Chapter 6
Intellectual Need

Guershon Harel

Abstract Most students, even those who desire to succeed in school, are intellectually aimless in mathematics classes because often they do not realize an intellectual need for what we intend to teach them. The notion of intellectual need is inextricably linked to the notion of epistemological justification: the learners' discernment of how and why a particular piece of knowledge came to be. This chapter addresses historical and philosophical aspects of these two notions, as well as ways teachers can be aware of students' intellectual need and address it directly in the mathematics classroom.

Years of experience with schools have left me with a strong impression that most students, even those who are eager to succeed in school, feel intellectually aimless in mathematics classes because we (teachers) fail to help them realize an *intellectual need* for what we intend to teach them. The main goal of this chapter is to define *intellectual need*, discuss its manifestations in mathematical practice, and demonstrate its absence and potential presence in mathematics instruction.

Intellectual need is inextricably linked to problem solving. Problem solving is usually defined as engagement in a problem "for which the solution method is not known in advance" (NCTM, 2000, p. 52). Alas, many of the situations students encounter in school satisfy this definition and yet do not constitute "true" problem solving because, from the students' perspective, these problems are often devoid of any intellectual purpose. Thus, another goal of this chapter is to advance the perspective, articulated by many other scholars (e.g., Brownell, 1946; Davis, 1992;

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G. Harel (✉)
University of California San Diego, 9500 Gilman Drive #0112, La Jolla,
San Diego 92093, CA, USA
e-mail: harel@math.ucsd.edu

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Hiebert, 1997; Schoenfeld, 1985; Thompson, 1985), that problem solving is not just a goal, but also the means—the only means—for learning mathematics.

The chapter is organized around five sections. The first section briefly outlines a set of underlying premises in which the concept of intellectual need resides. The second section defines the concept of *intellectual need* on the basis of these premises. The third section defines five categories of intellectual needs, describes their functions in mathematical practices, and offers concrete curricular implications. The fourth section introduces several common fundamental characteristics to these needs. They are fundamental because without them the concept of intellectual need is both pedagogically and epistemologically incoherent. The last section abstracts the themes of the paper into a definition of learning and a consequent instructional principle.

Underlying Premises

The perspective put forth in this paper is oriented within the Piagetian theory of equilibration and is part of a conceptual framework called *DNR-based instruction in mathematics* (*DNR*). *DNR* can be thought of as a system consisting of three categories of constructs: *premises*—explicit assumptions underlying the *DNR* concepts and claims; *concepts* oriented within these premises; and *instructional principles*—claims about the potential effect of teaching actions on student learning justifiable in terms of these premises and empirical observations. The initials *D*, *N*, and *R* stand for the three foundational instructional principles of the framework: *Duality*, *Necessity*, *and Repeated reasoning*. Here we only discuss the four *DNR* premises that are needed for our definition (see Fig. 6.1) of *intellectual need*: the *knowledge*

Premise	
Knowledge of Mathematics	Knowledge of mathematics consists of two related but different categories of knowledge: all the ways of understanding and ways of thinking that have been institutionalized throughout history.
Knowing	Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium.
Knowledge - Knowing Linkage	Any piece of knowledge humans know is an outcome of their resolution of a problematic situation.
Subjectivity	Any observations humans claim tohave made are due to what their mental structure attributes to their environment.

Fig. 6.1 Four DNR premises

(*of mathematics*) *premise*, the *knowing premise*, the *knowledge-knowing linkage premise*, and the *subjectivity premise*.[1]

Antecedent to the concepts of *way of understanding* and *way of thinking* referred to in the *Knowledge of Mathematics Premise* is the primary concept of *mental act*. Examples of mental acts include the acts of interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving. When one carries out a mental act, one produces a particular outcome. For example, when reading a string of symbols, a statement, or a problem, one of the mental acts a person carries out is the interpreting act, which, in turn, results in a particular meaning for it. Similarly, upon encountering an assertion, one may carry out the justification act and produce, accordingly, a particular justification. Such a product of a mental act is called a *way of understanding* associated with that act. Different individuals are likely to produce different ways of understanding associated with the same mental act. For example, students engaged in a dynamic geometry software activity may carry out conjecturing and justifying acts and, accordingly, produce different conjectures and justifications. Each conjecture and justification is a way of understanding—a product of the conjecturing act and justification act, respectively.

A common cognitive characteristic of a person's (or a community's) ways of understanding associated with a particular mental act is referred to as that person's *way of thinking* associated with that act. For example, a teacher or a researcher may infer (from a multitude of observations) one or more of the following characteristics: that a student's interpretations of arithmetic operations are characteristically inflexible, devoid of quantitative referents, or, alternatively, flexible and connected to other concepts; that a student's justifications of mathematical assertions are typically based on empirical evidence or, alternatively, based on rules of deduction. Each of these characteristics is a way of thinking. It is important to emphasize that in *DNR*, ways of understanding and ways of thinking are distinguished from their qualities. Namely, one's way of understanding or way of thinking can be judged as correct or wrong, useful or impractical in a given context. Of course, the goal is to help students gradually advance their ways of understanding and ways of thinking toward those that have been institutionalized in the mathematics community.

The *Knowing Premise* is after Piaget and is about the mechanism of knowing: that the means—the only means—of knowing is a process of assimilation and accommodation. Disequilibrium, or perturbation, is a state that results when one encounters an obstacle or fails to assimilate. It leads the mental system to seek equilibrium, that is, to reach a balance between the structure of the mind and the environment. Its cognitive effect in suitable emotional conditions is that the subject feels compelled "to go beyond his current state and strike out in new directions" (Piaget, 1985, p. 10). Equilibrium, on the other hand, is a state in which one perceives success in removing such an obstacle. In Piaget's terms, it occurs when one modifies

[1]These are four of the eight *DNR* premises (see Harel, 1998, 2008a, 2008b, 2008c).

his or her viewpoint (accommodation) and is able, as a result, to integrate new ideas toward the solution of the problem (assimilation).

The *Knowledge-Knowing Linkage Premise*, too, is inferable from Piaget, and is consistent with Brousseau's claim that "for every piece of knowledge there exists a fundamental situation to give it an appropriate meaning" (Brousseau, 1997, p. 42). The *Subjectivity Premise* orients our interpretations of the actions and views of the learner. Many scholars (e.g., Confrey, 1991; Dubinsky, 1991; Steffe & Thompson, 2000; Steffe, Cobb, & von Glasersfeld, 1988) have articulated essential implications of the *Subjectivity Premise* to mathematics curriculum and instruction.

These and the rest of the *DNR* premises (see Harel, 2008b, 2008c) were not conceived a priori, but emerged in the process of reflection on and exploration of justifications for the *DNR* concepts and claims.

Definition

With these premises at hand, we can now define the concept of *intellectual need* and its associated concept, *epistemological justification*. If K is a piece of knowledge possessed by an individual or a community, then, by the *Knowing-Knowledge Linkage Premise*, there exists a problematic situation S out of which K arose. S (as well as K) is subjective, by the *Subjectivity Premise*, in the sense that it is a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, his or her current knowledge. Such a problematic situation S, prior to the construction of K, is referred to as an individual's *intellectual need*: S is the need to reach equilibrium by learning a new piece of knowledge. Thus, intellectual need has to do with disciplinary knowledge being created out of people's current knowledge through engagement in problematic situations conceived as such by them. One may experience S without succeeding to construct K. That is, intellectual need is only a necessary condition for constructing an intended piece of knowledge, and, as discussed below, other motivational conditions are also necessary. Methodologically, however, intellectual need is best observed when we see that (a) one's engagement in the problematic situation S has led one to construct the intended piece of knowledge K and (b) one sees how K resolves S. The latter relation between S and K is crucial, in that it constitutes the geneses of mathematical knowledge—the perceived reasons for its birth in the eyes of the learner. We call this relation *epistemological justification*. An individual's or the institutionalized epistemological justification may not (and often does not) coincide with the historical epistemological justification. For example, many central concepts of real analysis—and some argue the entire field of real analysis (Bressoud, 1994)—were intellectually necessitated from Fourier's solution to Laplace's equation, $\frac{\partial^2 z}{\partial w^2} + \frac{\partial^2 z}{\partial x^2} = 0$. In particular, his solution as an infinite cosine series led, after major objections from the leading mathematician of the time, to a reconceptualization of the concept of function. Specifically, the expansion $f(x) = \frac{\pi}{4}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \cdots\right)$, whose graph oscillates between the −1 and 1, was not conceived as a function, because

> functions were polynomials; roots, powers, and logarithms; trigonometric functions and their inverses; and whatever could be built up by addition, subtraction, multiplication, division, or composition of these functions. Functions had graphs with unbroken curves. Functions had derivatives and Taylor series. Fourier's cosine series flew in the face of everything that was *known* about the behavior of functions. (Bressoud, 1994, p. 7).[2]

Thus, the historical epistemological justification for the concept of function is not necessarily that currently held by most mathematicians.

Categories of Intellectual Need

Although laying claim to neither completeness nor uniqueness, I offer five categories of intellectual needs: (1) *need for certainty*, (2) *need for causality*, (3) *need for computation*, (4) *need for communication*, and (5) *need for structure*. In modern mathematical practices these categories of needs are inextricably linked, which makes it difficult to discuss them in isolation. Despite this difficulty, they will be discussed in separate sections in an effort to demonstrate the existence of each need and to better elucidate their distinctions. Each of these sections is divided into two parts. The first part (a) defines the respective need, (b) discusses its cognitive primitives (preconceptualizations whose function is to orient us to the intellectual needs we experience when we learn mathematics[3]), and (c) illustrates its occurrence in the history of mathematics. The second part of each section discusses pedagogical considerations of the respective need.[4] None of these discussions intends to provide a comprehensive epistemological, historical, cognitive, or instructional account for any of these needs; rather, the goal is to describe the intended meaning for each need and illustrate its function in mathematical practice and its possible application in the teaching of mathematics. Nor are these discussions of equal length. The need for computation, for example, occupies the largest space due to its ubiquity in mathematical practice, on the one hand, and its special role in mathematics curricula, on the other hand.

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[2]See also Lakatos (1976, Footnote 3, pp. 19–20, Footnote 2, pp. 22–23, and Appendix 2, pp. 151–152) for an interesting discussion on a similar resistance "monstrous" conceptualization of function.

[3]Here and elsewhere in this chapter it is essential to understand the phrase "learn mathematics" in the sense described earlier, that is, in accordance with the *Knowledge of Mathematics Premise* and the definition of learning presented earlier.

[4]Since the discussion of pedagogical considerations follow the discussion of historical phenomena, it is important to state our belief that the intellectual necessity for a learner need not—and in most cases cannot—be the one that occurred in the history of mathematics.

Need for Certainty

Definition and function. When an individual (or a community) considers an assertion, he or she conceives it either as a *fact* or as a *conjecture*—an assertion made by a person who has doubts about its truth. The assertion ceases to be a conjecture and becomes a fact in his or her view once the person becomes certain of its truth. The *need for certainty* is the natural human desire to know whether a conjecture is true—whether it is a fact. When the person fulfills this need, through whatever means deemed appropriate by him or her, the person gains new knowledge about the conjecture.

We reserve the term *proving* for the mental act one carries out to achieve certainty about a conjecture, and *explaining* (to be discussed in the next section) for the mental act one carries out to understand the cause for a conjecture to be true or false. A person is said to have proved an assertion if the person has produced an argument that convinced him or her that the assertion is true. Such an argument is called *proof*. The proof someone produces may not be one that is acceptable by the mathematics community, but it is a proof for the person who has produced it. Hence, a proof is a way of understanding; it is a cognitive product of one's mental act of proving. A *proof scheme*, on the other hand, is a way of thinking; it is a collective cognitive characteristic of the proofs one produces. Proof schemes can be thought of as the means by which one obtains certainty. For example, a proof scheme may be empirical, where conviction is reached through perceptual or inductive observations (e.g., drawings, measurements, a series of examples, etc.), or deductive, where conviction is reached through application of rules of logic (see Harel (2008a) for a more thorough discussion).

Humans' instinctual desire to seek certainty is a cognitive primitive to the mathematical certainty reached through deductive proof schemes. Throughout history, proof schemes have not been static but varied from civilization to civilization, generation to generation within the same civilization, and community to community within the same generation (Kleiner, 1991). For example, the Babylonians merely prescribed specific solutions to specific problems, and so their proof schemes were mainly empirical. The deductive proof scheme—that is, the approach of establishing mathematical certainty by deducing facts from accepted principles—was first conceived by the Greeks and continues to dominate the mathematics discipline today.

Pedagogical considerations. Our subjectivity toward the meaning of proof does not imply ambiguous goals in the teaching of this concept. Ultimately, the goal is to help students learn to produce mathematical proofs and acquire mathematical proof schemes. A proof or a proof scheme is mathematical if it is consistent with those shared and practiced in contemporary mathematics. It is due to these schemes and practices that mathematicians trust the validation process of proofs established by the mathematics community. Clearly a mathematician is certain of a result when he or she proved it or read its proof. However, mathematicians are certain of numerous results, especially those outside their mathematical specialty, whose proofs they have not read. They accept a result if it has been validated by a mathematician they trust or has gone through a certification process by the community (e.g., published in a reputable journal). Auslander (2008) points out that this process of validation and certification “is an indication that we are part of a community whose members trust one another,” and that “mathematics could not be a coherent discipline, as opposed to a random collection of techniques and results, without [this process]” (p. 64).

These socio-mathematical norms for conviction are fundamentally different from the norms prevalent in the mathematics classroom. Strong evidence exists that students at all grade levels, and even school teachers, draw certainty from undesirable proof schemes, such as verification on the basis of specific examples (the *inductive proof scheme*), appearances in drawings (the *perceptual proof scheme*), forms in which a proof is conveyed (the *ritualistic proof scheme*), and teacher’s authority (the *authoritative proof scheme*) (Harel & Sowder, 1998, 2007). These behaviors are not surprising, given common teaching practices. Harel and Rabin (2010) identified a series of teaching practices that might account for the strong presence of these proof schemes among students. These practices include the following: the teacher’s answers to students’ questions mainly tell them how to perform a task and whether an action is correct or incorrect; the justification of the need for content taught is social rather than intellectual; and the teacher’s justifications are mainly authoritative, and those that are not authoritative are mainly empirical rather than deductive.

Beyond such detrimental teaching practices, other intuitively sound teaching practices aimed at changing students’ undesirable proof scheme have turned out to be largely ineffective. In particular, raising skepticism as to whether an assertion is true beyond the cases evaluated, and showing the limitations inherent in the use of examples through situations where an assertion is true for a very large number n of cases but untrue for the $n+1$ case, does not, in most cases, alter students’ proof-related behaviors. This observation was made repeatedly in my teaching experiments with undergraduate math and engineering students as well as in-service teachers. An explanation for this phenomenon rests on the recognition that doubts and conviction—and more generally disequilibrium and equilibrium—are interdependent. A person’s doubts about an observation cannot be defined independently of what constitutes certainty for him or her, and, conversely, a person’s certainty cannot be defined independently of what doubt is for that person. The presence of doubts necessarily implies the presence of conditions for their removal, and, conversely, a fulfillment of these conditions is necessary for attaining certainty. Thus, since the students viewed their actions of verifying an assertion in a finite number of cases as sufficient for removing their doubts about the truth of the assertion, the question of whether the assertion is true beyond the cases evaluated is unlikely to generate intellectual perturbation with the students. Moreover, since in most cases the teacher’s verification actions confirm what the students have already concluded, these actions add little or nothing to the students’ conviction about the truth or falsity of the assertion. The counterexample cases students (rarely) encounter, where assertions are true for a large number of cases but untrue for all cases, do not shake students’ confidence in their empirical methods of proving. This is so because students’ conditions for gaining certainty have not been fulfilled; the attempt to bring students to doubt their empirical proving methods is done by a method those students do not accept in the first place.

The experience of disequilibrium cannot be described independently of its corresponding experience of equilibrium, and, therefore, as a form of perturbational experience, intellectual need cannot be determined independently of what satisfies it. An important implication of this observation is that curriculum developers and teachers must think hard as to what constitutes perturbation and equilibrium for students and how to enculturate them into a milieu of *mathematical* perturbations and equilibriums. The rest of this chapter is an attempt to make a contribution toward defining the content of this milieu.

Need for Causality

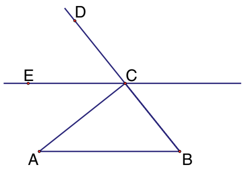
Definition and function. Certainty is achieved when an individual determines (by whatever means he or she deems appropriate) that an assertion is true. Truth alone, however, may not be the only aim for the individual, and he or she may desire to know *why* the assertion is true—the cause that makes it true. Thus, the *need for causality* is one's desire to *explain*, to determine a cause of a phenomenon. "Mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain" (Steiner, 1978, p. 135). For many, the role of mathematical proofs goes beyond achieving certainty—to show that something is true; rather, "they're there to show … why [an assertion] is true," as Gleason, one of the solvers of Hilbert's Fifth Problem (Yandell, 2002, p. 150), points out. Two millennia before him, Aristotle, in his *Posterior Analytic*, asserted,

> We suppose ourselves to possess unqualified scientific knowledge of a thing, as opposed to knowing it in the accidental way in which the sophist knows, when we think that we know the cause on which the fact depends as the cause of the fact and of no other. (p. 4)

Like with certainty, humans' instinctual desire to explain phenomena in their environments serves as a cognitive primitive to mathematical justification. The distinction between achieving certainty and finding causality in mathematics was the focus of a debate during the sixteenth and seventeenth centuries. Some philosophers of this period argued that mathematics is not a perfect science because mathematics is concerned with mere certainty rather than cause: Mathematicians are satisfied when they arrive at a conclusion by logical implications but do not require the demonstration of the *cause* of their conclusion (Mancosu, 1996). These philosophers point, for example, to Euclid's proof of Proposition 1.32 (the sum of the three interior angles of any triangle ABC is equal to 180°). Consider Euclid's proof of this proposition (Fig. 6.2).

In this proof, these philosophers argue, the cause of the property that is proved is absent. The two facts to which the proof appeals—the one about the auxiliary segment *CE* and the one about the external angle *ACD*—cannot be the true cause of the property, for the property holds whether or not the segment *CE* is produced and the angle *ACD* considered. A causal proof, according to these philosophers, gives not just evidence of the truth of the theorem but of the *cause* for the proposition's truth.

Proof by contradiction was another example of a noncausal proof in the eyes of these philosophers. When a statement "*A* implies *B*" is proved by showing how not



Construct CE parallel toAB . Then the alternate angles BAC and ACE are congruent and the corresponding angles ABC and ECD are congruent. Hence, $m(\sphericalangle BAC) + m(\sphericalangle ABC) + m(\sphericalangle ACB) = 180°$.

Fig. 6.2 Euclid's proof for the triangle angle sum theorem

B (and *A*) leads logically to an absurdity, we do not learn anything about the causal relationship between *A* and *B*. Nor, continued these philosophers to argue, do we gain any insight into how the result was obtained. Proofs by contradiction continued to be controversial until the late nineteenth and early twentieth centuries. In 1888, for example, Hilbert astonished the mathematical community of the time when he proved the Gordan conjecture: There is a finite "basis" from which all algebraic invariants of a given polynomial form could be constructed by applying a specified set of additions and multiplications. It was more the form of Hilbert's solution than the shear success in solving an open problem that was controversial. Hilbert didn't *find* a basis that everyone had been searching for; he merely proved that if we accept Aristotle's law of the excluded middle ("Any statement is either true or its negation is true") then such a basis had to exist, whether we could produce it or not.

> At first this result was greeted with disbelief. Gordan said, "Das ist nicht Mathematic. Das ist Theology." Cayley at first failed to grasp the proof. Lindemann thought the proof unheimlich ("uncomfortable, sinister, weird"). Only Klein got it right away: "Wholly simple and, therefore, logically compelling." Within the next five years organized opposition disappeared, and this was the result that initially made Hilbert's reputation. (Yandell, 2002, p. 12)

Why was Hilbert's use of proof by contradiction so controversial? After all, he was not the first to use this method of argument? According to Yandell, previous uses had not dealt with a subject of such obvious calculational complexity. A pure existence proof does not produce a specific object that can be checked—one had to trust the logical consistency of the growing body of mathematics to trust the proof. The presence of an actual object that can be evaluated provides more than mere certainty; it constitutes a cause (in the Aristotelian sense) for the observed phenomenon.

The philosophical stance about the scientific nature of understanding and its implication that mathematical proofs must conform to the Aristotelian definition of science seems to have played a role, perhaps implicitly, in Grassmann's (1809–1877) work. According to Lewis (2004), when Grassmann published his theory of extension (*Ausdehnungslehre*) in 1844, and again, in a modified version, in 1862, it went unnoticed, partly due to its novel and large-scale discoveries, and partly due to its novel method of presentation. The latter is of particular relevance to our discussion about the need for certainty versus the need for causality. Grassmann insisted on a presentation that met the highest standards of rigor, on the one hand, and provided the reader with a clear understanding of the epistemological justifications behind his concepts and proofs, on the other. Grassmann's insistence on such a presentation, according to Lewis, goes beyond pedagogical considerations to help the reader grasp his new concepts and techniques; rather, Grassmann "appears to regard the pedagogical involvement as an essential part of the justification of mathematics as a science" (p. 19).

Recall that *proving* and *explaining* are two different, yet related, mental acts: the first is carried out to remove doubts, and the second to determine cause. Accordingly, a *proof* is a way of understanding associated with the mental act of proving, and an *explanation* is a way of understanding associated with the mental act of explaining. Often, when facing a particular assertion or arriving at a conjecture, one may carry out the two mental acts of proving and explaining together, resulting in a single product that is both a proof and an explanation—it removes doubts about the truth of the assertion and provides a reason, or a cause, for its truth. The issue of what makes a proof a causal proof (i.e., proof *and* explanation) was addressed by Steiner (1978). He distinguishes between proofs that prove and proofs that explain, but his distinction is a priori, independent of the individual's conceptions. This distinction and its corresponding ontological position are adopted by Hanna (1990), who argues that proofs by mathematical induction, for example, are proofs that prove but do not explain. Our position is different. We hold that it is the individual's scheme of doubts, truths, and convictions in a given context that determines whether an argument is a proof or an explanation.

Pedagogical considerations. This historical analysis, together with the findings discussed earlier about the ineffectiveness of some intuitively sound teaching practices, led to a pedagogical lesson regarding the transition from undesirable proof schemes, especially the empirical proof schemes, to deductive proof schemes. The idea is to shift students' attention from *certainty* to *cause*. Rather than justifying the need for deductive proofs by raising questions about the logical legitimacy of empirical proofs—which, as indicated earlier, turned out to have little or no perturbational effect—we turned students' attention to the cause (or causes) that makes an assertion true or false. By repeatedly attending to explanations as well as to proofs, we aimed at enculturating students into the habit of seeking to understand cause, not only attaining certainty. To illustrate how this can be done, consider the following episode: A group of in-service secondary teachers participating in a professional development summer institute were given the Quilt Problem (Fig. 6.3).

A company makes square quilts. Each quilt is made out of small congruent squares, where the squares on the main diagonals are black and the rest are white. The cost of a quilt is calculated as follows: Materials: \$1.00 for each black square and \$0.50 for each white square; Labor: \$0.25 for each square. To order a quilt, one must specify the number of black squares, or the number of white squares, or the total number of squares on the following order form:

Number of Black Squares	Number of White Squares	Total of Squares

April, Bonnie, and Chad ordered three identical quilts. Each of the three filled out a different order form. April entered the number of black squares in the Black Cell. The other two entered the same number as April's, but accidentally Bonnie entered her number in the Whites Cell, and Chad entered his number in the Total Cell. April was charged \$139.25. How much money were Bonnie and Chad charged?

Fig. 6.3 The Quilt Problem

Fig. 6.4 Nina's equations

$$\text{Price} = \#\text{Black} + \frac{(\#\ \text{White})}{2} + \frac{(\#\text{Black} + \#\ \text{White})}{4}.$$

$$139.25 = 2x - 1 + \frac{(x-1)^2}{2} + \frac{\left((2x-1)+(x-1)^2\right)}{4}.$$

Fig. 6.5 Nina's pattern for the number of white squares

Size	# White Squares
1	0
3	4 (4·1)
5	16 $(3\cdot 1)+(1\cdot 4) = 4(4) = 4(2\cdot 2) = 4\cdot\left(\frac{5-1}{2}\right)^2$
7	36 $4(9) = 4(3\cdot 3) = 4\cdot\left(\frac{7-1}{2}\right)^2$
…	
x	$4\cdot\left(\frac{x-1}{2}\right)^2 = 4\cdot\frac{(x-1)^2}{4} = (x-1)^2$

The teacher participants worked in small groups on the problem for some time, and then each group presented its solution (whether complete or partial). Nina,[5] a teacher participant in the institute, presented her group's solution. The solution considers two cases: an even-sized quilt and an odd-sized quilt. Our discussion here pertains to the odd-sized case, but for the sake of completeness the even-sized case is also presented.

Nina noted that viewing each partial cost in terms of units of \$0.25 excludes the possibility that the quilt is of an even-sized dimension, for if the dimension were even then each partial cost, and therefore the total cost as well, would comprise an even number of \$0.25 – units. But the total cost (\$139.25) comprises an odd number of these units. For the odd-sized case, Nina first wrote the two lines in Fig. 6.4: where x is the size of the quilt (the number of squares on each side). Following this, she proceeded to solve the algebraic equation. When asked by one of the teacher participants in the class why the number of whites is $(x-1)^2$, Nina responded by presenting a table (see Fig. 6.5).

Nina indicated that this table was the result of an effort by her group to express the number of white squares as a function of the quilt's size. Based on these special cases, the group concluded that for an x-sized quilt, the number of white squares is $4\left(\frac{x-1}{2}\right)^2$, or $(x-1)^2$.

In the discussion that followed this presentation, it was clear that the class as a whole was impressed by Nina's solution, and was convinced that the generalization was valid. Rather than dwelling on the question of how we know the pattern

[5]Pseudonyms.

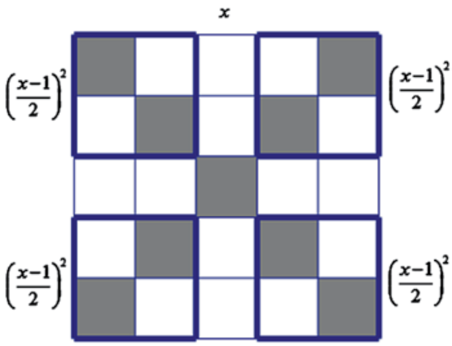


Fig. 6.6 The drawing that accompanied John's solution

continues to be valid for all positive odd integers, the instructor presented an alternative solution offered by John, one of the teacher participants in an earlier summer institute. In John's solution, x is an odd number representing the size of the quilt (the number of square on its side). Removing the middle row and middle column of squares (those containing the black square shared by the main diagonals) leaves four "subsquares" having $\left(\frac{x-1}{2}\right)^2$ squares, including black squares from the diagonals (Fig. 6.6). So, excluding the row and column previously removed, there would be a total of $4\left(\frac{x-1}{2}\right)^2$, or $(x-1)^2$, squares in the four subsquares. Since each black square along the diagonal corresponds to one square that had been removed by eliminating the row and column containing the center black square, the number of white squares in an x-sized quilt remains $(x-1)^2$.

The teacher participants had been impressed by Nina's solution and they were equally impressed by John's solution. The general consensus among the teachers was that both solutions are convincing, but John's solution has an added value; it reveals the *reason* (i.e., the *cause*) for *why* the number of white squares is $(x-1)^2$.[6]

Our experience from these professional development institutes and other teaching experiments is that through repeated experiences such as the one described here—of comparing empirical solutions (such as Nina's) with causal solutions (such as John's)—learners gradually come to the realization that one type of reasoning is of more intellectual value than the other. Whereas empirical reasoning provides them with certainty (because of their robust *empirical proof scheme*), causal reasoning provides them with both certainty and enlightenment (understanding of cause). We observed a change in the teacher participants' argumentation for

[6]Other solutions were offered by the class. For example, one solution examined all the possible cases for the size of the quilt, and another solution simply calculated the number of white squares by subtracting the number of black squares from the total number of squares.

ascertainment and persuasion after having gone through this experience for an extended period of time. Thus, shifting the focus from certainty to causality seems to have effected the teacher participants' schemes of doubts and, in turn, their proof schemes. Though they continued to produce empirical proofs, they also sought casual justifications.[7]

Need for Computation

Definition and function. After Piaget, quantifying is the act of transforming a sensation (i.e., a perceptual action scheme—visual, auditory, tactile, etc.) into a quantity—a measurable sensation. For example, the sensation *fastness* is transformed into *speed*; *heaviness* into *weight*; *extent* into *length*, *area*, or *volume*; *pushing or pulling* into *force*; *rotational twist* into *torque*; *hotness* into *heat* (i.e., *thermal energy*), etc. Some sensations might be difficult to quantify; "texture," "taste," "pain," "happiness," and "instructional quality" are examples. The quantification process involves assigning a unit of measure to a quantity; for example, "mph," "gram," "Newton meter," and "square meter" are unit measures assigned to the quantities "speed," "weight," "torque," and "area," respectively. As can be seen from these examples, often quantification is a nested act: one quantity is constructed from previously formed quantities.

Sensations such as *fastness* and *heaviness* constitute cognitive primitives to the need to quantify, which is one expression of the *need to compute*. Another expression is the act of determining a missing quantity from a set of quantitative constraints, as when, for example, one seeks to determine the dimensions of a right triangle from its area and the ratio of two of its sides. Collectively, these two expressions of the need to compute manifest humans' desire to accurately compare different sensations, determine their interrelationships, and, in turn, better understand and control their own physical and social environment.

The need to compute is not the invention of modern mathematics. The Babylonians (around 2000 B.C.) engaged in problems that required determining the value of a quantity from other given quantities. For example, they invented procedures for solving what we now view as quadratic equations (e.g., how to find the side of a square when the difference between the area and the side is given). This practice of computing continued to develop in different cultures throughout history, and it led gradually to the development of symbolic algebra, and, in turn, to new mathematical concepts (such as complex numbers, equations, and polynomials) and a system of symbols to represent these concepts. These invented symbols necessitated the creation of new concepts. For example, the Babylonian numerical system

[7]This transition involved interesting cognitive disequilibria, which are not discussed in this paper.

is a positional notation system (i.e., utilizes the principle of "place value") but denotes all the multiples $k60^{\pm n}$ by the same sign (e.g., the string of symbols, ٢ ٢   ٢ ٢, might mean $2(60)+2$, $2(60)^2+2(60)$, or $2(60)^3+2(60)^2$). The computational ambiguity of this system necessitated the conception of zero as a number and the introduction of this number into calculation. This conception, in turn, led to the creation of the numerical system of our present time, which removed the ambiguity and advanced the computational effectiveness of the place value system used by Babylonians. Another example of how the need for computation led to the creation of symbols, and, in turn, to the creation of new concepts, is from a later period. The Leibnitzian notation Df, $D^2 f$, etc. was needed to display the number of successive differentiations, but it also suggested the possibility of extending the meaning of $D^{\alpha} f$ for negative and fractional α. Davis and Hersh (1981) point out that this invention contributed powerfully to the development of abstract algebra in the mid-nineteenth century. In addition, this notation may have necessitated, or at least helped to advance, the object conception of function (in the sense of Dubinsky, 1991), namely, that f, in addition to being a process that assigns to a given input-number a single output-number, is itself an operand (an input) for another process, D. Overall, the nature of computing evolved rather slowly. As late as the fifteenth century, mathematicians lacked the ability to compute with symbols independent of their spatial referents—and encountered major difficulties as a result. For example, a major obstacle in justifying the formula for the roots of the cubic equation was the inability to figure out the identity $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$, whose proof required dissection of a cube in three-dimensional space (Tignol, 1988). Only later, with the work of Cardono (1501–1576), was the formula for the cubic equation justified by means of *symbolic algebra*—specifically, by transforming different forms of cubic equations into systems of equations.

To *compute by means of symbolic algebra* reflects two inseparable abilities: (a) the ability to represent a situation symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their reference, and (b) the ability to pause at will during the manipulation process in order to probe into the referential meanings for the symbols involved in the manipulation. The attempt to form a referential meaning need not always succeed or even occur. What matters is that the person who carries out the manipulation has the ability to investigate, when needed, the referential meaning of any symbol and transformation involved. In this paper, the *need to compute* is in the sense of this definition, in that it refers to one's desire to quantify, determine a missing object or construct an object (e.g., a number, geometric figure, function, or matrix), determine the property of an object or relations among objects, etc. by means of symbolic algebra. It also includes the need to find more efficient computational methods, such as one might need to extend computations to larger numbers in a reasonable "running time."

Historically, the practice of manipulating symbols without *necessarily* examining their meanings played a significant role in the development of mathematics. For example, during the nineteenth century a significant work was done in differential and difference calculus using a technique called "operational method," a method whose results are obtained by symbol manipulations without understanding their meaning, and in many cases in violation of well-established mathematical rules (see, for example, the derivation of the Euler–MacLaurin summation formula for approximating integrals by sums, in Friedman (1991)). Mathematicians sought meaning for the operational method, and with the aid of functional analysis, which emerged early in the twentieth century, they were able to justify many of its techniques.

Computing by means of symbolic algebra marked a revolutionary change in the history of mathematics. In particular, it provided a conceptual foundation for the critical shift from "results of operations" as the object of study to the operations themselves as the object of study. While the Greeks restricted their attention to attributes of spatial configurations and paid no attention to the operations underlying them, nineteenth-century mathematics investigated the operations, their algebraic representations, and their structures. For example, Euclidean constructions using only a compass and straightedge were translated into statements about the constructability of real numbers, which, in turn, led to observations about the structure of constructible numbers. A deeper investigation into the theory of fields led to the understanding of why certain constructions are possible whereas others are not. The Greeks had no means to build such an understanding, since they did not attend to the nature of the operations underlying Euclidean construction. Thus, by means of symbolic algebra and analytic geometry, mathematicians realized that all Euclidean geometry problems can be solved by a single approach, that of reducing the problems into equations and applying algebraic techniques to solve them. Euclidean straightedge-and-compass constructions were understood to be equivalent to equations, and hence the solvability of a Euclidean problem became equivalent to the solvability of the corresponding equation(s) in the constructible field.

Pedagogical considerations. The need for computation, perhaps the most powerful need in the context of school mathematics, is rarely utilized adequately. For example, after learning how to multiply polynomials, secondary-school students typically learn techniques for factoring polynomials, and then how to apply these techniques to simplify rational expressions. Judging from the students' perspective, the tasks of multiplying and factoring polynomials and simplifying rational expressions are intellectually purposeless. They learn to transform one form of expression into another without a clear understanding of the mathematical purpose such transformations serve and the circumstances under which one form of expression is more advantageous than another. A case in point is the way the quadratic formula is taught. Some algebra textbooks present the quadratic formula before the method of completing the square. Seldom do students see an intellectual purpose for the latter method (i.e., to solve quadratic equations and to derive a general formula for their solutions), rendering completing the square problems intellectually purposeless to most students. An alternative approach that would intellectually necessitate such problems builds on what the students know: Assuming that the students have already learned how to solve equations of the form $(x+T)^2 = L$, the teacher's action would be geared toward helping them manipulate the quadratic equation $ax^2 + bx + c = 0$ with a goal in mind—that of transforming the latter equation form into the former known equation form but maintaining the solution set unchanged. The intellectual gain is that students learn that algebraic expressions are reformed for a reason.

Often problems used to introduce a new concept do not demonstrate the intellectual benefit of the concept at the time of its introduction. For example, some high-school mathematics texts introduce the idea of using equations to solve word problems through trivial, one-step addition or multiplication word problems (see Harel, 2009). This approach is contrived, and is unlikely to intellectually necessitate this idea since students can easily solve such problems with tools already available to them. To make this point clearer, it is worth presenting an alternative approach—one that is more likely to intellectually necessitate algebraic tools to solve word problems. In this alternative approach, students first learn to solve nontrivial word problems with their current arithmetic tools. For example, they can reason about problems of the following kind directly, without any explicit use of variables.

> Towns A and B are 280 miles apart. At 12:00 PM, a car leaves A toward B, and a truck leaves B toward A. The car drives at 80 m/h and the truck at 60 m/h. When will they meet?

Students can do so by, for example, reasoning as follows:

> After 1 h, the car drives 80 miles and the truck 60 miles. Together they drive 140 miles. In 2 h, the car drives 160 miles and the truck 120 miles. Together they drive 280 miles. Therefore, they will meet at 2:00 PM.

Through this kind of reasoning, students develop the habit of building coherent images for the problems—a habit they often lack.

These problems can then be gradually modified (in context, as well as in quantities) so as to make them harder to solve with arithmetic tools alone, whereby necessitating the use of algebraic tools. For example, varying the distance between the two towns through the sequence of numbers, 420, 350, 245, and 309, results in a new sequence of problems with increasing degree of difficulty. Students still can solve these problems with their arithmetic tools but the problems become harder as the relationship between the given distance and the quantity 140 (the sum of the two given speeds) becomes less obvious. For example, for the case where the distance is 245 miles, the time it takes until the two vehicles meet must be between 1 and 2 h, and so one might search through the values 1 h and 15 min $\left(80\frac{75}{60}+60\frac{75}{60}\overset{?}{=}245\right)$, 1 h and 30 min $\left(80\frac{90}{60}+60\frac{90}{60}\overset{?}{=}245\right)$, and 1 h and 45 min $\left(80\frac{105}{60}+60\frac{105}{60}\overset{?}{=}245\right)$, and find that the last value is the time sought for. This activity of varying the time needed can give rise to the concept of variable (or unknown) and, in turn, to the equation, $80x+60x=245$. Granted, this is not the only approach to intellectually necessitate the use of algebraic tools for solving word problems. However, whatever approach is used, it is critical to give students ample opportunities to repeatedly reason about problems with their current arithmetic tools and to gradually lead them to incorporate new, algebraic tools. The goal is for students to learn to build coherent mental representations for the quantities involved in the problem and to intellectually necessitate the use of equations to represent these relationships. An added value of this approach is the development of computational fluency with numbers (especially fractions).

The inadequate use of the need for computation is prevalent in undergraduate mathematics as well. For example, typically, linear algebra textbooks introduce the pivotal concepts of “eigenvalue,” “eigenvector,” and “matrix diagonalization” with statements such as the following:

> The concepts of “eigenvalue” and “eigenvector” are needed to deal with the problem of factoring an $n \times n$ matrix A into a product of the form XDX^{-1}, where D is diagonal. The latter factorization would provide important information about A, such as its rank and determinant.
>
> The concepts of “eigenvalue” and “eigenvector” are needed to deal with the problem of computing a higher order of power of a given matrix, to study the long-term behavior of linear systems.
>
> The concepts of “eigenvalue” and “eigenvector” are needed to deal with a problem that arises frequently in application of linear algebra—that of finding values of a scalar parameter λ for which there exists $x \neq 0$ satisfying $Ax = \lambda x$, where A is a square matrix.

Each of these introductory statements aims at pointing out to the student an important problem. While the problem is intellectually intrinsic to its poser (a university instructor), it is most likely to be alien to the students, since a student in an elementary linear algebra course is unlikely to realize from such statements the true nature of the problem, its mathematical importance, and the role the concepts to be taught (“eigenvalue,” “eigenvector,” and “diagonalization”) play in determining its solution.

An alternative approach, based particularly on students’ intellectual need for computation, is through linear systems of differential equations. In what follows, I briefly outline part of a unit in a linear algebra course I have taught numerous times, some of which as teaching experiments. The goal of the unit is to necessitate fundamental ideas of the Eigen Theory, from the basic concepts of eigenvalue, eigenvector, diagonalization, and their related theorems up to the Jordan Theorem (i.e., “Every vector is a linear combination of generalized eigenvectors.”) and its related Jordan Canonical Form. The unit begins with an investigation of the linear system of differential equations:

$$\begin{cases} AY(t) = Y'(t) \\ Y(0) = C \end{cases} \qquad (*)$$

(Here A is a square matrix, and the matrix A and the vector C, the initial condition vector, are over the complex field.) Obviously, this system and its representation in a matrix form do not emerge in a vacuum, but out of a context established in previous units. The investigation consists of a series of stages. Here I focus on the first several stages that lead up to the concept of diagonalization.

In the first stage of the investigation, we help students analogize system (*) to the scalar case:

$$\begin{cases} ay(t) = y'(t) \\ y(0) = c \end{cases} \qquad (**)$$

(Here a and c are real numbers.) The students are familiar with this equation and its (unique) solution, $y(t) = c\mathrm{e}^{at}$, from their calculus classes. To refresh their memory of this topic, we assign them (prior to the start of the unit on Eigen Theory) a few problems involving this equation and the exponential function power series. Students notice the similarity in form between (*) and (**), and accordingly offer the analogous expression, $Y(t) = C\mathrm{e}^{At}$, as a solution to system (*). It takes some prompting from the instructor for the students to attend to the meaning of the objects and operations involved in this expression. After some discussion, the students offer to rewrite the product At as tA, and ask about the meaning of the phrase "e to the power of a matrix." At this point, students' attention is centered on this phrase, and so questions concerning the dimension of the matrix and whether the product $C\mathrm{e}^{tA}$ is meaningful are not raised. Often, but not always, students suggest that e^{B} is the matrix with the entries $(\mathrm{e}^{B})_{i,j} = \mathrm{e}^{B_{i,j}}$. With this definition at hand,[8] the instructor provides a special case of system (*) and asks the students to verify whether the expression $Y(t) = C\mathrm{e}^{At}$ is a solution to the system, as they have conjectured. In this process, the students first realize the need to reverse the order of the product $C\mathrm{e}^{At}$ into $\mathrm{e}^{At}C$, and then conclude that the revised expression $Y(t) = \mathrm{e}^{tA}C$, under their definition of the matrix-valued exponential function, is not a solution to system (*). Consequently, students conclude that the solution to system (*) must be of different form from the one they offered; it does not occur to them to seek a different definition for the matrix-valued exponential function.

The second phase of the investigation commences with the instructor suggesting a different approach for defining this function. He reminds the students of the definition of the real-valued function e^{b} as a power series (a topic they reviewed in the preceding unit). Some students suggest analogizing e^{B} to e^{b}; namely, that analogous to $\mathrm{e}^{b} = \sum_{i=0}^{\infty}(1/i!)b^{i}$, we define $\mathrm{e}^{B} = \sum_{i=0}^{\infty}(1/i!)B^{i}$. Again, despite the use of the term "define," students do not view the latter equality as a definition but as a formula, perhaps because the former equality was derived from a Taylor expansion rather stated as a definition. Nor do they raise any concern about the convergence of the series. Furthermore, only when the instructor asks the class to compute e^{B} for a particular simple 2×2 matrix B do the students realize that e^{B} is meaningless unless B is a squared matrix, and consequently they observe that e^{B} too is a squared matrix. With this new definition at hand, the instructor leads the class in the process of verifying that $Y(t) = \mathrm{e}^{tA}C = \sum_{i=0}^{\infty}(t^{i}/i!)A^{i}C$ is a solution to system (*). As with the question of convergence, the question of uniqueness too is never addressed in this class.

In the third phase of the investigation, the instructor returns to the above solution in its expansion form ($Y(t) = \mathrm{e}^{tA}C = \sum_{i=0}^{\infty}(t^{i}/i!)A^{i}C$) and points out the following critical observation: If $AC = \lambda C$ for some scalar λ, then the solution to system (*) is easily computable. Specifically, it is $Y(t) = \mathrm{e}^{\lambda t}C$, for under this condition $Y(t) = \sum_{i=0}^{\infty}(t^{i}/i!)A^{i}C = \sum_{i=0}^{\infty}(t^{i}/i!)\lambda^{i}C = \sum_{i=0}^{\infty}((\lambda t)^{i}/i!)C = \mathrm{e}^{\lambda t}C$. This

[8] The use of the term "definition" here should not imply that the students' intention was to *define*—in the mathematical sense of the term—the concept "e to the power of a matrix" (see the discussion on *definitional reasoning*).

observation necessitates attention to the relationship $AC = \lambda C$, and therefore a name: C is called an eigenvector of A and λ its corresponding eigenvalue. Following a few examples of solving system (*), the instructor (and in a few cases a student) raises the question about the computability of the solution in cases where the condition vector is not an eigenvector of the coefficient matrix. The instructor suggests looking at the case where C is not an eigenvector of A but it is a linear combination of eigenvectors of A. We proceed to show that in this case too the solution to system (*) is easily computable. Specifically $Y(t) = \sum_{i=1}^{n} a_i e^{\lambda_i t} C_i$, where $C = \sum_{i=1}^{n} a_i C_i$ and $AC_i = \lambda_i C_i$. This result is then used to conclude that if the coefficient matrix has a basis of eigenvectors then for any condition vector the solution to system (*) is easily computable. Such a matrix, therefore, is of a computational significance, and hence it warrants attention. This concludes the third phase of the investigation.

The content of the next phases depends on the level of the course. For an elementary linear algebra course, the proceeding phases deal with the factorization of matrices with a basis of eigenvectors (i.e., *diagonalization*) and change of basis. For the more advanced linear algebra course, the proceeding phases continue the investigation of the computability of the solution to system (*). The investigation leads up to the Jordan Theorem (and its related Canonical Form), which yields the interesting results that the solution to system (*) is always easily computable.

All the alternative approaches discussed here demonstrate how both conditions (a) and (b) in our definition of computing by means of symbolic algebra are implemented. It is never the case that every single symbol in the manipulation process is referential. Rather, it is only in critical stages (viewed as such by the person who carries the symbol manipulations) that one forms, or attempts to form, referential meanings. One does not usually attend to interpretation in the middle of symbol manipulations unless one encounters a barrier or recognizes a symbolic form that is of interest to the problem at hand. Thus, for most of the process the symbols are treated as if they have a life of their own. It is in this sense that symbol manipulation skills should be understood and, accordingly, be taught.

Needs for Communication

Definition and function. In mathematics, the *need for communication* refers collectively to two reflexive acts: *formulating* and *formalizing*. Formulating is the act of transforming strings of spoken language into algebraic expressions (i.e., expression amenable to computation by means of symbolic algebra as discussed in the preceding section). Formalization is the act of externalizing the exact intended meaning of an idea or a concept or the logical basis underlying an argument. A cognitive primitive of these two acts is the act of conveying and exchanging ideas by means of a spoken language and gestures, which are defining features of humans.

In modern mathematics the acts of formulation and formalizations are reflexive in that as one formalizes a mathematical idea it is often necessary to formulate it, and, conversely, as one formulates an idea one often encounters a need to formalize it.

Historically, however, the need for formulation seems to have emerged well after the need for formalization. At least in the Western world the need for formalization began with Greeks, whereas that of formulating with Viete (1540–1603) and Stevin (1548–1620). These two scholars are viewed by historians as milestones in the evolution of the need for formulation, and, in turn, in the evolution of the need for formalization beyond the Greeks. Until then, the exchange of mathematical ideas was largely colloquial (i.e., idiomatic and conversational). The Babylonians (around 2000 B.C.), for example, used only text to exchange problems and procedures for their solutions, as can be seen in one of their tablets:

> I have subtracted from the area the side of my square: 14.30 [meaning, the result is 14.30]. [To solve], divide 1 into two parts: 30. Multiply 30 and 30: 15. You add to 14.30, and 14.30.15 has the root 29.30. You add to 29.30 the 30 which you have multiplied by itself: 30, and this is the side of the square. (Tignol, 1988, p. 7).

The arithmetic here is in base 60, so, for example, 14.30 in base 10 is $14 \times 60 + 30 = 870$. Tignol points out that the "Babylonians had no symbol to indicate the absence of a number or to indicate that certain numbers are intended as fractions. For instance, when 1 is divided by 2, the result which is indicated as 30 really means 30×60^{-1}, i.e., 0.5" (p. 7).

Three and a half millennia later, Cardano began to *formulate* the notation of equations. For example, the equation $x^2 + 2x = 48$ is written by Cardano as, "*1. quad. P : 2 pos. aeq. 48* (*quad*. for 'quadratum'; *pos*. for 'positiones' and *aeq*. for 'aequatur')" (Tignol, 1988, p. 36). An essential characteristic of this type of notation is that *its syntax is in the form of a spoken language*. Of course, it is both simplistic and wrong to bundle a span of over three millennia of development of mathematical notation into a single characteristic. This is not our intention here. Rather, we merely aim at pointing to one of the features of the notational conventions of the time: the use of immediate, natural tools of a spoken language to exchange ideas. Remarkably, this level of notation was sufficient to attain major achievements, the most notable of which is the solution of the cubic equation.

Pedagogical considerations. Spoken knowledge is an essential means for the development of the need for (mathematical) communication. Consider the following example: Students may be satisfied with their intuitive explanation of why $\lim_{x\to\infty} 1/x = 0$, which typically is communicated through a statement such as the following: "$\lim_{x\to\infty} 1/x = 0$ because the larger x gets the closer $1/x$ is to 0." A teacher whose goal is to help students see a need to formulate and formalize their concept of *limit* might proceed, upon hearing this explanation, by writing it on the board along with the graphs of $f(x) = 1/x$ and $g(x) = -1$ (see Fig. 6.7).

Then the teacher may point out to the students that based on their own statement one can rightly argue $\lim_{x\to\infty} 1/x = -1$, because, by their own words, "the larger x gets the closer $1/x$ is to -1." This exchange may, as our experience confirms, result in a conflict for the students, whereby they see a need to formulate and formalize their idea of limit.

The reflexive nature between the need for formulation and the need for formalization is best captured by Thompson (1992), in analyzing the use of concrete

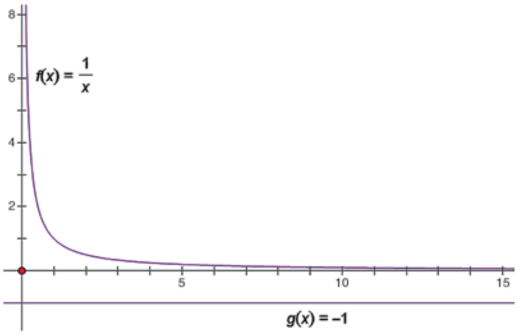


Fig. 6.7 The graphs of $f(x) = 1/x$ and $g(x) = -1$

materials in elementary mathematics instruction: “When students are aware of reciprocal relationships between notation and reasoning they may be more inclined to concentrate on their reasoning when experiencing difficulty and concentrate less on performing correct notational actions” (p. 124). Thompson places students’ use of concrete materials in the context of their development of the use of notation to express their reasoning. He points out that authoritative need deprives students from the opportunity to see a (intellectual) need for formalization:

> Students’ reenactment of a prescribed procedure does not give them opportunities to construct constraints in their meanings and reasoning—they meet constraints only because they are obliged to adhere to prescription, and it matters little that the prescriptions entail use of concrete materials. In reenacting prescribed procedures, students do not experience constraints as arising from tensions between their attempts to say what they have in mind and their attempts to be systematic in their expressions of it. (p. 124)

Conversely, it is the need for formalization that compels students to formulate (or reformulate) their symbolic system:

> As students come to be systematic in their expressions of reasoning and make a commitment to express their reasoning within their system, that same systematicity places constraints on the reasoning they wish to express. When students are aware of the constraining influence exerted by their arbitrary use of notation, they may feel freer to modify their standard uses of notation to express better what they have in mind. (p. 124)

Repeated application of the need for formulation and formalization is necessary to advance students’ conception of the notion of *mathematical definition*. This conception is associated with *definitional reasoning*—a way of thinking by which one defines objects and proves assertions in terms of mathematical definitions. A mathematical definition is a description that applies to all objects to be defined and only to them. A crucial feature of this way of thinking is that with it one is compelled to conclude logically that there can be only one mathematical definition for a concept within a given theory; namely, if D_1 and D_2 are such definitions for a concept C, then D_1 is a logical consequence of D_2, or vice versa; otherwise, C is not well defined. Typically, students’ definitions of concepts are not mathematical, even if the concepts were defined to them mathematically. Understanding the notion of mathematical definition and appreciating the role and value of mathematical definitions in proving is a developmental process, which is not achieved for most students until adulthood (if at all). Many students even in advanced grades do not possess definitional reasoning. For instance, in Van Hiele's (1980) model, only in the highest stage of geometric reasoning are students' definitions of Euclidean objects mathematical (see Burger & Shaughnessy, 1986). Definitional reasoning is largely absent among college students as well, even among undergraduate mathematics and engineering majors (Harel, 1999). For example, when asked to define "invertible matrix," many linear algebra students stated a series of equivalent properties (e.g., "a square matrix with a non-zero determinant," "a square matrix with full rank," etc.) rather than a definition. The fact that they provided more than one such property is an indication that they were not thinking in terms of mathematical definition.

Need for Structure

Definition and function. The *need for structure* is the need to reorganize the knowledge one has learned into a logical structure. A critical element in this definition is the verb "to reorganize," and, by implication, its source verb "to organize." The verb "to organize" implies an action on something that already exists, and the verb "to reorganize" implies that something has already been organized. Accordingly, the need for structure is not a forward need; that is, one does not feel intellectually compelled to learn new knowledge in a particular order and from that fit a predetermined structure; rather, one assimilates knowledge into one's existing structure, and reorganizes it if and when one perceives a need to do so. The nature of the structure into which one organizes one's own knowledge is idiosyncratic and depends entirely on one's past experience. Such a structure is unlikely to be logically hierarchical, and even mathematicians are unlikely to involuntarily organize their knowledge into a systematic logical structure. Thus, the term "reorganize" in the above definition recognizes that individual learners or communities of learners first organize the mathematical knowledge they learn in a form determined by their existing cognitive structures; later they may meet the need to *reorganize* what they have learned into a logical structure. The history of Euclidian geometry illustrates this point. Perhaps the most recognized mathematical structure is *Euclid*'*s Elements*, a geometrical edifice organized in a logical structure where each assertion depends on the previous ones. Relevant to our discussion here are two historical observations. First, the development of the theorems in the *Euclid*'*s Elements* did not follow a systematic logical progression, as it is laid out in this treatise, but evolved largely unsystematically over several centuries. Second, it was the need to organize this accumulated body of knowledge that led to the production of the logical structure of axioms, definitions, and propositions, as we know it; it was the need to perfect this structure that, in turn, led to the two-millennium-long attempt to prove the parallel postulate.

The need for structure often leads to the discovery of unifying principles (e.g., *associativity*), common elements to different systems (e.g., the *identity element*), invariants (e.g., for the quadratic form, $ax^2 + bx + cy^2$, the form $b^2 - 4ac$ remains unchanged under rotations and scalar changes of the axes), and similarities or analogies of form, which may, in turn, lead to recognizing isomorphism between different systems. It also often leads to a unification of scattered ideas into a single concept. "Convergence," as was formalized by Cauchy in 1827, is an example of knowledge reorganization. The particularities of convergence were well known and widely used prior to this time, but Cauchy's formalization reorganized and unified this knowledge into a single concept: "convergence."

Sometimes the need for structure compels us to define objects in a particular way. For example, we define $x^0 = 1$ for $x \neq 0$ in order for the familiar law of exponents to hold for nonzero bases, x. Specifically $1 = \frac{x}{x} = \frac{x^1}{x^1} = x^{1-1} = x^0$. On the other hand, 0^0 is excluded in this definition because it leads to the ill-founded statement, $0^0 = \frac{0}{0}$. The proposed definition $0^0 = 1$ is not forced by any demands of consistency with laws of exponents. That being said, mathematicians frequently adopt the convention that $0^0 = 1$ anyway, in order, for example, to make the binomial theorem and Taylor's theorem valid for zero values of a variable.

Another important aspect of the need for structure is the need to make connections—for example, the need to analogize structures, problems, and solutions to problems. H. Bass (personal communication, May 15, 2012) calls these aspects *theory building*. Our earlier discussion concerning Eigen Theory provides an example for the need to analogize structures. We have seen how students successfully analogized between two structures: from a scalar differential equation to a matrix differential equation, and from a real-valued exponential function to a matrix-valued exponential function.

As to the role of analogy in mathematical practice, this topic has been debated widely in the literature in cognitive psychology and mathematics education (see, for example, English, 1997; Simon & Hayes, 1976). For the sake of completeness, however, I briefly discuss here one example. Consider the problem, "In how many ways can 8 identical chocolate bars be distributed into three groups, where none of the bars are to be broken and each group must contain at least one bar?" A tenth-grade student solved the problem by analogizing it to what was to him a simpler problem. He began by saying something to the effect that when the eight bars are placed in a row, seven spaces (one space between two bars) are created. Each choice of two spaces among the seven will determine one possible distribution. For example, if the second and seventh spaces are selected, the corresponding distribution is as follows: one group consists of two bars, the second group of five bars, and the third group of one bar. Thus, the student reduced the original problem into a different, familiar problem—in how many ways can two objects (spaces in our case) be chosen among seven objects? The student then easily determined the answer to be $\binom{7}{2} = \frac{7!}{(7-2)!2!} = 21$.

Pedagogical considerations. As is evident throughout the history of mathematics, the rigor of a logical structure—that is, the level of scrupulousness in which a mathematical argument is examined—is not absolute, but a process of continual development. Intellectual need applies here too. It is a vital guide in determining the level of rigor suitable for a particular group of students. The question is always whether students, given their current knowledge and mathematical maturity, can see a need for an idea we intend to teach them. Often students are asked to provide justifications to claims they view as self-evident. This is particularly true for certain properties of the real numbers and geometrical objects. We observed, for example, a ninth-grade teacher, teaching algebra and geometry, who requires his students to accompany each assertion written on the left-hand side of two-column proofs by a reason on the right-hand column. Students in his geometry class were required to justify the assertion "$AB \cong AB$" by the phrase "reflexive property" and the assertion "If $\angle ABC = 30^\circ$ and $\angle CBD = 45^\circ$, then $\angle ABD = 75^\circ$" by "additive property." Similarly, students in his class were required to justify the assertion "$a + b = b + a$" by the phrase "commutative property," "$(a+b)+c = a+(b+c)$" by "associative property," and "$(-1)b = -b$" by "multiplying by -1 property." It turned out that both the teacher and his students viewed these assertion as obvious (ones that require no justification) but all felt compelled to follow rules; the students had to follow rules imposed by their teacher, and the teacher those imposed by the textbook. Thus, the task to justify was alien to the teacher and to his students, and the tasks added no understanding of logical structure or rigor.

The requirement to justify operations on real numbers in terms of basic properties such as "commutativity," "associativity," and "identity" is not exclusive to secondary school mathematics; it is also common in elementary mathematics. Here, too, the task to justify is commonly *alien* to both the teachers and students. For example, a fifth-grade teacher assigned the problem: "Use properties to find n in the following equations: (1) $55 + 8 = n + 55$, (2) $8 + (2+3) = (n+2)+3$, and (3) $17 + 0 = n$." The properties referred to in this assignment are the commutative, associative, and identity properties. Students were expected to solve the three problems by resorting to these three properties, respectively. The attention of many of these fifth graders was focused solely on the teacher's demand to use these properties rather than on the quantitative meaning of the equations. There were students who solved each of these problems directly (e.g., in Problem 1, some students first added 55 and 8 to get 63, and then looked for and found a number whose sum with 55 is 63), and then accompanied their answer by the property they guessed to be the one expected by the teacher ("commutative property," in Problem 1). From the students' point of view, the task to use the properties to find the unknown n is likely to have been intellectually alien (merely to satisfy the teacher's will) rather than intellectually intrinsic (to solve a problem they find intellectually puzzling). The teacher's justification for the task she assigned, too, was intellectually alien: "So that students will do well when tested on these properties."

Geometry is perhaps the only place in high-school mathematics where a relatively complete and rigorous mathematical structure can be necessitated. Deductive geometry can be treated in numerous ways and in different levels of rigor. Deciding what constitutes an "adequate level of rigor" is crucial, of course. Earlier works, especially the work by Van Hiele (1980), suggest that subtle concepts and axioms, such as those related to "betweenness" and "separation," must be dealt with intuitively. However, the progression from definitions and intuitive axioms to theorems and from one theorem to the next must be coherent, be logical, and exhibit a clear mathematical structure. In passing, I speculate that a program that sequences its instructional unit so that *neutral geometry* (a geometry without the parallel postulate) precedes *Euclidean geometry* (a geometry with the parallel postulate) would enhance students' understanding of the concept of logical structure.

Unfortunately, some current high-school geometry textbooks amount to empirical observations of geometric facts; they have little or nothing to do with deductive geometry (for an extended detailed review, see Harel (2009)). There is definitely a need for intuitive treatment of geometry in any textbook, especially one intended for high-school students. But the experiential geometry presented in these texts is hardly utilized to develop geometry as a deductive system. In one of the texts reviewed, most assertions appear in the form of conjectures and most of the conjectures are not proved deductively. It is difficult, if not impossible, to systematically differentiate which of the conjectures are postulates and which are theorems. It is difficult to learn from these texts what a mathematical definition is or to distinguish between a necessary condition and sufficient condition. Another text presents the entire mathematical content through problems (an approach we support wholeheartedly) but fails to convey a clear mathematical structure. It is not clear which assertions are to be proved and which are not, and which are needed for the deductive progressions and which are not. Only one who knows the development in advance is likely to identify a deductive structure for the material from the set of problems in a given lesson. And to identify such a structure, it is necessary to go over the entire set of problems, including the homework problems. If, for example, one skips certain problems on uniqueness of perpendicularity, an important piece of the structure would be missing. Similarly, other problems appear as homework problems and yet they are needed for the development of a logical progression. Furthermore, even if all the problems are assigned and solved correctly, without a guide as to how these problems, together with some problems from the lesson, form a logical structure, it is difficult, if not impossible, to organize the material within a deductive structure.

Summary

We have identified five categories of intellectual need: (1) the *need for certainty* is the need to prove, to remove doubts. One's certainty is achieved when one determines, by whatever means one deems appropriate, that an assertion is true. Truth alone, however, may not be the only need of an individual, who may also strive to explain *why* the assertion is true. (2) The *need for causality* is the need to explain—to determine a cause of a phenomenon, to understand what makes a phenomenon the way it is. This need does not refer to physical causality in some real-world situation being mathematically modeled, but to logical explanation within the mathematics itself. (3) The *need for computation* includes the need to quantify and to calculate values of quantities and relations among them by means of symbolic algebra. (4) The *need for communication* consists of two reflexive needs: *the need for formulation*—the need to transform strings of spoken language into algebraic expressions—and the *need for formalization*—the need to externalize the exact meaning of ideas and concepts and the logical justification for arguments. (5) The *need for structure* includes the need to reorganize knowledge learned into a logical structure.

As was indicated earlier, in modern mathematical practices these five needs are inextricably linked, and the reason for discussing them in different sections was merely to demonstrate the existence of each and to explicate their distinctions. The need for computation, in particular, is strongly connected to other needs. For example, the need to compute the roots of the cubic equations led to advances in exponential notation, which, in turn, has helped to abolish the psychological barrier of dealing with the third degree "by placing all the powers of the unknown on an equal footing" (Tignol, 1988, p. 38).

Collectively, these five needs are ingrained in all aspects of mathematical practice—in forming hypotheses, proving and explaining proofs, establishing common interpretations, definitions, notations, and conventions, describing mathematical ideas unambiguously, etc. They have driven the historical development of mathematics and characterized the organization and practice of the subject today. In modern mathematical practice different needs often occur concurrently. DNR-based instruction is structured, so these same needs drive student learning of specific topics, *and* by realizing the different needs that drive mathematical practice, students are likely to construct a global understanding of the epistemology of mathematics as a discipline. The notion of intellectual need is related to the Realistic Mathematics Education (Gravemeijer, 1994) dictum that students must engage in mathematical activities that are real to them, for which they see a purpose. Initially, this may mean problems arising in the "real" (nonmathematical) world, but as students progress, mathematics becomes part of their world and "self-contained" or "abstract" mathematical problems become equally real. Thus, again, what stimulates intellectual need depends on the learner at any given time.

Fundamental Characteristics of Intellectual Need

This discussion of intellectual need is unfinished without addressing its fundamental characteristics. Without these characteristics, the concept of intellectual need is devoid of instructional value and lacks sufficient epistemological basis. The decision to postpone the presentation of these characteristics to the end, after an extensive discussion of the definitions, functions, and pedagogical implications of the five categories of intellectual needs, was purely pedagogical (to first allow for the formation of a solid concept image for the concept definition of intellectual need).

Subjectivity

Intellectual needs are subjective. When we talk about intellectual need we always refer to the need of the learner, not the need of a teacher or an observer. There should be no ambiguity about the sources of intellectual need—it is a learner's conception, not a teacher's conception. And since intellectual need depends on the learner's background and knowledge, what constitutes an intellectual need for one particular population of students may not be so for another population of students. This view is rooted in the Subjectivity Premise and entailed from the very definition of intellectual need. Without it, the concept of intellectual need, as well as other central concepts of *DNR* such as ways of understanding and ways of thinking, loses its substance. In particular, the pedagogical discussions discussed previously would be devoid of instructional value should one lose sight of where intellectual needs reside.

Innateness and Cognitive Primitives

The five needs discussed here are not claimed to be exhaustive or final; additional or different categories might be found. Further, and more important, these categories are not static constructs; rather, they have developed over millennia of mathematics practice and are likely to continue to develop in the future. This historical fact leads to the hypothesis that intellectual needs are learned, not innate. If accepted, as we do, this hypothesis has two consequences. The first consequence is pedagogical. Intellectual needs cannot be taken for granted in mathematics teaching. A continual and sustained instructional effort is necessary for students' mathematical behaviors to become oriented within and driven by these needs.

The second consequence is epistemological. If intellectual needs are learned, not innate, then by the *Knowledge-Knowing Linkage Premise*, they evolve out of resolutions of problematic situations. But then one is compelled to conclude that the learning of an intellectual need A requires the occurrence of an intellectual need B, which in turn requires the occurrence of an intellectual need C and so on, ad infinitum. To resolve this puzzle, we need a second conjecture: intellectual needs have cognitive primitives, whose role is to orient us to the intellectual needs we experience when we learn mathematics. In this respect, they are like subitizing (Kaufman et al., 1949), the ability to recognize the number of briefly presented items without actually counting, whose function is to orient us to recognize numerosity as a property that can be measured (English & Halford, 1995). For example, as we have discussed earlier, the *need for communication* occurs in mathematical practice when one is compelled to express ideas in a form and syntax that is amenable to computation by means of symbolic algebra, or when one is compelled to externalize the exact intended meaning of a concept and its logical basis (as when we ensure that a concept is well defined). A cognitive primitive to this need is the act of conveying and exchanging ideas by means of a spoken language and gestures, which is a defining feature of humans.

Interdependency

Intellectual need cannot be determined independently of what satisfies it. Human's experience of disequilibrium cannot be described independently of its corresponding experience of equilibrium, and, therefore, as a form of perturbational experience, intellectual need cannot be determined independently of what satisfies it.[9] For example, to understand the nature of one's doubts about a particular assertion, it is necessary to understand what evidence would be sufficient for that person to remove these doubts. And, conversely, to understand why a person is certain about an assertion, it is necessary to understand what caused him or her to doubt the assertion before he or she became certain of its truth.

Intellectual Need Versus Affective Need

Often there is confusion between intellectual need and application. Cognitively and pedagogically, the term "application" refers to those problematic situations that aim at helping students solidify mathematical knowledge they have already learned. Intellectual need problems, on the other hand, aim at eliciting knowledge students are yet to learn. This does not mean that problems from other fields cannot serve as intellectual need problems. As we know from history, many mathematical concepts emerged from the need to solve problems in fields outside mathematics.

One's engagement in a problem can be purely affective (e.g., self-interest) or social (e.g., to cure diseases, clean the environment, develop forensic tools to achieve justice, etc.). Affective need is different from intellectual need. While intellectual need has to do with *the epistemology of a discipline*, affective need has to do with *people's desire*, *volition*, *interest*, *self-determination*, *and the like*. Affective need is the drive to initially engage in a problem and pursue its solution. As such, it is strongly linked to social and cultural values and conventions. For example, by and large, students accept the obligation to attend school to learn, an obligation rooted in the cultural values and social conventions of the society in which we live. This need may manifest itself in different but interrelated ways. First, there is the need that originates from external expectation, explicit or implicit, by authoritative figures, such as teachers, parents, and society in general. This need is particularly dominant in current teaching practices and is utilized through a complex system of rewards and punishments (e.g., grades, contests, etc.). Second, there is the need driven by causes of self-advancement, such as a desire to advance one's social stature or improve

[9] More precisely, intellectual need cannot be determined independently of what *hypothetically* satisfies it. The added qualification ("hypothetically") is needed, for otherwise this claim would mean that the experience of disequilibrium over famous unsolved problems such as the Riemann hypothesis would not be describable.

one's economic conditions. Third, there is the need that stems from a desire to advance societal causes, such as technological, political, environmental, and social justice causes. Such causes might be less global, as when one might go into medicine because a sibling has some complex medical condition. Common to these types of need is a sense of a social obligation, to an authority, to oneself within a community, or to the society in general. Affective needs thus belong to the field of motivation, which addresses conditions that activate and boost (or, alternatively, halt and inhibit) learning in general. Undoubtedly questions about the fulfillment of such conditions are of paramount importance, but these are beyond the scope of this chapter.

Local Intellectual Need Versus Global Intellectual Need

By the *Knowledge-Knowing Linkage Premise* any piece of mathematical knowledge is an outcome of a resolution to a problematic situation. These situations, however, do not usually occur haphazardly, but emerge along paths toward a resolution of a major problem. Such a problematic situation, understood as such by an individual, is referred to as a *global intellectual necessity*. A problem that emerges along the way to solve a major problem is referred to as a *local intellectual need*. This is a rough characterization, of course, since it is not uncommon that some of these intermediate problems become themselves major milestones, or global necessities. The pedagogical goal is that students develop a general image of the overall problem toward which all activities relate. I illustrate this point with two examples.

Linear algebra. A curriculum in elementary linear algebra can be developed in numerous ways. What is said here is not to advocate one way over another. Rather, the goal is to illustrate the application of global necessity in teaching elementary linear algebra. If, for example, one decides to teach this topic from a matrix theory perspective, one might start with systems of equations, both linear and nonlinear. Systems of equations, if understood by the students as quantitative constraints on a set of unknowns, constitute a need for computation—the need to determine the value of the unknowns by means of symbolic algebra. Students entering their first course in linear algebra are familiar with systems of equations and understand their importance (in solving word problems, for example). Once this need is in place—and our experience suggests that undergraduate students do realize this need—students can be brought to appreciate the importance of a special kind of systems of equations, those whose equations are linear. This can be done in different ways, for example by showing how the solution of certain nonlinear systems cannot be found accurately but can be approximated by suitable linear systems, or by showing how many application problems can be modeled by linear systems. The leading questions would constitute global need. Such questions include the following: Given a linear system, how do we solve it? Are there ways to solve linear systems systematically—algorithmically, that is? Can we determine, without necessarily solving the system, if the system has a solution? If the system is solvable, how many solutions does it have? Can the system have a finite number of solutions? If yes, what are the necessary and sufficient conditions for this to happen? When the system has infinitely many solutions (a situation students should observe early on), can all the solutions be listed? The need for formulation then is applied to translate these questions in formal terms involving central concepts, such as "linear combination," "linear independence," "basis," etc. What is crucial here is that students come to understand that any new concept is formed to advance investigation of these questions (see Harel, 1998). Once the scalar case (i.e., systems whose unknown are numbers) is completed, one can turn to systems of differential equations. As we discussed earlier, we introduced the global need for Eigen Theory through the question whether it is always the case that the solution of a linear system of differential equations with an initial condition is easily computable. Through it we necessitated fundamental concepts of the Eigen Theory, from the basic concepts of eigenvalue, eigenvector, diagonalization, and their related theorems up to the Jordan Theorem and its related Jordan Canonical Form.

Rate of change. The concept of rate of change can be necessitated around the *need to model reality*. When seeking a function to model a natural phenomenon, the data typically available consist of how the phenomenon changes. Thus, one of the main purposes of examining rates of change is to use some information about the rate to gain information about a function, a purpose which is often masked in traditional calculus courses. We (Harel, Fuller, Rabin, & Stevens (n.d.)) have designed a sequence of problems consistent with this purpose as a global necessity. We began with a set of problems on functions—in particular, problems in which the objective is to describe a physical situation (e.g., At any time, what is the population?). One of our primary goals was that students understand functions as models of reality. In these problems, attending to rate of change is necessary for determining a model. The need to determine a model, in turn, necessitates an in-depth study of rates of change—in particular, an exploration of average rate of change, which leads naturally to an intuitive notion of instantaneous rate of change. The *need for communication*—in this case the need to communicate to others a precise definition of "approaches" and "arbitrarily close"—demands the formalization of our intuitive notion (i.e., the definition of the derivative). With the definition of the derivative in hand, we prove properties of functions that follow from properties of their derivatives. Many of these properties are intuitive, but the *need for certainty* (to know that something is true) demands formal proof. Truth alone, however, is not our only aim; we desire students to know *why* something is true, and thus appeal to the *need for causality*.

Concluding Remark

In its current form, *DNR* is primarily concerned with the intellectual components, not with the motivational components, of perturbation, though its definition of *learning* incorporates intellectual needs and affective needs, as well as the ways of understanding and ways of thinking currently held by the learner. Specifically,

> *Learning* is a continuum of disequilibrium-equilibrium phases manifested by (a) *intellectual needs* and *affective needs* that instigate or result from these phases and (b) *ways of understanding* or *ways of thinking* that are utilized and newly constructed during these phases. (Harel, 2008b, p. 897)

Learning in *DNR*, thus, is driven by exposure to problematic situations that result in a learner experiencing perturbation, or disequilibrium in the Piagetian sense. The drive to resolve these perturbations has both psychological and intellectual components. The psychological components pertain to the learner's *motivation*, whereas the intellectual components pertain to *epistemology*—the structure of the knowledge domain in question, both for the learner as an individual and as the domain developed historically and is viewed by experts today.

In essence, this chapter deals with the question of how instruction can help students experience the need to construct an epistemological justification for the knowledge we intend to teach them. The basis for this question is the stipulation, rooted in the *DNR* premises, that the responsibility of curriculum developers and teachers is to intellectually necessitate the mathematical knowledge intended for students to learn. Elsewhere I formulated this stipulation as an instructional principle, called the *necessity principle*: "For students to learn the mathematics we intend to teach them, they must see a need for it, where 'need' means *intellectual need*, not *social* or *cultural need*" (Harel, 2008b, p. 900). The pedagogical considerations of the different intellectual needs are rooted in this fundamental principle. In all, this principle translates into the following four concrete instructional steps:

1. Recognize what constitutes a global intellectual need for a particular population of students, relative to a particular subject (e.g., in linear algebra such a need might be solving systems of equations).
2. Translate this need into a set of general questions formulated in terms that students can understand and appreciate.
3. Structure the subject around a sequence of problems whose solutions contribute to the investigation of these questions. These problems, in turn, serve as local necessities for the emergence of particular concepts needed to advance the investigation at hand.
4. Help students elicit the concepts from solutions to these problems.

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