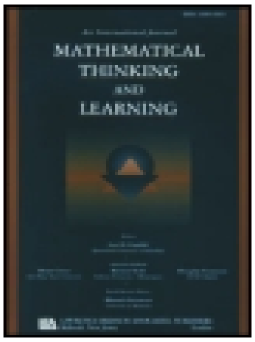

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Advanced Mathematical Thinking

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In this article we propose the following definition for *advanced mathematical thinking*: Thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses. We argue that this definition is not necessarily tied to a particular kind of educational experience; nor is it tied to a particular level of mathematics. We also give examples to illustrate the distinction we make between advanced mathematical thinking and *elementary mathematical thinking*. In particular, we discuss which kind of thinking may be required depending on the size of a mathematical problem, including problems involving infinity, and the types of models that are available.

Over the past two decades, the study of "advanced mathematical thinking" has attracted increased interest, even though there is little agreement on what is meant by "advanced mathematical thinking." A broad definition of advanced mathematical thinking (AMT) might include thinking that seems to be "advanced" for one's age or grade level. By this definition, very young students who offer insightful comments or work that seems to be beyond the ability of most children their age could be said to be employing AMT. For example, the often-repeated story about young Carl Friederich Gauss, who surprised his teacher when he quickly summed the first

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100 positive integers by inventing an algorithm, would certainly represent "advanced" thinking for a child his age. At the other extreme, one might more narrowly define advanced mathematical thinking as being associated only with cutting-edge research in mathematics.

Many mathematics educators have used the phrase "advanced mathematical thinking" as a characterization that would most often describe certain kinds of student thinking in collegiate level mathematics or some of the mathematical thinking at the professional level of mathematics. In our definition we focus on the phenomenon that seems to first occur during a mathematics student's experience in undergraduate mathematics when he or she first begins to deal with abstract concepts and deductive proof. At this time students often recognize that many of the thinking skills that contributed to their success in calculus courses no longer work in courses such as introductory real analysis or abstract algebra.

The purpose of this article is to define advanced mathematical thinking in a way that loosely links AMT to this transitional period in a mathematics student's education. We illustrate our definition with examples of mathematical situations, contrasting what we think of as advanced mathematical thinking with more elementary mathematical thinking (EMT).

EARLIER DEFINITIONS OF ADVANCED MATHEMATICAL THINKING

In the following statement, Robert and Schwarzenberger (1991) described learning in advanced mathematics courses as different from learning in elementary mathematics courses:

> There is a quantitative change [from elementary mathematics to advanced mathematics]: more concepts, less time, the need for greater powers of reflection, greater abstraction, fewer meaningful problems, more emphasis on proof, greater need for versatile learning, greater need for personal control over learning. The confusion caused by new definitions coincides with the need for more abstract deductive thought. Taken together these quantitative changes engender a qualitative change, that characterizes the transition to advanced mathematical thinking. (p. 133)

It is unclear in their statement how much of the difference they noted is due to pedagogical and curricular issues that are traditionally associated with college-level teaching rather than to the mathematics itself. Furthermore, to some degree, some of the changes described above occur throughout one's learning experience in mathematics. For instance, the transition in middle school from arithmetic to algebra could be said to involve the need for greater powers of reflection, greater abstraction, and fewer meaningful problems.

Tall (1992) linked his notion of advanced mathematical thinking to formal mathematics. He characterized AMT as consisting of "two important components: precise mathematical definition (including the statement of axioms in axiomatic theories) and logical deductions of theorems based upon them." He went on to say,

> The move to more advanced mathematical thinking involves a difficult transition, from a position where concepts have an intuitive basis founded on experience to one where they are specified by formal definitions and their properties reconstructed through logical deductions. (Tall, 1992, p. 495)

We propose a definition that directly addresses the "thinking." Although it is true that advanced mathematics students and professional mathematicians work with concepts that are "specified by formal definitions and their properties reconstructed through logical deduction" (Tall, 1992, p. 495), the thinking that is required is not always what we define as advanced mathematical thinking. Edwards (1997, 1999) found that students could often successfully (and perhaps superficially) reason from formal definitions if these definitions did not conflict with their previous mathematical understandings and experiences. The difficulties for some students began when they were reasoning about concepts that were not physically accessible to them and their intuitions and the definitions conflicted.

Edwards (1997) gave an example of a student called Stephanie, who at one point was asked to complete some tasks using the following definition for infinite decimal.

Definition. Let $c_1, c_2, \ldots, c_n, \ldots$ be an infinite sequence of integers with $0 \le c_i \le 9$. The number, $\sup\{ .c_1 c_2 \ldots c_n ; n = 1,2,3,\ldots\}$ is denoted by $.c_1 c_2 \ldots c_n\ldots$ and is called an infinite decimal.

First, Stephanie addressed the possible equivalence of .333… and 1/3. She successfully argued that 1/3 equals sup {.3, .33, .333, ….} because it is the smallest possible upper bound for the sequence; thus, 1/3 must equal .333…. But when faced with the possible equivalence of .999… and 1, she said that this one was not possible. She explained that it was clear that 1/3 and .333… are the same number because when you divide 3 into 1 you get .333…, but "if you divide 1 into 1 you don't get .999…!" (Edwards, 1997, p. 20). It is clear that Stephanie was not really reasoning from the definition, but instead from her earlier experience of changing a fractional representation to a decimal representation by dividing the numerator by the denominator. Her intuitions collided with her understanding of the mathematical definitions and her intuitions prevailed.

DEFINING ADVANCED MATHEMATICAL THINKING

Our definition of AMT shares some characteristics of earlier definitions. We propose to define advanced mathematical thinking as follows: Advanced mathematical thinking is thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses.[1] Our definition has close connections with what Saunders Mac Lane (1981) wrote concerning the source of mathematics. Mac Lane stated that mathematics comes from the application of logic and rigor to human activities. For example, he said that counting is the source of arithmetic and number theory; and estimating is the source of probability, measure theory, and statistics. Mac Lane, however, did not differentiate between activities that can be accomplished with the use of the five senses and those that cannot. Thus, he did not set up the categorization of elementary mathematical thinking and advanced mathematical thinking.

Although the impetus for creating our definition was prompted by our observations of students undergoing the transition from calculus to more advanced mathematics courses, it is not necessarily tied to a particular kind of educational experience; nor is it tied to a particular educational level of mathematics. We do not claim that there is some point in one's educational experience (or in one's mathematical career) when elementary mathematical thinking ends and advanced mathematical thinking begins. In our view, AMT resides on a continuum of mathematical thought that seems to transcend, but does not ignore, the procedural experiences or intuitions of elementary mathematical thinking.

Both conditions—the deductive and rigorous reasoning and the inaccessibility of the mathematical notions to our senses—are necessary in order for thinking to be considered AMT. While the limit in real analysis is a notion whose full understanding requires deductive and rigorous reasoning about an inaccessible process, we could not claim that all thinking about limits is advanced. In calculus courses students are often required to evaluate limits, however this activity does not necessarily require advanced mathematical thinking because it can often be reduced to an automated symbolic manipulation. On the other hand, deductive and rigorous reasoning is required of students in high school geometry, however the ideas, or the representations of the ideas, about which the students are asked to reason in the context of high school geometry are usually accessible to them through examples in the physical world. Thus the important characteristic of our definition of advanced mathematical thinking is the combination of the need for deductive and rigorous reasoning about concepts and the fact that these concepts are not accessible to the individual through the five senses.

It is important to note that we recognize that the use of the word *advanced* is problematic. We may seem to imply that thinking that is not AMT is somehow inferior, and that is not our intent. However, to remain consistent with earlier literature we continue to use the phrase.

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[1]We refer to the conventional five senses of hearing, sight, touch, taste, and smell. Some are probably more applicable than others.

In what follows we hope to make clear the distinction we see between elementary mathematical thinking and advanced mathematical thinking and to ask and answer questions that are related to our definition.

SIZE OF THE PROBLEM

There are many mathematical questions that when posed one way might only require EMT, but when posed in a different way would require AMT. Consider, for example, the following question.

> Given an m by n rectangle, where m and n are integer inches, tiled by one-inch squares, how many tiles must the diagonal of the rectangle intersect?

A student can solve this problem by carefully constructing models of rectangles of different measure and looking for a pattern in the answers for each example. The student may realize that the examples divide into two equivalence classes, depending upon whether or not m and n are relatively prime. He or she can then create a general formula for solving this problem for any integer values of m and n. By our definition this activity would involve EMT because actual paper and pencil models can be created to assist the problem solver.

Suppose, however, that the question were written as follows.

> Given a rectangular parallelepiped measured in inches as a by b by c, where a, b, and c are integers, and sectioned by one-inch cubes, how many cubes must be intersected by the diagonal line that connects the front lower left corner to the back upper right corner?

Thinking about this new problem is more complex, however because it is still possible to create a model and to reason from particular examples, the thinking required to solve this problem would still not be considered AMT by our definition. Could we say that thinking about this problem without a model would be AMT? Possibly, but a more definitive situation would be to extend the problem into multidimensional space where one would have to reason from examples and models in lower dimensional spaces. Any models in the higher dimensions would be purely mental and not accessible to the five senses.

In some sense it seems that the "size" of the problem in this example is an important factor in determining whether or not a successful solution requires AMT. Once this problem goes beyond three dimensions it is no longer directly accessible to our senses and although one may reason from the lower dimensional situations, one must eventually reason in higher dimensions to attack the problem. We propose then that size could be a determining factor in deciding what is accessible to our five senses and thus which situation would require EMT and which would require AMT. For this problem, three dimensions or fewer would require only EMT, but greater than three dimensions might require AMT.

We cannot say, however, that reasoning about any figure that has three or fewer dimensions would never require AMT. In *What is Mathematics?* (1941) Courant and Robbins have depicted a simple, closed curve that is so twisty, it is impossible using only one's senses to tell whether a given point is inside or outside the region bounded by the curve. Using AMT, however, one could understand that following a straight line from the given point one need merely count the intersections with the curve in moving to a region that is clearly outside.

PROBLEMS INVOLVING INFINITY

Problems involving infinity—either infinite processes or objects with infinite cardinality—also contain this aspect of size. We suggest that considering the countably infinite natural numbers may not require AMT, but comparing $|\mathbf{N}|$ with $|2\mathbf{N}|$ may require AMT. The ability to understand that there is a one-to-one relationship between **N** and **2N** is probably not available through experience in the physical world. In their article, Brown, McDonald, and Weller (2009) proposed a theoretical description of how a student might construct an understanding of infinite iterative processes such as might define the explicit one-to-one correspondence between these two sets. While the construction is based on finite iterative processes of producing one object from a prior object, at some point the individual must realize that this iterative process holds for all of the infinite objects. This requires the individual to see the infinite process as being complete, even though there is no final step, and no final object is obtained. We argue that this, in and of itself, may not require AMT. It is the ability to transcend this process and reason accurately about the entirety of the process and what is obtained from the completion of the process that exemplifies AMT. We illustrate this in the following.

In their article, Brown et al. (2004) examined students' reasoning about whether or not the following equality holds.

$$\cup_{k=1}^{\infty} P(\{1,2,\ldots,k\}) = P(N),$$

where P denotes the power set of the given set and N is the set of natural numbers. A student they called Emily could conceive of the infinite union as being complete, even though she obviously could not access it explicitly through physical models. However, she was still unable to rigorously prove whether the equality held or not. In particular, with respect to the infinite union, Emily said that "if you infinitely union sets, eventually you've got to union the infinite set I would think…. When do you ever reach infinity?" Ultimately she was not able to show that the equality does not hold, unlike the student described in the following section.

The next step in the construction proposed by Brown et al. (2004) clearly requires AMT. Trying to find the parallel to finite processes, the individual feels pressure to ask a question such as "What does one have when this process is complete?" This requires the individual to see the completed process as a totality, that is, all infinite objects are present in the individual's mind at a moment in time. This attempted action of asking such a question may lead the individual to encapsulate the process into a cognitive object,[2] thus creating what Brown et al. call a "transcendent object" that he or she places in relation to the completed infinite iterative sequence. This transcendent object is outside of the process and is not produced by the process; it is the product of the encapsulation of the process. The student referred to as Tobi in Brown et al. understood the nature of all of the steps of the infinite union, and understood the nature of the transcendent object. Ultimately, she was able to state that the left hand side "is the union of an infinite number of finite sets." We believe she was engaged in AMT because she not only conceived of the infinite process as complete, something clearly not accessible to her through any physical model, but she also rigorously reasoned about the nature of the sets present in the transcendent object of the process and how this object compared with another object, namely P(N).

Likewise by our definition, deductive and rigorous reasoning about the infinite process involved in the concept of limit requires AMT, even though we often see students talking about limits without using AMT. In dealing with limits, students often struggle with the human need to make sense of things by attempting to carry out a process that is impossible to see to the end. Students who view the concept of limit as a dynamic process (meaning a process of getting closer and closer to a limit, but not the object that is the limit) or an unreachable bound, for example, are demonstrating in this instance a failure to use AMT as they are not transcending the finite physical models available to them.

For example, a student trying to reason about

$$\lim_{x\to 1} \frac{x-1}{x^2-1}$$

might plug in x = 1.1, then x = 1.01, then x = 1.001, and so on and get a sense that the limit appears to be approaching the value ½. The student may even say that the limit is ½, but this still does not demonstrate a use of AMT. Rather an individual is engaged in AMT if he or she sees the limit as a coordinated pair of processes, the domain process of approaching 1 and the range process of approaching ½, coordinated by the function (Cottrill et al., 1996). In addition, he or she should see that

[2]Even though a student may try to perform the action on the process, he or she may not actually succeed in encapsulating the process into an object. The attempt at performing an action on a process spurs a student to encapsulation, but it may not be enough to guarantee encapsulation.

these processes can be completed even though no last step is reached and no last object is produced, see the processes as totalities, and ultimately see that the limit is the transcendent object associated with the encapsulation of the range process (i.e., the limit is exactly equal to ½ even though ½ is never actually produced by the process).

RIGOROUS AND DEDUCTIVE REASONING

We have said that to qualify as AMT, one's reasoning must be rigorous and deductive. Certainly careful deductive reasoning is also rigorous but we believe that requiring strictly deductive reasoning would be too limiting. When is reasoning rigorous enough to qualify for the label of AMT?

This leads to an interesting question. Was Isaac Newton using advanced mathematical thinking when he invented calculus? Bishop Berkeley's characterization of Newton's "ghosts of departed quantities" suggests that at least he found Newton's rigor lacking in what Berkeley saw as a seemingly casual use of infinitesimals (cf. Boyer, 1985). To be sure, the definition of limit that we use today was unavailable to Newton (although he came very close to it[3]) and the language Newton used to describe his ideas was awkward even for his time, but his ideas were developed in a rigorous way. He developed a rigorous method of analyzing infinite series and in the invention of calculus applied this method of analysis to the age-old problems of finding rates of change and areas under curves.

Newton's Binomial Method linked the operations on finite polynomial expressions to those on infinite series. Although Pascal had already developed a method for finding the coefficients in a binomial expansion of $(a + b)^n$, where n equals a positive integer (using what is now known as Pascal's Triangle), it was Newton who developed a generalized formula applicable to binomial expansions where n is any positive or negative rational number. Pascal's method could deal with binomials such as $(a + b)^3$, that when expanded terminates after four terms. Newton's method could deal with the expansion of expressions such as $(a + b)^{-3}$ and $(a + b)^{1/3}$ for which the series expression on the right hand side of the equation never terminates. Newton was able to reason about these infinite series using a finite series model and to prove that his formulas worked. He then combined his binomial theorem with his method of *fluxions* to build differential and integral calculus. The creation of accessible models to represent seemingly inaccessible concepts plays an important role in AMT. The following example illustrates the complexity involved in using "imperfect" models.

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[3]In Newton's treatise *Philosophiae Naturalis Principia*, he writes of "Quantities, and the ratios of quantities, that in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal" (Boyer, 1985, p. 436).

THE USE OF MODELS

In mathematics when one reasons about objects that are not accessible to the five senses, it is often necessary to create models that depict one or more, but not all, of the characteristics of the desired object. There are, for instance, "imperfect" models that attempt to depict the Klein bottle in this way. The mathematician uses these models to assist in creating a mental model that becomes real to her or him. Since these mental models cannot be directly communicated to others, there are potential difficulties especially when designing models for use in the teaching of mathematics.

In a recent course taught by one of the authors, students were engaged in reasoning about familiar geometric shapes such as triangles and straight lines in Euclidean, spherical, and hyperbolic spaces. For the most part, the models that the students used to represent Euclidean and spherical space accurately depicted the represented spaces. The models for the hyperbolic plane were created by crocheting an object in which the number of stitches from one row to the next was increased by a constant ratio of 5:6 (see Henderson, 2001, pp. 49–51). The resulting objects resembled ballerinas' tutus (see Figure 1) and were necessarily "imperfect" representations of the hyperbolic plane.

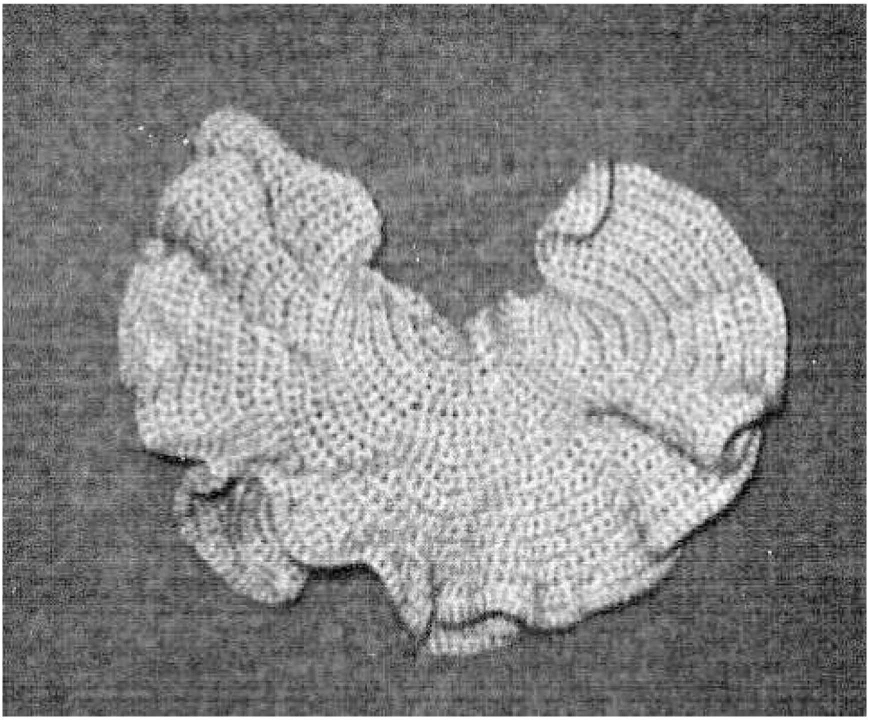

FIGURE 1 Crocheted model of hyperbolic plane.

The instructor's intent was that these models would assist students in creating better mental models from which to reason. This was no easy task, but many students experienced some level of success over the duration of the course. For example, a student called Tia, talked about how she used the model in an interview following the completion of the course. Tia said,

> I knew that the [crocheted model] wasn't exactly hyperbolic space, but I had to just use it to find out what a straight line would be and from there I thought about what hyperbolic space in my head… sort of based on what I saw in [the model]…. I tried to find the logical conclusion of what things would have to be true if these were the straight lines.

We claim Tia was employing AMT in her use of the crocheted model for hyperbolic space.

Another student, called Jim, however, was not able to see beyond the actual model. Jim seemed grounded by elementary mathematical thinking. He talked about the "wobbly" and "stretchy" nature of the hyperbolic plane and at one point he wrote on an exam that there were no "real" symmetries on the hyperbolic plane because "its shape is always changing." During class Jim worked in a group with another student (we will call him Mark) who seemed to have produced a sufficient mental model of the hyperbolic plane. Mark tried many times to share his mental model with Jim, but because it was only real for Mark and not accessible to Jim through his senses, Jim was not able to benefit from it.

CONCLUSION

Exemplary mathematical thinking may occur at any age of student and level of mathematics, but the particular notion that we describe as advanced mathematical thinking occurs only under certain conditions involving rigorous and deductive reasoning about mathematical objects that are unavailable to our five senses. This definition can help mathematics educators focus on the difficult transition period as students move from calculus to more abstract and theoretical courses in mathematics. In our view it embodies the essence of the difficulty that students experience.

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