitoring influence the solution path of the solver.

Orchestrating the many facets of problem solving when confronting a novel problem may be as complex as skiing off-course down a steep and slippery slope. In one split second as she pushes off, the skier analyzes the possible paths before her, assaying factors such as moguls, slope, snow conditions, and her own skills and limitations. With little hesitation, she accesses a vast reservoir of techniques, knowledge, and past experiences to imagine the moves required to navigate each possible path, and – employing *transformational reasoning* – assesses the likely outcome of selecting each one. So too with our mathematicians: their ability to play out possible solution paths to explore the viability of different approaches appears to have contributed significantly to their efficient and effective decision making and resultant problem-solving success. It is precisely this sort of thinking that we refer to as transformational reasoning, as described by Simon (1999); we note that the thinking we have described has similarities to what Mason and Spencer (1999) speak of as "knowing to act in the moment." Fundamental to "knowing to act in the moment" is the use of mental imagery to imagine a future situation. Further explorations of these reasoning patterns in problem solving are needed.

Our study also exposed the central role of well-connected conceptual knowledge in the problem-solving process. Although past studies have identified metacognitive control as a primary attribute affecting an individual's ability to access knowledge when needed (Schoenfeld, 1992; DeFranco, 1996), previous studies have not provided a detailed characterization of the interplay between metacognition and conceptual knowledge. Among the mathematicians we studied, well-connected conceptual knowledge appears to have influenced all phases of the problem-solving process (orientating, planning, executing, and verifying). Our results also support that the ability to access useful mathematical knowledge at the right moment during each of the problem-solving phases is highly dependent on the richness and connectedness of the individual's conceptual knowledge. We call for future work to investigate this claim.

Consistent with the findings of DeBellis (1998) and Hannula (1999), the results of our study support the notion that local affective pathways play a powerful role in the problem-solving process. Our findings suggest that the effective management of frustration and anxiety, using a variety of coping mechanisms, was an important factor in these mathematicians' persistent pursuit of solutions to complex problems. We suggest that future work investigate the role of these management behaviors in students.

The multiple dimensions and diverse components of the *MPS Framework* suggest that learning to become an effective problem solver requires the development and coordination of a vast reservoir of reasoning patterns, knowledge, and behaviors, and the effective management of both resources and emotional responses that surface during the problem-solving process, as well as a great deal of practice and experience. We suggest that future research examine the longitudinal development of problem-solving behaviors in students, and we call for curriculum developers and instructors to focus increased attention on promoting problem-solving behaviors in their students.

We believe that our study has brought additional clarity to the problem-solving process and has successfully consolidated much of the existing body of the problem-solving literature. We hope that the insights that we report may be useful for others' attempts to promote effective problem-solving behaviors in students. It is also our hope that both our findings and the *MPS Framework* will be useful for future problem-solving studies.

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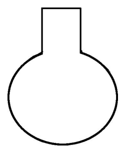
APPENDIX I: INTERVIEW PROBLEMS

The Bottle Problem

Imagine this bottle filling with water. Construct a rough sketch of the graph of the height as a function of the amount of water that is in the bottle.



The Paper Folding Problem

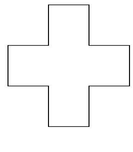
A square piece of paper ABCD is white on the frontside and black on the backside and has an area of 3 in.^2 Corner A is folded over to point A′ which lies on the diagonal AC such that the total visible area is $1/2$ white and $1/2$ black. How far is A′ from the fold line?

The Mirror Number Problem

Two numbers are “mirrors” if one can be obtained by reversing the order of the digits (i.e., 123 and 321 are mirrors). Can you find: (a) Two mirrors whose product is 9256? (b) Two mirrors whose sum is 8768?

The Pólya Problem

Each side of the figure below is of equal length. One can cut this figure along a straight line into two pieces, then cut one of the pieces along a straight line into two pieces. The resulting three pieces can be fit together to make two identical side-by-side squares, that is a rectangle whose length is twice its width. Find the two necessary cuts.



The Car Problem

If 42% of all the vehicles on the road last year were sports-utility vehicles, and 73% of all single car rollover accidents involved sports-utility vehicles, how much more likely was it for a sports-utility vehicle to have such an accident than another vehicle?

APPENDIX II: DECOMPOSITION OF THE PROBLEM-SOLVING CYCLE

Orienting—sense making, organizing, and constructing

Effort and energy is put forth to read and understand the problem.
Effort is put forth to make sense of information in a table, graph, diagram, or text.
Information is organized.
Goals and given are established.
Goals and givens are represented by symbols, tables, and charts.
Diagrams are constructed.

Planning—conjecturing, imagining, and testing

Mathematical concepts, knowledge, and facts are accessed and considered.
Various solution approaches are considered.
A conjecture is formulated.
Various solution approaches are considered.
The unfolding of a solution approach(es) is(are) imagined.
An approach is determined.

Executing—computing and constructing

Selection and implementation of various procedures and heuristics.
Constructs logically connected mathematical statements.
Carries out computations.
Evidence of sense making/attempts to fit new information with existing schemata.
Validity of conjecture is considered.

Checking—verifying

Results are tested for their reasonableness.
Decision is made about validity of answer.
The problem solver cycles back or cycles forward based on results from checking.

Monitoring

Considers the efficiency and effectiveness of the various methods.
Considers the efficiency and effectiveness of cognitive activities.
Effort is put forth to stay mentally engaged.

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